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ABSTRACT

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Assortative Matching in a Nontransferable World*

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April 2002

Abstract

Progress in the application of matching models to environments in which the utility between matching partners is not fully transferable has been hindered by a lack of characterization results analogous to those that are known for transferable utility. We present sufficient conditions for matching to be monotone that are simple to express and easy to verify. We illustrate their application with some examples that are of independent interest.

1 Introduction

Matching models are convenient tools for studying a wide range of issues in economics, such as income distribution, contractual choice, group lending, or household behavior. When applying these models, the first task of analysis is to characterize the matching outcomes, that is to determine the attributes of matched partners. As well as being a source of testable predictions, such a characterization is usually crucial to further analysis.

Much is known about this characterization when the utility between matched partners is fully transferable. For instance, if the total payoff to the match is supermodular in the partners' attributes, then matching involves segregation (matched partners are always identical) in one-sided models and

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positive assortative matching (the type of the first partner is increasing in the type of the second) in two-sided models. If instead the payoff is submodular, there will be negative assortative matching (the type of the first partner is decreasing in the type of the second) in both one- and two-sided models. Recently, results for other forms of so-called monotone matching have also been obtained for the transferable utility case (Legros-Newman 2002).

But in many applications, the utility between partners is not fully transferable ("nontransferable," in the parlance): partners may be risk averse with limited insurance possibilities, or incentive problems may restrict the way in which the joint output can be shared. As Becker (1973) pointed out long ago, rigidities that prevent partners from costlessly dividing the gains from a match may change the matching outcome, even if the level of output is still supermodular in type.

While interest in the nontransferable case is both long-standing and lively (see for instance Farrell-Scotchmer, 1988; Rosenzweig-Stark, 1989; and more recently, Ackerberg-Botticini, forthcoming; and Chiappori-Salanié, forthcoming), there is as yet little theoretical guidance for characterizing the equilibrium matching pattern. As progress in the application of matching models to nontransferable environments is likely to be hindered by this gap, it is highly desirable to have sufficient conditions for monotone matching analogous to those that exist for transferable utility.

In this paper we present some – the first general results on this question, to our knowledge. These conditions are simple to express, intuitive to understand, and, we hope, tractable to apply. Indeed we illustrate their use with some examples that are of some independent interest.

The class of models we consider are those in which the utility possibility frontier for any pair of agents, which for the most part we take to be the primitive of the model, is a strictly decreasing function. After introducing the model and providing formal definitions of the monotone matching patterns, we review the logic of the classical transferable utility result, for a close examination of that logic leads us to propose our "generalized difference conditions," which suffice to guarantee monotone matching for any distribution of types. We illustrate their use by studying a simple model of risk sharing within households.

Since it is often easier to verify local properties of functions than global properties, we also present local conditions for monotone matching that imply our generalized difference conditions. The local condition is also intuitive and revealing and is applied to a model in which principals are matched to agents. Finally we discuss the connection of our difference conditions to supermodularity of the frontier function.

2 Preliminaries

The economy is populated by a continuum of agents who differ in type, which is taken to be a real-valued attribute such as skill, wealth, or risk attitude. In the *two-sided* model, agents are also distinguished by a binary "gender" (man-woman, firm-worker, etc.). Payoffs exceeding that obtained in autarchy, which we normalize to zero for all types, are generated only if agents of opposite gender match. In the *one-sided* model, there is no gender distinction, but positive payoffs still require a match (in neither case is there any additional gain to matching with more than one other agent). For simplicity, we will assume that the measure of agents on each side of a two-sided model is equal. The type space A is a compact subset of the real line (or such a set crossed with $\{0, 1\}$ in the two-sided case), and the number of types may be finite or infinite. Either way, we think of there being a continuum of each type.

The object of analytical interest to us is the utility possibility frontier (since in equilibrium agents will always select an allocation on this frontier) for each possible pairing of agents. This frontier will be represented by a *function* $\phi(a, b, v)$ which denotes the maximum utility generated by a type a in a match with a type b who receives utility v. We shall sometimes refer to the first argument of ϕ as "*own type*" and the third argument as "*payoff.*"

Typically, ϕ may be generated in part by choices made by the partners after they match. We assume throughout that this function is continuous and strictly decreasing in v and continuous in the types. If $\phi(a, b, v)$ can be written $f(a, b) - v$, we have transferable utility (TU); otherwise, we have nontransferable utility (NTU).

The maximum equilibrium payoff that a could ever get in a match with b is $\phi(a, b, 0)$, since b would never accept a negative payoff. By slight abuse of notation, if $v > \phi(b, a, 0)$, we will define $\phi(a, b, v) = 0$. Note that $\phi(a, b, v)$ is still strictly decreasing in $[0, \phi(b, a, 0)]$ and that $\phi(b, a, \phi(a, b, v)) = v$ for all v in this interval. In general, $\phi(a, b, v) \neq \phi(b, a, v)$.

The notation reflects two further assumptions of matching models, namely (1) that the payoff possibilities depend only on the types of the agents and not on their individual identities; and (2) the utility possibilities of the pair of agents do not depend on what other agents in the economy are doing, i.e., there are no externalities across coalitions.[1]

[1] Of course the equilibrium *payoffs* in one coalition will depend on the other coalitions, in general.

2.1 Equilibrium

We use the core as our equilibrium concept. The equilibrium specifies the way types are matched – the focus of this paper – and the payoff to each type. Specifically, an equilibrium consists of a matching correspondence $\mathfrak{M} : A \rightarrow A$ that specifies the type (s) to which each type is matched, and a payoff allocation $u^* : A \rightarrow \mathbb{R}$ specifying the equilibrium utility achieved by each type. The key property it satisfies is a stability or no-blocking condition: if u^* is the equilibrium payoff allocation, then there is no a, b and v such that $\phi(a, b, v) > u^*(a)$ and $v > u^*(b)$. Equilibria always exist under our assumptions[2].

2.2 Descriptions of Equilibrium Matching Patterns

A match is a measurable correspondence

$$\mathfrak{M}^* : A \rightrightarrows A.$$

$\mathfrak{M}^*$ is symmetric: $a \in \mathfrak{M}^*(b)$ implies $b \in \mathfrak{M}^*(a)$. Let

$$\overline{A} = \{a \in A : \exists b \in \mathfrak{M}^*(a) : a \geq b\}$$

be the set of larger partners. Obviously, $\overline{A}$ depends on $\mathfrak{M}^*$, but we suppress this dependence in the notation. Note that in the case of two-sided matching, we identify $\overline{A}$ with one of the sides.

Symmetry of $\mathfrak{M}^*$ implies that the correspondence $\mathfrak{M}$

$$\mathfrak{M} : \overline{A} \rightrightarrows A, \text{where } b \in \mathfrak{M}(a) \Longleftrightarrow b \in \mathfrak{M}^*(a) \; \& \, a \geq b,$$

completely characterizes the assignment. The coalitions generated by $\mathfrak{M}^*$ can then be written as ordered pairs $(a, b) \in \overline{A} \times \mathfrak{M}((\overline{A})$. Our descriptions of matching patterns will be in terms of the properties of the graph of $\mathfrak{M}$. Note that for a one-sided model, the graph of $\mathfrak{M}$ is the portion of the graph of $\mathfrak{M}^*$ that is on or below the 45^0 line.

When $\mathfrak{M}$ is a monotone correspondence, matching is *monotone*. We consider only a few types of monotone matching patterns in this paper. For sets $X, X' \subset \mathbb{R}$,write $X \succeq X'$ if $x \in X$ and $x' \in X'$ implies $x \geq x'$. An equilibrium satisfies *segregation* if $\mathfrak{M}(a) = \{a\}$ for all a. It satisfies *positive assortative*

[2] The facts that there is a continuum of agents and that the only coalitions that matter are of size two at most technically make the core here a special case of the f-core. See Kaneko-Wooders (1996) for definitions and existence results — with a continuum of types, they also assume that the slopes of the frontiers are uniformly bounded away from zero, a condition that is satisfied if the marginal utility of consumption at autarchy is not infinite.

matching (PAM) if for all $a, b \in \overline{A}$, $[a > b \Rightarrow \mathfrak{M}(a) \succeq \mathfrak{M}(b)]$), and *negative assortative matching* (NAM) if for all $a, b \in \overline{A}$, $[a > b \Rightarrow \mathfrak{M}(b) \succeq \mathfrak{M}(a)]$. In one sided models, an alternative way to say that there is NAM is that whenever we have types $a > b \geq c > d$, $\langle a, c \rangle, \langle b, d \rangle$ and $\langle a, b \rangle, \langle c, d \rangle$ are ruled out as possible matches (while $\langle a, d \rangle, \langle b, c \rangle$ is permitted).

Note that while segregation only occurs in one-sided models, PAM and NAM can occur in both one- and two-sided models. However, in this paper, when we refer to PAM, we shall be referring exclusively to two-sided models.

For brevity, we will say that an economy is *segregated* (*positively, negatively matched)*, if all equilibria are payoff equivalent to one with segregation (positive, negative matching).

3 Sufficient Conditions for Monotone Matching

Before proceeding, let's recall the nature of the conventional transferable utility result and why it is true, as that will provide us with guidance to the general case. In the TU case, only the total payoff $f(a, b)$ is relevant. The assumption that is often made about f is that it satisfies *increasing differences* (ID): whenever $a > b$ and $c > d$, $f(c, a) - f(d, a) \geq f(c, b) - f(d, b)$. Why does this imply positive assortative matching (segregation in the one-sided case), irrespective of the distribution of types? Usually, the argument is made by noticing that the total output among the four types is maximized (a condition of equilibrium in the TU case, but not, we should emphasize, in the case of NTU) when a matches with b and c with d: this is evident from rearranging the ID condition.

However, it is more instructive to analyze this from the equilibrium point of view. Suppose that a and b compete for the right to match with c rather than d. The increasing difference condition says that a can outbid b in this competition, since the incremental output produced if a were to switch to c exceeds that when b switches from d to c. In particular, this is true whatever the level of utility v that d might be receiving: (rewrite ID as $f(c, a) - [f(d, a) - v] \geq f(c, b) - [f(d, b) - v]$: this is literally the statement that a's willingness to pay for c, given that d is getting v, exceeds b's). The key observation then is that whatever d gets, a outbids b to match with c. Thus a situation in which a matched with d and b with c is never stable: a will be happy to offer more to c than the latter is getting with b (this assumes that b prefers to be with c than with d – else b can upset the match himself – so if b is getting v' with c, $f(c, b) - v' < f(c, b) - [f(d, b) - v]$ follows from

$v' > f(d, b) - v$). The ID result is distribution free: the type distribution will affect the payoffs, but the argument given above says that a matches with c and b with d regardless of what these might be.

Now the easy thing about the TU case is that if a outbids b at one level of v, he does so for all v. Such is not the case with NTU. Our sufficient condition will have to explicitly require that a can always outbid b, something which is necessary to make things work in the TU case as well but which is automatically taken care of by the very structure of TU. To be explicit that the condition must be satisfied for all v may seem stringent, but the nature of the result sought, namely monotone matching regardless of the distribution, is also strong. At the same time, since it includes TU as a special case, it is actually weaker!

The distinguishing feature of NTU models is that the division of the surplus between the partners can no longer be separated from the level that they generate. Switching to a higher type partner may not be attractive if it is also more costly to transfer utility to a high type, that is, if the frontier is steeper. A sufficient condition for PAM is that not only is there the usual complementarity in the production of surplus, but also there is a complementarity in the transfer of surplus – frontiers are flatter, as well as higher, for high types. This will perhaps be more apparent from the local form of our conditions.

3.1 Generalized Difference Conditions

Let $a > b$ and $c > d$ and suppose that d were to get v. Then the above reasoning would suggest that a would be able to outbid b for c if

$$\phi(c, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v). \tag{1}$$

The LHS is a's willingness to pay (in utility terms) for c rather than d, given that d receives v (a then receives $x = \phi(a, d, v)$, so c would get $\phi(c, a, x)$ if matched with a). Intuitively, $\phi(c, a, \phi(a, d, v))$ is the amount of extra utility that a can give to c, over what he is getting in a match with d when d gets v,and the RHS is the counterpart expression for b.Thus the condition says that a can outbid b in an attempt to match with c instead of d.

If this is true for any value of v then we expect that an equilibrium will never have a matched with d while b is matched with c. But this is all that is meant by PAM: a's partner can never be smaller than b's. In the case of one sided models, taking $c = a$ and $d = b$ gives us segregation: everyone's partner is identical to himself.

Before proving our main result, we shall need to establish that equilibria in this environment satisfy an equal treatment property: all agents of the same type receive the same equilibrium payoff. The reason that an argument needs to be made is that this is not a general property of the core in NTU models.[3] But strictly decreasing frontiers ensure it is satisfied.

Lemma 1 *(Equal Treatment) All agents of the same type receive the same equilibrium payoff.*

Proof. Suppose that there are two agents i and j of type a getting different utilities, $v > v'$, and that the partner of agent i is of type b. Then the b gets $\phi(b, a, v) < \phi(b, a, v')$,where the inequality follows from the fact that ϕ is strictly decreasing in v. Thus there exists $\epsilon > 0$ such that $\phi(b, a, v' + \epsilon) > \phi(b, a, v)$;(j, b) can therefore block the equilibrium, a contradiction. ■

This allows us to refer to payoffs simply by v_a etc. without ambiguity.

When satisfied by any v, $a > b$, and $c > d$, condition (1) is called *Generalized Increasing Differences* (GID).[4] The concept is illustrated in Figure 1. The frontiers for the matched pairs $\langle d, b\rangle$, $\langle b, c\rangle$, $\langle c, a\rangle$, and $\langle a, d\rangle$ are plotted in a four-axis diagram. The compositions in (1) are indicated by following the arrows around from a level of utility v for d. Note that the utility c ends up with on the "a side" exceeds that on the b side of the diagram.

Our main result states that GID is sufficient for segregation (PAM in the two-sided case). There is an analogous condition, *Generalized Decreasing Differences* (GDD), for NAM.

Proposition 1 *(1) A sufficient condition for segregation in one-sided models and PAM in two-sided models is generalized increasing differences (GID): whenever $a > b$, $c > d$, and for all $v \in [0, \phi(d, a, 0)]$, we have $\phi(c, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v))$.*

[3] Suppose there are two types, a and b,with the measure of the b's exceeding that of the a's. If an a and a b match, each gets a payoff of exactly 1, while unmatched agents or agents who match with their own type get 0. There is no means to transfer utilty. Then any allocation in which every a is matched to a b, with the remaining b's unmatched, is in the core. But some b's get 1 while others get 0, violating equal treatment.

[4] The designation generalized increasing differences may be justified as follows. Let T be a well-ordered set with $\geq$ as the order. Let G be a (possibly partially) ordered group with operation $*$ and order $\succeq$. We are interested in maps from $\psi : T^2 \to G$.

When $G = \mathbb{R}$, $\succeq\, = \,\geq$, and $* =$ real addition, then the standard notion of increasing differences can be written as

$$t > t' \text{ and } s > s' \text{ implies } \psi(t, s) * \psi(t', s)^{-1} \succeq \psi(t, s') * \psi(t', s')^{-1}.$$

Generalized Increasing Differences (GID) just corresponds to the case in which $G =$ monotone functions from $\mathbb{R}$ to itself, $\succeq\, =$ the pointwise order, and $* =$ functional composition.

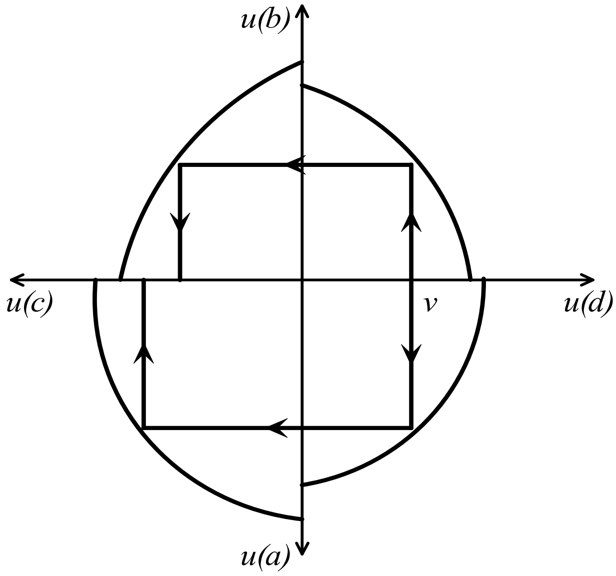

Figure 1: Generalized increasing differences.

(2) A sufficient condition for NAM is generalized decreasing differences (GDD): whenever $a > b$, $c > d$, *and for all* $v \in [0, \phi(d, b, 0)]$, *we have* $\phi(c, b, \phi(b, d, v) \geq \phi(c, a, \phi(a, d, v))$.

Proof. Here we consider only the one-sided cases; the two-sided cases are similar. For segregation, suppose that instead we have a positive measure of heterogeneous matches of the form $\langle a, b\rangle$ and that the equilibrium is not payoff equivalent to segregation. Then a must strictly prefer being matched to b rather than being matched to an a : $v_a = \phi(a, b, v_b) > \phi(a, a, v_a)$,where the other a's payoff is also v_a by equal treatment. Hence, $v_a > \phi(a, a, \phi(a, b, v_b))$. Similarly, the fact that b doesn't want to switch to a implies $v_b > \phi(b, b, v_b))$. Composing the "inverse" functions $\phi(a, b, \cdot)$ with this inequality yields $v_a < \phi(a, b, \phi(b, b, v_b))$. It then follows that $\phi(a, a, \phi(a, b, v_b)) < \phi(a, b, \phi(b, b, v_b))$ which contradicts GID condition (taking $c = a$ and $d = b$ there), and we conclude that the economy is segregated.

For one-sided NAM, it suffices to rule out as possible equilibrium matches $(\langle a, b\rangle, \langle c, d\rangle)$ and $(\langle a, c\rangle, \langle b, d\rangle)$ whenever $a > b \geq c > d$. Suppose to the contrary that $\langle a, b\rangle$ and $\langle c, d\rangle$ is part of a stable match that is not payoff equivalent to a negative one. Then $\phi(a, b, v_b) > \phi(a, d, v_d)$ (a prefers b to d) and $v_b > \phi(b, c, v_c) = \phi(b, c, \phi(c, d, v_d))$ (b prefers a to c). Apply $\phi(b, a, \cdot)$ to the first inequality, to get $v_b < \phi(b, a, \phi(a, d, v_d))$. Thus, $\phi(b, c, \phi(c, d, v_d)) < \phi(b, a, \phi(a, d, v_d))$, contradicting GDD. If instead $\langle a, c\rangle$ and $\langle b, d\rangle$ are stable, we have $\phi(a, c, v_c) > \phi(a, d, v_d) \implies v_c < \phi(c, a, \phi(a, d, v_d))$ and $v_c > \phi(c, b, \phi(b, d, v_d))$, which again contradicts GDD. ■

We now apply this result to a model of risk sharing within households. Although risk sharing within households has attracted considerable attention in the development literature and economics of the family, we are not aware of any attempts to establish formally what the pattern of matching among agents with differing risk attitudes would be, something which is obviously important for empirical identification.

Example 1 *(Risk sharing). Consider a one-sided household production model in which ouput is random, with a finite number of possible outcomes $w_i > 0$ and associated probabilities π_i. All agents are expected utility maximizers who are identical except for initial wealth. The utility of income is $\ln(a+x)$, where type $a \in [1, \bar{a}]$ is initial wealth (or it can be interpreted as an index of absolute risk aversion: $\rho_a(x) = \frac{1}{a+x}$ is strictly decreasing in a for all x). The only risk sharing possibilities in this economy lie within a household consisting of two agents. When partners match, their (explicit or implicit) contract specifies how each realization of the output will be shared between them.*

The utility possibility frontier for a match between and a and a b is generated by solving the optimal risk sharing problem:

$$\phi(a, b, v) \equiv \max_{\{x_i\}} \Sigma_i \pi_i \ln(a + w_i - x_i) \ s.t. \ \Sigma_i \pi_i \ln(b + x_i) \geq v. \tag{2}$$

The first-order condition (Borch's rule) is $\frac{1}{a+w_i-x_i} = \lambda \frac{1}{b+x_i}$, where λ is the multiplier on the constraint, from which one solves for the optimal sharing rule:

$$x_i = (w_i + a + b)e^{v - \Sigma_i \pi_i \ln(w_i + a + b)} - b.$$

This yields

$$\phi(a, b, v) = \ln(1 - e^{v - \Sigma_i \pi_i \ln(w_i + a + b)}) + \Sigma_i \pi_i \ln(w_i + a + b).$$

We claim that the GDD is satisfied. Let $a > b$ and $c > d$, and let Σ_{ab} denote $\Sigma_i \pi_i \ln(w_i + a + b)$. Then

$$\begin{aligned} \phi(c, a, \phi(a, d, v) &= \ln(1 - e^{\ln(1 - e^{v - \Sigma_{ad}}) + \Sigma_{ad} - \Sigma_{ac}}) + \Sigma_{ac} \\ &= \ln(1 - e^{\Sigma_{ad} - \Sigma_{ac}} + e^{v - \Sigma_{ac}}) + \Sigma_{ac} \end{aligned}$$

and

$$\phi(c, b, \phi(b, d, v) = \ln(1 - e^{\Sigma_{bd} - \Sigma_{bc}} + e^{v - \Sigma_{bc}}) + \Sigma_{bc}.$$

Now,

$$\phi(c, a, \phi(a, d, v)) \ < \ \phi(c, b, \phi(b, d, v))$$

if and only if

$$(1 - e^{\Sigma_{ad} - \Sigma_{ac}} + e^{v - \Sigma_{ac}})e^{\Sigma_{ac}} < (1 - e^{\Sigma_{bd} - \Sigma_{bc}} + e^{v - \Sigma_{bc}})e^{\Sigma_{bc}},$$

that is if $e^{\Sigma_{ac}} - e^{\Sigma_{ad}} < e^{\Sigma_{bc}} - e^{\Sigma_{bd}}$. *But this is just the requirement that the function* $e^{\Sigma_{ab}}$ *satisfies decreasing differences, which it clearly does, since* $\frac{\partial^2}{\partial a \partial b} e^{\Sigma_{ab}} = -e^{\Sigma_{ab}} Var(\frac{1}{w+a+b}) < 0$. *Thus GDD is indeed satisfied, and we conclude that* in the risk-sharing economy with logarithmic utility, agents will always match negatively in wealth. *This is of course intuitive: the most risk averse share risk with the least risk averse, while the moderately risk averse share with each other.*

3.2 A Local Condition

Often it is easier to check whether a condition holds locally than globally. We now provide a set of local conditions which suffice for monotone matching. In addition to being computationally convenient, these conditions illuminate the "complementarity in transferability" property alluded to above. In this section we suppose that $\phi(x, y, v)$ is twice differentiable (except of course at $v = \phi(y, x, 0)$).

Proposition 2 *(1) A sufficient condition for segregation (or PAM) is that for all* $x, y \in A \times A$ *and* $v \in [0, \phi(y, x, 0))$, $\phi_{12}(x, y, v) \geq 0$, $\phi_{13}(x, y, v) \geq 0$ *and* $\phi_1(x, y, v) \geq 0$.

(2) A sufficient conditions for NAM is that for all $x, y \in A \times A$ *and* $v \in [0, \phi(y, x, 0))$, $0 \geq \phi_{12}(x, y, v)$, $0 \geq \phi_{13}(x, y, v)$ *and* $\phi_1(x, y, v) \geq 0$.

Proof. We show that the local conditions imply the generalized difference conditions. Fix v, $a > b$ and $c > d$, and consider the case (1) for segregation (the other case is similar). Then $\phi_{12} \geq 0$ implies that for any $x \in [d, c]$

$$\phi_1(x, a, \phi(b, d, v)) \geq \phi_1(x, b, \phi(b, d, v));$$

$\phi_1 \geq 0$ implies $\phi(a, d, v) \geq \phi(b, d, v)$, and $\phi_{13} \geq 0$ in turn yields

$$\phi_1(x, a, \phi(a, d, v)) \geq \phi_1(x, a, \phi(b, d, v)),$$

so that $\phi_1(x, a, \phi(a, d, v)) \geq \phi_1(x, b, \phi(b, d, v))$. Integrating both sides of this inequality over x from d to c then gives

$$\phi(c, a, \phi(a, d, v)) - \phi(d, a, \phi(a, d, v)) \geq \phi(c, b, \phi(a, b, v)) - \phi(d, b, \phi(b, d, v));$$

Noting that $\phi(d, a, \phi(a, d, v)) = \phi(d, b, \phi(b, d, v)) = v$ gives us GID. ■

Obviously, with TU, $\phi_{13} = 0$, so this reduces to the standard condition in that case. The extra term reflects the fact that changing the type results in a change in the slope of the frontier, so the extra utility available to her is the extra she contributes adjusted by the change in slope. For segregation, the idea is that higher types can transfer utility to their partners more easily (ϕ_3 is less negative, hence flatter).

The conditions imply that the total possible transfer of utility is everywhere increasing in type ($\frac{d}{da}\phi_1(x, a, \phi(x, a, v)) = \phi_{12} + \phi_{13} \cdot \phi_1$). Indeed, this is a necessary implication of GID. To see this, take $a > b$ and $c > d$ and note that GID is equivalent to $\phi(c, a, \phi(a, d, v)) - \phi(d, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v)) - \phi(d, b, \phi(b, d, v))$. Dividing by $c - d$ and taking limits as $c \to d$ yields $\phi_1(d, a, \phi(a, d, v)) \geq \phi_1(d, b, \phi(b, d, v))$. Dividing by $a - b$ and letting $a \to b$ yields $\phi_{12}(d, b, \phi(b, d, v)) + \phi_{13}(d, b, \phi(b, d, v)) \cdot \phi_1(b, d, v) \geq 0$.

Weaker sufficient conditions can be found, but as they involve compositions of ϕ and its partial derivatives, they appear to be no easier to apply than GID and GDD, so we omit them.

Finally, note that the condition $\phi_1 \geq 0$ is less restrictive then might first appear: in a model in which instead $0 \geq \phi_1$ everywhere, one can redefine the type space with the "reverse" order; then the cross partial ϕ_{12} retains its sign, while ϕ_{13} and ϕ_1 reverse sign and Proposition 2 can be applied.

The following example is based on Newman (1999).

Example 2 *(Matching principals and agents). There is a continuum of risk-neutral principals with type indexed by $p \in (\frac{1}{2}, 1)$, and an equal measure of agents with type index $a > 1$. The principal's type p indicates the probability that his agent's effort e, which can either be 1 or 0, is correctly detected. All tasks are equally productive, yielding expected revenue π, and every principal wishes to implement $e = 1$. All agents derive utility $\ln y$ from income y; their type represents initial wealth.*

As this is a two sided model, one needs to compute ϕ from both points of view. The frontier for a principal of type p who is matched to an agent of type a is given by

$$\phi(p,a,v) = \max \ \pi - pw_1 - (1-p)w_0$$
$$s.t. \ p\ln(a+w_1) + (1-p)\ln(a+w_0) - 1 \geq v$$
$$p\ln(a+w_1) + (1-p)\ln(a+w_0) - 1 \geq (1-p)\ln(a+w_1) + p\ln(a+w_0),$$

where w_1 and w_0 are the wages paid in case the signal of effort is 1 or 0 respectively. The second inequality is the incentive compatibility condition that ensures the agent takes high effort.

The frontier for an agent of type a matched to a principal of type p who gets v is

$$\phi(a,p,v) = \max \ p\ln(a+w_1) + (1-p)\ln(a+w_0) - 1$$
$$\pi - pw_1 - (1-p)w_0 \ s.t. \geq v$$
$$p\ln(a+w_1) + (1-p)\ln(a+w_0) - 1 \geq (1-p)\ln(a+w_1) + p\ln(a+w_0),$$

The solution to these problems yields

$$\phi(p,a,v) = \pi + a - e^{v+1}[pe^{\frac{1-p}{2p-1}} + (1-p)e^{\frac{p}{2p-1}}]$$

and

$$\phi(a,p,v) = \frac{1-p}{2p-1} + \ln\left(\frac{\pi + a - v}{pe^{\frac{1}{2p-1}} + 1 - p}\right)$$

Intuition might suggest that since wealthier agents are less risk averse, they should be matched to tasks for which the signal quality is poor, since these tasks are effectively riskier. This intuition is incomplete, and indeed misleading, as the following application of Proposition 2 indicates.

It is straightforward to verify that when own type is a principal,

$$\phi_1 = \left(e^{\frac{p}{2p-1}} - e^{\frac{1-p}{2p-1}} + \frac{p}{(2p-1)^2}e^{\frac{1-p}{2p-1}} + \frac{1-p}{(2p-1)^2}e^{\frac{p}{2p-1}}\right)e^{v+1} = \phi_{13} > 0,$$

that when own type is an agent, $\phi_1 = \frac{1}{\pi+a-v} > 0$ *and* $\phi_{13} = \left(\frac{1}{\pi+a-v}\right)^2 > 0,$*and that* $\phi_{12} = 0$ *in either case. Thus the* agents with lower risk aversion (higher wealth) are matched to principals with higher quality signals, *i.e. more observable tasks. This result may appear surprising, since empirically we tend to associate* less *observable tasks to wealthier workers. The intuition is that incentive compatibility entails that the amount of risk borne by the agent increases with wealth* ($w_1 - w_0 = \frac{(e^{\frac{1}{2p-1}}-1)(\pi+a-v)}{pe^{\frac{1}{2p-1}}+1-p}$ *); when this effect is rapid enough, as it is with logarithmic utility, it swamps the decline in risk* *aversion. Wealthier agents therefore have higher cost for given v;it is best to transfer this to them along a flatter frontier, i.e. to assign them to the better signals.*

This example is instructive because the entire effect comes from the non-transferability of the problem. There is no direct "productive" interaction between principal type and agent type ($\phi_{12} = 0$);only the complementarity between type and transferability plays a role in determining the match.

Finally, as is apparent from their derivation, the local conditions are stronger than generalized difference conditions, even restricting to smooth frontier functions. This is of practical as well as logical interest: as we saw, Example 1 satisfies GDD, from which we concluded there is negative matching in wealth. But in spite being smooth, $\phi(a,b,v) = \ln(1 - e^{v-\Sigma_i \pi_i \ln(w_i+a+b)}) + \Sigma_i \pi_i \ln(w_i + a + b) \equiv \ln(1 - e^{v-\Sigma_{ab}}) + \Sigma_{ab}$ doesn't satisfy our local condition:

$$
\begin{aligned}
\phi_1 &= \frac{1}{1-e^{v-\Sigma_{ab}}}\frac{\partial \Sigma_{ab}}{\partial a} > 0, \\
\phi_{12} &= \frac{1}{(1-e^{v-\Sigma_{ab}})^2}\left((1-e^{v-\Sigma_{ab}})\frac{\partial^2 \Sigma_{ab}}{\partial a \partial b} - e^{v-\Sigma_{ab}}\left(\frac{\partial \Sigma_{ab}}{\partial a}\right)^2\right) < 0,
\end{aligned}
$$

yet

$$
\phi_{13} = \frac{e^{v-\Sigma_{ab}}}{(1-e^{v-\Sigma_{ab}})^2}\frac{\partial \Sigma_{ab}}{\partial a} > 0.
$$

3.3 Lattice Theoretic Conditions

Proposition 2 can be weakened by considering (possibly) nondifferentiable functions that are supermodular in pairs of variables.

Proposition 3 *(1) A sufficient condition for segregation (PAM in two sided models) is that ϕ is supermodular in types, increasing in own type, and supermodular in own type and payoff.*
(2) A sufficient condition for NAM is that ϕ is submodular in types, increasing in own type and submodular in own type and payoff.

Proof. Consider case (1); the other case is similar. Take v, $a > b$ and $c > d$. Supermodularity in own type and partner's utility, along with increasing in own type implies $\phi(c,a,\phi(a,d,v)) + \phi(d,a,\phi(b,d,v)) \geq \phi(c,a,\phi(b,d,v)) + \phi(d,a,\phi(a,d,v))$, or $\phi(c,a,\phi(a,d,v)) - \phi(d,a,\phi(a,d,v)) \geq \phi(c,a,\phi(b,d,v)) - \phi(d,a,\phi(b,d,v))$. But the right hand side of the latter inequality weakly exceeds $\phi(c,b,\phi(b,d,v)) - \phi(d,b,\phi(b,d,v))$ by supermodularity in types. Thus

$\phi(c,a,\phi(a,d,v))-\phi(d,a,\phi(a,d,v)) \geq \phi(c,b,\phi(b,d,v))-\phi(d,b,\phi(b,d,v))$, and since $\phi(d,a,\phi(a,d,v)) = \phi(d,b,\phi(b,d,v)) = v$, $\phi(c,a,\phi(a,d,v)) \geq \phi(c,b,\phi(b,d,v))$, which is GID. ■

It is evident from this proposition that a stronger sufficient condition for segregation (or PAM) is that ϕ itself is a supermodular function that is increasing in own type. Indeed, given v, $a > b$ and $c > d$, put $x = (d, a, \phi(a, d, v))$ and $y = (c, b, \phi(b, d, v))$in the defining inequality $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \geq \phi(\mathbf{x}) + \phi(\mathbf{y})$. Then since $\phi(a,d,v) \geq \phi(b,d,v)$, $\mathbf{x} \vee \mathbf{y} = (c, a, \phi(a, d, v))$, $\mathbf{x} \wedge \mathbf{y} = (d, b, \phi(b, d, v))$,and we have

$$\phi(c,a,\phi(a,d,v)) + \phi(d,b,\phi(b,d,v)) \geq \phi(d,a,\phi(a,d,v)) + \phi(c,b,\phi(b,d,v)),$$

which is just GID since $\phi(d,b,\phi(b,d,v)) = \phi(d,a,\phi(a,d,v) = v$.

The principal interest of this observation is that it enables us to offer sufficient conditions for monotone matching expressed in terms of the fundamentals of the model, rather than in terms of the frontiers (such results leading to our local conditions would be much harder to come by).

The frontier can be expressed fairly generally as

$$\begin{aligned} \phi(a,b,v) &= \max_{x,x'} U(x,a,b) \\ \text{s.t. } U(x',b,a) - v &\geq 0 \\ (x,x') &\in F(a,b). \end{aligned}$$

Here $F(a,b) \subset X$, a (sub)lattice of some $\mathbb{R}^n$, is the set of choices available to types (a,b). (In matching models, where the cardinality properties of the frontier are important, it makes sense to think of the payoff functions as coming from a one-parameter family – then monotone transformations of a single type's payoff cannot be performed independently of the others.) A sufficient condition for ϕ to be increasing in own type is that U is increasing in type and F is continuous and increasing (in the set inclusion order) in own type. A sufficient condition for ϕ to be strictly decreasing is that U is strictly monotone.

We also need the set $S = \{(a,b,v,x,x') | a \in A, b \in A, v \in \mathbb{R}, (x,x') \in F(a,b)\}$ to form a sublattice. Then an application of Theorem 2.7.2 of Topkis (1998) yields

Corollary 1 *If payoffs functions are supermodular (submodular), strictly increasing in choices, and increasing in type; choice sets are continuous and increasing in own type; and the set of types, payoffs and feasible choices forms a sublattice, then the economy is segregated in the one-sided case and positively matched in the two-sided case (negatively matched).*

Topkis's theorem tells us that under the stated hypotheses, ϕ will be supermodular (submodular); since it is also increasing in own type by the hypotheses on F and U, the result follows.

As a practical matter, the usefulness of this corollary hinges on the ease of verifying that the sets S and F have th required properties. In many cases it may be more straightforward to compute the frontiers and apply Propositions 1, 2, or 3. Note, for example, that since the frontier function in the risk-sharing example is not submodular despite the fact that the objective function is, the choice-parameter set S is not a sublattice. In the prinicpal-agent example, the feasible set F is not increasing in own type when the type is that of an agent.

4 Discussion

We have presented some general sufficient conditions for monotone matching in nontransferable utility models. These have an intuitive basis and appear to be reasonably straightforward to apply.

One question that arises is whether there are also necessary conditions for monotone matching. Such a condition, the "segregation principle," is indeed obtainable for segregation. For each type, the *segregation payoff* as the (equal treatment) payoff an agent of that type generates in a match with an identical agent. Then segregation occurs regardless of the distribution of types if for all pairs of types, there is no point in the utility possibility set that Pareto dominates the vector of segregation payoffs; otherwise, there is always some distribution for which the economy is not segregated. This result is very general: it applies even when the frontiers are not strictly decreasing functions. Whether there are tractable necessary conditions for other matching patterns remains an open question.

Other forms of monotone matching not discussed here have been identified in the literature (Legros-Newman, 2002). These include one-sided PAM (which includes segregation as a special case) and another form of one-sided PAM, median matching. Sufficient conditions for these are easily generated as weakenings or modifications of the basic GID condition.

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