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ABSTRACT

Conventional and Unconventional Monetary Policy with Endogenous Collateral Constraints*

We consider the effects of central-bank purchases of a risky asset, financed by issuing riskless nominal liabilities (reserves), as an additional dimension of policy alongside "conventional" monetary policy (central-bank control of the riskless nominal interest rate), in a general-equilibrium model of asset pricing and risk sharing with endogenous collateral constraints of the kind proposed by Geanakoplos (1997). When sufficient collateral exists for collateral constraints not to bind for any agents, we show that central-bank asset purchases have no effects on either real or nominal variables, despite the differing risk characteristics of the assets purchased and the ones issued to finance these purchases. At the same time, the existence of collateral constraints allows our model to capture the common view that large enough central-bank purchases would eventually have to effect asset prices. But even when central-bank purchases raise the price of the asset, owing to binding collateral constraints, the effects need not be the ones commonly assumed. We show that under some circumstances, central-bank purchases relax financial constraints, increase aggregate demand, and may even achieve a Pareto improvement; but in other cases, they may tighten financial constraints, reduce aggregate demand, and lower welfare. The latter case is almost certainly the one that arises if central-bank purchases are sufficiently large.

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One of the more notable developments in central banking since the global financial crisis has been an increase in the diversity of types of market transactions through which central banks have sought to influence financial conditions. Before the crisis, it had become common to think of monetary policy as a uni-dimensional decision: the periodic reconsideration of the central bank's operating target for a single, short-term (typically overnight) nominal interest rate. Over the past five years, instead, a number of leading central banks have made almost no changes in their policy rates — having taken those rates to levels viewed as their effective lower bounds by the beginning of 2009, while additional monetary easing continued to be desired — yet have been quite active on other dimensions, making dramatic changes in both the size and composition of their balance sheets.

Here we undertake a theoretical analysis of the effects of these alternative dimensions of monetary policy, in a general-equilibrium asset-pricing framework in which assets with different risk characteristics co-exist and earn different rates of return in equilibrium. We introduce a central bank with effective control over short-term nominal interest rates, that can determine the general level of prices (of goods and services in terms of money) through this "conventional" monetary policy; but we also allow the central to engage in open-market purchases and sales of the various types of assets with differing risk characteristics that are traded in the marketplace, and consider the extent to which allowing for variations in the size and composition of the balance sheet, holding interest-rate policy fixed, provide useful additional dimensions of policy.

It is important to note that we do not here seek to model central-bank *credit policies:* lending by a central bank to specific types of borrowers at below-market rates, either because it wishes to subsidize certain activities or institutions, or because private intermediation has become highly inefficient, as during the most severe phase of the recent financial crisis. The policies with which we are concerned, such as the Federal Reserve's asset-purchase programs since the fall of 2010, involve open-market purchases of assets that are traded on highly liquid markets, and are aimed at achieving macroeconomic goals by influencing financial conditions for the economy as a whole, rather than at providing credit for specific borrowers or categories of borrowers. Our model is therefore one in which financial markets are efficient, in the sense that all traders are able to purchase the same set of assets, at prices that are independent of the identity of the purchaser and of the quantity purchased, and that the spread between the price paid by a buyer and that received by the seller of assets

is assumed to be negligible; and all central-bank trades are assumed to occur at these well-defined market prices.[1]

There is, however, one important respect in which we shall assume that financial markets are not frictionless in the sense of Arrow and Debreu, and this is important for the consequences of "unconventional" monetary policy: we shall assume, as in Geanakoplos (1997) and Araújo et al. (2002), that all privately issued financial claims (as opposed to physical assets or government liabilities) must be collateralized. Moreover, rather than assuming collateral requirements (and hence borrowing limits) of an arbitrary form, we endogenize the collateral requirements, as in the models of Geanakoplos (1997) and Geanakoplos and Zame (2013).[2] Sharp increases in collateral requirements were a notable feature of the recent financial crisis (as discussed, for example, by Adrian and Shin, 2010; Brunnermeier, 2009; and Gorton and Metrick, 2012). This makes it of particular interest to ask how collateral constraints matter for the effects of both conventional and unconventional monetary policies.

Our main conclusions can briefly be summarized. We find that pure changes in the central bank's balance sheet, in the absence of any change in the short-term nominal interest rate, can affect asset prices, the allocation of resources and the general level of prices; hence they do constitute a potentially useful independent dimension of policy. However, these effects depend critically upon the way in which and degree to which collateral constraints bind in equilibrium; hence the allowance for collateral constraints is crucial to our results. We show that when collateral is sufficiently abundant for no households' collateral constraints to bind, central-bank asset purchases are *irrelevant,* affecting neither the equilibrium prices of financial assets nor the money prices of goods and services nor the allocation of resources.

And even when collateral constraints bind, the effects of asset purchases depend critically on the particular *way* in which they bind; for example, we show that central-bank purchases of the risky good used as collateral will loosen private borrowers'

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[1]Models such as those of Cúrdia and Woodford (2011) or Gertler and Karadi (2011, 2013) instead consider central-bank purchases of assets that many investors cannot directly purchase themselves, because only certain specialized intermediaries (with limited capital and constraints on their access to financing) have the expertise required to evaluate them. These are more obviously appropriate as models of programs such as the Fed's "credit easing" policies (Bernanke, 2009) during the acute phase of the financial crisis, rather than its more recent asset-purchase programs.

[2]See Araújo et al. (2005) and Araújo (2014) for an alternative approach to the endogenization of collateral requirements.

collateral constraints under some circumstances, but tighten them under others. The conditions that determine which will be the case are somewhat complex; but one quite general observation is that acquisition of a sufficiently large fraction of the total supply of the collateral good by the central bank makes it almost inevitable that the collateral constraints of a non-trivial part of the population will be tightened by the central bank's policy. There are, however, conditions under which central-bank asset purchases can improve the situation of all parties, and thus achieve a Pareto improvement relative to an inefficient initial status quo; we offer both analytical sufficient conditions for this to be the case and a numerical illustration.

We introduce conventional monetary policy (i.e., interest-rate policy) into the model of collateral-constrained equilibrium proposed by Geanakoplos and Zame (2013) and Araújo *et al.* (2012) in section 1, and show that in our model conventional monetary policy has relatively standard effects. We then turn in section 2 to the effects of central-bank asset purchases. We first establish an irrelevance proposition for the case when collateral is sufficiently abundant, but then discuss why the same argument will not continue to be valid when the collateral constraint binds for at least some households. We further distinguish between two different ways in which the collateral requirement may constrain a household's decisions, and the different effects of asset-purchase policies upon the household's situation in these two cases.

The general-equilibrium effects of asset purchases on financial and macroeconomic equilibrium when collateral constraints bind are then developed in more detail in section 3, focusing on a case of particular interest, in which the collateral requirement limits the degree to which "natural buyers" of the risky asset are able to leverage themselves to take a longer position in this asset. Section 4 explores the consequences of an alternative way in which investors may be constrained, namely the case of a binding constraint on their ability to short the risky asset; it especially highlights the characteristic distortions that result when too large a fraction of the supply of the asset used as collateral comes to be held by the central bank. Section 5 summarizes our conclusions.

1 A Monetary Model with Endogenous Collateral Constraints

Here we present a finite-horizon general-equilibrium model with endogenous collateral constraints, along the lines of Geanakoplos and Zame (2013) and Araújo *et al.* (2012), but with a nominal unit of account, the value of which is determined by conventional monetary policy, and a central bank that is not subject to the same collateral constraint as private actors. We use the model to examine the effects of two independent dimensions of monetary policy, interest-rate policy ("conventional monetary policy") and central-bank asset purchases ("unconventional policy").

We consider a pure exchange economy over two time periods $t = 0, 1$, with uncertainty about the state of nature in period 1 denoted by the subscript $s \in \mathcal{S} = \{1, \ldots, S\}$. The economy consists of a finite number of households denoted by the superscript $h \in \mathcal{H} = \{1, \ldots H\}$ which can each consume two goods or commodities each period. One good is a non-durable consumer good, while the other is a durable good, which yields a service flow in both periods; the service flow from the durable (which might be thought of as housing) is not perfectly substitutable with non-durable consumption, and is possibly risky in period 1. The importance of the durable good in our model is as the only acceptable collateral in private loan contracts, discussed below; hence the supply of durables will be an important determinant of the scarcity of collateral.[3]

Because the durable good is assumed to be the only possible form of collateral, it is possible that the households that choose to hold the durable at the end of period 0 will differ from those that choose to consume the services of the durable in period 0. We therefore assume the existence of a market for "rental" of the durable (i.e., consumption of its service flow) in addition to purchases of it as an asset to hold until the next period. There are then effectively three goods each period (in addition to various types of financial assets) — the non-durable good (good 1), the service flow

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[3]Our results do not really depend on the assumption that the asset used as collateral is a real good that provides a service flow. What is crucial for our results is that the one-period return on the asset used as collateral is not completely riskless; thus it is important that it is not nominal (one-period) government debt. Many of our conclusions here about central-bank purchases of the risky durable good would apply equally to central-bank purchases of longer-term nominal government debt, in a multi-period model in which longer-term debt is used as collateral for short-term borrowing.

from the durable (good 2), and the durable good itself, held as an asset (good 3) — though utility is obtained from the consumption of only the first two of these goods.

Each household has an initial endowment $e_1^h \geq 0$ of the non-durable and $e_3^h \geq 0$ of the durable in period 0, and an initial endowment $e_{s1}^h \geq 0$ of the non-durable in state s of period 1. In addition, it is endowed with $e_3^h \geq 0$ units of the durable good and $d^h \geq 0$ units of government debt in period 0. (There are no further period-1 endowments of these assets.) Each household has a preference ordering defined over consumption plans $x^h = (x_l^h, x_{1l}^h, \ldots, x_{Sl}^h) \in \mathbb{R}_+^{2(S+1)}$ specifying the household's consumption of each of goods $l = 1, 2$ in each of the states. To simplify the analysis, we shall assume that households have identical preferences, and each seek to maximize expected utility

$$u^h = u(v(x_1^h, x_2^h)) + \sum_{s=1}^{S} \pi_s u(v(x_{s1}^h, x_{s2}^h)), \tag{1.1}$$

where $\pi_s > 0$ is the (commonly agreed) probability of occurrence of state s, $v(x_1, x_2)$ is a homogeneous-degree-one aggregator of the two goods (an increasing, concave function of its two arguments), and that $u(v)$ is an isoelastic utility function, so that

$$u'(v) = v^{-\gamma} \tag{1.2}$$

for some $\gamma \geq 0$.[4] Thus in the examples considered here, the heterogeneity of households (and hence the role of financial exchange) follows solely from their differing endowment patterns, and not from any differences in preferences or beliefs. This provides an especially clear basis for judgments about the welfare consequences of alternative policies, as the preferences used by each household to evaluate outcomes for itself are ones shared with everyone else.

1.1 Monetary Policy in a Finite-Horizon Model

We assume the existence of a supply $d \equiv \sum_{h=1}^{H} d^h$ of riskless nominal government debt, issued prior to the monetary policy decisions (taken in period 0) with which

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[4]Several of our more general characterizations of equilibrium below do not depend on preferences of this special form, or even on the assumption of identical preferences. But restricting attention to this special case simplifies the analysis by making intra-period relative prices independent of policy and other determinants of the intertemporal allocation. In particular, this allows more detailed characterizations of collateral-constrained equilibria.

we are concerned in this model. A unit of government debt is a promise to deliver one unit of money (the economy's nominal unit of account) in period 1, regardless of the state s reached at that date, and we assume that there is no doubt about the government's ability and intention to raise the tax revenue necessary in period 1 to pay off this debt. We let q_0 denote the price (in units of money) at which a unit of government debt trades in period 0.

We also assume the existence of a central bank that can acquire assets in period 0, financing its open-market purchases by issuing riskless nominal liabilities (reserves) of its own. These reserves are the economy's unit of account (called "money" in the previous paragraph); thus a price p_3 for the durable good in period 0 means p_3 units of reserve balances at the central bank in period 0. If the central bank chooses to acquire d^{CB} units of public debt and x_3^{CB} units of the durable good, it creates

$$M = q_0 d^{CB} + (p_3 - p_2)x_3^{CB} \qquad (1.3)$$

units of reserves. (Note that the effective cost of a unit of the durable is only $p_3 - p_2$, since the central bank can rent the durable in period 0; alternatively, we may suppose that the central bank purchases the durable after the period-0 rental income has already been collected by the initial owner.) We shall restrict attention to policies under which at least one element of the vector (d^{CB}, x_3^{CB}) is positive (while both elements are non-negative), so that $M > 0$. Allowing the central bank to separately vary d^{CB} and x_3^{CB} means that we can separately consider the effects of variation in the *size* of the balance sheet and its *composition*. Moreover, purchases of the durable allow us to consider the effects of purchases of assets with different risk exposure than the liabilities issued to purchase them.

Reserves held at the central bank pay a riskless nominal return i; that is, one unit of reserves held after trading in period 0 becomes a claim to $1 + i$ units of reserves in period 1, regardless of the state s. This riskless nominal interest rate is a policy variable, that may be freely set by the central bank; this choice represents "conventional monetary policy" in our model. Note that the central bank is free to set the interest rate on its liabilities at whatever level it likes, given that the unit of account is only *defined* in terms of balances held at the central bank, and the only link between the unit of account in two successive periods arises from the central bank's willingness to deliver future money in exchange for money held now on particular

6

terms.[5] Under the assumption that $M > 0$, so that some amount of reserves earning the return i must be voluntarily held, in any equilibrium (defined below) i will also have to be the rate of return on any *other* riskless nominal asset that may be traded, including riskless government debt. Hence in equilibrium $q_0 = (1+i)^{-1}$, and monetary policy determines "the" riskless nominal interest rate.

There is, however, an important constraint on the central bank's ability to freely choose the value of i, under typical institutional arrangements. This is that it is not possible to choose a value of i less than zero, if people are also free to exchange reserves for currency that offers a riskless nominal return of zero. In practice, non-interest-earning currency typically coexists with reserve balances at the central bank paying a positive interest rate, because of certain special uses for currency (not modeled in this paper); but the fact that holders of reserves always have the right to convert them into currency at a fixed parity prevents the central bank from driving the riskless rate below zero by paying a negative interest rate on reserves.[6]

In our model, there are no special uses of currency, and so currency will not be held in the case that the interest on reserves is positive. But even though currency will not be issued or held in any of our equilibria corresponding to policies $i > 0$, the *possibility* of requesting currency matters, because it implies that the central bank cannot choose a value of i less than zero. To economize on notation, we simply assume that the central bank's monetary liabilities all pay the same interest rate i, but that this rate must satisfy the constraint $i \geq 0$.

If the central bank acquires some of the durable ($x_3^{CB} > 0$), and the nominal value of the durable differs across states of the world in period 1 (as we shall assume in all of the equilibria considered below), then the value of the central bank's assets will not exactly equal the value of its liabilities in all states. We assume that any such balance-sheet earnings of the central bank are transferred to the Treasury, and

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[5]We abstracting from the existence of a liquidity premium associated with the use of reserves in payments, as in the "cashless economy" of Woodford (2003, chap. 2). The analysis here also applies to an economy in which the supply of reserves is maintained at all times at such a high level as to satiate the economy in reserves, as in the "floor system" for the implementation of monetary policy used by the Norges Bank (the central bank of Norway) over the past decade (Bowman *et al.*, 2010).

[6]In fact, the existence of small positive holding costs for currency mean that a slightly negative interest rate on reserves is possible; but this does not change the fact that the existence of currency puts a floor on the central bank's interest-rate target. For simplicity, we abstract from holding costs of currency here, and treat the lower bound as exactly zero.

reduce the taxes that must be collected to retire the government debt in period 1; correspondingly, any balance-sheet losses of the central bank are made up by the Treasury, and increase the taxes that must be collected in that state. The revenues required to retire the public debt (and pay off any losses of the central bank) are raised through lump-sum taxation. The share of taxes raised from each household h is $\theta^h \geq 0$, assumed to be the same for each state s, where $\sum_h \theta^h = 1$. Hence the tax obligation of household h in state s (in nominal units) is $\theta^h(\mu - p_{s3}x_3^{CB})$, where

$$\mu \equiv (1+i)M + (d - d^{CB}) \tag{1.4}$$

is the total public supply of riskless nominal assets (in terms of their value at maturity in period 1).

Finally, monetary policy also specifies the value of the nominal unit of account (in terms of real goods) in each state s in period 1. In an infinite-horizon model, there would be no need for such an additional dimension of policy; we could simply specify monetary policy as a choice of i, d^{CB} and x_3^{CB} each period. But in such a model, expectations about the value of the money that the central bank promises to deliver in the following period when it promises to pay the interest rate i will be determined by expectations about monetary policy in that subsequent period (and thereafter).[7] Here, instead, period 1 is a terminal period, in which there are no further decisions about interest-rate policy or the supply of bank reserves to make; but we nonetheless suppose that the value of the nominal unit of account in any state in period 1 can be made higher or lower by central-bank policy at that time. Technically, we suppose that the central bank redeems all nominal quantities remaining in accounts with it at the end of period 1 trading in terms of a specified (positive) number of units of the non-durable good per unit of money, as would occur under a "commodity money" scheme (though here there is convertibility only at the terminal date). Thus for each state s, the price p_{s1} of good 1 in units of money is fixed by monetary policy. A complete specification of monetary policy in our model is then given by the variables $(i, d^{CB}, x_3^{CB}, \{p_{s1}\}_{s=1}^S)$, with the implied supply of reserves given by (1.3).

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[7]See Woodford (2003, chap. 2) for illustration of how the price level (or exchange value of money) can be determined in each of an infinite sequence of periods purely by interest-rate policy in each of the sequence of periods.

1.2 Private Borrowing with Endogenous Collateral Requirements

We also allow for trading in privately issued financial claims; but contrary to what is assumed in the Arrow-Debreu [A-D] model or in standard models of general equilibrium with incomplete asset markets [GEI],[8] we do not assume that households can issue arbitrary quantities of financial claims as long as they are able to deliver the promised amount in each possible state of the world. Instead, we assume that borrowing must be collateralized, as in the models of Geanakoplos (1997) and Araújo *et al.* (2002), though the collateral requirements are determined endogenously (by what people will pay for private financial claims that are collateralized to a greater or lesser extent), rather than specified exogenously (for example, by law or social custom). We first introduce the notation that we use to describe collateralized borrowing, and then discuss what it means for the collateral requirements to be endogenously determined.

We assume that any privately issued financial claim specifies a quantity of money that must be repaid (independently of the state s) in order to extinguish the debt, and also a quantity of the durable good that must be held by the borrower (i.e., issuer of the claim) as collateral for the debt, and that can be seized by the lender (i.e., holder of the claim) in the event of default (i.e., non-payment of the specified amount of money). We also assume that the claim gives the holder no rights to assets of the issuer *except* the right to seize the assets pledged as collateral for the loan in the event of default; and it gives the issuer the right to discharge the claim (preventing seizure of the collateral) by paying the specified amount of money. Different types of private financial claims may simultaneously be traded, that are collateralized to different extents; thus there may be both "prime" and "subprime" loans collateralized by housing, where in our model the difference relates to the value of the collateral relative to the size of the loan, and not to any personal characteristics of the borrowers. But we assume a competitive equilibrium in which arbitrary quantities of a given type of financial claim can be purchased at a given per-unit price; hence we may without loss of generality normalize each of the types of private financial claims so that "one unit" of the claim promises delivery of one unit of money at maturity.

Thus we assume trading in a variety of types of privately issued financial claims $j \in \mathcal{J}$. Each asset j promises delivery of one unit of money in period 1, regardless of

[8]See Geanakoplos and Zame (2013) for discussion of these alternative model structures.

the state s. The collateral requirement for asset j is denoted $C_j \geq 0$; any issuer must hold C_j units of the durable in period 0 per unit sold of asset j. Given the possibility of default, the actual payoff of asset j in state s is $\min(1, p_{s3}C_j)$ in units of money, where p_{s3} is the price of the durable (in units of money) in state s of period 1. We let q_j denote the price (in units of money) at which assets of type j trade in period 0.

Thus far, we have supposed that the set of assets that may be issued and the collateral requirement associated with each of them is given; but in fact, these can be endogenously determined. As first proposed by Geanakoplos (1997) and developed more thoroughly by Geanakoplos and Zame (2013), we may actually suppose that competitive markets exist in which *all* possible collateralized financial claims are traded, though the equilibrium quantities issued of most of these securities will be zero. (The "market-determined" collateral requirements will then simply be those values of collateral for which the existence of such a market is not redundant.)

In the present example, the set of possible private financial claims corresponds to different possible values of C_j. Moreover, one can show that it suffices to assume trading in a particular finite set of assets, $j = 1, \ldots, S$, such that

$$C_j = 1/p_{j3} \tag{1.5}$$

for each j; that is, asset j is a claim with a collateral requirement such that if state j is realized in period 1, the value of the collateral will exactly equal the face value of the debt. In the case of any equilibrium for an economy with a set of private financial claims that includes the S types (1.5), but possibly other types as well, there necessarily exists a corresponding equilibrium for an economy with only the S markets (1.5) open, in which the prices of all goods and assets traded in the restricted economy are the same as in the original equilibrium, and the consumption allocation is also the same. (See Proposition 1 of Araújo *et al.*, 2012.)

Because of this result, we do not reduce the set of equilibria by assuming that only (at most) the set of S assets defined above are traded.[9] From now on, we assume that $\mathcal{J} = \{1, \ldots, S\}$ and $C_s = 1/p_{s3}$ for each j. These are our endogenously determined collateral requirements, as in Araújo *et al.* (2012).

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[9]In fact, asset 1 is also redundant, as shown by Lemma 6 in the Appendix. We nonetheless retain a market for asset 1 in our notation for the general case, in order to preserve a simple association between the number of the asset and the state in which the value of the collateral just suffices to allow repayment in full of the debt.

1.3 Equilibrium

Let p_1, p_2, p_3 denote the prices (in units of money) of the non-durable, the service flow from the durable, and the durable good respectively in period 0, and similarly let p_{s1}, p_{s2}, p_{s3} be the prices of the same three goods in state s in period 1. In fact, we necessarily have have $p_{s3} = p_{s2}$ in each state s (as there is no reason to acquire the durable in period 1 other than to enjoy the period 1 service flow). We can also simplify notation by observing that government debt and reserves issued by the central bank will be perfect substitutes, so that in equilibrium $q_0 = (1+i)^{-1}$, and households are indifferent as to how much of their holdings of publicly issued riskless assets are of one type or the other. We then have $2S + 3$ goods prices to determine (where we omit the redundant prices $\{p_{s3}\}$ from the price vector), along with the S privately-issued financial asset prices. Each household h chooses a consumption plan x^h and a portfolio described by a vector $\psi^h \in \mathbb{R}_+^S$ of asset purchases (lending), a vector $\varphi^h \in \mathbb{R}_+^S$ of asset issuance (borrowing), a quantity $\mu^h \geq 0$ of post-trade holdings of publicly-issued riskless assets (measured in units of their value at maturity in period 1), and a quantity $x_3^h \geq 0$ of post-trade holdings of the durable good. Note that we must separately specify financial asset purchases and issuances (rather than simply net trades, as in a GEI model), because of the need to satisfy the collateral requirements, that are increased by issuance of financial claims but not reduced by purchases of such claims. These are the prices and quantities that we seek to determine.

Given prices and financial conditions described by $p \in \mathbb{R}_{++}^{2S+3}$, $q \in \mathbb{R}_+^S$, $C \in \mathbb{R}_+^S$, and $q_0, i \geq 0$, household h chooses a consumption plan and portfolio $(x^h, \psi^h, \varphi^h, \mu^h, x_3^h)$ that solve the problem

$$\max_{x^h \geq 0, \, \psi^h \geq 0, \, \varphi^h \geq 0, \, \mu^h \geq 0, \, x_3^h \geq 0} u^h(x^h) \qquad \text{s.t.} \qquad (1.6)$$

$$p_1(x_1^h - e_1^h) + p_2(x_2^h - x_3^h) + p_3(x_3^h - e_3^h) + q \cdot (\psi^h - \varphi^h) + (1+i)^{-1}(\mu^h - d^h) \leq 0,$$

$$p_{s1}(x_{s1}^h - e_{s1}^h) + p_{s2}(x_{s2}^h - x_3^h) - \sum_{j=1}^{S}(\psi_j^h - \varphi_j^h)\min\{1, \, p_{s2}C_j\}$$
$$+\theta^h(\mu - p_{s2}x_3^{CB}) - \mu^h \leq 0, \qquad \forall s \in \mathcal{S}$$

$$x_3^h \geq \sum_{j=1}^{S} \varphi_j^h C_j,$$

11

where u^h is given by (1.1) and μ is determined by (1.3) and (1.4). A competitive equilibrium is then defined as usual as a situation in which each household's plan is optimal and markets clear. Our concept of competitive equilibrium with endogenous collateral constraints involves the additional requirement that the set of privately issued assets include all non-redundant financial assets of the kind discussed above.

Definition 1 *Let an economy $\mathcal{E}$ be defined by endowments $(e_1^h, e_3^h, \{e_{s1}^h\}_{s \in \mathcal{S}})$ for each $h \in \mathcal{H}$ and a monetary policy specification $(i, d^{CB}, x_3^{CB}, \{p_{s1}\}_{s \in \mathcal{S}})$. Then an equilibrium for the economy $\mathcal{E}$ is a vector $[(\overline{x}, \overline{\psi}, \overline{\varphi}, \overline{\mu}, \overline{x_3}); (\overline{p}, \overline{q}); \overline{C}]$ consistent with the monetary policy specification, such that in addition*

(i) *for each $h \in \mathcal{H}$, $(\overline{x}^h, \overline{\psi}^h, \overline{\varphi}^h, \overline{\mu}^h, \overline{x_3^h})$ solves problem (1.6), given prices $(\overline{p}, \overline{q})$, the interest rate i, and collateral requirements $\overline{C}$;*

(ii) $\sum_{h=1}^H \overline{x}_1^h = \sum_{h=1}^H e_1^h$;

(iii) $\sum_{h=1}^H \overline{x}_2^h = \sum_{h=1}^H e_3^h$;

(iv) $\sum_{h=1}^H \overline{x}_3^h + x_3^{CB} = \sum_{h=1}^H e_3^h$;

(v) $\sum_{h=1}^H \overline{x}_{s1}^h = \sum_{h=1}^H e_{s1}^h$ *for each $s \in \mathcal{S}$;*

(vi) $\sum_{h=1}^H \overline{x}_{s2}^h = \sum_{h=1}^H e_3^h$ *for each $s \in \mathcal{S}$;*

(vii) $\sum_{h=1}^H (\overline{\psi}^h - \overline{\varphi}^h) = 0$;

(viii) $\sum_{h=1}^H \overline{\mu}^h = \mu \equiv d + (1+i)(p_3 - p_2)x_3^{CB}$; *and*

(ix) $\overline{C}_s = 1/p_{s2}$ *for each $s \in \mathcal{S}$.*

Here condition (ix) reflects the endogenous determination of the collateral requirements (1.5).

A useful general observation about equilibrium in this model concerns the market for riskless (fully collateralized) private debt securities (asset S).[10]

Lemma 1 *There exists no equilibrium in which $\overline{q}_S < 1/(1+i)$. Moreover, if in equilibrium, some household h holds a quantity of collateral x_3^h that exceeds the quantity required to satisfy the household's collateral constraint, then $\overline{q}_S = 1/(1+i)$. Finally, if*

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[10]The proofs of all numbered lemmas and propositions are given in Appendix A.

in equilibrium, $\overline{q}_S > 1/(1+i)$, no units of asset S are issued in equilibrium, and the market is inessential, in the sense that the same equilibrium could be obtained if the market were to be closed.

The significance of this result is to show that if riskless private debt exists, it must promise the nominal interest rate i set by monetary policy. Hence our model is one in which the central bank has effective control of the riskless (one-period) nominal interest rate in private transactions (as well as the nominal interest yield on government debt, as already noted), subject to the constraint that it must choose a value $i \geq 0$.

1.4 Effects of Conventional Monetary Policy

We first consider the effects of "conventional" monetary policy, by which we mean changes in the nominal interest-rate target i, while holding fixed the size and composition of the central-bank balance sheet.[11] In our flexible-price model, we obtain the following simple result.

Proposition 1 *For a given economy $\mathcal{E}$ specified by the endowment pattern, let period-1 monetary policy commitments $\{p_{s1}\}_{s \in \mathcal{S}}$ and the balance-sheet variables (d^{CB}, x_3^{CB}) be fixed, but consider alternative interest-rate policies $i \geq 0$. Such variations in interest-rate policy have no effect on the equilibrium allocation of resources $\overline{x}$, on any relative prices $(p_2/p_1, p_3/p_1, p_{s2}/p_{s1}, q_j/p_1)$, or on any real rates of return $((1 + i)p_1/p_{s1}, p_{s3}/(p_3 - p_2) \cdot p_1/p_{s1}, \min\{1, p_{s2}C_j\} \cdot p_1/q_j p_{s1})$. That is, if there is an equilibrium associated with a given value of i, then for any other value of the interest rate (leaving unchanged the other dimensions of monetary policy), there exists a corresponding equilibrium, in which the allocation, relative prices, and real rates of return are the same, as are all period 1 prices, while period 0 prices vary inversely with $1 + i$.*

This result makes it clear that interest-rate policy can be used to determine the general level of prices in period 0, and indeed that any price level below a certain upper bound (the one achieved by the "loosest" possible policy, $i = 0$) is achievable by an appropriate choice of interest-rate policy. Moreover, interest-rate policy has an

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[11]Note that no changes in the balance sheet are required to implement the bank's desired interest-rate target, because of the possibility of varying the rate of interest paid on reserves.

effect on prices of the conventional sign: a "tightening" of current policy (raising i) is disinflationary (lowers the period 0 prices of all goods). Similarly, interest-rate policy can be used to control aggregate demand, in the sense of achieving a given volume of aggregate nominal expenditure

$$Y \equiv \sum_{h=1}^{h} [p_1 x_1^h + p_2 x_2^h], \tag{1.7}$$

in period 0, since this quantity also varies inversely with $1 + i$.

It is true that in our flexible-price endowment economy, variations in aggregate demand affect only the general level of prices, and not real activity. In an extension of the model to allow for sticky prices and endogenous output, however, conventional monetary policy would also affect equilibrium output.[12] And even in an endowment economy, the equilibrium allocation of resources would generally be affected if we were instead to suppose that households are initially endowed with nominal claims that promise to pay a fixed nominal amount in period 0, rather than assuming (as above) that their initial endowments of nominal financial claims consist only of government debt maturing in period 1; in that case, a change in the period-0 price level would (except in special cases) redistribute real income among the households. We do not pursue the equilibrium implications of such redistributive effects of conventional policy here, as there would be little novelty to such an analysis. It suffices for our purposes in this paper to have a simple benchmark for the effects of conventional monetary policy against which we can compare the effects of "unconventional" policies, i.e., variations in the size and composition of the balance sheet unrelated to any change in the interest-rate target.

2 Collateral Constraints and the Effects of Unconventional Policy

We now consider the additional dimensions of policy that result from possible variations in the size and composition of the central bank's balance sheet, unrelated to any change in the interest-rate target. In our simple framework, there are two such additional dimensions to consider: variations in the *size* of the balance sheet, and

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[12]We leave the analysis of this extension of the model for a future paper.

hence in the supply of reserves M, that need not be associated with any change in the amount of *risk* on the central bank's balance sheet, if M is increased by purchasing riskless government debt ("quantitative easing" in the original sense of the term); and variations in the quantity of risky durables x_3^{CB} held by the central bank, that need not be associated with any change in the supply of reserves, if the risky asset is substituted for riskless government debt. We consider each of these additional dimensions of policy in turn.

2.1 Irrelevance Results for Central-Bank Asset Purchases

A first simple result concerns the effects of open-market purchases or sales of government debt, resulting in corresponding increases or decreases in the supply of bank reserves (the monetary base).

Proposition 2 *Let interest-rate policy, the terminal-period price-level targets, and the central bank's purchases of the risky durable be fixed, but consider variations in the central bank's purchases d^{CB} of riskless public debt, and corresponding variations in the supply of reserves M implied by (1.3). Then the equilibrium values of all real and nominal variables listed in Definition 1 are independent of the value of d^{CB} (and hence also independent of the value of M), to the extent that variations in the supply of reserves occur through open-market operations of this kind.*

Of course, this result depends on the fact that in our model "government debt" means *short-term* debt, indeed of the same maturity as reserves held at the central bank (corresponding to very short-maturity Treasury bills). Longer-term Treasury securities would *not* generally be riskless, in terms of their short-term holding returns, and so open-market purchases of them do not represent a substitution of equivalent assets; but this type of open-market operation is effectively the purchase of a risky asset, of the kind that we take up next, rather than purchase of a riskless asset, as considered in this proposition.

We next consider the effect of variations in x_3^{CB}, the central bank's purchases of the risky durable. It is convenient to parameterize this as $x_3^{CB} = \omega e_3$, where $0 \leq \omega \leq 1$ indicates the fraction of the total supply of the durable that is held by the central bank. In the (generic) case that p_{s2} is not the same in all states s, in this case the asset purchased is not a perfect substitute for the liabilities issued to finance

15

the central bank's purchases. Yet even in this case, there need not be any effects of central-bank purchases on either real or nominal variables, though the conditions required for the irrelevance result are now more restrictive than in Proposition 2.

Proposition 3 *In the case that the central bank's share of the risky durable is $0 \leq \bar{\omega} < 1$, suppose there is an equilibrium in which each household h holds a quantity x_3^h of the durable that exceeds the quantity required to satisfy the household's collateral constraint. Then for any ω satisfying $\bar{\omega} < \omega < 1$ and*

$$(\omega - \bar{\omega})e_3 \ \leq \ \min_h \ \frac{x_3^h - \sum_j \varphi_j^h C_j}{\theta^h}, \tag{2.1}$$

additional central-bank purchases that increase the central bank's share to ω result in an equilibrium in which all prices are unchanged (both goods prices and asset prices), and the consumption allocation $\{x^h\}_{h \in \mathcal{H}}$ is similarly unchanged.

Thus in this case, we obtain an irrelevance result for central-bank asset purchases in the spirit of Wallace (1981), though we do not assume A-D financial markets, as Wallace does. Proposition 3 demonstrates the fallacy in a common way of discussing the effects of asset purchases. Central banks often appeal, in their explanations of the effects that they expect their asset-purchase programs to have, to a theory of "portfolio balance effects":[13] if the central bank holds less of certain assets and more of others, then the private sector is forced (as a requirement for equilibrium) to hold more of the former and less of the latter, and (according to this theory) a change in the relative prices of the assets should be required to induce the private parties to change the portfolios that they prefer. In order for such an effect to exist, it is thought to suffice that private parties not be perfectly indifferent between the two types of assets, owing to differences in their pattern of state-contingent payoffs.

But Proposition 3 shows that this is not the case. The flaw in the "portfolio-balance" theory is a simple one. The fact that the central bank takes some risk (say, real-estate risk) onto its own balance sheet, and allows the private investors instead to hold securities that pay as much in the event of a real-estate crash as in other states, does not make the risk disappear from the economy. The central bank's earnings on its portfolio will be lower in the crash state as a result of the asset exchange, and this will mean lower earnings distributed to the Treasury, which will in turn mean

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[13]See, for example, Gagnon *et al.* (2010).

that higher taxes will have to be collected by the government from the private sector in that state; so households' after-tax income will be just as dependent on the real-estate risk as before. This is why the asset pricing kernel does not change, in the case illustrated by Proposition 3, and why asset prices are unaffected by the open-market operation.

In fact, households that correctly understand the fiscal implications of the asset-purchase policy have a motive to change their own portfolios (assuming unchanged prices) in ways that *exactly offset* the transactions of the central bank. If household h bears fraction θ^h of the fiscal consequences, this creates a hedging motive for a portfolio shift that offsets exactly θ^h of the central bank's trades (selling fraction θ^h of the durables purchased by the central bank, and increasing its holdings of riskless assets by fraction θ^h of the increase in central-bank liabilities);[14] summing over all households, the central bank's transactions are exactly offset.

We can thus already give an answer to the question whether central-bank asset purchases have effects that are equivalent to those achieved by a cut in the short-term nominal interest rate in the case of conventional monetary policy. When Proposition 3 applies, the answer is obviously no. In this case, neither policy would have any effect on *real quantities;* but interest-rate policy would still be able to influence the general price level (for example, to head off unwanted deflation, as long as it is not constrained by the zero lower bound), while asset purchases would have no effect on equilibrium prices *or* quantities. (Nor, in the case described by Proposition 3, is there any effect of asset purchases on financial market prices, while conventional monetary policy influences not just the riskless rate but the equilibrium interest rates on the various types of risky private debts as well.)

The validity of Proposition 3 depends, however, on the assumption that all households have more collateral than they need to satisfy their collateral constraints. The interest of the result therefore depends on this being a possibility. The following result indicates that such a situation can indeed occur.

Proposition 4 *Consider an economy in which all households are identical, both as to their preferences and their endowments, and pay an equal share of taxes ($\theta^h = 1/H \ \forall h$) as well. Then for any specification of central-bank policy with $\omega < 1$, there is an equilibrium in which each household holds durables in excess of the quantity*

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[14]See the discussion of Figure 1(b) below for an illustration.

required to satisfy its collateral requirement.

This result shows that it is possible to have an economy for which the hypothesis of Proposition 3 holds. Proposition 4 might seem to refer to an extremely special case, but in fact the result that the collateral constraints do not bind in equilibrium will continue to be true for any economy with an endowment pattern *close enough* to one satisfying the assumptions of Proposition 4. Thus there will be an open set of endowment specifications satisfying the hypothesis of Proposition 3. But while robust examples can be constructed to which the irrelevance result of Proposition 3 applies, it is equally possible to construct robust examples of economies in which central-bank asset purchases *do* affect financial conditions — and affect the equilibrium allocation of resources, not just prices, as we now explain.

2.2 The Equilibrium Intertemporal Allocation of Expenditure

The case in which collateral is insufficient, so that collateral constraints bind for at least some households, is more complex to analyze, and the effects of central-bank asset purchases depend on the precise way in which the constraints bind. We simplify our analysis by restricting attention to the case of two equiprobable states in period 1 ($\pi_s = 1/2$ for $s = 1, 2$).

Our analysis is also simplified by assuming preferences of the homothetic form (1.1). The homotheticity of the aggregator function $v(x_1, x_2)$ implies that in any period, and any state of the world, each household chooses the same *relative* consumption x_1/x_2 as any other household, determined purely by the relative price p_2/p_1, regardless of the intertemporal allocation of expenditure that may be chosen. This has the following useful implication.

Lemma 2 *If each household has preferences of the form (1.1), then in any equilibrium, the equilibrium relative price of the two goods is given by*

$$\frac{p_2}{p_1} = \frac{v_2(e_1, e_3)}{v_1(e_1, e_3)} \tag{2.2}$$

where $e_l \equiv \sum_h e_l^h$ for $l = 1, 3$ are are the aggregate endowments of the two goods in that period and state of the world.

Thus the relative prices p_2/p_1, p_{12}/p_{11}, p_{22}/p_{21}, can each be determined from the economy's endowment pattern alone. These must therefore be independent of policy, and can be solved for without having to solve for the intertemporal allocation or asset prices. Hence we can treat them as already known, in solving the rest of the model (though not necessarily the same over time or across states).

It is then possible to define an indirect utility function

$$\tilde{u}(c) = \max_{x_1, x_2} \quad u(x_1, x_2) \qquad st. \quad x_1 + \left(\frac{p_2}{p_1}\right)x_2 \le c$$

for the initial date 0, where c is the value of total *expenditure* (in units of the non-durable good) in a given period and state of the world. The definition of the indirect utility function depends on the value of (p_2/p_1), but this is independent of policy. We can define a corresponding indirect utility function $\tilde{u}_s(c)$ for each of the possible states s at date 1.[15] Preferences can then be defined over intertemporal expenditure plans: each household chooses a plan (c^h, c_1^h, c_2^h) to maximize

$$U^h = \tilde{u}(c^h) + \frac{1}{2}\tilde{u}_1(c_1^h) + \frac{1}{2}\tilde{u}_2(c_2^h). \tag{2.3}$$

This allows us to write the model entirely in terms of the intertemporal allocation of *expenditure*, without any further reference to endowments or consumption of the two individual goods.

The way in which alternative portfolio choices affect the household's intertemporal allocation of expenditure can be represented by a vector of intertemporal transfers $\mathbf{y}^h$, where for each $s \in \mathcal{S}$, element y_s^h of this vector indicates the value of the household's portfolio (in units of the non-durable good) in state s of period 1, and hence the amount by which the household's real expenditure can exceed the value of its endowment (net of taxes) in that state. In terms of the notation used above,

$$y_s^h = \frac{1+i}{p_{s1}}\mu^h + \frac{1}{p_{s1}}\sum_j(\psi_j^h - \varphi_j^h) + \frac{p_{s2}}{p_{s1}}x_3^h$$

for each state s. In the case that $S = 2$, it is especially simple to characterize the set of feasible intertemporal transfers, and the cost (in terms of reduced period-0 expenditure) of achieving any given transfer vector. Here we consider only the

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[15]While we have assumed that the direct utility function $u(c)$ is the same at each date and in each state, the relative price may differ across periods and states of the world (because of differing relative endowments), so that the indirect utility function may differ as well.

generic case in which $p_{12} \neq p_{22}$, so that the durable good is not equivalent to a riskless nominal asset. Using the convention proposed in section 1.2 for the ordering of states, we shall therefore assume (without loss of generality) that $p_{12} > p_{22}$.

Lemma 3 *Consider an economy with $S = 2$ in which all households have homothetic preferences of the form (1.1), and suppose that $p_{12} > p_{22}$ and that the endogenous collateral requirements are given by (1.5). Then the set of feasible intertemporal transfers for any household h (abstracting from any limit on the household's budget in period 0) consists of those vectors $\mathbf{y}^h$ such that*

$$p_{21} y_2^h \leq p_{11} y_1^h \tag{2.4}$$

and

$$y_2^h \geq 0. \tag{2.5}$$

Moreover, the cost (in units of the period-0 non-durable good) of a portfolio achieves these intertemporal transfers is given by $\mathbf{a}' \mathbf{y}^h$, where $\mathbf{a}$ is a vector of state prices $a_1, a_2 > 0$, consistent with the market valuations of riskless debt and the durable good, in the sense that

$$a_1 \left(\frac{1+i}{p_{11}} \right) + a_2 \left(\frac{1+i}{p_{21}} \right) = \frac{1}{p_1}, \tag{2.6}$$

$$a_1 \left(\frac{p_{12}}{p_{11}} \right) + a_2 \left(\frac{p_{22}}{p_{21}} \right) = \left(\frac{p_3 - p_2}{p_1} \right). \tag{2.7}$$

Thus the consequences of financial market conditions for the intertemporal transfers that are feasible for any household can be summarized by the two state prices a_1, a_2, that are uniquely defined in the case of any equilibrium as the unique solution to the two linear equations (2.6)–(2.7), given the equilibrium price of goods and assets.[16] The household decision problem can then be written in a more compact form, as the choice of a plan $(c^h, c_1^h, c_2^h, y_1^h, y_2^h)$ to maximize (2.3) subject to the constraints

$$c^h + a_1 y_1^h + a_2 y_2^h \leq e^h + \frac{p_3 - p_2}{p_1} e_3^h + \frac{1}{p_1} d^h \tag{2.8}$$

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[16]See Fostel and Geanakoplos (2013) for further discussion of the possibility of characterizing households' budget constraints in terms of state prices, in the case that (as here) there are only two possible states in the second period and only a single durable good that can be used as collateral. Note that the state-price characterization is not always possible outside this special case.

$$c_s^h \leq e_{s1}^h + y_s^h - \theta^h \frac{\mu - p_{s2}\omega e_3}{p_{s1}}, \quad \text{for } s = 1, 2 \tag{2.9}$$

and (2.4)–(2.5); where $e^h \equiv e_1^h + (p_2/p_1)e_3^h$ is the value of the household's "total non-durable endowment" in period 0, if we split the endowment of the durable good into the period 0 service flow (counted as part of the "total non-durable endowment") and treat only the value of the asset after the period 0 service flow as an asset endowment; and μ is defined in (1.4).

Budget constraints (2.8)–(2.9) are the same as in an A-D model; only the additional constraints (2.4)–(2.5) make our model different. In the absence of the latter two constraints, it would be possible to combine (2.8)–(2.9) into a single intertemporal budget constraint, that makes no reference to the elements of y^h. But we find it useful to write the separate period budget constraints as above in our model, since the collateral constraints (2.4)–(2.5) are more conveniently written in terms of the vector y^h.

The household's problem can be written still more compactly if we represent the household's portfolio choice not by the vector $\mathbf{y}^h$, but instead by the vector $\tilde{\mathbf{y}}^h$ with elements

$$\tilde{y}_s^h \equiv y_s^h + \theta^h \left\{ \left(\frac{p_{s2}}{p_{s1}} \right) - \frac{(1+i)p_1}{p_{s1}} \left[a_1 \left(\frac{p_{12}}{p_{11}} \right) + a_2 \left(\frac{p_{22}}{p_{21}} \right) \right] \right\} \omega e_3,$$

indicating the net amount by which the household's budget is increased in state s by the *sum* of the net returns on its portfolio *and* the fiscal consequences for the household of the central bank's balance-sheet policy. Thus $\tilde{\mathbf{y}}^h$ represents a household's effective vector of intertemporal transfers, when one counts both the transfers that the household arranges itself and those that the central bank arranges "for it" (whether desired or not). Since the fiscal consequences of the central bank's policy are assumed to be known to the household when it chooses its portfolio, the vector $\tilde{\mathbf{y}}^h$ can also be treated as a choice of the household.

In terms of this alternative notation, the household's budget constraints (2.8)–(2.9) can be alternatively written as

$$c^h + a_1 \tilde{y}_1^h + a_2 \tilde{y}_2^h \leq e^h + a_1 f_1^h + a_2 f_2^h, \tag{2.10}$$

$$c_s^h \leq g_s^h + \tilde{y}_s^h, \tag{2.11}$$

21

using the notation

$$f_s^h \equiv (\frac{p_{s2}}{p_{s1}})e_3^h + (\frac{1}{p_{s1}})d^h,$$

$$g_s^h \equiv e_{s1}^h - \theta^h \frac{d}{p_{s1}},$$

for $s = 1, 2$. With this change of notation, the only endowments that need to be specified are $(e^h, f_1^h, f_2^h, g_1^h, g_2^h)$, all of which can be specified independently of policy. These give the value of household h's endowment (in units of real expenditure) in each state at each date, and also indicate how the value of its period-0 endowment depends on the endogenous state prices. Once this notation is adopted, there need no longer be any reference to either goods prices or asset prices (except for the state prices $\mathbf{a}$) in stating the household's problem.

We can solve equations (2.10)–(2.11) for the expenditure allocation implied by any choice of the vector $\tilde{\mathbf{y}}^h$, and substitute this into the objective (2.3) to obtain an indirect utility function $U^h(\tilde{\mathbf{y}}^h; \mathbf{a})$ that is independent of the central bank's balance-sheet policy (except through the effects of such policy on the equilibrium state prices $\mathbf{a}$). In terms of this alternative notation, the collateral constraints (2.4)-(2.5) take the form

$$p_{21}\tilde{y}_2^h \leq p_{11}\tilde{y}_1^h - \theta^h[p_{12} - p_{22}]\omega e_3, \tag{2.12}$$

$$\tilde{y}_2^h \geq -\theta^h \phi(\mathbf{a})\omega e_3, \tag{2.13}$$

where

$$\phi(\mathbf{a}) \equiv \frac{a_1(p_{12} - p_{22})}{a_1 p_{21} + a_2 p_{11}} > 0 \tag{2.14}$$

is a homogeneous degree zero function of the vector a. Note that $\phi(\mathbf{a})$ is a known function, given the data $(p_{12}/p_{11}, p_{22}/p_{21})$ that are determined by the endowments, and p_{21}/p_{11} that is determined by monetary policy.

We can then define equilibrium more compactly as follows.

Definition 2 *Given a two-state economy $\mathcal{E}$ with homothetic preferences of the form (1.1), and a policy specified by $(p_{11}, p_{21}, i, \omega)$,[17] an equilibrium is a vector of state prices $\mathbf{a}$ and a vector of total intertemporal transfers $\tilde{\mathbf{y}}^h$ for each h, such that*

(i) for each h, $\tilde{\mathbf{y}}^h$ maximizes $U^h(\tilde{\mathbf{y}}^h; \mathbf{a})$ subject to the constraints (2.12)-(2.13); and

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[17]We omit the choice of d^{CB} or M from the specification of policy, as these are irrelevant for equilibrium determination, according to Proposition 2.

(ii) for each $s = 1, 2$,

$$\sum_{h=1}^{H} \tilde{y}_s^h = \sum_{h=1}^{H} f_s^h. \tag{2.15}$$

Once we determine state prices **a** that satisfy these equilibrium conditions, the equilibrium values of all other goods and asset prices are then uniquely determined as well. Solving (2.6) for the implied value of p_1 allows us to determine how both conventional and unconventional monetary policy affect the general level of prices in period 0. Since aggregate nominal expenditure Y will vary in proportion with p_1, we can similarly determine how both policies affect aggregate demand. If we define the *expected real rate of return* on riskless nominal assets as

$$1 + r \equiv p_1(1 + i)\Big[\frac{1}{2}\frac{1}{p_{11}} + \frac{1}{2}\frac{1}{p_{21}}\Big], \tag{2.16}$$

then solving for p_1 also allows us to solve for r. We can similarly define the expected real return on the risky durable r^{dur} as

$$1 + r^{dur} \equiv \left(\frac{p_1}{p_3 - p_2}\right)\left[\frac{1}{2}\frac{p_{12}}{p_{11}} + \frac{1}{2}\frac{p_{22}}{p_{21}}\right], \tag{2.17}$$

and solving (2.6)–(2.7) for p_1 and p_3 allows us to determine how balance-sheet policy affects this rate of return as well.

2.3 Collateral Constraints and the Effects of Open-Market Operations

This more compact reformulation of the model in the two-state case provides insight into the source of the irrelevance result in Proposition 3, and into the difference that binding collateral constraints should make. A simple geometrical exposition may help to clarify the way in which central-bank asset purchases affect the set of intertemporal expenditure allocations that are possible.

Panel (a) of Figure 1 shows the feasible set of intertemporal transfers y for a given household[18] as a grey region, where y_1 is on the horizontal axis, and y_2 on the vertical axis. (Alternatively, Figure 1(a) shows the attainable vectors $\tilde{y}$ for the

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[18]We dispense with the superscript h in this discussion, as we discuss the budget constraints of a single household.
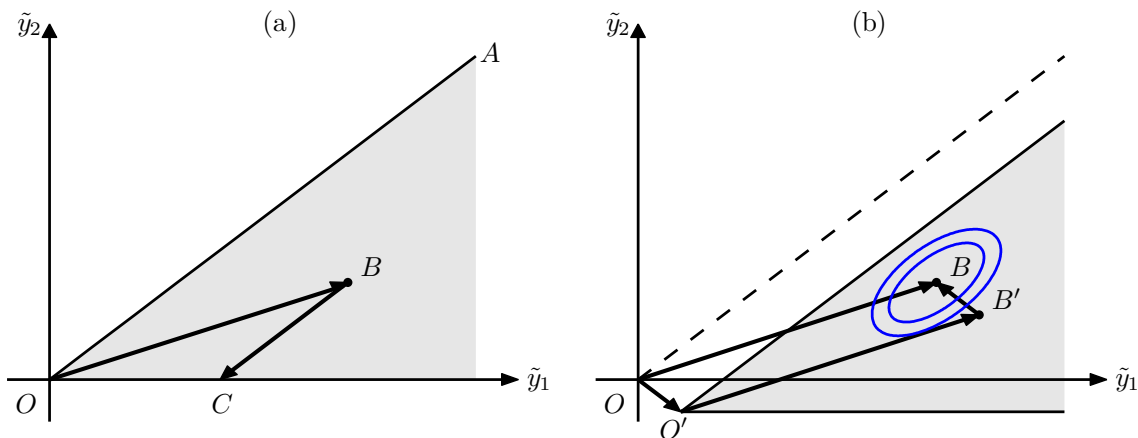
Figure 1: How central-bank purchases shift the set of feasible vectors $\tilde{y}$ of intertemporal transfers.

case of no central-bank purchases of durables, $\omega = 0$.) Ray $\overrightarrow{OA}$ represents transfers of purchasing power to period 1 that are possible by holding different amounts of riskless assets (only);[19] ray $\overrightarrow{OB}$ instead represents transfers that are possible by holding risky durables (only). (Ray $\overrightarrow{OB}$ is clockwise relative to $\overrightarrow{OA}$ under the assumption that $p_{12} > p_{22}$.) Points in the region between these two rays are attainable by holding a positive quantity of each of the two assets.

Points in the grey region below ray $\overrightarrow{OB}$ are instead attainable only by holding a positive quantity of durables and issuing riskless debt (collateralized by the durables). For example, point C can be achieved by holding a quantity of durables corresponding to vector $\overrightarrow{OB}$ and then issuing debt corresponding to vector $\overrightarrow{BC}$. The lower bound of this region is determined by the collateral constraint (2.5).

Figure 1(b) instead shows how the attainable set of vectors $\tilde{y}$ shifts as a result of central-bank durables purchases $\omega > 0$. The change in the value of $\tilde{y}$ corresponding to $y = 0$ (no holdings of any assets by the household, nor any borrowing) is shown by the vector $\overrightarrow{OO'}$, representing household h's "share" of the central bank's trades. This points outside the original grey region because the quantity of riskless liabilities issued by the central bank to finance its purchases is greater than the maximum amount that a household would be able to issue using the durables as collateral.[20]

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[19]This ray is the diagonal if $p_{11} = p_{21}$, i.e., the price level target in period 1 is independent of the state.

[20]The geometry of Figure 1 should make it clear that central-bank asset purchases can allow

24

Every value of y is mapped into a value of $\tilde{y}$ obtained by adding to y the vector $\overrightarrow{OO'}$, so the entire attainable region (again shown as the grey region) is linearly translated down and to the right. The indirect utility function $U(\tilde{\mathbf{y}}; \mathbf{a})$ is not affected by the change in ω, however. The iso-utility curves can be drawn in the plane, and remain fixed as ω varies. These iso-utility curves are shown as ellipses in the figure; in the case shown, point B represents the highest possible value of U.

Since point B is in the interior of the grey region when $\omega = 0$ (panel (a)), this is the intertemporal expenditure plan that the household will choose, achieved through the portfolio represented by vector $\overrightarrow{OB}$. When the central bank purchases durables in the amount indicated in panel (b), the attainable part of the plane shifts, but point B remains in the interior of the grey region, so the household still prefers exactly the same pattern of intertemporal expenditure (assuming no change in the state prices), and can still achieve it. However, the portfolio choice required to support this plan is no longer represented by vector $\overrightarrow{OB}$, but instead by $\overrightarrow{O'B}$. Relative to the portfolio that it would have chosen in the absence of the central-bank purchases ($\overrightarrow{OB}$, or equivalently, its parallel translation $\overrightarrow{O'B'}$, the household makes additional net trades $\overrightarrow{B'B}$, in order to achieve its desired intertemporal expenditure plan. This is the additional hedging demand created by the central bank's purchases.

Note that the change in the household's desired portfolio $\overrightarrow{B'B}$ is exactly the additive inverse of the vector $\overrightarrow{OO'}$, representing the household's share θ^h of the central bank's trades. Hence in the absence of any change in asset prices, the household chooses to undo fraction θ^h of the central bank's trades. If *each* household is in a situation like that depicted in Figure 1(b), as assumed in Proposition 3, then the aggregate additional trades of the households will exactly offset the central bank's trades, and markets will continue to clear at the same prices as before. Hence the conclusion of Proposition 3: there is no change in asset prices, no change in goods prices, and no change in the equilibrium allocation of resources.

This result depends, however, on the assumption that the collateral constraint does not restrict the household's intertemporal expenditure plan, either before or after the central bank's purchases. This need not be the case. Households might be constrained by the collateral constraint, in either of two ways, depicted in the two panels of Figure 2.

a household to achieve intertemporal allocations that would not otherwise be feasible for it *only* because the central bank is not subject to the same kind of collateral constraint as households.
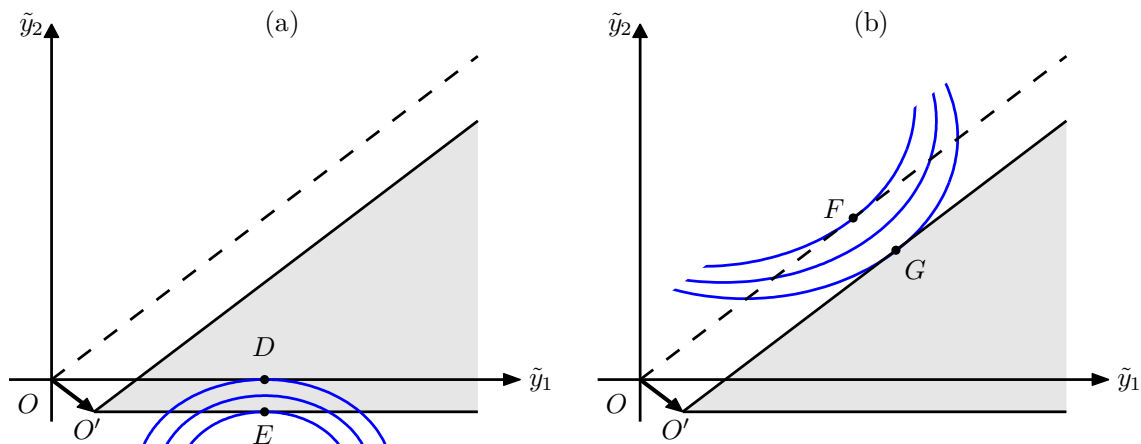
Figure 2: Two ways in which a household's collateral constraint might bind.

In the case shown in Figure 2(a), the household's preferred intertemporal transfers in the absence of central-bank purchases is shown by point D; this is not the household's unconstrained optimum, but represents the highest indifference curve that the household can reach while remaining in the grey region. Such a household would like to borrow more in order to take a longer position in the risky durable. In this case, if the central bank purchases durables, then if asset prices do not change, the attainable region shifts as shown, and the household's constrained optimum will now be point E. Effectively, the central bank borrows on the household's behalf, and so relaxes the collateral constraint for such a household.

Alternatively, a household's situation could be the one shown in Figure 2(b). In this case, the household's preferred intertemporal transfers when $\omega = 0$ are shown by point F. In this case, the household would like to short the durable, but is prevented by the collateral constraint. In this case, if the central bank purchases durables, then if asset prices do not change, the household's constrained optimum will now be point G. Once again, the household does not undo the central bank's trades, owing to the binding collateral constraint — but in this case, because it *cannot*. Effectively, the household's collateral constraint is *tightened* in this case, rather than being relaxed.

These examples illustrate how collateral constraints can invalidate the argument relied upon to establish Proposition 3. In either case, constrained households will fail to adjust their portfolios so as to offset their "share" of the central bank's trades, and may adjust their portfolios little at all; the aggregate effect, if some households

26

are constrained while others are not, will thus typically be an excess demand for the durable good and an excess supply of riskless assets, at unchanged asset prices. One should then expect the central bank's purchases to raise the equilibrium price of the asset that it purchases (the durable good), as we illustrate through both analytical and numerical examples below.

Yet even this simple partial-equilibrium discussion should indicate that the effects are more complex than common discussions of central-bank asset purchases assume. First of all, there *need not* be effects of asset purchases on asset prices; this only occurs when collateral is sufficiently scarce (relative to the degree of asymmetry in the situations of different economic agents) for collateral constraints for a sufficient number of traders. Second, even when collateral constraints bind, there are a variety of ways in which central-bank asset purchases can interact with them. The asset purchases may effectively relax the collateral constraints, as in Figure 2(a), but they might equally well tighten them further, as in Figure 2(b). And third, the mere fact that the central bank's purchases succeed in raising the price of the asset (when they do) is not necessarily informative as to whether financial constraints are eased by the policy. For both in Figure 2(a) and in Figure 2(b), the central bank's policy creates excess demand for the durable at unchanged prices, and so is likely to increase the price of the durable. But in one case the excess demand is created by *loosening* the constraint on a household's ability to hold more risk correlated with the return on the durable, while in the other case, it is created by *tightening* the constraint on a households' ability to short such risk.

It is also important to recognize that the welfare effects of the asset purchases cannot be simply read off from these partial-equilibrium diagrams. The figures show how a household's level of expected utility would change in each case *if prices were not to change,* but in the cases where collateral constraints bind, prices must change in order for markets to clear. As we show below, in some cases the welfare effects of the resulting price changes will more than offset the partial-equilibrium effects shown in Figure 2.

3 Effects of Asset Purchases When Leverage Constraints Bind

A full consideration of the effects of central-bank asset purchases requires that we go beyond the partial-equilibrium analysis presented above, and also consider the endogenous price changes that result when collateral constraints bind for at least some households. In this section, we consider such effects while restricting our attention to equilibria of a particular type: ones in which the collateral constraint of each households either binds in the way shown in Figure 2(a), or does not bind at all. We focus on the situation in which the collateral constraints bind in the way shown in Figure 2(a) — that is, in which constraint (2.13) binds rather than (2.12) — because, as shown in the figure, this is the case in which the asset purchases would increase the welfare of the constrained households in the absence of asset-price changes. The case in which the constrained households are *leveraged* households — who wish to borrow more in order to acquire even more of the risky durable, but are unable to owing to the collateral constraint — is also of particular interest because authors such as Adrian and Shin (2010) and Geanakoplos (2010) emphasize, in their models of the role of financial constraints in asset pricing, the role of variations in degree to which the "natural buyers" of risky assets are able to leverage themselves in order to acquire as much of these assets as they would like.

It is not possible, however, for constraint (2.13) to bind for everyone. For if (2.13) binds, the household chooses a portfolio that transfers no income to state 2 in period 1 ($y_2^h = 0$); such a household must issue the maximum quantity of debt allowed by the collateral requirement given its holdings of durables, and hold no riskless assets. But *everyone* cannot issue debt while no one chooses to hold such assets. (And there must be a positive aggregate capacity to issue debt, since households in aggregate must hold a positive quantity of durables, as long as $\omega < 1$.) Hence in the case of only two types, we consider equilibria in which one household is constrained, and one not. We first consider the conditions required for such an equilibrium, and then ask, when these conditions are satisfied, what the effects of increased central-bank holdings of durables will be.

3.1 Equilibrium When Only the Leverage Constraint Binds

We first note some general properties of collateral-constrained equilibria in which only constraint (2.13) binds (on some households), while constraint (2.12) binds for no one. These results do not depend on the restriction to an economy with only two household types, though they do rely on the special form of preferences (1.1)–(1.2). Note that (1.2) implies that the indirect utility functions used in (2.3) are such that

$$\tilde{u}'(c) = \alpha c^{-\gamma}, \quad \tilde{u}'_s(c) = \alpha_s c^{-\gamma} \tag{3.1}$$

for some coefficients $\alpha, \alpha_1, \alpha_2 > 0$ that depend on the relative supplies of durables and non-durables in the different states.

In the case that no households are short-sale constrained, we can establish the following.

Lemma 4 *Consider a two-state economy with homothetic preferences. If an equilibrium exists in which constraint (2.12) does not bind for any household, then the equilibrium value of state price a_1 must equal*

$$\overline{a}_1 = \frac{1}{2}\left(\frac{\alpha_1}{\alpha}\right)\left[\frac{e_1 + (p_2/p_1)e_3}{e_{11} + (p_{12}/p_{11})e_3}\right]^\gamma,$$

where $e_1 \equiv \sum_h e_1^h, e_{11} \equiv \sum_h e_{11}^h$. Thus the state price a_1 will be unaffected by policy (either conventional or unconventional monetary policy), to the extent that the variation in policy does not change the fact that constraints (2.12) do not bind.

This simple result is already enough to allow us to establish some useful conclusions about the possible effects of monetary policy on asset prices. Policy can influence real asset prices and real rates of return only insofar as it changes the equilibrium value of a_2, keeping a_1 fixed at the value indicated in the above lemma. This gives us a one-parameter family of possible equilibrium outcomes.

Proposition 5 *In an economy with homothetic preferences and two states in period 1, suppose that for any policy in some set under consideration, an equilibrium exists in which constraint (2.12) does not bind for any household, though constraint (2.13) may bind for some. Suppose also that the period 1 price-level commitments $\{p_{s1}\}_{s \in \mathcal{S}}$ are the same for all policies in the set. Then if any policy change (whether in interest-rate policy or in the central bank's asset purchases) raises (lowers) the real price of the*

durable p_3/p_1 in period 0 must also lower (raise) the expected real return on riskless debt r; and while it also lowers (raises) the expected real return r^{dur} on the durable, it increases (decreases) the spread

$$\hat{r}^{dur} - \hat{r} \equiv \log \frac{1 + r^{dur}}{1 + r}.$$

Suppose further that only the central bank's asset-purchase policy is changed, while the interest-rate target i remains fixed. Then a policy that raises (lowers) the real price of the durable in period 0 must lower (raise) the general price level in period 0 (i.e., the money prices of both non-durables and rental of the services of durables). More-over, the general price level must fall (rise) by a greater amount, in percentage terms, than the increase (decrease) in the real price of durables, so that the nominal price of the durable good in period 0 must also fall (rise). Thus an asset-purchase policy that increases (decreases) the nominal price of the durable in period 0 must increase (decrease) the equilibrium real return r on riskless nominal debt, reduce (increase) the size of the spread $\hat{r}^{dur} - \hat{r}$ between the expected real returns on durables and those on riskless debt, and increase (decrease) aggregate nominal expenditure on goods and services, resulting (in our flexible-price endowment economy) in an increase (decrease) in the general level of prices.

Thus to the extent that an asset-purchase policy is able to raise the nominal price of the asset purchased by the central bank, consequences necessarily follow for both the equilibrium real returns on other assets, and for aggregate nominal spending. This suggests that the concern of central banks with policies intended to raise the prices of particular assets, as a way of influencing macroeconomic conditions more generally, is not misguided. However, it is worth noting that the effects allowed by Proposition 5 are rather different than those implied by the "portfolio balance" theory typically relied upon by central banks as a theory of these policies.

According to the "portfolio balance" theory, the central bank's purchase of assets that are more exposed to a particular type of risk than are assets in general — in this case, the risk of a low return in state 2, the state in which the return on durables is relatively low compared to that on riskless nominal debt, and hence to that on the economy's aggregate portfolio as well — should lower the market risk premium associated with that type of risk, and hence lower the risk premium for holding the type of assets purchased by the central bank. It is generally supposed that this

30

reduction in the risk premium should also reduce the expected real return on the risky asset purchased by the central bank, since there is less reason for the riskless real rate to be influenced by the purchase of risky assets; and it is this reduction in the expected real return on risky assets that is relied upon to increase aggregate demand.

It remains to be analyzed whether asset purchases by the central bank should indeed reduce the risk premium associated with the assets purchased; below, we give conditions under which this will be true, though they are not as general as might be expected. But even granting that they do, it is already evident from Proposition 5 that the conventional story does not match what happens in our model. An asset-purchase policy that reduces the spread $\hat{r}^{dur} - \hat{r}$ would have to reduce a_2; such a policy would indeed reduce aggregate nominal expenditure, according to the proposition, but it would be associated with an *increase* rather than a decrease in the expected real return r^{dur} on the risky asset, and a decrease rather than an increase in the asset's real price. Thus the conventional account would not be correct, either about the implications of the reduction in the spread for the expected real return on the risky asset purchased by the central bank, or about the role of this return in explaining the effects on aggregate demand.

In order to consider how central-bank asset purchases should affect a_2 (and hence the asset prices and returns just discussed), it is useful to further simplify our definition of equilibrium for the special case under consideration. We can write each household's intertemporal allocation of expenditure problem as one with two stages: first, the optimal division of total lifetime expenditure between the part c_2^h allocated to state 2 of period 1 on the one hand, and another part,

$$c_{01}^h \equiv c^h + \overline{a}_1 c_1^h$$

indicating the total present value of expenditure allocated to period 0 and to state 1 of period 1; and second, the optimal allocation of the quantity c_{01}^h between the two dates. The latter problem is (by hypothesis) unaffected by the collateral constraint, and depends only the state price a_1 which is independent of policy. A household's optimal expenditure in state 1 of period 1 will be given by $c_1^h = \chi c_{01}^h$, where

$$\chi \equiv \frac{\sum_h k_1^h}{\sum_h e^h + \overline{a}_1 \sum_h k_1^h} > 0,$$

introducing the notation $k_s^h = f_s^h + g_s^h$, for all h and s. We can similarly write c^h as a fixed fraction of c_{01}^h, and express the maximized value of

$$\tilde{u}(c^h) + \frac{1}{2}\tilde{u}_1(c_1^h)$$

as $(1/2)\tilde{u}_{01}(c_{01}^h)$, where the indirect utility function is of the form $\tilde{u}'(c) = \alpha_{01}c^{-\gamma}$ where α_{01} is another positive constant.

We can then write the first sub-problem as the choice of (c_{01}^h, c_2^h) to maximize

$$U^h = \frac{1}{2}\left[\tilde{u}_{01}(c_{01}^h) + \tilde{u}_2(c_2^h)\right], \tag{3.2}$$

subject to the constraints

$$c_{01}^h + \bar{a}_2 c_2^h \leq e^h + \bar{a}_1 k_1^h + \bar{a}_2 k_2^h, \tag{3.3}$$

$$c_2^h \geq g_2^h - \theta^h \phi(\bar{a}_2)\omega e_3. \tag{3.4}$$

Here (3.4) is an alternative expression of the leverage constraint (2.13); $\phi(a_2)$ is simply the function $\phi(a)$ defined earlier, in which the value $\bar{a}_1$ defined in Lemma 4 has been substituted for a_1. The solution to this two-stage problem will be one in which the short-sale constraint (2.12) does not bind if and only if the solution (c_{01}^h, c_2^h) to the first-stage problem is one for which

$$p_{21}c_2^h \leq \chi p_{11}c_{01}^h + (p_{21}e_{21}^h - p_{11}e_{11}^h) - \theta^h(p_{12} - p_{22})\omega e_3. \tag{3.5}$$

We can then state necessary and sufficient conditions for an equilibrium in which constraint (2.12) binds for no households.

Definition 3 *A state price $\bar{a}_2$ and intertemporal expenditure plans $(\bar{c}_{01}^h, \bar{c}_2^h)$ for each of the $h \in \mathcal{H}$ describe an equilibrium in which the short-sale constraint (2.12) binds for no households if*

(i) *for each $h \in \mathcal{H}$, the plan $(\bar{c}_{01}^h, \bar{c}_2^h)$ maximizes the function U^h defined in (3.2), subject to the constraints (3.3)–(3.4);*

(ii) *markets clear in state 2, so that*

$$\sum_{h=1}^{H} c_2^h = \sum_{h=1}^{H} k_2^h; \tag{3.6}$$

and

32

(iii) inequality (3.5) is satisfied for each $h \in \mathcal{H}$.

We need not add a corresponding market-clearing relation for aggregate expenditure in the initial period and in state 1, as this is guaranteed by condition (ii) and Walras' Law. We shall also say that we have an *equilibrium neglecting short-sale constraints* if conditions (i)–(ii) are satisfied (but possibly not condition (iii)). This modified equilibrium concept has the advantage of being more easily characterized, as shown below.

3.2 Effects of Asset Purchases with One Constrained Household

Explicit calculations of the effects of central-bank asset purchases are especially simple if we further restrict ourselves to the case of an economy made up of households of only two types ($h = 1, 2$), assumed to exist in equal numbers.[21] In the case of only two households, the possible equilibrium allocations of expenditure, in any equilibria of the kind defined in Definition 3 can be represented using an Edgeworth Box diagram. In Figure 3, the allocation between the two households of expenditure in the initial period and in state 1 is indicated on the horizontal axis: movement to the right indicates an increasing value of c_{01}^1, and a corresponding decreasing value of c_{01}^2, since in any feasible allocation these must sum to $\sum_h e^h + \bar{a}_1 \sum_h k_1^h$, a quantity independent of policy. Similarly, the allocation between the two households of expenditure in state 2 of period 1 is indicated on the vertical axis: movement upward indicates an increasing value of c_2^1, and a corresponding decreasing value of c_2^2, since these must sum to $\sum_h k_2^h$, a quantity that is also independent of policy.

The preferences of each household can be depicted by indifference curves in the plane, representing the level curves of the indirect utility function U^h defined in (3.2). In the figure, the indifference curves of household 1 are the ones that are

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[21]Note that there is no loss of generality in assuming that the number of households of the two types are equal since, in the case of homothetic preferences, the only thing that matters for equilibrium is the *share* of the aggregate endowment of each good that is controlled by households of a given type, and not the *number* of households among whom the endowment is divided. Thus when we refer to parameters such as e_3^1/e_3^2, they should be understood to specify the relative quantities owned by households of the two types in aggregate, and not the relative size of the endowments of individuals.
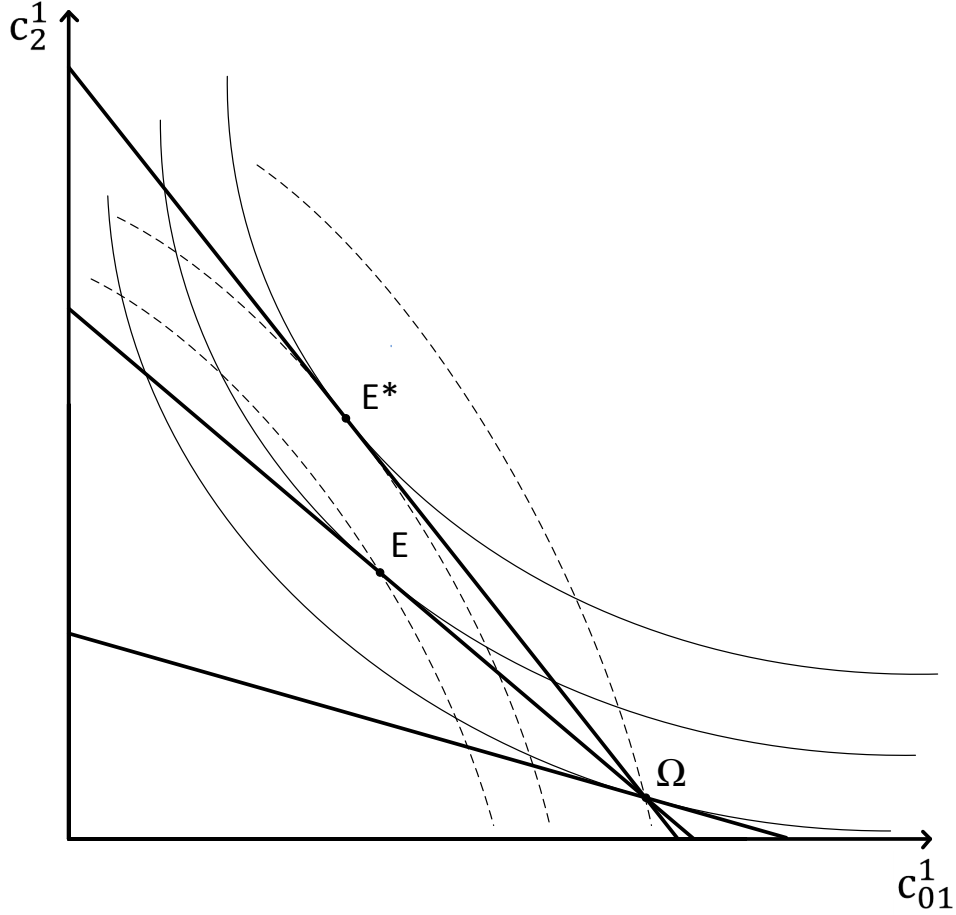
Figure 3: Possible equilibria in the case of two households and two states, shown in an Edgeworth Box diagram. The equilibria at Ω, E and E^* correspond to differing degrees of tightness of the leverage constraint of household 2.

concave upward (solid curves), and indifference curves that are higher and farther to the right represent higher expected utility for this household. The indifference curves of household 2 are the ones that are concave downward (dashed curves), and indifference curves that are lower and farther to the left represent higher expected utility for household 2.

Equilibria neglecting short-sale constraints can be characterized using this diagram. The budget constraint (3.3) corresponds to a straight line with slope $-1/\overline{a}_2$, passing through the endowment point Ω, which represents the allocation $c_{01}^h = e^h + \overline{a}_1 k_1^h, c_2^h = k_2^h$ for each household. The location of this point is unaffected by

changes in $\bar{a}_2$ or the central bank's balance sheet. A household for which the leverage constraint (3.4) does not bind must choose an expenditure plan on this line where the line is tangent to one of its indifference curves. A household for which the leverage constraint binds, instead, must choose the point on the line which reaches the highest indifference curve that is attainable given the lower bound on c_2^h implied by the leverage constraint. (Three possible budget lines, corresponding to different values of $\bar{a}_2$, are shown in the figure.)

An equilibrium in which the leverage constraint binds for neither type must correspond to point E^* in Figure 3, as this is the unique point with the property that (i) the indifference curves of the two types are tangent to each other at this point, and (ii) the common tangent line to the two indifference curves passes through point Ω. (It corresponds to the A-D equilibrium of this economy, which can easily be shown to be unique given our assumption of homothetic preferences.) Point E^* will represent an equilibrium neglecting short-sale constraints if when $\bar{a}_2$ has the value implied by the slope of the budget line through point E^*, constraints (3.4) are satisfied for both households. Specifically, the lower bound for c_2^1 must correspond to a vertical height not higher than point E^*, while the lower bound for c_2^2 must correspond to a vertical height not lower than point E^*. Since these constraints depend on the value of ω, this condition may be satisfied for some values of ω but not for others.

In the generic case (illustrated in the figure), the A-D equilibrium will not coincide with the endowment point Ω. If so, the slopes of the indifference curves of the two types through point Ω will be unequal; moreover, one must have an indifference curve steeper than $\overrightarrow{\Omega E^*}$, and the other an indifference curve that is flatter. Without loss of generality, let us suppose that

$$\frac{k_2^1}{e^1 + \bar{a}_1 k_1^1} < \frac{k_2^2}{e^2 + \bar{a}_1 k_1^2}, \tag{3.7}$$

so that household 1 has the flatter indifference curve through point Ω, as shown in the figure.

An equilibrium in which the leverage constraint binds for household 2 only is illustrated by point E in Figure 3. At this point, household 1's indifference curve is tangent to the budget line passing through the point, so this represents an allocation that household 1 would choose (if $\bar{a}_2$ takes the value implied by the slope of the budget line) if not constrained by its leverage constraint. Household 2, instead, would prefer to move up and to the left on the budget line, as it could reach higher indifference

curves in that case. However, point E can be a constrained optimum for household 2, if its leverage constraint requires c_2^2 to be no lower than the value corresponding to point E. Thus point E represents an equilibrium neglecting short-sale constraints, if the leverage constraints imply a lower bound for c_2^1 somewhere below the value corresponding to the vertical height of point E, while they imply a lower bound for c_2^2 exactly equal to the corresponding to the vertical height of E. Since the heights of these lower bounds depend on ω, there will be at most one precise value of ω for which point E will be an equilibrium. The value of ω that is required is the unique value that causes (3.4) to hold with equality for household 2 at the allocation represented by point E.

All equilibria of this kind must therefore correspond to points (like E) that lie on the "offer curve" of household 1, the set of pairs (c_{01}^1, c_2^1) that maximize U^1 subject to the budget constraint (3.3), for some value of the state price $a_2 > 0$. This curve passes through points Ω, E and E^* (in the case shown in the figure) as the value of a_2 is progressively reduced (tilting the slope of the budget line passing through Ω). Each point on the offer curve between Ω and E^* corresponds to a different lower bound for c_2^2 that would be required to support this allocation as an equilibrium neglecting short-sale constraints, and hence to a different value of ω that causes (3.4) to hold with equality for household 2 at that point. Thus the diagram illustrates the way in which a changing quantity of risky durables on the balance sheet of the central bank can sweep out a one-parameter family of alternative equilibrium allocations, corresponding to points on the offer curve of household 1.

Standard results on the properties of offer curves then allow us to establish several properties of this family of possible equilibrium allocations. For any value of a_2, let $c_2^1(a_2)$ denote household 1's desired level of expenditure in state 2 in the case of state price a_2. The value of a_2 for which the offer curve passes through the A-D equilibrium (point E^*) is given by

$$a_2^* = \frac{\alpha_2}{\alpha_{01}} \left(\frac{\sum_h e^h + \bar{a}_1 \sum_h k_1^h}{\sum_h k_2^h} \right)^{\gamma};$$

thus $-1/a_2^*$ is the slope of the line $\overrightarrow{\Omega E^*}$ in the figure. Similarly, the value of a_2 for which household 1's offer curve passes through the endowment point Ω is given by

$$a_2^{**} = \frac{\alpha_2}{\alpha_{01}} \left(\frac{e^1 + \bar{a}_1 k_1^1}{k_2^1} \right)^{\gamma} > a_2^*.$$

36

(The budget line with this slope is also shown in the figure, as the flattest of the three dark straight lines passing through point Ω.) Moreover, $c_2^{1*} \equiv \hat{c}_2^1(a_2^*)$, household 1's expenditure in state 2 in the A-D equilibrium, will necessarily be greater than k_2^1, as also shown in the figure. We can then establish the following general result about equilibria neglecting short-sale constraints.

Proposition 6 *Consider a two-state model with two types of households with homothetic preferences. Suppose that the endowment allocation is not Pareto optimal, and let household 1 be identified by the inequality (3.7). Let the value of $i \geq 0$ be fixed, but consider alternative possible balance-sheet policies.*

Then for any value of c_2^1 in the interval

$$k_2^1 \leq c_2^1 \leq c_2^{1*}, \tag{3.8}$$

there is a unique value of a_2 in the interval $a_2^* \leq a_2 \leq a_2^{**}$ such that $\hat{c}_2^1(a_2) = c_2^1$. If in addition these values (c_2^1, a_2) satisfy the bounds

$$\underline{c} \leq c_2^1 < \underline{c} + \theta^2 \phi(a_2) e_3, \tag{3.9}$$

where

$$\underline{c} \equiv \left(\frac{p_{22}}{p_{21}}\right) e_3 - \theta^2 \left(\frac{d}{p_{21}}\right) + e_{21}^1,$$

then there exists an asset-purchase policy $0 \leq \omega < 1$ for which the point on household 1's offer curve corresponding to the values (c_2^1, a_2) represents an equilibrium neglecting short-sale constraints, in which household 2's borrowing is constrained by the leverage constraint (3.4), except in the limiting case in which $c_2^1 = c_2^{1*}$, but household 1 is unconstrained.

In the case of any $c_2^1 < c_2^{1*}$, the unique value of ω consistent with this equilibrium is

$$\omega = \hat{\omega}(c_2^1) \equiv \frac{c_2^1 - \underline{c}}{\theta^2 \phi(a_2)} e_3, \tag{3.10}$$

while for the case $c_2^1 = c_2^{1*}$, any value of ω in the interval $[\hat{\omega}(c_2^{1*}), 1)$ is consistent with the equilibrium. The value of $\bar{a}_2$ associated with each of these possible equilibria is a monotonically decreasing function of c_2^1. Moreover, a higher value of c_2^1 is associated with a lower value of the real price p_3/p_1 for the durable, with the consequences for other asset prices and rates of return stated in Proposition 5.

37

This result establishes conditions under which there will exist a continuum of distinct real allocations of resources, each of which corresponds to an equilibrium neglecting short-sale constraints under an appropriate choice of ω. (Only one of these, however, corresponds to an equilibrium in which the leverage constraints do not bind for either household; thus the possibility of obtaining different real allocations and different equilibrium asset prices through variation in the central bank's asset purchases depends on the fact that the leverage constraint binds for household 2.) These will also correspond to distinct possible equilibria of the model with collateral constraints, as long as the additional inequality constraints (3.5) do not bind. This must be checked in addition to the conditions stated in Proposition 6; but since these are inequalities, it is possible for a non-empty interval of values of c_2^1 to satisfy both of them, as we verify through a numerical example below.

Under somewhat stronger assumptions, we can sign the relationship between the change in the central bank's balance sheet and the changes in the endogenous variables that are related to one another in Proposition 6.

Lemma 5 *If preferences are of the form (1.1)–(1.2) with $\gamma \leq 1$, then the value of ω defined by (3.10) is an increasing function of c_2^1.*

In this case, over the range of values for c_2^1 satisfying the hypotheses of Proposition 6, increases in central-bank holdings of the durable are associated with relaxations of the leverage constraint of the constrained household (household 2) — i.e., a reduction of the lower bound for c_2^2 — as in the partial-equilibrium analysis shown in Figure 2(a). Hence increasing ω results in a movement up the offer curve (away from the endowment point Ω and toward the Pareto-optimal equilibrium E^*), which must be associated with a decrease in $\bar{a}_2$.

In such a case, we can give a clear answer to our questions about the effects of central-bank purchases on both asset prices and goods prices. If ω is increased while i is held constant, then — over the range of variation in ω for which an equilibrium exists in which constraints (3.5) do not bind — $\bar{a}_2$ must fall. Proposition 5 then implies that the real price of the durable p_3/p_1 falls, while its nominal price p_3 rises; that the expected return r^{dur} rises, along with the expected return r on riskless debt, but that the spread $\hat{r}^{dur} - \hat{r}$ decreased; and that the money prices of goods and services in period 0 increase, so that aggregate nominal expenditure in period 0 also increases.

3.3 Welfare Consequences of Asset Purchases

We have shown in the previous section that under certain conditions, central-bank purchases of the durable have a variety of effects on real and nominal variables. This means that this dimension of policy is not irrelevant, under circumstances where the leverage constraints of some economic agents bind in equilibrium. Moreover, our results show that the effects of asset-purchase policy are distinct from those of interest-rate policy. According to Proposition 1, changes in i have no effect on any real variables or relative prices, and only change the general level of prices in period 0. Our results above show, instead, that under certain conditions, central-bank asset purchases change a variety of relative prices and real rates of return, in addition to their effects on the nominal prices of goods and services in period 0.

But in judging how best to use this additional dimension of policy, it is important to consider not merely whether asset prices are affected, by how these price changes affect the welfare of economic agents. In fact, the mere fact that central-bank purchases of the durable can loosen a household's leverage constraint does not always imply that the household benefits from such a policy. Consider the shift from equilibrium E to equilibrium E^* in Figure 3, which results from an increase in central-bank holdings of the durable (under the assumption made in Lemma 5), that reduces the lower bound on c_2^2 for household 2. In this example, household 2 is the one whose collateral constraint binds in equilibrium, and the constraint is relaxed — indeed, it ceases to bind, if purchases are sufficient to shift the equilibrium all the way to point E^*. In the absence of any price changes, the situation of household 2 would be the one depicted in Figure 2(a), and the household would clearly benefit. But in fact, in the case shown in Figure 3, the expected utility of household 2 is *reduced* by the policy. This results from the adverse effect on household 2 of the price changes resulting from the policy: these leveraged investors suffer an income loss when the real market price of the debt that they issue falls by more than does the real market price of the risky assets that they purchase, and this loss more than offsets the gain from relaxation of the leverage constraint.

On the other hand, household 1 benefits from the policy change, even though household 1's collateral constraint does not bind. The income effect of the price changes is positive for household 1, for the same reason that it is negative for household 2. One's conclusion about the desirability of the policy change will therefore

39

depend on the relative weight placed on the welfare of households in the two situations.

In fact, the effects of central-bank asset purchases on the welfare of the constrained household depend on how sharply this household is constrained by its leverage constraint; that is, on how close the equilibrium allocation is to the A-D allocation (the allocation in the limiting case in which the leverage constraint no longer binds).

Proposition 7 *In any equilibrium neglecting short-sale constraints of the kind described in Proposition 6, the expected utility of household h is given by $\hat{U}^h(c_2^1)$, the value of the function U^h defined in (3.2) evaluated at the point in the Edgeworth Box that is the unique point on the offer curve of household 1 with this value of c_2^1. The function $\hat{U}^1(c_2^1)$ is a monotonically increasing function of c_2^1 over the entire range (3.8); thus if asset purchases by the central bank relax the leverage constraint of household 2 (as under the hypothesis of Lemma 5), raising the equilibrium value of c_2^1, they necessarily increase the welfare of household 1. The function $\hat{U}^2(c_2^1)$, instead, is non-monotonic. In particular, it is necessarily monotonically increasing for values of c_2^1 close enough to k_2^1, but monotonically decreasing for values of c_2^1 close enough to c_2^{1*}. Over the entire range (3.8), it is on average increasing, since $\hat{U}^2(c_2^{1*}) > \hat{U}^2(k_2^1)$.*

This result shows that the fact that in moving from equilibrium E to equilibrium E^* in Figure 3, the result that household 2 is harmed by the policy that relaxes its leverage constraint is no error in the drafting of the figure; this is necessarily the case if the asset-purchase policy moves the economy to the A-D equilibrium E^* from any sufficiently nearby equilibrium E in which household 2's leverage constraint binds. However, the proposition also implies that it is possible for an increase in the central bank's holdings of the durable to increase the welfare of *both* types of households. This possibility is illustrated by a movement from equilibrium Ω to equilibrium E in Figure 3. Note that point Ω is also a potential leverage-constrained equilibrium, corresponding to the case in which household 2's leverage constraint requires c_2^2 to be at least as large as k_2^2.

Central-bank purchases of the durable can move the equilibrium from point Ω to point E, again by reducing the lower bound for c_2^2 implied by household 2's leverage constraint (3.4), though not by enough for household 2's leverage constraint to cease to bind. In the case shown in the figure, equilibrium E is strictly preferred by both households to the original equilibrium Ω; hence there would be a clear benefit

from central-bank asset purchases in this case. This result depends on the indifference curves of household 2 being a good deal steeper than those of household 1, in both equilibria; in other words, the Lagrange multiplier associated with the leverage constraint for household 2 is substantial in the situation depicted.

Figure 3 only illustrates the possibility of a Pareto improvement to the extent that the equilibria neglecting short-sale constraints shown in the figure are actually equilibria of the model with collateral constraints; that is, that the short-sale constraints (3.5) are satisfied for both households in the allocations corresponding to both points Ω and E. We show through numerical examples in the next section that Pareto improvements of this kind can indeed occur.

4 Distortions Resulting from Central-Bank Monopolization of Collateral

In the previous section, we have emphasized the possibility of equilibria in which the collateral constraints bind for some households in the way shown in Figure 2(a) — what we have called a binding leverage constraint — rather than binding in the way shown in Figure 2(b), the case of a binding short-sale constraint. This does not mean, however, that the short-sale constraint cannot also be relevant in equilibrium; our numerical examples below show that either or both of the two types of constraints may bind, depending on parameter values. Indeed, it is worth remarking that sufficiently *large* asset purchases by the central bank will almost certainly create a situation in which many households are constrained in the way shown in Figure 2(b).

When ω approaches 1, so that most of the durable is held by the central bank, equilibrium will necessarily involve many households holding more riskless assets than durables, so that they will be at a position not far from the upper boundary of the grey region shown in Figures 1 and 2. (Recall that the upper boundary corresponds to portfolios made up solely of riskless assets.) Equilibrium will require asset prices that lead households to choose points in that region; and assuming some degree of heterogeneity in the endowment patters of the different households, it will almost certainly be the case that many households are driven entirely to the boundary (so that they would like to short the durable, at the equilibrium prices, but are unable to), while the (now very expensive) durable is held only by those households with

41

the greatest desire to shift more income into state 1 than into state 2. Hence except in very special cases (such as the one assumed in Proposition 4), as $\omega \to 1$, one will eventually have an equilibrium in which the short-sale constraint binds for many households, while none may be constrained in the way shown in Figure 2(a).

This will mean that while the central bank will still be able to further increase the price of the durable by purchasing more of it, these effects will surely be achieved by tightening traders' financial constraints, rather than relaxing them. Moreover, this tightening of financial constraints will necessarily reduce welfare for many (though not necessarily all) households. If nearly all collateral is held by the central bank, risk-sharing between households ceases to be possible, as does borrowing; households can only obtain an expenditure pattern different from that determined by their endowments by accumulating riskless assets. In addition to preventing mutually beneficial trades, the fact that the policy raises the price of the durable good redistributes period 0 income from households with shares of the aggregate endowment of durables less than θ^h to households with shares greater than θ^h. The latter benefit from this redistribution, but the former are hurt. Thus central-bank asset purchases on too large a scale will necessarily have significant costs, owing to the impairment of the functioning of financial markets that predictably results from an induced scarcity of collateral.

We illustrate this point with two numerical examples. In each of the examples, there are two states in period 1, and the economy has two types of household $h = 1, 2$, each with a utility function of the form

$$u^h(x) = \sum_{l=1}^{2} \log(x_l) + \frac{1}{2} \sum_{s=1}^{2} \sum_{l=1}^{2} \log(x_{sl}). \qquad (4.1)$$

Note that preferences of this form are an example of the general form (1.1)–(1.2) assumed above, corresponding to the value $\gamma = 1$. This the stronger preference hypothesis of Lemma 5 is also satisfied, so that all of our analytical results above apply to the examples considered here (and in the Appendix).

In our first example, we assume that both households have equal endowments of both the non-durable and durable goods in period 0, and that initial endowments of government debt (that are in any event very small) and tax shares are equal as well.[22] Thus the two household types differ only with regard to the distribution

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[22]See the discussion of "Example 1" in the Appendix for further details of the numerical speci-

of their endowments of the non-durable good in period 1. To make this difference especially stark, we suppose that in state 1, only household 1 has a positive non-durable endowment, while in state 2, only household 2 has a positive non-durable endowment.[23]

We assume a period-1 monetary policy commitment to achieve the same inflation rate regardless of the state, so that $p_{11} = p_{21}$; hence the riskless nominal contracts are also riskless in real terms (in units of the non-durable good). We assume instead that the aggregate non-durable good endowment in state 1 is 15/7 times the aggregate endowment of the durable good, while in state 2 it is only 6/7 times the durable endowment; this implies that $p_{12}/p_{11} = 15/7$ in state 1, while $p_{22}/p_{21} = 6/7$ in state 2. Thus the nominal value of the durable in state 2 is only 40 percent of its value in state 1. The collateral requirement for debt that defaults in state 2 but not in state 1 is accordingly $C_1 = 7/15$, while the collateral requirement for riskless debt is $C_2 = 7/6$.

In Figure 4, we consider how equilibrium varies as the value of ω varies from zero to 1, fixing the endowment patters as above.[24] Panel (a) shows how p_1 and p_3 vary as ω increases; the graphs show the amount by which the log of each price changes, relative to the equilibrium prices when the central bank holds none of the durable. (Thus the vertical distance between the two curves also shows how the log of the relative price p_3/p_1 changes.) Panel (b) shows how the expected utilities of the two household types vary as a result of changes in the allocation of risk; here the expected utility of each is measured relative to its expected utility when the central bank holds none of the durable.

In this numerical example, the short-sale constraint of household 1 binds for all values of ω, but no other collateral constraints bind. (Household 1 would like to short the durable and instead hold a long position in the riskless asset, because its endowment risk is positively correlated with the return on the durable.[25]) As illustrated by

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fication. As shown by Lemma 7 in the Appendix, our qualitative conclusions about the effects of central-bank purchases depend only on the households' shares of the aggregate endowment, and not on the size of the aggregate endowments in each state.

[23]The implications of alternative assumptions about the two households' relative endowments in the two period-1 states are treated in the Appendix.

[24]We use the algorithm described in Schommer (2013) to numerically solve for the collateral-constrained equilibrium associated with each possible parameter configuration.

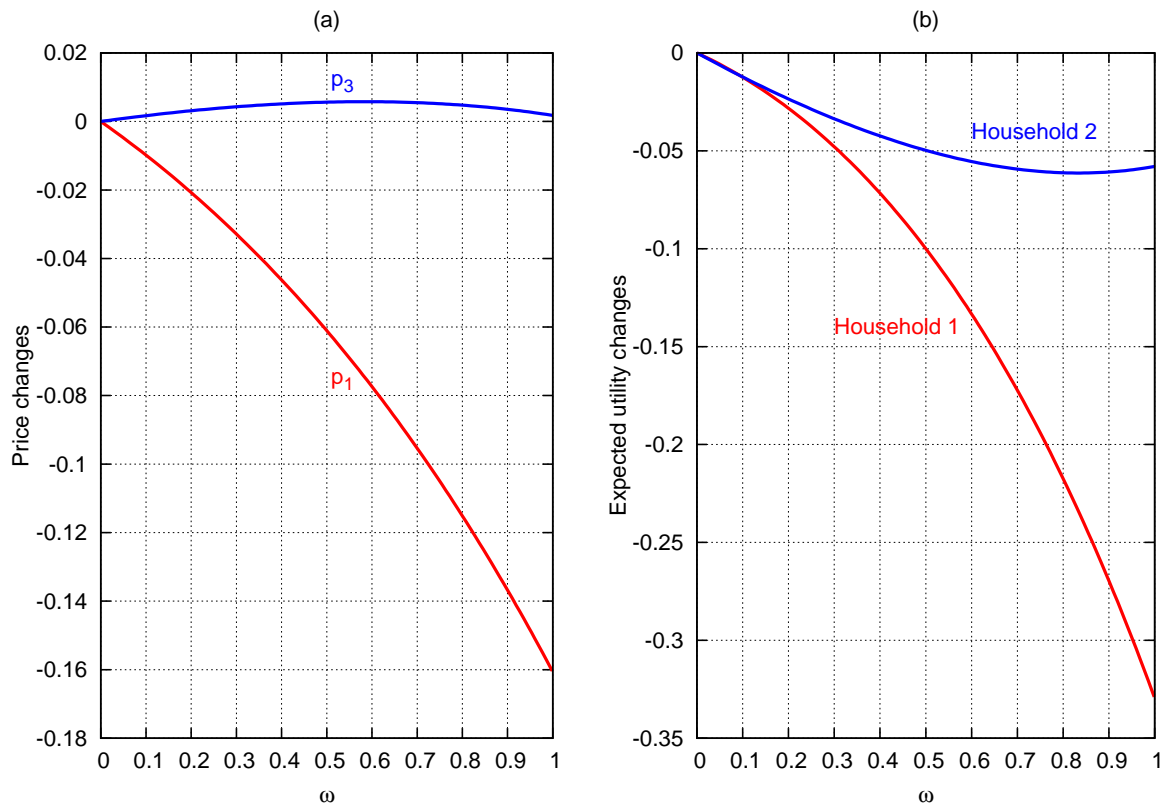[25]See further discussion of this example in the Appendix.

Figure 4: Effects of variation in ω for Example 1: (a) effects on prices; (b) welfare consequences.

Figure 2(b), it follows that central-bank purchases of the durable will further tighten the short-sale constraints of households of type 1, rather than loosening financial conditions. In this example, further central-bank purchases of the durable also progressively *reduce* aggregate nominal spending, as shown by the monotonic decline in p_1 in panel (a). Nonetheless, the price p_3 of the durable is *increased*, until central-bank holdings of the durable reach nearly 60 percent of the total supply. Thus the mere fact that asset purchases raise the price of the asset is not sufficient to imply that such purchases increase aggregate demand.

Nor does it suffice for one to conclude that welfare is increased. As shown in panel (b) of Figure 4, the welfare of household 1 is monotonically decreasing in ω in this example, over the entire feasible range of values. The welfare of household 2 is also decreasing as ω increases, until the central bank already holds more than 80 percent of the total supply, though for very high values of ω, further asset purchases raise these

households' level of expected utility somewhat. Thus everyone's welfare is reduced, in all of the cases in which the policy raises the price of the durable; however, at least some can benefit, when the policy *reduces* the asset price to a sufficient extent. Even in this last case, however, the reduction in the welfare of type 1 as ω increases is greater than the increase in the welfare of type 2; and the equilibria with very high values of ω are in any event Pareto dominated by those with low values of ω (even if they are not Pareto dominated by the equilibria associated with only *slightly* lower values of ω). Hence in this example, central-bank asset purchases are clearly undesirable.

In the case of alternative endowment patterns, some households may instead be "natural buyers" of the risky asset who are constrained (by the collateral requirement) in their ability to make as large a leveraged position in this asset as they would otherwise wish, as in Figure 2(a). Consider an example in which aggregate endowments are the same as in the previous example, and period-0 endowments of the non-durable good are again equal for the two types; but suppose now that only households of type 2 are initially endowed with the durable. Let us also assume now that in state 1, only household 2 has a positive non-durable endowment, while the non-durable endowments are equal for the two types in state 2. We also now assume asymmetric tax shares: $\theta^1 = 0.9, \theta^2 = 0.1$.[26]

The fact that household 1 has a period-1 endowment only in state 2 means that holding the durable will allow this type of household to hedge its endowment risk, so that these households become "natural buyers" of the asset. The fact they have no initial endowment of the durable, and only a tiny initial endowment of government debt, means furthermore that they need to borrow in order to finance purchases of the risky asset. Under our parametric assumptions, if we start from a situation in which the central bank holds sufficiently little of the durable, households of type 1 wish to purchase so much of the durable that their leverage constraint binds. No other constraint binds, and so we have a leverage-constrained equilibrium of the kind characterized in section 3. Central-bank purchases of the durable then relax the leverage constraint of the type 1 households, with the consequences described
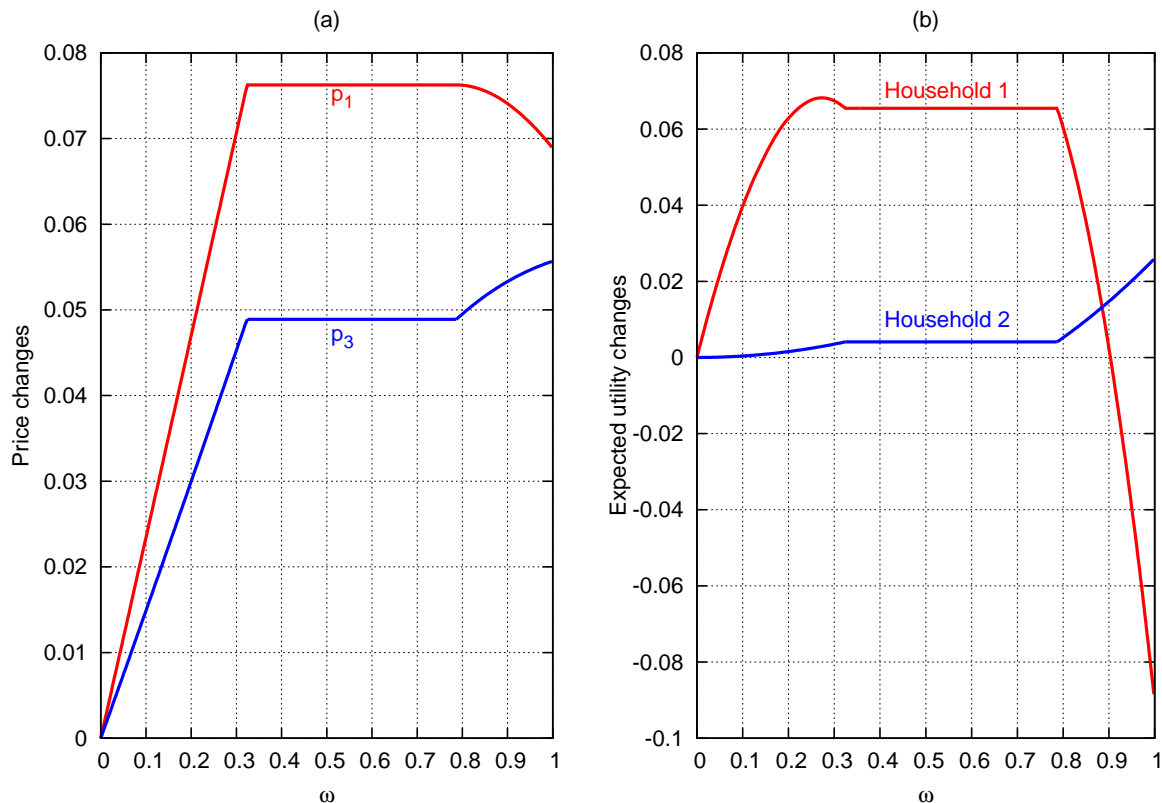
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Figure 5: Effects of variation in ω for Example 2: (a) effects on prices; (b) welfare consequences.

earlier.[27]

The effects of variation in ω over its entire feasible range are shown in Figure 5, using the same format as Figure 4. As long as the central bank owns less than 32 percent of the total supply of the durable, the leverage constraint for household 1 continues to bind, and no other financial constraints bind.[28] Over this range, in accordance with Proposition 6), p_1 and p_3 both increase with further central-bank purchases, though the relative price p_3/p_1 falls, as shown in Figure 5(a) by the fact that $\ln p_3$ increases less steeply than does $\ln p_1$.

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[26] The numerical assumptions made are described in greater detail in the Appendix.

[27] The larger tax share for type 1 in this example amplifies the effects of central-bank asset pur-chases on the collateral constraints of this type; recall that the shifts in the feasible regions shown in Figure 2 are proportional to θ^h.

[28] The dependence of this result on the exact nature of the second-period endowment pattern is explored in the Appendix.

Initially (until the central bank owns about 28 percent of the durables), the leverage constraint of household 1 binds to a sufficient extent for the welfare of both types to be increased by a modest increase in central-bank holdings of the durable, so that a Pareto improvement is achieved.[29] (The effects of asset purchases are like those resulting from a movement from equilibrium Ω to equilibrium E in Figure 3, but with the roles of the two households reversed.) But as the leverage constraint of household 1 is relaxed to a sufficient extent (and the A-D allocation is approached), the adverse income effect of the relative-price change dominates the benefit to household 1 of relaxation of the constraint. The expected utility of household 1 then decreases with additional central-bank purchases, though the expected utility of household 2 continues to rise, in accordance with Proposition 7.

Yet even in this example — constructed so as to illustrate the possibility of a relaxation of financial constraints through central-bank asset purchases — further purchases, beyond a certain point, cease to have this effect. Once the central bank owns more than 32 percent of the durable, neither household's leverage constraint binds any longer, and the A-D allocation results. In this case, neither prices nor the allocation of resources are affected by further central-bank purchases (up until the bank owns 79 percent of the total supply), in accordance with Proposition 3, resulting in flat regions of the plots in both panels of Figure 5.

And if the central bank continues to increase its share beyond 79 percent, the collateral constraint of household 1 binds again — but now in the way shown in Figure 2(b); that is, it is the short-sale constraint (2.12) that now binds. Because of the central bank's losses on its large holdings of the durable in state 2, tax obligations are substantially higher in state 2 than in state 1; and because of the effects of this on after-tax income, household 1 eventually no longer wishes to hold the durable as a hedge, and instead would prefer to short the durable (or issue debt on which it could default in state 2), if the collateral constraint did not prevent this.

In the numerical example, the short-sale constraint eventually binds for household 1 rather than household 2, because of the assumed distribution of tax obligations: households of type 1 are assumed to pay 90 percent of the taxes, and therefore are more strongly affected by the central bank's balance-sheet risk. While the equilibrium allocation of resources does not change as ω increases from 32 percent to 79 percent,

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[29]The range of second-period endowment patterns for which a Pareto improvement results is also explored in the Appendix.

household 1's holdings of the durable steadily decline, as the amount of this asset needed to achieve its desired balance of after-tax income between states 1 and 2 falls, reaching zero as ω reaches 79 percent.

Beyond this point, further central-bank purchases cause household 1's short-sale constraint to bind ever more tightly. As in the example shown in Figure 4, central-bank asset purchases reduce aggregate demand (and hence the equilibrium price level p_1) in this case, even though they succeed in increasing the price of the durable p_3, as shown in Figure 5(a). Moreover, the welfare of household 1 is reduced by the tighter financial constraint, as seen in Figure 5(b). Household 2 continues to benefit from the higher relative price of the durable, as household 2 sells all of the durables purchased by the central bank; but household 1's budget suffers, as household 1 bears a disproportionate share of the burden of paying for the central bank's losses on the transactions that have been so profitable for household 2. Household 1's losses are a consequence of two factors: the income redistribution to household 2, but also the progressive reduction in risk-sharing between the two households, as household 1 is forced to accept an after-tax income pattern that is skewed further toward greater income in state 1 (the state in which the central bank's risky assets pay off well) than is that of household 2. Though in this example household 2 continues to benefit from additional central-bank purchases, even when ω is already large, household 1 suffers a substantial welfare loss.

Thus even in the case of an endowment pattern for which central-bank asset purchases of a modest size are clearly beneficial, it remains the case that *too large* a quantity of asset purchases by the central bank will be harmful. In fact, in the example shown in Figure 5, it is the "natural buyers" of the risky asset who are eventually harmed — to such an extent that their welfare is lower for high values of ω than if there had been no asset purchases by the central bank at all.

5   Conclusions

We have considered the consequences of central-bank purchases of a risky asset, which is also the asset used as collateral for private debt contracts, in a general-equilibrium asset pricing model with endogenous collateral constraints. In our model, this asset is a durable good. In practice, central banks are less likely to directly hold real assets, such as real estate, than to hold securities that represent claims to income flows from

the real assets. But what is important for our analysis is the type of risk exposure that we allow the central bank to take onto its balance sheet. Also, in our model, the durable good is the only acceptable form of collateral for private borrowing, and central banks certainly do acquire risky assets, such as longer-term Treasury bonds, that are commonly used as collateral in financial transactions.

We have shown that it is possible for purchases of such an asset by the central bank to increase its equilibrium price, as has been the intention of recent central-bank asset-purchase programs. Yet as elementary as such a conclusion might seem, we have found that it will not obtain under all circumstances. In our model, if there exists a sufficient level of collateral for no household's collateral constraint to bind in equilibrium, central-bank asset purchases will have no effect on equilibrium asset prices, as the fiscal consequences of the changes in the central bank's state-contingent revenues provide households with a hedging motive to adjust their portfolios in ways that, in aggregate, will perfectly offset the trades by the central bank.[30] Moreover, even when this is not true, owing to a greater degree of heterogeneity in the situations of different households, the mere fact that collateral constraints bind and that central-bank purchases alter financial conditions does not imply that the price of the asset purchased by the central bank will necessarily *increase.* It is possible, instead, for it to decrease.[31] And even when purchases increase the *nominal* price of the asset (p_3), they do not necessarily increase its *real* price (p_3/p_1).[32] To the extent that the goal of policy is to lower real yields on assets in order to encourage borrowing and discourage saving, asset purchases fail to achieve the desired goal in the latter case, even though collateral constraints bind.

We have also shown that the effects of asset purchases are not *equivalent* to those of adjusting the central bank's nominal interest-rate target by a certain amount. This means that the mere fact that a central bank is prevented from lowering the nominal interest rate as much as it would wish to, owing to the zero lower bound, does not suffice to imply that asset purchases are desirable. On the other hand, the non-equivalence of these two types of policies also means that the mere fact that interest-rate policy *is* available (because the lower bound has not been reached) does

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[30]See Proposition 3 above.

[31]See, for example, the case in which the central bank owns more than 60 percent of the total supply of the asset, in Figure 4(a) above.

[32]See, for example, Proposition 6, and the case illustrated in Figure 5(a), when the central bank owns less than 32 percent of the total supply.

not necessarily imply that there is no reason to consider asset purchases. In principle, multiple objectives can be more fully achieved when multiple (non-equivalent) policy instruments are available. In particular, asset-purchase policies may be of interest because they can affect the size of distortions associated with financial constraints, and hence the efficiency of risk sharing, *in addition* to their consequences for aggregate demand. When a central bank is free to adjust policy along both dimensions independently, it may make sense to use unconventional policy mainly to influence the allocation of risk, while the consequences of the central bank's asset purchases for aggregate demand are offset by a suitable adjustment of the interest-rate target.

However, it is important to note that the effects of unconventional policy on the market price of the asset acquired by the central bank is not sufficient information from which to draw a conclusion as to whether the policy will be successful at "easing financial conditions," increasing aggregate demand, or preventing unwanted disinflation or deflation. When collateral constraints bind, one cannot say in general whether purchases of the risky asset by the central bank will loosen households' borrowing constraints, or instead *tighten* them. This depends on whether the constraints bind in the way shown in Figure 2(a) or in the way shown in Figure 2(b).[33] It follows that even when asset purchases increase the real price of the asset, one cannot conclude that the corresponding intertemporal marginal rate of substitution (IMRS) is reduced for everyone in the economy; for if the increase in the real price of the asset is associated with a tightening of the short-sale constraints of some households (as in the example shown in Figure 4 for values of ω less than 0.7, or the example shown in Figure 5 for values of ω greater than 0.8), then the wedge between these households' IMRS and the reciprocal of the asset price increases, so that an increase in the asset price need not imply a decrease in every household's IMRS.

We have also shown that asset purchases do not necessarily raise p_1, the general price level in period 0,[34] even when they increase the real price of the durable.[35] This means that when the central bank is unable to use conventional interest-rate policy

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[33]Of course, it is possible for constraints of both types to bind in a given equilibrium, as some of our numerical examples illustrate.

[34]Recall that under our assumption of homothetic preferences, p_1 and p_2 must change in the same proportion, so that the change in $\log p_1$ is also the change in the log of an index of the prices of both non-durable goods and the services received from durable goods, *i.e.,* the entire household consumption basket in period 0.

[35]See Figure 4, and Figure 5 for the case of ω greater than 0.8.

to prevent deflation, or unwanted disinflation, due to the zero lower bound on i, a resort to asset purchases will not necessarily be of any help — these may lower the equilibrium price level still further.[36] In such a case, central-bank asset purchases also lower aggregate nominal expenditure (on goods and services, as opposed to assets) in period 0 — the "aggregate demand" that interest-rate cuts are intended to increase.

Because we assume an endowment economy, a reduction in nominal aggregate demand has no consequences for the aggregate quantity of goods that are produced and consumed; but in an extension of the model with an endogenous supply of goods in period 0 and sticky wages or prices, the reduction of nominal aggregate demand can translate into reduced output — again, the opposite of what a cut in the nominal interest rate (if one is possible) would achieve.[37] This it is not always even approximately correct to view asset purchases as a substitute for an interest-rate cut, that can be used even when an interest-rate cut is precluded by the zero lower bound. Moreover, while the conditions that determine which types of effects asset purchases will have are complex, our numerical examples suggest that asset purchases typically reduce aggregate demand (and lower the price level) when some households are short-sale constrained, and none are leverage-constrained — and this case is quite likely to arise once the central bank owns a sufficiently large share of the total supply of the asset.

These conclusions make the welfare consequences of central-bank asset purchases complex to assess. In our flexible-price model, there are no effects of monetary policy (whether conventional or unconventional) on output, nor are there any consequences of changes in the general price level for household utility. Hence our analysis of the consequences of policy for household utility takes account only of the consequences of policy for the efficiency of the equilibrium consumption allocation, owing to changes in the size of the financial wedges that separate the IMRS of differently situated households, and for the redistributions of income among households that may result from changes in equilibrium asset prices. We have seen that financial constraints may be either tightened or loosened by central-bank asset purchases, depending on the way in which households are constrained. If households are prevented from is-

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[36]Of course, if there is sufficient collateral for households' constraints not to bind, asset purchases have no effect on the price level of either sign — just as they have no effect on other asset prices, or on the equilibrium allocation of resources.

[37]We leave the analysis of this extended model for a separate paper.

suing as much riskless debt as they would like, central-bank asset purchases relax this constraint and hence reduce the associated distortion (Figure 2(a)); but if instead households are preventing from shorting the risky asset, central-bank purchases tighten this constraint and increase the associated distortion (Figure 2(b)). In the former case, at least one type of household benefits from the more efficient allocation of resources, but some may be hurt, owing to the redistributive effects of price changes (the passage from E to E^* in Figure 3); only under certain more special circumstances will the purchases result in a Pareto improvement (as in the passage from Ω to E in Figure 3). When short-sale constraints bind, the welfare of at least some households must be reduced by the increase in financial distortions as a result of central-bank purchases; and while at least some households may benefit from the associated price changes (as illustrated by the high-ω region of Figure 5), in other cases the welfare of all households will be reduced (as illustrated by Figure 4).

These conclusions about welfare do not take account of any desire on the part of the central bank to influence aggregate spending or the general level of prices. If interest-rate policy can be used to offset the policy's effects on aggregate demand, these may not be the consequences of interest in any event (except in order to determine how interest-rate policy must be adjusted in light of the asset purchases). But when interest-rate policy is constrained by the zero lower bound, this argument will not apply; and in fact, the unconventional policies undertaken by central banks recently have primarily been motivated the hope that these policies can increase aggregate demand and prevent unwanted disinflation or deflation. In order to analyze the desirability of unconventional policy under such circumstances, we should consider not only the effects of household purchases on the utilities of the various households in our model, but *also* the effects of aggregate expenditure (or the general price level, p_1) in period 0. In the case that only leverage constraints bind (in our two-household model), we have seen that asset purchases raise p_1, which is a further benefit of the policy in this case. (In an extended model with nominal rigidities, the conditions required for a Pareto improvement are likely to be somewhat weaker than in the analysis here, as all households could benefit from higher utilization of productive capacity in period 0.) But when (only) short-sale constraints bind instead, our examples indicate that asset purchases *reduce* aggregate demand and the general level of prices — which would be an additional *negative* effect of the policy, under circumstances where aggregate demand is already insufficient, owing to a binding

lower bound on the nominal interest rate.

It thus matters greatly, in judging the likely benefits of central-bank asset purchases, which sorts of financial constraints bind, and to what extent. The mere fact that aggregate demand is judged to be insufficient in the absence of such purchases (or given the quantity of purchases that have already been made) is not a sufficient ground for expecting additional purchases to have a desirable effect. First, the effects of the purchases on the degree to which financial constraints bind *also* matters for welfare, apart from the consequences for aggregate demand; and second, the effects of asset purchases on aggregate demand cannot be predicted, without taking into account the way in which they will tighten or loosen the financial constraints of differently situated parties. Nor can these questions be answered simply by observing whether central-bank purchases succeed in raising the market price of the assets purchased; the price p_3 of the asset purchased by the central bank may increase either in a case in which financial constraints are loosened and aggregate demand is increased (Figure 5(a) when $\omega < 0.32$), or in a case in which financial constraints are tightened and aggregate demand is reduced (Figure 4(a) when $\omega < 0.59$, or Figure 5(a) when $\omega > 0.79$). Thus our analysis suggests that such policies should be undertaken only on the basis of a careful analysis of the consequences of the policies for the allocation of risk through the financial system, and not simply on the basis of an assessment of the current output or unemployment gap and of the degree to which central-bank purchases seem to affect market prices.

While our model's general implications for the effects of asset purchases on financial constraints are difficult to summarize, one fairly simple conclusion is worth noting. Regardless of whether asset purchases on a modest scale relax financial constraints or tighten them, our model implies that continued asset purchases by the central bank will *eventually* result in a situation where many households are constrained in their ability to short the asset acquired by the central bank. Once a sufficient fraction of the total supply of the asset is held by the central bank, it becomes almost inevitable that the primary effect of further purchases will be to *tighten* financial constraints, rather than to loosen them, and to *contract* aggregate demand, rather than to increase it. Thus even under circumstances where asset purchases on a sufficiently modest scale are clearly beneficial (as in the numerical example considered in Figure 5), at some point further asset purchases of the same kind become counter-productive from both microeconomic (efficient risk-sharing) and macroeco-

nomic (aggregate-demand management) perspectives. Central banks would thus do well to avoid the trap of thinking that if asset-purchase policies have proven useful, even larger-scale purchases must always be better.

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