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ABSTRACT

All-pay auctions with certain and uncertain prizes

We study all-pay auctions with multiple prizes. The players have the same value for all the certain prizes except for one uncertain prize for which each player has a private value. We characterize the equilibrium strategy and show that if the number of prizes is smaller than the number of players, independent of the ranking of the uncertain prize, a player's probability to win as well as his expected utility increases in his value for this prize. We demonstrate that a stochastic dominance relation between two distribution functions of the players' private values may increase but also even decrease the players' ex-ante expected utility as well the players' expected total effort. Also, increasing the number of prizes may decrease the players' ex-ante expected utility. Thus, we may conclude that a larger number of prizes does not necessarily benefit the players in a contest.

JEL Classification: D44, D82, J31 and J41
Keywords: all-pay auctions, contests and uncertain prizes

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1 Introduction

In the 2016 UEFA European Championship, the final tournament will be contested by 24 teams instead of the usual 16 teams. With this expansion to 24 teams, the teams have a much greater chance for qualifying for the final than previously. However, one of the implications of the results of the model studied in this paper is that this might not necessarily be the case. We show that in a contest with multiple prizes, if we add some new prizes, independent of their values, the participants' ex-ante expected utility may not increase. Thus, contestants may prefer a lower number of prizes in a contest.

We consider the all-pay auction to model a multiple prize contest. Different types of contests have been modeled by the all-pay auction including sports competitions, political lobbying, job promotions, and R&D races. In this kind of contest each player submits a bid (effort) and the player who submits the highest bid wins the contest, but, independently of success, all players bear the cost of their bids. All-pay auctions have been studied either under complete information where each player's type (valuation for winning the contest) is common knowledge or under incomplete information where each player's type is private information and only the distribution from which the players' types is drawn is common knowledge.[1] In this paper, we study all-pay auctions both under complete and incomplete information where the players' values for all the prizes are certain except for one prize which is uncertain. We show that the players' behavior in such an environment might be significantly different from environments where all the values for all the prizes are either certain or uncertain.

There are many real-life examples of contests where some values of the prizes are certain and others are not. For instance, in the 2014 FIFA World Cup qualification, the top four teams of the South American's group automatically qualified, whereas the fifth-placed team proceeded to the inter-confederational play-off against the fifth-placed team from Asia. In this case, while the prize

[1] For all-pay auctions under complete information see Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993, 1996, 2012), Che and Gale (1998), and for all-pay auctions under incomplete information see Hillman and Riley (1989), Amann and Leininger (1996), Krishna and Morgan (1997) and Moldovanu and Sela (2006).

for the top four teams is certain, the prize of the fifth-placed team is uncertain since the identity of the fifth-placed team from Asia is not known before the beginning of the tournament. Another example is the US voting procedure where the highest polling candidate is elected in a general election or else nominated in a primary election. In the state primary election, the candidate who receives the higher number of delegates is selected as the party's presidential nominee. In fact, in many cases, depending on the choice of the party's presidential nominee, one of his opponents in the state's primary election becomes the Vice President if the presidential nominee is elected in the general election. Thus, we can say that the value of the first prize in this contest (the party's nominee) is certain while the value of the second prize (the party's vice-nominee) is uncertain.

We analyze all-pay auctions with n players and $m, m \leq n$ prizes where the players have the same values for all the certain prizes except for one uncertain for which each player has a private value. The prizes can be ranked according to their values where the uncertain prize has the k-th highest value, $k = 2, ..., m$, namely, all the players' values for this prize are lower than the $k-1$–th highest prize and higher than the $k+1$–th highest prize. It is important to note that in a contest with n participants the marginal effect of each of the m prizes on the participants' total effort is different. If we increase the value of the 1-th highest prize, it is well known that the total effort will increase. On the other hand, if we increase the n-th highest prize (the lowest one when the number of prizes is equal to the number of players, i.e., $m = n$) the expected total effort will decrease since the values of all the other n-1 higher prizes are actually decreasing compared with the lowest prize. The effect of increasing each of the other prizes on the expected total effort is ambiguous since it relatively changes the other prizes with both lower and higher values. According to this argument, the characterization of the equilibrium strategies in our model when the uncertain prize is the k-th highest prize, $k = 2, ..., m$, is not clear at all. Nonetheless, our characterization of the equilibrium strategy shows that if the number of prizes is smaller than the number of players, $m < n$, independent of the ranking of the uncertain prize, a player's probability to win as well as his expected utility increases in his value for this prize. This result holds also when the number of prizes is the same as the number of players, i.e., $n = m$, where the uncertain prize is not the lowest one. Otherwise, if this prize is the lowest one, we obtain that a player's probability of winning decreases, but his expected utility increases in his value for this prize.

We first analyze the effect of the distribution of the players' private values on their ex-ante expected utility. It turns out that a stochastic dominance relation between two distribution functions may, on the one hand, increase the players' ex-ante expected utility, but, on the other hand, for other distribution functions may decrease it. Furthermore, we show that depending on the location (ranking) of the uncertain prize a stochastic dominance relation may increase but also even decrease the players' expected total effort. In other words, the players' expected total effort in a contest might be lower than in another one with expected higher values of prizes.

We then analyze the effect of the number of prizes on the players' expected utilities. Usually for each player a larger number of prizes improves his (ex-ante) expected utility. However, we show that this intuitive result does not hold in our model. Namely, if the uncertain prize is not the lowest one, then the ex-ante expected utility of a player might be either smaller or larger than that in an all-pay auction with the same number of players and with an additional prize. In particular, we identify conditions under which increasing the number of prizes decreases the players' ex-ante expected payoff. This unique result is not common in standard all-pay auctions under either complete or incomplete information. The intuition for this result is that in our model by adding new prizes the location (ranking) of the uncertain prize changes and accordingly the players' efforts change as well. Thus, our results imply that the optimal location of the uncertain prize may vary according to the distribution of the players' private values.

1.1 Related literature

Moldovanu et al. (2008) studied a model of a uniform price multi-object auction with n buyers and a single, monopolist seller with variable supply where the seller decides on the supply k, and then the buyers decide what to bid. These authors showed that if the distribution of the players' abilities is convex (concave), then the optimal supply in the auction with n buyers is larger (smaller) than $n/2$. By the revenue equivalence theorem, this result holds in our model as well, and indicates that in order to maximize the players' expected total effort the contest designer has to offer an optimal number of prizes which is smaller than the number of players. However, in contrast to our model, in the model of Moldovanu et al. (2008) the players' ex-ante expected utility always increases in the number of prizes.

The papers most related to our work include Moldovanu and Sela (2001) and Moldovanu et al. (2012) both of which study the optimal allocation of prizes in all-pay auctions under incomplete information. Moldovanu and Sela (2001) analyzed an all-pay auction framework with incomplete information in order to find out the optimal allocation of several non-negative prizes (rewards) in contests where the designer has a fixed budget. They showed that when the cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single prize but when they are convex, several positive prizes may be optimal. Moldovanu et al. (2012) studied an optimal contest design in situations where the designer can reward high performance players with positive prizes and punish low performance players with negative ones. They identified conditions under which, even if punishment is costly, punishing the player with the lowest effort is more effective than rewarding the player with the highest effort. They also studied the optimal number of punishments in the contest when punishment is costless. Similarly to these papers and other recent ones on contest theory (for example, Hoppe et al. 2009, 2011 and Moldovanu et al. 2007, 2012) our present paper is based on Barlow and Proschan's (1966, 1975) elegant theory on stochastic order relations among differences of order statistics.

Our paper is also related to the extensive literature on tournaments, initiated originally by Lazear and Rosen (1981). That literature has shown how prizes based on rank orders of performance can be effectively used to provide incentives in labor tournaments (see Green and Stokey (1983), and Nalebuff and Stiglitz (1983)). Akerlof and Holden (2012) further extended the analysis of Lazear and Rosen to the case with multiple prizes. They linked the optimal prize structure to the form of utility functions, and showed that the prize difference between two adjacent top players is often smaller than the prize difference between two adjacent bottom players.

The rest of the paper is organized as follows: Section 2 presents the model. In Sections 3 we analyze the equilibrium strategies when the numbers of prizes is either smaller than or equal to the number of players. In Section 4 we explicitly calculate the players' expected utility and their expected total effort. In Section 5 we analyze the effect of the distributions of the players' private values on their expected utilities as well as their expected total effort. In Section 6 we analyze the effect of the number of prizes on the players' expected utilities. Section 7 concludes.

2 The Model

We consider an all-pay auction with n risk-neutral players. The contest designer allocates $m, m \leq n$ prizes to the n players, $v_{n-m+1} \leq v_{n-m+2} \leq \cdots \leq v_n$. The players have the same values for all the certain prizes except the prize $v_{n-j+1}, j = 2, ..., m$ which is uncertain such that for this prize each player has a private value that is drawn independently from the interval $[v_{n-j}, v_{n-j+2}]$ according to a distribution function F_{n-j+1} that is common knowledge. We assume that F_{n-j+1} has a continuous density $f_{n-j+1} = dF_{n-j+1} > 0$.

Each player i makes an effort x_i. The player with the highest effort wins the highest prize v_n; the player with the second highest effort wins the second highest prize v_{n-1}; and so on until all the prizes are allocated where the player with the j-th highest effort wins the uncertain prize. The utility of player i who submits effort x_i is $v_s - x_i$ if he wins prize $s, s = n-m+1, ..., n$. Each player i chooses his effort in order to maximize his expected utility (given the other competitors' efforts and the values of the different prizes).

Throughout this paper, we use the following notation:

1. $A_{k,n}$ denotes the k-th order statistic out of n independent variables independently distributed according to F. Note that $A_{n,n}$ is the highest order statistic.

2. $F_{k,n}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j [1 - F(x)]^{n-j}$ denotes the distribution of $A_{k,n}$, and $f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1}[1 - F(x)]^{n-k} f(x)$ denotes its density.

3. $E_F(k,n)$ denotes the expected value of $A_{k,n}$, where we set $E_F(0,n) = 0$. Note that $E_F(n,n)$ is the expectation of the highest order statistic.

3 The equilibrium analysis

In the following equilibrium analysis we consider two cases.

3.1 Case A: The number of prizes is smaller than the number of players

We consider a symmetric equilibrium where $\beta(a)$ denotes the strategy for the player with type a; namely, the value of the prize v_{n-j+1} ($j = 2, ..., m$) for this player is equal to a where a is distributed according to F. We also assume that $\beta(a)$ is strictly increasing in which case we can formulate the player's optimization problem as follows: player i with value a chooses to behave as a player with value s in order to solve the following problem:

$$\max_{s} \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}}^{n} F^{i,n}(s)v_i + aF^{n-j+1,n}(s) - \beta(s)$$

where $F^{i,n}(s)$ denotes the probability that a player's type s ranks exactly i-th lowest among n random variables distributed according to F. It is easy to verify that

$$F^{i,n}(s) = \frac{(n-1)!}{(i-1)!(n-i)!}[F(s)]^{i-1}[1-F(s)]^{n-i},\ i = 1, 2, ..., n.$$

Define $F_{n,n-1}(s) \equiv 0$ and $F_{0,n-1}(s) \equiv 1$ for all $s \in [0, 1]$. Then, it is immediate that $F^{i,n}(s) = F_{i-1,n-1}(s) - F_{i,n-1}(s)$. Therefore, we can rewrite the player's maximization problem as

$$\max_{s} \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}}^{n} [F_{i-1,n-1}(s) - F_{i,n-1}(s)]\, v_i + [F_{n-j,n-1}(s) - F_{n-j+1,n-1}(s)]\, a - \beta(s)$$

In equilibrium, the above maximization problem must be solved by $s = a$. Then, the solution of the resulting differential equation with boundary condition $\beta(v_{n-j}) = 0$ is given by

$$\beta(a) = \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}}^{n} [F_{i-1,n-1}(a) - F_{i,n-1}(a)]\, v_i + \int_{v_{n-j}}^{a} x\, [f_{n-j,n-1}(x) - f_{n-j+1,n-1}(x)]\, dx \qquad (1)$$

Note that

$$\beta'(a) = \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}}^{n} [f_{i-1,n-1}(a) - f_{i,n-1}(a)]\, v_i + a\, [f_{n-j,n-1}(a) - f_{n-j+1,n-1}(a)] > 0.$$

Given these equilibrium efforts, the expected utility of the player with type a is given by

$$\begin{aligned} U(a) &= \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}}^{n} F^{i,n}(a)v_i + aF^{n-j+1,n}(a) - \beta(a) \qquad (2) \\ &= \left[F_{n-j,n-1}(a) - F_{n-j+1,n-1}(a)\right]a - \int_{v_{n-j}}^{a} x\left[f_{n-j,n-1}(x) - f_{n-j+1,n-1}(x)\right]dx \\ &= \int_{v_{n-j}}^{a} \left[F_{n-j,n-1}(x) - F_{n-j+1,n-1}(x)\right]dx \end{aligned}$$

The last equality in (2) follows by integration by parts. Note also that type $a = v_{n-j}$ wins the prize v_{n-j+1} with probability zero and his equilibrium effort is $x = 0$. Therefore his utility is zero. We also have that

$$U'(a) = \left[F_{n-j,n-1}(a) - F_{n-j+1,n-1}(a)\right] > 0$$

Thus, we obtain

Proposition 1 *In an all-pay auction with n players and m prizes, $m < n$, where the value of the uncertain prize $v_{n-j+1}, j = 2, ..., m$ is private information, and the values of all the other prizes are certain, the equilibrium effort of type $a \in [v_{n-j}, v_{n-j+2}]$ is given by*

$$\beta(a) = \sum_{\substack{i=n-m+1 \\ i\neq n-j+1}}^{n} \left[F_{i-1,n-1}(a) - F_{i,n-1}(a)\right]v_i + \int_{V_{n-j}}^{a} x\left[f_{n-j,n-1}(x) - f_{n-j+1,n-1}(x)\right]dx$$

Then, a player's probability to win as well as his expected utility increase in his value for the uncertain prize.

The result of Proposition 1 according to which the equilibrium efforts are monotonically increasing in the value of the uncertain prize is not completely intuitive given that this prize is not the highest one and therefore its effect on the players' efforts is ambiguous. In other words, it is not clear whether a player has an advantage or a disadvantage when he has a higher value for the uncertain prize. Nonetheless, the players' efforts increase in their value for this prize.

3.2 Case B: The number of prizes is the same as the number of players

Similarly to the analysis of the equilibrium strategies in the previous section we obtain that

Proposition 2 *In an all-pay auction with n players and n prizes where the value of the uncertain prize $v_{n-j+1}, j = 2, ..., n-1$ is private information and the values of all the other prizes are certain, the equilibrium effort of type $a \in [v_{n-j}, v_{n-j+2}]$ is given by*

$$\beta(a) = \sum_{\substack{i=1 \\ i \neq n-j+1}}^{n} \left[F_{i-1,n-1}(a) - F_{i,n-1}(a)\right] v_i + \int_{V_{n-j}}^{a} x \left[f_{n-j,n-1}(x) - f_{n-j+1,n-1}(x)\right] dx - v_1$$

Then, a player's probability to win as well as his expected utility increase in his value for the uncertain prize.

In this case, the expected utility of a player with type a is

$$U(a) = v_1 + \int_{v_{n-j}}^{a} \left[F_{n-j,n-1}(x) - F_{n-j+1,n-1}(x)\right] dx$$

Suppose now that the value of the lowest prize v_1 is uncertain. Then, we write the player's maximization problem as

$$\max_s \sum_{i=2}^{n} \left[F_{i-1,n-1}(s) - F_{i,n-1}(s)\right] v_i + \left[1 - F_{1,n-1}(s)\right] a - \beta(s)$$

In equilibrium, the above maximization problem must be solved by $s = a$, and then the solution of the resulting differential equation with boundary condition $\beta(0) = 0$ is given by

$$\beta(a) = \sum_{i=2}^{n} \left[F_{i-1,n-1}(a) - F_{i,n-1}(a)\right] v_i - \int_0^a x f_{1,n-1}(x) dx$$

Therefore, we obtain

$$U(a) = \left[1 - F_{1,n-1}(a)\right] a + a F_{1,n-1}(a) - \int_0^a F_{1,n-1}(x) dx$$

By integration by parts we have

$$U(a) = a - \int_0^a F_{1,n-1}(x) dx < a$$

This is a contradiction, however, since each player does not have an incentive to participate in the contest. Hence, the equilibrium strategy $\beta(a)$ cannot be increasing. Therefore, since the equilibrium strategy decreases in the player's type, we write the player's maximization problem as

$$\max_s \sum_{i=2}^{n} \left[F_{n-i,n-1}(s) - F_{n-i+1,n-1}(s)\right] v_i + F_{n-1,n-1}(s) a - \beta(s)$$

In equilibrium, the above maximization problem must be solved by $s = a$, and then the solution of the resulting differential equation with boundary condition $\beta(v_2) = 0$ is given by

$$\begin{aligned}
\beta(a) &= \sum_{i=2}^{n} \left[F_{n-i,n-1}(a) - F_{n-i+1,n-1}(a)\right] v_i + \int_0^a x f_{n-1,n-1}(x)dx - \int_0^{v_2} x f_{n-1,n-1}(x)dx \qquad (3) \\
&= \sum_{i=2}^{n} \left[F_{n-i,n-1}(a) - F_{n-i+1,n-1}(a)\right] v_i + \int_{v_2}^a x f_{n-1,n-1}(x)dx
\end{aligned}$$

Note that

$$\beta'(a) = \sum_{i=2}^{n} \left[f_{n-i,n-1}(a) - f_{n-i+1,n-1}(a)\right] v_i + a f_{n-1,n-1}(a) < 0$$

Then, the expected utility of the player with type a is given by

$$\begin{aligned}
U(a) &= F_{n-1,n-1}(a)a - \int_{v_2}^{a} x\left[f_{n-1,n-1}(x)\right]dx \qquad (4) \\
&= F_{n-1,n-1}(a)a - F_{n-1,n-1}(a)a + v_2 + \int_{v_2}^{a} F_{n-1,n-1}(x)dx \\
&= v_2 - \int_a^{v_2} F_{n-1,n-1}(x)dx > a
\end{aligned}$$

Thus, we obtain

Proposition 3 *In an all-pay auction with n players and n prizes where the value of lowest prize v_1 is private information and the values of all the other prizes are certain, the equilibrium effort of type $a \in [0, v_2]$ is given by*

$$\beta(a) = \sum_{i=2}^{n} \left[F_{n-i,n-1}(a) - F_{n-i+1,n-1}(a)\right] v_i + \int_{v_2}^{a} x f_{n-1,n-1}(x)dx \qquad (5)$$

Then, the player's probability of winning decreases but his expected utility increases in his value for the uncertain prize.

In contrast to the results of Propositions 1 and 2, the result of Proposition 3, according to which the equilibrium efforts are monotonically decreasing in the value of the lowest prize, is quite intuitive. The reason is that if each player gets a prize, then the higher the players' value for the lowest prize is, the lower is the incentive of this player to participate in the contest.

4 Expected utility and expected total effort

Using the players' equilibrium strategies given in the previous section, we can now analyze the players' ex-ante expected utility and their expected total effort. We assume again that the number of players is larger than the number of prizes, namely, $n > m$. By (2), if the prize $v_{n-j+1}, j = 2, ..., m$ is uncertain, the ex-ante expected utility of a player is given by

$$U_{n-j+1} = \int_{v_{n-j}}^{v_{n-j+2}} \int_{v_{n-j}}^{a} \left[F_{n-j,n-1}(x) - F_{n-j+1,n-1}(x)\right] dx f(a) da \tag{6}$$

According to Aboutahoun and Al-otabi (2009), we have

$$E_F(j,n) - E_F(j-1,n-1) = \begin{pmatrix} n-1 \\ j-1 \end{pmatrix} \int_{v_{n-j}}^{v_{n-j+2}} F^{j-1}(x)(1-F(x))^{n-j+1} \tag{7}$$

Then, by integration by parts in (6) and (7) we obtain

Proposition 4 *In an all-pay auction with n players and m prizes where the value of the the uncertain prize $v_{n-j+1}, j = 2, ..., m$ is private information and the values of all the other prizes are certain, the ex-ante expected utility of a player is given by*

$$U_{n-j+1} = \left[E_F(n-j+1,n) - E_F(n-j,n-1)\right] \tag{8}$$

The players' expected total effort when v_{n-j+1} is uncertain is given by

$$R_{n-j+1} = n \int_{v_{n-j}}^{v_{n-j+2}} \beta(a) f(a) da$$

By (3), we obtain

$$\begin{aligned} R_{n-j+1} &= n \int_{v_{n-j}}^{v_{n-j+2}} [\sum_{\substack{i=n-m+1 \\ i \neq n-j+1}}^{n} \left[F_{i-1,n-1}(a) - F_{i,n-1}(a)\right] v_i \\ &\quad + \int_{v_{n-j}}^{a} x \left[f_{n-j,n-1}(x) - f_{n-j+1,n-1}(x)\right] dx f(a) da \end{aligned}$$

Rearranging yields

$$R_{n-j+1}=\sum_{\substack{i=n-m+1\\ i\neq n-j+1}}^{n} v_i + n\int_{v_{n-j}}^{v_{n-j+2}}\int_{V_{n-j}}^{a} x\left[f_{n-j,n-1}(x)-f_{n-j+1,n-1}(x)\right]dxf(a)da$$

Note that

$$\begin{aligned}
& n\int_{v_{n-j}}^{v_{n-j+2}}\left[\int_{v_{n-j}}^{a} xf_{r,n-1}(x)dx\right]f(a)da \\
=\ & n\left[F(a)\int_{v_{n-j}}^{a} xf_{r,n-1}(x)dx\right]_{v_{n-j}}^{v_{n-j+2}} - n\int_{v_{n-j}}^{v_{n-j+2}} F(a)af_{r,n-1}(a)da \\
=\ & n\int_{v_{n-j}}^{v_{n-j+2}} a\left(1-F(a)\right)f_{r,n-1}(a)da,
\end{aligned}$$

where the first equality follows from integration by parts. We further observe that

$$n\left(1-F\left(a\right)\right)f_{r,n-1}(a)=\left(n-r\right)f_{r,n}(a).$$

Therefore,

$$n\int_{v_{n-j}}^{v_{n-j+2}}\left[\int_{v_{n-j}}^{a} xf_{r,n-1}(x)dx\right]f(a)da=\left(n-r\right)E\left(r,n\right).$$

Thus, for all $j=2,...,m$, we obtain

Proposition 5 *In an all-pay auction with n players and m prizes where the value of the uncertain prize $v_{n-j+1}, j=2,....,m$ is private information and the values of all the other prizes are certain, the players' expected total effort is*

$$\begin{aligned}
R_{n-j+1} &= \sum_{\substack{i=n-m+1\\ i\neq n-j+1}}^{n} v_i + jE_F(n-j,n)-(j-1)E_F(n-j+1,n) \qquad (9)\\
&= \sum_{\substack{i=n-m+1\\ i\neq n-j+1}}^{n} v_i + E_F(n-j+1,n)-j(E_F(n-j+1,n)-E_F(n-j,n))
\end{aligned}$$

5 Utility and revenue comparisons under different distribution functions

In this section we analyze the effect of the distributions of the players' private values on their expected utilities and their expected total effort. For this purpose, we will use the following well-known definitions:

Definition 1 *Let F be a distribution on $[0,k]$ with density f. The* hazard rate *of F is given by the function $\lambda(x) \equiv f(x)/[1-F(x)]$, $x \in [0,k)$. F is said to have an increasing (decreasing) failure-hazard rate (IFR (DFR)) if $\lambda(x)$ is increasing (decreasing) in x. The* reverse hazard rate *of F is given by the function $\widetilde{\lambda}(x) \equiv f(x)/F(x)$, $x \in [0,k)$. F is said to have an increasing (decreasing) reverse hazard rate (IRFR (DRFR)) if $\widetilde{\lambda}(x)$ is increasing (decreasing) in x.*[2]

Definition 2 *For any distributions F and G and hazard rates λ_F and λ_G, respectively, F dominates G in terms of the hazard rate if $\lambda_F(x) \leq \lambda_G(x)$ for all $x \geq 0$. Similarly, for any distributions F and G and reverse hazard rates $\widetilde{\lambda}_F$ and $\widetilde{\lambda}_G$, respectively, F dominates G in terms of the reverse hazard rate if $\widetilde{\lambda}_F(x) \leq \widetilde{\lambda}_G(x)$ for all $x \geq 0$. We also say that F first-order stochastically dominates G if $F(x) \leq G(x)$ for all $x \geq 0$.*

In the following, we denote by U^F_{n-j+1} the players' ex-ante expected utility when the uncertain prize is v_{n-j+1} and the values of this prize are distributed according to F. Note that dominance in terms of the hazard rate implies first-order stochastic dominance. The following result shows that dominance in terms of the hazard rate between two distribution functions may improve the players' ex-ante expected utility. On the other hand, it may deteriorate it for any other two distribution functions.

Proposition 6 *Consider an all-pay auction with n players and m prizes where the value of the uncertain prize $v_{n-j+1}, j = 2, ..., m$ is distributed according to either F or G where F dominates G in terms of the hazard rate. Then,*

1. If F or G has DFR, the ex-ante expected utility of a player under F is larger than under G, i.e., $U^F_{n-j+1} \geq U^G_{n-j+1}$.

2. If F or G has IRFR, the ex-ante expected utility of a player under F is smaller than under G, i.e., $U^F_{n-j+1} \leq U^G_{n-j+1}$.

[2] The exponential, uniform, normal, power (for $\alpha \geq 1$), Weibull (for $\alpha \geq 1$), gamma (for $\alpha \geq 1$) distributions are *IFR*, while the exponential, Pareto, Weibull (for $0 < \alpha \leq 1$), gamma (for $0 < \alpha \leq 1$) are *DFR* (see Barlow and Proschan (1975)).

Proof. According to Arnold (1977) we have

$$E_F(j-1,n-1) = \frac{j-1}{n}E_F(j,n) + \frac{n-j+1}{n}E_F(j-1,n)$$

Then, by (8), we obtain

$$\begin{aligned} U^F_{n-j+1} &= [E_F(n-j+1,n) - E_F(n-j,n-1)] \\ &= \frac{j}{n}\left[E_F(n-j+1,n) - E_F(n-j,n)\right] \end{aligned}$$

Thus,

$$U^F_{n-j+1} - U^G_{n-j+1} = \frac{j}{n}\left[E_F(n-j+1,n) - E_F(n-j,n)\right] - \left[E_G(n-j+1,n) - E_G(n-j,n)\right]$$

According to Hu and Wei (2001), if F dominates G in terms of the hazard rate and F or G has DFR, the RHS of the above equation is non-negative. But if F dominates G in terms of the hazard rate and F or G has $IRFR$, the RHS of the above equation is non-positive. ■

In the case when the number of prizes m is the same as the number of players n, and the uncertain prize is the lowest one, v_1, we require only first-order dominance in order to ensure that the players' ex-ante expected utility increases or decreases.

Proposition 7 *Consider an all-pay auction with n players and n prizes where the value of the lowest prize v_1 is distributed according to either F or G, and F first-order stochastically dominates G. Then, the players' ex-ante expected utility under F is equal to or larger than that under G, i.e., $U^F_1 \geq U^G_1$.*

Proof. By (4), we get

$$\begin{aligned} U^F_1 - U^G_1 &= \int_0^{v_2}\left[V_2 - \int_a^{v_2} F_{n-1,n-1}(x)dx\right] f(a)da - \int_0^{v_2}\left[v_2 - \int_a^{v_2} G_{n-1,n-1}(x)dx\right] g(a)da \\ &= \int_0^{v_2}\left[\int_a^{v_2} G_{n-1,n-1}(x)\right] g(a)da - \int_0^{v_2}\left[\int_a^{v_2} F_{n-1,n-1}(x)dx\right] f(a)da \end{aligned}$$

and when integrating by parts we get

$$
\begin{aligned}
U_1^F - U_1^G &= \int_0^{v_2} \left[G_{n-1,n-1}(a)\right] G(a)da - \int_0^{v_2} \left[F_{n-1,n-1}(x)\right] F(a)da \\
&= \int_0^{v_2} G^n(a) - F^n(a)da
\end{aligned}
$$

The assumption that $G(x) \geq F(x)$ completes the proof. ■

We now analyze the effect of the distributions of the players' private values on their expected total effort. We denote by R^F_{n-j+1} the players' expected total effort when the uncertain prize is v_{n-j+1} and the values of this prize are distributed according to F. Then, we have

Proposition 8 *Consider an all-pay auction with n players and m prizes where the value of the uncertain prize $v_{n-j+1}, j = 2, ..., m$ is distributed according to either F or G where F dominates G in the terms of the hazard rate. Then, if F or G has $IRFR$, the players' expected total effort under F is larger than under G, i.e., $R^F_{n-j+1} \geq R^G_{n-j+1}$.*

Proof. By (9), we have

$$
\begin{aligned}
R^F_{n-j+1} - R^G_{n-j+1} &= E_F(n-j+1,n) - E_G(n-j+1,n) \\
&\quad -j(((E_F(n-j+1,n) - E_F(n-j,n)) - ((E_G(n-j+1,n) - E_G(n-j,n)))
\end{aligned}
$$

According to Hu and Wei (2001), if F dominates G in terms of the hazard rate and F or G has $IRFR$ then

$$
((E_F(n-j+1,n) - E_F(n-j,n)) - ((E_G(n-j+1,n) - E_G(n-j,n)) \leq 0
$$

Since F dominates G in terms of the hazard rate then

$$
E_F(n-j+1,n) - E_G(n-j+1,n) \geq 0
$$

Thus, we obtain that $R^F_{n-j+1} - R^G_{n-j+1} \geq 0$. ■

The following example illustrates that even when F dominates G in the terms of the hazard rate, the players' expected total effort under F might be smaller than under G.

Example 1 *Let* $F(x) = \frac{x^{0.5}-a^{0.5}}{b^{0.5}-a^{0.5}}$ *and* $G(x) = \frac{x^{\frac{1}{3}}-a^{\frac{1}{3}}}{b^{\frac{1}{3}}-a^{\frac{1}{3}}}$ *where* $b > a > 0$. *Note that* F *dominates* G *in terms of the hazard rate. Suppose that* $a = 0.1, b = 0.3$. *Then, we have*

$$E_F\,(1,8) = \int_{0.1}^{0.3} x \frac{8!}{(1-1)!(8-1)!}\left[\frac{x^{0.5}-(0.1)^{0.5}}{(0.3)^{0.5}-(0.1)^{0.5}}\right]^{1-1}\left[1-\frac{x^{0.5}-(0.1)^{0.5}}{(0.3)^{0.5}-(0.1)^{0.5}}\right]^{8-1}\left(\frac{-0.5}{x^{0.5}((0.1)^{0.5}-(0.3)^{0.5})}\right)$$

$= 0.117\,46$

$$E_F\,(2,8) = \int_{0.1}^{0.3} x \frac{8!}{(2-1)!(8-2)!}\left[\frac{x^{0.5}-(0.1)^{0.5}}{(0.3)^{0.5}-(0.1)^{0.5}}\right]^{2-1}\left[1-\frac{x^{0.5}-(0.1)^{0.5}}{(0.3)^{0.5}-(0.1)^{0.5}}\right]^{8-2}\left(\frac{-0.5}{x^{0.5}((0.1)^{0.5}-(0.3)^{0.5})}\right)$$

$= 0.136\,11$

$$E_G\,(1,8) = \int_{0.1}^{0.3} x \frac{8!}{(1-1)!(8-1)!}\left[\frac{x^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}{(0.3)^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}\right]^{1-1}\left[1-\frac{x^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}{(0.3)^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}\right]^{8-1}\left(\frac{-1}{3x^{\frac{2}{3}}\left(\sqrt[3]{0.1}-\sqrt[3]{0.3}\right)}\right)$$

$= 0.116\,10$

$$E_G\,(2,8) = \int_{0.1}^{0.3} x \frac{8!}{(2-1)!(8-2)!}\left[\frac{x^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}{(0.3)^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}\right]^{2-1}\left[1-\left(\frac{x^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}{(0.3)^{\frac{1}{3}}-(0.1)^{\frac{1}{3}}}\right)\right]^{8-2}\left(\frac{-1}{3x^{\frac{2}{3}}\left(\sqrt[3]{0.1}-\sqrt[3]{0.3}\right)}\right)$$

$= 0.133\,6$

By (9), if the number of players is $n = 8$ *and the value of the uncertain prize* v_2 *is distributed on the interval* $[0.1, 0.3]$, *then*

$$\begin{aligned} R_2^F - R_2^G &= 6(E_G\,(2,8) - E_F\,(2,8)) + 7((E_F\,(1,8) - E_G\,(1,8)) \\ &= 6(0.1336 - 0.13611) + 7(0.11746 - 0.1161) = -0.005\,54 \end{aligned}$$

Thus, we obtain that $R_2^F - R_2^G < 0$ *although* F *dominates* G *in terms of the hazard rate.*

6 Utility comparison under different numbers of prizes

In all-pay auctions with multiple prizes under incomplete information it is usually the case that if a contest designer adds new prizes to the current set of prizes, the players' ex-ante expected utility strictly increases. As we show below, however, this does not hold for our model, the reason being that by adding new prizes the location of the uncertain prize changes. Since it is not clear what the optimal location for the uncertain prize is, the additional prizes may either increase or decrease the players' ex-ante expected utility. In the following, in an all-pay auction with n players and $m, m < n$ prizes where the uncertain prize is v_{n-j+1}, the ex-ante expected utility of a player is denoted by $U_{n-j+1,m,n}$. If we add a prize $v > v_{n-j+2}$ it is denoted by $U_{n-j,m+1,n}$ and if we add a prize $v < v_{n-j}$ it is denoted by $U_{n-j+1,m+1,n}$. Then, we obtain

Proposition 9 *In an all-pay auction with n players and $m, m < n$ prizes, if F has IFR (DFR), the ex-ante expected utility of a player is smaller (larger) than or equal to that in an all-pay auction with the same number of players and with an additional prize.*

Proof. By (8), if we add a new prize with a lower value than the players' values for the uncertain prize, i.e., $v < v_{n-j}$, then the players' ex-ante expected utility is unchanged as follows:

$$U_{n-j+1,m} = [E_F(n-j+1,n) - E_F(n-j,n-1)] = U_{n-j+1,m+1}$$

On the other hand, if we add a new prize with a higher value than the players' values for the uncertain prize, i.e., $v > v_{n-j+2}$, then the players' ex-ante expected utility is

$$U_{n-j,m+1} = [E_F(n-j,n) - E_F(n-j-1,n-1)]$$

By Arnold (1977), we have

$$E_F(j-1,n-1) = \frac{j-1}{n}E_F(j,n) + \frac{n-j+1}{n}E_F(j-1,n)$$

Thus, the players' ex-ante expected utility before and after adding the new prize is given by

$$\begin{aligned} U_{n-j+1,m} &= \frac{j}{n}\left[E_F(n-j+1,n) - E_F(n-j,n)\right] \\ U_{n-j,m+1} &= \frac{j+1}{n}\left[E_F(n-j,n) - E_F(n-j-1,n)\right] \end{aligned} \tag{10}$$

By Barlow and Porschan (1966), if F has IFR (DFR) then $(n-i+1)\left(E_F(i,n) - E_F(i-1,n)\right)$ is decreasing (increasing) in $i \leq n$ for a fixed n. Thus, by (10), if F has IFR, $U_{n-j+1,m} - U_{n-j,m+1} \leq 0$ and if F has DFR, $U_{n-j+1,m} - U_{n-j,m+1} \geq 0$. ■

According to Proposition 9, if F has IFR the players' ex-ante expected utility increases if some new prizes are added. Now, in order to examine how strong the effect of adding a new prize is, we also add another player. In contrast to the additional prize, it is clear that the additional player decreases the players' ex-ante expected utility for any distribution function. Below we examine which effect is stronger: that of the negative effect of an additional player or that of the positive effect of an additional prize.

Proposition 10 *In an all-pay auction with* n *players and* $m, m < n$ *prizes, if* F *has* $IRFR$ *(*IFR*) then the ex-ante expected utility of a player is smaller (larger) than or equal to that in an all-pay auction with* $m+1$ *prizes and* $n+1$ *players when the the value of the additional prize is larger (smaller) than the players' values for the uncertain prize.*

Proof. By (8), if we add a player and a prize with a higher value than the players' values for the uncertain prize, i.e., $v > v_{n-j+2}$, then the players' ex-ante expected utility before and after adding the prize and the player is

$$\begin{aligned} U_{n-j+1,m,n} &= \frac{j}{n}\left[E_{F_j}(n-j+1,n) - E_{F_j}(n-j,n)\right] \\ U_{n-j+1,m+1,n+1} &= \frac{j+1}{n+1}\left[E_F(n-j+1,n+1) - E_F(n-j,n+1)\right] \end{aligned}$$

By Hu and Wei (2001), *if* F *has IRFR then* $E_F(j-1,n-1) - E_F(i-1,n-1) \geq E_F(j,n) - E_F(i,n)$ for every $j \geq i$. Thus, we obtain that if F has $IRFR$ then

$$\frac{j}{n}\left[E_F(n-j+1,n) - E_F(n-j,n)\right] \leq \frac{j+1}{n+1}\left[E_F(n-j+1,n+1) - E_F(n-j,n+1)\right]$$

Therefore,

$$U_{n-j+1,m+1,n+1} \geq U_{n-j+1,m,n}$$

Now if we add a player and a prize with a lower value than the players' values for the uncertain prize, i.e., $v < v_{n-j}$, then the players' ex-ante expected utility before and after adding the prize and the player is

$$\begin{aligned} U_{n-j+1,m,n} &= \frac{j}{n}\left[E_F(n-j+1,n) - E_F(n-j,n)\right] \\ U_{n-j+1,m+1,n+1} &= \frac{j}{n+1}\left[E_F(n-j+2,n+1) - E_F(n-j+1,n+1)\right] \end{aligned}$$

By Hu and Wei (2001), *if* F *has IFR then* $E_F(j-1,n-1) - E_F(i-1,n-1) \geq E_F(j,n) - E_F(i,n)$. Thus, we obtain that if F has IFR then $U_{n-j+1,m,n} \geq U_{n-j+1,m+1,n+1}$. ■

Suppose now that the number of players and prizes are the same and that the smallest prize v_1 is uncertain where each player's value for this prize is distributed on $[0, v_2]$. Then we obtain that, independent of the form of the distribution function, the effect of an additional prize and a player is always positive.

Proposition 11 *In an all-pay auction with the same number of players and prizes (n) where the lowest prize is uncertain, the ex-ante expected utility of a player is smaller than or equal to that in an all-pay auction with an additional player and a prize with a higher value than the players' values for the lowest prize.*

Proof. By (4), the ex-ante utility of a player is given by

$$U_{1,n,n} = \int_0^1 (v_2 - \int_a^{v_2} F_{n-1,n-1}(x)dx)f(a)da$$

If we add a player and a prize $v > v_2$, the ex-ante utility of a player is given by

$$U_{1,n+1,n+1} = \int_0^1 (v_2 - \int_a^{v_2} F_{n,n}(x)dx)f(a)da$$

Thus,

$$\begin{aligned} U_{1,n,n} - U_{1,n+1,n+1} &= \int_0^1 \int_a^{v_2} (F_{n,n}(x) - F_{n-1,n-1}(x))dx)f(a)da \\ &= \int_0^1 \int_a^{v_2} (F(x)^n - F(x)^{n-1})dx)f(a)da < 0 \end{aligned}$$

■

7 Concluding remarks

This paper studies all-pay auctions under both complete and incomplete information by considering a model with multiple prizes where the players' values for the prizes are certain except for one uncertain prize for which the players' values are private information. The uniqueness of our model is that a stochastic dominance relation between two distribution functions of the players' private values may increase the players' ex-ante expected utility, but, on the other hand, the same stochastic dominance relation between other distribution functions may decrease it. Similarly, a stochastic dominance relation between two distribution functions of the players' private values may increase the players' expected effort, but, on the other hand, the same stochastic dominance relation between other distribution functions may decrease it. Furthermore, an additional number of prizes, independent of their values for the players, does not necessarily increase the players' ex-ante expected utility. From this we may conclude that players may prefer a lower number of prizes in the contest.

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