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ABSTRACT

Equilibrium Cross-Section of Returns*

We explicitly link expected stock returns to firm characteristics such as firm size and book-to-market ratio in a dynamic general equilibrium production economy. Despite the fact that stock returns in the model are characterized by an intertemporal CAPM with the market portfolio as the only factor, size and book-to-market play separate roles in describing the cross-section of returns. These firm characteristics appear to predict stock returns because they are correlated with the true conditional market β of returns. These cross-sectional relations can subsist after one controls for a typical empirical estimate of market β . This lends support to the view that the documented ability of size and book-to-market to explain the cross-section of stock returns is not necessarily inconsistent with a single-factor conditional CAPM model.

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1 Introduction

The cross-sectional properties of stock returns have attracted considerable attention in recent empirical literature in financial economics. One of the best known studies, by Fama and French (1992), uncovers the relations between factors such as book-to-market ratio and firm size and stock returns, which appear to be inconsistent with the standard Capital Asset Pricing Model (CAPM). Despite their empirical success, these simple statistical relations have proved very hard to rationalize and their precise economic source remains a subject of debate.¹ The challenge posed by the Fama and French (1992) findings to traditional structural models has created a significant hurdle to the understanding of more complex, dynamic properties of the cross-section of stock returns.

In this work we construct a stochastic dynamic general equilibrium one-factor model in which firms differ in characteristics such as size, book value, investment and productivity among others, thus establishing an explicit economic relation between firm level characteristics and stock returns. We show that the simple structure of our model provides a parsimonious description of the firm level returns and makes it a natural benchmark for interpreting many empirical regularities.

Our findings can be summarized as follows. First, we show that our one-factor equilibrium model can still capture the ability of book-to-market and firm value to describe the cross-section of stock returns. These relations can subsist after one controls for typical empirical estimates of conditional market β . This lends support to the view that the documented ability of size and book-to-market to explain the cross-section of stock returns is *not*

¹Cochrane (1999), Campbell (2000) and Campbell, Lo and MacKinlay (1997) review the related literature. Various competing interpretations of observed empirical regularities include, among others, Berk (1995), Berk, Green and Naik (1999), Fama and French (1993, 1995, 1996), Jagannathan and Wang (1996), Kothari, Shanken, and Sloan (1995), Lakonishok, Shleifer, and Vishny (1994), Lettau and Ludvigson (1999), Liew and Vassalou (2000), Lo and MacKinlay (1988) and MacKinlay (1995).

necessarily inconsistent with a single-factor conditional CAPM model and provides a possible rationalization for the Fama and French (1992) findings. Second, we also establish a number of additional properties of the cross-section of stock returns with important implications for optimal dynamic portfolio choice. In particular, we find that cross-sectional dispersion in individual stock returns is related to the aggregate stock market volatility and business cycle conditions. In addition, we show that the size and book-to-market return premia are inherently conditional in their nature and likely counter-cyclical.

Our theoretical approach builds on the work of Berk, Green, and Naik (1999). These authors construct a two-factor partial equilibrium model based on ideas of time-varying risks to explain cross-sectional variations of stock returns associated with book-to-market and market value. They show that their calibrated model is able to capture several of the Fama and French (1992) findings. Our work differs along several important dimensions. First, ours is a single-factor model in which the conditional CAPM holds. We can then identify separate roles of size and book-to-market without appealing to multiple sources of risk. Second, the simple structure of our model allows us to illustrate the role of β mismeasurement in generating the cross-sectional relations between the Fama and French's factors and stock returns. Finally, the general equilibrium nature of our model allows us to present a self-consistent account of the business cycle properties of firm level returns.

Our work is also related to a variety of recent papers that explore the asset pricing implications of production and investment in an equilibrium setting. Examples of this line of research include Bossaerts and Green (1989), Cochrane (1991 and 1996), Jermann (1998), Kogan (2000a and 2000b), and Naik (1994). To the best of our knowledge, however, ours is the first work aiming directly at explaining the cross-sectional variations of stock returns from a structural general equilibrium perspective.

The rest of the paper is organized as follows. Section 2 describes the model economy and its competitive equilibrium and derives an explicit analytical relation between the systematic risk of stock returns and firm characteristics. Sections 3 and 4 examine the quantitative implications of our model. Section 5 concludes.

2 The Model

In this section we develop a general equilibrium model with heterogeneous firms to characterize individual returns and link them to underlying firm characteristics. There are two types of agents: firms and households. We keep the household sector very standard, summarized by a single representative household which makes the optimal consumption and portfolio allocation decisions. The heart of the model is the production sector, where a continuum firms are engaged in production of the consumption good. Each firm operates a number of individual projects of different characteristics. This firm level uncertainty is crucial to obtain a non-degenerate equilibrium cross-sectional distribution of firms, a necessary condition for our analysis in sections 3 and 4. Subsection 2.1 details the structure of the economy, while subsection 2.2 describes the equilibrium aggregate asset prices and establishes the link between systematic risk of stock returns and firm characteristics.

2.1 The Economy and the Competitive Equilibrium

Technology

Production of the consumption good (numeraire) in this economy takes place in basic productive units, which we label *projects*. Projects have three individual features: scale, productivity, and cost. They expire at a randomly chosen time, defined by a project-specific Poisson process with common arrival rate δ .

Let \mathcal{I}_t denote the set of all projects existing at time t and let i be the index of an individual project and s denote the time of creation, or vintage. We make two simplifying assumptions with respect to the scale of the project, k_{it}^s . First, the scale of a project is determined when the project is created and it remains fixed throughout the life of the project. Second, all projects of the same vintage have identical scale. Given these assumptions, and when there is no possibility of confusion, we will use only $k_i = k_{it}^s$ to denote the scale of project i created at time $s(i) \leq t$.

Project's productivity is driven by an exogenous stochastic process X_{it} , resulting in a flow of output at rate $X_{it}k_i$. Specifically, we define $X_{it} = \exp(x_t)\epsilon_{it}$, where x_t is a systematic, economy-wide productivity measure common for all projects, while ϵ_{it} is the idiosyncratic, project-specific component. Furthermore, we assume that x_t follows a linear mean-reverting process

$$dx_t = -\theta_x(x_t - \bar{x})dt + \sigma_x dB_{xt} \tag{1}$$

and ϵ_{it} is driven by a square-root process

$$d\epsilon_{it} = \kappa(1 - \epsilon_{it})dt + \sigma_\epsilon \sqrt{\epsilon_{it}} dB_{it} \tag{2}$$

where B_{xt} and B_{it} are standard Brownian motions.² Naturally we will assume that the idiosyncratic productivity shocks of all projects are independent of the economy-wide productivity shock, i.e., $dB_{xt}dB_{it} = 0$ for all i . We will place one further restriction on the

²The process in (1) is chosen to possess a stationary long-run distribution with constant instantaneous volatility, so that aggregate stock returns are not heteroscedastic by assumption. The idiosyncratic component in (2) follows a different type of process. It also has a stationary distribution, but it is heteroscedastic. Since our focus in this paper is on the systematic component of stock returns, such heteroscedasticity is not problematic. The advantage of (2) is that the conditional expectation of ϵ_{it} is an exponential function of time and a linear function of the initial value ϵ_{i0} , which facilitates computation of individual stock prices. An additional advantage of this process is that its unconditional mean is independent of κ and σ_ϵ , which simplifies the calibration.

correlation structure of the shocks below. Initial productivity of new projects is unobserved and drawn from the long-run distribution implied by (2).

While specific nature of processes (1) and (2) is convenient but not essential to our purposes, the assumption of mean-reversion in productivity shocks is very important. This assumption, however, is supported by both aggregate and cross-sectional evidence. At the aggregate level, mean-reversion implies that the growth rate of output is not exploding, which is consistent with standard findings in the economic growth literature (e.g., Kaldor (1963)). At the firm level, this assumption is required to obtain a stationary equilibrium distribution of firms. This is consistent with the cross-sectional evidence on firm birth and growth, suggesting that growth rates decline with age and size (e.g., Hall (1987) and Evans (1987)).

Finally, projects of the same vintage differ in their unit cost, measured in terms of consumption goods as e_{it} . Specifically, a potential new project i can be adopted at time s with investment cost of $e_{is}k_i$, where k_i is the scale of all new projects at time s .

Together, our assumptions about productivity and cost imply that all new projects are *ex-ante* identical in terms of expected future output, differing only in their cost. As we will see below, these assumptions guarantee that individual investment decisions can be aggregated into a stochastic growth model with adjustment costs. In addition to its computational appeal, this feature is useful in providing a realistic setting for aggregate asset pricing (e.g., Jermann (1998)).

Firms

Firms in our economy are infinitely lived. We assume that the set of firms \mathcal{F} is exogenously fixed and let f be the index of an individual firm. Each firm owns a finite number of individual projects. While we do not explicitly model entry and exit of firms, a firm can

have zero projects, thus effectively exiting the market, and a new entrant can be viewed as a firm that begins operating its first project.

We make a further assumption that the idiosyncratic productivity shocks ϵ_{it} are firm-specific. Formally, let \mathcal{I}_{ft} denote the set of projects owned by firm f at time t and let $f(i)$ denote the index of the firm owning project i . If (ongoing) projects i and j belong to the same firm, then dB_{it} and dB_{jt} are perfectly correlated, otherwise they are independent. Mathematically,

$$dB_{it}dB_{jt} = \begin{cases} dt, & j \in \mathcal{I}_{f(i),t} \\ 0, & j \notin \mathcal{I}_{f(i),t} \end{cases} \quad (3)$$

Firms are financed entirely by equity and outstanding equity of each firm is normalized to one share. We denote individual firm's stock price by V_{ft} . Stocks represent claims on the dividends, paid by firms to shareholders, and equal to the firm's output net of investment costs.³ We specify the shareholders' problem below.

While they do not control the scale or productivity of their projects, firms do make *investment* decisions by selecting which new projects to operate. Specifically, firms are presented with potential new projects over time. If a firm decides to invest in a new project, it must incur the required investment cost, which in turn entitles it to the permanent ownership of the project. These investment decisions are irreversible and investment cost cannot be recovered at a later date.⁴ If the firm decides not to invest in a project, the project disappears from the economy.

The arrival rate of new projects is independent of the individual firm's past investment decisions. Specifically, all firms have an equal probability of receiving a new project in every

³Instead of assuming that investment is financed by retaining earnings, one can make an equivalent assumption that investment is financed by new equity issues. The exact form of financing has no effect on the firm market value.

⁴Otherwise the assumption that initial productivity is unobserved would not matter.

period. This assumption guarantees that large firms do not adopt more projects than small firms, which is again consistent with the evidence on firm size and growth.⁵ Moreover, it also implies that the decision to accept or reject a project has no effect on the individual firm's future investment opportunities.

Hence, current investment decisions do not depend on the nature of a specific firm — they are determined exclusively by the cost of new projects relative to the present value of projects' cash flows. Given these assumptions, the optimal investment decision of a firm faced with project i at time s is to invest if

$$V_{it}^a = E_t \left[\int_0^\infty e^{-\lambda s} M_{t,t+s} (e^{-\delta s} k_i X_{t+s}) ds \right] \geq e_{it} k_i \quad (4)$$

where V_{it}^a is the net present value of the future stream of cash flows associated with the project and $M_{t,t+s}$ is the equilibrium stochastic discount factor between periods t and $t+s$, equal to the inter-temporal marginal rate of substitution of the representative household.⁶ Note also that we have used the fact that the idiosyncratic productivity component ϵ_{it} is independent of all other processes in the economy and that, for any new project, ϵ_{it} is drawn from the steady state distribution of process (2). Hence, $E_t [X_{t+s} \epsilon_{it+s}] = E_t [\epsilon_{it+s}] E_t [X_{t+s}] = E_t [X_{t+s}]$.

⁵All that is required is that project arrival is less than proportional to firm size. This is the simplest way of meeting this requirement and it seems the natural one to start with. Results for alternative assumptions are substantially similar and are available upon request.

⁶Our treatment of the firm's problem can be related to the Arbitrage Pricing Theory of Ross (1976). Even though cash flows of individual projects and firms are not spanned by a small number of traded assets, their idiosyncratic components are perfectly diversifiable. Therefore, the only stochastic components of cash flows and returns that are priced by the market are those associated with market-wide risk factors, which are common to all firms. In our model, x_t is the only systematic risk factor, which in equilibrium is spanned by the market portfolio. Thus, the associated risk premium is uniquely determined by absence of arbitrage. Alternatively, in the framework of a representative household, consumption-based asset pricing model, the aggregate consumption process can be used as a single systematic risk factor which is sufficient for pricing all risky assets (e.g., Breeden (1979)).

Proposition 1 (*Optimal firm investment*) *A new project i is adopted if and only if*

$$e_{it} \leq \bar{e}_t = \bar{e}(x_t)$$

where $\bar{e}(x)$ is an (endogenously determined) function of the state variable.

Proof Given the stochastic process for aggregate productivity shocks (1), it follows that the present value of project's cash flows per unit production scale equals

$$\frac{V_{it}^a}{k_i} = E_t \left[\int_0^\infty e^{-\lambda s} M_{t,t+s} (e^{-\delta s} X_{t+s}) ds \right]$$

which in turn depends only on the current state of the economy x_t . Equation (4) implies then that a new project is adopted if and only if

$$e_{it} \leq V_{it}^a(x_t)/k_i = \bar{e}_t = e(x_t)$$

■

Proposition 1 establishes a simple, but crucial, property that optimal investment decisions by firms at any time t are independent of the firms' identity and only rely on the unit cost of new projects. This then allows for the simple aggregation results below. Note that this result hinges on the convenient assumption that projects are ex-ante identical in their productivity.

The value of the firm can then be viewed as a sum of two components, the present value of output from existing projects and the present value of dividends (output net of investment) from future projects. Using the terminology from Berk et al. (1999), the former component represents the value of *assets-in-place*, V_{ft}^a , while the second can be interpreted as the value of *growth options*, V_{ft}^o . We can then compute the value of a firm's stock as a sum of these

two components

$$V_{ft} = V_{ft}^a + V_{ft}^o \quad (5)$$

where the value of assets in place can be constructed as

$$V_{ft}^a = \int_{\mathcal{I}_{ft}} V_{it}^a di \quad (6)$$

Finally, it is useful for future use to define the book value of a firm as the sum of book values of the firm's (active) individual projects and the book value of a project is defined as the associated investment cost $e_{is} k_{it}^s$.

Heterogeneity and Aggregation

To facilitate aggregation, we assume that there exists a large number (a continuum) of firms in the economy. In our informal construction we appeal to the law of large numbers, which simplifies the analysis and clarifies economic intuition, albeit at a cost of some mathematical rigor.⁷ Thus, one might view the results based on the law of large numbers as an approximation to an economy with a very large number of firms.

Let $\int_{\mathcal{I}_t} \cdot di$ and $\int_{\mathcal{F}} \cdot df$ denote aggregation operators over projects and firms respectively.

The aggregate scale of production in the economy, K_t , is

$$K_t \equiv \int_{\mathcal{I}_t} k_i di = \int_{-\infty}^t k_i^s \left(\int_{\mathcal{I}_t} \chi_{\{i: s(i) \in [\tau, \tau + d\tau]\}} di \right) ds$$

where $\chi_{\{\cdot\}}$ denotes the indicator function and $\int_{\mathcal{I}_t} \chi_{\{i: s(i) \in [\tau, \tau + d\tau]\}} di$ is the number (measure) of projects created during $[\tau, \tau + d\tau)$ that remain in existence at time t . Similarly, aggregate

⁷Feldman and Gilles (1985) formalize the law of large numbers in economies with countably infinite numbers of agents by aggregating with respect to a finitely-additive measure over the set of agents. Judd (1985) demonstrates that a measure and the corresponding law of large numbers can be meaningfully introduced for economies with a continuum of agents.

output Y_t is given by

$$\begin{aligned}
Y_t &= \int_{\mathcal{I}_t} X_{it} k_i di = \int_{-\infty}^t k_i^s \left(\int_{\mathcal{I}_t} X_{it} \chi_{\{i: s(i) \in [\tau, \tau+d\tau]\}} di \right) ds \\
&= \exp(x_t) \int_{-\infty}^t k_i^s \left(\int_{\mathcal{I}_t} \chi_{\{i: s(i) \in [\tau, \tau+d\tau]\}} \epsilon_{it} di \right) ds \\
&= \exp(x_t) \int_{-\infty}^t k_i^s \left(\int_{\mathcal{I}_t} \chi_{\{i: s(i) \in [\tau, \tau+d\tau]\}} di \right) ds = \exp(x_t) K_t
\end{aligned} \tag{7}$$

where the fourth equality follows from the law of large numbers, since by (2) random variables ϵ_{it} s are identically distributed with unit mean and are independent across a continuum of firms, with each firm owning a finite number of projects. Equation (7) is consistent with our interpretation of x_t as the aggregate productivity shock.

New potential projects are continuously arriving in the economy. To ensure balanced growth, we assume that the arrival rate of new projects is proportional to the total scale of existing projects in the economy K_t and independent of project unit cost. Formally, the arrival rate (measured by production scale) of new projects with cost less than e_t equals $ZK_t e_t$. Alternatively, $ZK_t e_t dt$ is the total scale of projects with the cost parameter less than e_t arriving between t and $t + dt$. The parameter Z governs the quality of the investment opportunity set. Given our definition of the arrival rate, the total scale of projects in the economy evolves according to

$$dK_t = -\delta K_t dt + ZK_t \bar{e}_t dt \tag{8}$$

where δ is the rate at which existing projects expire. The aggregate investment spending, I_t , is then given by

$$I_t = I(\bar{e}_t) \equiv \int_0^{\bar{e}_t} e_{it} ZK_t de_{it} = \frac{1}{2} ZK_t \bar{e}_t^2 \tag{9}$$

Aggregate dividends are defined as the aggregate output net of aggregate investment, or

$$D_t = Y_t - I_t \tag{10}$$

In addition, we define the value of the aggregate stock market V_t , which is the market value of a claim on aggregate dividends, as

$$V_t = \int_{\mathcal{F}} V_{ft} df \tag{11}$$

Finally, given (10) and (11) we can define the process for cumulative aggregate stock returns as

$$\frac{dR_t}{R_t} = \frac{dV_t + D_t dt}{V_t} \tag{12}$$

Households

There is a single consumption good in the economy, which is produced by the firms. The economy is populated by identical competitive households, who derive utility from the consumption flow C_t . The entire population can then be modeled as a single representative household. We assume that this household has standard time-separable isoelastic preferences:

$$E_0 \left[\frac{1}{1-\gamma} \int_0^\infty e^{-\lambda t} C_t^{1-\gamma} dt \right] \tag{13}$$

Households do not work and derive income from accumulated wealth only.⁸ We let W_t denote the individual wealth at time t . Financial markets in our model consist of risky stocks and an instantaneously riskless bond in zero net supply that earns a rate of interest r_t . Financial markets are perfect: there are no frictions and no constraints on short sales or borrowing.

The representative household then maximizes her expected utility of consumption (13),

⁸Since labor is not productive, this assumption is innocuous.

subject to the constraints

$$dW_t = -C_t dt + W_{bt} r_t dt + W_{st} \frac{dR_t}{R_t} \quad (14)$$

$$W_t = W_{bt} + W_{st} \quad (15)$$

$$W_t \geq 0 \quad (16)$$

where W_{bt} and W_{st} is the amount of wealth invested in the bond and stocks, respectively.⁹ The returns processes on bonds, r_t , and stocks, R_t , are taken as exogenous by households and will be determined in equilibrium. The nonnegative-wealth constraint (16) is used to rule out arbitrage opportunities, as shown in Dybvig and Huang (1989).¹⁰

The Competitive Equilibrium

With the description of the economic environment complete we are now in a position to state the definition of the competitive equilibrium.

Definition 1 (*Competitive equilibrium*) *A competitive equilibrium is summarized by stochastic processes for optimal household decisions C_t^* , W_{bt}^* , W_{st}^* , and firm investment policy \bar{e}_t^* , such that*

(a) *Optimization*

⁹We are assuming that households invest directly in the aggregate stock market portfolio. Combined with the assumption that firms' value is computed using the economy-wide stochastic discount factor to discount their dividends, this formulation is not restrictive and allowing households to invest in individual securities would lead to identical implications for equilibrium prices and policies.

¹⁰To make sure that the wealth process is well defined by (14), we assume that both the consumption policy and the portfolio policy are progressively measurable processes, satisfying standard integrability conditions:

$$\int_0^{\tau_n} C_t + \left| W_{bt} r_t + W_{st} \frac{E_t [dR_t]}{R_t} \right| dt < \infty$$

$$\int_0^{\tau_n} \left\langle W_{st} \frac{dR_t}{R_t}, W_{st} \frac{dR_t}{R_t} \right\rangle < \infty$$

for a sequence of stopping times $\tau_n \nearrow \infty$, where $\langle \cdot, \cdot \rangle_t$ denotes the quadratic variation process.

(i) Given security returns, households maximize their expected utility (13), subject to constraints (14–16);

(ii) Given the stochastic discount factor

$$M_{t,t+s} = e^{-\lambda s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma$$

firms maximize their market value (5).

(b) *Equilibrium*

(i) *Goods market clears:*

$$C_t^* = D_t = Y_t - I_t \tag{17}$$

(ii) *Stock market clears:*

$$W_{st}^* = V_t = \int_{\mathcal{F}} V_{ft} df \tag{18}$$

(iii) *Bond market clears:*

$$W_{bt}^* = 0 \tag{19}$$

■

The following proposition establishes that the optimal policies \bar{e}_t^* and C_t^* can be characterized as the solution to a system of one differential equation and one algebraic equation.

Proposition 2 (*Equilibrium allocations*) *The competitive equilibrium allocations of consumption C_t^* and investment \bar{e}_t^* can be computed by solving the equations*

$$\bar{e}^*(x) = [c^*(x)]^\gamma p(x) \tag{20}$$

and

$$c^*(x) = \exp(x) - \frac{1}{2}Z[\bar{e}^*(x)]^2 \quad (21)$$

where function $p(x)$ satisfies

$$\frac{\exp(x)}{[c^*(x)]^\gamma} = [\lambda + (1 - \gamma)\delta + \gamma Z\bar{e}^*(x)]p(x) + \theta_x(x - \bar{x})p'(x) - \frac{1}{2}\sigma_x^2 p''(x) \quad (22)$$

and

$$\bar{e}_t^* = \bar{e}^*(x_t)$$

$$C_t^* \equiv c^*(x_t)K_t$$

Proof See Appendix A.1. ■

2.2 Asset Prices

With the optimal allocations computed we can now easily characterize the asset prices in the economy, including the risk-free interest rate and both the aggregate and firm-level stock prices.

Aggregate Prices

The following proposition summarizes the results for the equilibrium values of the risk-free rate and the aggregate stock market value.

Proposition 3 (*Equilibrium asset prices*) *The instantaneous risk-free interest rate is determined by:*

$$r_t = -\frac{E_t[dM_{t,t+dt} - 1]}{dt} = \lambda + \gamma[Z\bar{e}^*(x_t) - \delta] + \gamma\frac{[A(c^*(x_t))]}{c^*(x_t)} - \frac{1}{2}\gamma(\gamma + 1)\sigma_x^2\left[\frac{c^*(x_t)'}{c^*(x_t)}\right]^2 \quad (23)$$

where $A(c(x))$ satisfies

$$A(c(x)) = -\theta_x (x - \bar{x}) c'(x) + \frac{1}{2} \sigma_x^2 c''(x)$$

The aggregate stock market value, V_t , can then be computed as

$$V_t = E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma C_{t+s}^* ds \right] = (c_t^*)^\gamma \psi(x_t) K_t \quad (24)$$

where function $\psi(x)$ satisfies the differential equation

$$\lambda \psi(x) = [c^*(x)]^{1-\gamma} + (1-\gamma) [Z\bar{e}^*(x) - \delta] \psi(x) - \theta_x (x - \bar{x}) \psi'(x) + \frac{1}{2} \sigma_x^2 \psi''(x)$$

Proof See Appendix A.3. ■

While the exact conditions are somewhat technical, the intuition behind them is quite simple. As we would expect, the instantaneous risk-free interest rate is completely determined by the equilibrium consumption process of the representative household, and its implied properties for the stochastic discount factor. Also, the aggregate stock market value represents a claim on the the future stream of aggregate dividends paid out by firms. In equilibrium, however, these must equal the consumption of the representative household.

In addition to the definition above, value of the stock market can also be viewed as a sum of two components, the present value of output from existing projects and the present value of dividends (output net of investment) from all future projects. The value of assets-in-place is given by

$$V_t^a = E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma \left(\int_{\mathcal{I}_t} X_{it+s} e^{-\delta s} k_i di \right) ds \right] \quad (25)$$

Using arguments similar to (7), we can restate this as

$$V_t^a = K_t E_t \left[\int_0^\infty e^{-(\lambda+\delta)s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma \exp(x_{t+s}) ds \right] = K_t (c_t^*)^\gamma p(x_t) \quad (26)$$

where $p(x_t)$ is defined by (22) above. By definition then, the value of aggregate growth options can be constructed as

$$V_t^o = V_t - V_t^a \quad (27)$$

Firm-Level Stock Prices

Valuation of individual stocks is straightforward once the aggregate market value is computed. First, note that as we have seen above, the value of a firm's stock is the sum of assets-in-place and growth options, where the value of assets-in-place is the sum of present values of output from all projects currently owned by the firm. The value of an individual project i is given by the following Proposition.

Proposition 4 (*Project valuation*) *The present value of output of a project i is given by*

$$V_{it}^a = E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma (e^{-\delta s} k_i X_{it+s}) ds \right] = \frac{k_i}{K_t} \left[\tilde{V}_t^a (\epsilon_{it} - 1) + V_t^a \right] \quad (28)$$

where \tilde{V}_t^a is defined as

$$\tilde{V}_t^a \equiv K_t E_t \left[\int_0^\infty e^{-(\lambda+\delta+\kappa)s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma \exp(x_{t+s}) ds \right]$$

Proof See Appendix A.4. ■

Given the result in Proposition 4, the value of assets in place for the firm, V_{ft}^a , can be constructed as

$$V_{ft}^a = \int_{\mathcal{I}_{ft}} \frac{k_i}{K_t} \left[\tilde{V}_t^a (\epsilon_{it} - 1) + V_t^a \right] di \quad (29)$$

Now since future projects are distributed randomly across the firms with equal probabilities, all firms will derive the same value from growth options. Clearly then this implies that the value of growth options of each firm, V_{ft}^o , equals

$$V_{ft}^o = \frac{1}{\int_{\mathcal{F}} 1 df} V_t^o \quad (30)$$

We can then join these two components to obtain the total value of the firm, V_{ft} , as

$$V_{ft} = \int_{\mathcal{I}_{ft}} \frac{k_i}{K_t} \left[\tilde{V}_t^a (\epsilon_{it} - 1) + V_t^a \right] di + \frac{1}{\int_{\mathcal{F}} 1 df} V_t^o \quad (31)$$

By relating individual firm value to market aggregates, the decomposition (31) is extremely useful as it implies that the instantaneous market betas of individual stock returns can also be expressed as a weighted average of market β s of three *economy-wide* variables, V_t^a , \tilde{V}_t^a , and V_t^o . Proposition 5 formally establishes this property.

Proposition 5 (Market betas of individual stocks) *Firm market β s are described by*

$$\beta_{ft} = \tilde{\beta}_t^a + \frac{V_{ft}^o}{V_{ft}} \left(\beta_t^o - \tilde{\beta}_t^a \right) + \frac{K_{ft}}{V_{ft}} \left(\frac{K_t}{V_t^a} \right)^{-1} \left(\beta_t^a - \tilde{\beta}_t^a \right) \quad (32)$$

where

$$K_{ft} = \int_{\mathcal{I}_{ft}} k_i di$$

and

$$\beta_t^a = \frac{\partial \log(V_t^a) / \partial x}{\partial \log(V_t) / \partial x}, \quad \tilde{\beta}_t^a = \frac{\partial \log(\tilde{V}_t^a) / \partial x}{\partial \log(V_t) / \partial x}, \quad \beta_t^o = \frac{\partial \log(V_t^o) / \partial x}{\partial \log(V_t) / \partial x} \quad (33)$$

Proof Since the market beta of a portfolio of assets is a value-weighted average of betas of its individual components, the expression for the value of the firm (31) implies that

$$\begin{aligned}\beta_{ft} &= \left(1 - \frac{V_{ft}^o}{V_{ft}}\right) \beta_{ft}^a + \frac{V_{ft}^o}{V_{ft}} \beta_t^o \\ &= \left(1 - \frac{V_{ft}^o}{V_{ft}}\right) \left((1 - \pi_{ft}) \tilde{\beta}_t^a + \pi_{ft} \beta_t^a\right) + \frac{V_{ft}^o}{V_{ft}} \beta_t^o\end{aligned}$$

where

$$\pi_{ft} = \frac{K_{ft}}{V_{ft}^a} \left(\frac{K_t}{V_t^a}\right)^{-1}$$

Simple manipulation then yields (32).

■

Stock Returns and Firm Characteristics

Proposition 5 is extremely important. It shows that the weights on the “aggregate” betas, β_t^a , $\tilde{\beta}_t^a$, and β_t^o , depend on economy-wide variables like K_t/V_t^a , and V_t^o , but also, and more importantly on firm-specific characteristics such as the size, or value, of the firm, V_{ft} , and the ratio of the firm’s production scale to its market value, K_{ft}/V_{ft} .

The second term in (32) creates a relation between size and β , as the weight on the beta of growth options, β_t^o , depends on the value of the firm’s growth options relative to its total market value. Firms with small production scale derive most of their value from growth options and their betas are close to β_t^o . Since all firms in our economy have identical growth options, the cross-sectional dispersion of betas due to the loading on β_t^o is captured entirely by the size variable V_{ft} . Large firms, on the other hand, derive a larger proportion of their value from assets in place, therefore their betas are close to a weighted average of β_t^a and $\tilde{\beta}_t^a$.

The last term in (32) also shows that part of the cross-sectional dispersion of market

betas is explained by the firm-specific ratio of the scale of production to the market value, K_{ft}/V_{ft} , captured empirically to certain extent by the firm's book-to-market ratio.¹¹ To see the intuition behind this result consider two firms, A and B , with the same market value. Assume that firm A has larger scale of production but lower productivity than B . As a result, the two stocks would differ in their systematic risk due to the differences in the distribution of cash flows from the firms' existing projects. By assumption, such a difference is not reflected in the firms' market value, but it would be captured by the ratio K_{ft}/V_{ft} . Thus, while firm size captures the component of firm's systematic risk attributable to its growth options, the book-to-market ratio serves as a proxy for risk of existing projects.

Note that in this model the cross-sectional distribution of expected returns is determined entirely by the distribution of market β s, since returns on the aggregate stock market are perfectly correlated with the consumption process of the representative household (and hence the stochastic discount factor, e.g., Breeden (1979)). Thus, if conditional market β s were measured with perfect precision, no other variable would contain additional information about the cross-section of returns.

However, equation (32) implies that if for any reason market β s were mismeasured (e.g. because the market portfolio is not correctly specified), then firm-specific variables like firm size and book-to-market ratios could appear to predict the cross-sectional distribution of expected stock returns simply because they are related to true conditional β s. In section 4 we generate an example within our artificial economy of how mismeasurement of β s can lead to a significant role of firm characteristics as predictors of returns.

¹¹The ratio K_{ft}/V_{ft} can also be approximated by other accounting variables, e.g., by the earnings-to-price ratio.

3 Aggregate Stock Returns

In this section we evaluate our model’s ability to reproduce key qualitative and quantitative features of empirical data. While it is not the objective of this paper, it seems natural to ensure that the model is reasonably consistent with the well documented aggregate findings before examining its cross-sectional implications. Thus, our methodology follows the approach of Kydland and Prescott (1982) and Long and Plosser (1983). First, we calibrate the model parameters using the unconditional moments of aggregate stock returns and the moments of the aggregate consumption process. We then provide evidence on other aggregate-level properties of the model regarding the predictability of aggregate stock returns by the book-to-market ratio documented by Pontiff and Schall (1998).

3.1 Calibration

We first calibrate the aggregate-level preference and technology parameters. The values of γ , λ , δ , \bar{x} , and Z are chosen to match approximately the unconditional moments of the key aggregate variables. Table 1 reports the parameter values used in simulation and Table 2 compares the moments of some key aggregate variables in the model with corresponding empirical estimates. For completeness, we report two sets of moments from the model: population moments and sample moments. Population moments are estimated by simulating a 300,000-month time series; the sample moments are computed based on 200 simulations, each containing 70 years worth of monthly data.¹² In addition to point estimates and standard errors, we also report 95% confidence intervals based on empirical distribution functions from 200 simulations. Population moments are close to their empirical counterparts and almost all the moments of historical series are within the 95% confidence intervals in

¹²The 70-year sample length is comparable to that of CRSP, which is the historical data set used in generating the two (Data) columns in Table 2.

the (Sample) columns.

Our model is able to capture the historical level of the equity premium, while maintaining plausible values for the first two moments of the risk-free rate. These results are due to the combination of sufficiently high risk aversion ($\gamma = 15$) of the representative household and a small amount of predictability in the consumption process (e.g., Kandel and Stambaugh (1991)).¹³ Based on these results, we conclude that our model provides a satisfactory fit of the aggregate data.

To further illustrate the properties of our model, we plot some key economic variables against the state variable x in Figure 1. Panel A shows that the optimal investment policy, \bar{e}^* , increases with x . In equilibrium, \bar{e}^* equals the present value of cash flows from a new project of unit size, V^a/K , which is increasing in productivity parameter x . Similarly, the market value per unit scale of a typical project, V/K , is increasing in x , as shown in Panel B. According to Panel C, the value of assets-in-place as a fraction of the total stock market value decreases slightly with x . Most of the time, assets-in-place account for 75–80% of the stock market value in the model. Finally, Panel D compares the instantaneous stock market betas, β^a and β^o . The beta of growth options is higher than that of assets in place.

3.2 Quantitative Results

We now examine some additional quantitative implications of the model for the relationship between aggregate returns and other aggregate variables. Table 3 Panel A reports the means, standard deviations, and 1- to 5-year autocorrelations of the dividend yield and book-to-market ratio. We estimate these statistics by repeatedly simulating 70 years of monthly data, a sample size similar to that used in Pontiff and Schall (1998). The Data

¹³Note that we are not arguing that this is the precise mechanism behind the observed equity premium and other aggregate-level properties of asset prices. The only objective of this analysis is to verify that our cross-sectional results are not undermined by unreasonable aggregate-level properties of the model.

rows report the mean and standard deviation of the book-to-market ratio to be 0.668 and 0.23 respectively, the values taken from Pontiff and Schall, Table 1 Panel A. Our model produces similar values of 0.584 and 0.19. The autocorrelations of the book-to-market ratio are decreasing with the horizon, matching the pattern observed in the data. However, the ratio is more persistent in the model compared to the data, as indicated by higher magnitude of autocorrelations. The model also reproduces the decreasing pattern of autocorrelations of the dividend yield data. While the standard deviation of dividend yield is close to the empirical value, the average level exceeds the number reported by Pontiff and Schall (1998). Panel B in Table 3 examines the performance of the book-to-market ratio as a predictor of stock market returns. The slope in the regression of monthly value-weighted market returns on one-period lagged book-to-market ratios based on the model is 1.75%. The empirical value of 3.02% is within the 95% confidence interval around the simulation-based estimate. The adjusted R^2 s are also comparable. The same analysis at annual frequency produces similar results.

It is also important to note that, in the model, instantaneous stock market returns are perfectly correlated with consumption growth and the stochastic discount factor. As a result, asset returns are characterized by a single-factor intertemporal CAPM. To determine how closely monthly stock returns satisfy the ICAPM with the market portfolio being the only factor, we regress market returns on the contemporaneous realization of the stochastic discount factor, given by $(C_{t+\Delta t}/C_t)^{-\gamma}e^{-\lambda\Delta t}$. As expected, the regression shows that 96% of the variation in market return can be explained by variation in the stochastic discount factor. The unconditional correlation between the stochastic discount factor and the market return is -0.98 and the conditional correlation between the two is, effectively, -1 . Thus, even at the monthly frequency, a single-factor ICAPM is, theoretically, highly accurate.

In this respect our environment differs crucially from Berk, Green, and Naik (1999). By construction then, stock returns in their model cannot be described using market returns as a single risk factor, allowing variables other than market β s to play an independent role in predicting stock returns.

4 The Cross-Section of Stock Returns

This section establishes our key quantitative results. After outlining our numerical procedure, subsection 4.3 documents the ability of the model to replicate the empirical findings about the relation between firm characteristics and stock returns. It also establishes that these findings disappear after one controls for the theoretically correct measure of systematic risk. Subsection 4.4 describes the conditional, or cyclical, properties of firm level returns.

4.1 Calibration

To examine the cross sectional implications of the model we must choose the parameters of the stochastic process for firm-specific productivity shocks, κ and σ_ϵ . We restrict these values by two considerations. First, we want to be able to generate empirically plausible levels of volatility of individual stock returns, which directly affects statistical inference about the relations between returns and firm characteristics. Second, we also want the cross-sectional correlation between firm characteristics, i.e., the logarithms of firm value and book-to-market ratio, to match the empirically observed values. The value, and particularly the sign of this correlation, are critical in determining the *univariate* relations between firm characteristics and returns implied by the multivariate relation (32), due to the well-known omitted variable bias.

We can accomplish these goals by setting value of $\kappa = 0.51$ and $\sigma_\epsilon = 2.10$. These values

imply an average annualized volatility of individual stock returns of approximately 25% and a correlation between size and book-to-market variables of about -0.26 , the number reported by Fama and French (1992). Panel D of Figure 1 shows the behavior of $\tilde{\beta}^a$ implied by our choice of κ . In particular, $\tilde{\beta}^a$ is lower than the market beta of assets-in-place and is increasing in the state variable x .

According to equation (32), there exists a cross-sectional relation between the market β s of stock returns and firm characteristics. The sign of this relation depends on the aggregate-level variables $\beta_t^o - \tilde{\beta}_t^a$ and $\beta_t^a - \tilde{\beta}_t^a$ in (32). Under the calibrated parameter values, the long-run average values of $\beta_t^o - \tilde{\beta}_t^a$ and $\beta_t^a - \tilde{\beta}_t^a$ are 0.67 and 0.21 respectively.

These numbers suggest then a negative relation between market β s and firm size and a positive one between β s and book-to-market. Since size and book-to-market are negatively correlated in our model, coefficients in univariate regressions of returns on these variables should have the same sign as partial regression coefficients in a joint regression, i.e., returns should be negatively related to size and positively related to book-to-market. To further evaluate the quantitative significance of these effects, we repeatedly simulate a panel data set of stock returns based on our model and apply commonly used empirical procedures on the simulated panel.

We follow the empirical procedures used by Fama and French (1992). First, we present some descriptive statistics of the simulated panel in Tables 4 and 5, providing an informal summary of the relations between returns, size, and book-to-market. Our main results are presented in Tables 7, 8, and 9, where we detail the cross-sectional relations between stock returns and firm characteristics.

4.2 Simulation and Estimation

In our simulations, the artificial panel consists of 360 months of observations for 2,000 firms. This panel size is comparable to that in Fama and French (1992), who used an average of 2,267 firms for 318 months. We also adhere to Fama and French’s timing convention in that we match the accounting variables at the end of the calendar year $t-1$ with returns from July of year t to June of year $t+1$. Moreover, we use the value of the firm’s equity at the end of calendar year $t-1$ to compute its book-to-market ratios for year $t-1$, and we use its market capitalization for June of year t as a measure of its size.¹⁴ Further details of our simulation procedure are summarized in Appendix B.

Some of our tests use estimates of market β s of stock returns, which are obtained using the empirical procedure of Fama and French (1992).¹⁵ Their procedure consists of two steps. First, *pre-ranking* β s for each firm at each time period are estimated based on previous 60 monthly returns. Second, for each month stocks are sorted into ten portfolios by market value. Within each size portfolio, stocks are sorted again into ten more portfolios by their pre-ranking β s. The *post-ranking* β s of each of these 100 portfolios are then calculated using the full sample. All portfolios are formed using equal weights and all β s are calculated by summing the slopes of a regression of portfolio returns on market returns in the current and prior months. In each month, we then allocate the portfolio β s to each of the stocks within the portfolio. To highlight the fact that these post-ranking β s are estimated, we will refer to them as Fama and French- β s.

Following Fama and French (1992), we form portfolios at the end of June each year and the equal-weighted returns are calculated for the next 12 months. In each of these sorts,

¹⁴In this aspect our simulation procedure differs from that of Berk et al. (1999), since they use a straightforward and intuitive timing convention (one-period-lag values of explanatory variables), which does not however agree with the definitions in Fama and French (1992).

¹⁵For details of the beta estimation procedure, we refer readers to Fama and French (1992).

we form 12 portfolios. The middle 8 portfolios correspond to the middle 8 deciles of the corresponding characteristics, with 4 extreme portfolios (1A, 1B, 10A, and 10B) splitting the bottom and top deciles in half. We repeat the entire simulation 100 times and average the results of the sorting procedure across the simulations. In tables 4, 5 and 6, Panel A is taken from Fama and French (1992) and Panel B is computed based on the simulated panels.

4.3 Size and Book-to-Market Effects

Tables 4 and 5 report post-ranking average returns for portfolios formed by a one-dimensional sort of stocks on firm size and book-to-market. When portfolios are formed on firm value (Table 4), the simulated panel exhibits a negative relation between size and average returns, similar to the one observed empirically.¹⁶ Table 5 presents average returns for portfolios formed based on ranked values of book-to-market ratios. Similar to the historical data, our simulated panels on average also show a positive relation between book-to-market ratios and average returns. Thus, one-dimensional sorting procedures indicate cross-sectional relations between Fama and French factors and returns that are similar to those in the historical data.

Table 7 shows a summary of our results from the Fama-MacBeth (1973) regressions of stock returns on size, book-to-market, and conditional market β s.¹⁷ For comparison, we also report empirical findings of Fama and French (1992) and simulation results of Berk et al. (1999) in columns 2 and 3 of the same table.

Our first univariate regression shows that the logarithm of firm market value appears to contain useful information about the cross-section of stock returns in our model. The relation

¹⁶The level of average returns is higher in Panel A than in Panel B. This difference is due to the fact that we are modeling real returns in our model, while FF report the properties of nominal historical returns.

¹⁷For each simulation, we compute the slope coefficients as the time series average coefficients over the 360-month cross-sectional regressions, and the t -statistics are these averages divided by the standard deviations across the 360 months, which provide standard Fama-MacBeth (1973) tests for statistical significance of regression coefficients. We then average the results across 100 simulations. The market β s are exact conditional β s computed based on our theoretical model.

between returns and the size variable is significantly negative. The average slope coefficient as well as the corresponding t -statistic implied by the model are close to their empirical values reported by Fama and French (1992). Panel A of Figure 2 shows the histogram of realized t -statistics across simulations. The empirical value is well within the body of realizations produced by the model. Our second univariate regression confirms the importance of book-to-market ratio in explaining the cross-sectional properties stock returns. While our slope coefficient is smaller than the one obtained by Fama and French (1992), our estimate is also positive on average. Panel B of Figure 2 shows that the coefficient of book-to-market is often significant at traditional levels, however, the model is not able to produce the t -statistics as high as that reported by Fama and French (1992).

Next, we regress returns on size and book-to-market jointly. On average our coefficients have the same signs as in Fama and French (1992) and Berk et al. (1999) as returns exhibit negative dependence on size and positive dependence on book-to-market. While our average size slope and the corresponding t -statistic are close to the empirical values, the average slope on book-to-market is smaller than in Fama and French (1992). Panel C of Figure 2 illustrates the range of t -statistics in a joint regression of returns on size and book-to-market that could be obtained if the historical data were generated by our model. We present the results in the form of a scatter plot, where each point corresponds to a realization of two t -statistics obtained in a single simulation. The empirically observed t -statistic on the size variable is comparable to typical realizations produced by the model. However, the t -statistic on book-to-market is usually somewhat lower than in Fama and French (1992).

The first three regressions in Table 7 conform to the intuition derived from our theoretical relation (32) that size and book-to-market are related to systematic risks of stock returns and therefore have explanatory power in the cross-section. However, within our theoretical

framework, firm characteristics add no explanatory power to the conditional market β s of stock returns.¹⁸ To illustrate this point, we regress returns on size while controlling for market β . The fourth row of Table 7 shows that the average coefficient on size and the corresponding t -statistic are close to zero.

Fama and French (1992) find that the estimated market β s show no explanatory power when used individually or jointly with Fama and French factors. This could be because in practice returns on the market portfolio are not perfectly correlated with the stochastic discount factor and additional risk factors are necessary to describe expected returns. Such mechanism lies beyond the scope of our *single-factor* model. To reconcile our results with poor empirical performance of Fama and French- β s one must take into account the fact that so far we have been using the exact conditional β s, which are not observable in practice. Instead, β s must be estimated, which leaves room for measurement error. Potential sources of errors are, among others, the fact that the market-proxy used in estimation is not the mean-variance efficient portfolio (Roll (1977)) or the econometric methods employed in estimation do not adequately capture the *conditional* nature of the pricing model (e.g., Ferson, Kandel and Stambaugh (1987), Jagannathan and Wang (1996), Campbell and Cochrane (2000), and Lettau and Ludvigson (2000)). Our artificial economy provides an example of how significance of firm characteristics as predictors of returns can persist due to β mismeasurement.

In our simulations we use the true market portfolio. However, in the model conditional market β s are time-varying, which can potentially lead to measurement problems. To illustrate the impact of β mismeasurement, we apply Fama and French (1992) estimation

¹⁸Theoretically, market β s are sufficient statistics for instantaneous expected returns in our model. As shown in section 3, even at monthly frequency, the market portfolio is almost perfectly correlated with the stochastic discount factor.

procedure to our simulated data. First, we form 100 portfolios by sorting on size and then on pre-ranking β s. Table 6 provides evidence on the relation between β s and average returns. After stocks have been sorted by size, the second-pass β sort produces little variation in average returns. Table 8 shows results of the joint regression of returns on firm value and Fama and French- β . On average, the size variable remains negative and significant, while the average t -statistic on Fama and French- β is close to zero. The scatter plot in Panel D of Figure 2 shows that the t -statistic on Fama and French- β is usually less than 1.96, while the coefficient on size would often appear significant. In a univariate regression, the slope coefficient and the t -statistic on Fama and French- β reported in Table 8 are relatively low compared to those on the exact conditional β , as reported in Table 7.

Table 9 presents a measure of estimation noise in Fama and French- β , the average correlation matrix of the true conditional β s, Fama and French- β s, size, and book-to-market. For every simulation, we calculate the correlations between true β , Fama and French- β , book-to-market, and size every month and then report the averages of the correlation coefficients and their corresponding standard deviations across simulations. Table 9 shows that size is highly negatively correlated with the exact conditional β . The correlation between Fama and French- β and the true β is lower. Not surprisingly, size serves as a more accurate measure of systematic risk than Fama and French- β and hence outperforms it in a cross-sectional regression. Moreover, imperfect correlation between the true β and Fama and French- β in our model lowers the coefficient and the t -statistic in the univariate regression of returns on Fama and French- β s due to the errors-in-variables bias. This illustrates how mismeasurement of β can have an effect on all of the cross-sectional results, bringing out firm characteristics such as size and book-to-market as predictors of expected returns.

Sensitivity Analysis

Finally, it is interesting to take some measure of the sensitivity of our findings to choices of the key parameters, κ and σ_ϵ , governing the cross-sectional properties of stock returns. Tables 10 and 11 report the results of these experiments.

We consider two alternative combinations of parameters. First, we look at the effects of increasing the cross-sectional dispersion of stock returns to 30%, which corresponds to a value for σ_ϵ of 2.82. The results are reported in the columns labeled “High Variance” of Tables 10 and 11. Next, we study the effects of changing the persistence of the idiosyncratic productivity shocks by raising the value of κ to 0.4, while keeping the cross sectional variance of returns at 25%. The “Low Persistence” columns show the results of these simulations.

Comparison between columns 2 and 3 in Table 10 and 11 shows that the inference from the benchmark model carries, without any significant change, both to the High Variance and the Low Persistence variants of the model, as both the signs and significance of all the coefficients are preserved. Our main results appear to be quite robust with respect to perturbations of main parameter values.¹⁹

4.4 Business Cycle Properties

The theoretical characterization of stock prices and systematic risk, as given by (31) and (32), highlights the fact that the properties of the cross-section of stock prices and stock returns depend on the current state of the economy. This dependence is captured by the

¹⁹As the most drastic robustness check, we have also considered an alternative notion of equilibrium. Instead of computing the competitive equilibrium in our economy, we look for the Pareto-optimal allocation of resources and compute all prices based on the resulting consumption process of the representative household. The efficient allocation cannot be supported directly by the competitive equilibrium, but it can be sustained by an appropriately designed system of taxes and subsidies. We then reproduce all our tables under the new definition of equilibrium. All of our conclusions remain unchanged. The cross-sectional relations between systematic risk and size and book-to-market, as well as the analysis of noise in empirical β estimates are very similar to those under the original model. We interpret this exercise as an indication of the robustness of the economic mechanism behind our results.

economy-wide variables V_t^a , \tilde{V}_t^a , and V_t^o and their market β s. Thus, our model also gives rise to a number of predictions about the variation of the cross-section of stock prices and returns over the business cycle. These properties of the cross-section of stock returns may have important implications for optimal dynamic portfolio choice.

Firm Characteristics

To help understand the relation between the cross-section of firm characteristics and the business cycle, we first characterize the cross-sectional dispersion of firm market values. To this end, let $\mathbf{var}(h)$ denote the variance of the cross-sectional distribution of a firm-specific variable h . According to our characterization of firm market value (31), it follows immediately that

$$\mathbf{var}\left(\frac{V_{ft}}{V_t}\right) = \left(\frac{\tilde{V}_t^a}{V_t}\right)^2 \mathbf{var}\left(\frac{k_i}{K_t}\right) \mathbf{var}(\epsilon_{it}) + \left(\frac{V_t^a}{V_t}\right)^2 \mathbf{var}\left(\int_{\mathcal{I}_{ft}} \frac{k_i}{K_t} di\right) \quad (34)$$

The right-hand side of (34) captures the cross-sectional dispersion of relative firm size. This dispersion can be attributed to: (i) the cross-sectional variation of project-specific productivity shocks ϵ_{it} ; (ii) the cross-sectional differences in project-specific and firm-specific production scale, and (iii) economy-wide variables V_t^a/V_t and \tilde{V}_t^a/V_t .

The first two components are likely to have only minor impact on cross-sectional dispersion. The first, $\mathbf{var}(\epsilon_{it})$, should be independent of time in a stationary equilibrium, given (2). The contribution of the second source of heterogeneity, captured by $\mathbf{var}\left(\int_{\mathcal{I}_{ft}} k_i/K_t di\right)$ and $\mathbf{var}(k_i/K_t)$, is clearly path-dependent in theory, since the scale of new projects depends on the current aggregate scale of production K_t . Intuitively however this dependence is fairly low when the average life-time of individual projects is much longer than the average length of a typical business cycle.²⁰

²⁰Note that the average project life is about $1/\delta = 25$ years, given our calibration.

It falls then on the aggregate components, characterized by $V^a(x_t)/V(x_t)$ and $\tilde{V}^a(x_t)/V(x_t)$, to determine the cross-sectional variance in market value. Given the properties of our environment, it is easy to see that this implies that the cross-sectional dispersion of firm size is counter-cyclical, that is, it expands in recessions and it becomes compressed in expansions. We can see this by looking at Panel D of Figure 1. Since the market β s of V_t^a and \tilde{V}_t^a are less than one, the ratios V_t^a/V_t and \tilde{V}_t^a/V_t should be negatively related to the state variable x_t . Figure 3 confirms this finding.

To quantify this relation, we simulate our artificial economy over a 200-year period and compute the cross-sectional standard deviation of the logarithm of firm values and book-to-market ratios on a monthly basis. Since the state variable x_t is not observable empirically, we choose to capture the current state of the economy by the price-to-dividend ratio of the aggregate stock market.²¹

Figure 3 presents scatter-plots of the cross-sectional dispersion of firm characteristics against the logarithm of the aggregate price-dividend ratio. In both cases the relation is clearly negative. Note that cross-sectional dispersion is not a simple function of the state variable. This is partially due to the fact that we are using a finite number of firms and projects in our simulation, therefore our theoretical relations hold only approximately. Moreover, as suggested by the above theoretical argument, such relations are inherently history-dependent.

Stock Returns

Next we study how the cross-sectional distribution of actual stock returns depends on the state of the aggregate economy. First, we analyze the degree of dispersion of returns, $RD_t = \sqrt{\mathbf{var}(R_{ft})}$, where R_{ft} denotes monthly returns on individual stocks. We construct

²¹In the model, the unconditional correlation between x_t and $\log(V_t/D_t)$ is approximately 99.3%

a scatter-plot of RD_t versus contemporaneous values of the logarithm of the aggregate price-dividend ratio.

According to Figure 5, our model predicts a negative contemporaneous relation between return dispersion and the price-dividend ratio. This can be attributed to the counter-cyclical nature of both aggregate return volatility, as shown in Panel A of Figure 4, and of the dispersion in conditional market β , as shown in Panel B.

Since investment in our model is endogenously pro-cyclical, an increase in aggregate productivity shock leads to an increase in the scale of production as well as an increase in stock prices. On the other hand, since investment is irreversible, the scale of production cannot be easily reduced during periods of low aggregate productivity, increasing volatility of stock prices.²²

The counter-cyclical dispersion of conditional β s follows from the characterization of the systematic risk of stock returns (32) and the pattern observed in Figure 1, Panel D. During business cycle peaks, the dispersion of aggregate β s, i.e., β_t^a , $\tilde{\beta}_t^a$, and β_t^o , is relatively low, contributing to lower dispersion of firm-level market β s. This effect is then reinforced by the counter-cyclical behavior of dispersion of firm characteristics.

An interesting empirical finding by Stivers (2000) is the ability of return dispersion to forecast future aggregate return volatility, even after controlling for the lagged values of market returns. We conduct a similar experiment within our model, by simulating 1000 years of monthly stock returns and regressing absolute values of aggregate market returns on lagged values of return dispersion and market returns. As in Stivers (2000), we allow for different slope coefficients depending on the sign of lagged market returns. As shown in Table 12, both lagged market returns and return dispersion predict future conditional volatility of

²²Qualitatively, the impact of the irreversibility on conditional volatility of stock returns in our model is similar to that in Kogan (2000a, 2000b).

returns. Return dispersion retains significant explanatory power even after controlling for market returns in the regression. This is due to the fact that lagged market returns provide only a noisy proxy for the current state of the economy, and return dispersion contains independent information such as the current dispersion of market β s.

Conditional Size and Book-to-Market Effects

The fact that dispersion of returns on individual stocks in our model changes counter-cyclically suggests that the size and book-to-market effects analyzed in subsection 4.3 are also conditional in nature.

To capture this cyclical behavior of cross-sectional patterns in returns and its implications for dynamic portfolio allocation, we analyze the conditional performance of alternative size- and value-based strategies. Specifically, we simulate 1,000 years of monthly individual stock returns and then form zero-investment portfolios by taking a long position in bottom-size-decile stocks and a short position in top-size-decile stocks, as sorted by size, with monthly rebalancing. We also construct alternative portfolios by doing the opposite for book-to-market deciles. We then regress portfolio returns on the logarithm of the aggregate price-dividend ratio.

Our model predicts an average annualized value (book-to-market) premium of 1.45% and an average annualized size premium of 1.93%. Moreover, both strategies exhibit significant counter-cyclical patterns in their expected returns. In particular, we find that a 10% decline in the price-dividend ratio below its long-run mean implies approximately a 16.8% and 15.7% increase in expected returns on the size and book-to-market strategies, respectively, measured as a fraction of their long-run average returns.

5 Conclusion

This paper analyzes a general equilibrium production economy with heterogeneous firms. In the model, the cross-section of stock returns is explicitly related to firm characteristics such as size and book-to-market. Firms differ in the share of their total market value derived from their assets, as opposed to future growth opportunities, which is captured by their characteristics. Since these two components of firm value have different market risk, firm characteristics are closely related to market β .

To the best of our knowledge, our paper is the first to explain the cross-section of stock returns from a general equilibrium perspective. Our model demonstrates that size and book-to-market can explain the cross-section of stock returns because they are correlated with the true conditional β . We also provide an example of how empirically estimated β can perform poorly relative to firm characteristics due to measurement errors.

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A Proofs and Technical Results

A.1 Proof of Proposition 2

The equilibrium conditions imply that the optimal firm investment policy $\bar{e}^*(x)$ satisfies the condition

$$V_{it}^a = E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma \left(e^{-\delta s} k_i X_{t+s} \right) ds \right] = \bar{e}^*(x) k_i \quad (\text{A1})$$

where we impose that optimal consumption decisions are used in determining the stochastic discount factor in equilibrium. In words, optimality of firms' investment decisions requires that the most expensive project undertaken has a present value of cash flows equal to its cost.

Using the fact that k_i is independent of t and equation(8), we obtain that:

$$\begin{aligned} \bar{e}^*(x) k_i &= (C_t^*)^\gamma k_i E_t \left[\int_0^\infty e^{-(\lambda+\delta)s} \frac{X_{t+s}}{(C_{t+s}^*)^\gamma} ds \right] = \\ &= (C_t^*)^\gamma k_i E_t \left[\int_0^\infty e^{-(\lambda+\delta)s} \frac{X_{t+s}}{(c_{t+s}^*)^\gamma K_{t+s}^\gamma} ds \right] = \\ &= (C_t^*)^\gamma k_i E_t \left[\int_0^\infty e^{-(\lambda+\delta)s} \frac{X_{t+s}}{(c_{t+s}^*)^\gamma K_t^\gamma \exp\left(\int_0^s -\gamma\delta + \gamma Z \bar{e}_\tau^* d\tau\right)} ds \right] = \\ &= (c_t^*)^\gamma k_i p(x_t) \end{aligned}$$

or, as in equation (20)

$$\bar{e}^*(x) = (c_t^*)^\gamma p(x_t)$$

where the Feynman-Kac theorem implies then that $p(x)$ satisfies the differential equation:²³

$$[\lambda + (1 - \gamma)\delta + \gamma Z \bar{e}^*(x)] p(x) - \mathcal{A}[p(x)] - \frac{\exp(x)}{[c^*(x)]^\gamma} = 0$$

and $\mathcal{A}[p(x)]$ is the infinitesimal generator of the diffusion process x_t :

$$\mathcal{A}[p(x)] \equiv -\theta_x(x - \bar{x})p(x) + \frac{1}{2}\sigma_x^2 p''(x)$$

In addition, optimal consumption and investment policies are also related by the resource constraint (17). Using equations (7) and (9) this can be easily transformed into equation (21)

$$c^*(x) = \frac{Y_t}{K_t} - \frac{I_t}{K_t} = \exp(x) - \frac{1}{2}Z [\bar{e}^*(x)]^2$$

thus completing the proof of the Proposition.

²³See, for example, Duffie (1996) Appendix E.

A.2 Computation of Equilibrium

We solve for the equilibrium iteratively. First, we use equation (21) to eliminate $c(x)$ in (22). We then approximate the resulting differential equation for $p(x)$ with a system of linear equations upon discretizing the state space of x :

$$[\lambda + (1 - \gamma)\delta + \gamma Z\bar{e}_i]p_i = \widehat{\mathcal{A}}(p)_i + \frac{\exp(x_i)}{\left[\exp(x_i) - \frac{1}{2}Z(\bar{e}_i)^2\right]^\gamma}$$

where $\widehat{\mathcal{A}}(p)$ is the finite-difference approximation to the infinitesimal generator $\mathcal{A}(p)$. We then solve this system together with (20). We do this by using the following iterative procedure:

$$\begin{aligned} p_i^{(n+1)} &= p_i^{(n)} + \Delta t^{(n)} \left[\frac{\exp(x_i)}{\left[\exp(x_i) - \frac{1}{2}Z(\bar{e}_i^{(n)})^2\right]^\gamma} + \widehat{\mathcal{A}}(p)_i^{(n)} - [\lambda + (1 - \gamma)\delta + \gamma Z\bar{e}_i^{(n)}] p_i^{(n)} \right] \\ \bar{e}_i^{(n+1)} &= \bar{e}_i^{(n)} - \Delta t^{(n)} \left[\frac{\bar{e}_i^{(n)} - p_i^{(n)} \left[\exp(x_i) - \frac{1}{2}Z(\bar{e}_i^{(n)})^2\right]^\gamma}{1 + \gamma Z\bar{e}_i^{(n)} p_i^{(n)} \left[\exp(x_i) - \frac{1}{2}Z(\bar{e}_i^{(n)})^2\right]^{\gamma-1}} \right] \end{aligned}$$

where the step-size $\Delta t^{(n)}$ is adjusted to ensure convergence.

A.3 Proof of Proposition 3

Let $m_t = (C_t^*)^{-\gamma}$. Then $M_{t,t+s} = e^{-\lambda s} m_{t+s}/m_t$ and by Ito's Lemma,

$$\begin{aligned} M_{t,t+dt} - 1 &= \frac{\partial M}{\partial s} \Big|_{s=0} dt + \frac{\partial m_t}{m_t \partial C_t^*} dC_t^* + \frac{1}{2} \frac{\partial^2 m_t}{m_t \partial (C_t^*)^2} [dC_t^*]^2 \\ &= -\lambda m_t dt - \frac{\gamma}{C_t^*} m_t dC_t^* + \frac{1}{2} \frac{\gamma(\gamma+1)}{(C_t^*)^2} m_t [dC_t^*]^2 \end{aligned}$$

Thus,

$$E[M_{t,t+dt} - 1] = -\lambda dt - \frac{\gamma}{C_t^*} E[dC_t^*] + \frac{1}{2} \frac{\gamma(\gamma+1)}{(C_t^*)^2} [dC_t^*]^2$$

Next, since $C_t^* = K_t c^*(x_t)$, another application of Ito's Lemma yields

$$\begin{aligned} E[dC_t^*] &= c^*(x_t) dK_t + K_t E[dc^*(x_t)] = c^*(x_t)[Z\bar{e}^*(x_t) - \delta]K_t dt + K_t \mathcal{A}[c^*(x_t)] dt \\ [dC_t^*]^2 &= K_t^2 [c^*(x_t)']^2 \sigma_x^2 \end{aligned}$$

where $\mathcal{A}[c^*(x)] \equiv \mu_x c^*(x)' + \frac{1}{2} \sigma_x^2 c^*(x)''$. As a result,

$$r_t = \lambda + \gamma [Z\bar{e}^*(x_t) - \delta] + \gamma \frac{\mathcal{A}[c^*(x)]}{c^*(x_t)} - \frac{1}{2} \gamma(\gamma+1) \sigma_x^2 \left(\frac{c^*(x_t)'}{c^*(x_t)} \right)^2$$

Second, the value of the aggregate stock market, V_t , can be computed as

$$\begin{aligned} V_t &= E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma C_{t+s}^* ds \right] = (C_t^*)^\gamma E_t \left[\int_0^\infty e^{-\lambda s} (c_{t+s}^*)^{1-\gamma} K_{t+s}^{1-\gamma} ds \right] \\ &= (C_t^*)^\gamma E_t \left[\int_0^\infty e^{-\lambda s} (c_{t+s}^*)^{1-\gamma} K_t^{1-\gamma} \exp \left(\int_0^s - (1-\gamma) \delta + (1-\gamma) Z \bar{e}_\tau^* d\tau \right) ds \right] \\ &= (c_t^*)^\gamma \psi(x_t) K_t \end{aligned}$$

where, $\psi(x_t)$ is defined by

$$\psi(x_t) \equiv E_t \left[\int_0^\infty e^{-\lambda s} (c_{t+s}^*)^{1-\gamma} \exp \left(\int_0^s - (1-\gamma) \delta + (1-\gamma) Z \bar{e}_\tau^* d\tau \right) ds \right]$$

which, by Feynman-Kac theorem, satisfies the following differential equation:

$$\lambda \psi(x) = [c^*(x)]^{1-\gamma} + (1-\gamma) [Z \bar{e}^*(x) - \delta] \psi(x) - \theta_x(x - \bar{x}) \psi'(x) + \frac{1}{2} \sigma_x^2 \psi''(x)$$

A.4 Proof of Proposition 4

The present value of output from a specific project i , denoted V_{it}^a , is given by

$$\begin{aligned} V_{it}^a &= E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma \left(e^{-\delta s} k_i X_{it+s} \right) ds \right] \\ &= k_i (C_t^*)^\gamma \int_0^\infty e^{-(\lambda+\delta)s} E_t \left[\frac{X_{t+s}}{(C_{t+s}^*)^\gamma} \right] E_t [\epsilon_{it+s}] ds \end{aligned}$$

where the last equality follows from mutual independence of X_t and ϵ_{it} . The square-root process (2) has the property

$$E_t [\epsilon_{it+s}] = \epsilon_{it} e^{-\kappa s} + (1 - e^{-\kappa s})$$

which implies that

$$\begin{aligned} V_{it}^a &= k_i (C_t^*)^\gamma \int_0^\infty e^{-(\lambda+\delta)s} E_t \left[\frac{X_{t+s}}{(C_{t+s}^*)^\gamma} \right] [\epsilon_{it} e^{-\kappa s} + (1 - e^{-\kappa s})] ds \\ &= \frac{k_i}{K_t} K_t (C_t^*)^\gamma \left[\int_0^\infty e^{-(\lambda+\delta+\kappa)s} E_t \left[\frac{X_{t+s}}{(C_{t+s}^*)^\gamma} \right] (\epsilon_{it} - 1) ds + \int_0^\infty e^{-(\lambda+\delta)s} E_t \left[\frac{X_{t+s}}{(C_{t+s}^*)^\gamma} \right] ds \right] \\ &= \frac{k_i}{K_t} \left[\tilde{V}_t^a (\epsilon_{it} - 1) + V_t^a \right] \end{aligned}$$

where \tilde{V}_t^a is defined as

$$\tilde{V}_t^a \equiv K_t E_t \left[\int_0^\infty e^{-(\lambda+\delta+\kappa)s} \left(\frac{C_t^*}{C_{t+s}^*} \right)^\gamma \exp(x_{t+s}) ds \right]$$

B Computation

B.1 Discretization and Simulation

We use a finite number of firms in the numerical implementation. While the number of firms is fixed, the total number of projects in the economy is time-varying and stationary. The dynamics of k_i^t , the scale of new projects at time t , is as follows: $k_i^t = \frac{K_t}{\varphi}$ where the constant φ is defined by $\varphi = \frac{\delta N^*}{ZE[\bar{e}_t]}$,²⁴ where N^* is the long run number of projects in the economy and $f(x)$ is the unconditional (steady-state) density of x_t .

In the simulation, time increment is discrete. The unit cost of a new project are spaced out evenly over the interval $[0, \bar{e}_t]$. The investment of individual firm at time t is computed as the total amount the firm spends on its new projects at time t . The dividend paid out by a given firm during period t is defined as the difference between the cash flows generated by the firm's existing projects and its investment. Finally, the individual firm's book value is measured as the cumulative investment cost of the firm's projects that remain active at time t .

In our simulation, we first generate 200 years worth of monthly data, to allow the economy to reach steady state. After that, we repeatedly simulate a 420-month panel data set consisting of the cross-sectional variables (360 months of data constitute the main panel and 60 extra months are used for pre-ranking β estimation).

B.2 Quality of the Aggregation

We appeal to the law of large number in our theoretical analysis of the economy. Discretization of the economy introduces approximation error, the magnitude of which we evaluate by comparing the aggregate series to their exact analytical counterparts. We simulate the corresponding quantities for 10,080 firms over 420 months and record the aggregation results, the corresponding theoretical values, and the difference between the two. In all cases, the difference between these variables and their analytical counterparts is very close to zero.²⁵ We thus conclude that the quality of aggregation in our simulation is sufficiently high.

²⁴On average, projects expire at the total rate δN^* . The arrival rate of new projects is $ZE[\bar{e}_t]\varphi$. To ensure stationarity of the total number of projects in the economy, we thus set $ZE[\bar{e}_t]\varphi = \delta N^*$.

²⁵Complete results are available upon request.

Table 1 : Parameter Values Used in Simulation

The table lists the values of all model parameters used in simulation: the risk aversion coefficient (γ), the time preference parameter (λ), the rate of project expiration (δ), the long run mean of the aggregate productivity variable (\bar{X}), the quality of investment opportunities (Z), the volatility (σ_x) and the rate of mean-reversion (θ_x) of the productivity variable, the rate of mean-reversion (κ) and the volatility (σ_ϵ) of the idiosyncratic productivity component.

Parameters	γ	λ	δ	\bar{x}	Z	σ_x	θ_x	κ	σ_ϵ
Values	15	0.01	0.04	$\log(0.01)$	0.50	0.08	0.275	0.51	2.10

Table 2 : Moments of Key Aggregate Variables

This table reports unconditional means and standard deviations of consumption growth ($C_{t+1}/C_t - 1$), real interest rate (r_t), equity premium ($\log R_t - \log r_t$), and the mean of the Sharpe ratio ($E(\log R_t - \log r_t)/\sigma(\log R_t - \log r_t)$). The numbers reported in columns denoted (Data) are from Campbell, Lo, and MacKinlay (1997). The numbers reported in columns denoted (Population) are population moments. These statistics are computed based on 300,000 months of simulated data. The two columns denoted (Sample) report the finite-sample properties of the corresponding statistics. We simulate 70-year long monthly data sets, which is comparable to the sample length typically used in empirical research. Simulation is repeated 200 times and the relevant statistics are computed for every simulation. Then we report the averages across the 200 replications. The numbers in parenthesis are standard deviations across these 200 simulations and the two numbers in brackets are 2.5% and 97.5% percentiles of the resulting empirical distribution, respectively. All numbers except those in the last three rows are in percentages.

	Data Mean	Data Std	Population Mean	Population Std	Sample Mean	Sample Std
$C_{t+1}/C_t - 1$	1.72	3.28	0.85	3.22	0.84 (0.28) [0.22 1.33]	3.06 (0.26) [2.56 3.50]
r_t	1.80	3.00	1.30	4.33	1.34 (1.30) [-0.63 4.23]	3.98 (0.85) [2.55 5.73]
$\log R_t - \log r_t$	6.00	18.0	6.00	14.34	5.89 (1.32) [2.97 8.13]	15.28 (1.73) [11.80 18.58]
Sharpe Ratio	0.33		0.42		0.39 (0.11) [0.17 0.62]	

Table 3 : Book-To-Market As a Predictor of Market Returns

This table examines our model’s ability to match the empirical regularities documented by Pontiff and Schall (1998). Panel A reports means, standard deviations, and autocorrelations of dividend yield (DIV) and book-to-market ratio (B/M), both from historical data and from simulation output. The numbers in columns denoted (Data) are from last two rows in Table 1 Panel A of PS. Panel B reports the properties of the regression of value-weighted market returns, both at monthly and annual frequency, on one-period lagged book-to-market. The columns denoted (Data) are from Table 2 of PS. In both Panels, the columns denoted (Model) report the statistics from 200 simulations, each of which has the same length as that of the data set used in PS. The numbers in parenthesis are standard deviations across 200 simulations and the two numbers in brackets are 2.5th and 97.5th percentiles, respectively. All numbers, except autocorrelations and adjusted R^2 s, are in percentages.

Panel A: Means, Standard Deviations, and Autocorrelations								
	Source	mean	std	1 year	2 yrs	3 yrs	4 yrs	5 yrs
DIV	Data	4.267	1.37	0.60	0.36	0.26	0.23	0.25
	Model	6.407	0.97	0.69	0.46	0.31	0.19	0.11
		(0.321)	(0.22)	(0.08)	(0.14)	(0.17)	(0.18)	(0.18)
		[5.789 7.084]	[0.61 1.45]	[0.51 0.82]	[0.17 0.70]	[-0.05 0.61]	[-0.16 0.51]	[-0.22 0.45]
B/M	Data	0.668	0.23	0.68	0.43	0.23	0.08	0.00
	Model	0.584	0.19	0.88	0.80	0.73	0.68	0.64
		(0.052)	(0.04)	(0.03)	(0.07)	(0.09)	(0.12)	(0.13)
		[0.495 0.707]	[0.12 0.28]	[0.81 0.93]	[0.63 0.89]	[0.48 0.86]	[0.38 0.84]	[0.31 0.83]

Panel B: Regressions on Book-To-Market				
	Data	Data	Model	Model
	slope	adj.- R^2	slope	adj.- R^2
monthly	3.02	0.01	1.75	0.00
			(0.79)	(0.00)
			[0.68 3.65]	[0.00 0.01]
annual	42.18	0.16	19.88	0.04
			(10.46)	(0.04)
			[6.57 46.09]	[0.00 0.14]

Table 4 : Properties of Portfolios Formed on Size

At the end of June of each year t , 12 portfolios are formed on the basis of ranked values of size. Portfolios 2-9 cover corresponding deciles of the ranking variables. The bottom and top 2 portfolios (1A, 1B, 10A, and 10B) split the bottom and top deciles in half. The breakpoints for the size portfolios are based on ranked values of size. Panel A is from Fama and French (1992) Table II, Panel A. Panel B is constructed from the simulated panel. The average returns are the time-series averages of the monthly equal-weighted portfolio returns, in percent. $\log(V_f)$ and $\log\left(\frac{B_f}{V_f}\right)$ are the time-series averages of the monthly average values of these variables in each portfolio. β is the time-series average of the monthly portfolio post-ranking β s.

Panel A: Historical Data												
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
Return	1.64	1.16	1.29	1.24	1.25	1.29	1.17	1.07	1.10	0.95	0.88	0.90
β	1.44	1.44	1.39	1.34	1.33	1.24	1.22	1.16	1.08	1.02	0.95	0.90
$\log(V_f)$	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
$\log(B_f/V_f)$	-0.01	-0.21	-0.23	-0.26	-0.32	-0.36	-0.36	-0.44	-0.40	-0.42	-0.51	-0.65

Panel B: Simulated Panel												
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
Return	0.74	0.72	0.72	0.71	0.71	0.70	0.69	0.68	0.67	0.65	0.62	0.57
β	1.05	1.05	1.03	1.02	1.02	1.01	1.00	0.99	0.97	0.95	0.89	0.89
$\log(V_f)$	4.23	4.40	4.48	4.53	4.56	4.60	4.64	4.68	4.73	4.82	4.95	5.21
$\log(B_f/V_f)$	-0.92	-0.86	-0.84	-0.84	-0.84	-0.85	-0.86	-0.89	-0.96	-1.08	-1.24	-1.50

Table 5 : Properties of Portfolios Formed on Book-to-Market

At the end of June of each year t , 12 portfolios are formed on the basis of ranked values of book-to-market, measured by $\log\left(\frac{B_f}{V_f}\right)$. The pre-ranking β 's use 5 years of monthly returns ending in June of t . Portfolios 2-9 cover deciles of the ranking variables. The bottom and top 2 portfolios (1A, 1B, 10A, and 10B) split the bottom and top deciles in half. The breakpoints for the book-to-market portfolios are based on ranked values of book-to-market equity. Panel A is from Fama and French (1992) Table IV, Panel A. Panel B is from the simulated panel. The average returns are the time-series averages of the monthly equal-weighted portfolio returns, in percent. $\log(V_f)$ and $\log\left(\frac{B_f}{V_f}\right)$ are the time-series averages of the monthly average values of these variables in each portfolio. β is the time-series average of the monthly portfolio post-ranking β s.

Panel A: Historical Data												
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
Return	0.30	0.67	0.87	0.97	1.04	1.17	1.30	1.44	1.50	1.59	1.92	1.83
β	1.36	1.34	1.32	1.30	1.28	1.27	1.27	1.27	1.27	1.29	1.33	1.35
$\log(V_f)$	4.53	4.67	4.69	4.56	4.47	4.38	4.23	4.06	3.85	3.51	3.06	2.65
$\log(B_f/V_f)$	-2.22	-1.51	-1.09	-0.75	-0.51	-0.32	-0.14	0.03	0.21	0.42	0.66	1.02

Panel B: Simulated Panel												
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
Return	0.61	0.67	0.69	0.71	0.71	0.71	0.71	0.72	0.72	0.71	0.72	0.72
β	0.95	0.98	1.01	1.02	1.02	1.03	1.03	1.03	1.02	1.02	1.02	1.02
$\log(V_f)$	4.92	4.66	4.53	4.47	4.45	4.45	4.46	4.47	4.48	4.49	4.51	4.52
$\log(B_f/V_f)$	-1.54	-1.28	-1.15	-1.04	-0.97	-0.92	-0.87	-0.82	-0.77	-0.72	-0.66	-0.58

Table 6 : Average Returns For Portfolios Formed on Size (Down) and then β (Across)

Panel A is identical to FF (1992) Table I Panel A, in which the authors report average returns for 100 size- β portfolios using all NYSE, AMEX, and NASDAQ stocks from July 1963 to December 1990 that meet certain CRSP-COMPUSTAT data requirements. Panel B is produced using our simulated panel data set. The portfolio-sorting procedure is identical to that used in FF (1992). In particular, portfolios are formed yearly. The breakpoints for the size deciles are determined in June of year t using all the stocks in the panel. All the stocks are then allocated to the 10 size portfolios using the breakpoints. Each size decile is further subdivided into 10 β portfolios using pre-ranking β s of individual stocks, estimated with 5 years of monthly returns ending in June of year t . The equal-weighted monthly returns on the resulting 100 portfolios are then calculated for July of year t to June of year $t+1$. The pre-ranking β s are the sum of the slopes from a regression of monthly returns on the current and prior month's market returns. The average return is the time-series average of the monthly equal-weighted portfolio returns, in percent. The (ALL) column shows statistics for equal-weighted size-decile (ME) portfolios and the (ALL) row shows statistics for equal-weighted portfolios of the stocks in each β group.

Panel A: Average Monthly Returns (in Percent) from FF (1992)											
	All	Low- β	β -2	β -3	β -4	β -5	β -6	β -7	β -8	β -9	High- β
All	1.25	1.34	1.29	1.36	1.31	1.33	1.28	1.24	1.21	1.25	1.14
Small-ME	1.52	1.71	1.57	1.79	1.61	1.50	1.50	1.37	1.63	1.50	1.42
ME-2	1.29	1.25	1.42	1.36	1.39	1.65	1.61	1.37	1.31	1.34	1.11
ME-3	1.24	1.12	1.31	1.17	1.70	1.29	1.10	1.31	1.36	1.26	0.76
ME-4	1.25	1.27	1.13	1.54	1.06	1.34	1.06	1.41	1.17	1.35	0.98
ME-5	1.29	1.34	1.42	1.39	1.48	1.42	1.18	1.13	1.27	1.18	1.08
ME-6	1.17	1.08	1.53	1.27	1.15	1.20	1.21	1.18	1.04	1.07	1.02
ME-7	1.07	0.95	1.21	1.26	1.09	1.18	1.11	1.24	0.62	1.32	0.76
ME-8	1.10	1.09	1.05	1.37	1.20	1.27	0.98	1.18	1.02	1.01	0.94
ME-9	0.95	0.98	0.88	1.02	1.14	1.07	1.23	0.94	0.82	0.88	0.59
Large-ME	0.89	1.01	0.93	1.10	0.94	0.94	0.89	1.03	0.71	0.74	0.56
Panel B: Average Monthly Returns (in Percent) from Simulated Panel											
	All	Low- β	β -2	β -3	β -4	β -5	β -6	β -7	β -8	β -9	High- β
All	0.69	0.68	0.69	0.69	0.69	0.69	0.69	0.69	0.68	0.68	0.69
Small-ME	0.73	0.73	0.73	0.73	0.73	0.73	0.73	0.73	0.74	0.73	0.74
ME-2	0.72	0.72	0.73	0.72	0.72	0.70	0.73	0.72	0.72	0.72	0.71
ME-3	0.71	0.70	0.72	0.71	0.72	0.72	0.70	0.72	0.71	0.70	0.71
ME-4	0.71	0.70	0.71	0.70	0.71	0.69	0.71	0.72	0.71	0.72	0.70
ME-5	0.70	0.71	0.69	0.71	0.70	0.71	0.71	0.70	0.70	0.71	0.71
ME-6	0.70	0.69	0.68	0.70	0.71	0.70	0.70	0.71	0.71	0.68	0.67
ME-7	0.69	0.68	0.68	0.69	0.68	0.69	0.70	0.67	0.67	0.70	0.68
ME-8	0.67	0.64	0.68	0.68	0.66	0.69	0.70	0.68	0.66	0.66	0.68
ME-9	0.65	0.65	0.67	0.65	0.64	0.64	0.68	0.64	0.62	0.62	0.67
Large-ME	0.59	0.56	0.59	0.59	0.61	0.61	0.58	0.62	0.58	0.60	0.59

Table 7 : Exact Regressions

This table lists summary statistics for the coefficients and the t -statistics of Fama-MacBeth regressions using exact conditional β on the simulated panel sets. The dependent variable is the realized stock return and independent variables are market β , the logarithm of the market value ($\log(V_t)$), and the logarithm of the book-to-market ratio ($\log(B_t/V_t)$). The column denoted (FF) gives the empirical results obtained by Fama and French (1992), Table III, using the historical returns of 2,267 firms over 318 months. The column denoted (BGN) gives the results obtained by Berk et al. (1999). The column denoted (Model) reports the results from our model. The coefficients in the columns are in percentage terms. The numbers in parenthesis are their corresponding t -statistics. Both coefficients and t -statistics are averaged across 100 simulations.

	FF	BGN	Model
$\log(V_t)$	-0.15 (-2.58)	-0.035 (-0.956)	-0.139 (-2.588)
$\log[B_t/V_t]$	0.50 (5.71)	– –	0.079 (1.845)
$\log(V_t)$	-0.11 (-1.99)	-0.093 (-2.237)	-0.127 (-2.476)
$\log[B_t/V_t]$	0.35 (4.44)	0.393 (2.641)	0.043 (1.119)
β	-0.37 (-1.21)	0.642 (2.273)	1.076 (2.602)
$\log(V_t)$	-0.17 (-3.41)	0.053 (1.001)	0.038 (0.601)
β	– –	– –	0.916 (2.992)
$\log[B_t/V_t]$	– –	– –	0.010 (0.257)
β	0.15 (0.46)	0.377 (1.542)	0.932 (3.052)

Table 8 : Fama-French Regressions

This table lists summary statistics for the coefficients and the t -statistics of Fama-MacBeth regressions using exact conditional β on the simulated panel sets. The dependent variable is the realized stock return and independent variables are market β , the logarithm of the market value ($\log(V_t)$), and the logarithm of the book-to-market ratio ($\log(B_t/V_t)$). The column denoted (FF) gives the empirical results obtained by Fama and French (1992), Table III, using the historical returns of 2,267 firms over 318 months. The column denoted (BGN) gives the results obtained by Berk et al. (1999). The column denoted (Model) reports the results from our model. The coefficients in the columns are in percentage terms. The numbers in parenthesis are their corresponding t -statistics. Both coefficients and t -statistics are averaged across 100 simulations.

	FF	BGN	Model
$\log(V_t)$	-0.15 (-2.58)	-0.035 (-0.956)	-0.139 (-2.588)
$\log[B_t/V_t]$	0.50 (5.71)	– –	0.079 (1.845)
$\log(V_t)$	-0.11 (-1.99)	-0.093 (-2.237)	-0.127 (-2.476)
$\log[B_t/V_t]$	0.35 (4.44)	0.393 (2.641)	0.043 (1.119)
β	-0.37 (-1.21)	0.642 (2.273)	0.100 (0.318)
$\log(V_t)$	-0.17 (-3.41)	0.053 (1.001)	-0.126 (-2.091)
β	0.15 (0.46)	0.377 (1.542)	0.572 (2.081)

Table 9 : Cross-Sectional Correlations

We calculate the cross-sectional correlations of exact conditional β , FF- β , book-to-market, and size for every simulated panel every month and then report the average correlations across 100 simulations. The numbers in parentheses are cross-simulation standard deviations.

	True β	FF- β	$\log[B_t/V_t]$	$\log(V_t)$
True β	1	0.597 (0.031)	0.322 (0.023)	-0.764 (0.012)
FF- β		1	0.269 (0.035)	-0.761 (0.041)
$\log[B_t/V_t]$			1	-0.268 (0.019)
$\log(V_t)$				1

Table 10 : Exact Regressions — Sensitivity Analysis

This table lists summary statistics for the coefficients and the t -statistics of Fama-MacBeth regressions using exact conditional β . The dependent variable is the realized stock return. Independent variables are market β , size measured as the log market value ($\log(V_t)$), and the log of book-to-market ratio ($\log(B_t/V_t)$). The column denoted (FF) gives the empirical results obtained by Fama and French (1992), Table III, using the actual returns of 2,267 firms over 318 months. The column denoted (Benchmark) reports the regression results for the benchmark model, the same as the last column in Table 8. The column denoted (High Variance) reports the results from the model with perfect correlated shocks within each firm but with the calibrated parameter values $\kappa=0.51$ and $\sigma_\epsilon=2.82$ such that $\sigma_f=30\%$, which is higher than the benchmark case when $\sigma_f=25\%$. The column denoted (Low Persistence) reports the results from the model with perfect correlated shocks within each firm but with the calibrated parameter values $\kappa=0.40$ and that σ_f remains at the benchmark level of 25%. However, the persistence level is now lower. The regression coefficients are in percentage terms. The numbers in parenthesis are t -statistics.

	FF	Benchmark	High Variance	Low Persistence
$\log(V_t)$	-0.15 (-2.58)	-0.138 (-2.583)	-0.134 (-2.246)	-0.133 (-2.669)
$\log[B_t/V_t]$	0.50 (5.71)	0.079 (1.866)	0.084 (1.667)	0.085 (2.205)
$\log(V_t)$	-0.11 (-1.99)	-0.126 (-2.474)	-0.120 (-2.115)	-0.120 (-2.502)
$\log[B_t/V_t]$	0.35 (4.44)	0.043 (1.157)	0.040 (0.887)	0.043 (1.286)
β	-0.37 (-1.21)	1.026 (2.477)	1.000 (2.032)	0.938 (2.561)
$\log(V_t)$	-0.17 (-3.41)	0.029 (0.449)	0.027 (0.344)	0.024 (0.402)
β	—	0.892 (2.933)	0.891 (2.604)	0.831 (2.992)
$\log[B_t/V_t]$	—	0.013 (0.355)	0.010 (0.204)	0.010 (0.313)
β	0.15 (0.46)	0.913 (3.007)	0.914 (2.682)	0.846 (3.086)

Table 11 : Fama-French Regressions — Sensitivity Analysis

This table lists summary statistics for the coefficients and the t -statistics of Fama-MacBeth regressions using Estimated Portfolio β . The dependent variable is the realized stock return. Independent variables are market beta β , size measured as the log market value ($\log(V_t)$), and the log of book-to-market ratio ($\log(B_t/V_t)$). The column denoted (FF) gives the empirical results obtained by Fama and French (1992), Table III, using the actual returns of 2,267 firms over 318 months. The column denoted (Benchmark) reports the regression results for the benchmark model, the same as the last column in Table 8. The column denoted (High Variance) reports the results from the model with perfect correlated shocks within each firm but with the calibrated parameter values $\kappa=0.51$ and $\sigma_\epsilon=2.82$ such that $\sigma_f=30\%$, which is higher than the benchmark case when $\sigma_f=25\%$. The column denoted (Low Persistence) reports the results from the model with perfect correlated shocks within each firm but with the calibrated parameter values $\kappa=0.40$ and that σ_f remains at the benchmark level of 25%. However, the persistence level is now lower. The regression coefficients are in percentage terms. The numbers in parenthesis are t -statistics.

	FF	Benchmark	High Variance	Low Persistence
$\log(V_t)$	-0.15 (-2.58)	-0.138 (-2.583)	-0.134 (-2.246)	-0.133 (-2.669)
$\log[B_t/V_t]$	0.50 (5.71)	0.079 (1.866)	0.084 (1.667)	0.085 (2.205)
$\log(V_t)$	-0.11 (-1.99)	-0.126 (-2.474)	-0.120 (-2.115)	-0.120 (-2.502)
$\log[B_t/V_t]$	0.35 (4.44)	0.043 (1.157)	0.040 (0.887)	0.043 (1.286)
β	-0.37 (-1.21)	0.087 (0.273)	0.018 (0.030)	0.080 (0.269)
$\log(V_t)$	-0.17 (-3.41)	-0.126 (-2.112)	-0.131 (-1.955)	-0.123 (-2.203)
β	0.15 (0.46)	0.557 (2.031)	0.488 (1.625)	0.556 (2.162)

Table 12 : Cross-Sectional Return Dispersion As a Predictor of Market Volatility

This table illustrates the intertemporal relation between market volatility and the lagged cross-sectional return dispersion (RD). The volatility is measured by the absolute value of the market excess return. Variations of the following model are estimated:

$$|R_t^e| = a + b_1 RD_{t-1} + b_2 \mathbf{1}_{\{R_{t-1}^e < 0\}} RD_{t-1} + c_1 |R_{t-1}^e| + c_2 \mathbf{1}_{\{R_{t-1}^e < 0\}} |R_{t-1}^e| + \epsilon_t$$

where $|R_t^e|$ is the absolute value of the market excess return, RD_t is the cross-sectional standard deviation of the individual stock returns, $\mathbf{1}_{\{R_{t-1}^e < 0\}}$ is a dummy variable that equals one when the market excess return is negative and zero otherwise, and ϵ_t is the residual. All t -statistics are adjusted with respect to heteroskedasticity and autocorrelation using Newey-West procedure. For the F -test on joint restrictions, the p -values are in parentheses. Panel A is from Stivers (2000) who uses 400 firm returns from July 1962 to December 1995. Panel B is generated as the average coefficients and statistics across repetitive simulations.

Panel A: Results from Stivers (2000)							
Coefficients	b_1	b_2	c_1	c_2	Joint $b_1 = b_2 = 0$	Joint $c_1 = c_2 = 0$	$R^2(\%)$
Restricted $b_1 = b_2 = 0$			-0.11 (-2.06)	0.314 (4.37)		12.53 (0.00)	5.91
Restricted $c_1 = c_2 = 0$	0.255 (3.04)	0.229 (4.41)			20.32 (0.00)		9.24
Full Model	0.365 (3.61)	0.111 (1.40)	-0.157 (-2.94)	0.221 (1.84)	10.08 (0.000)	2.69 (0.069)	10.45
Panel B: Simulation Results							
Coefficients	b_1	b_2	c_1	c_2	Joint $b_1 = b_2 = 0$	Joint $c_1 = c_2 = 0$	$R^2(\%)$
Restricted $b_1 = b_2 = 0$			0.010 (0.151)	0.051 (0.705)		1.221 (0.404)	0.76
Restricted $c_1 = c_2 = 0$	0.969 (2.774)	0.069 (1.564)			5.820 (0.065)		3.01
Full Model	1.198 (3.09)	-0.008 (-0.138)	-0.083 (-1.203)	0.172 (1.487)	6.206 (0.038)	2.038 (0.282)	4.10

Figure 1 : Some Key Variables in Competitive Equilibrium

This Figure plots some key aggregate variables as functions of the common productivity shock $\log(X)$, the sole exogenous state variable at the aggregate level. Panel A shows \bar{e}^* or equivalently V^a/K in (26). Panel B shows the ratio of total market value to aggregate capital stock, V/K , and Panel C shows the ratio of aggregate value of assets-in-place to total market value, V^a/V . Panel D shows three aggregate level β s, β^a (solid line), $\tilde{\beta}^a$ (dashed-dotted line), and β^o (dashed line), defined in (33).

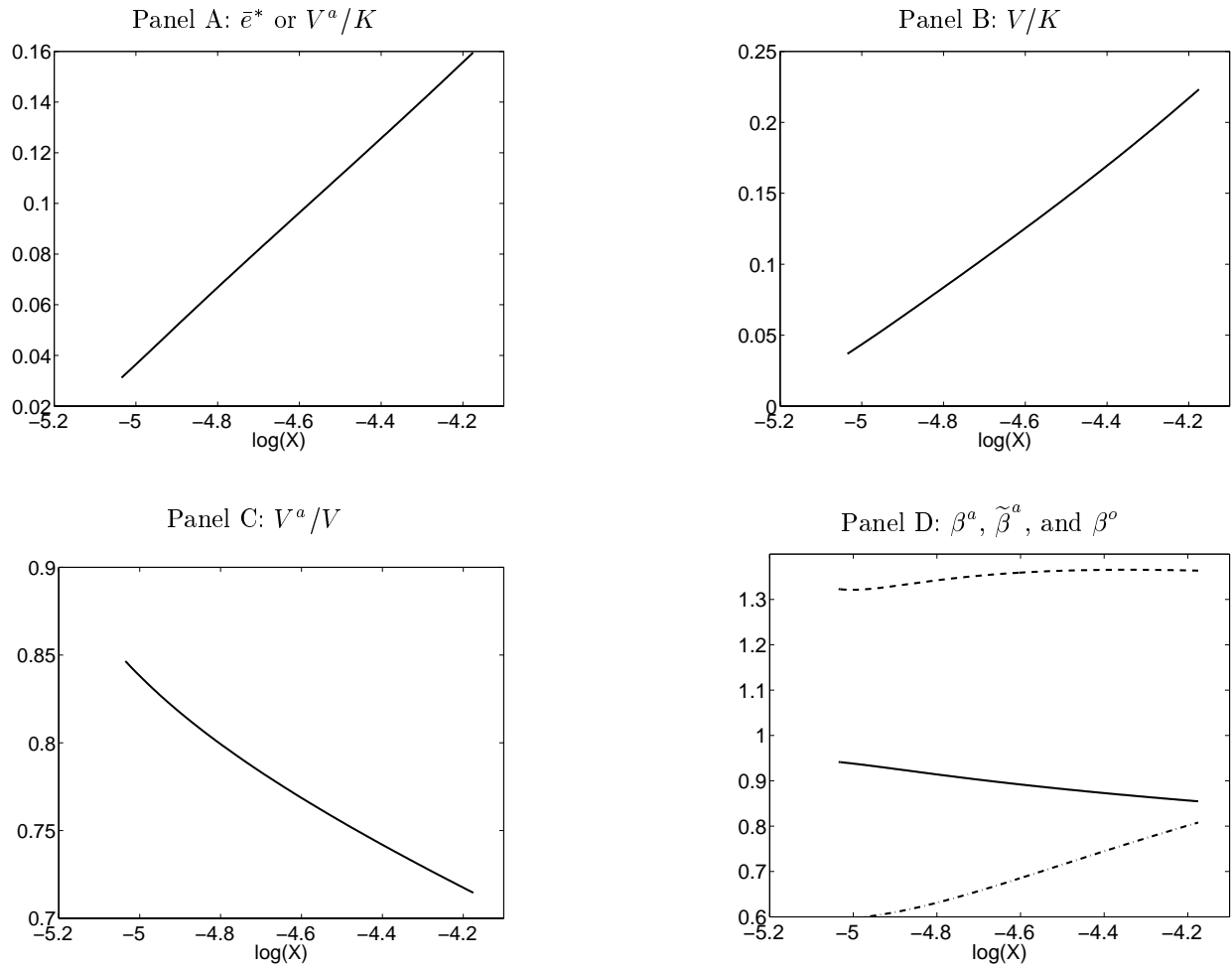
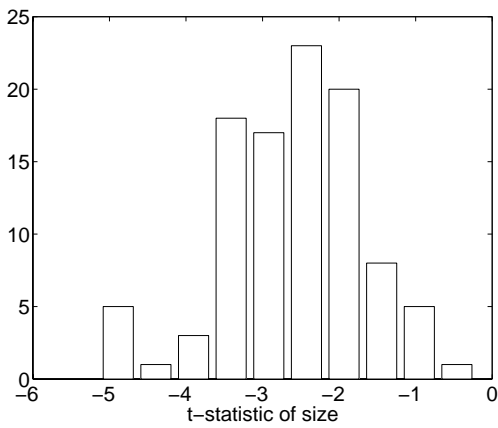


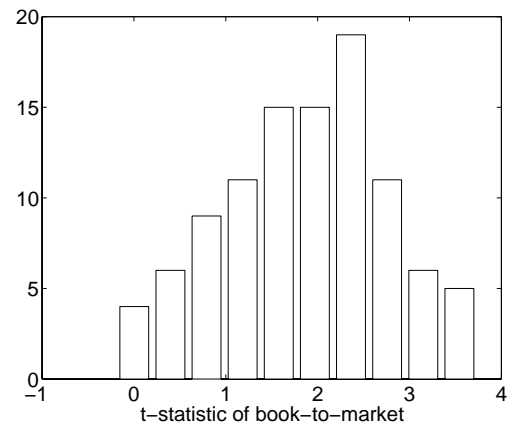
Figure 2 : Size and Book-to-Market in Cross-sectional Regressions

This Figure illustrates the size and book-to-market effects in the cross-sectional regressions of stock returns. Panel A shows the histogram of t -statistic of univariate regressions of returns on size and Panel B shows the histogram of t -statistic of univariate regressions of returns on book-to-market across 100 simulations. Panel C reports the scatter plot of t -statistics on size and book-to-market and Panel D reports the scatter plot of t -statistics on size and $FF-\beta$ in a joint regression of returns.

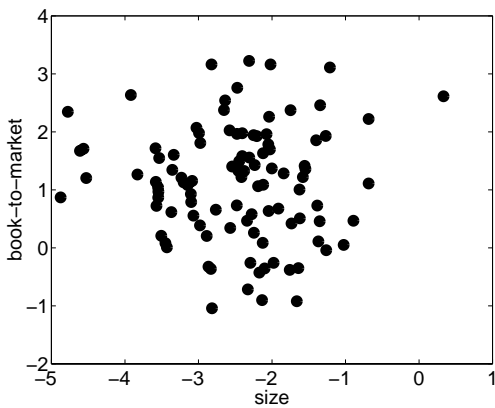
Panel A: Histogram of t -statistic on Size



Panel B: Histogram of t -statistic on Book-to-Market



Panel C: Size and Book-to-Market: t -statistics



Panel D: Size and $FF-\beta$: t -statistics

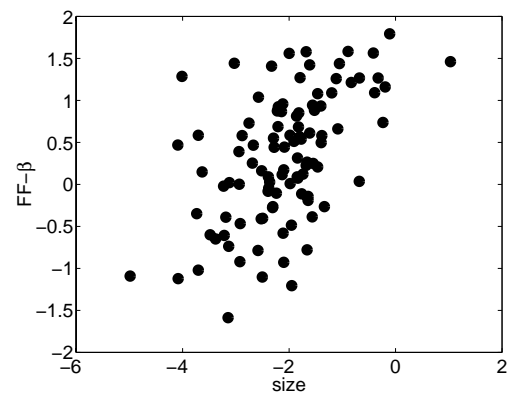


Figure 3 : Business Cycle Properties: I

This Figure illustrates the business cycle properties of some aggregate and cross-sectional variables. Panel A shows V^a/V (the solid line) and \tilde{V}^a/V (the dashed line) as functions of x . Panel B shows the relation between log price-dividend ratio, $\log(V/D)$, and x . Panel C reports the size ($\log(V_f)$) dispersion as a function of $\log(V/D)$ and Panel D shows the dispersion of book-to-market ($\log(B_f/V_f)$) as a function of $\log(V/D)$.

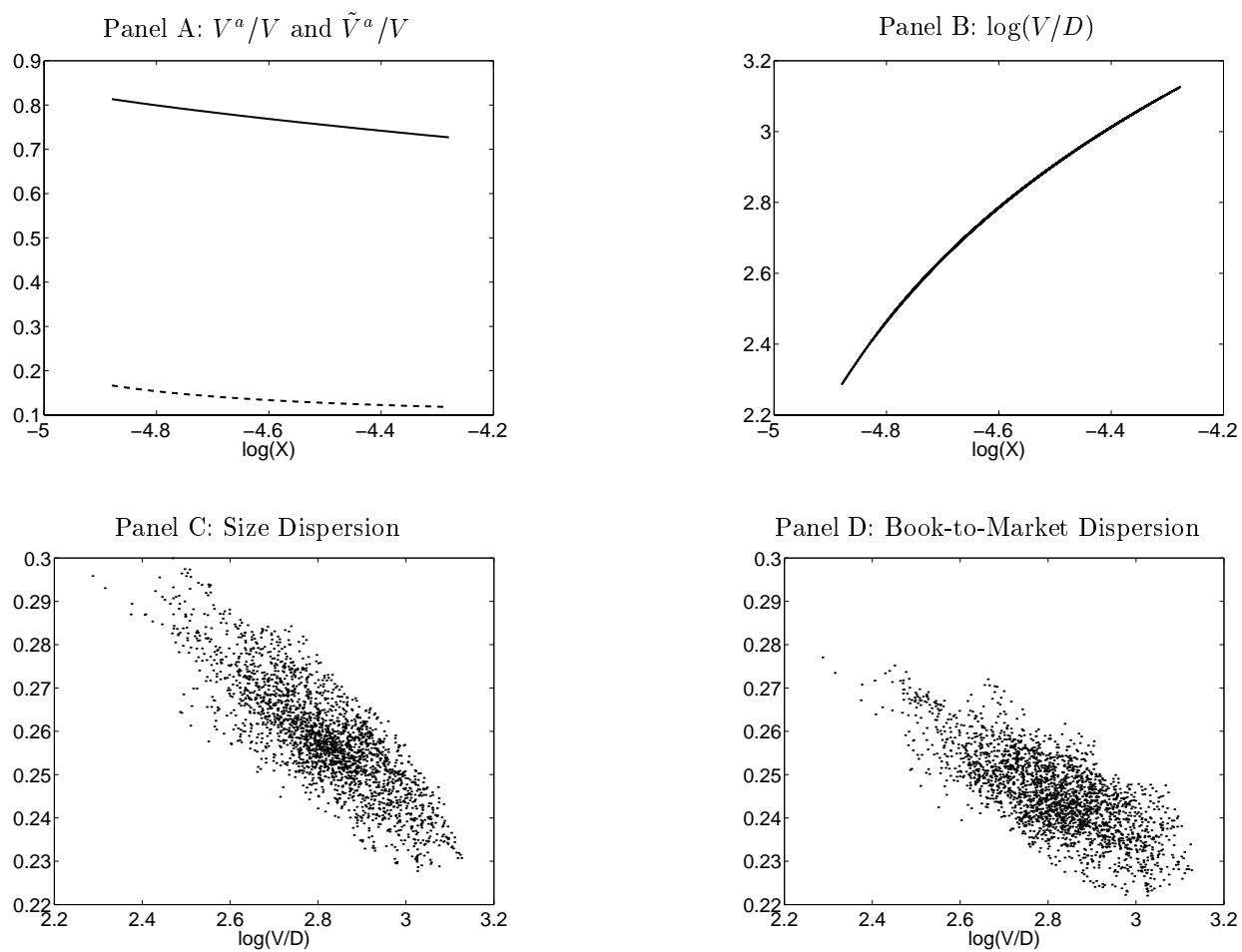


Figure 4 : Business Cycle Properties: II

This Figure illustrates the business cycle properties of some aggregate and cross-sectional variables. Panel A shows the relations between market volatility and $\log(V/D)$ and Panel B shows the relation between beta (β_f) dispersion as a function of $\log(V/D)$.

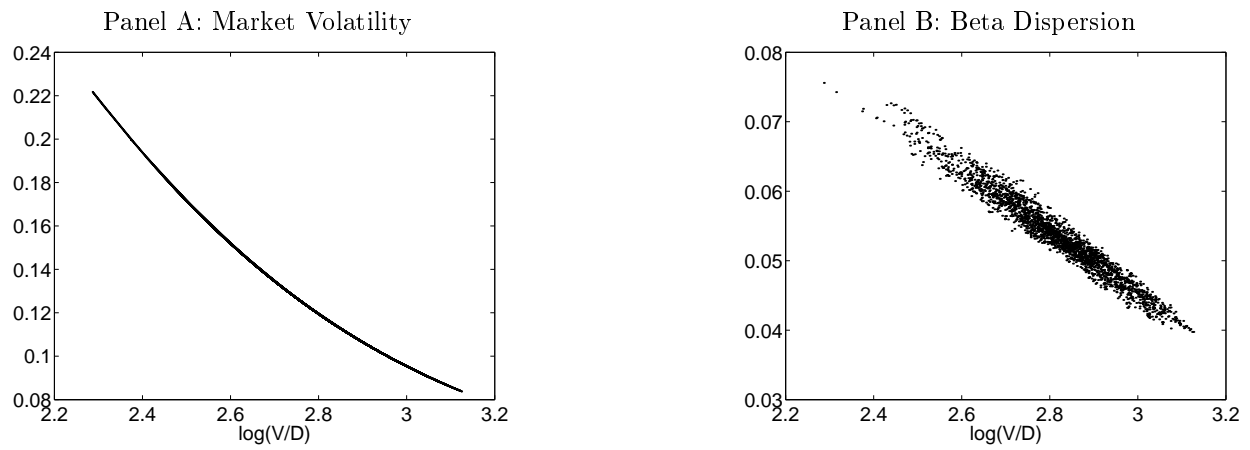


Figure 5 : Return Dispersion over Business Cycle

This Figure illustrates the business cycle variation of return dispersion by plotting it against $\log(V/D)$.

