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Patrick Legros and Andrew Newman

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Patrick Legros, ECARES, Université Libre de Bruxelles, Université de Liège and CEPR
Andrew Newman, University College London

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Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: http://www.cepr.org

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ABSTRACT

Monotone Matching In Perfect And Imperfect Worlds*

We study frictionless matching models in large production economies with and without market imperfections and/or incentive problems. We provide necessary and sufficient distribution-free conditions for monotone matching which depend on the relationship between what we call the segregation pay-off – a generalization of the individually rational pay-off – and the feasible set for a pair of types. Imperfections have two distinct effects that are relevant for equilibrium matching patterns: they can overwhelm the complementarity properties of the production technology and they can introduce non-transferabilities that make equilibrium matching inefficient. We also use our framework to reveal the source of differences in the comparative static properties of some models in the literature and to explore the effects of distribution on the equilibrium matching pattern.

JEL Classification: D30, D51, L10, L22, L23
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Patrick Legros
ECARES
Université Libre de Bruxelles
Avenue F D Roosevelt 39
1050 Brussels
BELGIUM
Tel: (32 02) 650 4219
Fax: (32 02) 650 4475
Email: plegros@ulb.ac.be

Andrew Newman
Department of Economics
University College London
Gower St
London WC1E 6BT
UK
Tel: (44 171) 504 5871
Fax: (44 171) 916 2775
Email: a.newman@ucl.ac.uk

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NON-TECHNICAL SUMMARY

Matching models have proved to be remarkably adaptable to a wide range of problems. Early applications considered environments in which there were no market imperfections: the only departure from standard Arrow-Debreu assumptions was the presence of indivisibility in agents' characteristics that make a matching problem relevant. But many of the more recent applications, including for example community stratification, education financing, international trade, organizational design, or the market for ownership and control, also involve some sort of imperfection arising from a missing market or an information asymmetry.

The main insight of the early literature was a fundamentally monotonic result. In the presence of complementarities there is positive assortative matching: more able individuals are assigned to more productive tasks or to more able individuals. Monotone matching patterns of this kind are compelling both because of their empirical appeal and because they greatly facilitate computation of the equilibrium. However, it is unclear to what extent the connection between complementarity and positive assortative matching carries over to the more general environments that have attracted recent attention. Our purpose here is to (1) provide conditions for monotone matching to occur in equilibrium in some of these more general environments and (2) show, by studying some applications, how these conditions facilitate computation of equilibria (including those in the classical models) and help with an assessment of the impact of imperfections on matching.

Our analysis proceeds by studying an object called the 'surplus', defined as the difference between the utility possibility set of a matched pair and the segregation pay-off vector, the latter consisting of the utility levels that each type would get in the equilibrium of an economy consisting solely of that type. The surplus is a natural measure of the 'gains from trade' (more precisely gains from a heterogeneous match), and its properties can tell us a lot about the equilibrium outcome.

We derive the following main results. (1) For the case of transferable utility, in which case the surplus is a real valued function, the match will be positively assortative of all type distributions if and only if the surplus satisfies 'weak increasing differences', a strict weakening of the standard increasing difference condition. (2) Two-sided matching models, in which there is a gender as well as a type, have strong invariance properties compared with one-side models; this is explained by showing that they are equivalent to a particular form of one-side model with a particular structure. (3) For the case of non-transferrable utility, we provide a readily verifiable sufficient condition (the ‘spiralling condition’), that has a simple diagrammatic representation, for positive assortative matching.

We then employ these results to analyse several extended examples, which are simplified versions of models from the recent literature. Market imperfections have two distinct effects that are relevant to matching. First, as we illustrate with a model of production with a credit market imperfection, they may swamp the complementarity properties of the production technology, resulting in matches that may be non-monotonic; more generally the monotonic nature of the match may depend on the distribution of types. Thus, even when it is known that technology is complementary, positive assortative matching is not a universal prediction of matching models,

Second, imperfections reduce transferability *within* coalitions. We apply the spiralling condition to a model of risk-sharing within households, showing that there will be negative assortative matching by wealth. We also study a production model with moral hazard in which a very strong form of positive matching, known as segregation, wherein agents match only with agents of their own type, emerges despite the fact that it is not efficient: a social planner could increase the economy’s output by reassigning people to other types. The incentive problem prevents the winners in such a reassignment from compensating the losers, which is why it doesn’t happen in equilibrium. The result arises only from the failure of transferability within coalitions and has nothing to do with there being too few towns for the number of types or other ‘external’ effects. The example also shows that positive assortative matching may not constitute reliable evidence of an efficient matching process.

1 Introduction

Ever since Roy [22] and Tinbergen [27] used them to study the distribution of earnings, matching (or assignment) models have proved to be remarkably adaptable to the study a wide range of problems.[1] Early applications of these models (as well as some more recent ones) tended to be to environments in which there were no market imperfections: the only departure from standard Arrow-Debreu assumptions was the presence of an indivisibility in agents' characteristics that make a matching problem relevant. Many more recent applications — including for example community stratification, education financing, international trade, organizational design, or the market for ownership and control ([2], [9], [5],[16], [18]) — also involve some sort of imperfection arising from a missing market or an information asymmetry.

Among the main insights of the early literature was a fundamental monotonicity result. In the presence of complementarities there is positive assortative matching: more able individuals are assigned to more productive tasks or to more able individuals. Monotone matching patterns of this kind are compelling both because of their empirical appeal and because they greatly facilitate computation of the equilibrium. Indeed, in the minds of most economists, the connection among efficiency, positive assortative matching and complementarities is probably *the* main idea of the matching literature.

But while this connection has been established for cases in which there are no market imperfections, it is unclear to what extent it carries over to the more general environments that have attracted recent attention. Some examples suggest that complementarities in the production technology alone need not entail positive assortative matching.[2] Moreover, the presence of market imperfections leads to the possibility that matches may not be efficient, at least in the sense of maximizing social surplus. Our purpose here is to provide necessary and sufficient conditions for monotone matching in some of these more general environments. These conditions facilitate computation of equilibria (including those in classical environments) and help with an assessment of the impact of imperfections on matching.

The standard argument for positive assortative matching goes something

[1] Some other classic references are Becker [1], Gale-Shapley [10], Roth-Sotomayor [21] , and Sattinger [24]. Sattinger [25] provides a fine survey of the "classical" literature.

[2] For instance, Legros-Newman [16] study a model of firm formation and find that when capital markets are perfect, matches are segregated, while when capital markets are imperfect, there may be negative assortative matching.

like this. Consider two workers of abilities $a_H > a_L$ and two firms with productivities $b_H > b_L$. All agents have payoffs which are linear in income. Complementarity in production means that the output gain when a_L switches from b_L to b_H is smaller than that gained when a_H makes the same switch. Therefore, a_H can outbid a_L for the more productive firm and, in an efficient (or competitive or core) allocation of such an economy, a_H matches with b_H and a_L matches with b_L : we get positive assortative matching.

Notice two crucial assumptions. The obvious one is complementarity, which arises quite naturally in production situations; most neoclassical production technologies display this property. But what matters for positive assortative matching is complementarity in the joint *payoffs*; this immediately leads us to ask what happens in the presence of market imperfections, when payoffs and output are not always the same thing. Suppose, for instance that outside financing of a project is possible only if the joint output exceeds a certain minimum level (many models of imperfect financial markets have this or a similar property); then negative assortative matching (in which the high firm matches with the low worker and vice versa) may be the outcome of competition and may even be optimal. Thus imperfections may affect the complementarity properties of the joint payoff in ways that overwhelm the effects of the technology.

Second, and less obvious perhaps, is the implicit assumption that there is full transferability of utility between the partners in a match. In order for a_H to outbid a_L for b_H, she may have to pay to b_H the full marginal gain from matching with b_H rather than b_L. But if there are incentive problems (suppose it is harder to detect more able workers when they shirk, so inducing effort requires paying them a large rent), transferring too much income to the high productivity firm may destroy her incentives, so she may only be able to match with the lower (and cheaper) firm. The outcome might be inefficient in the sense that a social planner could generate higher total output by forcing a new match and requiring high productivity firms to accept less than their equilibrium income.

In order to handle these two effects of market imperfections — changes to the complementarity properties of the joint payoff and reductions in transferability — the appropriate object to look at is the utility possibility set for each possible partnership: in other words, we are interested in the characteristic function of a cooperative game representing the matching problem. To describe the outcome on this game (we will use the core as our equilibrium concept) requires that we characterize the equilibrium match and the corresponding equilibrium utilities of all the agents.

In practice, such a characterization is difficult. Nevertheless, as we will show, a lot of information about the comparative statics in technological and informational parameters of equilibrium can be obtained by analyzing changes in the feasible utility sets and in what we call the segregation payoffs. The segregation payoff for an individual is the equilibrium payoff to this individual in an economy consisting solely of individuals of the same type as himself.

Our analysis follows a simple economic logic. One normally thinks of an equilibrium as a situation in which individuals' current benefits exceed their outside options. Occasionally, some individuals' equilibrium outside option is equal to the segregation payoff, but this will not be true in general. Nevertheless, we use the segregation payoff as a lower bound on the outside option and compare it to the utility possibility obtained in different matches. Doing so we have a natural concept of "gains from trade" (more precisely gains from a heterogeneous match). It is the comparative static of these gains from trade that will tell us much about the equilibrium outcome. For example, it helps to indicate situations in which individuals of very different types match together — despite strong complementarities in the production technology — because one type has a very low segregation payoff.

Within this framework, we provide necessary and sufficient conditions for monotone matching to occur for any distribution of characteristics. All of them rely on the same kind of logic: positive matching requires that there be no negative matches (i.e. if $a > b > c > d$, we cannot have a matched with d and b with c). Thus, if four types are matched in a negative way, it must be possible for two of them to improve upon the payoffs they are getting in the negative match. This simple observation places restrictions on the characteristic function which are relatively easy to verify. For instance, in the case of transferable utility, we obtain a direct weakening — known as weak increasing differences — of the standard complementarity conditions on the joint payoff (Proposition 4). In the nontransferable utility case, our necessary and sufficient condition for positive matching (Condition P and Proposition 6) describes those characteristic functions which will not admit a stable negative match.

Perhaps the most novel result applies to the nontransferable utility case in which the Pareto frontiers of the utility possibility sets for all pairs of types are strictly decreasing. In this instance, one can describe these frontiers by invertible functions. A sufficient condition for positive matching then turns out to be that the image of any positive real number under certain fourfold compositions of these functions is always less than that number (Proposition 10). This condition is fairly amenable to analytic and/or numerical verification and should therefore help to render tractable a wider class of matching models (for instance, those with risk averse agents).

We employ these results to analyze several extended examples, some of which are taken from the recent literature and some of which are new. We study two imperfect markets examples, making use of our results on monotone matching to help compute equilibria. More substantively, we illustrate the points made above about the two effects of imperfections on matching. In one case, a financial market imperfection swamps the complementarity in production, resulting for instance in situations in which nonmonotonic matching (mixtures of positive and negative assortative matching) may occur. Thus, positive assortative matching is not a universal prediction of matching models, even when it is known that technology is complementary.

Another feature of this example is that despite being derived from a market imperfection, payoffs remain fully transferable within coalitions. A consequence of this fact is that matching, though no longer necessarily monotonic, is always optimal.

A second model we study illustrates the reduced transferability effect of imperfections. In the context of a production model with moral hazard, a very strong form of positive matching, known as segregation, in which agents match only with agents of their own type, emerges despite the fact that it is not efficient: a social planner could increase the economy's output by reassigning people to other types. The incentive problem prevents the winners in such a reassignment from compensating the losers, which is why it doesn't happen in equilibrium. Similar arguments have been made for example with respect to school choice [2], [9]. Here though, the inefficiency stems solely from lack of transferability *within* coalitions, and has nothing to do with there being too few towns for the number of types or other "external" effects. The example also shows that positive assortative matching may not constitute reliable evidence of an efficient matching process.

Of course the utility of our results is not limited to economies with imperfections. They can be helpful in understanding some comparative static properties of different "perfect-world" models. For example, consider Becker's [1] model of the marriage market. There are two tasks, 1 and 2; if a_i is the ability of the individual performing task i, output is $h(a_1, a_2)$; ability is complementary: h has positive cross partial derivatives throughout. Individuals are distinguished on the basis of their gender $i = 1, 2$; men perform task 1, women perform task 2. It is well known that the matching in this model is always positive assortative. Moreover, for a given distribution of abilities, the equilibrium matching pattern is invariant to the choice of h as long as h has the complementarity property.

Kremer [14] and Kremer and Maskin [15] use a similar model to study income distribution. The first paper uses $h(a_1, a_2) = a_1 a_2$ while the second uses $h(a_1, a_2) = \max\{a_1^2 a_2, a_2^2 a_1\}$. In the first case there is perfect segregation: in equilibrium, each firm consists of a single type of worker. In the second case, firms will not be segregated; in particular, if the support of the distribution is tight enough, the best worker will match with the median worker and the others will match in a positive assortative way. The change in the matching pattern can only have come from the difference in the production function. As we know, this cannot happen in Becker's model.

Why the dramatic difference in comparative statics? In both models, a worker's willingness to pay for a partner depends on the difference between what he achieves with a partner and his segregation payoff (we call this difference, when positive, the "surplus"). But there is an important distinction between the two models. In Becker's case, if two people of the same gender match together they receive a payoff of zero irrespective of their abilities. Therefore the gains from a heterogeneous match relative to the segregation payoffs are fully described by the output function h. Every man would like to match with the ablest woman, but it is the ablest man who is willing to pay the most. This fact is independent of the specific form of h. By contrast, in Kremer-Maskin, the segregation payoff is positive, and so the surplus varies with ability and with the choice of h in nonmonotonic ways. Therefore the individuals do not unanimously rank the other individuals and the pattern of matching will be more complex and more sensitive to the specifics of the technology and type distribution. We show that a general property of two-sided matching models (such as Becker's) is that matching will be invariant to changes in technology as long as complementarity is preserved. We also derive a sufficient condition (the "single trough surplus condition") for one-sided matching models (such as Kremer's) which lead to a similar pattern. The latter condition is more vulnerable to changes in the technology, which helps explain why the one-sided models have more complex comparative statics than the two-sided models.

Another issue that has attracted some attention recently is the dependence of the pattern of matching on the distribution of types [15]. In fact, it is clear in general that the match *must* depend on the distribution, if only in the sense that the correspondence $m(a)$ which sends a type a into the type(s) with which it matches will not be invariant to the distribution (think of the example above in which say b_L increases slightly to $\hat{b}_L < b_H$: $m(a_L)$ will change from b_L to $\hat{b}_L$). Of course, requiring that $m(a)$ be invariant is very demanding (Condition S below is necessary and sufficient for this kind of invariance). At the other extreme, we might only require that monotonicity of the match be preserved, for which our Conditions P and N are necessary and sufficient. In between, these conditions don't help directly, but reveal a lot about the structure of particular models and make them easier to solve. To this end, we study the model of Kremer and Maskin, showing how their main result on the effect of distribution of the degree of segregation is easily understood as a consequence of the shape of the surplus functions. Their model always has monotone matching, however, and the dependence of matching on distribution is reflected in cardinal measures of changes in the matching map $m(a)$. A more striking dependence of matching on distribution occurs in our imperfect financial market example: changes to the type distribution can cause the match to go from positive to negative assortative, and typically there will be a nonmonotonic mix of the two.

2 Theory

2.1 Notation

The economies we study have a continuum of agents who are designated by the set $I = [0,1] \times [0,1]$ with Lebesgue measure. The description of a specific economy includes an assignment of individuals to types via a map $\tau : I \longrightarrow T$, where the "type space" T is taken to be a compact subset of some Euclidean space with the usual order. The map τ is measurable. We also assume that any two agents with the same first coordinate get assigned the same type by τ: if $i = (x, y)$ and $j = (x, \hat{y})$, then $\tau(i) = \tau(j)$. The type assignment τ induces a distribution of types which may have finite or continuous support; we shall be concerned with both cases depending on context.

This somewhat unconventional construction is appropriate for two reasons. First, the core is the equilibrium concept that we will use, and this is defined in terms of individuals rather than types. Moreover, in the environments we shall be considering, defining an equilibrium directly in terms of types is awkward because we cannot guarantee that all agents of a given type get the same payoff (i.e. there is no "equal treatment property"). Secondly, we use the two-dimensional set of agents because we require that there is a continuum of agents of every type so that the segregation-payoff reasoning is logically consistent. Throughout the discussion, however, we shall talk interchangeably in terms of either individuals or types matching together, blocking allocations, etc. as convenience and clarity dictate.

We will follow much of the literature in restricting attention to matches of size two (some of our results generalize to multiperson matches, as we will indicate); the next step then is to specify what the payoff possibilities are for a pair of individuals. The most general approach would be to simply posit that there is such a set with abstract properties. For definiteness, we shall specify a somewhat restricted class for the reader to keep in mind; our examples will mostly come from this class.

In many applications, the individuals are assumed to be risk-neutral income maximizers who can feasibly share the output of their joint production in any way. The level of output they can generate depends on their type according to a (possibly stochastic) "production function" $h(t_1, t_2, \theta)$, where $\theta \in \mathbb{R}^l$ are parameters reflecting aspects of the technology: we shall often be interested in studying how the pattern of matching varies with changes in this parameter. Thus the set of utilities that a pair of individuals with types t_1 and t_2 can generate would be described as

$$V(t_1, t_2) = \{(v_1, v_2) \in \mathbb{R}^2 | v_1 + v_2 \leq h(t_1, t_2, \theta)\}.$$

The notation reflects the fact that the utility possibilities of the pair of agents do not depend on what other agents in the economy are doing: *there are no externalities across coalitions.*[3] We shall maintain this assumption throughout.

Since we are interested in studying how market imperfections affect matching outcomes, we shall need a more general framework. This is easily accom-

[3] Of course the equilibrium payoffs in one coalition will depend on the other coalitions, in general. The restriction on externalities may exclude certain types of imperfections from the analysis (e.g. community formation models such as [3] and [9] in which there are congestion effects), but we believe our approach has some relevance to those cases.

modated by considering utility possibility sets of the form

$$
\begin{aligned}
V(t_1, t_2) &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1) \\
\{(v_1, v_2) &\in \mathbb{R}^2 | \, \exists x, q : \left\{ \begin{array}{l} v_1 + v_2 \leq h(q, x; t_1, t_2, \theta) - g(q, x; t_1, t_2, \theta, \phi) \\ v_i \geq f(q, x; t_i, \phi), \;\; i = 1, 2 \end{array} \right. \} \\
& \cup \{(0, 0)\}.
\end{aligned}
$$

We have added some extra variables: $q \in Q \subseteq \mathbb{R}^m$ represents possible technological or organizational choice variables for the coalition (supposed contractible and/or publicly observable), $x \in X \subseteq \mathbb{R}^n$ are unobservable or noncontractible choices that can be made by the individual partners, and $\phi \in \mathbb{R}^p$ are parameters representing costs associated with asymmetries of information within the coalition or between the coalition and the outside world.

The function g has been subtracted from the original production function to account for costs to the coalition arising from certain market imperfections. For instance, g could represent the cost of monitoring the partners' effort. Or it could be a general representation of the cost of financial market imperfections: for example, one element of θ could be a fixed capital requirement k without which no output can be produced, the types could be wealth levels, and $\phi > 1$. Then one way to write a capital market imperfection would be

$$g(q, t, t', k, \phi) = \max\{k - t - t', \phi(k - t - t')\}.$$ [4]

Finally, the constraint(s) f are restrictions on how the output is shared and are typically present when there are incentive problems (of the hidden action or hidden information variety) within the coalition. (Note there are possibly as many constraints as there are choices in X, although usually only a few of these bind). For simplicity, we assume that the minimum payoff to one partner depends only on his own type, the observable choices q, the unobservable choices x that he might make, and (possibly) the information parameters ϕ. For example, if effort e is zero or one and not directly observable, $q \in [0, 1]$ a monitoring intensity measuring the probability of detecting a shirking partner, and a partner of type t incurs a disutility t if and only if he exerts effort, an incentive compatibility constraint might assume the form $v_i \geq e(\frac{t}{q} - t)$.

All choice sets are compact and h, g, and f are continuous in the choice variables; h and f are continuous and g lower semicontinuous (so that $h - g$ is upper semicontinuous) in types. These assumptions help to ensure existence of an equilibrium.

We assume that the payoff to an unmatched individual is zero and that a coalition of larger than two individuals cannot achieve anything that could not also be achieved by subcoalitions of size one or two. Notice that $(0,0) \in V(t,t')$ for any pair of types. What we have done then is to specify a game in characteristic function form, and by a slight abuse of the conventional definition, we will often refer to $V(\cdot,\cdot)$ as the characteristic function.

Of course, the characteristic function can be generated in other ways from the one we have described here. For instance, the f functions could be used to describe the utility possibility set for a situation (with or without imperfections) in which the partners are risk averse. As should be clear, Propositions 2, 6, and 9 apply to the more general case, and in fact, we will examine an instance of this in Section 2.4.4.

Consider two individuals of the same type t who are matched together. Define the *segregation payoff* of type t as the (unique) payoff $\underline{u}(t)$ such that $(\underline{u}(t), \underline{u}(t))$ is on the Pareto frontier of the convex hull of $V(t,t)$. (Sometimes $V(t,t)$ itself fails to be convex, in which case its Pareto frontier may not intersect the 45° line; this is why we use the convex hull.[5]) The segregation payoff has the interpretation of the minimum utility that an agent can expect to get: if two agents of a particular type get less than this, they can always match together and share the output equally (at least in expectation).

In partial equilibrium analyses (bargaining problems, principal-agent models), outside options are exogenously given and are crucial for predicting how gains from cooperation will be allocated across the individuals. In our framework however, the outside option of an individual will usually be his equilibrium payoff. There is therefore no obvious operational concept of outside option that can be used if one wants to understand the structure of equilibria without having to compute them. What we hope to show in this paper is that the segregation payoff is actually such an operational concept: it tells us a lot about the patterns of matching that can arise in equilibrium, and from this information computation of equilibrium is greatly simplified.

It will often be convenient to analyze economies using a modified characteristic function that captures the notion of the potential gains from "trade"

[5]Actually, it is enough to consider the intersection of the 45° line with the Pareto frontier of the comprehensive extension $V - \mathbb{R}^2_+$ of V in order to ensure that the segregation payoff is well defined.

(i.e., heterogeneous matching). Formally, let[6]

$$\begin{aligned} S(t,t') &= \{(0,0)\} \cup \left([V(t,t') - (\underline{u}(t), \underline{u}(t'))] \cap \mathbb{R}^2_+\right) \\ S(t) &= \{0\}. \end{aligned}$$

Denote by S^P its Pareto frontier.

Special notation is useful for the case in which utility is transferable within coalitions (in terms of (1), this is the case $f \equiv 0$). This case encompasses many of the perfect-markets examples already present in the literature. It also includes a number of imperfect markets examples (the imperfect financial market example below is one). If surplus is transferable, there exists σ such that the Pareto frontier can be expressed as $S^P = \left\{s \in \mathbb{R}^2_+ : s_1 + s_2 = \sigma\right\}$. Since σ depends on the types, we have a *surplus function* which is the maximum of 0 and σ and which we write $\sigma(t,t')$.[7] Observe that $\sigma(t,t) = 0$ for all t.

2.2 Equilibrium

The equilibrium specifies the way individuals are matched to each other, i.e., the way the set I is partitioned into coalitions. We use the core as the equilibrium concept: a partition can be part of an equilibrium if there exists a payoff structure that is feasible for that partition and such that it is not possible for any individuals to obtain a higher payoff by forming a coalition different from their equilibrium coalition.

Denote by $\mathbb{P}$ the set of *measure consistent* partitions of I into subsets of size two at most.[8]

[6] Therefore, if the segregation payoff vector lies outside the feasible set, we define the surplus set to be the zero vector. The fact that the surplus is zero captures the idea that there are no gains from trade.

[7] Thus, $\sigma(t,t') = \max\left\{0, H(t,t') - \frac{1}{2}[H(t,t) + H(t',t')]\right\}$, where $H(\cdot,\cdot)$ is the maximized value of net output $h(q,x;\cdot,\cdot,\theta) - g(q,x;\cdot,\cdot,\theta,\phi)$; $\frac{1}{2}H(t,t)$ is just the segregation payoff for t.

[8] Let $\mathcal{P}$ be a partition of I. Let $\mathcal{P}^2$ be the set of elements of $\mathcal{P}$ of size two. List the elements of every $P \in \mathcal{P}^2$ according to the lexicographic order $\succeq_L$on $\mathbb{R}^2$ (hence, write $P = (i,j)$ when $i \succeq_L j$). Let I^1 be the set of agents who are first and I^2 the set of agents who are second. $\mathcal{P}$ is measure consistent if $\lambda\left(I^1\right) = \lambda\left(I^2\right)$. This restriction rules out partitions in which say, all agents in $[0,1/3] \times [0,1]$ are matched one-to-one with all the agents in $(1/3,1] \times [0,1]$. See also Wooders [28] and Kaneko-Wooders [12].

Definition 1 *An equilibrium is a pair* $(\mathcal{P}, u)$ *consisting of a partition* $\mathcal{P} \in \mathbb{P}$ *and a utility allocation* $u : I \rightarrow \mathbb{R}$ *such that*

(i) u *is feasible: for almost all* $P = \{i, j\} \in \mathcal{P}$, $(u_i, u_j) \in V(\tau(i), \tau(j))$.

(ii) u *cannot be improved upon: there does not exist a pair of agents* $\{i, j\}$ *and a vector of payoffs* $(\hat{u}_i, \hat{u}_j) \in V(\tau(i), \tau(j))$ *such that* $(\hat{u}_i, \hat{u}_j) >> (u_i, u_j)$.

Our assumptions guarantee that an equilibrium always exists if $f \equiv 0$ or if the type distribution has finite support (see the Appendix and Gretsky-Ostroy-Zame [11], Kaneko-Wooders [12] and Wooders [28]).

We first note that all equilibria are constrained Pareto efficient. Indeed, since effective coalitions are finite, the grand coalition cannot achieve anything more than what two person coalitions can achieve. If there were a Pareto improvement, then the grand coalition could block the equilibrium payoff but then a two person coalition could also do it and this would violate the definition of an equilibrium.

In the case of transferable utility (that is the case in which the function f in (1) is identically zero),[9] something much stronger can be asserted, namely that the equilibrium match will maximize the aggregate net output (this includes in particular the case in which g is nonzero). In this case, any pair of matched agents who are part of an equilibrium will always maximize their joint net output $h - g$. Call this maximized value $H(a, b)$ when an a matches with a b (it always exists under our assumptions). Observe that if a and b are two types which are unmatched in equilibrium, then $u_a + u_b \geq H(a, b)$, else the pair (a, b) would block.[10] Now, if the equilibrium matching pattern $(a, m(a))$ fails to maximize aggregate net output ($m(a)$ is the (set of) type(s) that are assigned to type a), there is another measure consistent match $(a, \tilde{m}(a))$ which generates a higher aggregate; for at least some type $\hat{a}$ such that $\tilde{m}(\hat{a}) \neq m(\hat{a})$ we must then have $u_{\hat{a}} + u_{\tilde{m}(\hat{a})} < H(\hat{a}, \tilde{m}(\hat{a}))$, or the aggregate could not be higher. But then the pair $(\hat{a}, \tilde{m}(\hat{a}))$ would have blocked the original equilibrium. A similar argument can be made for the aggregate surplus,[11] and we have

[9] The terminology may be slightly confusing because we have assumed that agents are risk-neutral in income. However, incentive or liquidity constraints restrict how utility can be transferred from one agent to another, so it is important to bear in mind that transferability is a separate assumption.

[10] With transferable utility, there must be equal treatment, so there is no ambiguity in denoting the equilibrium utility u as dependent on type rather than on individual.

[11] In this case, note that in equilibrium, $u_a \geq \underline{u}(a)$. If $\tilde{m}(a)$ is a match which yields

Proposition 1 *If $f \equiv 0$, then in equilibrium (i) the match is efficient in the sense that given the type distribution, it maximizes aggregate net output; and (ii) aggregate surplus is also maximized.*

The optimality of equilibrium (i) under transferable utility is, of course, well known; what we want to emphasize here is that certain market imperfections can still be treated under the rubric of transferable utility and therefore lead to efficient outcomes. We will return to this point in Section 3.2.1. As we will also show there, result (ii) can be useful in computations.

2.3 Descriptions of Equilibrium

Since we are unconcerned with the identities of individuals in this economy apart from their type, we shall usually denote a coalition by the types of its members, i.e. a coalition consisting of individuals i and j will be written as (a, b), where $\tau(i) = a$ and $\tau(j) = b$. We now provide some definitions useful for characterizing equilibria.

The simplest (and strongest) form of monotone matching occurs when each agent matches only with someone like himself, a condition we refer to as *segregation.*

Definition 2 *An equilibrium $(\mathcal{P}, u)$ satisfies segregation (SEG) if for almost all $P \in \mathcal{P}$, $t = t'$ for $(t, t') \in P$.*

Definition 3 *An equilibrium $(\mathcal{P}, u)$ satisfies essential segregation (ESEG) if there exists another equilibrium $\left(\widehat{\mathcal{P}}, \widehat{u}\right)$ satisfying SEG such that $\widehat{u} = u$ almost everywhere. An economy is* segregated *if all equilibria are essentially segregated.*

Note that if an economy is segregated, the equilibrium payoff is essentially unique: in equilibrium, almost every individual obtains the segregation payoff for his type. For this reason, the segregation payoff provides a lower bound on the outside option of an individual in any equilibrium match.

higher surplus, we have $\int \sigma(a, m(a)) = \int u_a + u_{m(a)} - \underline{u}(a) - \underline{u}(m(a)) = \int u_a + u_{\tilde{m}(a)} - \underline{u}(a) - \underline{u}(\tilde{m}(a)) < \int \sigma(a, \tilde{m}(a))$. So there is a type $\hat{a}$ for which $\sigma(\hat{a}, \tilde{m}(\hat{a})) > u_{\hat{a}} + u_{\tilde{m}(\hat{a})} - \underline{u}(\hat{a}) - \underline{u}(\tilde{m}(\hat{a})) \geq 0$; Thus $H(\hat{a}, \tilde{m}(\hat{a})) - \underline{u}(\hat{a}) - \underline{u}(\tilde{m}(\hat{a})) > u_{\hat{a}} + u_{\tilde{m}(\hat{a})} - \underline{u}(\hat{a}) - \underline{u}(\tilde{m}(\hat{a}))$, or $H(\hat{a}, \tilde{m}(\hat{a})) > u_{\hat{a}} + u_{\tilde{m}(\hat{a})}$,which again contradicts no-blocking.

Segregation is an extreme kind of equilibrium outcome. When the set of types is one-dimensional, the literature has used a weaker concept than segregation, namely positive assortative matching. For the remainder of this paper then, we assume that the type space T is one dimensional, that is, a compact subset of the real line. Strictly speaking, this leaves out the two-sided matching models, but as we shall see these can be handled within the one-dimensional framework. We define positive assortative matching as follows.

Definition 4 *An equilibrium $(\mathcal{P}, u)$ satisfies positive assortative matching (PAM) if for almost any two equilibrium coalitions $P = (a, b)$ and $P' = (c, d)$ the following is true:*

$$\max(a, b) > \max(c, d) \Longrightarrow \min(a, b) \geq \min(c, d)$$

Note that segregation is a kind of positive assortative matching.

A third type of matching is "negative assortative," defined analogously:

Definition 5 *An equilibrium $(\mathcal{P}, u)$ satisfies negative assortative matching (NAM) if for almost any two equilibrium coalitions $P = (a, b)$ and $P' = (c, d)$ the following is true:*

$$\max(a, b) > \max(c, d) \Longrightarrow \min(a, b) \leq \min(c, d)$$

These two concepts are illustrated in Figure **??**. One can define *essential positive assortative matching* (EPAM), a *positively matched* economy, *essential negatively assortative matching* (ENAM), and a *negatively matched* economy in ways analogous to those done for segregation.

There is an equivalent characterization of positive assortative matching which is useful in some applications. Suppose that we describe any matched pair of types by listing the larger type first; hence when we write $\langle a, b \rangle$ we mean that $a \geq b$ and that a type a and a type b are matched. We can write $m(a)$ to indicate the type(s) with which a is matched. There is positive assortative matching in equilibrium if and only if the graph $\langle a, m(a) \rangle$ is upward sloping. Note that this graph lies weakly below the 45° line and that segregation occurs when the graph coincides with that line. Negative assortative matching corresponds to a downward-sloping graph.

While the terminology we use is standard, there seems not to be a consensus in the literature on what is meant, for instance, by "positive assortative

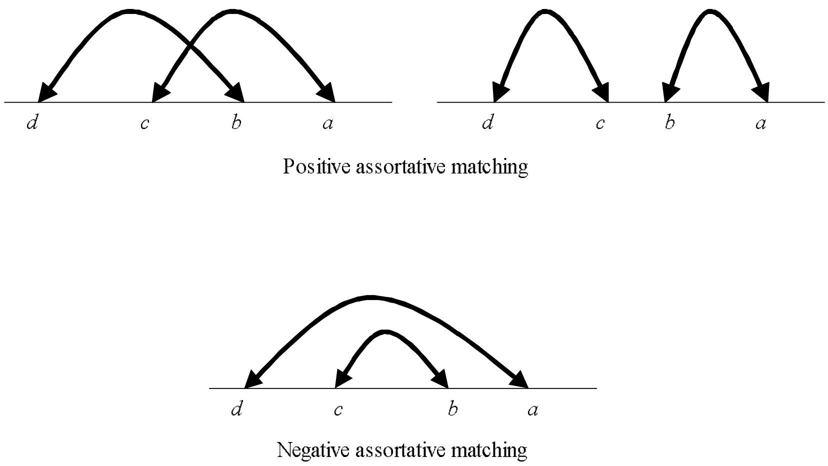

Figure 1:

matching." For example, Shimer and Smith [26] define it by requiring that the graph $(a, m(a))$ (note that the pairs are not necessarily ordered by size) form a lattice. This is a useful definition for the problem they are studying, namely matching under search frictions, but it is too weak to be of interest in the frictionless case. Moreover, the class of models they consider leads to *segregation* in the absence of search frictions, not PAM.

2.4 Distribution-Free Conditions for Monotone Matching

The Becker result suggests that it is possible to find conditions on the characteristic function such that we need know nothing about the type distribution in order to conclude that the economy satisfies PAM. Sometimes PAM is all one wants to know; in other instances (we shall study some below), knowing that PAM will be satisfied greatly facilitates computation of the equilibrium. Thus, our first goal is to provide characterization results for segregated and positively and negatively matched economies which can be checked without considering the particular type distribution. They are expressed in terms of conditions which depend only on the characteristic function and are therefore relatively easy to verify. Later, we shall examine some consequences of the failure of our necessary and sufficient conditions

2.4.1 Segregation

Our first condition is based on the observation that if we are always to have segregation, the segregation payoff vector must not lie "inside" the utility possibility set of any heterogeneous coalition.

Definition 6 (Condition S) *Let*

$$X = \{(t,t') \in T^2 : \exists v \in V(t,t'),\ v > (\underline{u}(t), \underline{u}(t'))\}.$$

Condition S is satisfied if X is empty.

X is the set of types for which there are gains from "trade," i.e. heterogeneous matching, meaning that it is possible for individuals of those types to match and Pareto improve relative to the segregation payoffs. An example of a model in which Condition S is satisfied is the one in [14], since $h(a,b) = ab$, $\underline{u}(a) = \frac{a^2}{2}$ and for any $a \neq b$, $h(a,b) < \frac{a^2}{2} + \frac{b^2}{2}$. Therefore $X = \emptyset$. Of course, that economy is segregated. In fact, we obtain the following general result.

Proposition 2 *(i) An economy is segregated if Condition S is satisfied. (ii) If Condition S is not satisfied, there is a type assignment τ such that the economy is not segregated.*

P roof. (i) Suppose that Condition S holds and that there is an equilibrium which violates ESEG. This means that a positive measure of agents are receiving more than their segregation payoffs. For this to be true, there must be heterogeneous matches (t,t'). In such matches, at least one of the agents is getting more than its segregation payoff; for stability, the other type must be getting at least its segregation payoff. But then there must exist $v \in V(t,t')$ such that $v > (\underline{u}(t), \underline{u}(t'))$, which contradicts Condition S.

(ii) Suppose that Condition S does not hold: there exists a pair of types (t,t') and a $v \in V(t,t')$ such that $v > (\underline{u}(t), \underline{u}(t'))$; clearly this is only possible if $t \neq t'$. Take the type assignment which puts an atom of size $1/2$ at t and an atom of size $1/2$ at t'. There is an equilibrium in which almost every coalition is composed of types (t,t') and the payoffs are given by v (or by $\hat{v} \in V(t,t')$ in case v is Pareto dominated by $\hat{v}$). These payoffs cannot be replicated by segregation, hence the economy is not segregated. ■

The result says simply heterogeneous coalitions may form only if there are "gains from trade" relative to the segregation payoffs; otherwise there can only be segregation. The same result is true for general matching problems as long as effective coalitions are finite: one merely has to modify Condition S to say that there is no finite set of heterogeneous types which can strictly Pareto improve relative to the corresponding segregation payoff vector. Moreover, neither the definition of segregation nor the result depend on the dimension of the type space.[12]

Note that if we want all equilibria in an economy to satisfy SEG, we need the segregation payoff vector of any heterogeneous coalition to be outside its feasible set, i.e., that $(\underline{u}(t), \underline{u}(t')) \in V(t,t')$ implies $t = t'$. Hence, Condition S can be restated to say that the set $\{(t,t') : S(t,t') = \{(0,0)\}\}$ has full measure. For the transferable utility case, Condition S can be written $\sigma(\cdot,\cdot) \equiv 0$.

There are, of course, conditions on the primitives of the model which lead to Condition S. For instance, suppose that we are in the standard perfect-markets setting without choice variables. If types are "truly" one-dimensional (so that agents aren't described by a "gender" in addition to their ability, as in Becker), it is natural to suppose that the production function is symmetric in type.[13] The following sufficiency result is known; we state it here for completeness.

Proposition 3 *Suppose that utility is transferable, the production function assumes the form $h(a,b)$, and it is symmetric in types: $h(a,b) = h(b,a)$. If h is also supermodular, the economy is segregated.*

P roof. If h is supermodular, it satisfies the inequality $h(x \vee y) + h(x \wedge y) \geq h(x) + h(y)$;[14] putting $x = (a,b)$ and $y = (b,a)$ and using symmetry then

[12] Condition S is also a sufficient condition for the existence of an equilibrium for *any* distribution of types, even without continuity assumptions on $V(\cdot,\cdot)$: any allocation in which each coalition consists of two individuals of the same type is clearly measure consistent and cannot be blocked.

[13] In fact, many "asymmetric" production functions can be made symmetric by specifying the type space appropriately. For instance, in the hospital-intern matching problem [21], we might have joint output equal to $h(a,b) = a^2 b$, where a is the hospital's productivity and b is the intern's ability. If one thinks of a hospital of type $\eta = (a,0)$ and an intern of type $\iota = (0,b)$, where the first component is hospital productivity and the second component is intern ability, then the type space is now $([\underline{a},\overline{a}] \times \{0\}) \cup (\{0\} \times [\underline{b},\overline{b}])$. Output can now be written as a symmetric function of η and ι, namely $\hat{h}(\eta,\iota) = \max\{h(\eta_1,\iota_2), h(\iota_1,\eta_2)\}$.

[14] For x, $y \in \mathbb{R}^2$ we denote by $x \wedge y$ the componentwise minimum of x and y : $x \wedge y = (\min\{x_1,y_1\}, \min\{x_2,y_2\})$; similarly, $x \vee y = (\max\{x_1,y_1\}, \max\{x_2,y_2\})$.

implies that $h(a,b) - \frac{1}{2}[h(a,a) + h(b,b)] \leq 0$; hence $\sigma(a,b) \equiv 0$ and the economy is segregated. ■

This result suggests that innocuous-looking restrictions on production functions may turn out to be undesirably strong. In this case, if one is to have (nontrivial) heterogeneous matching, one must rule out supermodular production functions, or else introduce imperfections.[15]

2.4.2 Positive Assortative Matching

The logic of our necessary and sufficient condition for positive assortative matching is similar to that for segregation: if we do not have PAM everywhere, there must be a negative pair of matches somewhere. For now, assume that utility is transferable, and consider four types $a > b \geq c > d$ and a negative match of the form (a,d) and (b,c) (there are other combinations to consider, but we simplify the argument by ignoring them for now; details are in the Appendix), which generates a total surplus of $\sigma(a,d) + \sigma(b,c)$. To ensure EPAM, this negative match must be generating the same payoffs that would be generated under PAM, or the negative match could be blocked by a rearrangement of the types. Since the surplus is transferable, it is necessary that the total surplus generated under the positive matches of these types be at least equal to that of the negative match. In other words, either $\sigma(a,b) + \sigma(c,d) \geq \sigma(a,d) + \sigma(b,c)$ or $\sigma(a,c) + \sigma(b,d) \geq \sigma(a,d) + \sigma(b,c)$. A rearrangement of these inequalities suggests the following

Definition 7 *The symmetric function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies weak increasing differences (WID) on a set $T \subset \mathbb{R}$ if for any four elements a, b, c, d of T, where $a > b \geq c > d$,*

$$F(b,c) - F(b,d) \leq F(a,c) - F(a,d)$$

or

$$F(b,c) - F(c,d) \leq F(a,b) - F(a,d).$$

[15] This conclusion does not change greatly if we introduce choice variables into the production function. For instance, if $h(q;a,b)$ is symmetric and supermodular in types and C^2, q is a real-valued choice, and h_{12} is single-signed (by symmetry h_{13} will be as well), then $H(a,b) = \max_{q \in Q} h(q;a,b)$ will be symmetric and supermodular and we get segregation.

Definition 8 (Condition PT) *The surplus σ satisfies Condition PT if it satisfies WID on T whenever $\sigma(a,d) > 0$.*

Condition PT is the necessary and sufficient condition we seek:

Proposition 4 *When the surplus is transferable: (i) An economy is positively matched if Condition PT is satisfied. (ii) If Condition PT is not satisfied, there is a type assignment τ such that the economy is not positively matched.*

P roof. Appendix.

The weak increasing difference condition resembles the familiar increasing difference (ID) condition discussed for instance in [20]: F satisfies ID if for all $a > b$ and $c > d$, $F(a,c) - F(a,d) \geq F(b,c) - F(b,d)$. WID is weaker since it requires comparison among four ordered elements (while the ID condition would not require that $a > c$).[16]

As is well known, increasing differences is equivalent for smooth functions to non-negative cross partial derivatives. Typically, however, σ will not be differentiable everywhere, even if it is derived from a smooth production function. In fact, σ will be smooth and satisfy PT only if it is identically zero, and hence satisfies Condition S. It can also be shown that any smooth function which satisfies WID also satisfies ID. But many production functions that are useful in matching applications (such as those used by Kremer-Maskin) are not smooth, and do not satisfy ID, although they do satisfy WID.

Often it is easier to check that the production function satisfies WID than that σ satisfies PT. Fortunately, we have

Proposition 5 *If h satisfies WID, then σ satisfies PT and the economy is positively matched.*

P roof. Since Condition PT is necessary and sufficient for the economy to be positively matched, it is enough to show that h implies the economy is positively matched. Suppose not, i.e. that for some distribution of types

[16] Another way to see this is to note that ID implies that

$$F(a,c) - F(a,d) \geq F(b,c) - F(b,d) \textbf{ and } F(a,b) - F(a,d) \geq F(c,b) - F(c,d)$$

whenever $a > b \geq c > d$.

there is an $a > b \geq c > d$ with matches (a, d) and (b, c) which are not payoff equivalent to a positive match. But since $h(a, d) + h(b, c) \leq h(a, c) + h(b, d)$ or $h(a, d) + h(b, c) \leq h(a, b) + h(c, d)$, at least one of the pairs of matches (a, c) and (b, d) or (a, b) and (c, d) generates at least as high total output. Thus either the negative match is payoff equivalent to a positive one or one of the pairs (a, c), (b, d), (a, b), or (c, d) can block it, a contradiction. ■

Notice that the converse is not true, since h could fail to satisfy WID at a point at which σ does not or at which σ is zero; an instance of this is given in the next example.

On the other hand, in some instances, the surplus function will be more informative than the production function about the properties of the equilibrium match. For instance, there are cases in which production functions are neither super- nor submodular and yet from the surplus computation it is easy to see that the economy must be segregated:

Example 1 *Let* $T = [4, 5]$ *and* $h(a, b) = A(\sqrt{a} + \sqrt{b}) - \epsilon \max\{a^3 b, b^3 a\}$, *where* $0 < \epsilon < \frac{A}{2000}$. *It is straightforward to verify that* h_a *and* h_b *are positive wherever they exist (which is everywhere except on the diagonal). And* $h_{ab} < 0$ *almost everywhere; hence* h *is not supermodular and doesn't satisfy WID. Nor is it submodular, since this would require* $h(a, b) + h(b, a) \geq h(a, a) + h(b, b)$; *but* $h(a, b) + h(b, a) - [h(a, a) + h(b, b)] \propto a^4 + b^4 - 2\max\{a^3 b, b^3 a\} \leq 0$ *on* T^2. *By the same token,* $\sigma(a, b) = \max\{0, h(a, b) - \frac{1}{2}[h(a, a) + h(b, b)]\} \equiv 0$ *there, and the economy is segregated.*

We have a condition analogous to PT for the case in which surplus is not transferable. Let $S^D = S \backslash S^P$ denote the set of Pareto dominated elements of S.

Definition 9 (Condition P) *Condition P is satisfied if for any four elements* $\{a, b, c, d\}$ *of* T, *where* $a > b \geq c > d$, $s \in S^P(a, d) \times S^P(b, c)$, *and* $S(a, d) \neq \{(0, 0)\}$ *one of the two conditions below is true. Either*

$$s \in S(a, b) \times S(c, d) \text{ or } s \in S(a, c) \times S(b, d) \tag{2}$$

or

$$\exists t \in \{a, d\}, \ \hat{t} \in \{b, c\} \text{ such that } (s(t), s(\hat{t})) \in S^D(t, \hat{t}) \tag{3}$$

As with Condition PT, Condition P says that for any *negative* match, either it is possible to reassign the types in a positive way that keeps all four types (at least) indifferent (2), or the match is not stable (3).

Proposition 6 *(i) An economy is positively matched if Condition P is satisfied. (ii) If Condition P is not satisfied, there is a type assignment τ such that the economy is not positively matched.*

P roof. (i) Suppose that Condition P holds. If an economy is not positively matched, there exist a, b, c, d where $a > b \geq c > d$ and payoffs s, where $s \in S^P(a,d) \times S^P(b,c)$,[17] such that the matches (a,d) and (b,c) are part of the equilibrium *and* it is not possible to obtain a positively matched reshuffling of these types which keeps the payoffs the same (2 is violated). Since there are no beneficial deviations from the equilibrium payoffs s, (3) is also violated, contradicting Condition P.

(ii) Suppose that Condition P is not satisfied. Since (2) is violated, $s \notin S(a,b) \times S(c,d)$ and $s \notin S(a,c) \times S(b,d)$. Since (3) is violated, for each $\hat{t} \in \{a,d\}$ and each $t \in \{b,c\}$, $\left(s\left(\hat{t}\right), s(t)\right) \notin S^D\left(\hat{t}, t\right)$. Consequently, it is not possible to replicate the payoffs s by a positive match between a, b, c and d. Consider τ such that there are four atoms at a, b, c and d of equal mass. The matches $(a,d), (b,c)$ together with the payoff s constitute an equilibrium and the economy is not positively matched. ■

Propositions 2, 4, and 6 show that if the characteristic function satisfies certain properties, the equilibrium matching pattern will (essentially) always assume a positive assortative form. But in economies in which this condition is violated, the outcome will be sensitive to type assignment map: *the equilibrium matching pattern will depend on the distribution of types.* We will illustrate this point in the Applications section below.

2.4.3 Negative Assortative Matching

Since the logic is similar, we state here without proof conditions for negative matching which are analogous to Conditions P and PT.

Definition 10 *The symmetric function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies weak decreasing differences (WDD) if for any four elements a, b, c, d of T, where $a > b \geq c > d$,*

$$F(b,c) - F(b,d) \geq F(a,c) - F(a,d)$$

[17] If $s \notin S^P(a,d) \times S^P(b,c)$, either (a,d) or (b,c) has an incentive to deviate from the proposed payoffs.

and

$$F(b,c) - F(c,d) \geq F(a,b) - F(a,d).$$

Note that this condition is still weaker than decreasing differences (and indeed is implied by it), since it applies only to quadruples of types with the specified order.

Definition 11 (Condition NT) *The surplus σ satisfies Condition NT if it satisfies WDD on T whenever $\sigma(a,d) > 0$.*

Proposition 7 *When the surplus is transferable: (i) An economy is negatively matched if Condition NT is satisfied. (ii) If Condition NT is not satisfied, there is a type assignment τ such that the economy is not negatively matched.*

Proposition 8 *If utility is transferable, and the production function is symmetric and strictly submodular on $\mathbb{R}^2$, the economy is negatively matched.*

There is no direct analog to Proposition 5 because strict WDD by itself is not enough to rule out segregation, which after all is a kind of positive assortative matching. However, if it is known on other grounds that matching will be heterogeneous, it is sufficient for ENAM only that the production function satisfy WDD.

Definition 12 *Condition N is satisfied if for any four elements $a > b \geq c > d$, the following conditions both hold:*

(i) if $s \in S^P(a,b) \times S^P(c,d)$, then either $[s \in S^D(a,d) \times S^D b,c)]$ or $[\exists t \in \{a,b\}, \exists \hat{t} \in \{c,d\}, \left(s(t), s(\hat{t})\right) \in S^D\left(t,\hat{t}\right)]$

(ii) if $s \in S^P(a,c) \times S^P(b,d)$, then either $[s \in S^D(a,d) \times S^D(b,c)]$ or $[\exists t \in \{a,c\}, \exists \hat{t} \in \{b,d\}, \left(s(t), s(\hat{t})\right) \in S^D\left(t,\hat{t}\right)]$.

Proposition 9 *(i) An economy is negatively matched if Condition N is satisfied. (ii) If Condition N is not satisfied, there is a type assignment τ such that the economy is not negatively matched.*

2.4.4 Some Sufficient Conditions for Monotone Matching with Nontransferable Utility

When utility is nontransferable but the Pareto frontier of the utility possibility set is strictly decreasing, there are alternative sufficient conditions for monotone matching that can be fairly easy to verify. We present these here and then show how they can be used by applying one of them to a model of risk sharing based loosely on Kihlstrom-Laffont [13] and Sadoulet [23].

Suppose then that for all pairs of types $(t, \hat{t})$, the Pareto frontier of $S\left(t, \hat{t}\right)$ is strictly decreasing (which it is trivially in case $S\left(t, \hat{t}\right) = \{(0,0)\}$). Since $S\left(t, \hat{t}\right)$ is bounded above, there exist two values $\phi_{t\hat{t}}(t)$ and $\phi_{t\hat{t}}\left(\hat{t}\right)$ corresponding to the least upper bound of the payoff that type t and $\hat{t}$ respectively can attain in $S\left(t, \hat{t}\right)$ (since the frontier is strictly decreasing, these occur only when the other type receives 0, i.e. $(0, \phi_{t\hat{t}}(t))$ and $\left(\phi_{t\hat{t}}\left(\hat{t}\right), 0\right)$ are elements of $S\left(t, \hat{t}\right)$.) This environment is of general interest since it corresponds, for instance, to the formation of households with risk-averse partners or to general bargaining problems.

We shall require the use of a function defined on all of $\mathbb{R}$ which is an extension of the frontier of $S\left(t, \hat{t}\right)$. Acceptable extensions of the frontier can be represented by a map:

$$\beta_{t\hat{t}}(\hat{s}) = \begin{cases} \max\left\{s : (s, \hat{s}) \in S\left(t, \hat{t}\right) \text{ with } s = s(t), \hat{s} = s\left(\hat{t}\right)\right\} & \text{if } \hat{s} \in \left[0, \phi_{t\hat{t}}\left(\hat{t}\right)\right] \\ \psi_{t\hat{t}}(\hat{s}) & \text{otherwise,} \end{cases} \tag{4}$$

where $\psi_{t\hat{t}}(\hat{s})$ is any strictly decreasing function from $\mathbb{R}$ onto itself which satisfies $\psi_{t\hat{t}}\left(\phi_{t\hat{t}}\left(\hat{t}\right)\right) = 0$, $\psi_{t\hat{t}}(0) = \phi_{t\hat{t}}(t)$ and $\psi_{t\hat{t}} \circ \psi_{\hat{t}t}(s) = s$. By construction, $\beta_{t\hat{t}}(\hat{s})$ is strictly decreasing in $\hat{s} \in \mathbb{R}$, and $\beta_{t\hat{t}}$ and $\beta_{\hat{t}t}$ are inverses: $\beta_{t\hat{t}} \circ \beta_{\hat{t}t}(s) = s$ for any s.

We now provide a sufficient condition for positive matching:

Definition 13 (Condition P*) *There exists an extension β such that for any $a > b \geq c > d$ with $\phi_{ad}(d) > 0$ and $s \in [0, \phi_{ad}(d)]$, one of the following holds:*

$$\text{either } s \geq \beta_{da} \circ \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s) \tag{5}$$

$$\text{or } s \geq \beta_{da} \circ \beta_{ac} \circ \beta_{cb} \circ \beta_{bd}(s); \tag{6}$$

and for negative matching:

Definition 14 (Condition N*) *There exists an extension β such that for any $a > b \geq c > d$ where $\phi_{ad}(d) > 0$, we have*

$$[s \leq \phi_{cd}(d)] \Longrightarrow [s \leq \beta_{da} \circ \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s)] \tag{7}$$
$$\textit{and } [s \leq \phi_{bd}(d)] \Longrightarrow [s \leq \beta_{da} \circ \beta_{ac} \circ \beta_{cb} \circ \beta_{bd}(s)]. \tag{8}$$

Proposition 10 *Suppose that the Pareto frontier of any surplus set is strictly decreasing. (i) If Condition P* holds, the economy is positively matched; (ii) If Condition N* holds, the economy is negatively matched.*

P roof. Appendix. ■

The basic logic of the proof is very simple. If we are to have a positively matched economy, we cannot have a negative match that is not payoff equivalent to a positive one. Suppose instead that there is such a negative match ((a, d) and (b, c)) with payoffs $s(t)$ ($t = a, b, c, d$); if the negative match is feasible, we have $s(a) = \beta_{ad}(s(d))$ and $s(b) = \beta_{bc}(s_c)$. Stability also requires that b (strictly) doesn't want to switch to a, given what a is currently getting: $s(b) > \beta_{ba} \circ \beta_{ad}(s(d))$. And c doesn't want to switch to d : $s(c) > \beta_{cd}(s(d))$; since $s(b) = \beta_{bc}(s(c))$, we have $s(b) < \beta_{bc} \circ \beta_{cd}(s(d))$. Thus, $\beta_{ba} \circ \beta_{da}(s(d)) < \beta_{bc} \circ \beta_{cd}(s(d))$, and, using the inverse operators, $s(d) < \beta_{da} \circ \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s(d))$. Similarly, the requirements that b doesn't want to switch to d, and c doesn't want to switch to a, imply $s(d) < \beta_{da} \circ \beta_{ac} \circ \beta_{cb} \circ \beta_{bd}(s(d))$. The negation of these necessary conditions for the existence of a stable negative match then yields Condition P*. A complete proof is in the Appendix.

Conditions P* (5) and N* (8) are illustrated graphically in Figure 2 in which the frontiers are plotted in four-axis diagrams. For instance, positive matching results if any path like that shown starting with s ends up at a point less than s. We expect that even when closed-form expressions for the frontiers are not available, Conditions P* and N* can feasibly be checked using numerical methods.

It is important to bear in mind that Conditions P* and N* are not necessary for monotone matching in the general, nontransferable utility case. In the case of transferable utility, however, they are necessary. To see this, notice that in this case, $\phi_{t\hat{t}}(t) = \phi_{t\hat{t}}(\hat{t}) = \sigma(t, \hat{t})$. Hence, if we choose $\psi_{t\hat{t}}(\hat{s}) = \phi_{t\hat{t}}(\hat{t}) - \hat{s}$ in (4), we have $\beta_{t\hat{t}}(\hat{s}) = \sigma(t, \hat{t}) - \hat{s}$. Observe that

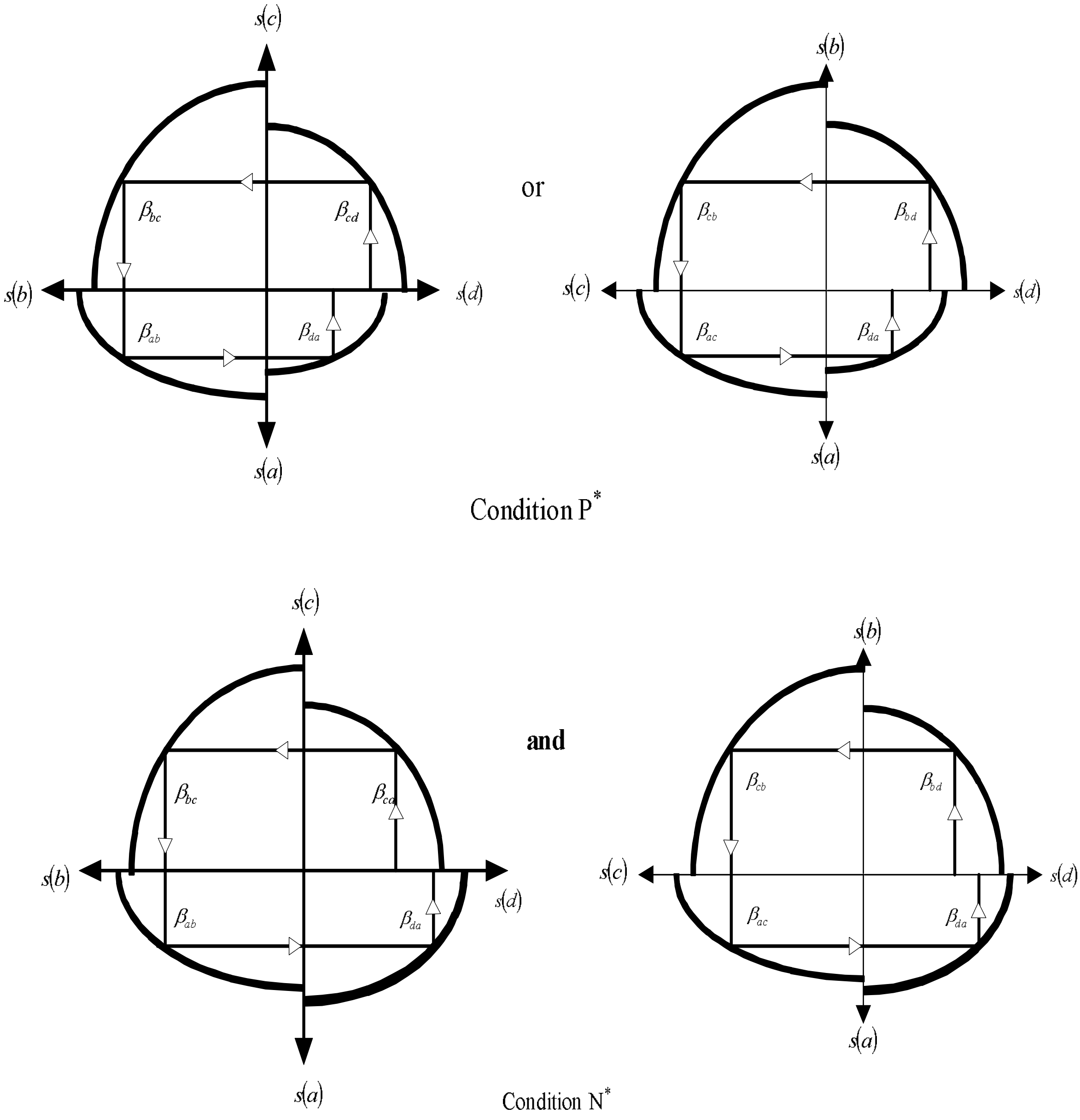

Figure 2: The Not Transferable Case

$\beta_{da} \circ \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s) = \sigma(a,d) - (\sigma(a,b) - (\sigma(b,c) - (\sigma(c,d) - s)))$. Hence, the first line in Condition P* is equivalent $\sigma(a,d) - \sigma(a,b) + \sigma(b,c) - \sigma(c,d) \leq 0$, which is the first line in Condition PT. Similarly, the second line of Condition P* is equivalent to the second line in Condition P. It follows that in this case, Condition P* is equivalent to Condition PT and is therefore necessary and sufficient for EPAM. Similarly, Condition N* is equivalent to Condition NT.

Note that Conditions P* and N* might hold only for certain extensions β but not for others. We provide below sufficient conditions that do not require the use of an extended frontier. These conditions require the additional (weak) condition that the feasible sets are monotonic in types.[18] The sufficient conditions that we obtain are stronger than the previous conditions but as the following example will show, are often enough to establish monotone matching.

Corollary 11 *Suppose that for any t and $\hat{t}$, there exists a real $\Sigma_{t\hat{t}} \in \mathbb{R}$, increasing in $(t,\hat{t})$, such that the Pareto frontier of $V(t,\hat{t})$ can be described by a bijective function $\gamma_{t\hat{t}} : (-\infty, \Sigma_{t\hat{t}}) \to (-\infty, \Sigma_{t\hat{t}})$.*

(i) If for any $t \neq \hat{t}$, $S(t,\hat{t}) \neq \{(0,0)\}$, and for any $a > b \geq c > d$ we have

$$v < \Sigma_{cd} \Rightarrow \gamma_{ba} \circ \gamma_{ad}(v) \geq \gamma_{bc} \circ \gamma_{cd}(v) \quad (9)$$

$$\text{and } v < \Sigma_{bd} \Rightarrow \gamma_{ca} \circ \gamma_{ad}(v) \geq \gamma_{cb} \circ \gamma_{bd}(v), \quad (10)$$

then the economy is negatively matched.

(ii) If for any $a > b \geq c > d$ we have

$$v < \Sigma_{ad} \Rightarrow \begin{cases} \gamma_{ba} \circ \gamma_{ad}(v) \leq \gamma_{bc} \circ \gamma_{cd}(v) \\ \text{or} \\ \gamma_{ca} \circ \gamma_{ad}(v) \leq \gamma_{cb} \circ \gamma_{bd}(v) \end{cases} \quad (11)$$

then the economy is positively matched.

P roof. Appendix. ■

We now apply this corollary to a simple example of risk sharing.

[18] Note that monotonicity of the feasible sets does not imply monotonicity of the surplus sets. For this reason, while it would be possible to obtain a similar looking result for surplus sets, the required condition of monotonicity of the surplus sets would exclude a large set of environments.

Example 2 *Production is risky with a finite number of possible outcomes* w_i *and associated probabilities* π_i. *Agents are expected utility maximizers who are identical except for their attitudes toward risk. The utility of income is* $u_a(x)$ $(u'_a > 0 > u''_a)$, *where type* $a \in [\underline{a}, \bar{a}]$ *is an index of absolute risk aversion:* $\rho_a(x) \equiv -\frac{u''_a(x)}{u'_a(x)}$ *is strictly decreasing in a for all x (below we shall use* $\log(x+a)$ *as the family of utility functions). The only risk sharing possibilities in this economy lie within (two-person) production units because observing the outcome of production is prohibitively costly except to the people directly involved. When parties match, they sign a contract which specifies how the output will be shared in each state (level of output).*

Optimal risk sharing between a type a and a type b is characterized by the solution to

$$\max_{\{x_i\}} \Sigma_i \pi_i u_a(w_i - x_i) \ s.t. \ \Sigma_i \pi_i u_b(x_i) \geq v_b.$$

The first-order condition (Borch's rule) is $u'_a(w_i - x_i) = \lambda u'_b(x_i)$, *where* λ *is the multiplier on the constraint. If* $a = b$ *and utility is equal for both partners, then* $\lambda = 1$, *from which* $x_i = w_i/2$. *Thus the segregation payoff is achieved by equal sharing in each state.*

Observe that any heterogenous pair can guarantee each member the segregation payoff, as equal sharing is always feasible. Since this will not constitute the optimal contract,[19] *the segregation payoff vector can be (strictly) Pareto dominated by every heterogeneous coalition:* $S(t, \hat{t}) \neq \{(0,0)\}$ *if* $t \neq \hat{t}$ *(so Condition S is violated), and in fact* ***matching will always be heterogeneous if there is more than one type in the economy.***

The question is what pattern it will assume. Intuitively, we might expect that the least risk averse will want to insure the most risk averse in order to extract a large risk premium; the moderately risk averse will be less willing to undertake this role because it would entail bearing too much risk, so they will match together instead. We now verify this intuition for the case of logarithmic utility.

Consider then the special case in which $u_a(x) = \ln(x + a)$ *(a can interpreted directly as a risk parameter or as an initial wealth level; either way, higher a means lower risk aversion, and the form of the optimal contract is*

[19] To see this, note that $\rho_a(x) < \rho_b(x)$ everywhere implies $\frac{u'_a(x)}{u'_b(x)}$ is strictly increasing in x. Since $\frac{u'_a(w_i - x_i)}{u'_b(x_i)}$ is constant at the optimum, $w_i - x_i = x_i$ for at most one value of w_i.

of course independent of the interpretation). Assume that $w_i \geq -\underline{a}$ for each i.

The optimal contract in this case is

$$x_i = b + (w_i + a - b)e^{v_b - \Sigma_i \pi_i \ln(w_i+a+b)},$$

from which

$$\gamma_{ab}(v_b) = \ln(1 - e^{v_b - \Sigma_i \pi_i \ln(w_i+a+b)}) + \Sigma_i \pi_i \ln(w_i + a + b).$$

Let Σ_{ab} denote $\Sigma_i \pi_i \ln(w_i+a+b)$. It is clear that Σ_{ab} is increasing in (a, b) and that the Pareto frontier γ_{ab} satisfies $\gamma_{ab}(v_b) \to -\infty$ as $v_b \to \Sigma_{ab}$. Applying the γ operator as in Corollary for types $a > b \geq c > d$, we obtain

$$\begin{aligned}
\gamma_{ca} \circ \gamma_{ad}(v_d) &= \ln(1 - e^{\Sigma_{ad} - \Sigma_{ac}} + e^{v_d - \Sigma_{ac}}) + \Sigma_{ac} \\
\gamma_{cb} \circ \gamma_{bd}(v_d) &= \ln(1 - e^{\Sigma_{bd} - \Sigma_{bc}} + e^{v_d - \Sigma_{bc}}) + \Sigma_{bc} \\
\gamma_{ba} \circ \gamma_{ad}(v_d) &= \ln(1 - e^{\Sigma_{ad} - \Sigma_{ab}} + e^{v_d - \Sigma_{ab}}) + \Sigma_{ab} \\
\gamma_{bc} \circ \gamma_{cd}(v_d) &= \ln(1 - e^{\Sigma_{cd} - \Sigma_{bc}} + e^{v_d - \Sigma_{bc}}) + \Sigma_{bc}.
\end{aligned}$$

Inequalities (9) and (10) then become

$$\begin{aligned}
e^{\Sigma_{ad}} + e^{\Sigma_{bc}} &> e^{\Sigma_{ab}} + e^{\Sigma_{cd}} \\
\text{and } e^{\Sigma_{ad}} + e^{\Sigma_{bc}} &> e^{\Sigma_{ac}} + e^{\Sigma_{bd}},
\end{aligned}$$

which are satisfied if and only if the function

$$F(a, b) \equiv e^{\Sigma_{ab}} = \prod_i (w_i + a + b)^{\pi_i}$$

satisfies WDD. But it is easily verified that in fact $\partial^2 F/\partial a \partial b < 0$;[20] thus WDD is satisfied (strictly), and we conclude by Corollary 11 (i) that ***in the risk-sharing economy with logarithmic utility, agents will always match negatively in wealth.***

[20] To see this, note that $\partial F/\partial a = \partial F/\partial b = \sum_i [\pi_i (w_i + a + b)^{\pi_i - 1} \prod_{j \neq i} (w_j + a + b)^{\pi_j}] = \sum_i [\frac{\pi_i}{w_i+a+b} \prod_j (w_j + a + b)^{\pi_j}] = F(a, b) \sum_i \frac{\pi_i}{w_i+a+b}$. Thus, $\partial^2 F/\partial a \partial b = \frac{\partial F}{\partial b} \sum_i \frac{\pi_i}{w_i+a+b} - F(a, b) \sum_i \frac{\pi_i}{(w_i+a+b)^2} = F(a, b)(\sum_i \frac{\pi_i}{w_i+a+b})^2 - F(a, b) \sum_i \frac{\pi_i}{(w_i+a+b)^2}$, which is negative by $F > 0$ and Jensen's inequality.

2.4.5 Comment

Distribution-free conditions for monotone matching are relatively easy to verify. In general, distribution will affect the match to some degree: $m(a)$ is independent of distribution if and only if Condition S is satisfied. Invariance of $m(a)$ is of course a very strong requirement; if one wants only that monotonicity be preserved as the distribution changes, then Condition P (or N) is necessary and sufficient. If one is using more refined measures of matching patterns (e.g. the degree of segregation), then since the matching map is unlikely to be invariant to the distribution, these measures are also unlikely to be invariant to the distribution.

In checking whether Conditions S, P or N apply for a particular characteristic function, it will often become apparent that changes in the technology and imperfection parameters θ and ϕ will affect $\underline{u}(\cdot)$ and $V(\cdot)$ very differently for different types. For instance, if ϕ measures a degree of capital market imperfection, increases in ϕ will typically lower the segregation payoffs of poor agents but may have no effect on those of wealthy ones; at the same time, the production possibilities for mixed coalitions, if one partner is wealthy enough, may also be unaffected. Thus for low ϕ we could have segregation, while for high ϕ we would have heterogeneous, even negative, matching: Conditions S will be satisfied at some parameter values, while for others it will not. The consequence is that the qualitative properties of the equilibrium matching pattern, even for a fixed type distribution, will vary across economies.

When the conditions for monotone matching are violated, the distribution will also play an important role in determining the qualitative nature of the outcome. In these more difficult cases, though our monotonicity conditions may be violated globally, they are often satisfied locally, and this information can be useful in computing the match. We shall demonstrate this point in the Applications section.

2.5 When Is the Match Invariant to Changes in Technology?

The previous subsection has been concerned with finding conditions on the characteristic function such that the "qualitative" property of the match is not dependent on the particular type distribution. That is, given an arbitrary distribution of types, any change to the characteristic function which preserves Condition P (N) will preserve positive (negative) assortative match-

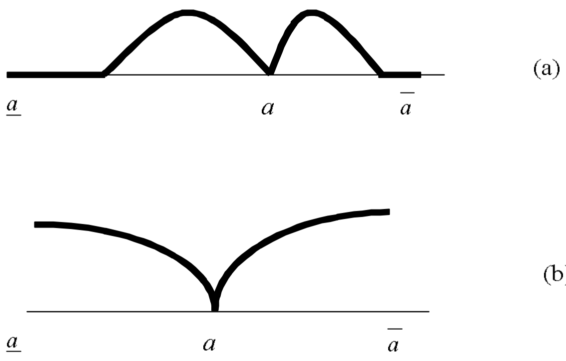

Figure 3: The Single Trough Condition

ing. But we may be interested in a stronger question: is there a class of characteristic functions all of which give rise to the *same* match? If this is the case, then changes in technology (or in the severity of imperfections) which preserve this class will leave the matching pattern invariant.

In this section we show that the answer to this question is yes, and with it we can clarify the differences in the comparative statics of the two-sided matching models such as that of Becker and of one-sided models such Kremer-Maskin's that we alluded to in the Introduction. We do not provide an exhaustive characterization of invariant matches, but will focus instead on a particular one that has appeared repeatedly in the literature, which we call *median matching.*

We consider the case of transferable utility. Let the support of the type distribution be contained in $[\underline{a}, \bar{a}]$. The surplus can be written in the form $\max\{0, h(a,b) - \frac{1}{2}[h(a,a) + h(b,b)]\}$. Fix one of the types (say a) and plot this as function of the other type. Typically, the picture will resemble Figure 3(a) (note that the surplus is always equal to 0 at a). Suppose, however, that as in Figure 3(b) the surplus achieves a unique minimum at a and is increasing on $(a, \bar{a}]$ and decreasing on $[\underline{a}, a)$. We say in this case that the *surplus is single-troughed for a.*

Definition 15 *The economy satisfies the single-trough surplus condition (STSC) if the surplus is single-troughed for all a in the support of the type distribution.*

The result, which is surprisingly easy to prove, is that if we have an atomless type distribution, the STSC holds, and the production function satisfies

WID, then the equilibrium matching pattern has a simple characterization in which the highest type matches with the median type, and all other match in such a way as to keep the "probability distance" between a type and his partner constant at 1/2. Because of full transferability and the atomless distribution, the matching correspondence $m(a)$ is in fact a function.

Proposition 12 *Suppose that the STSC holds, that h satisfies weak increasing differences and that the assignment map τ generates a continuous distribution of types $T(a)$. Then there exists an essentially unique equilibrium matching pattern in which for $a \in [a_m, \bar{a}]$, $T(a) - T(m(a)) = \frac{1}{2}$, where a_m is the median type.*

We call this matching pattern *median matching.*

P roof. Denote by $s(a) \equiv u(a) - \underline{u}(a)$ the amount of surplus that a obtains in equilibrium. Suppose that $m(\bar{a}) < a_m$. Since h satisfies WID, we have PAM, which implies that $m(a) < m(\bar{a})$ for $m(\bar{a}) < a < \bar{a}$. But this violates measure consistency, since more than one-half the population is matching with less than one-half the population. Thus $m(\bar{a}) \geq a_m$. A similar argument establishes that $m(\underline{a}) \leq a_m$. Suppose that these inequalities are strict. In equilibrium, we must have $\sigma(\bar{a}, m(\bar{a})) - s(m(\bar{a})) \geq \sigma(\bar{a}, m(\underline{a})) - s(m(\underline{a}))$, else $\bar{a}$ would try to match with $m(\underline{a})$. Thus $s(m(\underline{a})) - s(m(\bar{a})) \geq \sigma(\bar{a}, m(\underline{a})) - \sigma(\bar{a}, m(\bar{a})) > 0$, the last inequality following from the STSC. The same argument for $\underline{a}$ establishes that $s(m(\underline{a})) - s(m(\bar{a})) < 0$, a contradiction. Therefore $m(\underline{a}) = m(\bar{a}) = a_m$. To complete the argument for the remaining types, note that if $T(a) - T(m(a)) > \frac{1}{2}$ for $a > a_m$, then the measure of agents between a_m and a, who by PAM are matching with agents between $\underline{a}$ and $m(a)$, exceeds that of the latter set, which violates measure consistency. A similar violation of measure consistency occurs if $T(a) - T(m(a)) < \frac{1}{2}$. ■

An example of an economy which conforms to these hypotheses is the one in Kremer-Maskin, provided the support of the type distribution is tight enough. We shall return to this below.

Three observations are in order. First note that heavy use is made of the fact that we have PAM; this illustrates how our distribution-free conditions can simplify the computation of equilibrium.

Second, if the technology or production changes in such a way as to preserve WID and the STSC, the match will be unchanged because there is only one way to have median matching for a given distribution of types.

Third, any change in the type distribution that preserves WID and the STSC (for instance, small changes in the support of the distribution), preserves median matching, although this in general means that the types which match will change after the distribution changes. In particular, we have

Corollary 13 *Suppose that h satisfies WID and that the STSC holds on $[\underline{a}, \bar{a}]$. Then there is median matching for any continuous distribution with support in $[\underline{a}, \bar{a}]$.*

Of course, if the distribution changes by "stretching" the support enough, then STSC will not generally hold on all of the new distribution's support, and we will lose median matching (although because of WID we retain positive assortative matching). This is the main comparative static result of [15].

The STSC case is not the only one delivering median matching. Write $a \perp b$ when a and b are on opposite sides of the median (i.e. $a \leq a_m \leq b$ or $a \geq a_m \geq b$). Now consider the following class of transferable utility models, which we call class M:

1. Whenever $a > b \perp c > d$, σ satisfies one of the weak-increasing-difference inequalities $\sigma(b,c) - \sigma(b,d) \leq \sigma(a,c) - \sigma(a,d)$ or $\sigma(b,c) - \sigma(c,d) \leq \sigma(a,b) - \sigma(a,d)$.
2. For all a, $\sigma(a,b)$ is strictly positive when $a \perp b$
3. $\sigma(a,a') = 0$ whenever it is not the case that $a \perp a'$.

Proposition 14 *If the type distribution is atomless and σ belongs to class M, there is median matching.*

P roof. Observe that in this economy, $a \perp m(a)$ for almost every a; if not, then on each side of the median a positive measure of agents are matched with each other and getting zero; this can be blocked by having the agents find partners with whom they generate a strictly positive surplus on the other side of the median (measure consistency assures these potential partners exist). Matching is therefore positive assortative: if not, there is a negative match of the form (a,d) and (b,c), with $a > b \perp c > d$. But this will be blocked since the WID inequalities are satisfied. Now mimic the argument used in the proof of Proposition 12 to conclude that matching is median. ■

One reason for studying this class is that it can be used to understand many "two-sided" matching models, such as the marriage market model of Becker. Two-sided matching models have type spaces which are really two dimensional: there is a "gender" as well as an ability. A positive payoff is generated only when types from different sides match.

Consider Becker's model. The joint payoff to a couple consisting of a man of ability a and a women of ability b is $h(a,b)$; h is supermodular (and strictly increasing, although this is not essential). Joint payoffs are positive if and only if a man matches with a woman and can be divided in any way. The distribution of ability within each gender can be arbitrary. Suppose for simplicity that the total measure of women (supported on $[\underline{b},\bar{b}]$) equals that of men (supported on $[\underline{a},\bar{a}]$) and that the distributions are atomless. It is well-known that the outcome of this model has the men of ability $\bar{a}$ matching with the women of ability $\bar{b}$; as the men's ability decreases, so does the ability of their partners, until $\underline{a}$ matches with $\underline{b}$.[21]

The segregation payoff in this model is zero for every type, so the surplus function σ is just equal to the production function h. Now map this model into a one-dimensional model with a new type space $[\underline{x},\bar{x}]$, where type x is given by

$$x(t)=t,\ t\in[\underline{b},\bar{b}],\ x(t)=\bar{b}+t-\underline{a},\, t\in[\underline{a},\bar{a}].$$

That is, the women are the "left half" of the interval and the men become the right half. The types $\bar{b}$ and $\underline{a}$ are then identified and become the median type x_m. Define a new surplus function $\tilde{\sigma}(y,y')=\sigma(x^{-1}(y),x^{-1}(y'))$. It is clear that $\tilde{\sigma}$ is of class M and we therefore have median matching which is isomorphic to the match in the Becker model.

This construction, along with Proposition 14, shows why Becker's result is so strong. Changes to the production technology which preserve supermodularity of h will also preserve the WID condition, so there is no change to the match from such changes in the technology. The key property of the two-sided matching model is that same-side matches generate zero output. This implies (1) that the segregation payoff is identically zero, which makes the surplus coincide with the production function; and (2) that the surplus

[21] The monotonicity of $h(\cdot,\cdot)$ will play a role if the measure of men is not equal to the measure of women, in which case the lowest ability agents of the gender in excess supply will be unmatched; the way surplus is shared among the remaining types will be affected by this, but not the pattern of matching. The construction in the next paragraph is easily modified to allow for this possibility.

for types on opposite sides is positive because the production function is. Positivity of the production function for types on opposite sides of the median is of course a reasonably general property, but positivity of the surplus is not (neither is it generally equal to zero on the same side of the median), and the latter is crucial to median matching (or its two-sided counterpart).

If instead one allowed for positive payoffs to same-sex matches, there generally would not be median matching (it would no longer be automatic for types on the same side of the median not to match or for the surplus to be positive for types on opposite sides), and neither would the match be invariant to changes in technology. Indeed, the model based on Kremer-Maskin which we study in the next section is neither of class M, nor does it satisfy the STSC for all admissable technologies, and we show that technological change which preserves the WID property will generally change the match.

Somewhat less formally, if the distribution of abilities changes (keeping the measure of the two sides the same), we still retain median matching, which in particular means that for all distributions, the matching map $m(a)$ is monotonic, with $m(\underline{a}) = \underline{b}$, $m(\bar{a}) = \bar{b}$. This is admittedly a crude sort of invariance, since in general $m(a)$ *will* depend on the distribution for intermediate values of a. But the latter effect is perhaps not so conspicuous; insofar as two-sided models have tended to dominate the literature, we may have an explanation as to why the dependence of matching patterns on distribution has not been noticed until recently.

3 Applications

We now apply the above theory to the analysis of some examples. First, we consider the examples discussed by Kremer and Kremer-Maskin. The apparatus we have developed clarifies and generalizes the results they obtain and leads to some new comparative static results. The next two examples consider economies with imperfections. In both cases the production technology satisfies increasing differences so that the first-best version of these economies will display segregation. The first example considers a financial market imperfection which results in the violation of Conditions S and PT. We show that the matching configuration will be sensitive to the distribution of types and that in some instances the effects of the imperfections swamp the effects of the production technology. The second example considers production with an incentive problem. There it turns out that Condition S is still satisfied but that segregated matching may be inefficient.

3.1 A Perfect-Markets Example

In this subsection we will be interested mainly in how the matching pattern depends on a technological parameter θ. We shall make heavy use of the surplus concept developed in the previous section.

Let the type space be an interval $[\underline{a}, \overline{a}]$, $\underline{a} > 0$, and $h(a,b) = \max\left\{a^{\theta}b^{1-\theta}, b^{\theta}a^{-\theta}\right\}$ where $\theta \in (0,1)$ (if we make a change of variable $\hat{a} = \sqrt{a}$, i.e., if the type space is $\left[\sqrt{\underline{a}}, \sqrt{\overline{a}}\right]$, [14] corresponds to $\theta = \frac{1}{2}$ and [15] to $\theta = \frac{2}{3}$ for this type space). Note that the segregation payoff is $\frac{h(a,a)}{2} = \frac{a}{2}$. The idea is that the tasks are asymmetric, and that the two partners will be assigned to them in an output maximizing fashion. Since these are transferable utility models, we will use the surplus function $\sigma(a,b) = \max\left\{0, h(a,b) - \frac{1}{2}(a+b)\right\}$ to study these economies.

We note that when $\hat{\theta} = 1 - \theta$ the two economies will have the same equilibria (since the $\hat{\theta}$ production function is obtained from the θ production function by "renaming" the two tasks). Hence, it is enough to consider the case $\theta \in \left[\frac{1}{2}, 1\right)$. The (pre-) surplus function $h(a,b) - \frac{1}{2}(a+b)$ is concave in $b \leq a$ and attains its maximum at $M(a;\theta) = a\left(2(1-\theta)\right)^{\frac{1}{\theta}}$. Simple algebra shows that $M(a;\theta)$ is a decreasing function of θ. Simple algebra also shows that the presurplus function is an increasing function of θ.

A quick calculation shows that h satisfies WID for all $\theta \in \left[\frac{1}{2}, 1\right)$ so that by Proposition 5 the economy will always be positively matched. A somewhat longer calculation shows that h is supermodular if and only if $\theta = \frac{1}{2}$. Thus Proposition 3 tells us that the economy is segregated when $\theta = \frac{1}{2}$; alternatively, note that $M\left(a; \frac{1}{2}\right) = a$. As long as $\theta > \frac{1}{2}$, $M(a;\theta) < a$ and there exist $b < a$ such that $\sigma(a,b) > 0$. If we use a simple measure of segregation like the average difference between two matched types in equilibrium, this shows that when there is more technological bias (i.e., one of the tasks is more productive) the economy will be *less segregated* than when the two tasks are equally productive.

Moreover, ass $\theta \to 1$, $M(a;\theta) \to 0$; since $\underline{a} > 0$, it follows that the single-trough surplus condition is satisfied and that there is median matching when θ is large enough. Hence, when the relative productivities of the two tasks is the largest, segregation is the lowest.

As for changes in distribution, observe that if we hold θ fixed and instead change the distribution by "lengthening" the support, the economy can change from satisfying STSC to violating it. This kind of result is obtained in [15]. Seen in the light of the forgoing discussion, it is not hard to understand that matching patterns depend on the distribution of types. In Kremer and Maskin [15] segregation increases when the *support* of the type distribution increases *for a given value of* θ.

If agents can modify their types (say through costly education), then two effects are likely to happen in response to technological shocks. In the short term, when the type distribution is given, our effect is likely to dominate, i.e., technological bias *reduces* segregation. In the long run, when agents decide to "change" their types, the type distribution will also change. Whether or not the type distribution changes in the manner that Kremer-Maskin assume is still an open question.

3.2 Imperfect-Markets Examples

We consider two examples. The first introduces a financing constraint. The most obvious change is the differential effect of the imperfection on the segregation payoffs of different types. Put simply, high ability agents get the same segregation payoff with or without the financing constraint. Low ability agents suffer a large decrease in the segregation payoff with sufficiently imperfect financing. The result is possibly a significant change in the matching pattern (as well as in the aggregate output of the economy). The same kind of effect was present in the case of the technological changes studied in the previous subsection, although here the outcome varies more conspicuously with the type distribution: whereas in the previous example, matching is always positive assortative, here we will get mixtures of segregation, positive and negative assortative matching, depending on the distribution.

Such is not the case in the second example, which introduces a moral hazard problem into the production process. This reduces the segregation payoff for all types, but again those of the lowest ability are most severely affected. But a second effect now comes into play, which is not present in either of the other two examples: increasing information costs also reduce the *transferability* of utility. The moral hazard problem requires that payoffs exceed a positive lower bound for each partner. It turns out that this change in the characteristic function offsets the changes in the segregation payoffs in such a way as to keep the matching pattern unchanged: Condition S will always be satisfied. But now aggregate performance will no longer be optimal, even conditional on the information constraints and the distribution of types: total surplus could be increased by forcing matches to differ from their equilibrium form. The source of the failure of optimality of equilibrium is the restricted transferability introduced by the incentive problem, on which we comment below.

3.2.1 Production with an Imperfect Financial Market

Consider now a modification of the standard production model in which a fixed amount $k > 0$ of capital is required for production to take place; once this is invested, output depends on the ability of the firm's members according to $h(a,b) = ab$. We assume that the lowest ability $\underline{a}$ exceeds $\sqrt{k}$ (so it is always efficient to produce if the capital market is perfect). The cost of a unit of capital is normalized to one. All individuals have zero wealth, and therefore every partnership must access a capital market in order to finance their firm. This market, however, is imperfect. We model this imperfection starkly: the output of a firm must exceed ϕk, $\phi \geq 1$, in order for financing to be possible.[22] The joint output for a pair (a,b) can then be written as

$$H(a,b,k,\phi) = \begin{cases} ab - k, & \text{if } ab \geq \phi k \\ 0, & \text{if } ab < \phi k \end{cases}$$

A perfect capital market corresponds to $\phi = 1$. In this case, the economy will be segregated by ability, since when $\underline{a} \geq \sqrt{k}$ the segregation payoff $\underline{u}(a) = \frac{a^2-k}{2}$ is positive and the surplus function $\sigma(a,b) = \max\left\{0, -(a-b)^2\right\} = 0$. Note that the surplus function is constant for all k as long as $k \leq \underline{a}^2$. This outcome is independent of the initial distribution of types.

As ϕ increases, the market becomes less efficient, excluding more and more types from producing positive output on their own. We note that despite the financial market imperfection, utility is still transferable within each coalition. Proposition 1 therefore has two implications for this model.

[22] This kind of capital market imperfection can be derived by supposing that the partner in the firm, upon having to repay, may renege on their debt and escape with probability π a punishment which brings their income to zero. Lenders will make loans of size k only to those firms whose output h will exceed $k/(1-\pi)$, since only for these firms is repaying, which yields a payoff of $h - k$, more attractive than reneging, which yields πh. Thus, $\phi \equiv \frac{1}{1-\pi} = 1$ corresponds to $\pi = 0$; with larger values of π escape becomes more likely, until with $\pi = 1$, the market shuts down altogether ($\phi = \infty$).

First, the equilibrium match will maximize aggregate surplus, which will be helpful in computing the equilibrium below. Second, aggregate output is also maximized at equilibrium, which may be of more interest from the welfare point of view.[23]

The segregation payoff is

$$\underline{u}(a) = \begin{cases} \frac{a^2-k}{2} & \text{if } \ a^2 \geq \phi k \\ 0 & \text{if } \ a^2 < \phi k. \end{cases}$$

We divide the type space into two intervals, $A_- = [\underline{a}, \sqrt{\phi k})$ and $A_+ = [\sqrt{\phi k}, \overline{a}]$. The pre-surplus and surplus functions are

$$\begin{aligned} s(a,b) &= \begin{cases} -\frac{(a-b)^2}{2} & \text{if} \quad a,b \in A_+ \\ a\left(b-\frac{a}{2}\right)-\frac{k}{2} & \text{if} \quad a \in A_+, b \in A_- \\ 0 & \text{if} \quad a,b \in A_-. \end{cases} \\ \Rightarrow \quad \sigma(a,b) &= \begin{cases} \max\left\{0, a\left(b-\frac{a}{2}\right)-\frac{k}{2}\right\} & \text{if } a \in A_+, b \in A_- \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{12}$$

We note that *there can be heterogenous matching only between types in* A_+ and A_-; a and b match and generate a positive surplus only if the three conditions below are satisfied.

$$a \in A_+, b \in A_- \tag{13}$$

$$ab \geq \phi k \tag{14}$$

$$b \geq \frac{a}{2} + \frac{k}{2a}. \tag{15}$$

[23] How do we reconcile optimality of the equilibrium here with the well-known results that say that in the presence of financial market imperfections, equilibrium need not maximize aggregate output? Pecuniary externalities are one possible cause of an inefficient outcome, but these are precluded here because of the assumption that there are no externalities across coalitions. Even without externalities, policies which redistribute initial wealth may increase output. Proposition 1 says that matching will be efficient *given* the distribution of types; this means that there are no policies that involve a mere reassignment of matches away from the equilibrium ones that can increase output, although mean-preserving changes to the initial type distribution (plausible if type is interpreted to be wealth, less so perhaps if type is ability) might raise output. But purely "associational redistribution" ([6]) can play no efficiency-enhancing role here. In [8], a financial market imperfection leads to a possibly inefficient match. But there the imperfection does not reduce the total output that a student and a school can produce together, but rather limits the amount of surplus that a student can transfer to a school, and it therefore does not correspond to the case $f = 0$; in this sense it is related to the example in the next section.

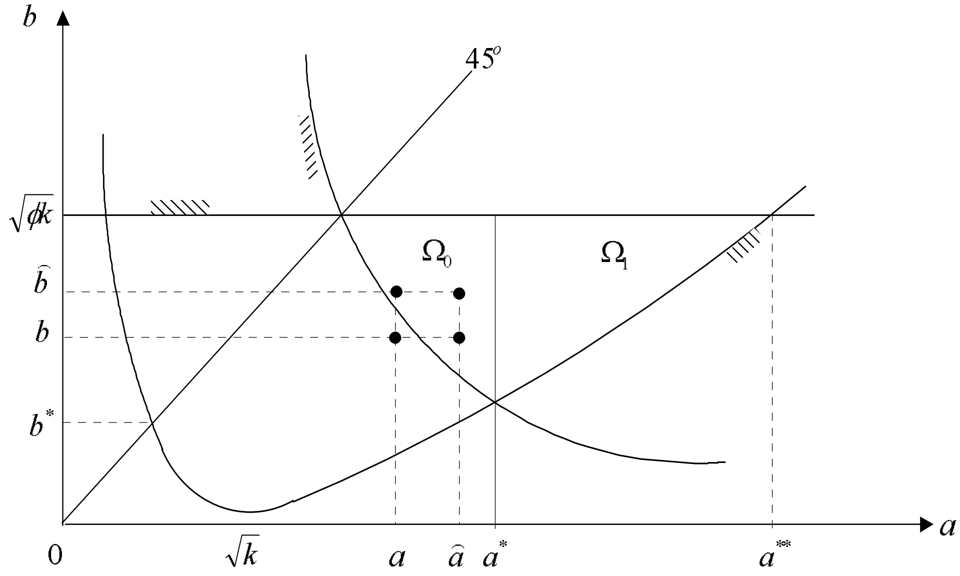

Figure 4:

(13) follows (12). Let Ω denote the set of pairs (a, b), with $a \in A_+$ and $b \in A_-$ which satisfy (14) and (15). This set is illustrated in Figure 4; the downward sloping boundary corresponds to the financing constraint (14) while the upward sloping boundary corresponds to the positive surplus constraint (15).[24] a^* is the value of a for which (14) and (15) bind; we have partitioned Ω into the sets Ω_0 — where $a < a^*$ — and Ω_1 where $a \geq a^*$.

It is useful to think of this problem as a two sided model, where the sides are A_+ and A_-. For any distribution τ on A, we can define a new distribution on the pairs (a, b), where $a \in A_+$ and $b \in A_-$; we are interested in the support of that new distribution; support that we denote by $A_+^\tau \times A_-^\tau$.

Let $\hat{a} > a > \hat{b} > b$; *if all pairs in* $\{a, \hat{a}\} \times \left\{b, \hat{b}\right\}$ belong to Ω, then the WID condition is satisfied since it coincides with the WID condition in the Becker model. In the figure, we have an example in which all pairs in $\{a, \hat{a}\} \times \left\{b, \hat{b}\right\}$ belong to Ω *except* the pair (a, b) (the point (a, b) does not satisfy the financing constraint $ab \geq \phi k$); in this case, the WID condition is $\sigma\left(\hat{a}, \hat{b}\right) -$

[24]Computations show that $a^* = \sqrt{(2\phi - 1)\,k}$, $b^* = \frac{\phi k}{a^*}$ and $a^{**} = \left(\sqrt{\phi} + \sqrt{\phi - 1}\right)\sqrt{k}$.

$\sigma(\hat{a}, b) \geq \sigma\left(a, \hat{b}\right)$ and this condition can be easily violated. Clearly, there also exist situations in which the WID condition is satisfied. For instance, if $(a, \hat{b})$ and $(\hat{a}, b)$ belong to Ω_1, then both $(\hat{a}, \hat{b})$ and (a, b) belong to Ω_1 (Ω_1 has a lattice structure but Ω_0 has not[25]) From these observations, the properties of the equilibrium matching will depend on the distribution of types. We first state conditions on the distribution under which the equilibrium matching is monotone; then we use these results to understand the effects on matching of changes in the distribution and in the parameter ϕ. All results are proved in the Appendix.

Proposition 15 *(i) If $A_+^{\tau} \times A_-^{\tau} \cap \Omega$ is a lattice, then the equilibrium satisfies PAM.*

(ii) Suppose that $A_+^{\tau} \subseteq [a^*, \overline{a}]$, then the equilibrium satisfies PAM.

Distributions satisfying the conditions of Proposition 15 are *bimodal*. We assume from now on that the distribution of types is *log-uniform*, i.e., that $\log a$ is uniformly distributed on $[\log \underline{a}, \log \overline{a}]$. We show that even if WID is satisfied for a non trivial subset of the set of types, the equilibrium match involves negative matching or segregation.

Proposition 16 *Suppose that the distribution of types is log-uniform on $[\underline{a}, \overline{a}]$. The unique equilibrium matching pattern is as follows.*

(i) If $\underline{a} \geq \sqrt{\phi k}$ or $\overline{a} \leq \sqrt{\phi k}$, there is segregation.

(ii) If $\underline{a} < \sqrt{\phi k} < \overline{a}$, there exists (a unique) $b_e \in \left[\underline{a}, \sqrt{\phi k}\right)$ such that the equilibrium matching is as follows: there is negative matching between $\left[b_e, \sqrt{\phi k}\right)$ and $\left(\sqrt{\phi k}, \frac{\phi k}{b_e}\right]$ and types outside these intervals segregate.

Comparative Statics

Using Propositions 15 and 16, we illustrate how the distribution of types can affect—here in a somewhat dramatic way—the equilibrium matching pattern. For descriptive purposes, call the interval $\left[\underline{a}, \sqrt{\phi k}\right)$ the "lower class", the interval $\left[\sqrt{\phi k}, \frac{\phi k}{b_e}\right]$ the "middle class" and the interval $\left(\frac{\phi k}{b_e}, \overline{a}\right]$ the "upper class", where b_e is as in Proposition 16. Starting from the log-uniform distribution on $[\underline{a}, \overline{a}]$ suppose that, keeping the average type in the economy constant, we decrease the measure of the middle class, by moving middle class agents to the other two classes in a mean preserving way. If the

[25] Recall that a subset $X \subset \mathbb{R}^2$ is a lattice if $x, y \in X$ implies that both $x \vee y$ and $x \wedge y$ are in X where $\vee$ is the "maximum" operator and $\wedge$ is the "minimum" operator.

middle class completely disappears, the support of the new distribution is $\left[\underline{a}, \sqrt{\phi k}\right) \cup \left(\frac{\phi k}{b_e}, \bar{a}\right]$. But $\left(\left[\underline{a}, \sqrt{\phi k}\right) \times \left(\frac{\phi k}{b_e}, \bar{a}\right]\right) \cap \Omega$ is a lattice, so from Proposition 15 there is PAM in equilibrium. In general, while the middle class still has positive measure, equilibrium matching will consist of a mixture of PAM, NAM and segregation.

It is also possible to obtain some results on the effects of changes in the parameters of the log-uniform distribution on a numerical measure of the matching patterns, namely the level of segregation (this is similar to the exercise carried out in [15]). Let $\mu(\phi, \underline{a}, \overline{a})$ be the measure of the interval $\left[b_e, \frac{\phi k}{b_e}\right]$, i.e., the interval of types that match in a negative fashion in Claim 16.[26]. A simple measure of the degree of segregation is $1 - \mu(\phi, \underline{a}, \overline{a})$. Keeping the assumption of log-uniformity, we can first make comparative statics on the bounds. The first result shows that the degree of segregation is not monotonic in the spread of the support; the second result shows a multiplicative shift in the support yields increased segregation. The proofs of these assertions are in the Appendix.

- For any $\underline{a} < \sqrt{\phi k}$, segregation first decreases for $\overline{a} \in \left(\sqrt{\phi k}, \min\left\{a^*, \frac{\phi k}{\underline{a}}\right\}\right)$ and then increases for $\overline{a} > \min\left\{a^*, \frac{\phi k}{\underline{a}}\right\}$
- Consider a family of log-uniform distributions on $[\alpha\underline{a}, \alpha\overline{a}]$, where $\underline{a} < b^*$, $\overline{a} \geq a^*$. Then, in equilibrium, segregation increases as α increases.

Finally, if we consider changes in ϕ, the degree of imperfection on the financial market, we obtain a non-monotonic relationship between ϕ and the degree of segregation

- Let $\phi^* = \frac{1}{2}\left(\left(\frac{\overline{a}}{\underline{a}}\right)^2 + 1\right)$; the index of segregation decreases with ϕ when $\phi \in (1, \phi^*)$ and increases with ϕ when $\phi \geq \phi^*$

Thus trends in the degree of segregation which could be explained by skill biased technical change or an increasingly unequal skill distribution might also be explained by improvements in the functioning of financial markets.

[26] Under log-uniformity, $\mu(\phi, \underline{a}, \overline{a}) = \frac{\log \phi k - 2 \log b_e}{\log \overline{a} - \log \underline{a}}$.

3.2.2 Production with an Incentive Problem

In contrast to the efficiency of matching in transferable utility environments (even those with financial market imperfections or other restrictions on transferability between coalitions and the outside world), inefficiencies may arise in matching environments when there are restrictions on transferability *within* coalitions.[27] In these instances, reassignment of matches away from their equilibrium values can raise aggregate output.

Consider the same production function as before, but now suppose that there is a moral hazard problem: each partner in a match must take some effort in order for output to be produced. The effort levels are low and high, with cost 1 if the high effort is chosen and zero otherwise. In order for partners of ability a and b to produce ab, both must take the high effort; otherwise output is zero. Effort is not observable unless it is monitored at a cost: if $c(\phi, q)$ is invested at the time of the match, the probability of detecting a partner if he takes the low effort is q (this probability is independent across partners, but the same q must be chosen for each partner). Thus $\phi \geq 0$ will index the severity of the moral hazard problem. We assume that $c(0, q) \equiv 0$, and that c is increasing in both arguments and convex in q.

Each partner receives a contract which specifies that he receives a payment y if he is not caught taking low effort, and 0 if he is.[28] Given the level of monitoring q, incentive compatibility then requires that $y - 1 \geq (1 - q)y$, or $y \geq \frac{1}{q}$. The net output generated by a firm with partners a and b and monitoring q is then $ab - c(\phi, q)$; but even though the partners are assumed to be risk neutral they cannot transfer this output to each other arbitrarily: each partner must receive at least $\frac{1}{q}$.[29]

For analyzing this problem it is convenient to consider the maximum payoff that an agent can achieve assuming his partner is incentive compatible,

[27] The observation that matching can be inefficient has already been made in the literature (e.g., [2], [9]), but in those cases it depends on spillovers across coalitions.

[28] If one takes the assumption of two partners literally, this is not the optimal contract, since the firm's output would typically serve as a signal of the partner's effort. We have in mind situations, such as those in large firms, where output information reveals little about individual effort and other (costly) signals must be employed instead. See [16] for a more general analysis.

[29] Thus in terms of (1), we have $X = \{0,1\} \times \{0,1\}$, $Q = [0,1]$, $h(q, x, t_1, t_2, \theta, \phi) = t_1 t_2 x_1 x_2$, $g(q, x, t_1, t_2, \theta, \phi) = c(\phi, q)$, $f(q, x, t_i, \phi) = x_i(\frac{1}{q} - 1)$.

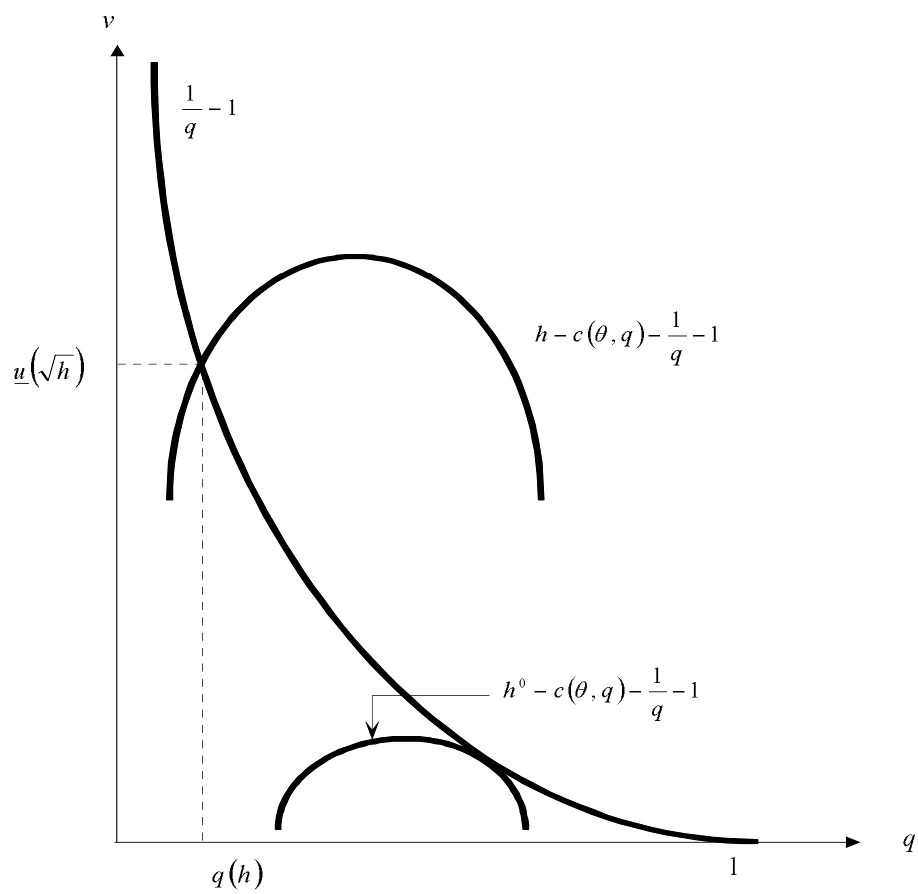

Figure 5:

considered as a function of q. This expression, $h - c(\phi, q) - \dfrac{1}{q} - 1$, with $\phi > 0$, is graphed for different values of h in Figure 5. Also shown is the incentive compatibility constraint $\dfrac{1}{q} - 1$: if $h = ab$, both a and b must get a payoffs at least this high if they are to be incentive compatible.

For $\phi = 0$, q is optimally set equal to 1. In this case, the first-best allocation with segregation is achieved in equilibrium (We assume it is efficient for all partnerships to produce: if abilities lie in the interval $[\underline{a}, \bar{a}]$, then $\underline{a} > \sqrt{2}$.)

Things can be rather different, however, if $\phi > 0$. Let $q\,(h)$ be the lower value of q, when it exists, at which the graph of $h - c(\phi, q) - \dfrac{1}{q} - 1$ intersects the graph of $\frac{1}{q} - 1$. When $q(a^2)$ exists (and lies in $[0,1]$), the segregation payoff of type a is $\frac{a^2 - c(\phi, q(a^2))}{2} - 1$. Clearly, there exists a unique h^0 such that the graphs of $h - c(\phi, q) - \dfrac{1}{q} - 1$ and $\frac{1}{q} - 1$ are tangent. Hence, when $a < \sqrt{h^0}$, $q\left(a^2\right)$ does not exist and the agents have a zero segregation payoff.[30]

The fact that the segregation payoff is zero for some types creates a situation parallel to the one with the imperfect financial market: high types might want to match with low ones because the latter have such poor outside opportunities and they might be "cheaper" than their more productive counterparts. In other words, we might conjecture that there would be heterogeneous matches in equilibrium. The first thing to check, then, is whether Condition S is violated. The surprising result is that, on the contrary, it is always satisfied:

Proposition 17 *The economy with moral hazard is segregated for all* ϕ.

P roof. Suppose instead that there is a heterogeneous match (a, b), with $a > b$. Let q be the level of monitoring they choose. Clearly, a has a positive segregation payoff (if not, then neither does b, and nothing can be gained if they match), and $q > q(a^2)$. Let y_a and y_b be the levels of compensation paid to each of the partners. If b has a positive segregation payoff and $q \geq q(b^2)$, then since for a heterogeneous match to occur we must have

$$y_a + y_b = ab - c(\phi, q) \geq \frac{a^2 - c\left(\phi, q\left(a^2\right)\right)}{2} + \frac{b^2 - c\left(\phi, q\left(b^2\right)\right)}{2},$$

we immediately conclude, since $c(\phi, \cdot)$ is increasing in q, that

$$0 > ab - \frac{a^2 + b^2}{2} \geq c(\phi, q) - \frac{c\left(\phi, q\left(a^2\right)\right) + c\left(\phi, q\left(b^2\right)\right)}{2} > 0,$$

a contradiction.

If instead $q < q(b^2)$ or b has a zero segregation payoff ($q(b^2)$ does not exist), then $b^2 - c\left(\phi, q\right) - \frac{1}{q} < \frac{1}{q}$; heterogeneous matching again requires that

$$ab - c\left(\phi, q\right) - y_b \geq \frac{a^2 - c\left(\phi, q\left(a^2\right)\right)}{2},$$

[30] If $c(\theta, 1)$ is finite, the set of types with zero segregation payoffs may be larger than $[\underline{a}, \sqrt{h^0}]$. This hardly affects the analysis, however; when it does, we shall point this out.

and, since $y_b \geq \frac{1}{q}$,

$$y_b > \frac{b^2 - c(\phi, q)}{2};$$

adding these two expressions and rearranging yields

$$0 > ab - \frac{a^2 + b^2}{2} > \frac{c(\phi, q) - c(\phi, q(a^2))}{2} > 0,$$

a contradiction. We conclude that no heterogeneous matches can occur.
■

Even though the matching configuration is unchanged when incentive problems are introduced, there is an important difference between the two cases from a welfare point of view: when ϕ is large enough, the equilibrium will not always be efficient in the sense of maximizing total output net of monitoring and effort costs. There are two sorts of reasons for this. First of all, since different types choose different levels of the monitoring technology, the average level of monitoring that will be used under heterogeneous matches might be lower than that used under segregation. More important, in equilibrium some types are "left out" of the economy, and more output could be generated if higher types were forced to match with them. In both cases, the source of inefficiency of equilibrium is the failure of full transferability.

To see this, suppose that $c(\phi, q) = \phi q$ and $\phi > 2$. A social planner trying to maximize total output net of monitoring and effort costs will always want the partners in a match to share output equally, since this minimizes the level of q they need to use. Thus, the level of q chosen by a partnership of an a and a b is given by the smaller solution to

$$ab - \phi q - \frac{1}{q} = \frac{1}{q},$$

provided it exists — one needs $ab \geq 8\phi$; otherwise the partnership generates zero output since it is not possible for both members to be incentive compatible. (Since $\phi > 2$, any partnership that is incentive compatible is also efficient in the sense that the output net of monitoring costs exceeds the disutility of effort). Solving for q, one finds that the maximized net output for a pair (a, b) is

$$H(a, b) = \frac{ab + \sqrt{a^2 b^2 - 8\phi}}{2}.$$

It is tempting to think of H as a "reduced-form" production function, but it differs from the usual notion of production function in that the level of output produced is dependent on a particular sharing rule (namely equal sharing) that the planner has imposed.

With this caveat in mind, note that H is symmetric but neither super- nor submodular; indeed a simple computation shows that its cross partial derivative is negative in the range $\sqrt{8\phi} < ab < \sqrt{4(\sqrt{5}+1)\phi}$ and positive for $ab > \sqrt{4(\sqrt{5}+1)\phi}$. As in the case of the financial market imperfection, incentive problems can cause the properties of the joint output to differ dramatically from those of the production technology.

Given this fact, it is clear that segregation is not the optimal matching pattern. In fact, if the support of the distribution lies in $[(8\phi)^{1/4}, (4(\sqrt{5}+1)\phi)^{1/4}]$, it is straightforward to show that the optimum consists of strictly negative matching.[31] Thus, even if all types in the economy are able to produce, the segregated outcome is not generally optimal simply because lower types, when segregated, (must) choose excessively high levels of monitoring. Aggregate performance is enhanced when high types match with them: the sacrifice in output is more than compensated by the reduction in monitoring costs.

A more significant increase in welfare can sometimes be obtained by forcing types with positive segregation payoffs to match with those with zero. In this case, even though the high types will now be in firms that are producing less output and incurring higher monitoring costs than they do under segregation, there are more active firms; the employment of previously unused resources can increase aggregate output very significantly.[32] Notice that this

[31] To see this, note that if the joint output function is symmetric and strictly submodular, it satisfies the WDD inequalities: we have

$H(x) + H(y) > H(x \vee y) + H(x \wedge y)$ by submodularity; putting $x = (a, d)$ and $y = (b, c)$ yields $H(a, d) + H(b, c) > H(a, c) + H(b, d)$; putting instead $y = (c, b)$ and using $H(c, b) = H(b, c)$ yields $H(a, d) + H(b, c) > H(a, b) + H(c, d)$. Thus whenever there are four types $a > b \geq c > d$, output is higher when they are negatively matched than when they are positively matched. Finally, putting $x = (a, b)$ and $y = (b, a)$ establishes that $H(a, b) > \frac{1}{2}[H(a, a) + H(b, b)]$ (and $H(c, d) > \frac{1}{2}[H(c, c) + H(d, d)]$), so $H(a, d) + H(b, c) > \frac{1}{2}[H(a, a) + H(b, b) + H(c, c) + H(d, d)]$, which shows that segegation is also dominated. The same sort of logic establishes Proposition 8.

[32] For instance, consider the example with a linear cost function, a uniform distribution on $[\underline{a}, \bar{a}]$, and parameter values $\phi = 4$, $\bar{a} = (4(\sqrt{5}+1)\phi)^{1/4}$, $a_0 = (8\phi)^{1/4}$, $\underline{a} = \frac{\sqrt{8\phi}}{\bar{a}}$.

In equilibrium, agents below a_0 are idle and net output is $\frac{1}{\bar{a}-\underline{a}} \int_{a_0}^{\bar{a}} (b^2 + \sqrt{b^4 - 8\phi} - 2) db =$

effect would remain if q were exogenously fixed independently of type, while the previous effect would disappear.

This example illustrates that the efficient (output maximizing) match need not occur in equilibrium when there is limited transferability due to incentive problems. We do not have a complete characterization of those nontransferable cases in which equilibrium fails to be optimal. However, there is the following partial characterization result.

Proposition 18 *Suppose that the output-maximizing match is (payoff-equivalent to) segregation. Then the economy is segregated.*

P roof. Suppose instead there is an equilibrium in which a positive measure of agents get strictly more than their segregation payoffs. Since in any equilibrium every type gets at least its segregation payoff, the aggregate equilibrium payoff strictly exceeds the aggregate of the segregation payoffs. But this contradicts the assumption that aggregate output is maximized by segregation. ■

In particular, if the planner's "reduced-form" production function $H(a, b)$ is symmetric and supermodular, then the optimum will involve segregation (the reasoning for this is analogous to that for Proposition 3), and therefore the equilibrium will too.

The converse to Proposition 18 is obviously not true, as our example shows. In that case, the reason that surplus maximizing matches are not achieved in equilibrium stems from the limited transferability introduced by incentive problems. An a who is forced to match with a b receives less than his segregation payoff; she cannot be compensated by the b because that would entail that the b end up with less than an incentive-compatible share of output (recall that the equilibrium is Pareto efficient, just not output maximizing). This would violate feasibility. Thus, this simple example illustrates how a conflict between "cake production" (maximizing the surplus generated by matches) and "cake division" (maximizing one's share of a given surplus) can lead to distortions in the pattern of matching.

3.85

Negative matching on $[a_0, \bar{a}]$ yields $\frac{1}{\bar{a}-\underline{a}}\int_{a_0}^{\bar{a}}(b(\bar{a}+a_0-b)+\sqrt{b^2(\bar{a}+a_0-b)^2-8\phi}-2)db =$ 3.91

But NAM on $[\underline{a}, \bar{a}]$ yields $\frac{1}{\bar{a}-\underline{a}}\int_{\underline{a}}^{\bar{a}}(b(\bar{a}+\underline{a}-b)+\sqrt{b^2(\bar{a}+\underline{a}-b)^2-8\phi}-2)db = 4.47$

In this case, while there is a gain to matching negatively where the reduced-form production function is submodular (namely on $[a_0, \bar{a}] \times [a_0, \bar{a}]$), this is small compared to the gain of employing the "unemployed" who are below a_0 (also in a negative assortative way).

More generally, the example demonstrates that even with knowledge of the production technology that indicates that SEG (or PAM) ought to be the outcome, and even with accompanying evidence to that effect, there need not be a presumption that such a matching pattern is efficient.

4 Conclusion

The foregoing analysis suggests that the conclusions that have been drawn from the basic matching model are vulnerable to misspecification. The financial market example indicates that market imperfections may significantly change the patterns of matching we observe, possibly making them very sensitive to the distribution of types. Or, as the moral hazard example indicates, they may have very little effect on matching, even if they have a large effect on economic outcomes. The general point is that the conclusions one can draw from observing a matching pattern — either about the underlying economic process or the efficiency of outcome — may be very limited. In particular, the properties of the production technology hardly suffice for making welfare evaluations on the basis of an observed matching pattern or in predicting the outcome of the match.

The next step in the agenda is the search for conditions on the distribution of types that will help characterize solutions for situations in which conditions like P are violated. The analysis in Section 3.2.1 indicates this may be a challenging task, but far from hopeless. In particular, optimization methods can be applied to transferable utility cases, which as we have seen, include a number of imperfect markets models.

This paper has focused on the effects of only one departure from the classical environment on two-person matching without externalities. There are two others which have received some attention recently, namely search frictions [26] and multidimensional type spaces [7]; we comment briefly on the latter.

On the positive side, as we have remarked, Condition S and Proposition 2 apply almost without modification to multidimensional type spaces (as well as to multiperson matches). It is possible to generalize the definitions of monotone matching to this case, although the typical incompleteness of orders on the type space tend to make fully satisfactory definitions hard to come by.[33]

[33] See our earlier working paper [17] for a discussion of this point.

The multidimensional case introduces other difficulties. First of all, there is a myriad of ways in which the various characteristics of the matching partners might enter into their joint output. A most natural way to proceed is to suppose that the characteristics can be summarized by a one-dimensional quantity (call it "talent"). Output then depends on talent in the usual way. Talent is not observable to the investigator, but (some of) the characteristics are (one thinks of athletes, whose height and weight might be easy to measure but whose athletic talent might require the appraisal of experts who match the athletes into teams).

But even in this restricted environment with two-person matches, it is easy to find cases in which matching satisfies PAM in talent but appears as NAM in every dimension observable to the investigator. The problem is that the joint distribution of characteristics leaves a degree of freedom that doesn't fully nail down the matching pattern. Avoiding this predicament requires a weaker, statistical definition of PAM and related restrictions on the joint distribution of characteristics. We discuss this in [19]. The point to emphasize is that one-dimensional models, like perfect-markets models, entail special assumptions, and it is important to have some idea just what those assumptions are and how strong they might be.

5 Appendix

5.1 A Note on Existence of Equilibrium

When $f \equiv 0$, utility is transferable. By the maximum theorem and the imposed conditions on the choice sets and the functions h and g, the maximized value of the joint payoff is upper semicontinuous in types; as shown in [11], this ensures that an equilibrium exists.

1. For the nontransferable case, things are slightly more involved. The first issue is comprehensiveness of the characteristic function,[34] which is generally essential to ensuring nonemptiness of the core. While a mild condition in the case of perfect markets (it amounts to being an assumption of free disposal), it is much less so in the presence of incentive and/or contractibility problems, since feasibility will often

[34] V is comprehensive if for any set P, $v \in V(P)$ implies that $v' \in V(P)$ whenever $v'_i \leq v_i$ for all $i \in P$.

entail that each agent receives a nonnegligible payoff (see Figure ?? for an example). None of our results depend on comprehensiveness (except in instances where it is already guaranteed by other assumptions).

For existence, the following construction suffices. Restrict attention to economies in which the type distribution has finite support (the issue with a continuous type distribution is possible failures of continuity of the characteristic function in type). Define the comprehensive extension of a set $V(\cdot)$ as the smallest comprehensive set containing $V(\cdot)$. The economy in which V is replaced by its comprehensive extension will have a nonempty core [12]. Moreover, there will always exist core allocations in the extended economy in which agents receive utility levels that are on the Pareto frontier of the original feasible set $V(\cdot)$. Such allocations satisfy feasibility, measure consistency, and the no blocking requirements of an equilibrium of the original economy, and so the original economy has an equilibrium.

When Condition S is satisfied and V is comprehensive, then the segregation payoff vector lies outside (or on the Pareto frontier) of $V(\cdot)$. Violations of this condition entail that the segregation payoff vector lies in the interior of $V(\cdot)$. But if V is not comprehensive, then segregation payoff vector *may* lie outside of $V(\cdot)$ and still entail a violation of Condition S.

5.2 Proof of Proposition 4

(i) Assume that Condition PT is satisfied. Consider $a > b \geq c > d$ and payoffs such that

$$\begin{aligned} s(a) + s(d) &= \sigma(a, d) \\ s(b) + s(c) &= \sigma(b, c). \end{aligned} \tag{16}$$

We will show that for any payoffs satisfying (16), a negative matching pattern $(\{a, d\}, \{b, c\})$ cannot be stable. Since PT is satisfied, one of the two WID inequalities holds. Assume without loss of generality that

$$\sigma(a, d) + \sigma(b, c) \leq \sigma(a, b) + \sigma(c, d). \tag{17}$$

If either $s(a) + s(b) < \sigma(a, b)$ or $s(c) + s(d) < \sigma(c, d)$, the negative matching pattern $(\{a, d\}, \{b, c\})$ is not stable when the payoffs are s. Hence, there can be negative matching in equilibrium only if $s(a) + s(b) \geq \sigma(a, b)$

and $s(c)+s(d)\geq\sigma(c,d)$. However, by using (17) it is immediate that both inequalities must be equalities. Hence, the payoffs s also sustain the positive match $(\{a,b\},\{c,d\})$.

(ii) Consider an economy with three atoms of equal mass $a>b>c$. We show that if PT is violated, i.e., if

$$\sigma(a,b)+\sigma(b,c)<\sigma(a,c) \tag{18}$$

PAM is violated in equilibrium. Consider payoffs s such that

$$\begin{aligned} s(a) &= \sigma(a,b)+\varepsilon \\ s(c) &= \sigma(a,c)-\sigma(a,b)-\varepsilon \\ s(b) &= 0. \end{aligned}$$

where $\varepsilon\in(0,\sigma(a,c)-\sigma(a,b))$. Then, $s(a)+s(c)=\sigma(a,c)$, $s(a)+s(b)>\sigma(a,b)$ and $s(b)+s(c)>\sigma(a,c)$ by (18). Hence the matching $(\{a,c\},\{b\})$ together with s constitutes an equilibrium that violates PAM.

Consider now any $a>b>c>d$ and suppose that PT is not satisfied. Hence, $\sigma(a,d)>0$, and,

$$\sigma(a,d)+\sigma(b,c) > \sigma(a,b)+\sigma(c,d) \tag{19}$$
$$\sigma(a,d)+\sigma(b,c) > \sigma(a,c)+\sigma(b,d) \tag{20}$$

Consider now an economy with four atoms of equal measure at a,b,c,d. We show that there is no equilibrium with positive matching; thus the equilibrium must exhibit negative matching.

There are positive matches in which some types segregate. Clearly, segregation of all types cannot be an equilibrium since $\sigma(a,d)>0$. The positive matches are the following

$$\begin{aligned} m_1 &= (\{a\}\{b\},\{c,d\}) \\ m_2 &= (\{a,b\},\{c\},\{d\}) \\ m_3 &= (\{a\},\{b,c\},\{d\}) \\ m_4 &= (\{a,b\},\{c,d\}) \\ m_5 &= (\{a,c\},\{b,d\}). \end{aligned}$$

Clearly, m_1 and m_2 are dominated by m_4 (if m_4 cannot be an equilibrium match, neither can m_1 and m_2). m_3 cannot be an equilibrium since $\sigma(a,d)>$

0 (this also explain the use of the condition $\sigma(a,d) > 0$). Hence, we are left with m_4 and m_5. The argument is the same for both, so we prove only the result for m_4.

Suppose by way of contradiction that m_4 is an equilibrium match for the economy. Then there is a payoff s such that

$$\begin{aligned} s(a) + s(b) &= \sigma(a,b) \\ s(c) + s(d) &= \sigma(c,d). \end{aligned}$$

However, by (19), $s(a) + s(b) + s(c) + s(d) < \sigma(a,d) + \sigma(b,c)$. Hence either $s(a) + s(d) < \sigma(a,d)$ or $s(b) + s(c) < \sigma(b,c)$ which means that for any payoff, there is beneficial deviation by a coalition. Since matching cannot be positive in equilibrium, the negative assortative matching $(\{a,d\}, \{b,c\})$ (this is the only other possible matching) is the unique equilibrium matching pattern.

5.3 Proof of Proposition 10

We prove part (i) only, as the proof of (ii) is similar. We observe that by monotonicity of $\beta_{a_i a_j}(s_j)$,

$$\begin{aligned} &\text{for any } \{a_1, a_2, a_3, a_4\}, \\ &\beta_{a_i a_j} \circ \beta_{a_j a_k}(s_k) \text{ is increasing in } s_k \\ &\beta_{a_i a_j} \circ \beta_{a_j a_k} \circ \beta_{a_k a_l}(s_l) \text{ is decreasing in } s_l. \end{aligned} \tag{21}$$

Suppose that Condition P is violated. I.e., consider a 4-tuple $\{a,b,c,d\}$ where $a > b \geq c > d$ and payoffs $s(a)$, $s(b)$, $s(c)$ and $s(d)$ with $s \in S^P(a,d) \times S^P(b,c)$ and such that the following conditions hold

(a) $s(a) \geq \beta_{ab}(s(b))$, $s(a) \geq \beta_{ac}(s(c))$, $s(d) \geq \beta_{db}(s(b))$, $s(d) \geq \beta_{dc}(s(c))$

(b) [either $s(a) > \beta_{ab}(s(b))$ or $s(d) > \beta_{dc}(s(c))$]

(c) [either $s(a) > \beta_{ac}(s(c))$ or $s(d) > \beta_{db}(s(b))$].

Note that since $s \in S^P(a,d) \times S^P(b,c)$,

$$\begin{aligned} s(a) &= \beta_{ad}(s(d)) \in [0, \phi_{ad}(a)] \\ s(b) &= \beta_{bc}(s(c)) \in [0, \phi_{bc}(b)] \end{aligned} \tag{22}$$

Condition (a) is implied by the core condition for $s \in S^P(a,d) \times S^P(b,c)$. Indeed, the original core condition implies that $(s(a), s(b)) \notin S^D(a,b)$; hence, it must be that the vector $(s(a), s(b))$ is "outside" the set $S^D(a,b)$. However, since both $s(a)$ and $s(b)$ are nonnegative, this implies that $(s(a), s(b))$ is also "outside" the extension of the set $S^D(a,b)$, i.e., that $s(a) \geq \beta_{ab}(s(b))$. Note that this is true *for any* extension. The other inequalities in (a)-(c) are derived following the same logic.

We show that (b) implies that $s(d) < \beta_{da} \circ \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s(d))$ and that (c) implies that $s(d) < \beta_{da} \circ \beta_{\alpha c} \circ \beta_{cb} \circ \beta_{bd}(s(d))$, hence that condition P* is violated.

In (b), suppose first that $s(a) > \beta_{ab}(s(b))$. By (22) and strict monotonicity of β, we have the following sequence

$$\begin{aligned} s(a) &> \beta_{ab}(s(b)) \\ &\Leftrightarrow \beta_{ad}(s(d)) > \beta_{ab}(s(b)) \\ &\Leftrightarrow \beta_{ad}(s(d)) > \beta_{ab} \circ \beta_{bc}(s(c)) \\ &\Leftrightarrow s(d) < \beta_{da} \circ \beta_{ab} \circ \beta_{bc}(s(c)) \\ &\Rightarrow s(d) < \beta_{da} \circ \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s(d)) \end{aligned}$$

If $s(a) = \beta_{ab}(s(b))$, then (b) implies that $s(d) > \beta_{dc}(s(c))$. We then have

$$\begin{aligned} s(d) &> \beta_{dc}(s(c)) \\ &\Leftrightarrow s(c) > \beta_{cd}(s(d)) \\ &\Leftrightarrow s(b) < \beta_{bc} \circ \beta_{cd}(s(d)) \\ &\Longrightarrow \beta_{ba}(s(a)) < \beta_{bc} \circ \beta_{cd}(s(c)) \\ &\Rightarrow s(a) > \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s(d)) \\ &\Rightarrow s(d) < \beta_{da} \circ \beta_{ab} \circ \beta_{bc} \circ \beta_{cd}(s(d)). \end{aligned}$$

Now, in (c), if $s(a) > \beta_{ac}(s(c))$, we have

$$\begin{aligned} s(a) &> \beta_{ac}(s(c)) \\ &\Longleftrightarrow s(d) < \beta_{da} \circ \beta_{ac}(s(c)) \\ &\Longleftrightarrow s(d) < \beta_{da} \circ \beta_{ac} \circ \beta_{cb}(s(b)) \\ &\Longrightarrow s(d) < \beta_{da} \circ \beta_{ac} \circ \beta_{cb} \circ \beta_{bd}(s(d)) \end{aligned}$$

If $s(a) = \beta_{ac}(s(c))$, then (c) requires that $s(d) > \beta_{db}(s(b))$, we then have,

$$\begin{aligned} s(d) &> \beta_{db}(s(b)) \\ &\Leftrightarrow s(b) > \beta_{bd}(s(d)) \\ &\Leftrightarrow s(c) < \beta_{cb} \circ \beta_{bd}(s(d)). \end{aligned}$$

Since $s(a) = \beta_{ac}(s(c))$, $\beta_{ad}(s(d)) = \beta_{ac}(s(c))$ the previous inequality and the fact that β_{ac} is strictly decreasing, imply

$$\begin{aligned} \beta_{ad}(s(d)) &= \beta_{ac}(s(c)) \\ &> \beta_{\alpha c} \circ \beta_{cb} \circ \beta_{bd}(s(d)) \\ &\Longleftrightarrow s(d) < \beta_{da} \circ \beta_{\alpha c} \circ \beta_{cb} \circ \beta_{bd}(s(d)). \end{aligned}$$

Therefore, (b) and (c) imply a violation of condition P* *for any extension* β. This proves that if there exists *one* extension β such that Condition P* holds, then Condition P also holds. This concludes the proof.

5.4 Proof of Corollary 11

We prove (i), as the proof of (ii) is similar. For any t and $\hat{t}$ and $v \leq \Sigma_{t\hat{t}}$, let $s = v - \underline{u}(\hat{t})$, $\phi_{t\hat{t}}(\hat{t}) = \Sigma_{t\hat{t}} - \underline{u}(\hat{t})$, and

$$\beta_{t\hat{t}}(s) = \begin{cases} \gamma_{t\hat{t}}\left(s + \underline{u}(\hat{t})\right) - \underline{u}(t) & \text{if } s \leq \phi_{t\hat{t}}(\hat{t}) \\ \phi_{t\hat{t}}(\hat{t}) - s & \text{otherwise} \end{cases}$$

Observe that β satisfies (4) since $\gamma_{t\hat{t}}\left(s + \underline{u}(\hat{t})\right)$ is strictly decreasing and since the boundary conditions are met. Also, for any $a > b \geq c > d$, $\phi_{ad}(d) > 0$. Therefore, the requirements in Condition N* are satisfied. We show that (9)-(10) imply (7)-(8) of Proposition 10 when applied to β.

Let $s \leq \phi_{cd}(d)$ and suppose that (7) does not hold for β. Since β is strictly decreasing, we have,

$$\beta_{ba} \circ \beta_{ad}(s) < \beta_{bc} \circ \beta_{cd}(s)$$

Using recursively the definition of β, we have:

$$\begin{aligned} \beta_{bc} \circ \beta_{cd}(s) &= \beta_{bc}\left(\gamma_{cd}(s + \underline{u}(d)) - \underline{u}(c)\right) \\ &= \gamma_{bc}\left(\gamma_{cd}(s + \underline{u}(d))\right) - \underline{u}(b) \\ &\text{and} \\ \beta_{ba} \circ \beta_{ad}(s) &= \beta_{ba}\left(\gamma_{ad}(s + \underline{u}(d)) - \underline{u}(a)\right) \\ &= \gamma_{ba}\left(\gamma_{da}(s + \underline{u}(d))\right) - \underline{u}(b). \end{aligned}$$

It follows that $\gamma_{ba} \circ \gamma_{ad}(s + \underline{u}(d)) < \gamma_{bc} \circ \gamma_{cd}(s + \underline{u}(d))$ which violates (9) for $v = s + \underline{u}(d)$ since $s \leq \phi_{cd}(d)$ implies that $s + \underline{u}(d) < \Sigma_{cd}$. Similar reasoning shows that (10) implies (8).

5.5 Proof of Proposition 15

(i) If $A_+^\tau \times A_0^\tau \cap \Omega$ is a lattice, then any two pairs that can be chosen out of Ω will satisfy WID. Since equilibrium matches outside Ω involve segregation, matching overall satisfies PAM. The condition basically requires the distribution of types to be *bipolar*: there is a "hole" in the support. (ii) is a special case of (i) since all gains from trade happen in Ω_1 and since Ω_1 is a lattice.

5.6 Proof of Proposition 16

Looking at a relaxed version of the surplus maximization program (i.e. one in which constraints that reflect the measure consistency requirement are ignored), we see that the surplus for (a,b) is maximum at $a = \frac{\phi k}{b}$, namely at the lowest value of a for which the financial constraint (14) is satisfied. So maximizing surplus pointwise by putting $m(b) = \frac{\phi k}{b}$ maximizes surplus in the relaxed program. Now observe that imposing the measure consistency constraint doesn't change anything: under log-uniformity it is satisfied with this choice of $m(b)$, so $m(b) = \frac{\phi k}{b}$ is indeed optimal. Thus the equilibrium consists of negative matching between $\left[b^*, \sqrt{\phi k}\right)$ and $\left(\sqrt{\phi k}, a^*\right]$ and segregation elsewhere: the "middle types" match negatively and the other types segregate. This result is generalized to the cases $\underline{a} > b^*$ or $\overline{a} < a^*$ in the following proposition.(i) is a direct implication of our earlier discussion. For (ii), let $\tilde{b} = \max\{\underline{a}, b^*\}$ and $\tilde{a} = \min\{\overline{a}, a^*\}$. If $\tilde{b} < \frac{\phi k}{\tilde{a}}$, define $b_e = \frac{\phi k}{\tilde{a}}$, $\tilde{b} \geq \frac{\phi k}{\tilde{a}}$, define $b_e = \tilde{b}$.

5.7 Proof of the Comparative Static Results in Section 3.2.1

Increasing the Lower Bound of the Support.

The measure of heterogeneous matches is, for $\underline{a} < b^*$,

$$\mu\left(\phi, \underline{a}, \bar{a}\right) = \begin{cases} 0 & \text{if } \bar{a} \leq \sqrt{\phi k} \\ \frac{2\log\bar{a} - \log(\phi k)}{\log\bar{a} - \log\underline{a}} & \text{if } \bar{a} \in \left[\sqrt{\phi k}, a^*\right] \\ \frac{\log(2\phi - 1) - \log\phi}{\log\bar{a} - \log\underline{a}} & \text{if } \bar{a} \geq \sqrt{\phi k} \end{cases} \tag{23}$$

On $\bar{a} \in \left(\sqrt{\phi k}, a^*\right)$, $\frac{\partial\mu(\phi,\underline{a},\bar{a})}{\partial\bar{a}}$ is proportional to $\log\left(\frac{\phi k}{\underline{a}^2}\right)$ which is positive since $\underline{a} < \sqrt{\phi k}$. Hence, the index of segregation $1-\mu$ *decreases* as $\bar{a}$ increases on this interval. On $\bar{a} > \sqrt{\phi k}$, $\mu(\phi, \underline{a}, \bar{a})$ is clearly decreasing in $\bar{a}$, and segregation *increases* for these values; in the limit, as $\bar{a} \to \infty$, $\mu(\phi, \underline{a}, \bar{a}) \to 0$. A similar argument can be used for any value of $\underline{a}$ as long as $\underline{a} < \sqrt{\phi k}$. (If $\underline{a} \geq \sqrt{\phi k}$, $\mu(\phi, \underline{a}, \bar{a}) = 0$ for any $\bar{a}$).

Shifting the Distribution

Since α is the parameter of interest, we write $\mu(\alpha)$ for $\mu(\phi, \alpha\underline{a}, \alpha\bar{a})$. Since there is a multiplicative shift of the support, the density of a is still $\frac{\log a}{\log \bar{a} - \log \underline{a}}$ on $[\alpha\underline{a}, \alpha\bar{a}]$. Direct application of Proposition 16 leads to the following values for $\mu(\alpha)$

$$\mu(\alpha) = \begin{cases} \frac{\log(2\phi-1)-\log\phi}{\log\bar{a}-\log\underline{a}} & \text{if } \alpha \leq \frac{\phi}{2\phi-1}\frac{a^*}{\underline{a}} \\ \frac{\log(\phi k)-2\log(\alpha\underline{a})}{\log\bar{a}-\log\underline{a}} & \text{if } \alpha \in \left[\frac{\phi}{2\phi-1}\frac{a^*}{\underline{a}}, \frac{\sqrt{\phi k}}{\underline{a}}\right] \\ 0 & \text{if } \alpha \geq \frac{\sqrt{\phi k}}{\underline{a}} \end{cases}$$

Clearly, $\mu(\alpha)$ is constant for $\alpha \leq \frac{\phi}{2\phi-1}\frac{a^*}{\underline{a}}$, and decreasing for $\alpha \in \left[\frac{\phi}{2\phi-1}\frac{a^*}{\underline{a}}, \frac{\sqrt{\phi k}}{\underline{a}}\right]$, yielding the result.

Increasing ϕ

To simplify the writing, let $\beta = \frac{\overline{a}}{\underline{a}}$. Since ϕ is the parameter of interest we write $\mu(\phi)$ for $\mu(\phi, \underline{a}, \overline{a})$. Without loss of generality, assume that $\underline{a} = \sqrt{k}$; hence, $\overline{a} = \beta\sqrt{k}$, where $\beta > 1$. Note that for $\phi > 1$, $\underline{a} < \sqrt{\phi k}$, hence as long as $\phi < \beta^2$, $\overline{a} > \sqrt{\phi k}$, and Ω is non empty, which implies that $\mu(\phi)$ is positive. $\phi^* = \frac{1}{2}\left(\beta^2 + 1\right)$ is the unique solution to the equation in ϕ, $\overline{a} = \sqrt{(2\phi - 1) k}$. Then, in the notation of Proposition 16, since $\underline{a} = \sqrt{k}$, $\underline{a}$ is less than $b^* = \frac{\phi\sqrt{k}}{\sqrt{2\phi-1}}$ for any $\phi > 1$ and it follows that $\tilde{b} = b^*$. Since $\overline{a} = a^*$ when $\phi = \phi^*$, $\tilde{a} = a^*$ if $\phi \leq \phi^*$ and $\tilde{a} = \overline{a}$ when $\phi \geq \phi^*$. Observing that $\log\overline{a} - \log\underline{a} = (\log\beta - 1)\frac{\log k}{2}$, and that $\log\overline{a} - \log\frac{\phi k}{\overline{a}} = (\log\beta - \log\phi)\log k$, it follows that

$$\mu(\phi) = \begin{cases} 2\frac{\log(2\phi-1)-\log\phi}{(\log\beta-1)\log k} & \text{if } \phi \in \left[1, \frac{1}{2}\left(\beta^2+1\right)\right] \\ 2\frac{\log\beta^2-\log\phi}{(\log\beta-1)\log k} & \text{if } \phi \in \left[\frac{1}{2}\left(\beta^2+1\right), \beta^2\right] \\ 0 & \text{if } \phi \geq \beta^2 \end{cases}$$

Clearly, $\mu(\phi)$ is decreasing (so that segregation $1 - \mu(\phi)$ is increasing) on $\left[\frac{1}{2}\left(\beta^2+1\right), \beta^2\right]$ and increasing on $\left(1, \frac{1}{2}\left(\beta^2+1\right)\right)$, yielding the result.

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