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CONSUMPTION–SAVINGS DECISIONS WITH QUASI-GEOMETRIC DISCOUNTING

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ABSTRACT

Consumption-Savings Decisions with Quasi-Geometric Discounting*

How do individuals with time-inconsistent preferences make consumption-savings decisions? We try to answer this question by considering the simplest possible form of the consumption-savings problem, assuming that discounting is quasi-geometric. A solution to the decision problem is then a subgame-perfect equilibrium of a dynamic game between the individual's 'successive selves'. When the time horizon is finite, our question has a well-defined answer in terms of primitives. When the time horizon is infinite, we are left without a sharp answer: we cannot rule out the possibility that two identical individuals in the exact same situation make different decisions! In particular, there is a continuum of dynamic equilibria even if we restrict attention to equilibria where current consumption decisions depend only on current wealth.

JEL Classification: C73, D90, E21
Keywords: indeterminacy, quasi-geometric discounting, time inconsistency

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NON-TECHNICAL SUMMARY

The purpose of this Paper is to study how an infinitely-lived, rational consumer with 'quasi-geometric' discounting would make consumption and savings decisions. We consider the idea that a consumer's evaluation of 'utils' at different points in time does not have to take the form of an aggregate with geometric weights. This idea was suggested first by Strotz (1956), and later elaborated on by Pollak (1968), Phelps and Pollak (1968), Laibson (1994, 1997) and others. Quasi-geometric discounting leads to time-inconsistent preferences: the consumer changes his mind over time regarding the relative values of different consumption paths. One version of this inconsistency takes the form of extreme short-term impatience. That formulation seems attractive based on introspection. The recent literature also emphasizes behavioural studies (such as Ainslie, 1992) as a motivation for a departure from geometric discounting. This literature documents 'preference reversals', and it generally argues that time inconsistency is as ubiquitous as risk aversion. This information is too important to dismiss: at the very least, there is no definite argument against a departure from geometric discounting, and since models with time inconsistency potentially can have very different positive and normative properties than standard models, they deserve to be studied in more detail. That is what we set out to do here.

We assume that time is discrete and that the consumer cannot commit to future actions. We interpret rationality as the consumer's ability to correctly forecast his future actions: a solution to the decision problem is required to take the form of a subgame-perfect equilibrium of a game where the players are the consumer and his future selves. We restrict attention to equilibria which are stationary: they are recursive, and Markov in current wealth; that is, current actions cannot depend either on time or on any other history than that summarized by current wealth.

The consumption–savings problem is of the simplest possible kind: there is no uncertainty, and current resources simply have to be divided into current consumption and savings. Utility is time-additive with quasi-geometric discounting, and the period utility function is strictly concave. We assume that the consumer operates a technology which has (weakly) decreasing returns in its input – capital (that is, savings from last period). A special case is that of an affine production function, where the return is constant; this special case can be interpreted as one with a price-taking consumer who has a constant stream of labour income and can save at an exogenous interest rate. We do not study interaction between consumers in this Paper.

Our main finding is one of indeterminacy of equilibria. That is, the restriction to Markov equilibria does not reduce the set of equilibria to a small number. First, there is indeterminacy in terms of long-run outcomes of the consumption–savings process: there is a continuum of stationary points to which the consumer’s capital holdings may converge over time. Second, associated with each stationary point is a continuum of equilibria. Put simply, our theorizing does not allow us to rule out the possibility that two identical consumers placed in the same environment make radically different decisions, both in the short and the long run.

What is the origin of the indeterminacy? Almost by definition, one important component is expectations: equilibria can be thought of as ‘expectation-driven’. Optimism and pessimism regarding your own future behaviour is a real phenomenon in our model. The expectation concerns future savings behaviour. In the time-consistent model, the expectations of future savings behaviour are not relevant, since there is agreement on that behaviour: an envelope theorem applies. If, instead, the consumer places a higher relative weight on consumption two periods from now than does his next-period self, then a high savings propensity of his next self is an added bonus from saving today. Therefore what he believes about this future savings propensity is important. One consumer may decide to save a lot because he expects himself to save a lot in the future, thereby giving a high return to saving today; another instead expects to consume a lot next period, thus lowering the incentives to save now. Another important component in our equilibrium construction is a discontinuous policy rule for savings. That is, we employ locally extreme savings propensities to make the construction alluded to above.

Our findings are significant in two mutually exclusive ways. First, as discussed above, they suggest a theory of optimism and pessimism, a theory that seems quite general (of course, with the nongeneric exception of exact geometric discounting). Second, they may be viewed as a critique of the non-geometric-discounting approach to understanding the phenomena documented in the behavioural literature: if the most basic framework is so fraught with indeterminacy, what prediction power does it have? Is there perhaps some fundamental modelling problem underlying it? This point is important, as an alternative modelling approach has recently been developed, one that is axiomatic and by definition leads to precise predictions: the recent work by Gul and Pesendorfer (1999), extended to a dynamic framework in Gul and Pesendorfer (2000). Our own view is not yet entirely settled; work comparing the two approaches is at its infancy and has not yet produced definitive results. In any case, the results in the present Paper document some fundamental features of one of the approaches.

The indeterminacy that we document in this Paper has not been noted in the existing literature on consumption–savings decisions with quasi-geometric discounting. Laibson (1994), and Bernheim, Ray and Yeltekin (1999) find indeterminacy in settings similar to the one studied in this Paper, but they rely on history-dependent (‘trigger’) strategies. In this Paper, we restrict ourselves to Markov equilibria in which current consumption decisions depend only on current wealth.

Harris and Laibson (2000) study a consumption–savings problem in which the agent faces a constant interest rate and stochastic labour income. Their framework is closely related to ours, which allows an affine production function as a special case. The difference is that we consider a deterministic environment; their analysis does not contain ours as a special case and it seems important to understand the deterministic case separately. In addition, we are able to provide an explicit characterization of equilibria near a stationary point; Harris and Laibson provide existence, but not uniqueness or explicit solutions.

Our findings of indeterminacy have the potential to be helpful in the applied literature using quasi-geometric discounting. There, numerical methods are often necessary for characterizing equilibria, and it turns out that there are fundamental problems in finding algorithms that succeed in producing accurate solutions, at least when the individual-specific uncertainty is limited. The results herein suggest an explanation for the numerical problems: the lack of convergence of algorithms appears to be cycling within the large set of equilibria.

Our basic framework uses recursive methods and allows capital to be any number on an interval of the real line. To illustrate the set-up, we parameterize the model – logarithmic utility and Cobb-Douglas production – and derive an analytical solution for this case. We then restrict the domain for capital to a finite grid. The discrete-domain case allows us to demonstrate and discuss our multiplicity results in a concrete and simple way. We also use it to study whether there are simple domination arguments to rule out all equilibria but one. We therefore spend some time analysing the simplest possible consumption–savings problem: capital can take on only two values, high and low. Finally, we use the discrete-domain case as a way of computing equilibria numerically.

We then construct equilibria analytically when capital is restricted to lie on an interval. The construction is not global; it applies to a restricted domain of the following nature. Given a conjectured stationary point, we show that there is a neighbourhood around this point such that, if capital is restricted to that neighbourhood, there exists a ‘step-function’ equilibrium. To the left of the stationary point, the policy rule – next period’s capital as a function of current capital – is a step function with infinite steps. To the right of the stationary point, the policy rule is flat. The construction works for any strictly concave utility function and for any (weakly) concave production function, provided that the discounting parameters lie in a certain range.

We also show that on any stationary point of a given equilibrium policy rule the consumer can obtain higher current utility by having expectations of higher future savings, and thereby increasing savings currently. In particular, this means that the analytical solution we used as illustration for the log/Cobb-Douglas case is dominated in utility – for at least some values of current capital – by an equilibrium with higher savings both in the long and in the short run.

1 Introduction

The purpose of this paper is to study how an infinitely-lived, rational consumer with "quasi-geometric" discounting would make consumption and savings decisions.[1] We consider the idea that a consumer's evaluation of "utils" at different points in time does not have take the form of an aggregate with geometric weights. This idea was suggested first by Strotz (1956), and later elaborated on by Pollak (1968), Phelps and Pollak (1968), Laibson (1994, 1997) and others. Quasi-geometric discounting leads to time-inconsistent preferences: the consumer changes his mind over time regarding the relative values of different consumption paths. One version of this inconsistency takes the form of extreme short-term impatience. That formulation seems attractive based on introspection. The recent literature also emphasizes behavioral studies (such as Ainslie (1992)) as a motivation for a departure from geometric discounting. This literature documents "preference reversals", and it generally argues that time-inconsistency is as ubiquitous as risk aversion. This information is too important to dismiss: at the very least, there is no definite argument against a departure from geometric discounting, and since models with time-inconsistency potentially can have very different positive and normative properties than standard models, they deserve to be studied in more detail. That is what we set out to do here.

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Our main finding is one of indeterminacy of equilibria. That is, the restriction to Markov equilibria does not reduce the set of equilibria to a small number. First, there is indeterminacy in terms of long-run outcomes of the consumption/savings process: there is a continuum of stationary

[1] We mean by the term quasi-geometric a sequence which is geometric from the second date and on. The term "quasi-hyperbolic" has been used in the literature with the same meaning—see, e.g., Laibson (1997). Laibson's use, presumably, is motivated by trying to mimic approximately a true (generalized) hyperbolic function, which is possible within a subset of the quasi-geometric class. Mathematically, however, quasi-geometric is clearly a more appropriate term, and since we are interested in this entire class, as opposed to the subset mimicking the hyperbolic case, we opt for this term.

points to which the consumer's capital holdings may converge over time. Second, associated with each stationary point is a continuum of equilibria. Put simply, our theorizing does not allow us to rule out the possibility that two identical consumers placed in the same environment make radically different decisions, both in the short and the long run.

What is the origin of the indeterminacy? Almost by definition, one important component is expectations: equilibria can be thought of as "expectations-driven". Optimism and pessimism regarding your own future behavior is a real phenomenon in our model. The expectations concern future savings behavior. In the time-consistent model, the expectations of future savings behavior are not relevant, since there is agreement on that behavior: an envelope theorem applies. If, instead, the consumer places a higher relative weight on consumption two periods from now than does his next-period self, then a high savings propensity of his next self is an added bonus from saving today. Therefore, what he believes about this future savings propensity is important. One consumer may decide to save a lot because he expects himself to save a lot in the future, thereby giving a high return to saving today; another instead expects to consume a lot next period, thus lowering the incentives to save now. Another important component in our equilibrium construction is a discontinuous policy rule for savings. That is, we employ locally extreme savings propensities to make the construction alluded to above.

Our findings are significant in two mutually exclusive ways. First, as discussed above, they suggest a theory of optimism and pessimism, a theory that seems quite general (of course, with the nongeneric exception of exact geometric discounting). Second, they may be viewed as a critique of the non-geometric-discounting approach to understanding the phenomena documented in the behavioral literature: if the most basic framework is so fraught with indeterminacy, what prediction power does it have? Is there perhaps some fundamental modeling problem underlying it? This point is important, as an alternative modelling approach has recently been developed, one that is axiomatic and by definition leads to precise predictions: the recent work by Gul and Pesendorfer (1999), extended to a dynamic framework in Gul and Pesendorfer (2000). Our own view is not yet entirely settled; work comparing the two approaches is at its infancy and has not yet produced definitive results.[2] In any case, the results in the present paper document some fundamental features of one of the approaches.

The indeterminacy that we document in this paper has not been noted in the existing literature on consumption-savings decisions with quasi-geometric discounting. Laibson (1994) and Bernheim, Ray, and Yeltekin (1999) find indeterminacy in settings similar to the one studied in this paper, but they rely on history-dependent ("trigger") strategies. In this paper, instead, we restrict ourselves to Markov equilibria in which current consumption decisions depend only on current wealth.

Harris and Laibson (2000) study a consumption-savings problem in which the agent faces a constant interest rate and stochastic labor income. Their framework is closely related to ours, which allows an affine production function as a special case. The difference is that we consider

[2]See Krusell, Kuruşçu, and Smith (2000).

a deterministic environment; their analysis does not contain ours as a special case, and it seems important to understand the deterministic case separately. In addition, we are able to provide an explicit characterization of equilibria near a stationary point; Harris and Laibson provide existence, but not uniqueness nor explicit solutions.

Our findings of indeterminacy have the potential to be helpful in the applied literature using quasi-geometric discounting. There, numerical methods are often necessary for characterizing equilibria, and it turns out that there are fundamental problems in finding algorithms that succeed in producing accurate solutions, at least when the individual-specific uncertainty is limited. The results herein suggest an explanation for the numerical problems: the lack of convergence of algorithms appears as cycling within the large set of equilibria.[3]

We lay out our basic framework, using recursive methods, in Section 2. That model allows capital to be any number on an interval of the real line. To illustrate the setup, we parameterize the model—logarithmic utility and Cobb-Douglas production—and derive an analytical solution for this case. In Section 3, we then restrict the domain for capital to a finite grid. The discrete-domain case allows us to demonstrate and discuss our multiplicity results in a concrete and simple way. We also use it to study whether there are simple domination arguments to rule out all equilibria but one. We therefore spend some time analyzing the simplest possible consumption-savings problem: capital can take on only two values, high and low. Finally, we use the discrete-domain case as a way of computing equilibria numerically.

In Section 4, we construct equilibria analytically when capital is restricted to lie on an interval. The construction is not global; it applies to a restricted domain of the following nature. Given a conjectured stationary point, we show that there is a neighborhood around this point such that, if capital is restricted to that neighborhood, there exists a "step-function" equilibrium. To the left of the stationary point, the policy rule—next period's capital as a function of current capital—is a step function with infinitely many steps. To the right of the stationary point, the policy rule is flat. The construction works for any strictly concave utility function and for any (weakly) concave production function, provided that the discounting parameters lie in a certain range.

We also show that on any stationary point of a given equilibrium policy rule, the consumer can obtain higher current utility by having expectations of higher future savings, and thereby by increasing savings currently. In particular, this means that the analytical solution we used as illustration for the log/Cobb-Douglas case in Section 2 is dominated in utility, for at least some values of current capital, by an equilibrium with higher savings both in the long and in the short run.

Section 5 concludes.

[3] See, e.g., Laibson (1997).

2 The setup

2.1 Primitives

Time is discrete and infinite and begins at time 0.[4] There is no uncertainty. An infinitely-lived consumer derives utility from a stream of consumption at different dates. We assume that the preferences of the individual at time t are time-additive, and that they take the form

$$U_t = u_t + \beta_1 u_{t+1} + \beta_2 u_{t+2} + \beta_3 u_{t+3} + \dots .$$

The variable u_t denotes the number of utils at time t; it is implicit that these utils are derived from a function $u(c_t)$, where c_t is consumption at time t. This formulation thus embodies an assumption of stationarity: the discounting at any point in time has the form $1, \beta_1, \beta_2, \dots$. The same consumer at $t+1$ thus evaluates utility as follows:

$$U_{t+1} = u_{t+1} + \beta_1 u_{t+2} + \beta_2 u_{t+3} + \beta_3 u_{t+4} + \dots .$$

Clearly, the lifetime utility evaluations at t and $t+1$ express different views on consumption at different dates, unless $\beta_{t+k+1}/\beta_{t+k}$ is the same for all t and k and equal to β_1, that is, unless discounting is geometric: $\beta_t = \beta^t$ for some β. We take the view here that geometric discounting is a very special case and that the a priori grounds to restrict attention to it are weak. We consider a very simple departure from geometric discounting: quasi-geometric discounting. Quasi-geometric discounting can be expressed with two parameters, β and δ. The weights on future utils are $1, \beta\delta, \beta\delta^2, \beta\delta^3, \dots$. That is, discounting is geometric across all dates excluding the current date:

$$U_t = u_t + \beta \left(\delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \dots \right).$$

The case where $\beta < 1$ corresponds to particular short-run impatience ("I will save, just not right now"), and $\beta > 1$ represents particular short-run patience ("I will consume, just not right now"). The case $\beta = 1$, of course, is the standard, time-consistent case.

It is straightforward to generalize quasi-geometric discounting: the weights would then be general for T periods, and geometric thereafter. Pure hyperbolic discounting corresponds to the case $\beta_t = 1/(t+1)$, which we do not consider here. In most of our analysis, we will restrict attention to $\delta < 1$, since our resources are bounded. With growing resources, it is possible to allow a δ larger than 1 if the utility function takes a certain form.

We assume that the period utility function $u(c)$ is strictly increasing, strictly concave, and twice continuously differentiable. The consumer's resource constraint reads

$$c + k' = f(k)$$

where k is current capital holdings, f is strictly increasing, concave and twice continuously differentiable. We will focus on the case where f is strictly concave, but this assumption is not essential for our main results.

[4] Barro (1997) studies a continuous-time model without uncertainty where the consumer's discounting is not exponential.

2.2 Behavior: modelling choices

How do we model the decision making? We use four principles:

1. We assume that the consumer cannot commit to future actions.
2. We assume that the consumer realizes that his preferences will change and makes the current decision taking this into account.
3. We model the decision-making process as a dynamic game, with the agent's current and future selves as players.
4. We focus on (first-order) Markov equilibria: at a moment in time, no histories are assumed to matter for outcomes beyond what is summarized in the current stock of capital held by the agent.

Some comments are in order. The first of the principles makes the problem different than the standard case. With commitment, decisions could be analyzed starting at time 0 in an entirely standard fashion (using recursive methods) and only the decisions across time 0 and the rest of time would be different. That decision would be straightforward given an indirect utility function representing utility at times 1, 2 and on. Moreover, commitment is not an unrealistic assumption. Notice that commitment to consumption behavior in practice would require a demanding monitoring technology and might be quite costly. 401(k) plans do not provide commitment to consumption, unless there are other restrictions, such as borrowing constraints. We do not consider such constraints here. One could consider how access to a costly monitoring technology would alter the analysis. We leave such an analysis as well to future work. Of course, the ability to overcome the commitment problem may be a crucial ability of a consumer, and it deserves to be studied more.

Our second principle is what we interpret rationality to mean in this framework. We would not want to abandon it: systematic prediction errors of one's own future behavior are not studied in the time-consistent model, and we do not want to study them here. Moreover, studying such prediction errors does not require time-inconsistent preferences.

Our third principle is the same as that suggested and adopted in the early literature on time-inconsistent preferences. Our fourth principle is more of a restriction than a principle: we do not study history-dependent equilibria with the hope of arriving at sharper predictions. There is perhaps also a sense in which we think bygones should be bygones on the level of decision-making. There is also existing work where bygones are not bygones: Laibson (1994) and Bernheim, Ray, and Yeltekin (1999) study similar models and allow history dependence. The set of equilibria can certainly be expanded in this way.

2.3 A recursive formulation

Assume that the agent perceives future savings decisions to be given by a function $g(k)$:

$$k_t = g(k_{t-1}).$$

Note that g is time-independent and only has current capital as an argument.

The agent solves the "first-stage" problem

$$W(k) \equiv \max_{k'} u(f(k) - k') + \beta\delta V(k'),$$

where V is the indirect utility of capital from next period on. In turn, V has to satisfy the "second-stage" functional equation

$$V(k) = u(f(k) - g(k)) + \delta V(g(k)).$$

Notice that successive substitution of V into the agent's objective generates the right objective if the expectations of future behavior are given by the function g.

A solution to the agent's problem is denoted $\tilde{g}(k)$. We have an equilibrium if the fixed-point condition $\tilde{g}(k) = g(k)$ is satisfied for all k.

The fixed-point problem in g cannot be expressed as a contraction mapping. For a given (bounded and continuous) g, it is possible to express the functional equation in V as a contraction mapping. However, continuity of g does not guarantee that V is concave, and it is not clear that the maximization over k' problem has a unique solution. This also implies that $\tilde{g}$ may be discontinuous.

A simple parametric example can be used as an illustration of the recursion. Suppose $u(c) = \log(c)$ and $f(k) = Ak^{\alpha}$, with $\alpha < 1$. Then it is straightforward to use guess-and-verify methods to solve for the following solution:

$$k' = \frac{\alpha\beta\delta}{1 - \alpha\delta(1-\beta)} Ak^{\alpha}$$

and

$$V(k) = a + b \log k$$

with steady state

$$k_{ss} = \left(\frac{\alpha\beta\delta A}{1 - \alpha\delta(1-\beta)} \right)^{\frac{1}{1-\alpha}}.$$

This solution gives a lower steady state than with $\beta = 1$.[5]

It is easy to check that, for this example, the time-consistent behavior thus solved for actually coincides, in the first period, with the behavior that would result in the commitment solution.

An algorithm for numerical computation of equilibria is suggested directly from our recursive problem: pick an arbitrary initial V, solve for optimal savings and obtain a decision rule, update V, and so on. This algorithm is similar to value function iteration for the standard time-consistent

[5] The coefficients a and b are given by $b = \alpha/(1-\alpha\delta)$ and $a = (\log(A-d) + \delta b \log(d))/(1-\delta)$, where $d = \beta\delta bA/(1+\beta\delta b)$.

problem. If the initial V is set to zero, it is also equivalent to how a finite-horizon problem would normally be solved. It turns out that this algorithm does not work here. Typically, it leads to cycling.[6] Similarly, an algorithm that starts with a guess on g, solves for V from the second stage condition, and then updates g (say, by a linear combination between g and $\tilde{g}$) also does not work: it produces cycles. These two algorithms produce cycles even when u and f are of the parametric form we discussed above—when g is known to be log-linear—and even an initial condition very close to the exact solution is given. As we will see, the analysis in the following sections suggests a reason for the apparent instability of these algorithms: there are other solutions to the fixed-point problem in g that are not continuous, and the function approximations we used in the above algorithms rely on continuity (for example, we use cubic splines).

We now turn to a version of our model with a discrete state space.

3 The case of a discrete domain

We now assume that capital can only take a finite number of values: $k \in \{k_1, k_2, \ldots, k_I\}$. We make the following assumptions:

1. *Consumption-savings:* $u_{21} > u_{11} > u_{12}$, $u_{21} > u_{22} > u_{12}$.
2. *Strict concavity of* u*:*
$$u_{ij} - u_{ik} > u_{i'j} - u_{i'k}.$$
for $i < i'$ and $j < k$.
3. *Impatience:* $\beta < 1$ and $\delta < 1$.

Define $\pi_{ij} \in [0,1]$ to be the probability of going from state i to state j. Given π (a set of π_{ij}'s), find the value function given uniquely by the V_i's solving the contraction

$$V_i = \sum_j \pi_{ij} \left(u_{ij} + \delta V_j \right)$$

for all i (this is a linear equation system). This gives $V(\pi)$. The fixed-point condition requires

$$\pi_{ij} > 0 \Rightarrow j \in \arg\max_k \; [u_{ik} + \beta\delta V_k(\pi)].$$

Proposition 1: *There exists a mixed-strategy equilibrium for the economy with discrete domain.*
Proof: This is shown with a straightforward application of Kakutani's fixed-point theorem. ■

It is also possible to show monotonicity of the decision rules:

[6] The algorithm may converge if g is approximated with very low accuracy (with few grid points, or with an inflexible functional form).

Proposition 2: *The decision rule is monotone increasing, that is, if positive probability is put on k at i and i' is larger than i, then the choice at i' cannot have positive probability on $j < k$.*
Proof: Suppose not.

$$u_{ik} + \beta\delta V_k \geq u_{ij} + \beta\delta V_j$$

and

$$u_{i'j} + \beta\delta V_j \geq u_{i'k} + \beta\delta V_k$$

can be combined into

$$u_{ij} - u_{ik} \leq u_{i'j} - u_{i'k}$$

which violates strict concavity. ■

Monotonicity is a very useful property for understanding the behavior of the consumer in this model. It will be used repeatedly below. The proof of monotonicity does not use discreteness, and therefore the monotonicity property also applies when the domain for capital is continuous.

3.1 The 2-state case

We study the simplest possible case in some detail: the case where capital can take only two values, 1 and 2 ($k_1 < k_2$). We will use the short-hand ij for an equilibrium where the decision in state 1 is to go to state i and the decision in state 2 is to go to state j, $i, j \in \{1, 2\}$; further, $i\pi$ refers to an equilibrium where there is mixing in state 2 (for some specific probability), and πj refers to mixing in state 1.

The characterization of the set of equilibria is that the parameter space (β, δ, and the u_{ij}'s) breaks into 6 regions:

Proposition 3: *Generically, there are six possible equilibrium configurations; each of the following characterizes a region:*

1. *A unique "no-saving" equilibrium: $1 \to 1$ and $2 \to 1$.*
2. *A unique "saving" equilibrium: $1 \to 2$ and $2 \to 2$.*
3. *A unique "status-quo" equilibrium: $1 \to 1$ and $2 \to 2$.*
4. *No pure-strategy equilibrium: $1 \to \pi$ and $2 \to 2$ (long-run saving).*
5. *Three equilibria:*
   (a) *$1 \to 1$ and $2 \to 1$ (no-saving).*
   (b) *$1 \to 2$ and $2 \to 2$ (saving).*
   (c) *$1 \to 1$ and $2 \to \pi$ (no-saving).*
6. *Three equilibria:*
   (a) *$1 \to 1$ and $2 \to 1$ (no-saving).*
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   (c) *$1 \to \pi$ and $2 \to 2$ (saving).*

Proof: See Appendix 1. ■

Notice that regions 1–3 are expected and standard; region 4 is a case where no pure strategy equilibrium exists; and the remaining two cases have multiplicity. We will discuss their interpretation below. As shown in the proof of Proposition 3, regions 4–6 disappear for $\beta = 1$.

When there is more than one equilibrium, there are three. Two of these are very different in character: they lead to different long-run outcomes. The third is a mixed version of one of the others, with the same long-run outcome (equilibrium 5c is very similar to 5a, and 6b to 6a). The essential character of each of the two equilibria is: if your future self is a saver, so are you; if not, then neither are you.

The idea that there are multiple solutions to a decision problem is conceptually disturbing: faced in a given situation, what will the consumer do? Our theory does not provide an answer, or, it says several things can happen. Identical consumers, apparently, can make different decisions, rationally, in the same situation.

Can we interpret this as there being room for "optimism" and "pessimism" to influence decisions? These terms should, if used appropriately, refer to utility outcomes, about which we have remained silent so far. The fact is that, in our 2-state economy, equilibria with long-run savings are better than those without: they give higher current life-time utility, independently of the starting condition, than no-saving equilibria.[7] In this sense, there is a free lunch here: just be optimistic, it is not associated with costs!

Of course, the free lunch aspect suggests a natural refinement of equilibria, one which has a renegotiation character: why stick to expectations which can be replaced with better ones? This refinement seems to work well in this application. However, it turns out that this refinement is problematic when there are more than two possible states for capital. The reason is that, in general, there are parameter regions (which become large when the number of states becomes large) where a utility ranking across equilibria does not exist. For example, state i might give equilibrium A higher utility than equilibrium B, whereas in state j the reverse is true; moreover, state i might lead to state j under equilibrium A. That is, if one picks equilibrium A now, one will want to change one's mind later. That means that this refinement is not time-consistent, and therefore not useful.[8]

This means, as far as we can tell, absent other useful refinement concepts, that there might be room for optimism and pessimism. Of course, these terms now have a more restricted meaning, since an equilibrium with optimism today (in terms of current utility) may imply pessimism in the future, and vice versa.

[7] In fact, all equilibria are ranked in this sense.

[8] Asheim (1997) proposes a concept called revision-proofness as a refinement to subgame-perfect equilibria and applies it in a context of time-inconsistent decision-making. He shows a specific example, similar to the present one and featuring a discrete state space, where a revision-proof equilibrium exists. Here, one would exist in the 2-state case, but not in general in the multi-state case.

3.2 More than 2 states

With more than two states, we resort to numerical methods for finding equilibria. To find equilibria, we either perform exhaustive search (which of course is a slow method, prohibitively so except for a very small number of states) or iterate on a fixed-point mapping from randomly selected initial conditions for π (this algorithm is fast, but will miss some equilibria, at least those which are "unstable").[9]

Several questions are relevant here:

- As the grid becomes finer, will the multiplicity expand, remain unchanged, or shrink?
- As the grid becomes extremely fine, and there is some hope that the solution approximates a continuous state-space solution, what are the properties of such a solution?
- If we restrict parameters to replicate log/Cobb-Douglas assumptions as closely as possible, will the analytical solution be found? Will other solutions continue to exist?

In general, the findings are: the multiplicity does not go away as the grid becomes finer, equilibria are not ranked in general, there is always some mixing when the grid is fine enough, the decision rules look "funny", and the analytical solution to the log/Cobb-Douglas case is not one of the equilibria that is produced by the algorithm.

We will illustrate the equilibrium features in Figures 1, 2, and 3. They depict the policy rule for capital, given current capital, and are constructed based on the case of 150 grid points. The parameters are chosen based on the log/Cobb-Douglas specification; the analytical decision rule is graphed in each of the figures. We found 30 equilibria in this case. If these are all the stable equilibria, there should be an odd number in addition.

The general features are as follows: decision rules seem smooth over some intervals, but have jumps. For a given decision rule, there is always a single stationary point. The stationary point is reached, from the right, by a flat section, and from the left, by a "creeping up along the 45-degree line". The "creeping up" actually occurs with mixing: these points are mixing the 45-degree line with a grid point above it. Mixing does not occur anywhere else. We will draw heavily on these features when we construct equilibria in the case of a continuous state space in the next section.

Comparing stationary points to the analytical case, Figure 1 has its stationary point above, Figure 3 below, and Figure 2 at the stationary point of the analytical solution. The equilibrium in Figure 1 actually dominates the other equilibria in the figures, but there is another equilibrium with which it cannot be ranked.

[9] In order to find a mixed-strategy equilibrium, the latter method iterates until near indifference, makes a specific guess of indifference and solves for the equilibrium given this guess, and finally checks all equilibrium conditions.

3.3 Additional comments

When the discrete-state model is solved backwards, that is, when a finite-horizon version of the model is solved, there is, as expected, a unique equilibrium for every time horizon. As the time horizon goes to infinity, there is sometimes no convergence in policy rules and value functions: a cycle is reached. "Sometimes" is always when there are many grid points. Intuitively, then, all equilibria we find with our other computational method have mixing, and mixing equilibria will not be found with backward-solving: they will not exist, generically, with a finite horizon. In the two-state case, for example, in region 4, where there is no pure-strategy equilibrium, there is lack of convergence. In regions 1–3, there is convergence to the unique pure-strategy equilibrium, and in regions 5 and 6 there is convergence to equilibrium (a): the no-saving equilibrium. Thus, the saving equilibrium seems to require an infinite horizon to be implementable.

4 The case of a continuous domain

We now assume that the domain is a part of the real line. For general u's and f's (which satisfy strict concavity, etc.), and for a $\beta < 1$, we find equilibria by construction. We are not able to construct these equilibria globally (unless we resort to numerical techniques), but instead restrict the domain. A typical domain will be $[\bar{k} - \epsilon, \bar{k} + b]$, where $\bar{k}$ is a stationary point, ϵ is a small positive number (defining a left-neighborhood of $\bar{k}$), and b a positive constant. The key properties of our decision rules are:

- Immediately to the right of the stationary point of a decision rule, the rule is flat over a range: if you don't expect your future self to save, you don't either.
- Immediately to the left of the stationary point of this rule, optimal behavior must be characterized by a step function.[10]

Moreover, there is a continuum of equilibria for any given primitives:

- There is a continuum of stationary points associated with distinct decision rules.
- For each stationary point, there is a continuum of step function equilibria leading in to it.

Figure 4 illustrates the set of equilibrium decision rules. Three rules are plotted in the figure; two lead in to the same stationary point, and the third leads to a higher stationary point.[11] We change the domain as we construct the continuum of equilibria with different stationary points. As

[10]When β exceeds 1, it is also possible to construct equilibria with the same methods: just exchange "left" for "right" in all definitions and vice versa (the step function will be to the right of the stationary point, and the flat section to the left).

[11]As discussed below, the decision rules approach but do not touch the 45-degree line: at the 45-degree line, the decision rules take a discrete jump upwards.

before, decision rules are monotone (weakly) increasing, but randomization is no longer part of the equilibria we construct.

We go through the construction of our equilibria in steps. The proof that the agent is maximizing given the constructed decision rule is contained in the proof of Proposition 4 in Appendix 2.

4.1 The flat part to the right

For the construction of an equilibrium decision rule, first select a stationary point $\bar{k}$. Then define a $\bar{\bar{k}} > \bar{k}$ to the left of which the decision rule is flat. Thus, b above is defined to be $\bar{\bar{k}} - \bar{k}$.

For the purpose of this discussion, restrict choices to $[\bar{k}, \bar{\bar{k}}]$ and show it is optimal to always choose $\bar{k}$ over this range (the value function V takes a simple, well-behaved form then). We will later verify that when the domain is extended left of $\bar{k}$, it is also not optimal to select points in that part of the domain.

The intuition behind the construction here is that $\bar{k}$ and $\bar{\bar{k}}$ are such that you go to a "corner": at $k' = \bar{k}$, the marginal benefit of saving is below the marginal cost of saving.

We thus have

$$V(k) = u(f(k) - \bar{k}) + \delta V(\bar{k}),$$

where

$$V(\bar{k}) = \frac{u(f(\bar{k}) - \bar{k})}{1 - \delta}.$$

By construction, V is differentiable and strictly concave over the restricted domain $[\bar{k}, \bar{\bar{k}}]$.

Given V and $k \in [\bar{k}, \bar{\bar{k}}]$, the agent solves:

$$\max_{k' \in [\bar{k}, \bar{\bar{k}}]} \; u(f(k) - k') + \beta\delta V(k').$$

The derivative of the objective function with respect to k' is given by:

$$D(k, k') \equiv -U'(f(k) - k') + \beta\delta U'(f(k') - \bar{k}) f'(k').$$

Given k, the optimal choice for k' is $\bar{k}$ provided that $D(k, k')$ is negative for all $k' \in [\bar{k}, \bar{\bar{k}}]$. Since, holding k fixed, D is decreasing in k', it suffices to check that D is negative at $k' = \bar{k}$. Thus, when $k = \bar{k}$, D is negative for all $k' \in [\bar{k}, \bar{\bar{k}}]$ if and only if

$$\beta\delta f'(\bar{k}) < 1.$$

This condition puts a lower bound on the stationary level of capital.

Since, holding k' fixed, D is increasing in k, the upper bound $\bar{\bar{k}}$ is pinned down by:

$$u'(f(\bar{\bar{k}}) - \bar{k}) = \beta\delta u'(f(\bar{k}) - \bar{k}) f'(\bar{k}).$$

For $k > \bar{\bar{k}}$, $D(k, \bar{k}) > 0$, implying that $\bar{k}$ is not the optimal choice.

When the domain and the decision rule are extended to the left, the construction will still make $\bar{k}$ the optimal choice, because V will not be differentiable at $\bar{k}$: it is continuous but has no left-derivative at that point. In particular, the step function decision rule to the left will generate a sufficient drop in utility that the consumer never wants to choose a point less than $\bar{k}$ if his current capital stock is at or above $\bar{k}$. Figure 5 illustrates the shape of the value function for the three equilibria depicted in Figure 4.

4.2 The step function to the left

We define a step decision rule as follows:

- There is a countably infinite number of steps, indexed by n, at $\{k_n\}_{n=0}^{\infty}$. At each of these step points, the step is taken with probability one.
- The future utility at step n is denoted v_n.
- For $k \in [k_n, k_{n+1})$, the future utility is

$$V(k) = u(f(k) - k_{n+1}) + \delta v_{n+1}.$$

Note that $V(k_n) = v_n$ and that V is strictly increasing and strictly concave on each open interval, but not left-continuous at the step points.

The $\{k_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ sequences satisfy two key conditions. First,

$$u(f(k_n) - k_n) + \beta\delta v_n = u(f(k_n) - k_{n+1}) + \beta\delta v_{n+1}. \tag{1}$$

This condition says that at steps you are indifferent between taking and not taking the step. Second,

$$v_n = u(f(k_n) - k_{n+1}) + \delta v_{n+1}. \tag{2}$$

This condition says that at a step you do take the step with probability one.[12]

Let $(\bar{k}, \bar{v})$ be a stationary point of the system of difference equations given by equations (1) and (2). It is easy to see that this dynamic system has a multiplicity of stationary points. In particular,

[12] It is important that the probability is one. Why cannot randomizing be part of optimal behavior when optimal behavior is given by our step function? At a point of indifference, we need to have

$$u(f(k_n) - k_n) + \beta\delta v_n = u(f(k_n) - k_{n+1}) + \beta\delta v_{n+1}.$$

This implies that $v_{n+1} > v_n$. We also have

$$v_n = \pi\left[u(f(k_n) - k_n) + \delta v_n\right] + (1-\pi)\left[u(f(k_n) - k_{n+1}) + \delta v_{n+1}\right] =$$
$$u(f(k_n) - k_{n+1}) + \delta v_{n+1} + \pi\left(u(f(k_n) - k_n) - u(f(k_n) - k_{n+1}) + \delta(v_n - v_{n+1})\right) =$$
$$u(f(k_n) - k_{n+1}) + \delta v_{n+1} + \pi\left(\delta - \beta\delta\right)(v_n - v_{n+1}) < u(f(k_n) - k_{n+1}) + \delta v_{n+1}.$$

where π is the probability of picking k_n. This means that $V(k)$ is not right-continuous at k_n if $\pi > 0$: the limit of $V(k_n + \epsilon) = u(f(k_n + \epsilon) - k_{n+1}) + \delta v_{n+1}$ as $\epsilon > 0$ goes to zero is $u(f(k_n) - k_{n+1}) + \delta v_{n+1} > v_n$. Since $V(k)$ makes a jump up at k_n, it cannot be optimal to put positive probability on k_n: a miniscule increase in savings above k_n would make the future benefits jump up discretely, and the current consumption loss would be miniscule.

equation (1) is satisfied at any stationary point, so the set of stationary points is determined by equation (2): $\bar{v} = u(f(\bar{k}) - \bar{k}) + \delta\bar{v}$, implying that

$$\bar{v} = \frac{u(f(\bar{k}) - \bar{k})}{1-\delta}.$$

We want to restrict attention to the set of stationary points for which the dynamic system (1) and (2) is locally stable, i.e., for which the sequence $\{k_n, v_n\}_{n=0}^{\infty}$ converges to $(\bar{k}, \bar{v})$. Equations (1) and (2) define an implicit function h that maps (k_n, v_n) into (k_{n+1}, v_{n+1}). The Jacobian matrix of first derivatives of h, evaluated at the stationary point $(\bar{k}, \bar{v})$, has one eigenvalue equal to 1 and one eigenvalue equal to

$$\frac{1-\beta\delta f'(\bar{k})}{\delta(1-\beta)}.$$

This eigenvalue is between 0 and 1 provided that $f'(\bar{k}) > 1 + \frac{1-\delta}{\beta\delta}$. This condition puts an upper bound on the set of admissible value for $\bar{k}$.

Under this condition, it is straightforward to modify standard results concerning the local stability of nonlinear difference equations (see, e.g., Scheinkman (1973)) to show that the dynamic system given by (1) and (2) has a one-dimensional stable manifold characterized by a continuously differentiable function $\varphi(k_0, v_0)$. In other words, given a stationary point $(\bar{k}, \bar{v})$, there exists a neighborhood N of this point such that the dynamic system (1) and (2) converges to the stationary point for any initial value $k_0 \in N$ provided that v_0 is chosen so that $\varphi(k_0, v_0) = 0$.

In summary, we can construct a continuum of decision rules for each stationary point, and the construction works for any stationary point $\bar{k} \in \Big((f')^{-1}(\frac{1}{\beta\delta}), (f')^{-1}(1 + \frac{1-\delta}{\beta\delta})\Big)$. This interval shrinks to a point as β goes to 1.

Throughout this section, we have assumed that f is strictly concave. Our results, however, also apply to the case in which $f(k) = Rk + w$, where R and w are constants. In this case, the agent, in effect, faces constant (exogenous) prices: a constant (gross) interest rate R and a constant wage w. When f is linear, any level of asset holdings can be a stationary point, provided that

$$1 + \frac{1-\delta}{\beta\delta} < R < \frac{1}{\beta\delta}$$

When $\beta = 1$, this range collapses to a point, yielding the familiar result that any level of asset holdings can be a stationary point provided that $R = \delta^{-1}$.

4.3 Optimality of the constructed decision rules

We now provide a formal proposition stating that the behavior posited in the constructed equilibria are optimal for the consumer. The proof is lengthy—it checks with "brute force" at any k that the posited behavior is better than any other behavior. The essential arguments rely on strict concavity of u and the restriction that $\bar{k}$ lie in the interval $\Big((f')^{-1}(\frac{1}{\beta\delta}), (f')^{-1}(1 + \frac{1-\delta}{\beta\delta})\Big)$.

Proposition 4: Assume that $\bar{k} \in \Big((f')^{-1}(\frac{1}{\beta\delta}), (f')^{-1}(1 + \frac{1-\delta}{\beta\delta})\Big)$. Then the decision rules constructed for $\bar{k}$ in Sections 4.1 and 4.2 represent optimal behavior. That is, there exists an interval

$A \equiv [\bar{k} - \epsilon, \bar{\bar{k}}]$, where $\epsilon > 0$ and $\bar{\bar{k}}$ satisfies $u'(f(\bar{\bar{k}}) - \bar{k}) = \beta\delta u'(f(\bar{k}) - \bar{k})f'(\bar{k})$, such that for each $k_0 < \bar{k}$ belonging to this interval, there exists a v_0 such that the sequence $\{k_n, v_n\}_{n=0}^{\infty}$ satisfying equations (1) and (2) is strictly increasing and converges to $(\bar{k}, \bar{v})$, where $\bar{v} = \frac{u(f(\bar{k})-\bar{k})}{1-\delta}$. Moreover, the value function

$$V(k) \equiv \begin{cases} u(f(k) - k_n) + \delta v_n & \text{if } k \in [k_{n-1}, k_n) \\ u(f(k) - \bar{k}) + \frac{\delta u(f(\bar{k})-\bar{k})}{1-\delta} & \text{if } k \in [\bar{k}, \bar{\bar{k}}] \end{cases}$$

satisfies

$$\arg\max_{k' \in A} \; [u(f(k) - k') + \beta\delta V(k')] = \begin{cases} k_n & \text{if } k \in [k_{n-1}, k_n) \\ \bar{k} & \text{if } k \in [\bar{k}, \bar{\bar{k}}] \end{cases}$$

Proof: See Appendix 2. ■

4.4 Utility comparisons across equilibria

Let $W(k)$ be the present-value function utility function associated with a given equilibrium; this function gives lifetime utility as of time 0 when the current asset holding is k. Numerical analysis (as well as evidence from the discrete case with more than two states) suggests that the present-value utility functions associated with different equilibria are generally not ranked across the entire state space, i.e., these functions can cross. Nonetheless, it is possible to rank equilibria in terms of present-value utility at selected points in the state space.

First, it is possible to show that the steady states are ranked in utility in the following sense: given a starting point in the interior of the range for which steady states exist, the utility from the equilibrium which stays at that point is lower than the utility from embarking on a path with capital accumulation and which leads to a higher steady state. This is demonstrated formally in the following proposition.

Proposition 5: Assume $\beta < 1$ and let capital be restricted to $[k^\star, \bar{k}]$, where both $k^\star$ and $\bar{k} > k^\star$ are within the bounds allowing steady states to exist. Consider a decision rule for which $k^\star$ is the long-run outcome. Then there exists a decision rule whose long-run outcome is $\bar{k}$ which gives higher utility starting at $k^\star$.[13]

Proof: See Appendix 3. ■

It is also true that for any starting point $k^\star$ in the flat part to the right of a given steady state $\bar{k}$ (provided that $k^\star$ is within the bounds in which steady states exist), the equilibrium whose steady state is $k^\star$ provides higher present-value utility than the equilibrium whose steady state is $\bar{k}$. This is demonstrated formally in the following proposition.

Proposition 6: Assume $\beta < 1$.[14] Consider a decision rule with steady state $k^\star$. Let $\bar{k}$ be smaller than $k^\star$ and such that there is a decision rule whose steady state is $\bar{k}$. In addition, suppose that $k^\star$

[13] If $\beta > 1$, then the conclusion of the proposition is reversed: the decision rule whose long-run outcome is $\bar{k}$ gives lower utility starting at $k^\star$.

[14] The statement of the proposition is reversed when $\beta > 1$.

is in the flat section to the right associated with the decision rule whose steady state is $\bar{k}$. Then, starting at $k^\star$, the decision rule whose steady state is $\bar{k}$ gives smaller utility than the decision rule whose steady state is $k^\star$.

Proof: See Appendix 3. ■

Notice that Proposition 5 holds whether or not the path which is constant is a smooth equilibrium (e.g., like the one we know exists in the log/Cobb-Douglas case). Therefore, we have shown that the loglinear equilibrium in the log/Cobb-Douglas case gives lower utility, at least on its stationary point, than (some) other equilibria.

It is not true, however, that equilibria with higher steady states necessarily give higher utility for all values of k. As an illustration, consider again the log-linear solution to the log/Cobb-Douglas case. Let us compare the utility for different values of k implied by this equilibrium to the utility levels implied by a step function equilibrium whose stationary point coincides with the stationary point of the log-linear equilibrium. Formally, we have

Proposition 7: Consider the case of logarithmic utility and Cobb-Douglas production. The utility at k associated with any step function equilibrium with stationary point $k^\star$, where $k^\star$ is the stationary point of the smooth solution, is larger (less) than that associated with the smooth solution if $k < (>)\ k^\star$.

Proof: See Appendix 3. ■

The present-value utility function $W(k)$ is continuous for our step function equilibria (it is continuous between steps, and continuous at steps by construction). Moreover, the function value at every k also varies continuously with the stationary point chosen. This means that if the log-linear solution to the log/Cobb-Douglas case is compared to a step-function solution with slightly lower stationary point than $k^\star$, then there will exist points to the left of $k^\star$ where this step-function solution yields strictly higher utility than the log-linear solution. That is, solution with a higher stationary point yields lower utility at those points in the state space.

4.5 Remarks

It is clear how one could extend the domain so as to solve for the step function equilibria more globally. The two key conditions defining the step function amount to a nonlinear difference equation which can be solved numerically. One would then need to verify that the agent is optimizing, which can be done numerically as well. At some point to the left, however, it would not be optimal for the agent to follow the steps (the proof of the main proposition uses the fact that k is close to $\bar{k}$ to guarantee optimality). Then, one could extend the decision rule to the left by constructing, piece by piece and iteratively from right to left, what must be optimal behavior; given that the capital stock is increasing there, all that is needed is the decision rule we have already solved for. A similar procedure can be used to extend the solution to the right of $\bar{\bar{k}}$.

The step function equilibria we have derived have consumption function counterparts which are quite jagged (lines with a slope of 1, followed by discontinuous downward drops, etc.). In fact, they are quite similar to those pictured in Laibson (1997), where the model is similar except for the presence of idiosyncratic uninsurable shocks and a finite horizon. A comment on computation is in order here. A variety of algorithms that we have implemented to compute equilibria in the infinite-horizon case (see Section 2.3 for a general description of two of them) fail to converge and instead tend to cycle. Although these algorithms differ in their details, they share the common feature that they iterate "backwards" from a terminal condition. Although we have no detailed insights into how the different algorithms for computing equilibria work, these methods seem to lead to step function equilibria (these equilibria seem "stable" with respect to the algorithm used). Thus one possible explanation for the numerical findings in Laibson (1997) is that he is using a backwards-iteration algorithm (which is the natural approach in a finite-horizon model) that does not converge but instead cycles through a sequence of step-function equilibria. Of course, the step function equilibria cannot be easily computed numerically, since most numerical methods rely on continuity, and this seems to be the reason for the lack of convergence.

As we show in Section 2.3, there is a "smooth" equilibrium (one with continuous decision rules) for specific choices of u and f. Whether smooth solutions exist in general is an open question. The optimal paths in our step function equilibria, however, are smooth, even though the decision rules are not. In this sense, the step function equilibria do not represent "unusual-looking" behavior. The propensities to consume at almost all points on the domain we use is one, and the propensity to save zero. However, the realization of any equilibrium occurs at the steps, except possibly the very first period (and, in a global solution which we have not solved for, for longer periods, before the step function is reached).

5 Conclusion

In this paper, we study the consumption-saving decisions of a consumer who has time-inconsistent preferences in the form of a departure from geometric discounting. Our analysis includes as a special case the simplest possible consumption-savings problem in which a price-taking consumer faces a constant exogenous interest rate and receives a constant stream of labor income. We make no restrictions on the period utility function save for concavity. When the time horizon is infinite, we find that the dynamic game played between the consumer's successive selves is characterized by a severe multiplicity of equilibria. This multiplicity arises even though we restrict attention to Markov equilibria. The multiplicity takes two forms. First, there is a continuum of stationary points for the consumer's asset holdings. Second, for each stationary point there is a continuum of paths leading into this stationary point. Since we study a deterministic environment, a key question is whether multiplicity of equilibria survives the introduction of uncertainty.

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Appendix 1

This appendix contains the proof of Proposition 3 in Section 3.1.

First, conditions for each type of equilibrium to exist are given below (it is implicit that the consumption-savings and strict concavity assumptions are required in addition to the stated conditions). After these conditions are given, the proof is provided.

- $\underline{1 \rightarrow 1 \text{ and } 2 \rightarrow 1.}$ We have, normalizing so that $u_{11} \equiv 1$,

  $$v_1 = 1 + \delta v_1$$

  and

  $$v_2 = u_{21} + \delta v_1$$

  which implies

  $$v_1 = \frac{1}{1-\delta}$$

  and

  $$v_2 = u_{21} + \frac{\delta}{1-\delta}.$$

  This equilibrium exists if

  $$1 + \beta\delta\frac{1}{1-\delta} \geq u_{12} + \beta\delta(u_{21} + \frac{\delta}{1-\delta})$$

  and

  $$u_{21} + \beta\delta\frac{1}{1-\delta} \geq u_{22} + \beta\delta(u_{21} + \frac{\delta}{1-\delta}).$$

  These expressions simplify to

  $$1 - u_{12} \geq \beta\delta(u_{21} - 1) \quad \text{and} \quad u_{21} - u_{22} \geq \beta\delta(u_{21} - 1).$$

  The latter of these implies the former, given the concavity assumption. Therefore this type of equilibrium exists if the latter is met.

- $\underline{1 \rightarrow 2 \text{ and } 2 \rightarrow 1.}$ This equilibrium cannot exist since it violates monotonicity.

- $\underline{1 \rightarrow 1 \text{ and } 2 \rightarrow 2.}$ We have

  $$v_1 = 1 + \delta v_1$$

  and

  $$v_2 = u_{22} + \delta v_2$$

  which implies

  $$v_1 = \frac{1}{1-\delta}$$

  and

  $$v_2 = \frac{u_{22}}{1-\delta}.$$

  This equilibrium exists if

  $$1 + \beta\delta\frac{1}{1-\delta} \geq u_{12} + \beta\delta\frac{u_{22}}{1-\delta}$$

and

$$u_{22} + \beta\delta\frac{u_{22}}{1-\delta} \geq u_{21} + \beta\delta\frac{1}{1-\delta}.$$

These expressions simplify to

$$1 - u_{12} \geq \beta\delta\frac{u_{22}-1}{1-\delta} \quad \text{and} \quad u_{21} - u_{22} \leq \beta\delta\frac{u_{22}-1}{1-\delta}.$$

- <u>$1 \to 2$ and $2 \to 2$.</u> We have

  $$v_1 = u_{12} + \delta v_2$$

  and

  $$v_2 = u_{22} + \delta v_2$$

  which implies

  $$v_1 = u_{12} + \frac{\delta}{1-\delta}u_{22}$$

  and

  $$v_2 = \frac{u_{22}}{1-\delta}.$$

  This equilibrium exists if

  $$u_{12} + \beta\delta\frac{u_{22}}{1-\delta} \geq 1 + \beta\delta\left(u_{12} + \frac{\delta}{1-\delta}u_{22}\right)$$

  and

  $$u_{22} + \beta\delta\frac{u_{22}}{1-\delta} \geq u_{21} + \beta\delta\left(u_{12} + \frac{\delta}{1-\delta}u_{22}\right).$$

  These expressions simplify to

  $$1 - u_{12} \leq \beta\delta(u_{22} - u_{12}) \quad \text{and} \quad u_{21} - u_{22} \leq \beta\delta(u_{22} - u_{12}).$$

  The former of these implies the latter, given the concavity assumption. Therefore this type of equilibrium exists if the former is met.

- <u>$1 \to \pi$ and $2 \to 1$.</u> This equilibrium cannot exist since it violates monotonicity.

- <u>$1 \to \pi$ and $2 \to 2$.</u> We have

  $$1 + \beta\delta v_1 = u_{12} + \beta\delta v_2$$

  and

  $$v_2 = u_{22} + \delta v_2$$

  which implies

  $$v_1 = \frac{u_{22}}{1-\delta} + \frac{u_{12}-1}{\beta\delta}$$

  and

  $$v_2 = \frac{u_{22}}{1-\delta}.$$

  The mixing probability satisfies

  $$v_1 = \pi(1 + \delta v_1) + (1-\pi)(u_{12} + \delta v_2),$$

implying

$$\pi = \frac{1}{1-\beta}\left(\frac{1}{\delta} + \beta\frac{u_{12}-u_{22}}{1-u_{12}}\right).$$

This equilibrium exists if

$$u_{22} + \beta\delta\frac{u_{22}}{1-\delta} \geq u_{21} + \beta\delta\left(\frac{u_{22}}{1-\delta} + \frac{u_{12}-1}{\beta\delta}\right),$$

which is unrestrictive since it is equivalent to concavity, $\pi \geq 0$, that is,

$$1 - u_{12} \geq \beta\delta(u_{22} - u_{12})$$

and $\pi \leq 1$, that is,

$$1 - u_{12} \leq \frac{\beta\delta(u_{22}-u_{12})}{1-\delta(1-\beta)}.$$

- <u>$1 \to 1$ and $2 \to \pi$.</u> We have

$$v_1 = 1 + \delta v_1$$

and

$$u_{21} + \beta\delta v_1 = u_{22} + \beta\delta v_2$$

which implies

$$v_1 = \frac{1}{1-\delta}$$

and

$$v_2 = \frac{1}{1-\delta} + \frac{u_{21}-u_{22}}{\beta\delta}.$$

The mixing probability satisfies

$$v_2 = \pi(u_{21} + \delta v_1) + (1-\pi)(u_{22} + \delta v_2),$$

implying

$$1 - \pi = \frac{1}{1-\beta}\left(\frac{1}{\delta} + \beta\frac{1-u_{21}}{u_{21}-u_{22}}\right).$$

This equilibrium exists if

$$1 + \beta\delta v_1 \geq u_{21} + \beta\delta v_2$$

which is automatically met since it is equivalent to concavity, and $1 - \pi \geq 0$, that is,

$$u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$$

and $1 - \pi \leq 1$, that is,

$$u_{21} - u_{22} \leq \frac{\beta\delta(u_{21}-1)}{1-\delta(1-\beta)}.$$

- <u>$1 \to 2$ and $2 \to \pi$.</u> This equilibrium cannot exist since it violates monotonicity.

- <u>$1 \to \pi$ and $2 \to \pi$.</u> This equilibrium also cannot exist since it violates monotonicity.

In each of the cases when conditions for existence are given it is straightforward to see that parameter values do exist such that the given conditions are met. We now turn to discussing the possible coexistence of equilibria for given parameter values. The possible equilibria are denoted 11, 12, 22, $\pi 2$, and 1π (referring to the decision in states 1 and 2, respectively). We assume in this section that β and δ are less than 1.

We now prove the proposition by methodically going through all possibilities. First we prove six facts.

- 22 does not coexist with any other equilibrium. 22 requires $1 - u_{12} \leq \beta\delta(u_{22} - u_{12})$. Let us consider each alternative equilibrium in turn.

  The condition for 11 is $u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$. Combining it with the condition for 22 we obtain
  $$1 - u_{12} - u_{21} + u_{22} \leq \beta\delta(u_{22} - u_{12} - u_{21} + 1)$$
  which is a contradiction given strict concavity and $\beta\delta < 1$.

  One of the conditions for the 12 equilibrium is that $(1-\delta)(1-u_{12}) + \beta\delta \geq \beta\delta u_{22}$. The condition for 22 is $\beta\delta u_{22} \geq 1 - u_{12} + \beta\delta u_{12}$. But these are inconsistent since $(1-\delta)(1-u_{12}) + \beta\delta - (1 - u_{12} + \beta\delta u_{12}) = -\delta(1-\beta)(1-u_{12}) < 0$.

  The $\pi 2$ equilibrium violates the 22 condition immediately if $\pi > 0$; if $\pi = 0$ it reduces to the 22 equilibrium.

  The 1π equilibrium, finally, requires $u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$, or $u_{22} \leq (1-\beta\delta)u_{21} + \beta\delta$ which is strictly less than $(1-\beta\delta)(1-u_{12}+u_{22}) + \beta\delta$. This implies $\beta\delta u_{22} < 1 - (1-\beta\delta)u_{12}$. But this is contradicted by the 22 condition. This completes the argument that the 22 equilibrium is the unique equilibrium if it exists.

- 12 does not coexist with $\pi 2$. The requirement that $\pi < 1$ for the $\pi 2$ equilibrium is $(1 - u_{12})(1 - \delta - \beta\delta) < \beta\delta(u_{22} - u_{12})$, which can be rewritten as $1 - u_{12} < \frac{\beta\delta}{1-\delta}(u_{22} - 1)$, which contradicts the first of the two conditions for the 12 equilibrium. If $\pi = 1$ the two equilibria are equivalent.

- If 11 and 12 are both equilibria, then so is 1π. It is sufficient to show that
  $$u_{21} - u_{22} \leq \frac{\beta\delta}{1 - \delta(1-\beta)}(u_{21} - 1),$$
  which is the second of the conditions for the 1π equilibrium, as the first condition is implied directly by the existence of the 11 equilibrium. This condition can be rewritten as $u_{21} - u_{22} \leq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$, which is identical to the second of the conditions needed for existence of the 12 equilibrium.

- If 1π exists, so does 11. The 1π case requires two conditions to hold, one of which is $u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$. But this condition is the only one required for the 11 equilibrium to exist.

- If 11 and 1π are both equilibria, then so is either 12 or $\pi 2$. The 12 equilibrium exists if $1 - u_{12} \geq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$, since the second condition under which 12 exists was just shown to be identical to the second condition under which 1π exists. If not, that is, if $1-u_{12} < \frac{\beta\delta}{1-\delta}(u_{22}-1)$, we need to show that $\pi 2$ exists. This condition can be rewritten as $1 - u_{12} < \frac{\beta\delta}{1-\delta(1-\beta)}(u_{22} - u_{12})$, which implies the second condition for $\pi 2$. It remains to show that the first condition for $\pi 2$, namely, $1 - u_{12} \geq \beta\delta(u_{22} - u_{12})$, is met. Suppose it is not. Then the only condition for the 22 equilibrium to exist is satisfied. But we showed above that the 22 equilibrium cannot coexist with any other equilibrium; in particular, it cannot coexist with 11 or 1π. This is a contradiction, so the $\pi 2$ equilibrium has to exist.

- If 11 and $\pi 2$ are both equilibria, then so is 1π. We need to show that the second condition for the 1π equilibrium, $u_{21} - u_{22} \leq \frac{\beta\delta}{1-\delta(1-\beta)}(u_{21} - 1)$, is met (the first one is implied directly since the 11 equilibrium exists). From above, we know that this expression can be rewritten as $u_{21} - u_{22} \leq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$. Now concavity implies that $u_{21} - u_{22} \leq 1 - u_{12}$. We also know, by the second condition for $\pi 2$ to exist, that $1 - u_{12} \leq \frac{\beta\delta}{1-\delta(1-\beta)}(u_{22} - u_{12})$, which can be rewritten as $1 - u_{12} \leq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$. Combining these two inequalities yields the desired result.

Going through all possible equilibrium sets, these six facts rule out everything except the six possibilities we claim exist. It is straightforward to verify that these six remaining cases are possible.

Appendix 2

This appendix contains the proof of Proposition 4 in Section 4.3. The existence of a one-dimensional stable manifold for the difference equation system (1) and (2) in a neighborhood of $\bar{k}$ follows from the remarks in Section 4.2 and from straightforward modifications of standard results concerning the local stability of nonlinear dynamic systems (see, e.g., Scheinkman (1973)). That the sequence $\{k_n, v_n\}_{n=0}^{\infty}$ is strictly increasing near $\bar{k}$ follows from the fact that the the Jacobian matrix associated with the system (1) and (2), evaluated at the stationary point, has one eigenvalue equal to one and one eigenvalue between 0 and 1.

To check the optimality of the proposed decision rule, a number of lemmas will be stated and proved. Each lemma considers a specific deviation from the proposed decision rule. We first consider deviations from the optimal choice at $\bar{k}$.

Lemma 1: At $\bar{k}$ it is not better to select a $k' \in (\bar{k}, \bar{\bar{k}}]$.
Proof: This is true given that $\bar{k}$ is less than $f^{-1}(1/(\beta\delta))$, as shown above: the marginal cost of increasing k' above $\bar{k}$ is above the marginal benefit for any k'. ■

Lemma 2: At $\bar{k}$ it is worse to select a $k' = k_n$ $(< \bar{k})$:

$$u(f(\bar{k}) - \bar{k}) + \beta\delta\bar{v} \geq u(f(\bar{k}) - k_n) + \beta\delta v_n.$$

Proof: We need to prove that

$$u(f(\bar{k}) - \bar{k}) - u(f(\bar{k}) - k_n) \geq \beta\delta(v_n - \bar{v}).$$

The left-hand side of this expression can be written

$$\sum_{s=0}^{N}[u(f(\bar{k}) - k_{n+s+1}) - u(f(\bar{k}) - k_{n+s})]$$

$$+u(f(\bar{k})-\bar{k})-u(f(\bar{k})-k_{n+N+1})]$$

which, since the last two terms cancel as N goes to ∞, equals

$$\sum_{s=0}^{\infty}[u(f(\bar{k})-k_{n+s+1})-u(f(\bar{k})-k_{n+s})].$$

The right-hand side of the expression, in turn, can be rewritten as

$$\beta\delta\sum_{s=0}^{\infty}[v_{n+s}-v_{n+s+1}],$$

since v_n goes to $\bar{v}$ as n goes to ∞. Using indifference on the steps, this expression becomes

$$\sum_{s=0}^{\infty}[u(f(k_{n+s})-k_{n+s+1})-u(f(k_{n+s})-k_{n+s})].$$

It is now clear that the left-hand side is no less than the right-hand side if

$$u(f(k_{n+s})-k_{n+s})-u(f(k_{n+s})-k_{n+s+1})\geq$$

$$u(f(\bar{k})-k_{n+s})-u(f(\bar{k})-k_{n+s+1}),$$

for each $s\geq 0$. But from the strict concavity of u these inequalities are all met (strictly), since $\bar{k}>k_{n+s}$ and $\{k_n\}$ is a strictly increasing sequence. ■

Lemma 3: At $\bar{k}$ it is worse to select a $k'\in(k_n,k_{n+1})$:

$$u(f(\bar{k})-\bar{k})+\beta\delta\bar{v}\geq u(f(\bar{k})-k')+\beta\delta V(k')\quad\forall k'\in(k_n,k_{n+1})$$

in which range

$$V(k')=u(f(k')-k_{n+1})+\delta v_{n+1}.$$

Proof: Straightforward given the structure of the proof of Lemma 10 below. ■

We now move on to consider deviations from the proposed rules at any point $k\geq\bar{k}$.

Lemma 4: At $k\geq\bar{k}$ it is worse to select a $k'>\bar{k}$:

$$u(f(k)-\bar{k})+\beta\delta\bar{v}\geq u(f(k)-k')+\beta\delta V(k'),$$

where

$$V(k')=u(f(k')-\bar{k})+\delta\bar{v}.$$

Proof: Parallels the proof of Lemma 1; that is, for any $k<\bar{\bar{k}}$, the marginal benefit of increasing k' above $\bar{k}$ is strictly below the marginal cost. For $k=\bar{\bar{k}}$, the marginal benefit equals the marginal cost. ■

Lemma 5: At $k>\bar{k}$ it is worse to select a $k'=k_n$:

$$u(f(k)-\bar{k})+\beta\delta\bar{v}\geq u(f(k)-k_n)+\beta\delta v_n.$$

Proof: Parallels the proof of Lemma 2; the fact that $k>\bar{k}$ strengthens the necessary inequalities. ■

Lemma 6: At $k \geq \bar{k}$ it is worse to select a $k' \in (k_n, k_{n+1})$:

$$u(f(k) - \bar{k}) + \beta\delta\bar{v} \geq u(f(k) - k') + \beta\delta V(k') \quad \forall k' \in (k_n, k_{n+1}),$$

in which range

$$V(k') = u(f(k') - k_{n+1}) + \delta v_{n+1}.$$

Proof: The proof makes use of the proof of Lemma 10; however, to show that

$$-u'(f(k) - k') + \beta\delta u'(f(k') - k_{n+1})f'(k') > 0$$

one more argument is necessary, since k is no longer close to k_n. This is straightforward, though, since the expression is strictly increasing in k. ■

Turning to starting points k less than $\bar{k}$, we first consider deviations at steps, i.e., at points $k = k_n$.

Lemma 7: At k_n it is worse to select a $k' = k_{n-s}$, $s > 0$:

$$u(f(k_n) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k_n) - k_{n-s}) + \beta\delta v_{n-s} \quad \forall s > 0.$$

Proof: By indifference at steps, we need to show that

$$u(f(k_n) - k_n) + \beta\delta v_n \geq u(f(k_n) - k_{n-s}) + \beta\delta v_{n-s},$$

or, that

$$u(f(k_n) - k_n) - u(f(k_n) - k_{n-s}) \geq \beta\delta(v_{n-s} - v_n).$$

The left-hand side of this expression can be written

$$\sum_{v=0}^{s-1}[u(f(k_n) - k_{n-v}) - u(f(k_n) - k_{n-v-1})]$$

and the right-hand side can be written

$$\beta\delta\sum_{v=0}^{s-1}[v_{n-v-1} - v_{n-v}]$$

which from indifference at steps equals

$$\sum_{v=0}^{s-1}[u(f(k_{n-v-1}) - k_{n-v}) - u(f(k_{n-v-1}) - k_{n-v-1})].$$

It suffices to show that, for each v in these sums,

$$u(f(k_n) - k_{n-v}) - u(f(k_n) - k_{n-v-1}) \geq u(f(k_{n-v-1}) - k_{n-v}) - u(f(k_{n-v-1}) - k_{n-v-1})].$$

But this inequality holds (strictly) for all v since u is strictly concave and $\{k_n\}$ is a strictly increasing sequence. ■

Lemma 8: At k_n it is worse to select a $k' = k_{n+s}$, $s > 1$:

$$u(f(k_n) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k_n) - k_{n+s}) + \beta\delta v_{n+s} \quad \forall s > 1.$$

Proof: The proof is similar to the proof of Lemma 7. We need to show that

$$u(f(k_n) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k_n) - k_{n+s}) + \beta\delta v_{n+s},$$

or, that

$$u(f(k_n) - k_{n+1}) - u(f(k_n) - k_{n+s}) \geq \beta\delta(v_{n+s} - v_{n+1}).$$

The left-hand side of this expression can be written

$$\sum_{v=1}^{s-1}[u(f(k_n) - k_{n+v}) - u(f(k_n) - k_{n+v+1})]$$

and the right-hand side can be written

$$\beta\delta\sum_{v=1}^{s-1}[v_{n+v+1} - v_{n+v}]$$

which from indifference at steps equals

$$\sum_{v=1}^{s-1}[u(f(k_{n+v}) - k_{n+v}) - u(f(k_{n+v}) - k_{n+v+1})].$$

Due to strict concavity of u and the sequence $\{k_n\}$ being strictly increasing,

$$u(f(k_n) - k_{n+v}) - u(f(k_n) - k_{n+v+1}) > u(f(k_{n+v}) - k_{n+v}) - u(f(k_{n+v}) - k_{n+v+1})],$$

for each $v > 0$, which suffices to show that the left-hand side exceeds the right-hand side. ■

Lemma 9: At k_n it is worse to select a $k' \geq \bar{k}$:

$$u(f(k_n) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k_n) - k') + \beta\delta V(k') \quad \forall k' \geq \bar{k},$$

where

$$V(k') = u(f(k') - \bar{k}) + \delta\bar{v}.$$

Proof: First consider $k' = \bar{k}$. Noting that

$$u(f(k_n) - k_{n+1} - u(f(k_n) - \bar{k}) = \sum_{s=1}^{\infty}[u(f(k_n) - k_{n+s}) - u(f(k_n) - k_{n+s+1})]$$

and that

$$\beta\delta(\bar{v} - v_{n+1}) = \sum_{s=1}^{\infty}(v_{n+s+1} - v_{n+s}) =$$

$$\sum_{s=1}^{\infty}[u(f(k_{n+s}) - k_{n+s}) - u(f(k_n) - k_{n+s+1}],$$

the result again follows, using concavity and $\{k_n\}$ being an increasing sequence for a term-by-term domination. Turning to k' values above $\bar{k}$,

we know from the construction of $\bar{\bar{k}}$ that

$$\mathrm{MC}(\bar{\bar{k}}, \bar{k}) = \mathrm{MB}(\bar{k}),$$

where

$$\text{MC}(k, k') \equiv u'(f(k) - k')$$

and

$$\text{MB}(k') \equiv \beta\delta V'(k') = \beta\delta u'(f(k') - \bar{k})f'(k'),$$

the last equation of which holds over the range $(\bar{k}, \bar{\bar{k}}]$. Since both the MC and MB functions are continuous over this range and the MC function is strictly decreasing in its first argument, $MC > MB$ for any $k' > \bar{k}$ whenever $k < \bar{\bar{k}}$ so the consumer is strictly better off setting $k' = \bar{k}$ than $k' > \bar{k}$. Since the first part of the proof of this lemma shows that the consumer prefers k_{n+1} over $\bar{k}$, the full proof is complete. ■

Lemma 10: At k_n it is worse to select a $k' \in (k_{n-s}, k_{n-s+1})$ for any s:

$$u(f(k_n) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k_n) - k') + \beta\delta V(k') \quad \forall s, k' \in (k_{n-s}, k_{n-s+1}),$$

in which range

$$V(k') = u(f(k') - k_{n-s+1}) + \delta v_{n-s+1}.$$

Proof: We will establish that the payoff function is strictly decreasing in k' over any interior range (k_{n-s}, k_{n-s+1}) when ϵ is small. From this fact it then follows, using Lemmata 7 and 8, that the points in the interior ranges cannot be optimal. The payoff function reads

$$u(f(k_n) - k') + \beta\delta V(k') = u(f(k_n) - k') + \beta\delta[u(f(k') - k_{n-s+1}) + \delta v_{n-s+1}]$$

when $k' \in (k_{n-s}, k_{n-s+1})$, since k_{n-s+1} is chosen from starting points in this region. Taking derivatives with respect to k', we have

$$-u'(f(k_n) - k') + \beta\delta u'(f(k') - k_{n-s+1})f'(k')$$

which takes on the sign of $\beta\delta f'(k') - 1$ when ϵ is small (recall that we restrict the feasible range of k' values to $[\bar{k} - \epsilon, \bar{\bar{k}}]$). Since $\beta\delta f'(\bar{k}) < 1$ is assumed, this expression is strictly positive for small values of ϵ. ■

Moving, finally, to points between steps to the left of $\bar{k}$, we can essentially use the proofs above.

Lemma 11: At a point $k \in (k_n, k_{n+1})$, it is worse to select a $k' = k_{n+s+1}$, $s \neq 0$:

$$u(f(k) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k) - k_{n+s+1}) + \beta\delta v_{n+s+1} \quad \forall s, k \in (k_n, k_{n+1}).$$

Proof: Uses the same kinds of arguments as the proofs of Lemmata 7, 8, and 10. ■

Lemma 12: At a point $k \in (k_n, k_{n+1})$, it is worse to select a $k' \geq \bar{k}$:

$$u(f(k) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k) - k') + \beta\delta V(k'), \quad k' \geq \bar{k}, k \in (k_n, k_{n+1}),$$

where

$$V(k') = u(f(k') - \bar{k}) + \delta\bar{v}.$$

Proof: The proof here parallels the proof of Lemma 9. ■

Lemma 13: At a point $k \in (k_n, k_{n+1})$, it is worse to select a $k' \in (k_{n+s}, k_{n+s+1})$ for any s:

$$u(f(k) - k_{n+1}) + \beta\delta v_{n+1} \geq u(f(k) - k') + \beta\delta V(k'), \quad k' \in (k_{n+s}, k_{n+s+1}), k \in (k_n, k_{n+1}),$$

where

$$V(k') = u(f(k') - k_{n+s+1}) + \delta v_{n+s+1}.$$

Proof: This proof closely follows that of Lemma 10. ■

With this, the proof of the proposition is complete.

Appendix 3

This appendix contains the proofs of Propositions 5, 6, and 7 in Section 4.4.

Proof of Proposition 5: The utility from the first rule is given by

$$u(f(k^\star) - k^\star) + \beta\delta v^\star,$$

where $v^\star \equiv u(f(k^\star) - k^\star)/(1-\delta)$. Let the second rule be given by a sequence of capital stock steps $\{k_n\}_{n=1}^\infty$ with $k_1 = k^\star$. The utility from the second rule is then given by

$$u(f(k^\star) - k_2) + \beta\delta v_2,$$

where v_n is defined by the difference equation system determining $\{k_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$. We need to show that

$$u(f(k^\star) - k_2) + \beta\delta v_2 - u(f(k^\star) - k^\star) - \beta\delta v^\star > 0.$$

First, note that $\beta\delta(v_2 - v^\star) = \beta\delta\left(\sum_{n=2}^\infty [v_n - v_{n+1}] + \bar{v} - v^\star\right)$, which in turn equals $\beta\delta(\bar{v} - v^\star) + \sum_{n=2}^\infty [u(f(k_n) - k_{n+1}) - u(f(k_n) - k_n)]$, where $\bar{v} \equiv u(f(\bar{k}) - \bar{k})/(1-\delta)$, from indifference at the steps. We therefore know that

$$u(f(k^\star) - k_2) - u(f(k^\star) - k^\star) + \beta\delta v_2 - \beta\delta v^\star \tag{3}$$

has to be equal to

$$u(f(k^\star) - k_2) - u(f(\bar{k}) - \bar{k}) + u(f(\bar{k}) - \bar{k}) - u(f(k^\star) - k^\star) +$$

$$\beta\delta(\bar{v} - v^\star) + \sum_{n=2}^\infty [u(f(k_n) - k_{n+1}) - u(f(k_n) - k_n)]$$

which in turn equals

$$\frac{1 - \delta(1-\beta)}{1-\delta}\left(u(f(\bar{k}) - \bar{k}) - u(f(k^\star) - k^\star)\right) +$$

$$\sum_{n=2}^\infty [u(f(k_n) - k_{n+1}) - u(f(k_n) - k_n)] + u(f(k^\star) - k_2) - u(f(\bar{k}) - \bar{k}).$$

Since $k_1 = k^\star$, this can be rewritten as

$$\frac{1 - \delta(1-\beta)}{1-\delta}\left(u(f(\bar{k}) - \bar{k}) - u(f(k^\star) - k^\star)\right) +$$

$$\sum_{n=1}^\infty [u(f(k_n) - k_{n+1}) - u(f(k_n) - k_n)] + u(f(k_1) - k_1) - u(f(\bar{k}) - \bar{k}).$$

Again using $k_1 = k^\star$, this expression becomes

$$\frac{\delta\beta}{1-\delta}\left(u(f(\bar{k})-\bar{k})-u(f(k^\star)-k^\star)\right)+\sum_{n=1}^{\infty}\left[u(f(k_n)-k_{n+1})-u(f(k_n)-k_n)\right]. \qquad (4)$$

Note that, using the same technique as before,

$$\beta\delta(v_1 - v^\star) = \beta\delta\left(\bar{v} - v^\star + \sum_{n=1}^{\infty}\left[v_n - v_{n+1}\right]\right) =$$

$$\beta\delta(\bar{v} - v^\star) + \sum_{n=1}^{\infty}\left[u(f(k_n)-k_{n+1})-u(f(k_n)-k_n)\right].$$

Solving for the sum and substituting back into our main expression (4), we obtain

$$\frac{\delta\beta}{1-\delta}\left(u(f(\bar{k})-\bar{k})-u(f(k^\star)-k^\star)\right)+\beta\delta(v_1-v^\star)-\beta\delta(\bar{v}-v^\star)=\beta\delta\left(v_1-v^\star\right).$$

Using the recursive definition of v_1 in terms of current utility and v_2, this can be rewritten as

$$\beta\delta\left\{u(f(k^\star)-k_2)-u(f(k^\star)-k^\star)+\delta(v_2-v^\star)\right\}.$$

The obtained quantity is still equal to the expression in (3), so
we obtain the following useful equation:

$$(1-\beta\delta)\left\{u(f(k^\star)-k_2)-u(f(k^\star)-k^\star)\right\}+\beta\delta(1-\delta)(v_2-v^\star)=0,$$

or

$$v_2 - v^\star = -\frac{1-\beta\delta}{\beta\delta(1-\delta)}\left\{u(f(k^\star)-k_2)-u(f(k^\star)-k^\star)\right\}.$$

Using this expression, we can evaluate our main expression in (3) to be

$$\left\{u(f(k^\star)-k_2)-u(f(k^\star)-k^\star)\right\}\left(1-\frac{1-\beta\delta}{1-\delta}\right)=$$

$$\left\{u(f(k^\star)-k_2)-u(f(k^\star)-k^\star)\right\}\delta(\beta-1)$$

which, since $\beta < 1$, is greater than zero if $\bar{k} > k^\star$, since then $k_2 > k^\star$. ■

Proof of Proposition 6: Starting at $k^\star$, the utility from the decision rule whose steady state is $k^\star$ is:

$$U(f(k^\star)-k^\star)+\frac{\beta\delta}{1-\delta}U(f(k^\star)-k^\star). \qquad (5)$$

Since $k^\star$ is in the flat section to the right of the decision rule associated with $\bar{k}$, the optimal path for this decision rule (starting at $k^\star$) is to jump immediately to $\bar{k}$ and then stay there. The utility associated with this path is:

$$U(f(k^\star)-\bar{k})+\frac{\beta\delta}{1-\delta}U(f(\bar{k})-\bar{k}). \qquad (6)$$

We want to show that (5) is greater than (6). To show this, we will show that (6) is an increasing function of $\bar{k}$. Note that (6) is equal to (5) if $\bar{k} = k^\star$. If (6) decreases as $\bar{k}$ decreases, then (5) becomes greater than (6) as $\bar{k}$ decreases.

To show that (6) is an increasing function of $\bar k$, take its derivative with respect to $\bar k$:

$$-U'(f(k^\star)-\bar k)+\frac{\beta\delta}{1-\delta}U(f(\bar k)-\bar k)(f'(\bar k)-1). \tag{7}$$

This derivative is positive if

$$\frac{U'(f(k^\star)-\bar k)}{U'(f(\bar k)-\bar k)}<\frac{\beta\delta}{1-\delta}(f'(\bar k)-1).$$

Note that

$$\frac{U'(f(k^\star)-\bar k)}{U'(f(\bar k)-\bar k)}<1.$$

In addition, the restriction $f'(\bar k)>1+\frac{1-\delta}{\beta\delta}$ implies that

$$\frac{\beta\delta}{1-\delta}(f'(\bar k)-1)>1.$$

■

Proof of Proposition 7: Let us restrict attention, first, to values of k above $\bar k$. On the stationary point, these two equilibria of course give the same utility. Their utility levels, as a function of k, can be written

$$W^s(k)\equiv\log(Ak^\alpha-\bar k)+\frac{\beta\delta}{1-\delta}\log(A\bar k^\alpha-\bar k)$$

for the step function solution and

$$W^l(k)\equiv\log((1-s)Ak^\alpha)+\frac{\alpha\beta\delta}{1-\alpha\delta}\log sAk^\alpha+\#$$

for the log-linear solution, where $s\equiv\frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)}$ is the savings rate and # is a constant (such that $W^s(\bar k)=W^l(\bar k)$). Taking derivatives, we have

$$(W^s)'(k)=\frac{\alpha Ak^{\alpha-1}}{Ak^\alpha-\bar k}$$

and

$$(W^l)'(k)=\frac{\alpha}{k}\frac{1-\alpha\delta(1-\beta)}{1-\alpha\delta}=\frac{\alpha}{k}\frac{1}{1-s}.$$

It is possible to rewrite $(W^s)'$ as follows:

$$(W^s)'(\bar k)=\frac{\alpha Ak^{\alpha-1}}{Ak^\alpha-sAk^\alpha\frac{\bar k}{sAk^\alpha}}=\frac{\alpha}{k}\frac{1}{1-sh(k)},$$

where $h(k)\equiv\frac{\bar k}{sAk^\alpha}$. Notice that $h(\bar k)=1$ and that $h'(k)<0$. Therefore, we have that

$$(W^s)'(k)\le(W^l)'(k)$$

for all $k\in[\bar k,\bar{\bar k}]$, with equality only at $\bar k$. This implies that the log-linear solution gives higher utility to the right of $\bar k$.

We can also show that the opposite relation holds for $k<\bar k$: there, the log-linear solution gives lower utility than that implied by any step function equilibrium with the same stationary point.

We need to show that the utility $W^s(k)$ is higher than $W^l(k)$ for k immediately to the left of $\bar{k}$. The function W^s is continuous, but not differentiable everywhere to the left of $\bar{k}$. In the interval (k_n, k_{n+1}), it is differentiable and strictly concave:

$$(W^s)'(k) = u'(f(k) - k_{n+1})f'(k).$$

At the step k_n, W^s has a left-derivative and a right-derivative, but the former is strictly below the latter:

$$(W^s)'_-(k_n) = u'(f(k_n) - k_n)f'(k_n)$$

and

$$(W^s)'_+(k_n) = u'(f(k_n) - k_{n+1})f'(k_n).$$

Since $W^s(\bar{k}) = W^l(\bar{k})$, it is sufficient to show that $(W^s)'_+(k) < (W^l)'(k)$ for k very near and smaller than $\bar{k}$. To do this, we have to compare

$$(W^s)'_+(k) = \frac{\alpha}{k}\frac{1}{1 - \frac{k_{n+1}}{Ak^\alpha}}.$$

to

$$(W^l)'(k) = \frac{\alpha}{k}\frac{1}{1-s}.$$

Clearly, we need to show that $\frac{k_{n+1}}{Ak^\alpha} < s \equiv \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)}$. This follows if we can show that

$$k_{n+1} < \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)}Ak_n^\alpha,$$

since $k \geq k_n$. Now we know that close to $\bar{k}$, $\{k_n\}$ almost satisfies (from the linearization)

$$k_{n+1} - \bar{k} = \lambda(k_n - \bar{k}),$$

where λ (the non-unitary eigenvalue) equals $\frac{1-\beta\delta f'(\bar{k})}{\delta(1-\beta)}$. This means that it is sufficient to show that

$$G(k_n) \equiv \lambda k_n + (1-\lambda)\bar{k} - \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)}Ak_n^\alpha < 0$$

for $k_n < \bar{k}$. To show this, first notice that since $\bar{k} = \left(\frac{\alpha\beta\delta A}{1-\alpha\delta(1-\beta)}\right)^{-1}$, λ simply equals α. The G function therefore can be written

$$G(x) = \alpha x + (1-\alpha)\bar{k} - \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)}Ax^\alpha.$$

Taking derivatives, we see that

$$G'(x) = \alpha\left(1 - \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)}Ax^{\alpha-1}\right).$$

Using the expression for $\bar{k}$, we see that $G'(\bar{k}) = 0$. But since $G'(x)$ is increasing, we conclude that $G'(k_n) < 0$ for $k_n < \bar{k}$, and the proof is complete. ■

Figure 1

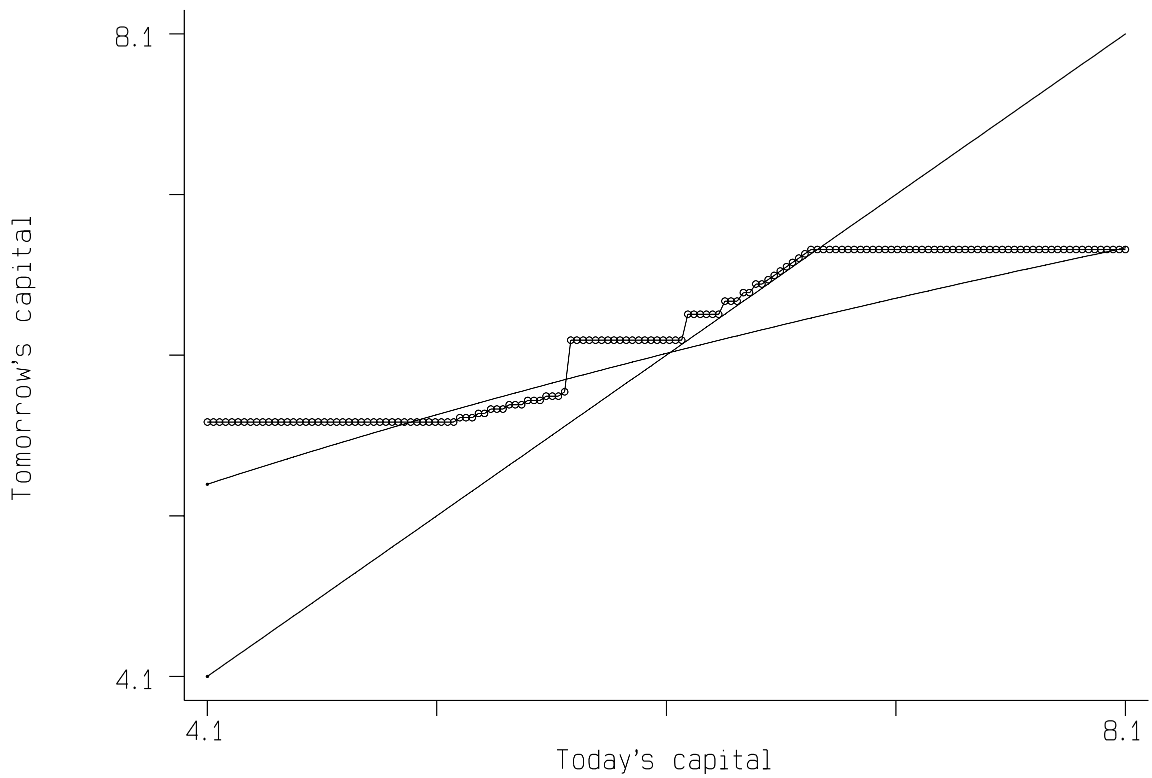

Figure 2

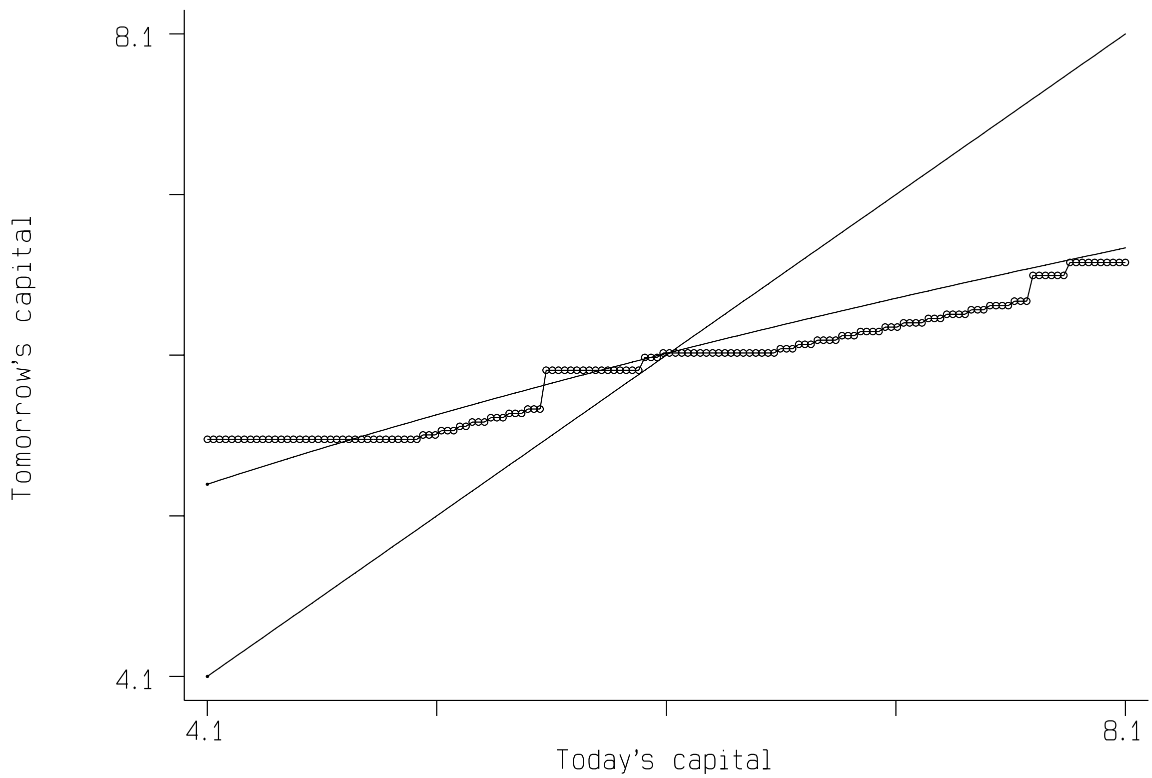

Figure 3

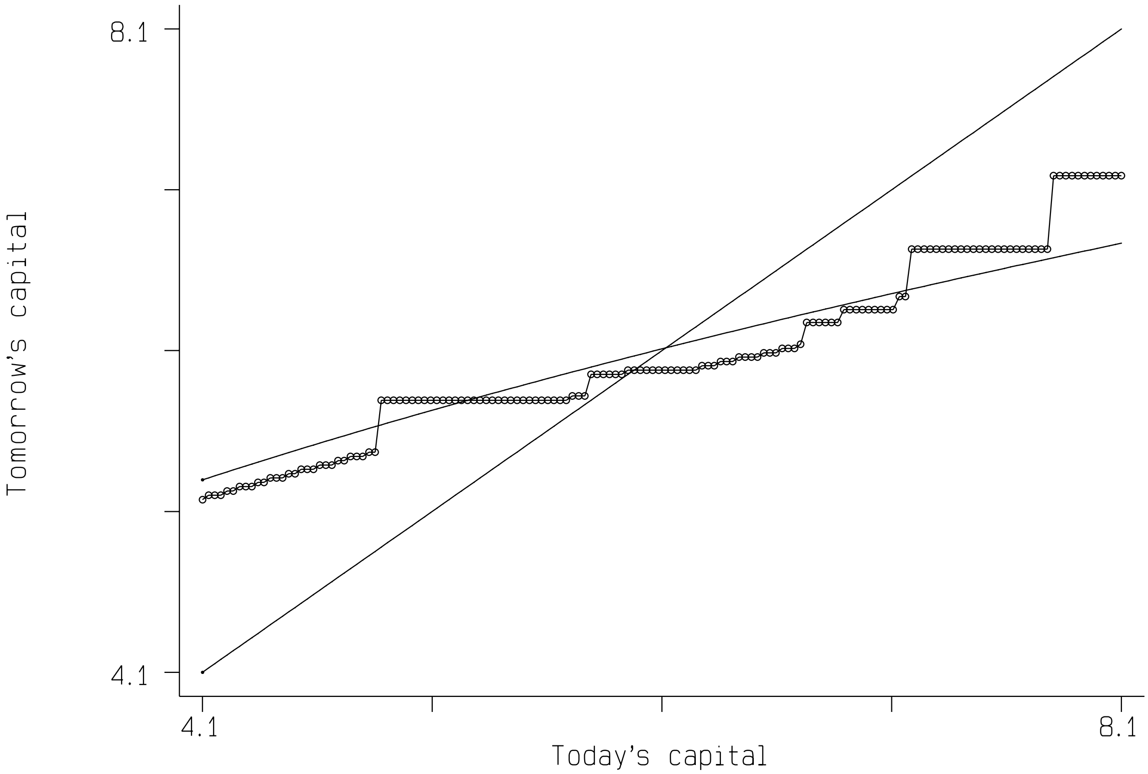

Figure 4: Three Equilibrium Decision Rules

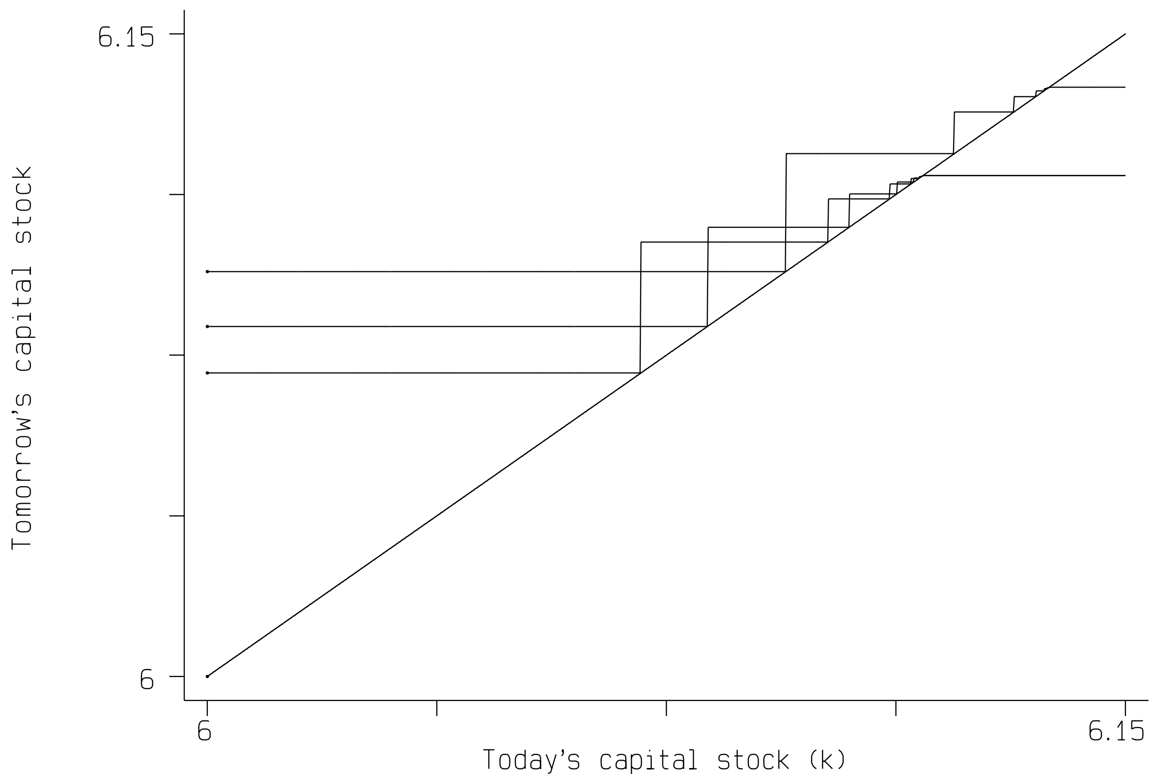

Figure 5: Three Value Functions