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ABOUT FUNDAMENTALS**

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ABSTRACT

Man or machine? Rational trading without information about fundamentals*

We present a model of quantitative trading as an automated system under human supervision. Contrary to previous literature we show that price-contingent trading is the profitable equilibrium strategy of large rational agents in efficient markets. The key ingredient is uncertainty about whether a large trader is informed about fundamentals. Even when uninformed, he still learns more from prices than market participants who still wonder about whether he is informed. Therefore, he will trade a non-zero quantity based on past prices, whose direction – trend-following or contrarian – depends on parameters. When informed, he will trade on that information and disregard the algorithm. One implication is that future order flow is predictable even if markets are semi-strong efficient by construction.

JEL Classification: D82, G12 and G14

Keywords: Kyle model, log-concavity, rational expectations and rational price-contingent trading

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1 Introduction

Why do agents trade in financial markets? Under which conditions do they trade in response to past price changes, either following the trend or acting contrarian? Do these trades threaten financial market stability? The unprecedented recent growth of quantitative trading by large financial institutions, as well as their use of proprietary automated algorithms (e.g., Osler (2003), Hendershott, Jones, and Menkveld (2011)) has raised fresh questions and concerns about the determinants of such trading and its effects on market stability, particularly in the wake of the recent financial crisis.

Conventional academic accounts view quantitative trading and algorithms almost exclusively as tools to implement quantitative models of portfolio selection in the tradition of Black, Scholes, and Merton, pretty much in the same way as the representative agents of those models would do, just much faster than any human could. Accordingly, Kirilenko and Lo (2013 p.52) define algorithmic trading as “the use of mathematical models, computers, and telecommunications networks to automate the buying and selling of financial securities”. In these accounts, quantitative trading is fully automated, aiming to rebalance portfolio positions in response to changes in market prices and quantities, which in turn are taken as given. Conversely, in these accounts little or no role is played by discretionary (“human”) trading, by algorithms’ secrecy, by research about asset fundamentals, by research about the market impact of trades, or by trend-following (trades in the direction of past prices, aka positive-feedback, or “momentum”) and contrarian strategies.

Practitioners on the other hand, while acknowledging the role of standard portfolio selection models, hold a very different view of quantitative trading and algorithms. In the practitioners’ view, algorithmic trading is primarily about implementing a combination of trend-following and contrarian trading strategies (e.g., see Chan (2013), Clenow (2013), Durenard (2013), and Narang (2013), among others).¹ In addition to that, Kissell (2014) emphasises the crucial importance that quants evaluate the market impact of algorithmic trading, and Narang (2013) underscores the role of fundamental research and stresses that algorithmic trading is never fully automated, as “there is almost never any attempt to eliminate human contributions to the investment process” (p.14-15). And in fact, practitioners are highly protective of their trading strategies, use proprietary algorithms and keep their exposures strictly secret.

This gap between the academics and practitioners’ view of algorithmic trading is at first

¹All these monographs describe algorithmic trading primarily in terms of trend-following and contrarian strategies. Further distinctions are then made between time series and cross-sectional momentum and mean reversion (Chan (2013)); and between data-driven, theory-driven and news-driven strategies (e.g., Durenard (2013), Narang (2013)). It is also worth noting that these strategies typically trade daily or weekly, which is a time horizon most appropriate to interpret our model. Durenard (2013) and Narang (2013) also consider intraday trading and high-frequency trading in the millisecond environment.

glance surprising—after all, a large body of literature has studied deviations from the neoclassical model and focused on trend-following and contrarian trading (e.g., Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), Hong and Stein (1999)). However, this literature typically associates trend-following and contrarian trading to portfolio losses by individual and retail investors stemming from behavioral biases, imperfect or bounded rationality, non-standard preferences (e.g., for liquidity), or institutional frictions (see Shleifer (2000) and Barberis and Thaler (2003) for surveys). Because quantitative trading by large financial institutions is systematically profitable, it is at odds with both the neoclassical paradigm—in which there is no scope for price-contingent trading—and with the recent finance literature on deviations from perfect rationality.

In this paper we attempt to take seriously the practitioners’ view of algorithmic trading by studying theoretically in a micro-founded model under which conditions a rational uninformed trader may want to systematically trade in response to past price changes. In the neoclassical paradigm there is no scope for such automated trading, as rational strategic agents with standard preferences trade only insofar as they have private information about the fundamental value of the asset being traded. Similar agents who are not informed and only observe market prices do not trade. The reason, as noted by Easley and O’Hara (1991), is that in standard rational expectations models any trading strategy that is contingent on observed prices would earn zero or negative profits against any rational, risk-neutral counterparty who observes the same prices. Crucially, in these standard models the types of all traders are public information—everybody knows if a trader is informed or uninformed about the fundamental.

We argue that price-contingent trading strategies naturally emerge as optimal and fully rational behavior in a setting with a single departure from an otherwise standard rational expectations framework. We relax the assumption that the types of all traders are public information, and introduce a large trader that may or may not be informed about the fundamental, whereby his type is not known to the market. By itself, such uncertainty about a trader’s type generates an information advantage for that same trader who knows his own type, which triggers price-contingent trading. The interpretation of this information advantage is a natural one: at any point in time a sophisticated financial institution such as a hedge fund knows better than the rest of the market which trading strategies it is pursuing and in general its portfolio positions and its exposures to all kinds of risks.

We aim to establish our results on price-contingent trading in a stylized setting with as small a departure as possible from the standard framework. Consistent with the practitioner’s account, we study a financial market where large traders have market impact.² There are two

²While the specific setting we consider is in the tradition of Kyle (1985), we discuss in Section 2 that our results are more general and apply whenever large rational traders have market impact, that is, move prices with their trades.

trading rounds, one risky asset, a large risk-neutral trader, type K , who is informed about the fundamental value of the asset, and noise traders who trade for reasons outside the model and uncorrelated with the fundamental. Our main innovation is to introduce another large risk-neutral trader, type P (for potentially "price-contingent"), who may be informed or not about the fundamental, and focus on P 's trading incentives.

Strategic traders K and P submit market orders before knowing the execution price and taking into account the expected price impact of their order. As standard in this class of models, we impose that prices are set such that the market is semi-strong efficient, i.e., prices reflect all public information, which includes the total order flow but does not include knowledge of P 's type. Such market efficiency condition is implemented through a hypothetical agent, the Market (often referred to as market maker in the literature). At the same time, the market is not strong-form efficient because traders will profit from private information. Indeed, K always holds private information about the fundamental, and P always knows his own type – informed (I) or uninformed (U) – while the Market does not know P 's type. This is true even when P does not directly observe the fundamental, which is crucial for our results.

In the first trading round P trades only if he knows the fundamental. This is because his only other information is the prior, which is publicly known. Our main result arises in the second period if P ends up being uninformed. In such case, P knows he has not traded in the previous round, so that the order flow was generated by K and the noise traders. By contrast, the Market learns from prices and order flow and updates the probability that P is uninformed, but in equilibrium still rationally weighs the possibility that the order flow reflected trades by informed P . As a result, $\text{price} = \mathbb{E}[\text{fundamental} | \text{public information}]$. The uncertainty about P 's type leads different agents to hold different expectations about the fundamental value upon observing date 1 order flow. Namely, we show that

$$\mathbb{E}[\text{fundamental} | \text{public info, type} = U] \neq \mathbb{E}[\text{fundamental} | \text{public info}] = \text{price},$$

simply because date 1 order flow is in general not independent of the number of informed traders. Therefore, uninformed P has incentives to trade at date 2. We start by assuming that noise trading is normally distributed, as it naturally stems from the central limit theorem when applied to a large number of small exogenous orders. The date 2 trading problem is non-trivial, as P 's expected payoff depends on how much the Market will learn about P 's type after observing date 2 order flow, which in turn depends on the unobservable noise trading shock. Still, we prove that there is a unique optimal pure strategy for P . This strategy is characterized by a non-zero trade contingent on past prices and proportional to the standard deviation of noise trading. As a result, when uninformed, trader P follows an automated trading strategy,

mapping past prices into a non-zero order.³

There are two ways to interpret this equilibrium, both of which consistent with the practitioners' view of quantitative trading. First, one can think of quantitative trading as a portfolio of fundamental-based strategies and automated strategies (e.g., Kissell (2014), Narang (2013)).⁴ Alternatively, one can think of quantitative traders writing down their automated algorithm at time 0, after which either there is no arrival of information and the algorithm proceeds as planned, or, upon the arrival of information that makes the algorithm outdated (e.g., rumors of a takeover bid), they disregard or override their algorithm and trade directly on their information about the fundamental (e.g., Narang (2013)). In addition, our model provides the novel insight, which the informal accounts of practitioners cannot generate, that it is the very possibility of informed trading that makes automated trading possible and profitable in equilibrium.

We then study under which conditions price-contingent trading is trend-following or contrarian. We already noted that the Market's expectation of the fundamental differs from that of trader P who knows his own type. As a result, from P 's perspective the Market will always make a 'mistake' in setting the sensitivity of prices to order flow. Either the price is too insensitive to the order flow, as the Market reacts too little to informed trading by K , thereby generating incentives for a trend-following strategy by P ; or the price is too sensitive to the order flow, as the Market reacts too much to a noise trading shock, thereby generating incentives for a contrarian strategy by P .

In the context of a symmetric three point distribution for the fundamental value, we find that with enough probability of "no news" and a small order flow the first effect prevails and P 's optimal strategy is trend-following. By contrast, with little probability of "no news" and a large order flow P 's optimal strategy is contrarian. P 's optimal strategy is contrarian also in the special case without trader K and with only trader P and noise traders. This is because in an environment without K the order flow can never reflect fundamental information if P is uninformed (the first effect is absent), while the Market still believes that there may be fundamental information (the second effect is present).

In section 6 we discuss the empirical implications. At the most basic level, our theory rationalizes why algorithmic trading is profitable on average, over and above standard remunerations for risk, as it is better able to chase information than the rest of the market. Of course, algorithms can occasionally end up chasing noise trading shocks, thereby incurring losses, both in

³When traders are informed the rationale for their trading is similar to Kyle (1985), even though there are additional complications because after observing the order flow the Market needs to update his beliefs both about the fundamental and the number of informed traders.

⁴Because algorithms can respond fast to changes in prices, one can view the time period between date 1 and 2 in our model to be relatively short compared with that between date 0 and 1.

our model and in real world episodes such as the Flash Crash of May 2010 and the Quant Melt-down of August 2007. In turn, such episodes of market turmoil are triggered by noise trading shocks and not by automated trading, both according to real world accounts (e.g., CFTC and SEC (2010)) and in our model. Indeed, we find that automated trading typically moves prices closer to fundamentals, consistent with the empirical evidence of Hendershott et al. (2011) and the practitioners' accounts in Kissell (2014), Durenard (2013), and others.

In terms of strategies, our theory of automated trading under human supervision most closely rationalizes trend-following strategies by Commodity Trading Advisors (CTAs) in futures markets (e.g., Clenow (2013); Baltas and Kosowski (2014)) and by AQR and other hedge funds in various asset classes; as well as contrarian strategies by various investors in equity markets (e.g. Lehmann (1990), Jegadeesh (1990)). Furthermore, in financial markets at sufficiently high frequencies that the likelihood of informed trading is essentially zero, our model predicts that any price-contingent trading should be contrarian.

Finally, our theory predicts that the order flow is predictable from past information, consistent with the empirical evidence of Biais, Hillion, and Spatt (1995), Ellul, Holden, Jain, and Jennings (2007), and Lillo and Farmer (2004). Remarkably, and contrary to previous literature, we obtain our prediction in a setting in which the market is semi-strong efficient by construction and therefore future returns are unpredictable. The reason is that in our setting, while the Market cannot be sure whether P is uninformed, still the Market knows that if P is uninformed he will trade in a predictable, price-contingent direction. Therefore, our results demonstrate that order flow predictability can be consistent with market efficiency.

Section 2 discusses some of the related literature. Section 3 outlines the setup. Section 4 presents the main results. Section 5 discusses some special cases and extension, as well as the generality of our results. Section 6 discusses the empirical implications, and Section 7 concludes.

2 Literature

The broad literature on asset pricing and learning in micro-founded financial markets is surveyed in Brunnermeier (2001) and Vives (2008), among others. Our work relates to the part of the literature that studies trading in markets with asymmetric information. Our results on the profitability of rational price-contingent trading require that informed traders be large, i.e., that their trades have market impact.⁵ Our model shows that rational traders with market impact and superior information about their own type can learn from prices better than average market participants. Another strand of the literature studies whether past prices contain useful

⁵We develop our model in a setting that generalizes the Kyle (1985) framework, but similar implications could be obtained in a Glosten and Milgrom (1985) framework in which trades arrive probabilistically and market makers observe individual trades (see also Back and Baruch (2004)).

information for a rational trader (e.g., Grossman and Stiglitz (1980), Brown and Jennings (1989)). However, in these papers there are no profits from uninformed trading in excess of the risk premium. It is worth noting that this is because uninformed traders do not have market impact in these models and the number/share of such traders is known to all market participants.

Our paper also relates to the literature on stock price manipulation, that is, the idea that rational traders may have an incentive to trade against their private information. Provided manipulation is followed by some (exogenously assumed) price-contingent trading, short run losses can be more than offset by long term gains (see Kyle and Viswanathan (2008) for a review). Somewhat closer to our work, Chakraborty and Yilmaz (2004a, 2004b) study the incentives of an informed trader when there is uncertainty about whether such trader is informed, or is a noise trader instead. If this trader turns out to be informed, he may choose to disregard his information and trade randomly, in order to build a reputation as a noise trader. In their model, uninformed traders are assumed to always act as noise traders and are never strategic and rational. Therefore, Chakraborty and Yilmaz do not analyze the trading incentives of rational agents when they are uninformed, which is our main focus.

Goldstein and Guembel (2008) show that if stock prices affect real activity then a form of trade-based manipulation such as short-sales by uninformed speculators can be profitable insofar as it causes firms to cancel positive NPV projects, and justifies ex post the "gamble" for a lower firm value. Such manipulation is possible because there is uncertainty about whether speculators are informed. In their setting, both uninformed trading and successful stock price manipulation stem from the feedback effect between stock prices and real activity. By contrast, in our paper there is price-contingent trading but no manipulation. Therefore, our results demonstrate that price-contingent trading does not make uninformed investors the inevitable prey of (potentially informed) speculators.⁶

Our work relates to the literature on rational herding (see Chamley (2004) for a review). Unlike our setting in which traders never disregard their private information, these models—e.g., Avery and Zemsky (1998)—characterize conditions under which, when information precision is uncertain, rational traders ignore their noisy private signal and follow the actions of other traders instead. In a recent paper Park and Sabourian (2011) generalize the setting of Avery and Zemsky (1998) and identify in a framework with a three-point prior the signal structures that gives rise to rational herd and contrarian-like behavior. In their setting, all strategic traders observe some relatively imprecise private signal about the fundamental. By contrast, in our setting strategic traders observe either a precise private signal or only quantifiable public information that they interpret better than the market.

⁶Allen and Gale (1992) also study a setting with manipulation but without price-contingent trading.

One notable strand of this herding literature has focused on understanding the stock market crash of October 1987 (Grossman (1988), Gennotte and Leland (1990), Jacklin, Kleidon, and Pfleiderer (1992) and Romer (1993)). A common theme of these papers is that market participants are assumed to have strongly underestimated the extent of portfolio insurance—i.e., positive-feedback trading—which in turn is assumed to be exogenous. Our focus instead is on deriving endogenously price-contingent trading, and characterize conditions under which it is trend-following as opposed to contrarian.

Finally, we should note that our model is most appropriate to understand quantitative strategies that trade daily or weekly, so that there is both some probability that trading reflects information, and some benefit from a relatively fast execution in response to changes in market prices. It is less appropriate for the millisecond environment in which high frequency traders may benefit from momentary imbalances between supply and demand. With this in mind, our paper is also related to a few recent papers that focus on the speed advantage of quantitative traders. Clark-Joseph (2013) studies a partial equilibrium model in which prices are exogenous, and finds empirical support for the idea that high-frequency traders learn from their own trades better than the rest of the market, very closely related to the ideas developed in our model. Biais, Foucault, and Moinas (2013) consider the decision of a financial institution to invest in a high-speed trading technology and derive conditions under which such investment is excessive from a social welfare standpoint; and Pagnotta and Philippon (2012) consider trading exchanges competing on speed to attract future trading activity. Unlike us, these papers do not focus on price-contingent trading.⁷

3 Setup

A single asset with a fundamental value θ is traded at date 1 and 2 and the fundamental is realized at date 3.⁸ As we are interested in understanding the incentives of large traders who have market impact, we adopt a setting similar to Kyle (1985). Namely, we assume that large strategic risk-neutral traders and non-strategic noise traders submit market orders before knowing the execution price. As in Kyle (1985) we assume that there is a large risk-neutral trader K , who learns the value of fundamental θ before date 1 and trades only in date

⁷Somewhat related are also Banerjee and Green (2013), who study a rational expectations model with myopic mean-variance investors who are learning about the presence of informed traders; and Li (2013), who studies a continuous time Kyle model in which the large trader may or may not be informed. None of these papers however considers price-contingent trading.

⁸For our main results in Sections 4.1 and 4.2 we consider the broad class of all symmetric distributions, as the exact distribution of the prior is not crucial. In our illustrative example on the direction of price-contingent trading, which we study in Section 4.3, we assume a symmetric three-point distribution. We also explore other distributions, including the normal prior in Appendix C.

1. The presence of this trader guarantees that the asset price always reflects at least some fundamental news.⁹ Our main innovation is to introduce another large risk-neutral trader, P , whose type/state, $R \in \{I, U\}$, is not known with certainty by the Market. In our baseline setting, we assume that state R is independent of the fundamental and noise trading. We denote

$$R = \begin{cases} I & \text{if } P \text{ is "informed" (i.e., knows } \theta) \\ U & \text{if } P \text{ is "uninformed" (i.e., does not know } \theta) \end{cases} .$$

The prior probability is $\Pr(I) = \eta$, where $0 < \eta < 1$.¹⁰ When the state is $R = I$, then P is identical to K and only trades at date 1. We are particularly interested in P 's trading incentives when the state is $R = U$. If he is uninformed, he still observes past information such as order flows and prices, and can trade on both dates. Provided that date 1 price is an invertible function of the order flow, then past order flow and prices have exactly the same information content - so if he trades at date 2, then he can be viewed as a "price-contingent trader".

While not crucial for our main results, we make the assumption that traders informed about the fundamental can only trade at date 1 for two reasons. First, this assumption captures the realistic feature that quantitative trading based on simple observable information such as past prices or order flows can be implemented relatively faster by an algorithm. This is in contrast to any trading based on fundamentals, which requires a more thorough analysis and is likely to require more time.¹¹ Accordingly, one can view the time interval between dates 1 and 2 as being relatively short compared to the time interval between dates 0 and 1. Second, this is also the most transparent setting to analyze the trade-off faced by a large trader who is uninformed about the fundamental. We abstract from the additional effects that arise from informed traders' incentives to split their orders, which are well understood in the literature.

It is natural to assume that P knows his own type. We also assume that K knows P 's type with certainty, but this is not crucial for our results.¹²

Strategic traders K and P do not take the (expected) asset price as given, but know that their market orders have a non-negligible impact on prices. Denote the market order by trader $J \in \{K, P\}$ in state $R \in \{I, U\}$ at date $t \in \{1, 2\}$ as h_t^{RJ} . If both traders are informed, $R = I$,

⁹The presence of a trader who is always informed allows to derive richer effects. See Section 5 for the special case where there is no trader K and there is uncertainty about the presence of any informed traders.

¹⁰We also assume that η is not arbitrarily close to one (see Appendix B). This realistically avoids a situation in which P is uninformed, but the Market is convinced that he is informed and is very reluctant to update his beliefs about P 's type.

¹¹We can also view P 's type as an outcome of P 's previous unobservable decision where he decided whether to invest in acquiring fundamental information about θ or to invest in a "machine" that allows him to trade faster.

¹²This assumption makes it ex ante harder for P to develop an information advantage as P never has more information than K . It also helps to highlight that for rational price-contingent trading to emerge as optimal, it is important that typical market participants (the Market) do not know large trader's types with certainty, even if (some) other sophisticated traders do.

then trader J solves

$$\max_{h_1^{IJ}} \pi_1^{IJ} = \mathbb{E} [h_1^{IJ} (\theta - p_1) | \theta, I], \quad (1)$$

where $J \in \{K, P\}$. If only K is informed about the fundamental, $R = U$, then K solves

$$\max_{h_1^{UK}} \pi_1^{UK} = \mathbb{E} [h_1^{UK} (\theta - p_1) | \theta, U], \quad (2)$$

and P solves¹³

$$\max_{h_1^{UP}} \pi_1^{UP} = \mathbb{E} [h_1^{UP} (\theta - p_1) + h_2^{UP} (\theta - p_2) | U] \quad (3)$$

$$\max_{h_2^{UP}} \pi_2^{UP} = \mathbb{E} [h_2^{UP} (\theta - p_2) | y_1, U].$$

The total order flow is

$$y_1 = h_1^{RK} + h_1^{RP} + s_1 \text{ for } R = \{I, U\} \quad (4)$$

$$y_2 = \begin{cases} s_2 & \text{if } R = I \\ h_2^{UP} + s_2 & \text{if } R = U, \end{cases}$$

where s_t is date $t \in \{1, 2\}$ demand by noise traders.¹⁴ We assume that noise traders demand is drawn from a normal distribution with mean zero and variance σ_s^2 , serially uncorrelated and independent of fundamental and state. We denote the probability density function with $\varphi_s(s_t)$ for $t \in \{1, 2\}$. While being a standard assumption in this literature, there is also a natural economic argument for this choice of distribution. In fact, we can interpret noise trading as the total demand by a large number of small traders who trade for idiosyncratic reasons unrelated to the fundamental (such as liquidity shocks, private values etc.). In such case, the normality of the distribution of noise trading follows directly from the central limit theorem.

Technically, a useful property of the normal distribution is that it is strictly log-concave, allowing us to use some general properties of log-concave functions.¹⁵ Log-concavity of noise trading also guarantees some desirable properties of the model, and we discuss generality further in Section 5.¹⁶

¹³We condition P 's expectation on the order flow (instead of the price or both), because date 1 order flow is always at least as informative as date 1 price. If price is monotonic in the order flow, then the two have the same information content.

¹⁴As usual, the presence of noise traders is needed to avoid the Grossman and Stiglitz's (1980) paradox about the impossibility of a fully revealing price in equilibrium.

¹⁵A function $f(\mathbf{x})$ (where x is a n -component vector) is log-concave if $\ln(f(\mathbf{x}))$ is concave. In the univariate and differentiable case, the following are equivalent: 1) $\partial^2 \ln(f(x)) / \partial x \partial x < 0$, 2) $f'(x) / f(x)$ is decreasing in x , 3) $f''(x) f(x) - (f'(x))^2 < 0$.

¹⁶Many other well known distributions are log-concave and symmetric. Notable examples include the beta

The equilibrium prices are set by a hypothetical agent, the Market, who observes the total order flow and implements the market efficiency condition.¹⁷ Namely, he sets period t price,

$$p_t = \mathbb{E} [\theta | \Omega_t^M], \quad (5)$$

where Ω_1^M is the information set available to the Market in $t \in \{1, 2\}$, which includes all publicly available information such as the current and past order flows. The Market knows all distributions, and observes the total order flow y_t before setting p_t . Crucially, he does not know the realization of P 's type, i.e., the value of R . Hence, $\Omega_1^M = \{y_1\}$ and $\Omega_2^M = \{y_1, y_2\}$. It is worth highlighting that in this setting the order flows provide noisy information about both the fundamental, θ , and P 's type, $R \in \{I, U\}$. This is in contrast to standard settings where all types are known with certainty and the total order flow only reveals information about the fundamental. Using the law of total expectations, we can expand the Market efficiency condition (5) as

$$\begin{aligned} p_1 &= \mathbb{E} [\theta | y_1] = Q_1 \mathbb{E} [\theta | y_1, I] + (1 - Q_1) \mathbb{E} [\theta | y_1, U] \\ p_2 &= \mathbb{E} [\theta | y_1, y_2] = Q_2 \mathbb{E} [\theta | y_1, y_2, I] + (1 - Q_2) \mathbb{E} [\theta | y_1, y_2, U], \end{aligned} \quad (6)$$

where $Q_1 \equiv \Pr(I | y_1)$ and $Q_2 \equiv \Pr(I | y_1, y_2)$ are the probabilities of P being informed conditional on the observed total order flows. We also use notation $p_1(y_1)$, $p_2(y_2)$, $Q_1(y_1)$ and $Q_2(y_2)$ to express these prices and probabilities as functions of contemporaneous order flows.

To summarize the setup, the timing of events is as follows:

- **date 0** - Nature draws $R \in \{I, U\}$ and θ . K and P learn R . If $R = I$, then both K and P learn θ . If $R = U$, only K learns θ .
- **date 1** - K , P , and noise traders submit market orders before knowing the price. The Market observes total order flow and sets the price p_1 based on the market efficiency condition (5).
- **date 2** - Noise traders submit market orders. If $R = U$, then P also submits a market order before knowing the price. The Market observes total order flow and sets the price p_2 based on the market efficiency condition (5).

(with parameters $\alpha = \beta > 1$) and truncated normal. See Bagnoli and Bergstrom (2005) for an overview and further examples of log-concave densities.

¹⁷In models based on Kyle (1985), this agent is frequently referred to as the "market maker". We prefer to call him the Market to emphasize that such agent proxies for the information observed by the whole market, as opposed to any individual broker. The market efficiency condition (5) can also be interpreted as the outcome of Bertrand competition between market makers or as the equilibrium outcome of a large number of small risk-neutral agents who take prices as given.

- **date 3** - uncertainty resolves and P and K consume profits given the realization of θ .

As standard in the literature we focus on equilibria in pure strategies by K and P .

4 Results

4.1 Importance of uncertainty about types

We start by highlighting why uncertainty about P 's type is crucial by considering a benchmark case where P is known to be uninformed with certainty, i.e., $\Pr(I) = \eta = 0$.

Proposition 1 *If $\eta = 0$: $\mathbb{E}[\theta|y_1, U] = \mathbb{E}[\theta|y_1] = p_1 = p_2$; P can never earn positive expected profits from trading; consequently P does not trade in date 1.*

Proof. See Appendix A. ■

Proposition 1 shows that in the special case in which $\eta = 0$, our model supports the Easley and O'Hara (1991) argument against the possibility of uninformed traders profiting from rational price-contingent trading. Indeed, in such a case uninformed P cannot earn positive profits, because prices already reflect all information that an uninformed P could have. To be more specific, at date 1 uninformed P 's best guess of the fundamental is the prior mean and any non-zero quantity traded would move prices and lead to an expected loss from trading at date 1 (this is because the price is an increasing function of the order flow). Therefore, it is optimal for P not to trade, which yields a zero profit. At date 2, uninformed P does learn new information from the order flow, but the information he obtains is exactly the same as the information that the Market has already obtained, $\mathbb{E}[\theta|y_1, U] = \mathbb{E}[\theta|y_1] = p_1$. Because prices will not change between date 1 and 2, he cannot earn positive profits from trading.¹⁸ It should be also noted that Proposition 1 holds for any symmetric prior and log-concave and symmetric noise trading. The latter is important as it guarantees that the price is an increasing function of the order flow.¹⁹

¹⁸Note that in our setting the Market knows that date 2 order flow is not informative and thus prices will not change, $p_2 = p_1$. As a result P would earn zero profits from any quantity traded. In this setting $h_2^{UP} = 0$, but also any other constant quantity traded by uninformed P at date 2 can be sustained as an equilibrium. However, the latter only holds because of risk neutrality of P . Only the equilibrium with $h_2^{UP} = 0$ could be sustained with even a very small degree of risk aversion.

¹⁹Log-concavity of noise trading implies that the likelihood function $f(y_1|\theta, U)$ has monotone likelihood ratio property (MLRP). As known from Milgrom (1981) MLRP guarantees that the expected value of the fundamental is increasing in the order flow due to first-order stochastic dominance. See Appendix A.

4.2 Date 2 problem

Assume $\eta > 0$ and notice that all date 1 quantities, $\mathbb{E}[\theta|y_1, R]$, p_1 and $Q_1 = \Pr(I|y_1)$ depend on y_1 and are known to P and the Market before date 2. Date 2 problem is only interesting if there is a difference between P 's and the Markets expectations about the fundamental ($\mathbb{E}[\theta|y_1, U] \neq p_1$ or equivalently $\mathbb{E}[\theta|y_1, U] \neq \mathbb{E}[\theta|y_1, I]$) and the Market has not fully learned P 's type ($Q_1 > 0$). For now, we conjecture that this is the case. We verify it later when analyzing the date 1 problem.

As there is no informed trading at date 2, it holds that conditional on a given state $R \in \{I, U\}$ and y_1 , the date 2 order flow only depends on θ through y_1 , which is already incorporated in prices and expectations and therefore $\mathbb{E}[\theta|y_1, y_2, R] = \mathbb{E}[\theta|y_1, R]$. Using (6) we obtain

$$p_2 = p_1 + \frac{(Q_1 - Q_2)}{Q_1} (\mathbb{E}[\theta|y_1, U] - p_1). \quad (7)$$

Clearly prices change between date 1 and 2 only if $Q_2 \neq Q_1$, which implies that they only change if the Market updates its beliefs about P 's type after observing date 2 order flow. Provided that the true state is $R = U$, the Market updates in the "correct" direction if $Q_2 < Q_1$. In such case prices increase (decrease) if $\mathbb{E}[\theta|y_1, U] > (<) p_1$. Using (7), we can restate P 's problem (3) as

$$\begin{aligned} \max_{h_2^{UP}} \pi_2^{UP} &= h_2^{UP} \mathbb{E}[Q_2|y_1, U] \frac{(\mathbb{E}[\theta|y_1, U] - p_1)}{Q_1} = \\ &= h_2^{UP} \left(\int_{-\infty}^{\infty} Q_2 (h_2^{UP} + s_2) \varphi_s(s_2) s_2 \right) \frac{(\mathbb{E}[\theta|y_1, U] - p_1)}{Q_1}. \end{aligned} \quad (8)$$

We can make some immediate observations. Suppose that $\mathbb{E}[\theta|y_1, U] > p_1$, i.e., uninformed P expects the fundamental to be higher than date 1 price. On the one hand, P can profit from trading any positive quantity. Ignoring the effect of his trade on $Q_2(\cdot)$ would make him to want to buy an infinitely large quantity of the asset at date 2. On the other hand, the term $\mathbb{E}[Q_2|y_1, U]$ captures the expected updating of P 's type by the Market. Because Q_2 depends on date 2 order flow, P knows that his trade will affect the Markets' beliefs about his type. Since these beliefs directly affect p_2 , one would expect the traditional trade-off between transaction size and information disclosure to be present, but to establish this formally we need to investigate further the properties of Q_2 .

As we focus on pure strategies and uninformed P 's trading strategy, we can see that the beliefs of the Market are characterized by the quantity it expects P to trade. Thus, consider that the market expects P to trade some quantity \bar{h}_2 , whereby \bar{h}_2 can take any value in \mathbb{R} . Then, from (4) $y_2 = \bar{h}_2 + s_2$ if $R = U$ and $y_2 = s_2$ if $R = I$, we can derive Q_2 by using Bayes'

rule, as

$$Q_2 = \frac{Q_1 f(y_2|y_1, I)}{Q_1 f(y_2|y_1, I) + (1 - Q_1) f(y_2|y_1, U)} = \frac{Q_1}{Q_1 + (1 - Q_1) r(y_2)}, \quad (9)$$

where

$$r(y_2) \equiv \frac{\varphi_s(y_2 - \bar{h}_2)}{\varphi_s(y_2)} \quad (10)$$

is the likelihood ratio and we used the fact that conditional on the state R the date 2 order flow is normally distributed with density $\varphi_s(\cdot)$.

Lemma 1 *The following properties hold for $Q_2 = \Pr(I|y_1, y_2)$*

1. Q_2 is decreasing (increasing) in y_2 for any $\bar{h}_2 > (<) 0$.
2. If $\bar{h}_2 > 0$ then $Q_2 > (<) Q_1$ for any $y_2 < (>) \frac{\bar{h}_2}{2}$. If $\bar{h}_2 < 0$ then $Q_2 > (<) Q_1$ for any $y_2 > (<) \frac{\bar{h}_2}{2}$.
3. $Q_2(0) = Q_1 \cdot \left(Q_1 + (1 - Q_1) \frac{\varphi_s(\bar{h}_2)}{\varphi_s(0)} \right)^{-1} = Q_1 \cdot \left(Q_1 + (1 - Q_1) \exp\left(-\frac{(\bar{h}_2)^2}{2\sigma_s^2}\right) \right)^{-1}$,
4. If $\bar{h}_2 > (<) 0$ then $\lim_{y_2 \rightarrow \infty} Q_2(y_2) = 0 (= 1)$ and $\lim_{y_2 \rightarrow -\infty} Q_2(y_2) = 1 (= 0)$.
5. $Q_2(y_2)$ is a log-concave function.

Proof. Part 1: Differentiating and simplifying we obtain $\partial Q_2 / \partial y_2 = -Q_2^2 (1/Q_1 - 1) r'(y_2)$. Because Lemma A.1 in Appendix A shows that log-concavity of φ_s implies the monotone likelihood ratio property (MLRP), i.e., $r'(y_2) > (<) 0$ for any $\bar{h}_2 > (<) 0$. This is because $\varphi_s(\tilde{y}_2 - \bar{h}_2) / \varphi_s(\tilde{y}_2) > (<) \varphi_s(y_2 - \bar{h}_2) / \varphi_s(y_2)$ for any $\tilde{y}_2 > 0$ and $\bar{h}_2 > (<) 0$. Parts 2-4 are straightforward from (9), (10) and the expression for the normal density. Part 5: Taking logs and differentiating, we obtain that $\frac{\partial^2 \ln(Q_2)}{\partial y_2 \partial y_2} = -\frac{(1-Q_1)^2 [(1/Q_1-1)r''(y_2)+r''(y_2)r(y_2)-(r'(y_2))^2]}{(Q_1+(1-Q_1)r(y_2))^2}$. It is sufficient to show that the likelihood ratio (10) is (at least weakly) log-convex.²⁰ Indeed from using the normal density in (10) we find that $\ln(r(y_2))$ is linear in y_2 and therefore weakly convex. ■

Part 1 of Lemma 1 implies that the Market updates its beliefs about P 's type (the state R) in a "sensible" manner. For example, if the Market believes that trader P in state $R = U$ trades a finite and positive quantity, then observing a higher order flow always leads the Market to assign a lower probability on P being informed. This also confirms that P indeed faces a meaningful trade-off - the presence of a profit opportunity gives P incentives to trade, but

²⁰ $r(y_2)$ is log-convex if $\ln(r(y_2))$ is convex. Equivalently, it must hold that $r''(y_2)r(y_2) - (r'(y_2))^2 \geq 0$. This, together with $r(y_2) > 0$ also implies that $r''(y_2) > 0$.

trading too aggressively will reduce P 's expected profit as he expects the Market will assign a higher probability on him being uninformed and to adjust the price accordingly.²¹ It is worth emphasizing that such realistic tradeoff is always guaranteed because the likelihood ratio (10) is monotone (for a similar argument, see also Milgrom (1981)).²²

While Bayesian updating itself guarantees that the Market updates its beliefs in the correct direction on average, we can see from part 2 of Lemma 1 that depending on the realized date 2 order flow, the Market can update the probability that P is informed, Q_2 , in the "correct" or "incorrect" direction. This is because the total order flow includes a random noise trading component. Namely, if the realized order flow is relatively small (i.e., less than half of the volume that the Market expects uninformed P to trade) or has an opposite sign to P 's expected trade, then the Market updates in the "correct" direction if the state is $R = I$ and in the "incorrect" direction if the state is $R = U$. It is also immediate from parts 2-4 of Lemma 1 that the Market never learns P 's type perfectly for finite order flows. Therefore, despite some learning about P 's type, it is clear from (8) that P would always earn positive profits from trading any finite quantity that has the same sign as the difference $\mathbb{E}[\theta|y_1, U] - p_1$.

Part 4 of Lemma 1 confirms that the Market's learning about P 's type is unbounded. This is necessary to guarantee that P has an incentive to trade a finite amount.²³

While the previous analysis gives some confidence that it may be optimal for uninformed P to trade a finite quantity in equilibrium, it is not yet clear whether P 's problem has a unique (interior) solution. Namely, from (8) P 's expected profit involves an integral over a non-trivial function $Q_2(\cdot)$ that depends on uninformed P 's demand and is always positive for $h_2^{UP} > (<) 0$ provided that $(\mathbb{E}[\theta|y_1, U] - p_1) > (<) 0$.

Lemma 2 *If $(\mathbb{E}[\theta|y_1, U] - p_1) > (<) 0$ then uninformed P 's expected profit (8) is strictly log-concave in $h_2^{UP} > (<) 0$.*

Proof. Assume without loss of generality that $(\mathbb{E}[\theta|y_1, U] - p_1) > 0$ and $h_2^{UP} > 0$. Taking logs of (8), we obtain $\ln(\pi_2^{UP}) = \ln(h_2^{UP}) + \ln(\mathbb{E}[Q_2|y_1, U]) + \ln(\mathbb{E}[\theta|y_1, U] - p_1) - \ln(Q_1)$ and $\partial^2 \ln(\pi_2^{UP}) / \partial h_2^{UP} \partial h_2^{UP} = -(h_2^{UP})^{-2} + \partial^2 \ln(\mathbb{E}[Q_2|y_1, U]) / \partial h_2^{UP} \partial h_2^{UP}$, which is negative if $\mathbb{E}[Q_2|y_1, U]$ is log-concave. By change of variables $y_2 = h_2^{UP} + s_2$, we can express

$$\mathbb{E}[Q_2|y_1, U] = \int_{-\infty}^{\infty} Q_2(y_2) \varphi_s(y_2 - h_2^{UP}) dy_2. \quad (11)$$

²¹If P trades h_2^{UP} , then the order flow is $y_2 = h_2^{UP} + s_2$ and $\mathbb{E}[Q_2|y_1, U] = \int_{-\infty}^{\infty} Q_2(h_2^{UP} + s_2) \varphi_s(s) ds$. It is clear that $\partial \mathbb{E}[Q_2|y_1, U] / \partial h_2^{UP} = \int_{-\infty}^{\infty} Q_2'(h_2^{UP} + s_2) \varphi_s(s) ds > 0$

²²The monotone likelihood ratio property holds for the whole family of log-concave distributions, to which the normal belongs (see Lemma A.1 in Appendix A).

²³Suppose instead that learning was bounded (i.e., Q_2 was always larger than some constant $\bar{Q}_2 > 0$) and consider a candidate equilibrium where P trades a finite amount. It is easy to prove that this cannot be an equilibrium as P would earn infinite profits by deviating to trade an infinite quantity. See also Section 5.

Using Theorem 6 of Prékopa (1973), restated as Theorem A.3 in Appendix A, a sufficient condition for (11) to be log-concave is that the function $Q_2(y_2) \varphi_s(y_2 - h_2^{UP})$ is log-concave in h_2^{UP} and y_2 . We can then derive $\partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial h_2^{UP} \partial h_2^{UP} = -\sigma_s^{-2}$ and $\partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial h_2^{UP} \partial y_2 = \partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial y_2 \partial h_2^{UP} = \sigma_s^{-2}$. As by part 5 of Lemma 1 $\partial^2 \ln(Q_2(y_2)) / \partial y_2 \partial y_2 < 0$, it is immediate that the Hessian²⁴ is negative definite, and therefore $\mathbb{E}[Q_2|y_1, U]$ and π_2^{UP} are log-concave. The proof for the case $(\mathbb{E}[\theta|y_1, U] - p_1) < 0$ and $h_2^{UP} < 0$ is similar. ■

Since any univariate log-concave function is also quasi-concave with a unique maximum, we can now state our main result²⁵:

Theorem 1 *Uninformed P's unique equilibrium strategy is to demand a finite amount*

$$h_2^{UP} = \bar{h}_2 = \begin{cases} \sigma_s \kappa & \text{if } \mathbb{E}[\theta|y_1, U] - p_1 > 0 \\ -\sigma_s \kappa & \text{if } \mathbb{E}[\theta|y_1, U] - p_1 < 0 \end{cases}, \quad (12)$$

where $\kappa > 1$ for any $Q_1 \in (0, 1)$ and κ depends on Q_1 only.

Proof. It is immediate from (8) that $h_2^{UP} < (>) 0$ when $\mathbb{E}[\theta|y_1, U] - p_1 > (<) 0$ cannot be optimal as it leads to strictly negative profits. By Lemma 2, the uninformed P's problem then has a unique maximum at a non-negative h_2^{UP} . Therefore it is sufficient to look at the first order condition only and then impose that in equilibrium beliefs must be consistent with the optimal strategy $h_2^{UP} = \bar{h}_2$. The first order condition, the expression for κ and the proofs of the statements that $\kappa > 0$ and only depends on Q_1 are in Appendix B. ■

Because κ only depends on Q_1 , it is most illustrative to present the solution²⁶ on a graph (see Figure 1). We find that whenever $\mathbb{E}[\theta|y_1, U] \neq p_1 = \mathbb{E}[\theta|y_1]$, it is generally optimal for uninformed P to trade at date 2. The volume traded by P is proportional to the standard deviation of noise trading and is increasing in Q_1 . Both effects are intuitive. When the order flow is more noisy (high σ_s), it is harder for the Market to update its beliefs about the state and it is less costly for uninformed P to trade more aggressively. Because the Market's posterior belief that the state is $R = I$, $Q_2 = \Pr(I|y_1, y_2)$, is increasing in Q_1 (its belief about P's type before date 2 trading), it is clear that a higher Q_1 also makes it less costly for an uninformed P to trade more aggressively as the Market is learning about his type more slowly. Overall P trades

²⁴The Hessian is $\begin{vmatrix} -\sigma_s^{-2} & \sigma_s^{-2} \\ \sigma_s^{-2} & \partial^2 \ln(Q_2(y_2)) / \partial y_2 \partial y_2 - \sigma_s^{-2} \end{vmatrix}$

²⁵An alternative proof of quasi-concavity is to require that the negative of the first derivative of the objective function is single crossing in h_2^{UP} . The sufficient conditions for single crossing under uncertainty have also been explored by Athey (2002) and Quah and Strulovici (2012). In both cases we can derive that log-concavity of $Q_2(\cdot)$ and $\varphi_s(\cdot)$ are sufficient for strict single crossing.

²⁶It is relatively easy to show that if $Q_1 = 0.5$, then $\kappa = \sqrt{2}$. The other values are derived using numerical integration.

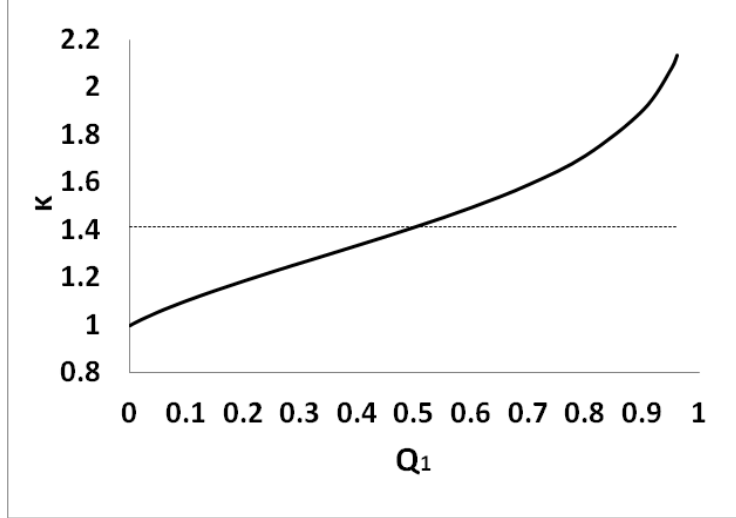


Figure 1: Uninformed P 's trading volume at date 2.

a finite quantity as he faces a trade-off between profiting from his superior information and revealing his type too much. This trade-off is fundamentally similar to the one in Kyle (1985), however differently from that setting P 's private information is not about the fundamental directly, but about his impact or lack of impact on date 1 price. Theorem 1 is in stark contrast to Proposition 1, because Theorem 1 demonstrates that price-contingent trading is profitable.

Before analyzing the date 1 problem, we can now formally define the two traditional types of price-contingent strategies within the context of our model.

Definition P 's date 2 strategy is called²⁷

- trend-following (momentum) for some y_1 if $y_1 > 0$ and $h_2^{UP} > 0$, or $y_1 < 0$ and $h_2^{UP} < 0$
- contrarian for some y_1 if $y_1 > 0$ and $h_2^{UP} < 0$, or $y_1 < 0$ and $h_2^{UP} > 0$.

Provided the price is monotonic in the order flow (as in all our examples below), it would be equivalent to define P 's strategy through date 1 price.

²⁷The words "momentum" and "contrarian" only refer to P 's strategy. They should not be confused with positive and negative autocorrelation in returns. By the assumption of efficient markets (5), there is zero autocorrelation by construction. See also Section 4.4.

4.3 Date 1 problem

As discussed above, we illustrate our date 1 results by assuming a symmetric three-point prior as follows:

$$\theta = \begin{cases} -\bar{\theta} & \text{wpr. } \frac{1-\gamma}{2} \\ 0 & \text{wpr. } \gamma \\ \bar{\theta} & \text{wpr. } \frac{1-\gamma}{2} \end{cases}, \quad (13)$$

where $0 \leq \gamma < 1$ and $\bar{\theta} > 0$. By varying the mass in the centre of distribution, γ , we cover distributions with very different shapes, i.e., how likely are "good" or "bad news" about the fundamental compared to "no news". The case with $\gamma = 0$ corresponds to a two-point distribution and is of special interest because it is a common assumption in the literature. We will show that assuming a two-point distribution is not without loss of generality in our setting - the size of the mass in the center of the prior distribution, relative to that in the tails, turns out to be crucial in determining the direction of price-contingent trading.

As we focus on pure strategies, the demand of K and P must be known to the Market for given realizations of the fundamental $\theta = \{-\bar{\theta}, 0, \bar{\theta}\}$. To shorten the argument, we limit our attention only to the cases where the Market's beliefs about traders' strategies have some natural properties given the symmetry of the distributions and the fact that P has no superior information at date 1. Namely, we conjecture that in state $R = U$, the informed trader K 's optimal demand is some real number \bar{g}_U if $\theta = \bar{\theta}$, zero if $\theta = 0$ and $h_1^{UK} = -\bar{g}_U$ if $\theta = -\bar{\theta}$; and uninformed trader P does not trade. In state $R = I$, the total demand by informed traders K and P is a real number \bar{g}_I if $\theta = \bar{\theta}$, zero if $\theta = 0$ and $-\bar{g}_I < 0$ if $\theta = -\bar{\theta}$. Given these beliefs, we can derive the expressions and main properties of $\mathbb{E}[\theta|y_1, R]$, the price, and Q_1 as described by the following lemma.

Lemma 3 *For the equilibrium price and conditional expectations of the fundamental, it holds that*

1. *The price is given by*

$$p_1(y_1) = \bar{\theta} \frac{M_n(y_1) - M_p(y_1)}{M_n(y_1) + M_p(y_1)} \quad (14)$$

where $M_n(y_1) \equiv \frac{1-\gamma}{2} \left(\eta \varphi_s(y_1 - \bar{g}_I) + (1-\eta) \varphi_s(y_1 - \bar{g}_U) + \frac{\gamma}{1-\gamma} \varphi_s(y_1) \right)$
and $M_p(y_1) \equiv \frac{1-\gamma}{2} \left(\eta \varphi_s(y_1 + \bar{g}_I) + (1-\eta) \varphi_s(y_1 + \bar{g}_U) + \frac{\gamma}{1-\gamma} \varphi_s(y_1) \right)$;

2. *The conditional expectation of the fundamental is*

$$\mathbb{E}[\theta|y_1, R] = \bar{\theta} \frac{\varphi_s(y_1 - \bar{g}_R) - \varphi_s(y_1 + \bar{g}_R)}{\varphi_s(y_1 - \bar{g}_R) + \varphi_s(y_1 + \bar{g}_R) + \frac{2\gamma}{1-\gamma} \varphi_s(y_1)}; \quad (15)$$

3. The updated probability of state $R = I$ is

$$Q_1(y_1) = \Pr(I|y_1) = \frac{\eta(\varphi_s(y_1 - \bar{g}_I) \frac{1-\gamma}{2} + \varphi_s(y_1) \gamma + \varphi_s(y_1 + \bar{g}_I) \frac{1-\gamma}{2})}{M_n(y_1) + M_p(y_1)}; \quad (16)$$

4. The price is increasing in the order flow, i.e., $p'_1(y_1) > 0$;

5. The price is symmetric around zero, i.e., $p_1(y_1) = -p_1(-y_1)$;

6. It holds that $\lim_{y_1 \rightarrow \infty} p_1(y_1) = \bar{\theta}$ and $\lim_{y_1 \rightarrow -\infty} p_1(y_1) = -\bar{\theta}$;

7. $\bar{\theta} - p_1(y_1) > 0$ for all (finite) y_1 ;

Proof. See Appendix B. ■

Lemma 3 confirms some reasonable and desirable properties of date 1 price, e.g., the price is increasing in the order flow, symmetric around zero and always between $-\bar{\theta}$ and $\bar{\theta}$. If the state is $R = U$, then the expected profit (2) of K can be written as

$$\pi_1^{UK} = h_1^{UK} \int_{-\infty}^{\infty} (\theta - p(h_1^{UK} + s_1)) \varphi_s(s_1) ds_1. \quad (17)$$

If the state is $R = I$ then the expected profit (1) of trader $J \in \{K, P\}$ can be written as

$$\pi_1^{IJ} = h_1^{IJ} \int_{-\infty}^{\infty} (\theta - p(h_1^{IK} + h_1^{IP} + s_1)) \varphi_s(s_1) ds_1. \quad (18)$$

As in Section 4.2, we cannot be immediately sure if the informed trader's expected profit has a unique maximum in own demand. The reason is that, unlike in the date 2 problem, we cannot be sure that the traders' objective function is log-concave. Namely, one sufficient condition for this would be that $\theta - p(y_1)$ is log-concave and this only holds for some parameters. However log-concavity is only a sufficient, but not a necessary condition for a unique maximum. What we need is that the trader's profit is quasi-concave in own demand, i.e., that $-\frac{\partial \pi_1^{RJ}}{\partial h_1^{RJ}}$ is a single crossing function of h_1^{RJ} . In Appendix B we prove that this is always the case for $\theta = 0$ and we identify some conditions where this is also the case for $\theta = \{-\bar{\theta}, \bar{\theta}\}$. Intuitively, a sufficient condition is that the expected slope of the price at high order flows is not too flat relative to that same slope at low order flows, or, at a minimum, the slope of the price does not decrease too rapidly in expectation. Provided that the informed trader's problem has a unique maximum in own demand, we can state

Proposition 2 *There is a pure strategy equilibrium at date 1, where the following holds:*

1. *Informed traders' demand is given by*

$$h_1^{UK} = \begin{cases} \bar{g}_U = \sigma_s \mu_U & \text{if } \theta = \bar{\theta} \\ 0 & \text{if } \theta = 0 \\ -\bar{g}_U = -\sigma_s \mu_U & \text{if } \theta = -\bar{\theta} \end{cases} \quad \text{and } h_1^{IK} = h_1^{IP} = \begin{cases} \frac{\bar{g}_I}{2} = \sigma_s \frac{\mu_I}{2} & \text{if } \theta = \bar{\theta} \\ 0 & \text{if } \theta = 0 \\ -\frac{\bar{g}_I}{2} = -\sigma_s \frac{\mu_I}{2} & \text{if } \theta = -\bar{\theta} \end{cases},$$

where μ_U and μ_I only depend on η and γ .

2. *Total demand by informed traders in the event of news ($\theta = \bar{\theta}$ or $\theta = -\bar{\theta}$) is always higher in absolute value in state $R = I$ compared to state $R = U$, i.e., $g_I > g_U$ (equivalently $\mu_I > \mu_U$).*

3. *In state $R = U$, the uninformed trader P does not trade at date 1, i.e., $h_1^{UP} = 0$.*

Proof. See Appendix B. ■

Proposition 2 states some intuitive properties of date 1 equilibrium. First, informed traders face the standard trade-off as in Kyle (1985) and Holden and Subrahmanyam (1992). On the one hand, whenever they have private information that indicates $\theta \neq 0$ they earn positive expected profits from trading, so they have an incentive to trade a high volume. On the other hand, they know that due to market impact, trading a high volume reveals information about the fundamental (and also—less importantly for these traders—about the state R) to the Market. Therefore, they trade a finite amount and the price will not adjust immediately to equal the fundamental value.

The trading volume is always proportional to the standard deviation of noise trading. This is because informed traders benefit on average at the expense of noise traders and more noise allows them to hide private information more easily. Because the equilibrium price is proportional to the fundamental (see (14)), the magnitude of the fundamental value does not affect the informed trader's optimal strategy, but clearly profits are higher if $\bar{\theta}$ is higher. By Proposition 2 we know that the optimal strategy only depends on two parameters that are between 0 and 1.²⁸ Figure 2 illustrates these dependences by plotting on the vertical axis μ_U and μ_I against η (on the left panel, assuming $\gamma = 0$) and against γ (on the right panel, assuming $\eta = 0.5$). These plots are qualitatively similar for different values of η and γ . First, if the prior probability of the state with two informed traders $R = I$ (i.e., $\Pr(I) = \eta$) is higher, then the informed traders are trading less aggressively. This is because $g_I > g_U$ and the Market expects more informed

²⁸While the solution does not generally have a nice analytical expression, we can find the benchmark cases corresponding to Kyle (1985) and to the two-traders version of Holden and Subrahmanyam (1992) by extending these to a different prior distribution. Namely if $\gamma = 0$ and $\eta = 0$, it holds that $\mu_U = 1$; if $\gamma = 0$ and $\eta = 1$, it holds that $\frac{\mu_I}{2} = \frac{\sqrt{2}}{2}$.

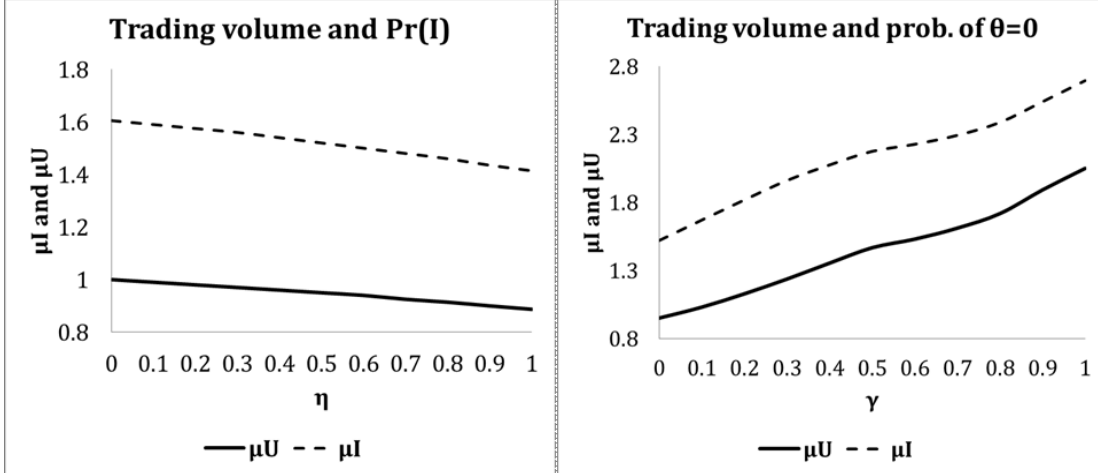


Figure 2: Informed trading volume for different values of η and γ .

trading and is updating its beliefs faster. This in turn increases the informed traders' market impact and reduces their willingness to trade aggressively. Second, if the prior probability of "no news" (γ) is higher, the informed traders trade more aggressively whenever they observe $\theta \neq 0$. This is because by Bayes' rule the Market is relatively reluctant to update its beliefs toward the more extreme realizations of the fundamental. This reluctance reduces the market impact of the informed traders and gives them incentives to trade more.

The most important part of Proposition 2 is part 3 which states that the total order flow by informed traders is different in the two states, as we can see when comparing the expressions for p_1 , $\mathbb{E}[\theta|y_1, I]$ and $\mathbb{E}[\theta|y_1, U]$. If the state is $R = U$, then P again obtains superior information exactly because he knows that he did not trade and we can explore the direction of his trade at date 2.

4.3.1 Direction of price contingent trading

We start by examining the two-point prior, i.e., for now we set $\gamma = 0$.

Proposition 3 *When the prior distribution of the fundamental is a symmetric two-point distribution, it holds that*

$$\mathbb{E}[\theta|y_1, U] < (>) \mathbb{E}[\theta|y_1, I], \text{ for any } y_1 > (<) 0.$$

Whenever $0 < \eta < 1$, the optimal strategy of P in state $R = U$ at date 2 is contrarian.

Proof. See Appendix B. ■

With a discrete two-point distribution we find that if the true state is $R = U$, i.e., P is uninformed, then P 's optimal strategy at date 2 is always contrarian. Note that when assuming a two-point prior we are focusing on an environment where any "news" about the fundamental is either "good" or "bad" and the Market always expects informed traders to trade. Any positive order flow is more likely to be associated with $\theta = \bar{\theta}$ compared to $\theta = -\bar{\theta}$. Furthermore, by part 2 of Proposition 2, we know that two informed traders would always trade a larger quantity in absolute value than one informed trader and therefore whenever the order flow is positive it holds that $\Pr(\bar{\theta}|y_1, I) > \Pr(\bar{\theta}|y_1, U)$ and $\Pr(-\bar{\theta}|y_1, I) < \Pr(-\bar{\theta}|y_1, U)$.²⁹ As the Market prices the asset considering that both states are possible, it tends to overprice the asset whenever the order flow is positive and the true state is $R = U$. Effectively the Market tends to underestimate the fact that it may have been a positive noise trading shock rather than the demand of the informed traders that generated a positive order flow.

The above conclusion is specific to a two-point distribution with no mass in the center. With a three-point prior, we can establish some more general properties about the direction of price-contingent trading.

Proposition 4 *When the prior distribution of the fundamental is a symmetric three-point distribution and $R = U$, then for any $0 < \eta < 1$ the following conditions hold for large and very small order flows*

1. *for order flows $|y_1| \geq \left| \frac{\bar{q}_I + \bar{q}_U}{2} \right|$, P always pursues a contrarian strategy at date 2.*
2. *for order flows y_1 in the neighborhood of zero (i.e., for $y_1 \rightarrow 0$), P pursues a trend-following strategy at date 2 iff the following condition holds.*

$$\frac{1 + \exp\left(\frac{\mu_I^2}{2}\right) \frac{\gamma}{1-\gamma}}{1 + \exp\left(\frac{\mu_U^2}{2}\right) \frac{\gamma}{1-\gamma}} > \frac{\mu_I}{\mu_U} \quad (19)$$

3. *Provided that (19) holds, there exists a threshold order flow in the interval of $(0, \frac{\bar{q}_I + \bar{q}_U}{2})$ below which P 's optimal strategy is trend-following and above which it is contrarian.*

Proof. See Appendix B. ■

Proposition 4 shows that with a three-point prior both trend-following and contrarian strategies are possible at date 2. We also gain further insights on how the characteristics of the prior distribution drive the direction of price-contingent trading.

²⁹This is straightforward to verify using part 3 of lemma 3 and the properties of log-concave functions in Appendix A.

Part 1 of Proposition 4 shows that, when the date 1 order flow is large in absolute value, then at date 2 uninformed P always pursues a contrarian strategy. The reason for this is similar to our argument about the two-point prior. Intuitively, high order flows in state $R = U$ are relatively more likely to be driven by high noise trading shocks compared to what the Market expects. For example, if the true state is $R = U$, then any order flow that exceeds \bar{g}_U must mean that there was a positive noise trading shock, while the Market will still consider order flows between \bar{g}_U and \bar{g}_I to be potentially reflecting small or even negative noise trading shocks. And this generates incentives for P to pursue a contrarian strategy.

Part 2 of Proposition 4 shows that if the probability of no-news is large enough, then at least for small order flows uninformed P 's optimal strategy at date 2 is trend-following. Namely, there is a threshold level for γ , above which this inequality (19) holds³⁰ and back-of-the-envelope calculations indicate that this threshold is quite low.³¹ This observation highlights the fact that for trend-following trading there should be at least some mass in the center of the distribution. The intuition for why at small order flows it is optimal for P to pursue a trend-following strategy again relates to part 2 of Proposition 2. Consider for example a small positive order flow. If the true state is $R = U$, then the Market is now reluctant to believe that it is driven by informed traders who observed $\bar{\theta}$ (as it considers the possibility that two informed traders who would trade a much larger quantity, \bar{g}_I in total, while the actual informed trading could have been at most \bar{g}_U) and sets the price relatively close to zero. Because uninformed P knows at date 2 that his trading did not contribute to date 1 order flow, he benefits from trend-following trading on average. What is crucially different in the situation where not just extreme events, but also moderate events regarding the fundamental are possible is that the Market tends to underestimate the probability that there were very good (or bad) news regarding the fundamental when it observes a small order flow, as it would expect two informed traders to always trade much more aggressively than one. This tendency to underestimate the possibility of big fundamental news is what gives incentives for P to pursue a trend-following strategy.

Figure (3) illustrates the equilibrium difference, $\mathbb{E}[\theta|y_1, U] - p_1$, (vertical axis) for different values of γ , assuming that $\eta = 0.5$. On the horizontal axis, there is always the date 1 order flow, y_1 and the shaded area point out the values of y_1 where P 's optimal strategy is trend-following. We can see that when γ is high enough, then there is a set of order flows around zero where $|\mathbb{E}[\theta|y_1, U]| > |p_1|$ and uninformed P 's optimal strategy at date 2 is trend-following. As the

³⁰Note that the right hand side of (19) is always bigger than 1 as $\mu_I > \mu_U$ by point 2 in Proposition 3. The right hand side is 1 if $\gamma = 0$, strictly increasing in γ and converges to $\exp(\mu_I^2) / \exp(\mu_U^2)$ when $\gamma \rightarrow 1$. We can also verify that $\exp(\mu_I^2) / \exp(\mu_U^2) > \mu_I / \mu_U$ at the limit. This is because $\exp(\mu_R^2) / \mu_R$ is strictly increasing in μ_R for any $\mu_R > 0.5$. Hence (19) will hold at $\gamma \rightarrow 1$ if $\mu_I > \mu_U > 0.5$. It can also be shown that μ_U is at its lowest when $\eta = 1$ and $\gamma = 0$, and from Figure (2) that in such case μ_U is noticeably higher than 0.5.

³¹For example, if $\eta = 0.5$ then the threshold is around $\gamma \approx 0.21$.

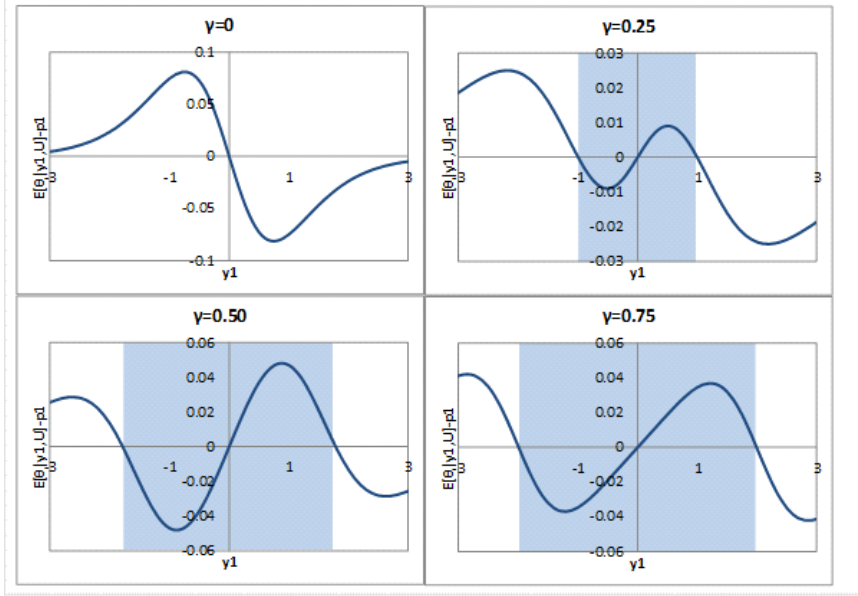


Figure 3: The difference $\mathbb{E}[\theta|y_1, U] - p_1$ for different values of γ .

informed trading volume is proportional to the standard deviation of noise trading, the values along the horizontal axis reflect the order flows normalized by the standard deviation of noise trading. We can see that at already $\gamma = 0.25$ order flows up to the magnitude of one standard deviation of noise trading will lead to trend-following trading. The magnitude of such order flows doubles if $\gamma = 0.75$. At very high order flows in absolute value, it always holds that $|\mathbb{E}[\theta|y_1, U]| < |p_1|$ and uninformed P 's optimal strategy at date 2 is contrarian.

The three-point distribution also allows to derive richer empirical implications. We find that price-contingent traders are likely to react differently when they observe order flows of different magnitude. It is plausible to expect that quantitative traders who typically trade in the direction of past price changes will adjust their behaviour and become contrarian at extreme order flows that are most likely driven by noise trading shocks.

4.4 Predictability of order flow and the effect of price-contingent trading on market efficiency.

Here we point out some natural consequences of equilibrium price-contingent trading under either the semi-strong or weak form of market efficiency.

Proposition 5 *While there is no predictability in returns, the order flow is predictable.*

Proof. The lack of predictability in returns is immediate and is due to imposing the efficient market condition (5). By construction $p_2 = \mathbb{E}[\theta|y_1, y_2]$ and $p_1 = \mathbb{E}[\theta|y_1]$, and by application of

the law of iterated expectations, it is clear that $\mathbb{E}[p_2 - p_1|y_1] = \mathbb{E}[\mathbb{E}[\theta|y_1, y_2] | y_1] - p_1 = \mathbb{E}[\theta|y_1] - p_1 = 0$. At the same time by Theorem 1 we know that if the state is $R = U$ then P trades at date 2 a known amount \bar{h}_2 . Therefore, $\mathbb{E}[y_2|y_1] = \Pr(I|y_1) \mathbb{E}[y_2|y_1, I] + \Pr(U|y_1) \mathbb{E}[y_2|y_1, U] = Q_1 \mathbb{E}[s_2|y_1, I] + (1 - Q_1) \mathbb{E}[\bar{h}_2 + s_2|y_1, U] = (1 - Q_1) \mathbb{E}[\bar{h}_2|y_1, U] \neq 0$. ■

In Kyle (1985) and Holden and Subrahmanyam (1992) and subsequent models that build on their framework, imposing the market efficiency condition implies *both* the lack of predictability of returns *and* the lack of predictability of the order flow. As discussed in Section 4.1, there is no profitable and predictable price-contingent trading and future order flow can only reflect unpredictable noise trading and informed trading. Matters differ considerably in our more general setting, because the Market cannot be perfectly sure of whether there is a price-contingent trader P or not, but the Market still knows that if there is one, he will trade in a predictable direction, described in Propositions 4 and 6. For example, if the optimal strategy is trend-following, the Market expects a positive order flow with some probability; if the actual order flow is zero, the prices fall.

It should also be noted that the type of price-contingent trading we analyze as emerging in a fully rational setting without other frictions, on average facilitates price discovery by moving prices closer to the fundamental. In state $R = U$, the best estimate of the fundamental conditional on all the information apart from the fundamental itself is $\mathbb{E}[\theta|y_1, U]$, and not $\mathbb{E}[\theta|y_1]$, so that uninformed P 's price-contingent trading on average pushes date 2 price p_2 closer to $\mathbb{E}[\theta|y_1, U]$.

Importantly, in our model there is also no sense in which contrarian trading is more stabilizing than momentum trading. For example, it is true that in our setting a rare situation can arise whereby prices change purely because of a noise trading shock and P 's optimal trend-following strategy moves prices further away from the fundamental, but similarly there can be a rare situation whereby following some draws of noise trading P 's optimal contrarian trading delays information about the fundamental from being reflected into prices. As a result, while both contrarian and momentum trading are on average stabilizing, both can end up pushing prices away from fundamentals.

5 Discussion of Special Cases and Extensions

In this Section we discuss some special cases and alternative assumptions and extensions. We examine the number of traders in Section 5.1, alternative distributions of the fundamental in Section 5.2, and other assumptions in Section 5.3.

5.1 Number of traders

We have assumed that in any state there is always an informed trader K . The reason was to guarantee that asset prices always reflect fundamental information at least from K , which P may learn from prices. The presence of trader K allows for a rich set of effects and generates a rationale for trend-following trading under some conditions. However, we should emphasise that uncertainty about P 's type alone is sufficient for rational price-contingent trading to emerge even if there is no fundamental information. Namely, assume that there is no trader K , but the Market still does not know P 's type as in the baseline model of Section 3. Consider a symmetric prior and normal noise trading.

Proposition 6 *If there is no informed trader K , then uninformed P will pursue and profit from a contrarian strategy at date 2.*

Proof. See Appendix B. ■

Proposition 6 further highlights why profitable price-contingent strategies emerge in a rational setting where traders' types are not perfectly observable by the average market participants, as it leads to "mistakes" in interpreting the order flow. In this example, prices would never change if the Market knew that P is uninformed. However, the possibility that the order flow may contain fundamental information is enough to make the Market sometimes change prices unnecessarily based on order flows. Naturally P 's optimal strategy in this example is always contrarian because uninformed P knows that the best guess about the fundamental is still the prior and would profit from speculating against any changes of prices.

It would be trivial to add more type K and type P traders. All the effects would be the same as long as the number of sophisticated traders of type K and P is finite. The reason why K and P trade finite amounts and earn returns on their information is because they have market impact and they are aware of it. If the number of type P traders were infinite, then they would be indistinguishable from the Market; if the number of type K traders were infinite, then in the limit prices would tend towards strong-form market efficiency, but also towards an information acquisition paradox in the spirit of Grossman and Stiglitz (1980). Second, it would also be possible to add more trading rounds in which uninformed P can trade. This would complicate the model as P would likely have a Kyle's (1985) type of incentive to split his orders and reveal information more slowly. However, it is intuitive that the Market will then still be imperfectly and slowly learning about the true state until the price eventually converges to $\mathbb{E}[\theta|y_1, R]$.

5.2 Alternative distribution of the fundamental and some more general properties

As we discussed in Sections 3, 4.1, and 4.2 above, our results on existence and uniqueness of an optimal price-contingent trading strategy at date 2 are general and do not depend on any particular prior distribution of the fundamental. However, we showed in Section 4.3 that the choice of prior distribution is crucial for determining the qualitative results, and most notably for the direction of price contingent trading. Therefore, for more concrete applications of the mechanisms identified in our paper to different assets and time frame, the assumptions about the fundamental should be predominately driven by economic arguments specific to the situation at hand. Here we extend our discussion about the robustness of our findings and of some important effects by considering more explicitly other prior distributions.

We focused on the examples of a two-point and a three-point prior. While the former is a common assumption in the literature (e.g., Cho and El Karoui (2000)), the latter allowed us to derive richer and more realistic results. Another common assumption in the literature is a normal prior, used for example in Kyle (1985) and Holden and Subrahmanyam (1992). Given the intuition we have built in Section 4.3 with the three-point distribution, it is intuitive to expect the presence of the force pushing towards trend-following trading, due to the sizable mass in the centre of the normal prior distribution. Furthermore, the normal prior has also infinite support and thin tails, which also implies that very high order flows are not necessarily driven by extreme noise trading, but may be still driven by very high draws of the fundamental. In Appendix C we explore the normal prior and show that in such case the optimal price-contingent strategy is trend-following. The reason is that the tendency for the market to underestimate the fundamental at a given order flow is always present and dominates the tendency to overreact to noise trading shocks. Namely, at any given fundamental two traders will always trade more than one, and it is also true that at a given level of demand by informed traders, the fundamental is always higher if there is one informed trader rather than two. This force leads the Market to setting prices too insensitive to the order flow, which makes trend-following trading profitable. While there is also a secondary effect where the Market underestimates the contribution of noise traders to the total order flow if the true state this $R = U$, we find that this latter effect never dominates. Appendix C includes the case where θ and R are independent as in our baseline setting, but we also give an example with a particular joint density of R and θ that keeps the date 1 equilibrium linear. Results in both cases are similar, but the latter allows for comparable equations to the aforementioned papers that are a special cases with $\eta = 0$ and $\eta = 1$ in our setting.

We can further derive some more general properties of our model with other prior distribu-

tions. For clarity, assume that the fundamental, θ , is a continuous variable;³² that noise trading is also continuous in the interval $[-\bar{s}, \bar{s}]$; and $f_s(s)$ is log-concave and symmetric. Provided that the price is increasing in the order flow and the date 1 problem is quasi-concave (with interior maximum)³³, there exists a pure strategy equilibrium where a set of properties are true, and we can derive some properties using the tools from the monotone comparative statics literature. Namely, we establish

Proposition 7 *If date 1 price is increasing in the order flow and the informed traders' problem is quasi-concave (with interior maximum), then:*

1. *The total demand of informed traders, $g_R(\theta)$ in state $R \in \{I, U\}$ is strictly increasing in θ .*
2. *It holds that $g_I(\theta) > (<) g_U(\theta)$ for any $\theta > (<) 0$.*

Proof. See Appendix B. ■

The most important implication of Proposition 7 is that, holding fixed any fundamental, two informed traders trade a higher quantity in equilibrium than one informed trader. This implies that conditional on date 1 order flow y_1 , the state R and the fundamental θ are not independent. In particular, $\Pr(U|\theta, y_1) = \frac{f_s(y_1 - g_U(\theta)) \Pr(U|\theta)}{f_s(y_1 - g_U(\theta)) \Pr(U|\theta) + f_s(y_1 - g_I(\theta)) \Pr(I|\theta)}$ is generally a function of θ . This allows to conclude that in general there will be a difference between uninformed P 's and the markets expectations. Namely, it holds that

$$\mathbb{E}[\theta|y_1, U] - p_1 = \frac{\mathbb{E}[\theta \cdot \Pr(U|\theta, y_1)|y_1]}{\Pr(U|y_1)} - \mathbb{E}[\theta|y_1] = \frac{Cov(\theta, \Pr(U|\theta, y_1))}{\Pr(U|y_1)},$$

where we used the market efficiency condition (5) and Bayes' rule $f(\theta|y_1, U) = \frac{f(\theta|y_1) \Pr(U|\theta, y_1)}{\Pr(U|y_1)}$. As we argued that $\Pr(U|\theta, y_1)$ is a function of θ , it is clear that θ and $\Pr(U|\theta, y_1)$ are not independent, which means that the covariance will generally not be zero. Therefore, our main result that price-contingent trading is profitable in a setting where there is uncertainty about traders' types is very general.

Also, given Proposition 7, the forces that affect the direction of the difference $\mathbb{E}[\theta|y_1, U] - p_1$ are also present more generally. As an example, assume that both noise trading and the fundamental θ are continuous and have support in $(-\infty, \infty)$ and noise trading has symmetric log-concave density. In such case, by Proposition 7 part 1, we know that $g_R(\cdot)$ is an invertible function. Therefore, in state R we obtain the following signal from the order flow only:

³²Similar arguments hold for a discrete fundamental.

³³Sufficient conditions for this are similar to those discussed in Appendix B.

given y_1 and unknown noise trading shock s_1^* , the fundamental is $\theta^* = g_R^{-1}(y_1 - s_1^*)$. The cumulative distribution function (c.d.f.) of θ conditional on the order flow only is therefore, $\Pr(\theta^* < \theta) = \Pr(g_R^{-1}(y_1 - s_1^*) < \theta) = 1 - F_s(y_1 - g_R(\theta))$, where F_s is the c.d.f. of noise trading. By part 2 of Proposition 7 and the fact that the c.d.f. is monotonically increasing, $1 - F_s(y_1 - g_U(\theta)) < (>) 1 - F_s(y_1 - g_I(\theta))$ for any $g_I(\theta) > (<) g_U(\theta) \iff \theta > (<) 0$. This confirms that the distribution of θ conditional on the order flow only is more dispersed in state $R = U$ compared to state $R = I$ for any order flow y_1 . The probability density function of the fundamental conditional on order flow only is $g'_R(\theta) f_s(y_1 - g_R(\theta))$. We find that the expected value conditional on the order flow only is then $\int_{-\infty}^{\infty} \theta g'_R(\theta) f_s(y_1 - g_R(\theta)) d\theta = \int_{-\infty}^{\infty} g_R^{-1}(y_\theta) f_s(y_1 - y_\theta) dy_\theta = \int_0^{\infty} g_R^{-1}(y_\theta) (f_s(y_1 - y_\theta) - f_s(y_1 + y_\theta)) dy_\theta$. It then holds that the difference in expectations is $\int_0^{\infty} (g_U^{-1}(y_\theta) - g_I^{-1}(y_\theta)) (f_s(y_1 - y_\theta) - f_s(y_1 + y_\theta)) dy_\theta > (<) 0$ if $y_1 > (<) 0$. The inequality follows from (25) and Lemma A.1 in Appendix A. Therefore, if $y_1 > 0$, then in state $R = U$ the order flow signal has a higher mean (which pushes toward trend-following trading) and a more dispersed distribution (which pushes toward contrarian trading) than in state $R = I$. Any Bayesian updating trades off these two effects, and which one dominates depends on the prior. If, additionally, the distributions are bounded then the same effects are present and there are additional effects due to these bounds - most importantly, large order flows in absolute value are always more likely in state $R = I$.³⁴

5.3 Normal noise trading and other assumptions

As discussed in Section 3, we view the noise traders in our model as capturing a large number of traders who trade for idiosyncratic reasons outside the main focus of our model. Therefore, the main argument for assuming normally distributed noise trading stems from the central limit theorem. However, technically, many realistic properties of our model rely on the less restrictive assumption of log-concave noise trading. Indeed, a log-concave distribution guarantees that the Market updates at date 2 in the "correct direction" - that is, in state $R = U$, if trader P submits a positive quantity in equilibrium, then higher order flows at date 2 always signal a higher posterior probability that the state is indeed $R = U$. It also guarantees that the expected value $\mathbb{E}[\theta|y_1, R]$ is increasing in date 1 order flow, which in turn often implies that also the price is increasing in order flow. Both of these properties hold because log-concavity implies the monotone likelihood ratio property (MLRP). These properties are realistic in the

³⁴If the prior has bounded support $[-\bar{\theta}, \bar{\theta}]$ and noise trading support is noticeably wider (such that any order flows can be generated by a noise trading shock), then conditional on the order flow only in state R the cdf of θ is $\frac{F_s(y_1 + g_R(\bar{\theta})) - F_s(y_1 - g_R(\bar{\theta}))}{F_s(y_1 + g_R(\bar{\theta})) - F_s(y_1 - g_R(\bar{\theta}))}$ and the probability density function is $\frac{g'_R(\bar{\theta}) F_s(y_1 - g_R(\bar{\theta}))}{F_s(y_1 + g_R(\bar{\theta})) - F_s(y_1 - g_R(\bar{\theta}))}$. With some algebra we can then identify the same effects, and find that informed trading at the highest fundamental, $g_R(\bar{\theta})$ in $R = \{I, U\}$ affect both the mean and the dispersion.

context of financial markets and guarantee that sophisticated large traders in our model face a meaningful trade-off in the spirit of Kyle (1985). Namely, an informed trader (either with superior information about the fundamental directly or indirectly due to superior knowledge of his own past actions) benefits from trading a higher volume due to positive expected returns, but trading a higher volume is costly due to market impact as it reveals more about his private information—whether about the fundamental or about his own type.

In our proofs we frequently relied only on log-concavity rather than on the explicit form of the normal density. For example, the determinants of the direction of price-contingent trading in the case of two- and three-point prior distribution hold for any (symmetric) log-concave noise trading.

It should also be noted that to avoid a situation whereby P at date 2 has an incentive to trade an infinite amount—which cannot occur in equilibrium—we need at least that the noise trading distribution, $f_s(\cdot)$, is such that the likelihood ratio $r(y_2) = \frac{f_s(y_2 - \bar{h}_2)}{f_s(y_2)}$ is unbounded, which is true for some, but not all log-concave densities.³⁵

Finally, one could also model more complex information structures where also informed traders can trade more frequently. It is well known from Kyle that traders with superior information and market power have incentives to split orders and not push prices immediately to be equal to fundamentals. For this reason allowing other large informed traders to trade at date 2 in parallel to P does not eliminate P 's gains from price-contingent trading. We have explored such setting and the results are available upon request.

6 Empirical Implications

We have presented a theory of algorithmic trading as an automated system under constant human supervision. In section 6.1 we discuss how our equilibrium helps understand basic features of algorithmic trading in the real world, and in section 6.2 we discuss the impact of algorithmic trading on market stability and market crashes.

6.1 Understanding Algorithmic Trading

Algorithmic trading is systematically profitable. While hedge funds and quantitative traders are shrouded in secrecy and systematic data is thus hard to come by, it is becoming increasingly evident that quantitative trading with algorithms generates large profits on a regular

³⁵By unbounded, we mean that $\lim_{y_2 \rightarrow \bar{s}} r(y_2) \rightarrow \infty$ for $0 < \bar{h}_2 < \bar{s}$. For example, not just the normal, but also the *Beta* (α, β) distribution with parameters $\alpha, \beta > 1$ is strictly log-concave and has an unbounded likelihood ratio. However, *Gamma* (k, θ) with $k > 0$ is strictly log-concave, but the likelihood ratio is not unbounded (Gamma is also not symmetric around zero).

basis.³⁶ These regular profits are hard to reconcile with a view of algorithmic trading as mere implementation of standard portfolio selection models, and do suggest the need to examine the micro-foundation of algorithmic trading strategies.

Our paper does offer such a micro-foundation of algorithmic trading. In our equilibrium, when trader P ends up uninformed about the fundamental, he trades a systematic non-zero quantity based on past prices, whose direction—trend-following or contrarian—depends on the parameters of the distribution. By contrast, when P ends up informed (e.g., rumors of a takeover bid), he trades on that information and disregards or override the algorithm (e.g., see the illustration based on rumors about the Merrill Lynch and Bank of America merger in Narang (2013, p.15-16)). Thus, in our equilibrium automated algorithmic trading needs constant human supervision: it is the very possibility of submitting an informed order at some point that makes algorithmic trading systematically profitable.³⁷ Interestingly, this view of automated trading under human supervision is consistent with the accounts of the source of hedge funds' profits that are typically found in the press.³⁸

Given the mechanics of automated trading under human supervision and its interplay with fundamental trading described above, our model most closely rationalizes price-contingent strategies at short horizons by Commodity Trading Advisors (CTAs) in futures markets and by various hedge funds in equity markets.³⁹ CTAs are popular recent investment vehicles that execute profitable trend-following strategies in futures markets at daily, weekly, and monthly frequencies (e.g., see Clenow (2013), and Baltas and Kosowski (2014)). By contrast, various hedge funds execute profitable contrarian strategies in equities at weekly (Lehmann (1990)) and monthly (Jegadeesh (1990)) frequency. Our model can account for both types of strategies, by noticing that futures markets are typically characterized by a distribution of the fundamental with substantial mass in the center, similar to our normal case examined in Section 5.2, or the three-point distribution with enough probability of no news of Section 4; and that equity markets at short horizons are rarely characterized by informed trading, so that they resemble the case examined in Proposition 6 without trader K . Along similar lines, in any financial

³⁶Quantitative hedge funds such as Citadel, CQS, Renaissance Technologies, and others, which implement multiple trading strategies with a strong emphasis on directional trading, feature regularly among the top performing hedge funds, e.g., see <http://media.bloomberg.com/bb/avfile/rMz9ZuocMhKo>.

³⁷Alternatively, one can also think of quantitative trading as being a portfolio of various trading strategies, some of which based on prices, such as trend following or contrarian strategies, and others based on fundamentals, such as for example 'value' or 'growth' strategies (albeit not based on publicly observed 'factors' but on proprietary research, e.g., see Kissell (2014) and Narang (2013)).

³⁸For example, Fortune Magazine (16 July, 2003) features an article on the success of Barclays Global Investors, which describes the fund's strategy as "They don't follow hunches; they follow computer models. /.../ This is modern quantitative investing. It falls somewhere between stock picking as it has been practiced for generations and the computer-driven "black box" techniques".

³⁹Some large and prominent funds that are known to use systematic trend-following and contrarian strategies are AQR, Renaissance Technologies, D.E. Shaw, Citadel, Barclays Global Investors.

market at such high frequencies that the likelihood of informed trading is essentially zero, our model predicts that any price-contingent trading should be contrarian.

Finally, one testable implication of our model is that the order flow should be predictable from past information even when returns are not. The reason is that, even if the Market does not know if trader P is informed, still the Market knows the direction of P 's trade when uninformed. Therefore, we expect a non-zero autocorrelation of the order flow, and we expect its sign to go hand in hand with the direction of price-contingent trading in equilibrium: contrarian trading should imply a negative autocorrelation, and trend-following trading should imply a positive autocorrelation. Importantly, these predictions differ from those of Barberis et al. (1998), Daniel et al. (1998), and Hong and Stein (1999), as in these models order flow predictability is a direct consequence of return predictability, which is ruled out in our model.

6.2 Market Quality and Crashes

In terms of the impact of the introduction of quantitative trading on various aspects of market quality such as volatility and liquidity, we find that quantitative trading is on average stabilizing, in the sense that price contingent trading typically moves prices closer to the fundamental, consistent with the empirical evidence of Hendershott et al. (2011) and the practitioners' accounts in Kissell (2014), Durenard (2013), and Narang (2013). However, there is a concern that in particular circumstances quantitative trading can propagate adverse negative shocks and generate instability, as in the Quant Meltdown of August 2007, and the Flash Crash of May 6, 2010. For example, the report on the "events of May 6," (CFTC and SEC (2010)) stated that a large 'mistaken' sell order triggered algorithms to start selling; soon after the volume of sell orders increased, and algorithms started to buy. Eventually, many algorithms incurred large losses and just stopped trading, so that the mismatch of supply and demand became so large that the entire system went to a halt for a few minutes.

Remarkably, while not specifically designed to describe these events, our model does capture some of their key features. First, algorithms did not trigger either episode—the trigger was a noise trading shock such as the 'mistake' by a large investor in 2010; and a series of large trades on the news of problems with subprime mortgages in 2007. Second, and consistent with our model, the initial response of algorithms in both cases was trend-following trading, as long as total order flow was 'small enough'. Third, and again consistent with our model, when total order flow became larger, algorithms started pursuing contrarian strategies. On the other hand, by its very design our model does not capture the failures of market efficiency that occurred when many algorithms just stopped trading and prices could no longer equate supply and demand. Most important, though, the events of August 2007 and May 2010 underscore a key feature of our model: quantitative trading through algorithms is profitable on average, as

it is better able to chase information than the rest of the market, but it can occasionally end up chasing noise trading shocks, thereby incurring losses.⁴⁰

7 Concluding Remarks

We have presented a theory of quantitative trading as an automated system under human supervision. We establish that price-contingent trading is the optimal strategy of large rational agents in a setting in which there is uncertainty about whether large traders are informed about the fundamental. We provide conditions under which price-contingent trading is trend-following (momentum) or contrarian in equilibrium. A robust implication of our results is that the order flow is predictable from current prices even if the market is semi-strong efficient and future returns are thus unpredictable.

Our model explains why hedge funds and other large financial institutions who engage in automated trading with algorithms are systematically profitable; and it explains why the secrecy of their algorithms, trading portfolios, and exposures is key to their success. By having market impact and by being relatively less known than other agents, hedge funds can learn any information that is reflected into prices better than any other investor who does not perfectly know their trading strategies and portfolios. As a result, hedge funds can successfully implement a broader range of strategies, such as trend-following and contrarian trading, than individual and retail investors without market impact that would lose money from those same strategies.

Quantitative trading with algorithms has recently come under attack in the popular press, and the profitability of these strategies has been occasionally attributed to illegal practices such as front-running.⁴¹ We have demonstrated that the superior systematic performance of trend-following and contrarian strategies needs not stem from illegal practices, as the simple and perfectly legitimate market impact of trades, together with uncertainty about access to fundamental information, is sufficient to generate systematic profits.

Of course, in the real world quantitative strategies can be a lot more sophisticated than our simple equilibrium momentum and contrarian strategies, and can use as input an array of quantifiable public information in addition to prices and order flows. One robust insight of our model is that quantifiable information can arise from superior knowledge of market participants' trading styles rather than economic fundamentals as traditionally thought. Extending our model to capture the additional nuances of real-world quantitative strategies would seem to be an interesting area for future research.

⁴⁰See Mendel and Shleifer (2012) for a related account of the Quant Meltdown of 2007.

⁴¹Even though such critique is most applicable to the subset of trading strategies at the highest frequencies in the millisecond environment, it is the full universe of quantitative trading that has lately come under scrutiny from the popular press.

A Background theorems and lemmas

Lemma A.1 *If $f_s(\cdot)$ is strictly log-concave, then it holds that*

$$\frac{f_s(x_2 - c_2)}{f_s(x_2 - c_1)} > \frac{f_s(x_1 - c_2)}{f_s(x_1 - c_1)} \text{ for any } x_2 > x_1 \text{ and } c_2 > c_1. \quad (20)$$

Proof. By definition of log-concavity it must hold that

$\alpha \ln(f_s(x_1 - c_2)) + (1 - \alpha) \ln(f_s(x_2 - c_1)) < \ln(f_s(\alpha(x_1 - c_2) + (1 - \alpha)(x_2 - c_1)))$ and
 $(1 - \alpha) \ln(f_s(x_1 - c_2)) + \alpha \ln(f_s(x_2 - c_1)) < \ln(f_s((1 - \alpha)(x_1 - c_2) + \alpha(x_2 - c_1)))$ for any
 $0 < \alpha < 1$. Let $\alpha = \frac{x_2 - x_1}{x_2 - x_1 + c_2 - c_1}$. Then $\ln(f_s(\alpha(x_1 - c_2) + (1 - \alpha)(x_2 - c_1))) = \ln(f_s(x_1 - c_1))$
and

$\ln(f_s((1 - \alpha)(x_1 - c_2) + \alpha(x_2 - c_1))) = \ln(f_s(x_2 - c_2))$. Adding up the inequalities, we obtain that $\ln(f_s(x_1 - c_2)) + \ln(f_s(x_2 - c_1)) < \ln(f_s(x_1 - c_1)) + \ln(f_s(x_2 - c_2))$. Exponentiating both sides and rearranging, we obtain (20). ■

Note that, in probability theory, this implies that if we interpret c as a signal about some random variable such that $x = c + s$, where the density $f_s(s)$ is strictly log-concave, then the conditional distribution of $f(x|c) = f_s(x - c)$ satisfies the strict monotone likelihood ratio property (MLRP).

Corollary A.1.1 *If $f_s(\cdot)$ is strictly log-concave and symmetric ($f_s(s) = f_s(-s)$), then for any $x > 0$, it holds that*

$$f_s(x - c) > (<) f_s(x + c) \text{ for any } c > (<) 0$$

Proof. For the case $c > 0$, let $x_2 = x$, $x_1 = -x$ and $c = c_2 > c_1 = 0$. By (20) $\frac{f_s(x-c)}{f_s(x)} > \frac{f_s(-x-c)}{f_s(-x)} = \frac{f_s(x+c)}{f_s(x)} \implies f_s(x - c) > f_s(x + c)$. For the case $c < 0$, let $x_2 = x$, $x_1 = -x$ and $c = c_1 < c_2 = 0$ to obtain that $\frac{f_s(x)}{f_s(x-c)} > \frac{f_s(-x)}{f_s(-x-c)} = \frac{f_s(x)}{f_s(x+c)} \implies f_s(x + c) > f_s(x - c)$. ■

For the next Lemma, assume that the prior distribution in state $R \in \{I, U\}$ is $f(\theta|R) \equiv f_{\theta R}(\theta) = f_{\theta R}(-\theta)$ in between $-\bar{\theta}$ and $\bar{\theta}$ (the case $\bar{\theta} = \infty$ is easy to incorporate in this framework). We consider a continuous prior, but a similar argument applies to a discrete prior. Assume that the informed trader's total demand at date 1 is symmetric around zero and strictly increasing, i.e., $h_1^{UK} = g_U(\theta)$ and $h_1^{IK} + h_1^{IP} = g_I(\theta)$, where $g_R(\theta) = g_R(-\theta)$ and $g'_R(\theta) > 0$ for $R \in \{0, 1\}$. Assume that uninformed P 's demand at date 1 is some constant \bar{h}_{1U} .

Lemma A.2 *If the noise trading distribution is log-concave and symmetric, $f_s(s_1)$, it holds for any prior distribution that $\mathbb{E}[\theta|\tilde{y}_1, R] - \mathbb{E}[\theta|y_1, R] > 0$ for any $\tilde{y}_1 > y_1$. It also holds that $\mathbb{E}[\theta|y_1, I] = \mathbb{E}[\theta|(-y_1), I]$ and $\mathbb{E}[\theta|(y_1 - \bar{h}_{1U}), U] = -\mathbb{E}[\theta|(-y_1 + \bar{h}_{1U}), U]$.*

Proof. The proof uses Milgrom (1981) and log-concavity of $f_s(s)$. Define the observable part of the order flow in state R as follows: $y_{1R} = y_1$ if $R = I$ and $y_{1R} = y_1 - \bar{h}_{1U}$ if $R = U$. It then

holds that $y_{1R} = g_R(\theta) + s_1$ for $R = \{I, U\}$. We first show that $\mathbb{E}[\theta|\tilde{y}_1, R] - \mathbb{E}[\theta|y_1, R] > 0$ for any $\tilde{y}_1 > y_1$. It is well known that $\mathbb{E}[\theta|\tilde{y}_1, R] > \mathbb{E}[\theta|y_1, R]$ if the cumulative distribution $F(\theta|\tilde{y}_1, R)$ dominates $F(\theta|y_1, R)$ in the sense of first order stochastic dominance, i.e., $F(\theta|\tilde{y}_1, R) \leq F(\theta|y_1, R)$ for all θ , with strict inequality for some θ . Given that the order flow $y_{1R} = g_R(\theta) + s_1$, $g_R(\cdot)$ is increasing, we know from Lemma A.1 that log-concavity of $f_s(\cdot)$ implies that $f_s(\tilde{y}_1 - g_R(\hat{\theta})) f_s(y_{1R} - g_R(\check{\theta})) > f_s(\tilde{y}_1 - g_R(\check{\theta})) f_s(y_{1R} - g_R(\hat{\theta}))$ for every $\hat{\theta} > \check{\theta}$. We can equivalently write this inequality as $f(\tilde{y}_1|\hat{\theta}, R) f(y_{1R}|\check{\theta}, R) > f(\tilde{y}_1|\check{\theta}, R) f(y_{1R}|\hat{\theta}, R)$ for every $\hat{\theta} > \check{\theta}$. The fact that the latter inequality implies first order stochastic dominance for any prior density $f(\theta|R)$ is Proposition 1 in Milgrom (1981). As in state $R = I$, $y_{1R} = y_1$, this immediately proves that $E[\theta|\tilde{y}_1, I] - E[\theta|y_1, I] > 0$ for any $\tilde{y}_1 > y_1$. For the state $R = U$, notice that because \bar{h}_{1U} is known, conditioning on y_1 is equivalent to conditioning on $y_{1U} = y_1 - \bar{h}_{1U}$ and it must always hold that $E[\theta|y_1, R] = E[\theta|y_{1R}, R]$ so we know that $E[\theta|\tilde{y}_1, U] - E[\theta|y_1, U] > 0$ for any $\tilde{y}_1 - \bar{h}_{1U} > y_1 - \bar{h}_{1U} \Leftrightarrow \tilde{y}_1 > y_1$. For the second part, note that we can express that $\mathbb{E}[\theta|y_{1R}, R] = \int_{-\bar{\theta}}^{\bar{\theta}} \theta f(\theta|y_{1R}, R) d\theta = \frac{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}$. Using the symmetry of $f_s(\cdot)$ and $f_{\theta R}(\theta)$, we then find that $\mathbb{E}[\theta - y_{1R}, R] = \int_{-\bar{\theta}}^{\bar{\theta}} \theta f(\theta - y_{1R}, R) d\theta = \frac{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(-y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} f_s(-y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta} = -\frac{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta} = -\mathbb{E}[\theta|y_{1R}, R]$. Using then the definition of y_{1R} proves the lemma. ■

Theorem A.3 (Prékopa (1973) Theorem 6) Let $f(\mathbf{x}, \mathbf{y})$ be a function of $n + m$ variables where \mathbf{x} is an n -component and \mathbf{y} is an m -component vector. Suppose that f is logarithmic concave in \mathbb{R}^{n+m} and let A be a convex subset of \mathbb{R}^m . Then the function of the variable \mathbf{x} :

$$\int_A f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

is logarithmic concave in the entire space \mathbb{R}^n .

Theorem A.4 (Chebyshev's integral inequality) Let $g, h : [a, b] \rightarrow \mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ be a probability density function. Suppose that g is monotonically increasing. Define $H_F : (a, b] \rightarrow \mathbb{R}$, $H_F(t) = \frac{\int_a^t h(x) f(x) dx}{\int_a^t f(x) dx}$. If $H_F(t) \leq H_F(b)$ for all $t \in (a, b]$ then

$$\int_a^b g(x) h(x) f(x) ds \geq \int_a^b g(x) f(x) dx \int_a^b h(x) f(x) dx$$

Proof. See for example Theorem 1 in Wagener (2006). Also see Mitrinović, Pečarić, and Fink (1993). ■

B Proofs

Proof of Proposition 1

As this statement does not rely on any specific distribution of the fundamental, we prove a more general case. Therefore, assume more generally that the prior distribution is symmetric (and prior mean exists) and noise trading distribution is log-concave. Assume that informed trader K 's optimal strategy is given by $g_U(\theta)$ and it holds $g_U(\tilde{\theta}) > g_U(\theta)$ for any $\tilde{\theta} > \theta$ and $g_U(\theta) = -g_U(-\theta)$.⁴²

By a straightforward application of Bayes' rule, it holds that $\eta = 0 \implies Q_1 = 0$ and $Q_2 = 0$. Also, notice that in this case date 2 order flow is not informative about the fundamental, i.e., $\mathbb{E}[\theta|y_1, y_2, U] = \mathbb{E}[\theta|y_1, U]$. From (6), we then find that $p_1 = \mathbb{E}[\theta|y_1] = \mathbb{E}[\theta|y_1, U] = p_2$. From (3), P 's expected date 2 profits are $\mathbb{E}[h_2^{UP}(\theta - p_2)|y_1, U] = h_2^{UP}(\mathbb{E}[\theta|y_1, U] - p_1) = 0$. Therefore, P earns zero profit at date 2 irrespective of the quantity he trades. For date 1, suppose that the Market sets the price under the belief that P trades some known quantity \bar{h}_{1U} at date 1. Given these beliefs, P chooses h_1^{UP} that corresponds to an order flow (see (4)) $y_1 = h_1^{UP} + s_1 + g_U(\theta)$. The order flow is uncertain at the time of P 's date 1 trading decision due to the presence of s_1 and θ . Using that P 's profit at date 2 is always zero and $\mathbb{E}[\theta|U] = 0$, (3) can be written as $\pi_1^{UP} = -h_1^{UP}\mathbb{E}[p_1|U]$. Denoting the prior distribution of the fundamental with $f_\theta(\theta)$, we find that $\mathbb{E}[p_1|U] = \int_s \int_\theta p_1(h_1^{UP} + s_1 + g_U(\theta)) f_s(s_1) f_\theta(\theta) ds_1 d\theta$.⁴³ The first derivative of the profit is $\partial\pi_1^{UP}/\partial h_1^{UP} = -\mathbb{E}[p_1|U] - h_1^{UP}(\partial\mathbb{E}[p_1|U]/\partial h_1^{UP})$. Because $p_1 = \mathbb{E}[\theta|y_1, U]$, by Lemma A.2 in Appendix A, the price is increasing in the order flow, and it holds that $-h_1^{UP}(\partial\mathbb{E}[p_1|U]/\partial h_1^{UP}) < (>) 0$ for any $h_1^{UP} > (<) 0$. Due to the symmetry of distributions and the fact that prices increase in the order flow, it also holds that $-\mathbb{E}[p_1|U] < (>) 0$ for any $h_1^{UP} > (<) 0$.⁴⁴ Therefore, P 's profit is maximized at $h_1^{UP} = 0$. In equilibrium, the beliefs of the Market must be consistent with P 's optimal strategy, i.e., it holds that $h_1^{UP} = \bar{h}_{1U} = 0$. In such a case, also $\pi_1^{UP} = 0$.

Proof of the remaining parts of Theorem 1

Assume that $\mathbb{E}[\theta|y_1, U] - p_1 > 0$. It is clear from (8) that the optimal demand h_2^{UP} cannot be negative. Because by Lemma 2 P 's problem at date 2 is log-concave, it is sufficient to

⁴²Proposition 6 in Section 5 proves that the total demand by informed traders must be increasing in θ due to supermodularity of the problem. Symmetry follows from the symmetry of all distributions.

⁴³A similar argument holds for a discrete distribution of the fundamental and/or of noise trading.

⁴⁴From Lemma A.2 in Appendix A it also holds that $p_1(y_1 - \bar{h}_{1U}) = -p_1(-y_1 + \bar{h}_{1U})$. Therefore, $\mathbb{E}[p_1|U] = \int_{\theta > 0} \int_{s_1 > 0} P_1(h_1^{UP}, s_1, \theta, \bar{h}_{1U}) f_s(s_1) f_\theta(\theta) ds_1 d\theta$, where $P_1(h_1^{UP}, s_1, \theta, \bar{h}_{1U}) \equiv p_1(h_1^{UP} + s_1 + g_U(\theta) + \bar{h}_{1U} - \bar{h}_{1U}) + p_1(h_1^{UP} + s_1 - g_U(\theta) + \bar{h}_{1U} - \bar{h}_{1U}) - p_1(-h_1^{UP} + s_1 - g_U(\theta) - \bar{h}_{1U} + \bar{h}_{1U}) - p_1(-h_1^{UP} + s_1 + g_U(\theta) - \bar{h}_{1U} + \bar{h}_{1U})$. As prices are increasing in the order flow, it holds that $P_1(h_1^{UP}, s_1, \theta, \bar{h}_{1U})$ is always non-negative and strictly positive for some values of $s_1, \theta > 0$ iff $h_1^{UP} > 0$.

explore the first order condition. Using (8), (9), (10), (11) and noticing that $\frac{\partial \varphi_s(y_2 - h_2^{UP})}{\partial h_2^{UP}} = \frac{y_2 - h_2^{UP}}{\sigma_s^2} \varphi_s(y_2 - h_2^{UP})$, we obtain that

$$\frac{\partial \pi_2^{UP}}{\partial h_2^{UP}} = \int_{-\infty}^{\infty} \frac{Q_1 \varphi_s(y_2)}{Q_1 \varphi_s(y_2) + (1 - Q_1) \varphi_s(y_2 - \bar{h}_2)} \left(1 - \frac{(h_2^{UP})^2}{\sigma_s^2} + \frac{h_2^{UP} y_2}{\sigma_s^2} \right) \varphi_s(y_2 - h_2^{UP}) dy_2 \quad (21)$$

Define $\kappa \equiv \frac{\bar{h}_2}{\sigma_s}$ and $z \equiv \frac{y_2}{\sigma_s}$, where $dy_2 = \sigma_s dz$, which implies that $\varphi_s(y_2 - \bar{h}_2) = \frac{1}{\sigma_s} \phi(z - \kappa)$ and $\varphi_s(y_2) = \frac{1}{\sigma_s} \phi(z)$, where $\phi(\cdot)$ is the p.d.f. of a standard normal. The optimal demand must solve $\frac{\partial \pi_2^{UP}}{\partial h_2^{UP}} = 0$ and it must hold in equilibrium that optimal demand $(h_2^{UP})^* = \bar{h}_2 = \kappa \sigma_s$. Using all this, in (21), we obtain that κ is the positive solution of

$$\int_{-\infty}^{\infty} \frac{Q_1 \phi(z)}{Q_1 \phi(z) + (1 - Q_1) \phi(z - \kappa)} (1 - \kappa^2 + \kappa z) \phi(z - \kappa) dz = 0, \quad (22)$$

which we know to be unique by Lemma 2. Because σ_s does not enter in (22), it also proves that P 's demand is proportional to σ_s and only depends on Q_1 . The proof for the case $\mathbb{E}[\theta|y_1, U] - p_1 < 0$ is similar and in such a case we need the unique negative solution of (22). It is easy to verify that if $\kappa > 0$ solves (22), then also $-\kappa > 0$ solves (22).

Next let us prove that $\kappa > 1$ by contradiction. Suppose instead that $0 < \kappa < 1$ solves (22). From (22), it must then be the case that $\kappa \int_{-\infty}^{\infty} z \frac{Q_1 \phi(z)}{Q_1 \phi(z) + (1 - Q_1) \phi(z - \kappa)} \phi(z - \kappa) dz < 0$. Using that $\phi(\cdot)$ is an even function, we can rewrite this as

$$\kappa \int_0^{\infty} z Q_1 \phi(z) \left(\frac{1}{Q_1 \frac{\phi(z)}{\phi(z - \kappa)} + (1 - Q_1)} - \frac{1}{Q_1 \frac{\phi(z)}{\phi(z + \kappa)} + (1 - Q_1)} \right) dz < 0$$

Because $\phi(\cdot)$ is log-concave, it holds that $\phi(z - \kappa) > \phi(z + \kappa)$ for all $z, \kappa > 0$ by Corollary A.1.1 from Appendix A. This implies that $\frac{1}{Q_1 \frac{\phi(z)}{\phi(z - \kappa)} + (1 - Q_1)} < \frac{1}{Q_1 \frac{\phi(z)}{\phi(z + \kappa)} + (1 - Q_1)}$. So all terms inside the integral are non-negative for all $z \geq 0$ (with strict inequality for $z > 0$), which leads to a contradiction and therefore $0 < \kappa < 1$ does not hold.

Proof of Lemma 3

For parts 1-3 note that (6) implies that, $p_1(y_1) = Q_1 \mathbb{E}[\theta|y_1, I] + (1 - Q_1) \mathbb{E}[\theta|y_1, U]$. By the law of total expectations $\mathbb{E}[\theta|y_1, R] = \bar{\theta} \Pr(\theta = \bar{\theta}|y_1, R) - \bar{\theta} \Pr(\theta = -\bar{\theta}|y_1, R)$ and by Bayes' rule $\Pr(\theta|y_1, R) = \frac{\frac{1-\gamma}{2} f(y_1|\theta, R)}{f(y_1|R)}$, where $f(y_1|\theta = \bar{\theta}, R) = \varphi_s(y_1 - \bar{g}_R)$; $f(y_1|\theta = 0, R) = \varphi_s(y_1)$; $f(y_1|\theta = -\bar{\theta}, R) = \varphi_s(y_1 + \bar{g}_R)$ and $f(y_1|R) = f(y_1|\theta = \bar{\theta}, R) \frac{1-\gamma}{2} + f(y_1|\theta = -\bar{\theta}, R) \frac{1-\gamma}{2} + \gamma f(y_1|\theta = 0, R)$. By Bayes' rule $Q_1(y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|I) + (1-\eta) f(y_1|U)}$. Combining all this proves parts 1-3. For part 4, note that $\partial \varphi_s(y_1 - c) / \partial y_1 = -\frac{y_1 - c}{\sigma_s^2} \varphi_s(y_1 - c)$ for any constant c , and therefore $M'_n(y_1) = -\frac{y_1}{\sigma_s^2} M_n(y_1) + M_{ng}(y_1)$ and $M'_p(y_1) = -\frac{y_1}{\sigma_s^2} M_p(y_1) - M_{pg}(y_1)$, where

$M_{ng}(y_1) \equiv \frac{1-\gamma}{2} \left(\eta \frac{\bar{g}_I}{\sigma_s^2} \varphi_s(y_1 - \bar{g}_I) + (1-\eta) \frac{\bar{g}_U}{\sigma_s^2} \varphi_s(y_1 - \bar{g}_U) \right) > 0$ and $M_{pg}(y_1) \equiv \frac{1-\gamma}{2} \left(\eta \frac{\bar{g}_I}{\sigma_s^2} \varphi_s(y_1 + \bar{g}_I) + (1-\eta) \frac{\bar{g}_U}{\sigma_s^2} \varphi_s(y_1 + \bar{g}_U) \right) > 0$. Using the above and differentiating, $p'_1(y_1) = 2\bar{\theta} \frac{M_{ng}(y_1)M_p(y_1) + M_{pg}(y_1)M_n(y_1)}{(M_n(y_1) + M_p(y_1))^2} > 0$. Parts 5-7 are straightforward when using (14), the expression of the normal density, and the fact that $\varphi_s(\cdot)$ is an even function that is always non-negative and positive at finite values.

Quasiconcavity of informed traders' date 1 problem.

Let us focus on the state $R = U$. If $\theta = 0$ then from (17), we obtain the first order condition $-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}} = \int_{-\infty}^{\infty} (p(h_1^{UK} + s_1) + h_1^{UK} p'(h_1^{UK} + s_1)) \varphi_s(s_1) ds_1 = \int_{-\infty}^{\infty} (p(y_1) + h_1^{UK} p'(y_1)) \varphi_s(y_1 - h_1^{UK}) dy_1 = 0$. It is clear that $h_1^{UK} = 0$ satisfies the first order condition as $p(y_1) \varphi_s(y_1)$ is an odd function of y_1 and therefore $-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=0} = 0$. Furthermore, the negative of the first derivative can be expressed as $-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}} = \int_0^{\infty} p(y_1) (\varphi_s(y_1 - h_1^{UK}) - \varphi_s(y_1 + h_1^{UK})) dy_1 + h_1^{UK} \int_{-\infty}^{\infty} p'(y_1) \varphi_s(y_1 - h_1^{UK}) dy_1$. Note that $\varphi_s(\cdot) \geq (>) 0$, for all (some) y_1 and h_1^{UK} , $p(y_1) \geq (>) 0$ for all (some) $y_1 \geq 0$ and $p'(y_1) \geq 0$ for all (some) y_1 . Furthermore, from Corollary A.1.1 in Appendix A, we know that $\varphi_s(y_1 - h_1^{UK}) > \varphi_s(y_1 + h_1^{UK})$ if and only if $h_1^{UK} > 0$ and $y_1 > 0$. Therefore, $-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}}$ is strictly single-crossing, which proves that the objective function (17) is quasiconcave and achieves the maximum at $h_1^{UK} = 0$.

If $\theta = \bar{\theta}$, then it is clear from (17) that $h_1^{UK} < 0$ cannot be the best response as it leads to negative expected profits, and there would be a profitable deviation to $h_1^{UK} = 0$. The negative of the first derivative is now $-\pi'(h_1^{UK}) = -\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}} = \int_{-\infty}^{\infty} (p(h_1^{UK} + s_1) + h_1^{UK} p'(h_1^{UK} + s_1) - \bar{\theta}) \varphi_s(s_1) ds_1 = \int_{-\infty}^{\infty} (p(y_1) + h_1^{UK} p'(y_1) - \bar{\theta}) \varphi_s(y_1 - h_1^{UK}) dy_1$. The solution on $-\pi'(h_1^{UK}) = 0$ is a unique maximum if $-\pi'(h_1^{UK})$ is a strictly single crossing function—that is $-\pi'(h) \geq 0$ implies that $-\pi'(\tilde{h}) > 0$ for any $0 < h < \tilde{h}$. Using the expression for $-\pi'(h_1^{UK})$ we require that $(\tilde{h} - h) \int_{-\infty}^{\infty} p'(y_1) \varphi_s(y_1 - \tilde{h}) ds_1 + \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - \tilde{h}) dy_1 > 0$. The first term is clearly positive. The second term can be written as $\int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h) dy_1$, where $\frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)}$ is increasing in y_1 due to log-concavity (see Lemma A.1).

Notice that if $hp'(y_1) + p(y_1) - \theta$ is single crossing in y_1 , we can prove that this integral is non-negative similarly to Lemma 5 and Extension to Lemma 5 in Athey (2002). Namely, suppose that $hp'(y_1) + p(y_1) - \theta$ is single crossing in y_1 then there exists $y_1 = \bar{y}$ such that $hp'(y_1) + p(y_1) - \theta < (>) 0$ for any $y_1 < (>) \bar{y}$. Furthermore, it is clear that $\frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} <$

($>$) $\frac{\varphi_s(\bar{y}-\tilde{h})}{\varphi_s(\bar{y}-h)}$ for any $y_1 < (\mathcal{G}) \bar{y}$. Then we obtain that

$$\begin{aligned} & \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h) dy_1 = \\ & \int_{-\infty}^{\bar{y}} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h) dy_1 + \\ & \int_{\bar{y}}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h) dy_1 > \\ & \frac{\varphi_s(\bar{y} - \tilde{h})}{\varphi_s(\bar{y} - h)} \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h) dy_1 = \frac{\varphi_s(\bar{y} - \tilde{h})}{\varphi_s(\bar{y} - h)} \cdot (-\pi'(h)) \geq 0, \end{aligned}$$

where the first inequality follows from the monotonicity of $\frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)}$. Overall in such case $-\pi'(h) \geq 0$ indeed implies that $-\pi'(\tilde{h}) \geq 0$. Note that a sufficient (but not necessary) condition for $hp'(y_1) + p(y_1) - \theta$ to be single crossing in y_1 is that $\frac{\theta - p(y_1)}{p'(y_1)}$ is decreasing in y_1 , i.e., $\theta - p(y_1)$ is log-concave.⁴⁵

We can also identify a somewhat more general sufficient condition for the term $\int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h) dy_1$ to be non-negative using Chebyshev's integral inequality. Namely, using Theorem A.5 from Appendix A, it holds that the sufficient condition for $\int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h) dy_1 \geq \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h) dy_1 = -\pi'(h)$ is that for every t

$$\frac{\int_{-\infty}^t (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h) dy_1}{\int_{-\infty}^t \varphi_s(y_1 - h) dy_1} \geq \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h) dy_1.$$

This condition can also be written as

$$\mathbb{E}[hp'(h + s_1) + p(h + s_1) | s_1 \leq t - h] \geq \mathbb{E}[hp'(h + s_1) + p(h + s_1) | s_1 > t - h].$$

As $\mathbb{E}[p(h + s_1) | s_1 \leq t - h] \geq \mathbb{E}[p(h + s_1) | s_1 > t - h]$ due to the fact that the price is increasing in the order flow, this condition essentially requires that the slope of $p(y_1)$ at high order flows is not too small compared to the slope at small order flows and is less restrictive than requiring $hp'(y_1) + p(y_1) - \theta$ to be single crossing.

While numerically both sufficient conditions clearly hold for a wide set of parameters, to the best of our knowledge there are no more mathematical results that we can apply to our setting

⁴⁵Using the results from Lemma 3, we can prove that this is indeed the case for η close to 0 or 1 and $\gamma = 0$.

to derive further analytical results. Overall, the necessary (and the least restrictive) condition for quasiconcavity is that if $-\pi'(h) > 0$ then

$$\left(\tilde{h} - h\right) \int_{-\infty}^{\infty} p'(y_1) \varphi_s(y_1 - \tilde{h}) ds_1 + \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1)) \left(\frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} - 1\right) \varphi_s(y_1 - h) dy_1 > 0,$$

which appears to always hold, at least numerically.

As the problem is symmetric, similar arguments apply for $\theta = -\bar{\theta}$ as well as for the quasiconcavity in own demand in the state $R = I$.

Proof of Proposition 2

We already know from the previous part that when $\theta = 0$, the unique solution is $h_1^{UK} = h_1^{IK} = h_1^{IP} = 0$. So let us focus on the case $\theta = \bar{\theta}$. Provided that the trader's problem has a unique maximum in own demand, we focus on the first order conditions.

$$\begin{aligned} -\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} &= \int_{-\infty}^{\infty} (p(y_1) + h_1^{UK} p'(y_1) - \bar{\theta}) \varphi_s(y_1 - h_1^{UK}) dy_1 = 0 \\ -\frac{\partial \pi_2^{IJ}}{\partial h_1^{IJ}} &= \int_{-\infty}^{\infty} (p(y_1) + h_1^{IJ} p'(y_1) - \bar{\theta}) \varphi_s(y_1 - h_1^{IK} - h_1^{IP}) dy_1 = 0 \end{aligned}$$

which by integration by parts can be also expressed as

$$\begin{aligned} -\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} &= \int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) \left(\frac{h_1^{UK} (h_1^{UK} - y_1)}{\sigma^2} - 1\right) \varphi_s(y_1 - h_1^{UK}) dy_1 = 0 \quad (23) \\ -\frac{\partial \pi_2^{IJ}}{\partial h_1^{IJ}} &= \int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) \left(\frac{h_1^{IJ} (h_1^{UK} + h_1^{IK} - y_1)}{\sigma^2} - 1\right) \varphi_s(y_1 - h_1^{IK} - h_1^{IP}) dy_1 = 0 \end{aligned}$$

It is straightforward to prove that K and P must trade the same quantity in equilibrium in state $R = I$ and that the solution is symmetric for $\theta = \bar{\theta}$ and $\theta = -\bar{\theta}$. In equilibrium the Market's beliefs must be consistent with optimal strategies, i.e., it must hold that $h_1^{IK} = h_1^{IP} = \frac{\bar{g}_I}{2}$ and $h_1^{UK} = \bar{g}_U$. Define $\mu_R \equiv \frac{\bar{g}_R}{\sigma_s}$ for $R \in \{I, U\}$ and $z \equiv \frac{y_2}{\sigma_s}$. Using the expression for normal density we can express $\varphi_s(y_2 - \bar{g}_R) = \frac{1}{\sigma_s} \phi(z - \mu_R)$, $\varphi_s(y_2) = \frac{1}{\sigma_s} \phi(z)$ and $\varphi_s(y_2 + \bar{g}_R) = \frac{1}{\sigma_s} \phi(z + \mu_R)$, where ϕ is the p.d.f. of a standard normal. Using (14), we then find that

$$\check{p}(z) \equiv p_1(z\sigma_s) = \bar{\theta} \frac{\eta \phi(z - \mu_I) + (1 - \eta) \phi(z - \mu_U) - \eta \phi(z + \mu_I) - (1 - \eta) \phi(z + \mu_U)}{\eta \phi(z - \mu_I) + (1 - \eta) \phi(z - \mu_U) + \eta \phi(z + \mu_I) + (1 - \eta) \phi(z + \mu_U) + \frac{2\gamma}{1-\gamma} \phi(z)}$$

that clearly does not depend on σ_s and it holds that $p_1(z\sigma_s) = -p_1(-z\sigma_s)$. Using these in (23) and equating $\frac{\partial \pi_2^J}{\partial h_1^J} = 0$ for $J \in \{K, P\}$; $\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} = 0$, we find that μ_I and μ_U are the positive

solutions of

$$\frac{\partial \pi_2^{IJ}}{\partial h_1^{IJ}} \Big|_{h_1^{IJ} = \bar{g}_I} = - \left(1 - \frac{\mu_I^2}{2}\right) \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) \phi(z - \mu_I) dz + \frac{\mu_I}{2} \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) z \phi(z - \mu_I) dz = 0 \quad (24)$$

$$\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} \Big|_{h_1^{UK} = \bar{g}_U} = - (1 - \mu_U^2) \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) \phi(z - \mu_U) dz + \mu_U \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) z \phi(z - \mu_U) dz = 0$$

For part 2 notice that from (24), we can express the first order condition of trader $J \in \{K, P\}$ in state $R = I$ as

$$\frac{\partial \pi_2^{IJ}}{\partial h_1^{IJ}} \Big|_{h_1^{IJ} = \bar{g}_I} = -\frac{1}{2} \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) \phi(z - \mu_I) dz - \frac{1}{2} \frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} \Big|_{h_1^{UK} = \bar{g}_I} = 0$$

By part 6 of Lemma 3 $(\bar{\theta} - p_1(y_1)) > 0$ for all finite y_1 . Therefore, also $(\bar{\theta} - \check{p}(z)) > 0$ for all finite z and $(\bar{\theta} - \check{p}(z)) \phi(z - \mu_I) \geq 0$ with strict inequality for some z . This implies that it must hold that

$$-\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} \Big|_{h_1^{UK} = \bar{g}_I} > 0.$$

Because $-\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}}$ is a single-crossing function and $\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} \Big|_{h_1^{UK} = \bar{g}_U} = 0$, it then follows that $\bar{g}_I > \bar{g}_U$.

For the uninformed trader's strategy, we need to verify that it is optimal for him to trade zero. We now verify that the first order condition of his problem indeed holds at zero. Define $\Delta(y_1) \equiv \mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]$ and $Q_{1U}(y_1) \equiv \mathbb{E}[Q_2|y_1, U]$. By (15) in Lemma 3 (and also by Lemma A.1 in Appendix A), it holds that $\Delta(y_1) = -\Delta(-y_1)$. Also, it is clear from (16) that it holds that $Q_1(y_1) = Q_1(-y_1)$. Using this in (9) and (11) we confirm that (11) $Q_{1U}(y_1) = Q_{1U}(-y_1)$.

Recalling uninformed P 's optimal trading strategy at date 2 from (12) in Theorem 1 and using (6) and (8), we can then find P 's expected profit at date 2 conditional on y_1 as

$$\pi_2^{UP} = \begin{cases} \sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) & \text{if } \Delta(y_1) > 0 \\ -\sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) & \text{if } \Delta(y_1) < 0 \end{cases}$$

Suppose that at date 1, uninformed P trades h_1^{UP} , then he also knows that the distribution of the total order flow is $f_s(y_1 - h_1^{UP} - \bar{g}_U)$ if $\theta = \bar{\theta}$; $f_s(y_1 - h_1^{UP})$ if $\theta = 0$ and $f_s(y_1 - h_1^{UP} + \bar{g}_U)$ if $\theta = -\bar{\theta}$. Using all this, $\mathbb{E}[\theta|U] = 0$, and we can use the law of iterated expectations to express

the expected profit of uninformed P before date 1 trading as

$$\begin{aligned}\pi_1^{UP} &= -h_1^{UP} \int_{-\infty}^{\infty} p_1(y_1) (m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP})) dy_1 + \\ &\int_{\Delta(y_1) > 0} \sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) (m(y_1 - h_1^{UP}) + m(y_1 + h_1^{UP})) dy_1,\end{aligned}$$

where $m(x) \equiv \frac{1-\gamma}{2} \varphi_s(x - \bar{g}_U) + \gamma \varphi_s(x) + \frac{1-\gamma}{2} \varphi_s(x + \bar{g}_U)$. Because of symmetry $\varphi_s(\cdot)$ it holds that $m(x) = m(-x)$ and $m'(x) = -m'(-x)$.

The first derivative of the profit is

$$\begin{aligned}\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}} &= -\int_0^{\infty} p_1(y_1) (m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP})) dy_1 \\ &+ h_1^{UP} \int_0^{\infty} p_1(y_1) (m'(y_1 - h_1^{UP}) + m'(y_1 + h_1^{UP})) dy_1 \\ &- \int_{\Delta(y_1) < 0} \sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) (m'(y_1 - h_1^{UP}) - m'(y_1 + h_1^{UP})) dy_1\end{aligned}$$

Replacing in $h_1^{UP} = 0$, we can now verify that $\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}|_{h_1^{UP}=0} = 0$. For the intuition that $h_1^{UP} = 0$ is also a global maximum, notice that $-\int_0^{\infty} p_1(y_1) (m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP})) dy_1 = -\int_{-\infty}^{\infty} (p_1(s_1 + h_1^{UP}) - p_1(s_1 - h_1^{UP})) m(s_1) ds_1$. Due to increasing prices, the first term is negative iff $h_1^{UP} > 0$. Also, using integration by parts, the second term is $h_1^{UP} \int_0^{\infty} p_1(y_1) (m'(y_1 - h_1^{UP}) + m'(y_1 + h_1^{UP})) dy_1 = -h_1^{UP} \int_0^{\infty} p_1'(y_1) (m(y_1 - h_1^{UP}) + m(y_1 + h_1^{UP})) dy_1$ and negative iff $h_1^{UP} > 0$. Both of these effects alone would guarantee that $-\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}$ is strictly single crossing at 0 as any trading by P at date 1 would lead to short term losses in expectations. The sign of the last term is ambiguous and reflects the fact that by deviating to a non-zero demand at date 1, P could affect the probability he expects the market to assign on him being informed at date 2 and the area where P would pursue different price-contingent strategies. However, it can be verified that this term is relatively small compared to the first two terms and $-\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}$ remains single crossing at 0. This is true at least as long as η is not too close to one.

Proof of Proposition 3

Assuming $\gamma = 0$, we obtain from (15) that $sgn(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]) = sgn\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(y_1 + \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)}\right) = sgn\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)}\right)$. Because $\varphi_s(\cdot)$ is log-concave and $\bar{g}_I > \bar{g}_U$ by part 2 in Proposition 2, it holds by the property of log-concave distributions in Lemma A.1 in Appendix A that $sgn\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)}\right) = -1$ if $y_1 > -y_1 \Leftrightarrow y_1 > 0$ and

$\text{sgn} \left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)} \right) = 1$ if $y_1 < -y_1 \Leftrightarrow y_1 < 0$. By (6) $\text{sgn} (\mathbb{E} [\theta|y_1, U] - \mathbb{E} [\theta|y_1, I]) = \text{sgn} (\mathbb{E} [\theta|y_1, U] - p_1)$ for any $0 < Q_1 < 1$, which is true for any $0 < \eta < 1$. Uninformed P 's optimal strategy at date 2, equation (12) in Theorem 1, and the definition of contrarian strategy in Section 4.2 complete the proof.

Proof of Proposition 4

To prove part 1 we use (15) to find that $\text{sgn} (\mathbb{E} [\theta|y_1, U] - \mathbb{E} [\theta|y_1, I]) = \text{sgn} \left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(y_1 + \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)} + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1 + \bar{g}_U)\varphi_s(y_1 + \bar{g}_I)} B(y_1) \right)$, where $B(y_1) \equiv \varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 + \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) + \varphi_s(y_1 + \bar{g}_I)$. Consider $y_1 > 0$ and let us focus on the sign of $B(y_1)$. Because $\varphi_s(\cdot)$ has a maximum at zero and is decreasing for any positive values, it also holds for any $y_1 > 0$ and $\bar{g}_I > \bar{g}_U$ that $-\varphi_s(y_1 + \bar{g}_U) + \varphi_s(y_1 + \bar{g}_I) < 0$. We can then prove that the necessary and sufficient condition for $\varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) \leq 0$ is that $y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2}$. Namely, defining $b \equiv y_1 - \frac{\bar{g}_I + \bar{g}_U}{2}$, it holds that $\varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) = \varphi_s(b + \frac{\bar{g}_I - \bar{g}_U}{2}) - \varphi_s(b - \frac{\bar{g}_I - \bar{g}_U}{2})$, which is indeed non-positive if and only if $b \geq 0$ (see Corollary A.1.1 in Appendix A and recall that $\bar{g}_I > \bar{g}_U$). Therefore, for any $y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2}$ it holds that $B(y_1) < 0$. From the proof of Proposition 3 (and Lemma A.1 in Appendix A), we already know that $\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(y_1 + \bar{g}_U)} < \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)}$ for any $y_1 > 0$. Therefore, $\text{sgn} (\mathbb{E} [\theta|y_1, U] - \mathbb{E} [\theta|y_1, I]) = \text{sgn} (\mathbb{E} [\theta|y_1, U] - p_1) = -1$ for any $y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2}$ and $0 < \eta < 1$. The definition of contrarian strategy in Section 4.2 completes this part of the proof. The proof for $y_1 \leq -\frac{\bar{g}_I + \bar{g}_U}{2}$ is similar due to symmetry.

To prove part 2, notice that the function determining the sign of $\mathbb{E} [\theta|y_1, U] - \mathbb{E} [\theta|y_1, I]$ can be expressed as

$$S(y_1) \equiv \left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(y_1 + \bar{g}_U)} - 1 \right) \left(1 + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1 + \bar{g}_I)} \right) - \left(\frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)} - 1 \right) \left(1 + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1 + \bar{g}_U)} \right),$$

which using the expression for the normal density becomes

$$S(y_1) = \left(\exp \left(\frac{2\bar{g}_U y_1}{\sigma_s^2} \right) - 1 \right) \left(1 + \frac{\gamma}{1-\gamma} \exp \left(\frac{2\bar{g}_I y_1 + \bar{g}_I^2}{2\sigma_s^2} \right) \right) - \left(\exp \left(\frac{2\bar{g}_I y_1}{\sigma_s^2} \right) - 1 \right) \left(1 + \frac{\gamma}{1-\gamma} \exp \left(\frac{2\bar{g}_U y_1 + \bar{g}_U^2}{2\sigma_s^2} \right) \right)$$

It is clear that $S(0) = 0$. Let us consider ε arbitrarily close to zero. By Taylor approximation, we find that $S(\varepsilon) = S'(0)\varepsilon$, where $S'(0) = \frac{2\bar{g}_U}{\sigma_s^2} \left(1 + \frac{\gamma}{1-\gamma} \exp \left(\frac{\bar{g}_I^2}{2\sigma_s^2} \right) \right) - \frac{2\bar{g}_I}{\sigma_s^2} \left(1 + \frac{\gamma}{1-\gamma} \exp \left(\frac{\bar{g}_U^2}{2\sigma_s^2} \right) \right)$. Using then $\bar{g}_R = \mu_R \sigma_s$, it is clear that $S'(0) > 0$ iff (19) holds, which by $\text{sgn} (\mathbb{E} [\theta|\varepsilon, U] - \mathbb{E} [\theta|\varepsilon, I]) = \text{sgn} (S(\varepsilon)) = 1 (-1)$ if $\varepsilon > (<) 0$ and definition from Section 4.2 implies a trend-following strategy.

If the condition (19) hold, then $S(\varepsilon) > 0$ for some small $\varepsilon > 0$, while $S\left(\frac{\bar{g}_I + \bar{g}_U}{2}\right) < 0$. As $S(y_1)$ is a continuous function of the order flow y_1 , there must exist an order flow in the interval of $(\varepsilon, \frac{\bar{g}_I + \bar{g}_U}{2})$, where $S(y_1)$ changes its sign.

Proof of Proposition 6

Recall from (12) and the definition of strategies in Section 4.2 that P will optimally pursue and profit from contrarian strategy if $\mathbb{E}[\theta|y_1, U] - p_1$ has an opposite sign compared to y_1 . Then from (6), it holds that $\mathbb{E}[\theta|y_1, U] - p_1 = Q_1(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I])$. It is clear that without the presence of any informed traders at date 1, no fundamental information can be contained in date 1 order flow in state $R = U$, i.e., with any symmetric prior $E[\theta|y_1, U] = E[\theta|U] = 0$. Therefore, $E[\theta|y_1, U] - p_1 = -p_1 = -Q_1 E[\theta|y_1, I]$. Also, similarly to the proof of Proposition 7, we know that $g(\theta)$ must be increasing in θ and symmetric $g(\theta) = g(-\theta)$. These facts and the log-concavity of noise trading together with Lemma A.2 in Appendix A ensure that $E[\theta|y_1, I]$ is an odd function that is increasing in y_1 and the sign of $E[\theta|y_1, U] - p_1$ is opposite to the sign of y_1 . Using Bayes' rule $Q_1 = \frac{\eta f(y_1|I)}{\eta f(y_1|I) + (1-\eta)f(y_1|U)} = \frac{\eta}{\eta + (1-\eta)\frac{f(y_1|U)}{f(y_1|I)}}$, where $\frac{f(y_1|U)}{f(y_1|I)} = \frac{\varphi_s(y_1)}{\int_{\theta} \varphi_s(y_1 - g(\theta))f(\theta)d\theta}$.⁴⁶ As $\frac{f(y_1|U)}{f(y_1|I)}$ is finite, it is clear that $\eta > 0$ is sufficient for $Q_1 > 0$.

Proof of Proposition 7

Part 1 follows from results in the monotone comparative statics literature that we can use to explore informed traders' profits (2) and (1). Denote in state $R = U$ trader K 's expected price when demanding h_1^{UK} as $p_E(h_1^{UK}) \equiv \int_{-\bar{s}}^{\bar{s}} p_1(h_1^{UK} + s_1) f_s(s_1) ds_1$. From Milgrom and Shannon (1994) it is known that $g_U(\theta) = \arg \max_{h_1^{UK}} h_1^{UK}(\theta - p_E(h_1^{UK}))$ is weakly increasing in θ if the trader's problem has increasing differences (which also implies the payoff is super-modular) in h_1^{UK} and θ . This is indeed true as for any $\tilde{\theta} > \theta$ and $\tilde{h}_1^{UK} > h_1^{UK}$, it holds that $\tilde{h}_1^{UK}(\tilde{\theta} - p_E(h_1^{UK})) - \tilde{h}_1^{UK}(\theta - p_E(\tilde{h}_1^{UK})) > h_1^{UK}(\tilde{\theta} - p_E(h_1^{UK})) - h_1^{UK}(\theta - p_E(h_1^{UK})) \iff (\tilde{h}_1^{UK} - h_1^{UK})(\tilde{\theta} - \theta) > 0$. From Edlin and Shannon (1998), it is also known that $g_U(\theta)$ is strictly increasing if the first derivative of the payoff (profit) is strictly increasing in θ , which is also true in our model, as $\partial h_1^{UK}(\theta - p_E(h_1^{UK})) / \partial h_1^{UK} = \theta - p_E(h_1^{UK}) - h_1^{UK} p'_E(h_1^{UK})$ is clearly increasing in θ . The proof is similar for the state $R = I$, where the same monotone comparative statics establish that K 's and P 's individual demand is increasing in θ , and so is the sum of their demands.

For part 2, notice that given the above assumptions, it is enough to only look at the first order conditions to find the unique equilibrium demands by all informed traders. Also, it is easy to verify that given θ , both K and P demand the same quantity in state $R = I$. We find that the

⁴⁶Without explicitly solving, we expect and impose in this expression that P does not trade in date 1 equilibrium if uninformed. Similar arguments would also hold if there existed an equilibrium where P trades at date 1.

equilibrium total informed demand $g_R(\theta)$ in state R solves

$$\begin{aligned}\theta &= \int_{-\bar{s}}^{\bar{s}} (p_1(s_1 + g_U(\theta)) + g_U(\theta) p_1'(s_1 + g_U(\theta))) f_s(s_1) ds_1 \\ \theta &= \int_{-\bar{s}}^{\bar{s}} \left(p_1(s_1 + g_I(\theta)) + \frac{g_I(\theta)}{2} p_1'(s_1 + g_I(\theta)) \right) f_s(s_1) ds_1\end{aligned}$$

As by part 1 $g_R(\theta)$ is invertible, it must also hold that

$$\begin{aligned}g_U^{-1}(y_\theta) &= \int_{-\bar{s}}^{\bar{s}} (p_1(s_1 + y_\theta) + y_\theta p_1'(s_1 + y_\theta)) f_s(s_1) ds_1 \\ g_I^{-1}(y_\theta) &= \int_{-\bar{s}}^{\bar{s}} \left(p_1(s_1 + y_\theta) + \frac{y_\theta}{2} p_1'(s_1 + y_\theta) \right) f_s(s_1) ds_1,\end{aligned}\tag{25}$$

where an order flow $y_\theta \equiv g_R(\theta)$. It is straightforward to verify that $g_R(\theta) = -g_R(-\theta)$ and clearly $y_\theta > 0$ iff $\theta > 0$. Because date 1 equilibrium price is increasing in the order flow, it holds that $g_U^{-1}(y_\theta) - g_I^{-1}(y_\theta) = \frac{y_\theta}{2} \int_{-\bar{s}}^{\bar{s}} p_1'(s_1 + y_\theta) f_s(s_1) ds_1 > (<) 0$ for any $y_\theta > (<) 0$. Taken $y_\theta = g_U(\theta) > 0$, we find that $g_U^{-1}(y_\theta) > g_I^{-1}(y_\theta) \iff \theta > g_I^{-1}(g_U(\theta)) \iff g_I(\theta) > g_U(\theta)$ for any $\theta > 0$. The case $y_\theta < 0$ is immediate by symmetry.

C Normal prior

We consider two cases. First, we explore a particular dependence structure of R and θ that guarantees the linearity of date 1 equilibrium and is thus directly comparable to the benchmark of Kyle (1985). Second, we explore a normal prior independent of the state that is comparable with our baseline setting.

C.1 Normal prior with a particular dependence structure

Assume that

$$\begin{aligned}f_{\theta R}(\theta) &= \Pr(R|\theta) f(\theta) = \Pr(R) f(\theta|R) = \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{\eta}{\sigma_I} \exp\left(-\frac{\theta^2}{2\sigma_I^2}\right) \right)^{\tilde{1}_\theta} \left(\frac{1-\eta}{\sigma_U} \exp\left(-\frac{\theta^2}{2\sigma_U^2}\right) \right)^{1-\tilde{1}_\theta},\end{aligned}\tag{26}$$

where $\tilde{1}_\theta$ is an indicator function that takes values $\tilde{1}_\theta = 1$ if $R = I$ and $\tilde{1}_\theta = 0$ if $R = U$ and where σ_I and σ_U are the standard deviations of the prior distribution in state $R = I$ and $R = U$ respectively. Furthermore, assume that $\sigma_I = \frac{3}{4}\sigma_U$. As shown shortly, this assumption will guarantee that date 1 order flow does not allow updating the types and therefore preserves

linearity.

We solve this problem using the standard technique. We conjecture and later verify that there is a rational expectations equilibrium where P does not trade in date 1 and that date 1 price is linear in the order flow, i.e.,

$$p_1 = \lambda_1 y_1, \quad (27)$$

where λ_1 is a constant to be solved for in the equilibrium.⁴⁷

Lemma C.1 *When the prior is given by (26), then there exists a rational expectations equilibrium where the following holds.*

1. *Informed traders' optimal demand in date 1 is*

$$h_1^{UK} = \frac{\theta}{2\lambda_1}; h_1^{IK} = h_1^{IP} = \frac{\theta}{3\lambda_1}$$

$$g_U(\theta) \equiv h_1^{UK} = \frac{\theta}{2\lambda_1} \text{ and } g_I(\theta) \equiv h_1^{IK} + h_1^{IP} = \frac{2\theta}{3\lambda_1}$$

2. *There is no updating about P 's type in the first trading round, i.e.,*

$$Q_1 = \Pr(I|y_1) = \eta.$$

3. *Equilibrium price is given by*

$$p_1 = \lambda_1 y_1,$$

where $\lambda_1 = \frac{\sigma_U}{2\sigma_s} \sqrt{\frac{(2-\eta)}{2}}$ and it holds that

$$\mathbb{E}[\theta|y_1, I] = \frac{3}{4-\eta} \lambda_1 y < p_1 < \frac{4}{4-\eta} \lambda_1 y_1 = \mathbb{E}[\theta|y_1, U]$$

4. *Uninformed P does not trade at date 1, $h_1^{UP} = 0$.*

Proof. Part 1: Given the conjectured price (27) and the total order flow (4), we find that in state $R = U$, the informed trader's expected profit (2) is given by $\mathbb{E}[h_2^{UK}(\theta - p_1)] = h_2^{UK}(\theta - \lambda_1 h_2^{UK})$ and K 's optimal demand is $h_1^{UK} = \frac{\theta}{2\lambda_1}$. From (27) and (4), trader J 's expected profit is $\mathbb{E}[h_2^{IJ}(\theta - p_1)] = h_2^{IJ}(\theta - \lambda_1 h_2^{IJ} - \lambda_1 h_2^{I\bar{J}})$, where $J, \bar{J} \in \{K, P\}$ and $J \neq \bar{J}$ and we find that the optimal demand is the same for K and P , and $h_1^{IK} = h_1^{IP} = \theta/3\lambda_1$. Part 2: The total order flow at date 1 in state $R = U$ is $y_2 = \theta/2\lambda_1 + s_1$. As $\theta|U \sim \mathcal{N}(0, \sigma_U^2)$ it holds that $y_2|U \sim \mathcal{N}(0, \sigma_U^2/4\lambda_1^2 + \sigma_s^2)$. The total order flow in state $R = I$ is $y_2 = \frac{2\theta}{3\lambda_1} + s_1$. Using that $\sigma_I =$

⁴⁷The problem can also be solved without immediately imposing this conjecture (see, e.g., Cho and El Karoui, (2000)).

$\frac{3}{4}\sigma_U$, $\theta|I \sim \mathcal{N}(0, \sigma_I^2) \sim \mathcal{N}(0, 9\sigma_U^2/16)$ and $y_2|I \sim \mathcal{N}(0, 4\sigma_I^2/9\lambda_1^2 + \sigma_s^2) \sim \mathcal{N}(0, \sigma_U^2/4\lambda_1^2 + \sigma_s^2)$. Clearly $f(y_1|U) = f(y_1|I)$, so by Bayes' rule $Q_1 = \Pr(I|y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|I) + (1-\eta)f(y_1|U)} = \eta$. Part 3: If $R = U$, then the signal that the Market obtains from the order flow is $2\lambda_1 y_2 = \theta + 2\lambda_1 s_1$, where $2\lambda_1 y_2|\theta \sim \mathcal{N}(\theta, 4\lambda_1^2 \sigma_s^2)$. As well known in the case of normally distributed prior and signals, the posterior is a precision-weighted average of the signals, hence we can simplify $\mathbb{E}[\theta|y_1, U] = \frac{\sigma_U^2}{4\lambda_1^2 \sigma_s^2 + \sigma_U^2} 2\lambda_1 y_2$. If $R = I$ then the signal that the Market obtains from the order flow is $\frac{3}{2}\lambda_1 y_2 = \theta + \frac{3}{2}\lambda_1 s_1$, where $\frac{3}{2}\lambda_1 y_2|\theta \sim \mathcal{N}(\theta, \frac{9}{4}\lambda_1^2 \sigma_s^2)$. Using $\sigma_I = \frac{3}{4}\sigma_U$ and simplifying, we find $\mathbb{E}[\theta|y_1, I] = \frac{\sigma_U^2}{4\lambda_1^2 \sigma_s^2 + \sigma_U^2} \frac{3}{2}\lambda_1 y_2$. Given this and $Q_1 = \eta$, (6), we obtain that $p_1 = \frac{\sigma_U^2[(3/2)\eta + 2(1-\eta)]}{4\lambda_1^2 \sigma_s^2 + \sigma_U^2} \lambda_1 y_2$. Equating coefficients with those in the conjectured prices (27), we find that $\lambda_1^2 = \sigma_U^2(2-\eta)/8\sigma_s^2$ and the positive solution of this proves the first part of the proposition. We then use the equilibrium value of λ_1 in the expressions of $\mathbb{E}[\theta|y_1, U]$ and $\mathbb{E}[\theta|y_1, I]$. For part 4, it is easy to verify that $h_1^{UP} = 0$ satisfies the first order condition. ■

We find from Lemma C.3. that while uninformed P optimally does not trade at date 1, the first trading round generates his information advantage for date 2 trading. We can see that for any non-zero order flow, there will be a difference between the expected value conditional on knowing the state, $\mathbb{E}[\theta|y_1, R]$, and date 1 equilibrium price. Furthermore, by the definition of P 's strategy in Section 4.2 and Theorem 1, we find that in this setting P always pursues a trend-following trading strategy in date 2, i.e., he buys at date 2 if date 1 order flow is positive (or equivalently, if the price has increased at date 1) and sells when date 1 order flow is negative (or equivalently, if the price has decreased).⁴⁸

Under this prior, uninformed P always perceives the price at date 2 to be too close to the prior mean. Indeed, the sensitivity of prices to order flow, λ_1 , in Theorem 1 is too low given that the true state is $R = U$. (If the Market knew the state, he would set $\lambda_1 = \frac{\sigma_U}{2\sigma_s}$ as in Kyle (1985)). This is the outcome of two competing effects. The first effect is related to the value of the fundamental revealed by the order flow alone. The fundamental is the inverse of total informed trader's demand at $y_1 - s_1$ $\theta = g_U^{-1}(y_1 - s_1) = 2\lambda_1(y_1 - s_1)$ and $\theta = g_I^{-1}(y_1 - s_1) = \frac{3}{2}\lambda_1(y_1 - s_1)$. It is clear that, conditional on the order flow *alone*, the expected value of the fundamental is higher in state $R = U$. This is true because the expected value $2\lambda_1 > \frac{3}{2}\lambda_1$. This effect tends to make the Market set the price too low at date 1. The second effect relates to the dispersion of the fundamental conditional on the order flow alone, which is higher in $R = U$ (because the variance $4\lambda_1^2 \sigma_s^2 > \frac{9}{4}\lambda_1^2 \sigma_s^2$). Because the Market uses Bayes' rule to set the price and puts higher weight on signals that are less noisy, this effect tends to make the Market set the price too high at date 1. With a normal prior given by (26), the first effect always dominates.

⁴⁸When we refer to price changes at date 1, we adopt the convention that date 0 price equals the prior $p_0 = \mathbb{E}[\theta] = 0$, which is consistent with our assumption of market efficiency.

C.2 Normal prior independent of the state

We now consider the prior density of the fundamental $f_\theta(\theta) = \frac{1}{\sqrt{2\pi}\sigma_p} \exp\left(-\frac{\theta^2}{2\sigma_p^2}\right)$ and assume that the fundamental is independent of the state, i.e., $f(\theta|R) = f_\theta(\theta)$ for $R \in \{I, U\}$. We can no longer conjecture that the price is linear in the order flow, because the Market will learn about the state from the order flow. Instead, we conjecture that the Market believes that total informed trading in state $R \in \{I, U\}$ is $g_R(\theta) = -g_R(-\theta)$ and P does not trade at date one if the state is $R = U$. Conditional on the state, $\mathbb{E}[\theta|y_1, R] = \int_{-\infty}^{\infty} \theta f(\theta|y_1, R) d\theta$ as we know the distributions and by Bayes' rule it holds that $f(\theta|y_1, R) = \frac{f(y_1|\theta, R)f(\theta|R)}{f(y_1|R)} = \frac{\varphi_s(y_1 - g_R(\theta))f_\theta(\theta)}{f(y_1|R)}$ and $f(y_1|R) = \int_{-\infty}^{\infty} \varphi_s(y_1 - g_R(\theta)) f_\theta(\theta) d\theta$. Also by Bayes's rule $Q_1 = \Pr(I|y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|U) + \eta f(y_1|I)}$. Using this in (6), we find that date one price is

$$p_1(y_1) = \frac{\int_{-\infty}^{\infty} \theta (\eta \varphi_s(y_1 - g_I(\theta)) + (1 - \eta) \varphi_s(y_1 - g_U(\theta))) f_\theta(\theta) d\theta}{\int_{-\infty}^{\infty} (\eta \varphi_s(y_1 - g_I(\theta)) + (1 - \eta) \varphi_s(y_1 - g_U(\theta))) f_\theta(\theta) d\theta}. \quad (28)$$

It is easy to verify that $p_1(y_1) = -p_1(-y_1)$.

To characterize the equilibrium, we also need the first order conditions of the trader's problem.⁴⁹ After taking the first order conditions in (2) and (1), it is easy to verify that it must hold that in state $R = I$, both informed traders trade the same optimal quantity $h_1^{IK*} = h_1^{UP*}$. Imposing then that the equilibrium beliefs must be consistent with the actual trades, we find after changing variables and simplifying that $g_U(\theta)$ and $g_I(\theta)$ solve

$$\begin{aligned} \theta &= \int_{-\infty}^{\infty} p_1(y_1) \left(1 - \frac{g_U^2(\theta)}{\sigma_s^2} + \frac{g_U(\theta) y_1}{\sigma_s^2}\right) \varphi_s(y_1 - g_U(\theta)) d\theta \\ \theta &= \int_{-\infty}^{\infty} p_1(y_1) \left(1 - \frac{g_I^2(\theta)}{2\sigma_s^2} + \frac{g_I(\theta) y_1}{2\sigma_s^2}\right) \varphi_s(y_1 - g_I(\theta)) d\theta. \end{aligned} \quad (29)$$

and it is straightforward to verify the symmetry of strategies: $g_R(\theta) = -g_R(-\theta)$. Furthermore, Proposition 2 applies and it must hold that $g_I(\theta) > g_U(\theta)$ for any $\theta > 0$. Equations (28) and (29) characterize the functions that determine equilibrium strategies and price. To derive the analytical solution, we can approximate $g_R(\theta)$ with a polynomial, derive the price (28) and change the constants in the polynomial until (29) holds.

For the numerical exercise assume that $\eta = 0.5$, which is the case in which there is most updating about the state R and hence the solution should in principle be most non-linear. Without loss of generality assume $\sigma_s = 1$ and $\sigma_p = 1$ (note that similarly to other settings in this paper, it can be verified that informed trading is proportional to the noise trading variance σ_s). It turns out that informed trader's strategies do not need to be approximated with a

⁴⁹In particular, it will be shown shortly that the price and equilibrium strategies are almost linear, and therefore quasiconcavity of the trader's problem is trivial to verify ex post.

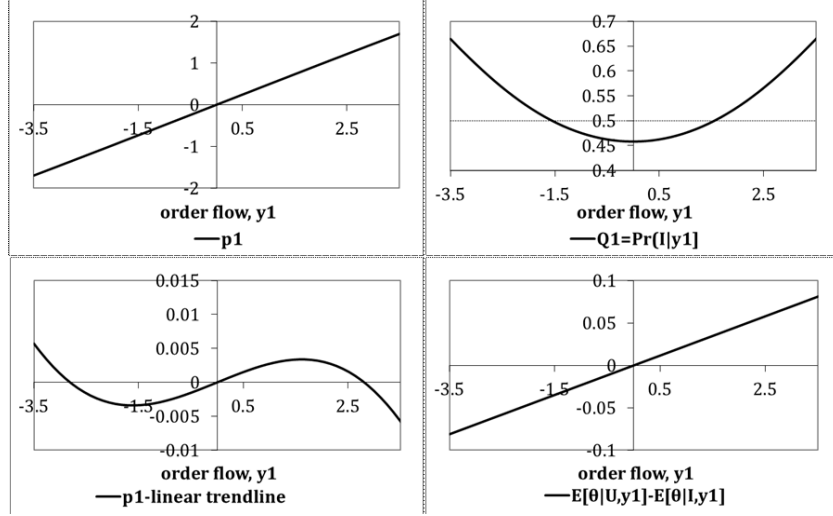


Figure 4: Date 1 price p_1 , updated probability $Q_1 = \Pr(I|y_1)$, and the difference in expectations $\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]$ as a function of the order flow.

high order polynomial, but are already very well approximated by a linear function, namely $g_U(\theta) \approx 1.0284 \cdot \theta$ and $g_I(\theta) \approx 1.3712 \cdot \theta$. Figure (4) presents the relevant results. The reason why trader's strategies are close to linear is that the price under linear strategies is "almost linear," i.e., the north-west panel of Figure (4) shows that it is hard to notice nonlinearity of price (the R^2 of the trendline is effectively 1) - only when we zoom in and show the difference between the price and a linear trendline (south-west panel), do we see that it is slightly non-linear. Because informed traders care about the expected price that they do not know when they submit their orders, these small nonlinearities have very little effect on their optimal strategies. On the north-east panel we see that there is some but limited updating of trader's types. Because two informed traders jointly trade more than one in absolute value, larger order flows in absolute value tend to signal a higher probability that the state is $R = I$. At small order flows, the Market tends to believe that the state is $R = U$, but even at zero, there is not much learning about P 's type and therefore P 's trading opportunities remain. Finally on the south-east panel, we see that the direction of P 's trading at date 2 is the same as in Section 4.3 - trend-following. For any $y_1 > (<) 0$, it holds that $\mathbb{E}[\theta|y_1, U] > (<) \mathbb{E}[\theta|y_1, I] \Leftrightarrow \mathbb{E}[\theta|y_1, U] > (<) p_1$.

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