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ABSTRACT

Modelling Scale Consistent VaR with the Truncated Lévy Flight*

Returns in financial assets show consistent excess kurtosis, indicating the presence of large fluctuations not predicted by Gaussian models. Mandelbrot (1963) first proposed the idea that price changes distributed according to a Lévy stable law. The unique feature of Lévy-stable distributions in general is that they are stable under addition. However, these distributions have power law tails that decay too slowly from the point of view of financial modelling. In recent studies the truncated Lévy flight has been shown to eliminate this problem and to be very promising for the modelling of financial dynamics. An exponential decay in the tails ensures that all moments are finite and the distribution is fat-tailed for short time scales and converges in a Gaussian process for increasing time scales, a feature observed in financial data. We propose a model with time-varying scale parameter (GARCH process) error terms that are truncated Lévy distributed. We determine the appropriate GARCH specification for each data set by conducting a specification test based on a generalization of the augmented GARCH process of Duan (1997). The Lévy flight includes a method for scaling up a single-day volatility to a multi-day volatility, precisely a α-root-of-time rule, where α is the characteristic parameter of the process. We use this rule to forecast future volatility and as a result estimate value-at-risk (VaR) several days ahead and compare it to the RiskMetricsTM (1996) approach, which is a special case of our method. We compare the models in in-sample and out-of-sample analyses for a sample of stock index returns.

JEL Classification: C22, C52, G10
Keywords: augmented GARCH process, in- and out-of-sample analysis, scale consistency, truncated Lévy flight, value-at-risk

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NON-TECHNICAL SUMMARY

In this Paper we consider the price dynamics of stock indices traded in a financial market. While financial applications involve many different time intervals, ranging from a few minutes (intraday) to a number of years, most techniques used in econometrics focus on modelling the fluctuations of price series in a single time interval. But the distribution that successfully explains daily price changes, for example, is unable to characterize the nature of hourly price changes. On the other hand, the statistical properties of monthly price changes are often not fully covered by a model for daily price changes. In order to describe the statistics of future prices of a financial asset, a distribution of price fluctuations for different time intervals is *a priori* needed, corresponding to different trading time horizons.

For derivative pricing and risk management a better description of the extreme events is crucial. There is a need for an alternative to the Gaussian with more weight in the tails and, just as importantly, more weight around the centre. A possible candidate is the Lévy distribution often studied in physics. Fitting the Lévy distribution to the data seems to provide a consistently better representation of price fluctuations. In practice it is possible to model the tails as fat as necessary by modifying only one parameter (the Gaussian distribution with thin tails is just a special case).

The problems with these kinds of distributions are the power law tails, which decay too slowly from the point of view of financial modelling and the fact that the distribution of price changes for larger time intervals converge in a Gaussian distribution model. What is observed on financial markets is a faster decay like the exponential decay of the Gaussian model. This problem can be overcome simply by taking the Lévy distribution in the central part and introducing a cut-off in the far tails that is faster than the Lévy power law tails. The Lévy distribution with a cut-off and exponentially declining tails was introduced by Mantegna and Stanley (1994) and is known as a truncated Lévy distribution (TLD). Koponen (1995) derived an analytical expression of the characteristic function. This cut-off ensures that the variance will be finite and the distribution converges to a Gaussian one in the end. To model financial prices over time the truncated Lévy flight (TLF) can be constructed by the sum of independent and identically distributed random variables described by a TLD. In contrast to fat-tailed distributions, the central limit theorem applies and the TLF approaches a Gaussian model. Lévy flights have been observed experimentally in physical systems and have been used to describe for instance the spectral random walk of a single molecule embedded in a solid. In these cases an unavoidable cut-off is always present.

In contrast to physical systems, hundreds of papers have shown that there is a strong non-IID clustering effect in financial data. Most of the studies so far performed a summability test of correlated unconditional return data and rejected the Lévy-stable hypothesis. But testing for stable properties requires IID data. For example, Ghose and Kroner (1995) found that summability holds for data generated from an IGARCH process with non-normal innovations. But of course the simulated data is correlated and not IID. Therefore if we want to justify the use of Lévy processes we first have to remove the autocorrelation in the data and second, assume the summability of the IID data. A solution to this problem would be a mechanism to 'filter out' the correlations in the data and test if the IID residual process is adequately described by the stable law.

Fortunately, the class of GARCH models has been very successful in modelling the significant volatility clustering and non-IID nature of the data. More specifically, the standard GARCH model produces a mean-reverting time-dependent volatility process that 'filters out' the correlations in the data and the remaining residuals are assumed to be IID. We develop a specification test based on the augmented GARCH process of Duan (1997) to derive the appropriate model for each data set and each distribution. We show that different choices for the innovations distribution lead to different results about the ability of the GARCH model to explain the autocorrelations in the data. In particular, we show that asymmetry in the volatility is not enough to capture the skewness in the data and a combination of a skewed distribution for the innovations and asymmetry in the volatility process leads to a much better fit in-sample and to better out-of-sample performance. In general, if the model residuals can be shown to be IID, we can assume the summability of the residuals. If we can assume that the residuals have stable properties we can use the alpha-root-of-time rule for scaling up single-day volatility to multi-day volatility.

A practical risk management tool for financial institutions nowadays is value-at-risk (VaR). It is prescribed by the Basle Committee to report to the (international) supervisor and also used as an internal management tool to see if traders are within their limits. VaR is defined as an amount lost on a portfolio with a given probability over a fixed number of days. The confidence level reflects 'extreme market conditions' with a probability of, for example, 99%. In other words in only 1% of the cases we will lose more than the reported VaR of our portfolio. Crucial for an accurate VaR estimate is the precise determination of the 'extreme events', which results from the distribution function of the underlying return series in the tails. Since the Lévy distribution overestimates and the Gaussian underestimates the fat tails, we expect the same behaviour for the true value-at-risk at high quantiles. To capture the real downside risk, a parametric distribution with fatter tails than the Gaussian and thinner tails than the Lévy should be used. It should be possible to fit the tails separately just to the extreme events. We show that especially for the maximum likelihood method it is crucial to use a skewed distribution to capture the downside risk. For this reason the truncated Lévy distribution is an attractive statistical tool for estimating value-at-risk. We use the truncated Lévy flight as a stochastic process for different time intervals and model the time-scaling behaviour of the quantiles.

1 INTRODUCTION

In this paper we consider the price dynamics of stock indices traded in a financial market.
While financial applications involve many different time intervals, ranging from a few minutes (intraday) to a number of years, most techniques used in econometrics focus on modeling the fluctuations of price series in a single time interval. But the distribution that successfully explains daily price changes, for example, is unable to characterize the nature of hourly price changes. On the other hand, the statistical properties of monthly price changes are often not fully covered by a model for daily price changes. In order to describe the statistics of future prices of a financial asset, one a priori needs a distribution of price fluctuations for different time intervals, corresponding to different trading time horizons.

For derivative pricing and risk management a better description of the large events is crucial. There is a need of an alternative to the Gaussian with more weight in the tails and also important more weight around the center. A possible candidate is the Lévy distribution often studied in physics. Fitting the Lévy distribution to the data seems to provide a consistently better representation of price fluctuations. In practice it is possible to model the tails as fat as necessary by modifying only one parameter (the Gaussian distribution with thin tails is just a special case).

The problems with these kinds of distributions are the power law tails, which decay too slowly from the point of view of financial modeling and the fact that the distribution of price changes for larger time intervals converge to a Gaussian. What is observed on financial markets is a faster decay like the exponential decay of the Gaussian. This problem can be simply overcome by taking the Lévy distribution in the central part and introducing a cutoff in the far tails that is faster than the Lévy power law tails. The Lévy distribution with a cutoff and exponentially declining tails was introduced by Mantegna/Stanley (1994) and is known as a truncated Lévy distribution (TLD). Koponen (1995) derived an analytical expression of the characteristic function. This cutoff ensures that the variance will be finite and the distribution converges to a Gaussian in the limit. To model financial prices over time the truncated Lévy flight (TLF) can be constructed by the sum of independent and identically distributed random variables described by a TLD. In contrast to fat-tailed distributions, the central limit theorem applies and the TLF approaches a Gaussian. Lévy flights have been observed experimentally in physical systems and have been used to describe for instance the spectral random walk of a single molecule embedded in a solid. In these cases an unavoidable cutoff is always present.

In contrast to physical systems, hundreds of papers have shown that there is a strong non-i.i.d. clustering effect in financial data. Most of the studies so far performed a summability test of correlated unconditional return data and rejected the Lévy-stable hypothesis. But testing for stable properties requires i.i.d. data. For example Ghose and Kroner (1995) found that summability holds for data generated from an IGARCH process with non-normal innovations. But of course the simulated data is correlated and not iid. Therefore if we want to justify the use of Lévy processes we first have to remove the autocorrelation in the data and second we can assume the summability of the i.i.d. data. A solution to this problem would be a mechanism to "filter out" the correlations in the data and test if the i.i.d. residual process is adequately described by the stable law.

Fortunately the class of GARCH models has been very successful in modeling the significant volatility clustering and non-i.i.d. nature of the data. More specifically, the standard GARCH model produces a mean reverting time dependent volatility process that "filters out" the correlations in the data and the remaining residuals are assumed to be i.i.d. We develop a specification test based on the augmented GARCH process of Duan (1997) to derive the appropriate model for each data set and each distribution. We show that different choices for the innovations distribution lead to different results about the ability of the GARCH model to explain the autocorrelations in the data. In particular, we show that asymmetry in the volatility is not enough to capture the skewness in the data and a combination of a skewed distribution for the innovations and asymmetry in the volatility process leads to a much better fit in-sample and to better out-of-sample performance.

In general, if the model residuals can be shown to be i.i.d., we can assume the summability of the residuals. If we can assume that the residuals have stable properties we can use the alpha-root-of-time rule for scaling up single day volatility to multi day volatility.

A good parametric description of the distribution of price changes especially in the tails is important for Value at Risk (VaR). Since the Lévy distribution overestimates and the

Gaussian underestimates the far tails, we expect the same behavior for the true Value at Risk at high quantiles. To capture the real downside risk, a parametric distribution with fatter tails than the Gaussian and thinner tails than the Lévy should be used. It should be possible to fit the tails separately just to the extreme events. We show that especially for the maximum likelihood method it is crucial to use a skewed distribution to capture the downside risk. For this reason the truncated Lévy distribution is an attractive statistical tool for estimating Value-at-Risk. We use the truncated Lévy flight as a stochastic process for different time intervals and model the time scaling behavior of the quantiles.

The outline of this paper is as follows. In section 2 we will describe the family of Lévy-stable distributions. Section 3 introduces the idea of a truncation in the tails of the Lévy distribution and summarizes the properties. Section 4 compares the Lévy distribution with a skewed version of the well-known Student t-distribution often used to model fat tails. Section 5 introduces the idea of scale invariant processes and defines the truncated Lévy flight used in this study. Section 6 describes the econometric framework for the study. The empirical results are summarized in section 7. In section 8 we use our models to derive VaR estimates for different time horizons. Finally, section 9 concludes.

2 THE LÉVY DISTRIBUTION

The characteristic function of the four-parameter family of Lévy distributions can be written

$$\psi_L(k) = i\mu k - C\,|\,k\,|^{\alpha}\,\{1 + i\beta\,\mathrm{sgn}(k)\tan(\pi\alpha/2)\}, \qquad 0 < \alpha \le 2.$$

α is the characteristic exponent determining especially the fatness of the tails, $\beta \in [-1,1]$ is an index of skewness, c is a scale parameter and μ is a location parameter. The density function is only known analytically when $\alpha=1$ (Cauchy distribution) and $\alpha=2$ (Gaussian distribution). However for the symmetric case the value of the density of the Lévy distribution is known at the origin and in the far tails.

In all other situations the density must be generated numerically. Accurate numerical values for the density ψ_L can be calculated Fourier-transforming the CF and evaluating the integral numerically. We use Romberg integration, which allows the prespecification of the tolerated error and in fact a calculation of the density as precise as necessary (see Lambert and Lindsey (1999)). The m-th moment of the distribution can be found from the CF using

$$\langle x^m \rangle = \left(i^m \frac{\partial^m}{\partial k^m} y_L(k) \right)\Bigg|_{k=0} .$$

The m-th moment is finite only for $m<\alpha$, for $\alpha<2$ the variance is infinite. For $\alpha=2$ all the moments are finite.

3 THE CUTOFF

Modeling financial returns with a distribution with infinite moments is frequently critized. One way to eliminate this problem is to reduce the weight in the tails, because the power-law decay with an exponent less than 2 is to slow to ensure that the second moment is finite. A possible solution is to introduce in the tails a function with a faster decay. One possible cutoff is the exponential function for which the CF has been developed (Koponen (1995)). Note the misprint in the original publication, the CF should read:

$$\psi_{TL}(k) = C\left\{\frac{\lambda^{\alpha} - \left(k^2 + \lambda^2\right)^{\alpha/2}}{\cos(\pi\alpha/2)}\cos\left(\alpha\arctan\left(\frac{|k|}{\lambda}\right)\right)\left[1 + i\,\mathrm{sgn}(k)\beta\tan\left(\alpha\arctan\left(\frac{|k|}{\lambda}\right)\right)\right]\right\}$$

$C > 0$ is a scale parameter, $0 < \alpha < 2$ (but $\alpha \neq 1$) is the characteristic exponent determining the shape of the distribution and λ is the so-called cutoff parameter, which determines the speed of the decay and as a result the cutoff region. The parameter β determines the skewness when $\beta \neq 0$. The distribution is skewed to the right when $-1 < \beta < 0$ and skewed to the left when $0 < \beta < 1$. The so-called truncated Lévy distribution shows less weight in the tails and nearly the same shape around the center. Nakao (2000) developed an alternative definition of the exponential truncation of the Lévy distribution. Apart from the scale parameter, the CF is identical to the CF given in Koponen (1995). The truncated Lévy density can also be obtained through an inverse Fourier transform.

For $l \to +0$ the TLD reduces to the Lévy distribution. In contrast to the Lévy distribution the exponential cutoff ensures that all moments exist. The m-th moment of the distribution can be found from the CF using

$$\left\langle x^m \right\rangle = \left(i^m \frac{\partial^m}{\partial k^m} \psi_{TL}(k) \right)\Bigg|_{k=0} .$$

The exponential cutoff appears ‘earlier’ in the tail for liquid markets (higher λ), and can be completely absent in less mature markets.

4 COMPARISON WITH A SKEWED STUDENT-T DISTRIBUTION

A popular distribution in the financial literature is the Student t-distribution. By varying the degrees of freedom it is possible to model every observed tail fatness. The Student t-distribution has the same power law tails as the Lévy distribution in the far tails with the degrees of freedom related to the characteristic exponent. In other words for a value 1<α<2 equal to the number of degrees of freedom the tails of both distribution are the same but the center is different. In contrast the numbers of degrees of freedom are not restricted. For θ>2 degrees of freedom the distribution behaves as a power-law for large arguments, but at the same time has a finite variance to ensure the validity of the CLT. The density with a scale parameter *a* is defined by:

$$f(u_t, 0, a^2, q) = \frac{1}{\sqrt{qp}} \frac{\Gamma((1+q)/2)}{a\Gamma(q/2)} \left[1 + \left(\frac{u_t^2}{a^2 q}\right)\right]^{-\left(\frac{q+1}{2}\right)},$$

which coincides with the Cauchy distribution for θ=1. In other words for α=θ=1 the Student t-distribution and the Lévy distribution are equal. If θ>2 the variance is defined by:

$$E\left(u_t^2\right) = \frac{a^2 q}{q-2}$$

We can rewrite the density function for a t variable with scale parameter σ and derive the Student-t distribution in a standardized version:

$$f(u_t, 0, 1, q) = \frac{1}{\sqrt{p(q-2)}} \frac{\Gamma((1+q)/2)}{\Gamma(q/2)} \left[1 + \left(\frac{u_t^2}{q-2}\right)\right]^{-\left(\frac{1+q}{2}\right)}.$$

The t-distribution tends towards a Gaussian in the limit θ→∞.

One feature of the degrees of freedom of the Student t-distribution is that it gives a restriction for the number of moments that exist for the distribution (comparable to the characteristic exponent of the Lévy distribution). A value of θ less than 4 reveals that only the first 3 moments exist. But for financial returns the kurtosis of the sample is well defined. A similar argument was used for a long time against the modeling of financial returns with the Lévy distribution. As a result there is a restriction that the parameter θ be greater than 4 and at the same time a restriction for the tail fatness.

The Student t distribution has often been used to account for the excess kurtosis in asset returns, but it cannot model the skewness. Paolella (1997) proposed the so-called t_3 distribution, an alternative to the t distribution with skewness. Mittnik et al. (2000) have shown that the skewed distribution outperforms the symmetric distribution in in-sample fit and out-of-sample density predictions.

The standardized density can be written as:

$$f_{t_3}(u_t,0,1,v,d,b)=\frac{1}{K}\begin{cases}\left[1+\left(\dfrac{-u_t^d b^d}{v}\right)\right]^{-\left(\frac{1+v}{d}\right)}, & if \quad u_t<0\\ \left[1+\left(\dfrac{u_t^d b^{-d}}{v}\right)\right]^{-\left(\frac{1+v}{2}\right)}, & if \quad u_t\geq 0\end{cases},$$

where $K=(b+b^{-1})d^{-1}v^{1/d}B(d^{-1},v)$ and B(.,.) is the beta function. Since the t_3 distribution nests among others the Student-t it can be seen as a generalization of the Student-t distribution. The distribution has power-law tails; therefore *vd* gives an upper limit for the number of moments that exist for the distribution.

5 SCALING PROPERTIES

In practice comparing the distributional properties of price increments at various time intervals provides insight into the temporal dependence structure of the time series. It is possible to reconstruct the distributions for different time intervals from the knowledge of the distribution for short time intervals only if we assume independent and identically distributed (iid) price changes. The Normal and the Lévy distributions plays a central role in this context, because they are stable under addition: the distribution of the sum of a large number of iid random variables belongs to the family of Lévy distributions (generalized Central Limit Theorem) (Gnedenko & Kolmogorov (1954)).

Analyzing the scaling behavior of financial fluctuations is just comparing the increments for shorter time scales τ and for longer time scales Nτ. This formally corresponds to summing N random variables. In the case of the Lévy distribution the characteristic function satisfies $N y_L(k) = y_L(N^{1/a} k)$ The distribution for various time scales for stationary and independent variables is related by a convolution relation:

$$P_{Nt} = P_t \otimes P_t \otimes ... \otimes P_t \ .$$

More generally the distribution P(x) of price changes on a time scale Nτ may be obtained from that of a shorter time scale τ by a rescaling of the variable:

$$P_L^N(x) = N^{-1/a} P_L^1\left(N^{-1/a} x\right),$$

where $P^N(x)$ denotes a N-times convoluted distribution of $P^1(x)$.

This relation means that the process x(t) is *self-similar* with a constant self-similar exponent $\frac{1}{a}$. The process is characterized by the scaling behavior of its moments. We can also refer to this as the scaling or fractal property of stochastic processes. The simplest case is the self-similar process with a constant scale factor which is uni-fractal in this sense. (Mandelbrot et al. (1997) introduced the idea of a multifractal process with a time depending random scale factor).

The scenario of a *scale invariant* price process is that of a Lévy flight with a characteristic exponent α which is the inverse of the self-similarity exponent. In other words if the random variable for short time intervals is distributed according to a Lévy distribution, then in the limit the distribution for long time intervals is also a Lévy with the same characteristic exponent α. The infinite variance of the Lévy distribution prevent that it converges to a Normal distribution.

In Mantegna/Stanley (1995) this scale invariant behavior is observed for short time scales with high frequency data. Cont et al. (1997a) showed that it breaks down for longer time scales. These observations have been explained as a structural break in terms of the truncated Lévy Flight. The scaling behavior is also observed for the variance and the kurtosis of price increments (Cont et al. (1997a)). A link can be established between the scaling behavior of the moments and high-order correlation functions of the time series (Cont (1997b)).

Since we introduced a cut-off for the CF of the truncated Lévy $y_{TL}(k, l)$, it is no longer self-similar or uni-fractal characterized by the criteria mentioned above, but the process is bi-fractal, the simplest version of a multi-fractal process (see Nakao (2000)). The convolution of the probability distribution can still be obtained by scaling both x and λ. The CF $y_{TL}(k, l)$ statisfies $N y_{TL}(k, l) = y_{TL}(N^{1/a} k, N^{1/a} l)$ and the N-times convoluted probability distribution satisfies $P_{TL}^{N}(x, l) = N^{-1/a} P_{TL}^{1}\left(N^{-1/a} x, N^{1/a} l\right)$.

C is the scale parameter and λ the cutoff-parameter of the so-called truncated Lévy flight (the discrete time alternative to the continuous time process) and therefore both are assumed to grow as $N^{1/a}$. For short time scales (daily) the process behaves like a Lévy flight, but converges towards a Gaussian for longer time scales (say monthly) (see Matacz (2000)). The scaling of λ, which means that for increasing time scales the cutoff is introduced earlier in the tails ensures that the process converges towards a Gaussian process instead of staying a Lévy flight.

To illustrate the RiskMetrics™ approach a Gaussian stochastic process x(t) can be defined by the CF

$$y_G(k) = -\frac{1}{2} s^2 t k^2 .$$

σ is the scale parameter of the process and therefore assumed to grow as $\sqrt{t}$. This is the scaling rule of a usual Gaussian random walk for the standard deviation and used in the standard RiskMetrics™ approach. Our approach can be interpreted as a generalization of the RiskMetrics™ methodology.

6 THE ECONOMETRIC FRAMEWORK

Volatilities can be well predicted with a parametric model such as GARCH. Traditional GARCH models (with Normal- or Student-t distributed error terms) were designed to capture clustering of large and small innovations, which can be modeled as serially correlated conditional variances when the variance exists. In the case of the standard Bollerslev GARCH (p,q) model (Bollerslev (1986)) the conditional variance σ^2_t can be approximated by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 (\varepsilon_{t-j})^2 \ ,$$

where ε_t is a realization from a mean zero, variance one density. The analogue of the standard deviation σ in the family of Lévy distributions is the scale parameter C. If we replace the standard deviation σ by the scale parameter C, we allow C_t to be serially correlated, which produces the volatility clustering.

A more general class of model, which is frequently used together with stable distributions (see Mittnik et al. (1999,2000) is the power-GARCH process of Ding, Granger and Engle (1993):

$$\sigma_t^\delta = \alpha_0 + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta (\left|\varepsilon_{t-j}\right| - \rho \varepsilon_{t-j})^\delta$$

Bollerslev's GARCH model (Bollerslev (1986)) correspond to δ=2 and ρ=0. The exponent δ is a parameter of the model and is estimated during the optimization routine. This model specification for p=q=1 leads to a different covariance stationarity condition. In our notation the condition becomes

$$\alpha_1 + E(|\,\varepsilon_t\,| - \rho \varepsilon_t)^\delta \ \beta_1 < 1$$

Mittnik et al. (1999) first derived a closed form expression for a symmetric power-GARCH process with Lévy-stable innovations. They show that as δ approaches the characteristic exponent of the Lévy distribution α, $E\left|\varepsilon_t\right|^\delta$ increases without bounds and leads to an explosive process, which is not covariance stationary. Most GARCH-stable models in the literature are estimated setting δ equal to α and therefore fail to address the correct measure of stationarity (the paper by Liu and Brorsen (1995) is the most recent example). The reason for this is that the characteristic exponent α of the Lévy- or the degrees of freedom of the Student-t distribution gives a restriction on the number of moments that exist. Since the stationary condition of an asymmetric power-GARCH process requires the calculation of the fractional moment, $\delta=\alpha$ or $\delta=\theta$ is the limit case and not defined. Apart from the case $\lambda\to0$ the moments of the truncated Lévy distribution are finite and the process would be covariance stationary for $\delta=\alpha$. We estimated the model and observed that the alternative definition leads to a different α parameter, because we are combining the power transformation in the volatility process and the parameter of the distribution. We concluded that the two parameters δ and α have to be estimated separately. The motivation for a model with a power transformation of the conditional standard deviation process and the asymmetric absolute residuals lies in the observed autocorrelation functions of $|r_t|^{\delta}$, see Ding, Granger and Engle (1993) (δ can be interpreted as a separate heteroskedasticity parameter (He and Teräsvirta (1999)).

Another feature of the model is the asymmetric response of volatility to positive and negative shocks, well known as the leverage effect and empirical studies have shown that it is crucial to include leverage parameters (Glosten, Jaganathan and Runkle (1989), Nelson (1991), Ding, Granger and Engle (1993) and Engle and Ng (1992)). Engle and Ng (1993) and Hentschel (1995) studied the asymmetry in terms of the news impact curve. The news impact on volatility can be modeled either with a rotation parameter (like in the power-GARCH or EGARCH) or with a shift parameter (like in the NGARCH). Also a combination of both is possible. In most empirical papers one or the other asymmetric GARCH specification is used without testing if a shift or a rotation is appropriate.

We use a specification test based on the augmented GARCH process of Duan (1997) and determine the type of asymmetry for each data set. Additionally there is no evidence that the appropriate GARCH specification for Gaussian innovations is necessarily the appropriate specification for fat tailed skewed innovation. In particular the asymmetric parameter is assumed to capture the skewness in the data if the innovations distribution is symmetric. Since we are dealing with skewed distributions the relationship between asymmetry in the volatility process and skewness for the innovations distribution is crucial.

For example allowing for skewness in the innovations distribution of the power GARCH process leads to a smaller leverage effect (smaller γ). Therefore we have to test which specification is appropriate for a particular data set and a particular distribution for the innovations.

Duan (1997) developed the augmented GARCH process, a family of parametric GARCH models containing most of the existing GARCH specifications. The augmented GARCH specification can be used to construct a specification test. A Lagrange multiplier test provides an extremely useful class of diagnostic tests. With the augmented GARCH as a general alternative, it can be used to check whether a given GARCH specification is appropriate (see Duan (1997) for details). The LM test deals with local alternatives and suggests, which model is best among the models analyzed. The complete augmented GARCH (1,1) model reads:

$$r_t = \mu_t + \sigma_t \varepsilon_t$$

$$\varepsilon_t \sim D(0,1)$$

$$f_t = a_0 + g_{1,t-1} f_{t-1} + g_{2,t-1}$$

$$\sigma_t = \begin{cases} |\delta\phi_t - \delta + 1|^{1/2\delta} & \text{if } \delta \neq 0 \\ \sqrt{\exp(\phi_t - 1)} & \text{if } \delta = 0 \end{cases}$$

$$g_{1,t-1} = a_1 + a_2 |e_t - b|^k + a_3 \max(0, b - e_t)^k$$

$$g_{2,t-1} = a_4 \frac{|e_t - b|^k - 1}{k} + a_5 \frac{\max(0, b - e_t)^k - 1}{k},$$

where the conditional location parameter μ_t can be additionally specified and the conditional scale parameter σ_t is assumed to vary over time. This is a kind of generalization of the original process developed by Duan (1997). Our model is not restricted to zero-mean and variance-one distributions, it also allows for location-zero and scale–one continuous distributions D(0,1). The finite stationary scale parameter and the sufficient condition for strict stationarity of the location adjusted augmented GARCH(1,1) process can be expressed in a general condition. The stationary scale parameter is equal to

$$\bar{\phi} = \frac{\alpha_0 + \alpha_4 E\left[\frac{|\varepsilon_t - b|^{\kappa} - 1}{\kappa}\right] + \alpha_5 E\left[\frac{\max(0, b - \varepsilon_t)^{\kappa} - 1}{\kappa}\right]}{1 - \alpha_1 - \alpha_2 E[|\varepsilon_t - b|^{\kappa}] - \alpha_3 E[\max(0, b - \varepsilon_t)^{\kappa}]}$$

$$\bar{\sigma} = \begin{cases} |\delta\bar{\phi} - \delta + 1|^{1/2\delta} & \text{if} \quad \delta \neq 0 \\ \sqrt{\exp(\bar{\phi} - 1)} & \text{if} \quad \delta = 0 \end{cases}.$$

Therefore the location-adjusted augmented GARCH(1,1) process is strictly stationary if

$$\begin{array}{ll} \alpha_1 + \alpha_2 E[|\varepsilon_t - b|^{\kappa}] + \alpha_3 E[\max(0, b - \varepsilon_t)^{\kappa}] \leq 1 & \text{if} \delta \neq 0 \\ \alpha_1 < 1 & \text{if} \delta = 0 \end{array}.$$

Most GARCH models can be verified as special case of the augmented GARCH process. For some specifications there exist a closed form solution of the unconditional scale parameter or the stationary condition, but otherwise they have to be evaluated numerically. For example for δ=0, κ=1, α_2=0, α_3=0, b=0 and α_0, α_1, α_4, α_5 free we derive the EGARCH (1,1) process of Nelson (1991). In contrast to other studies (Hentschel (1995), Franses and van Dijk (1996), Kaiser (1996) and Mittnik et al. (1999)) we experienced no estimation problems with the EGARCH as a special case of the augmented GARCH process.

In the statistical part of the paper we compare three different specifications for the innovations distribution: the Gaussian, a skewed Student-t distribution called t_3 and the skewed truncated Lévy (stable). The GARCH model with normally distributed innovations is nested into both cases with time varying volatility and skewed t_3 *or* truncated Lévy distributed error terms.

All models are estimated with maximum likelihood. The log-likelihood for a series of observations is equal to the sum of the conditional log-likelihood of each observation in the sample. The log-likelihood is given by

$$\ln L = \sum_t \left(\ln f(\varepsilon_t, 0,1) - \ln(\sigma_t)\right)$$

$f(e_t,0,1)$ is the standardized density and the term $-\ln(s_t)$ results from taking the log of the Jacobian of the transformation. The ML estimators follow the standard theory, so they are consistent and asymptotically normal with mean the parameter estimates and a variance-covariance matrix.

Accurate numerical values for the density of the Lévy distributions can be calculated Fourier-transforming the CF and evaluating the integral numerically. We used Romberg integration, which allows the prespecification of the tolerated error and in fact a calculation of the density as precise as necessary (see Lambert and Lindsey (1999)). Obtaining the density is necessary for the maximum likelihood estimation of the model parameter.

7 EMPIRICAL RESULTS

In this study we use daily closing prices for some major stock market price indices between January 1984 and April 2000. In particular we examine the S&P500, NASDAQ, FTSE 100, HANG SENG and the NIKKEI 225 from May 4, 1992, to April 3, 2000. The total number of trading days covered by the data is between 1955 (NIKKEI 225) and 2001 (S&P500 and NASDAQ). The data are obtained from DataStream.

We used the percentage daily logarithmic change 100 * ln (p_t / p_{t-1}), where p_t is the price index at time t.

The summary statistics in table I show that there is skewness and excess kurtosis in the data. The S&P 500, NASDAQ and FTSE 100 show negative skewness related to a fat left tail for the return distribution and the Asian indices show positive skewness related to a fatter right tail. An obvious solution is to use an asymmetric time varying volatility process or a skewed fat tailed distribution. We will show that we need both and that it is not enough to model the excess kurtosis with a fat tailed distribution (Student-t) and trying to capture the skewness with an asymmetric volatility process.

There are some possibilities to model the observed skewness and kurtosis in financial return data. First we can model it by estimating the tails of the empirical distribution separately with a tail estimator or using a skewed fat tailed distribution directly. Second we can model it by fitting an asymmetric time varying volatility process to the data, because the time varying volatility is capturing the excess kurtosis and the asymmetry is capturing the skewness. But the time varying volatility cannot capture all the excess kurtosis and essentially the asymmetry describes the negative correlation between excess returns and volatility; therefore it can reduce the skewness, but it cannot account for all the skewness in the data. Third and more general we can use an *appropriate* asymmetric time-varying volatility process and model the excess kurtosis and skewness separately with a skewed fat tailed distribution. There is already evidence that the inclusion of time-varying volatility (e.g. GARCH) and tail fatness (e.g. Student-t) improve the model. But we have to test if the inclusion of asymmetry and skewed tail fatness improves the model, because both try to model the same phenomenon. In general there are two possibilities to model the asymmetry: shift or rotation of the news impact curve. Hentschel (1995) studied extensively the news impact curves of various GARCH models and performed a specification test based on the likelihood ratio. In the standard GARCH process the news impact curve has the form of a parabola with a minimum at zero, therefore negative and positive excess returns have the same impact on volatility. If we shift the news impact curve, excess returns with different signs become different impact on volatility. The above-mentioned negative correlation between returns and volatility can be modeled with a shift of the news impact curve to the right or a clockwise rotation of the news impact curve. Hentschel showed that the differences in the conditional volatility estimates could be substantially among the various specifications. But how well do skewed distributions work together with asymmetric volatility processes modeled via shift or rotation of the news impact curve? Since the choice of a particular specification will affect the conditional volatility estimate, it will also affect VaR estimates, which is of our particular interest.

In order to determine the appropriate model we make use of a specification test based on the augmented GARCH process developed by Duan (1997), a family of parametric GARCH specification. We conduct a Lagrange multiplier (LM) test, using the augmented GARCH (1,1) as the general alternative. The (robust) LM statistics suggest which of our models analyzed is best.

The parameters estimates of the Gaussian model and a model with skewness and conditional leptokurtosis are reported in table II. We calibrated the models by setting the unconditional scale parameter equal to the sample scale parameter and predetermined α_0. The results show that our choices of model specifications are motivated by the data.

The significant positive parameters b (NGARCH and power GARCH with shift), α_3 (power GARCH with rotation), α_4,α_5 (EGARCH) and the improved log-likelihood compared to the standard GARCH show generally that there is a leverage effect in index returns. But in particular if we compare the Gaussian and t_3 specifications, the parameters capturing the leverage effect are different and for the t_3 distribution skewness can also be estimated. In general the skewness parameter is reduced if we add an asymmetry parameter to the model. The comparison of the linear GARCH and the asymmetric GARCH models for the t_3 distribution shows that the asymmetric volatility models filter out some skewness, but there is still skewness in the residuals. For example for the S&P 500 and the NASDAQ we estimated a (significant) skewness parameter and as a result the shift parameter capturing the leverage effect is reduced compared to the Gaussian. The other indices show the same results for a shift in the volatility process, but mixed results for a rotation. In general we can say that if the innovations distribution is significantly skewed to the left, the shift parameter is reduced (NGARCH and power GARCH with shift). For the rotation parameter this is not necessarily true. We can conclude that there's perfect interaction between the skewness parameter of the innovations distribution and the shift parameter in the volatility process, but there is no clear interaction between the skewness parameter and the rotation parameter.

The LM statistics of all models are presented in table III. It is rather clear from the table that none of the specifications work best in all cases. Nearly each data set needs another GARCH specification to derive the best results. In particular under the hypothesis of conditional normality, a GARCH specification with asymmetry modeled with a rotation of the news impact curve (EGARCH) works best for the Nikkei 225, and the other indices are best described by a shift of the news impact curve (among the models analyzed). For the FTSE 100 we derive no different conclusion under the Hypothesis of a skewed fat tailed distribution. But surprisingly for the other indices the results from the specification test change if we make other assumptions about the innovations distribution. Suddenly a model with rotation is better than the model with shift and vice versa. The S&P 500, NASDAQ and Nikkei 225 are now best described by a power-GARCH with rotation and the description of the Hang Seng index can be improved by introducing a power transformation. These results underline the usefulness of a GARCH specification test, in particular under the hypothesis of skewed fat tailed distributions. Table IV presents the parameter estimates for the appropriate model specifications and the t_3 and truncated Lévy distribution.

Since we model a different scale parameter for each model, we cannot compare them by using likelihood ratio test. But given the appropriate specification we can analyze the residuals of each model by comparing the QQ-plots. Figure II shows the QQ-plots for the residuals of the different models for the whole sample. For illustration we are reporting the results for the NASDAQ index. In general for the power GARCH-N model with shift, which is the appropriate one under conditional normality, the graphs show that the residuals are not normally distributed: there is an underestimation in both tails and a deviation from the empirical distribution in the center. For the model with a fat tailed distribution (Student-t) it can be shown (results not reported), that the fit in the center of the distribution is better, but for every data set the graph shows an underestimation in the left tail and an overestimation in the right tail, because the maximum likelihood method just reports an average degrees of freedom parameter for the whole distribution. Despite the asymmetric volatility parameter in the two models there is still skewness in the residuals. Using a skewed distribution like the t_3 or Lévy distributions should solve this problem. Our approach allows fitting the tails of the distribution separately to the extreme negative *and* positive events without loosing information in the center of the distribution. The t_3 specification (power-GARCH with rotation) leads to a better fit in both tails compared to the Student-t, but the underestimation in the left tail and the overestimation in the right tail is just reduced. The QQ-plot for the residuals for the truncated Lévy model and the same specification shows that despite the skewness in the data the fit in the tails is extremely precise, there is only a slight overestimation in the tails. The same is not true for the model without cutoff in the tails, because the power law tails of the Lévy distribution lead to an extreme overestimation of the extreme events (results not reported).

From the best performance in-sample we can not necessarily conclude that the model has also superior performance out-of-sample. A backtesting approach will answer the question if the model is also relevant for VaR calculations.

8 IN- AND OUT-OF-SAMPLE VaR ANALYSES

In this section we test the models in- and out-of-sample to see how well they work in practice.

The standard RiskMetrics™ EWMA method assumes that the transformed data, the residuals, are iid standard normally distributed. Both the skewness and excess kurtosis are generally reduce after transformation, but both are still significantly different from zero. It also shows that it is important to incorporate skewness and kurtosis into the model.

First assuming normality we can use the stable property of the Normal distribution and scale the standard deviation at time t by a square-root-of-time rule to get the multi-day values. Second it is well known that we can use this property to estimate the multi-day VaR by multiplying the one-day VaR by $\sqrt{T}$. Our study shows for a more sophisticated volatility process that it is not possible to guarantee the iid-ness and normality of the residuals. This misspecification results in volatility forecasts using the square-root-of-time scaling rule, which are inappropriate.

Our approach is a generalization of the popular RiskMetrics™ EWMA method, but we are able to provide a better description of the data. First we use the appropriate GARCH model from the specification tests for the particular data set to estimate the time dependent volatility process and to filter out the serial correlations in the data and second we forecast volatility by assuming the truncated Lévy flight for the residuals. We estimate the characteristic exponent α of the truncated Lévy distribution, instead of assuming normality and setting it equal to 2. The process incorporates a α-root-of-time rule as a scaling method for the scale parameter C, instead of assuming normality and using the square-root-of-time rule. Therefore the α-root-of-time rule is the scaling method for the scale parameter C *and* the cutoff parameter λ of the Truncated Lévy Flight and as a result our volatility forecasting method. This allows us to estimate VaR some days ahead.

The very good performance of our model in-sample does not necessary lead to nice out-of-sample results. We have to back test our model over a longer period of time.

We want to back test our model for all indices over a period May 1996 until March 2000 and compare it to the RiskMetrics™ EWMA approach. Every day we estimate the model using the last about 1000 trading days (that means exactly one half of each sample and a moving window) and forecasted the 99% (95%) VaR 5, 10 and 20 days ahead. Table V reports the out-of-sample results. The results are very promising: the VaR estimates we obtained by using our method compared to the RiskMetrics™ method are lower in low volatility periods and higher in high volatility periods, while on average over the whole period our method produces less violations of the expected VaR for all confidence intervals and horizons. The RiskMetrics™ method constantly underpredicts extreme events and this leads very often to an inappropriate number of violations. But in particular this underprediction is slightly reduced for lower confidence intervals or longer forecasting horizons. This is a well-known result for the Gaussian and the square-root-of-time scaling rule. On the other side the GARCH model with the truncated Lévy distribution and the alpha-root-of-time scaling rule leads to an appropriate number of violations for low *and* high confidence intervals and short *and* long forecasting horizon. This means not only that the fit in the tails of the distribution is very good, but also that the scaling rule captures the scaling behavior of the data very well and shows a convergence from a skewed leptokurtic distribution to a Gaussian for larger sampling intervals. This is actually the unique bi-fractal scaling behavior of the truncated Lévy flight.

9 CONCLUSIONS

In this paper we propose a generalization of the popular RiskMetrics^{TM} approach. Our approach is able to capture the observed conditional tail fatness and skewness in financial returns. Using the truncated Lévy flight for the innovations of a GARCH process we propose a new scaling rule to forecast volatility. The in-sample performance of the model as well as the out-of-sample performance for VaR calculations is very satisfactory.

There are plenty of practical benefits of this model, which can be explored in future research.

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Table I: In-Sample Analysis: Parameter estimates

	Parameter	S&P 500	NASDAQ	FTSE 100	HANG SENG	NIKKEI 225
SUMMARY STATISTICS	MEAN	0.064	0.098	0.044	0.056	0.007
	ST, DEV,	0.907	1.228	0.942	1.889	1.430
	MINIMUM	-7.113	-8.954	-4.140	-14.735	-5.957
	MAXIMUM	4.989	5.848	4.345	17.247	7.660
	SKEWNESS	-0.398	-0.676	-0.055	0.097	0.200
	KURTOSIS	9.224	7.942	4.748	11.870	5.462
	# OF OBSERVATIONS	2001	2001	2000	1964	1955

Table II: In-Sample Analysis: Estimates of the scale parameter equation

Index	Parameter		GARCH		Power GARCH (rotation)		NGARCH		EGARCH		Power GARCH (shift)	
S&P 500												
N (0,1)	γ		1.000		0.755		1.000		0.000		0.902	
		(SE)		(0.000)		(0.050)		(0.000)		(0.000)		(0.056)
		(SE)		(0.000)		(0.100)		(0.000)		(0.000)		(0.108)
	κ		2.000		1.511		2.000		1.000		1.804	
		(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
		(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	α1		0.940		0.907		0.898		0.984		0.888	
		(SE)		(0.006)		(0.011)		(0.008)		(0.002)		(0.010)
		(SE)		(0.020)		(0.032)		(0.042)		(0.008)		(0.044)
	α2		0.054		0.026		0.064		0.000		0.083	
		(SE)		(0.005)		(0.016)		(0.007)		(0.000)		(0.013)
		(SE)		(0.017)		(0.018)		(0.021)		(0.000)		(0.037)
	α3		0.000		0.149		0.000		0.000		0.000	
		(SE)		(0.000)		(0.021)		(0.000)		(0.000)		(0.000)
		(SE)		(0.000)		(0.055)		(0.000)		(0.000)		(0.000)
	α4		0.000		0.000		0.000		0.055		0.000	
		(SE)		(0.000)		(0.000)		(0.000)		(0.019)		(0.000)
		(SE)		(0.000)		(0.000)		(0.000)		(0.018)		(0.000)
	α5		0.000		0.000		0.000		0.162		0.000	
		(SE)		(0.000)		(0.000)		(0.000)		(0.018)		(0.000)
		(SE)		(0.000)		(0.000)		(0.000)		(0.055)		(0.000)
	b		0.000		0.000		0.660		0.000		0.610	
		(SE)		(0.000)		(0.000)		(0.126)		(0.000)		(0.115)
		(SE)		(0.000)		(0.000)		(0.118)		(0.000)		(0.114)
	Log-Likelihood		-2376.8		-2354.5		-2359.5		-2354.6		-2358.7	
t3 (0,1,v,d, b)	γ		1.000		0.663		1.000		0.000		0.842	
		(SE)		(0.000)		(0.100)		(0.000)		(0.000)		(0.105)
		(SE)		(0.000)		(0.153)		(0.000)		(0.000)		(0.153)
	κ		2.000		1.325		2.000		1.000		1.685	
		(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
		(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	α1		0.953		0.917		0.927		0.991		0.913	
		(SE)		(0.008)		(0.019)		(0.010)		(0.003)		(0.017)
		(SE)		(0.015)		(0.025)		(0.025)		(0.006)		(0.030)
	α2		0.053		0.038		0.062		0.000		0.092	
		(SE)		(0.010)		(0.030)		(0.011)		(0.000)		(0.027)
		(SE)		(0.016)		(0.028)		(0.015)		(0.000)		(0.038)
	α3		0.000		0.157		0.000		0.000		0.000	
		(SE)		(0.000)		(0.039)		(0.000)		(0.000)		(0.000)
		(SE)		(0.000)		(0.049)		(0.000)		(0.000)		(0.000)
	α4		0.000		0.000		0.000		0.057		0.000	
		(SE)		(0.000)		(0.000)		(0.000)		(0.029)		(0.000)
		(SE)		(0.000)		(0.000)		(0.000)		(0.018)		(0.000)
	α5		0.000		0.000		0.000		0.154		0.000	
		(SE)		(0.000)		(0.000)		(0.000)		(0.032)		(0.000)
		(SE)		(0.000)		(0.000)		(0.000)		(0.053)		(0.000)
	b		0.000		0.000		0.532		0.000		0.457	
		(SE)		(0.000)		(0.000)		(0.172)		(0.000)		(0.148)
		(SE)		(0.000)		(0.000)		(0.134)		(0.000)		(0.145)
	v		5.566		5.343		6.031		5.263		5.866	
		(SE)		(0.171)		(2.447)		(2.637)		(2.435)		(2.677)
		(SE)		(0.178)		(2.534)		(3.168)		(2.595)		(3.093)
	d		1.715		1.733		1.702		1.727		1.709	
		(SE)		(0.026)		(0.179)		(0.166)		(0.179)		(0.172)
		(SE)		(0.023)		(0.170)		(0.184)		(0.177)		(0.183)
	β		0.975		1.008		1.003		0.999		1.000	
		(SE)		(0.025)		(0.027)		(0.027)		(0.027)		(0.027)
		(SE)		(0.026)		(0.018)		(0.023)		(0.022)		(0.025)
	Log-Likelihood		-2316.6		-2303.5		-2309.4		-2304.1		-2308.6	

Table II: In-Sample Analysis: Estimates of the scale parameter equation

Index	Parameter	GARCH	Power GARCH (rotation)	NGARCH	EGARCH	Power GARCH (shift)
Nasdaq						
N (0,1)	γ	1.000	0.675	1.000	0.000	0.770
	(SE)	(0.000)	(0.083)	(0.000)	(0.000)	(0.088)
	(SE)	(0.000)	(0.080)	(0.000)	(0.000)	(0.085)
	κ	2.000	1.350	2.000	1.000	1.540
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	α1	0.859	0.818	0.828	0.976	0.805
	(SE)	(0.012)	(0.025)	(0.015)	(0.004)	(0.023)
	(SE)	(0.045)	(0.050)	(0.056)	(0.014)	(0.049)
	α2	0.123	0.138	0.136	0.000	0.189
	(SE)	(0.010)	(0.034)	(0.014)	(0.000)	(0.033)
	(SE)	(0.036)	(0.032)	(0.036)	(0.000)	(0.042)
	α3	0.000	0.124	0.000	0.000	0.000
	(SE)	(0.000)	(0.015)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.061)	(0.000)	(0.000)	(0.000)
	α4	0.000	0.000	0.000	0.182	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.024)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.044)	(0.000)
	α5	0.000	0.000	0.000	0.117	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.016)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.053)	(0.000)
	b	0.000	0.000	0.311	0.000	0.291
	(SE)	(0.000)	(0.000)	(0.047)	(0.000)	(0.044)
	(SE)	(0.000)	(0.000)	(0.097)	(0.000)	(0.091)
	Log-Likelihood	-2888.7	-2872.2	-2875.5	-2876.2	-2872.0
t3 (0,1,v,d, b)	γ	1.000	0.590	1.000	0.000	0.614
	(SE)	(0.000)	(0.119)	(0.000)	(0.000)	(0.126)
	(SE)	(0.000)	(0.086)	(0.000)	(0.000)	(0.093)
	κ	2.000	1.180	2.000	1.000	1.228
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	α1	0.908	0.864	0.905	0.993	0.859
	(SE)	(0.012)	(0.028)	(0.012)	(0.004)	(0.028)
	(SE)	(0.036)	(0.040)	(0.039)	(0.008)	(0.042)
	α2	0.113	0.161	0.113	0.000	0.198
	(SE)	(0.015)	(0.048)	(0.017)	(0.000)	(0.044)
	(SE)	(0.033)	(0.043)	(0.033)	(0.000)	(0.050)
	α3	0.000	0.060	0.000	0.000	0.000
	(SE)	(0.000)	(0.032)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.037)	(0.000)	(0.000)	(0.000)
	α4	0.000	0.000	0.000	0.201	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.042)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.050)	(0.000)
	α5	0.000	0.000	0.000	0.060	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.035)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.039)	(0.000)
	b	0.000	0.000	0.101	0.000	0.088
	(SE)	(0.000)	(0.000)	(0.084)	(0.000)	(0.068)
	(SE)	(0.000)	(0.000)	(0.088)	(0.000)	(0.058)
	v	6.592	5.773	6.260	5.059	6.190
	(SE)	(0.189)	(2.859)	(3.106)	(2.377)	(3.208)
	(SE)	(0.248)	(3.429)	(4.054)	(2.646)	(3.785)
	d	1.850	1.860	1.867	1.933	1.835
	(SE)	(0.021)	(0.194)	(0.191)	(0.207)	(0.190)
	(SE)	(0.020)	(0.237)	(0.267)	(0.227)	(0.229)
	β	0.877	0.889	0.890	0.879	0.883
	(SE)	(0.030)	(0.023)	(0.023)	(0.023)	(0.023)
	(SE)	(0.034)	(0.025)	(0.023)	(0.021)	(0.022)
	Log-Likelihood	-2838.5	-2830.8	-2837.6	-2831.6	-2831.7

Table II: In-Sample Analysis: Estimates of the scale parameter equation

Index	Parameter	GARCH		Power GARCH (rotation)		NGARCH		EGARCH		Power GARCH (shift)	
Ftse 100											
N (0,1)	γ	1.000		0.673		1.000		0.000		0.756	
	(SE)		(0.000)		(0.131)		(0.000)		(0.000)		(0.141)
	(SE)		(0.000)		(0.168)		(0.000)		(0.000)		(0.147)
	κ	2.000		1.346		2.000		1.000		1.512	
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	α1	0.951		0.930		0.944		0.993		0.921	
	(SE)		(0.007)		(0.019)		(0.006)		(0.002)		(0.017)
	(SE)		(0.011)		(0.022)		(0.014)		(0.003)		(0.028)
	α2	0.044		0.044		0.039		0.000		0.070	
	(SE)		(0.006)		(0.026)		(0.006)		(0.000)		(0.026)
	(SE)		(0.009)		(0.022)		(0.008)		(0.000)		(0.029)
	α3	0.000		0.069		0.000		0.000		0.000	
	(SE)		(0.000)		(0.020)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.031)		(0.000)		(0.000)		(0.000)
	α4	0.000		0.000		0.000		0.056		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.020)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.016)		(0.000)
	α5	0.000		0.000		0.000		0.070		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.018)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.024)		(0.000)
	b	0.000		0.000		0.561		0.000		0.544	
	(SE)		(0.000)		(0.000)		(0.159)		(0.000)		(0.152)
	(SE)		(0.000)		(0.000)		(0.158)		(0.000)		(0.151)
	Log-Likelihood	-2525.2		-2517.3		-2516.6		-2520.0		-2515.2	
t3 (0,1,v,d,b)	γ	1.000		0.707		1.000		0.000		0.690	
	(SE)		(0.000)		(0.161)		(0.000)		(0.000)		(0.166)
	(SE)		(0.000)		(0.176)		(0.000)		(0.000)		(0.200)
	κ	2.000		1.414		2.000		1.000		1.380	
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	α1	0.950		0.944		0.941		0.995		0.923	
	(SE)		(0.009)		(0.014)		(0.009)		(0.003)		(0.017)
	(SE)		(0.010)		(0.014)		(0.011)		(0.003)		(0.022)
	α2	0.082		0.043		0.071		0.000		0.107	
	(SE)		(0.014)		(0.027)		(0.013)		(0.000)		(0.028)
	(SE)		(0.015)		(0.021)		(0.012)		(0.000)		(0.031)
	α3	0.000		0.096		0.000		0.000		0.000	
	(SE)		(0.000)		(0.027)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.030)		(0.000)		(0.000)		(0.000)
	α4	0.000		0.000		0.000		0.069		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.029)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.022)		(0.000)
	α5	0.000		0.000		0.000		0.108		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.030)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.031)		(0.000)
	b	0.000		0.000		0.438		0.000		0.407	
	(SE)		(0.000)		(0.000)		(0.135)		(0.000)		(0.116)
	(SE)		(0.000)		(0.000)		(0.120)		(0.000)		(0.074)
	v	4.469		3.638		3.648		3.741		3.551	
	(SE)		(0.285)		(1.582)		(1.488)		(1.699)		(1.454)
	(SE)		(0.271)		(1.422)		(1.380)		(1.473)		(1.377)
	d	2.346		2.437		2.471		2.416		2.472	
	(SE)		(0.026)		(0.292)		(0.287)		(0.292)		(0.291)
	(SE)		(0.023)		(0.285)		(0.283)		(0.266)		(0.288)
	β	0.938		0.976		0.980		0.969		0.972	
	(SE)		(0.035)		(0.028)		(0.028)		(0.027)		(0.028)
	(SE)		(0.034)		(0.030)		(0.028)		(0.027)		(0.030)
	Log-Likelihood	-2515.0		-2508.0		-2507.1		-2510.0		-2505.7	

Table II: In-Sample Analysis: Estimates of the scale parameter equation

Index	Parameter	GARCH		Power GARCH (rotation)		NGARCH		EGARCH		Power GARCH (shift)	
Hang Seng											
N (0,1)	γ	1.000		0.686		1.000		0.000		0.882	
	(SE)		(0.000)		(0.102)		(0.000)		(0.000)		(0.123)
	(SE)		(0.000)		(0.087)		(0.000)		(0.000)		(0.120)
	κ	2.000		1.372		2.000		1.000		1.764	
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	$\alpha1$	0.887		0.883		0.876		0.977		0.875	
	(SE)		(0.011)		(0.012)		(0.010)		(0.003)		(0.011)
	(SE)		(0.021)		(0.018)		(0.021)		(0.007)		(0.021)
	$\alpha2$	0.099		0.070		0.094		0.000		0.106	
	(SE)		(0.009)		(0.015)		(0.009)		(0.000)		(0.016)
	(SE)		(0.017)		(0.018)		(0.013)		(0.000)		(0.019)
	$\alpha3$	0.000		0.100		0.000		0.000		0.000	
	(SE)		(0.000)		(0.013)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.028)		(0.000)		(0.000)		(0.000)
	$\alpha4$	0.000		0.000		0.000		0.129		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.020)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.023)		(0.000)
	$\alpha5$	0.000		0.000		0.000		0.128		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.018)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.034)		(0.000)
	b	0.000		0.000		0.393		0.000		0.371	
	(SE)		(0.000)		(0.000)		(0.065)		(0.000)		(0.064)
	(SE)		(0.000)		(0.000)		(0.110)		(0.000)		(0.123)
	Log-Likelihood	-3709.3		-3688.6		-3696.7		-3689.0		-3696.2	
t3 (0,1,v,d, b)	γ	1.000		0.630		1.000		0.000		0.840	
	(SE)		(0.000)		(0.154)		(0.000)		(0.000)		(0.185)
	(SE)		(0.000)		(0.097)		(0.000)		(0.000)		(0.121)
	κ	2.000		1.260		2.000		1.000		1.681	
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.000)		(0.000)
	$\alpha1$	0.901		0.892		0.880		0.988		0.882	
	(SE)		(0.015)		(0.018)		(0.016)		(0.004)		(0.017)
	(SE)		(0.020)		(0.017)		(0.023)		(0.005)		(0.023)
	$\alpha2$	0.104		0.083		0.106		0.000		0.123	
	(SE)		(0.019)		(0.028)		(0.019)		(0.000)		(0.030)
	(SE)		(0.021)		(0.020)		(0.019)		(0.000)		(0.022)
	$\alpha3$	0.000		0.114		0.000		0.000		0.000	
	(SE)		(0.000)		(0.027)		(0.000)		(0.000)		(0.000)
	(SE)		(0.000)		(0.034)		(0.000)		(0.000)		(0.000)
	$\alpha4$	0.000		0.000		0.000		0.140		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.034)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.025)		(0.000)
	$\alpha5$	0.000		0.000		0.000		0.132		0.000	
	(SE)		(0.000)		(0.000)		(0.000)		(0.034)		(0.000)
	(SE)		(0.000)		(0.000)		(0.000)		(0.039)		(0.000)
	b	0.000		0.000		0.408		0.000		0.375	
	(SE)		(0.000)		(0.000)		(0.111)		(0.000)		(0.105)
	(SE)		(0.000)		(0.000)		(0.105)		(0.000)		(0.122)
	v	17.510		18.498		14.369		12.060		14.838	
	(SE)		(10.145)		(22.798)		(14.768)		(11.187)		(15.367)
	(SE)		(11.158)		(26.439)		(17.864)		(12.371)		(18.262)
	d	1.428		1.447		1.457		1.543		1.452	
	(SE)		(0.123)		(0.148)		(0.153)		(0.162)		(0.151)
	(SE)		(0.124)		(0.156)		(0.167)		(0.165)		(0.162)
	β	0.970		0.996		0.995		0.991		0.993	
	(SE)		(0.046)		(0.025)		(0.025)		(0.025)		(0.026)
	(SE)		(0.050)		(0.020)		(0.022)		(0.017)		(0.021)
	Log-Likelihood	-3659.5		-3646.1		-3651.0		-3651.4		-3650.5	

Table II: In-Sample Analysis: Estimates of the scale parameter equation

Index	Parameter	GARCH	Power GARCH (rotation)	NGARCH	EGARCH	Power GARCH (shift)
Nikkei 225						
N (0,1)	γ	1.000	0.613	1.000	0.000	0.390
	(SE)	(0.000)	(0.117)	(0.000)	(0.000)	(0.076)
	(SE)	(0.000)	(0.225)	(0.000)	(0.000)	(0.161)
	κ	2.000	1.227	2.000	1.000	0.780
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	$\alpha1$	0.870	0.867	0.863	0.962	0.774
	(SE)	(0.014)	(0.015)	(0.015)	(0.004)	(0.026)
	(SE)	(0.035)	(0.058)	(0.036)	(0.023)	(0.108)
	$\alpha2$	0.090	0.079	0.086	0.000	0.214
	(SE)	(0.010)	(0.015)	(0.009)	(0.000)	(0.031)
	(SE)	(0.019)	(0.070)	(0.025)	(0.000)	(0.106)
	$\alpha3$	0.000	0.087	0.000	0.000	0.000
	(SE)	(0.000)	(0.016)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.045)	(0.000)	(0.000)	(0.000)
	$\alpha4$	0.000	0.000	0.000	0.116	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.012)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.070)	(0.000)
	$\alpha5$	0.000	0.000	0.000	0.122	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.017)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.056)	(0.000)
	b	0.000	0.000	0.426	0.000	0.527
	(SE)	(0.000)	(0.000)	(0.046)	(0.000)	(0.033)
	(SE)	(0.000)	(0.000)	(0.301)	(0.000)	(0.014)
	Log-Likelihood	-3396.9	-3385.7	-3387.8	-3383.2	-3380.1
t3 (0,1,v,d, b)	γ	1.000	0.725	1.000	0.000	0.723
	(SE)	(0.000)	(0.153)	(0.000)	(0.000)	(0.168)
	(SE)	(0.000)	(0.114)	(0.000)	(0.000)	(0.227)
	κ	2.000	1.451	2.000	1.000	1.446
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	$\alpha1$	0.919	0.919	0.897	0.986	0.885
	(SE)	(0.013)	(0.016)	(0.015)	(0.005)	(0.019)
	(SE)	(0.014)	(0.014)	(0.017)	(0.005)	(0.024)
	$\alpha2$	0.090	0.040	0.079	0.000	0.117
	(SE)	(0.015)	(0.021)	(0.013)	(0.000)	(0.030)
	(SE)	(0.015)	(0.016)	(0.013)	(0.000)	(0.036)
	$\alpha3$	0.000	0.138	0.000	0.000	0.000
	(SE)	(0.000)	(0.028)	(0.000)	(0.000)	(0.000)
	(SE)	(0.000)	(0.029)	(0.000)	(0.000)	(0.000)
	$\alpha4$	0.000	0.000	0.000	0.077	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.028)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.029)	(0.000)
	$\alpha5$	0.000	0.000	0.000	0.184	0.000
	(SE)	(0.000)	(0.000)	(0.000)	(0.034)	(0.000)
	(SE)	(0.000)	(0.000)	(0.000)	(0.036)	(0.000)
	b	0.000	0.000	0.683	0.000	0.600
	(SE)	(0.000)	(0.000)	(0.136)	(0.000)	(0.112)
	(SE)	(0.000)	(0.000)	(0.152)	(0.000)	(0.012)
	v	2.747	2.350	2.541	2.196	2.417
	(SE)	(0.198)	(0.521)	(0.538)	(0.506)	(0.532)
	(SE)	(0.265)	(0.794)	(0.832)	(0.701)	(0.839)
	d	2.120	2.256	2.230	2.300	2.255
	(SE)	(0.025)	(0.217)	(0.205)	(0.230)	(0.215)
	(SE)	(0.024)	(0.292)	(0.283)	(0.281)	(0.298)
	β	0.956	1.014	1.016	1.004	1.007
	(SE)	(0.047)	(0.027)	(0.027)	(0.028)	(0.028)
	(SE)	(0.048)	(0.029)	(0.029)	(0.027)	(0.026)
	Log-Likelihood	-3300.6	-3284.5	-3285.4	-3284.1	-3284.6

Table III: In-Sample Analysis: Augmented GARCH(1,1) based specification test

Index			GARCH	Power GARCH (rotation)	NGARCH	EGARCH	Power GARCH (shift)
S&P 500							
	N (0,1)	LM	42.9	17.8	10.9	22.1	9.8
		(p-value)	(0.0000)	(0.0013)	(0.0542)	(0.0005)	(0.0439)
		Robust LM	41.9	9.9	9.0	20.3	5.5
		(p-value)	(0.0000)	(0.0421)	(0.1075)	(0.0011)	(0.2362)
	t3 (0,1,v,d, b)	LM	37.4	9.7	19.0	13.4	21.4
	truncated stable (0,1, a , l ,b)	(p-value)	(0.0000)	(0.0454)	(0.0019)	(0.0198)	(0.0003)
		Robust LM	33.3	4.9	11.0	10.5	8.3
		(p-value)	(0.0000)	(0.3030)	(0.0504)	(0.0613)	(0.0799)
Nasdaq							
	N (0,1)	LM	35.1	1.2	10.8	15.8	0.8
		(p-value)	(0.0000)	(0.8825)	(0.0544)	(0.0076)	(0.9401)
		Robust LM	33.2	1.0	9.9	15.4	0.7
		(p-value)	(0.0000)	(0.9060)	(0.0779)	(0.0089)	(0.9472)
	t3 (0,1,v,d, b)	LM	19.6	2.1	17.8	8.3	3.9
	truncated stable (0,1, a , l ,b)	(p-value)	(0.0032)	(0.7260)	(0.0032)	(0.1415)	(0.4206)
		Robust LM	16.6	1.6	14.7	8.1	2.7
		(p-value)	(0.0110)	(0.8054)	(0.0119)	(0.1482)	(0.6159)
Ftse 100							
	N (0,1)	LM	20.8	5.7	3.6	17.7	4.6
		(p-value)	(0.0020)	(0.2214)	(0.6064)	(0.0034)	(0.3354)
		Robust LM	18.4	4.6	3.0	17.4	4.0
		(p-value)	(0.0054)	(0.3280)	(0.7041)	(0.0038)	(0.4090)
	t3 (0,1,v,d, b)	LM	26.3	17.7	8.5	12.8	6.6
	truncated stable (0,1, a , l ,b)	(p-value)	(0.0002)	(0.0014)	(0.1291)	(0.0249)	(0.1591)
		Robust LM	19.8	11.7	6.2	11.7	5.1
		(p-value)	(0.0031)	(0.0195)	(0.2858)	(0.0387)	(0.2727)
Hang Seng							
	N (0,1)	LM	43.6	13.0	14.6	25.2	14.4
		(p-value)	(0.0000)	(0.0115)	(0.0120)	(0.0001)	(0.0060)
		Robust LM	38.0	12.3	10.1	24.2	8.8
		(p-value)	(0.0000)	(0.0154)	(0.0736)	(0.0002)	(0.0662)
	t3 (0,1,v,d, b)	LM	43.3	19.0	19.4	23.5	18.7
	truncated stable (0,1, a , l ,b)	(p-value)	(0.0000)	(0.0008)	(0.0016)	(0.0003)	(0.0009)
		Robust LM	33.2	15.7	10.9	21.7	8.3
		(p-value)	(0.0000)	(0.0035)	(0.0534)	(0.0006)	(0.0807)
Nikkei 225							
	N (0,1)	LM	22.2	7.7	11.1	6.4	6.5
		(p-value)	(0.0011)	(0.1018)	(0.0497)	(0.2709)	(0.1652)
		Robust LM	16.8	4.9	8.4	4.3	5.2
		(p-value)	(0.0102)	(0.2954)	(0.1360)	(0.5008)	(0.2689)
	t3 (0,1,v,d, b)	LM	35.5	1.9	9.1	6.5	3.0
	truncated stable (0,1, a , l ,b)	(p-value)	(0.0000)	(0.7540)	(0.1050)	(0.2641)	(0.5500)
		Robust LM	31.1	1.3	7.4	6.3	1.8
		(p-value)	(0.0000)	(0.8609)	(0.1924)	(0.2769)	(0.7760)

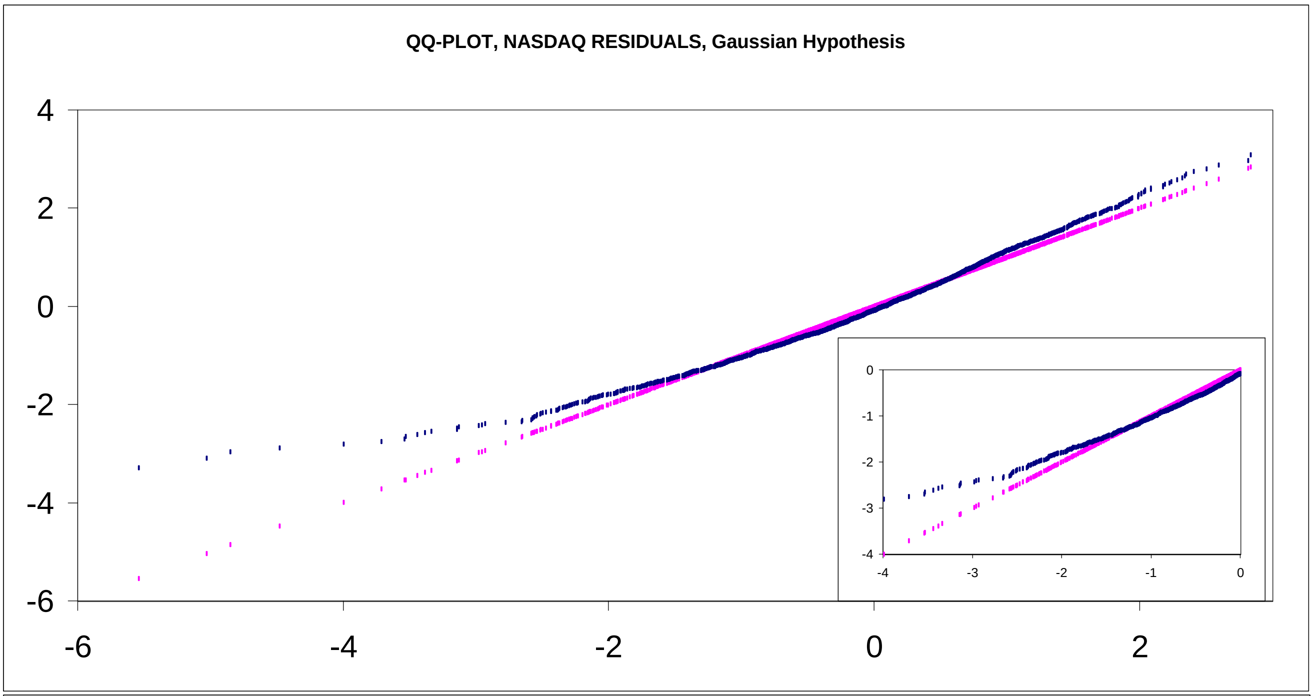
Table IV: In-Sample Analysis: Augmented GARCH(1,1) process and t_3 or truncated Levy hypothesis

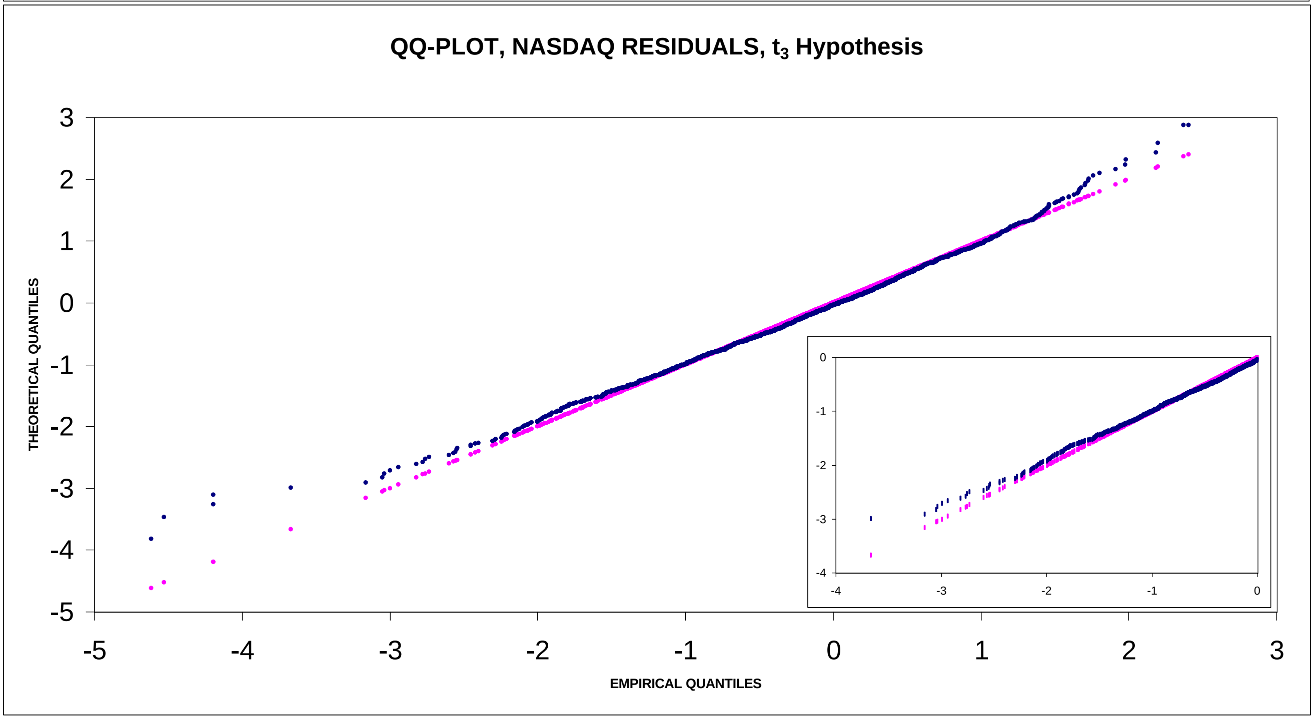
Index	Log-Likelihood	Parameter γ	κ	α1	α2	α3	α4	α5	b	ν / α	d / λ	β
S&P 500												
POWER GARCH, ROTATION												
t_3 (0,1,v,d, b)	-2303.5	0.6626	1.3253	0.9168	0.0383	0.1570	0.0000	0.0000	0.0000	5.3433	1.7325	1.0078
		(0.1002)	(0.0000)	(0.0193)	(0.0297)	(0.0388)	(0.0000)	(0.0000)	(0.0000)	(2.4472)	(0.1788)	(0.0266)
		(0.1532)	(0.0000)	(0.0254)	(0.0275)	(0.0493)	(0.0000)	(0.0000)	(0.0000)	(2.5337)	(0.1695)	(0.0177)
truncated stable (0,1, a,l,b)	-2304.5	0.8802	1.7604	0.9270	0.0080	0.0725	0.0000	0.0000	0.0000	1.6985	0.2349	0.2756
		(0.0334)	(0.0000)	(0.0146)	(0.0091)	(0.0197)	(0.0000)	(0.0000)	(0.0000)	(0.1049)	(0.1166)	(0.1327)
		(0.0577)	(0.0000)	(0.0180)	(0.0069)	(0.0258)	(0.0000)	(0.0000)	(0.0000)	(0.1846)	(0.1531)	(0.2483)
Nasdaq												
POWER GARCH, ROTATION												
t_3 (0,1,v,d, b)	*-2830.8*	0.5898	1.1796	0.8643	0.1607	0.0597	0.0000	0.0000	0.0000	5.7726	1.8600	0.8885
		(0.1189)	(0.0000)	(0.0276)	(0.0480)	(0.0322)	(0.0000)	(0.0000)	(0.0000)	(2.8585)	(0.1936)	(0.0230)
		(0.0864)	(0.0000)	(0.0402)	(0.0428)	(0.0371)	(0.0000)	(0.0000)	(0.0000)	(3.4286)	(0.0000)	(0.0251)
truncated stable (0,1, a,l,b)	*-2815.3*	0.790842	1.5817	0.876852	0.040436	0.070832	0.0000	0.0000	0.0000	1.813647	0.088733	0.918438
		(0.0519)	(0.0000)	(0.0220)	(0.0146)	(0.0216)	(0.0000)	(0.0000)	(0.0000)	(0.0445)	(0.0410)	(0.1407)
		(0.0488)	(0.0000)	(0.0288)	(0.0116)	(0.0220)	(0.0000)	(0.0000)	(0.0000)	(0.0362)	(0.0459)	(0.1200)
Ftse 100												
NGARCH, SHIFT												
t_3 (0,1,v,d, b)	*-2507.1*	1.0000	2.0000	0.9411	0.0713	0.0000	0.0000	0.0000	0.4381	3.6484	2.4710	0.9800
		(0.0000)	(0.0000)	(0.0088)	(0.0134)	(0.0000)	(0.0000)	(0.0000)	(0.1353)	(1.4876)	(0.2874)	(0.0282)
		(0.0000)	(0.0000)	(0.0110)	(0.0120)	(0.0000)	(0.0000)	(0.0000)	(0.1198)	(1.3795)	(0.2828)	(0.0282)
truncated stable (0,1, a,l,b)	*-2505.3*	1.0000	2.0000	0.9348	0.0188	0.0000	0.0000	0.0000	1.1163	1.9057	0.1639	0.4193
		(0.0000)	(0.0000)	(0.0101)	(0.0035)	(0.0000)	(0.0000)	(0.0000)	(0.2745)	(0.0653)	(0.1570)	(0.2889)
		(0.0000)	(0.0000)	(0.0113)	(0.0031)	(0.0000)	(0.0000)	(0.0000)	(0.2420)	(0.0548)	(0.1171)	(0.2889)
Hang Seng												
POWER GARCH, SHIFT												
t_3 (0,1,v,d, b)	*-3650.5*	0.8403	1.6806	0.8817	0.1226	0.0000	0.0000	0.0000	0.3750	14.8378	1.4524	0.9934
		(0.1847)	(0.0000)	(0.0171)	(0.0301)	(0.0000)	(0.0000)	(0.0000)	(0.1050)	(15.3670)	(0.1506)	(0.0256)
		(0.1209)	(0.0000)	(0.0227)	(0.0219)	(0.0000)	(0.0000)	(0.0000)	(0.1219)	(18.2621)	(0.1618)	(0.0210)
truncated stable (0,1, a,l,b)	*-3659.3*	0.7260	1.4520	0.8876	0.0642	0.0000	0.0000	0.0000	0.6443	1.6208	0.2401	0.0064
		(0.1212)	(0.0000)	(0.0195)	(0.0233)	(0.0000)	(0.0000)	(0.0000)	(0.1595)	(0.1133)	(0.1411)	(0.0773)
		(0.0750)	(0.0000)	(0.0303)	(0.0174)	(0.0000)	(0.0000)	(0.0000)	(0.2598)	(0.1116)	(0.0961)	(0.0804)
Nikkei 225												
POWER GARCH, ROTATION												
t_3 (0,1,v,d, b)	*-3284.5*	0.7254	1.4508	0.9186	0.0403	0.1380	0.0000	0.0000	0.0000	2.3501	2.2564	1.0142
		(0.1533)	(0.0000)	(0.0156)	(0.0214)	(0.0276)	(0.0000)	(0.0000)	(0.0000)	(0.5206)	(0.2167)	(0.0269)
		(0.1141)	(0.0000)	(0.0136)	(0.0164)	(0.0293)	(0.0000)	(0.0000)	(0.0000)	(0.7943)	(0.2920)	(0.0291)
truncated Lévy (0,1, a,l,b)	*-3282.9*	0.8642	1.7285	0.9141	0.0144	0.0531	0.0000	0.0000	0.0000	1.8087	0.0715	0.0300
		(0.1001)	(0.0000)	(0.0177)	(0.0085)	(0.0171)	(0.0000)	(0.0000)	(0.0000)	(0.0610)	(0.0629)	(0.1327)
		(0.1120)	(0.0000)	(0.0118)	(0.0152)	(0.0192)	(0.0000)	(0.0000)	(0.0000)	(0.0741)	(0.0700)	(0.0956)

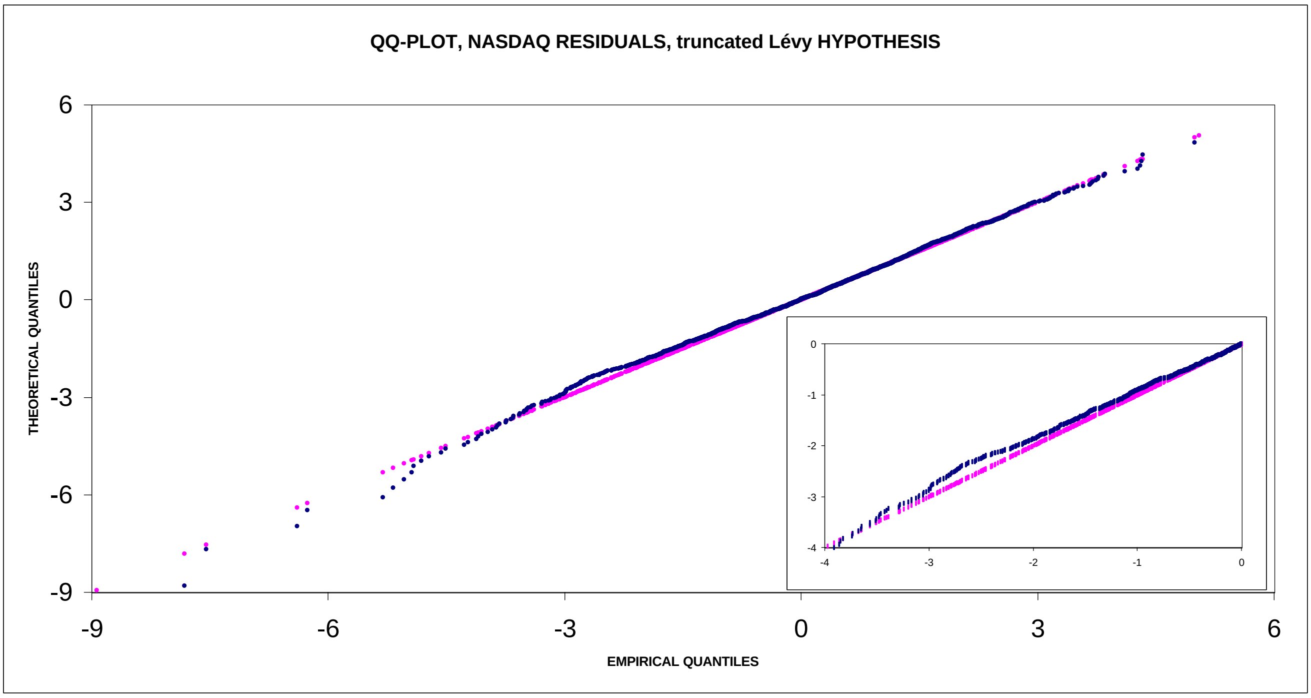
Table V: Out-of-Sample Analysis: Violations

	HORIZON (DAYS)	VaR	S&P 500	NASDAQ	FTSE 100	HANG SENG	NIKKEI 225
EWMA-N	5	99%	2.6%	3.3%	2.9%	3.2%	0.8%
		95%	5.9%	7.6%	6.1%	8.1%	5.0%
	10	99%	1.8%	2.7%	1.1%	2.4%	0.7%
		95%	5.2%	6.6%	4.9%	8.6%	4.4%
	20	99%	0.9%	2.3%	1.3%	4.8%	2.0%
		95%	4.6%	7.2%	4.3%	8.9%	6.3%
AUGMENTED GARCH-TRUNCATED LÉVY	5	99%	1.5%	1.2%	1.3%	1.3%	0.7%
		95%	5.2%	4.9%	5.3%	6.6%	4.0%
	10	99%	1.2%	1.3%	0.8%	1.5%	0.7%
		95%	4.3%	5.5%	4.0%	6.0%	3.8%
	20	99%	0.7%	1.4%	0.9%	1.4%	1.3%
		95%	3.7%	4.6%	3.9%	6.1%	4.6%

Notes. This table presents violations of the expected multi-day VaR (given an initial position of 100), where the model assumes 1% (5%) violations on average for the 99% (95%) level. The underlying models have been estimated every day for the second half of the period 05/1992 until 03/2000 on daily returns and a moving window.


QQ-PLOT, NASDAQ RESIDUALS, Gaussian Hypothesis


QQ-PLOT, NASDAQ RESIDUALS, t_3 Hypothesis
THEORETICAL QUANTILES
EMPIRICAL QUANTILES


QQ-PLOT, NASDAQ RESIDUALS, truncated Lévy HYPOTHESIS
THEORETICAL QUANTILES
EMPIRICAL QUANTILES