DISCUSSION PAPER SERIES

No. 9873

OPTIMAL ALLOCATIONS IN ROUND-ROBIN TOURNAMENTS

Alex Krumer, Reut Megidish and Aner Sela

INDUSTRIAL ORGANIZATION

Centre for Economic Policy Research

www.cepr.org

OPTIMAL ALLOCATIONS IN ROUND-ROBIN TOURNAMENTS

Alex Krumer, Ariel University
Reut Megidish, Sapir Academic College
Aner Sela, Ben-Gurion University of the Negev and CEPR

This Discussion Paper is issued under the auspices of the Centre's research programme in **INDUSTRIAL ORGANIZATION**. Any opinions expressed here are those of the author(s) and not those of the Centre for Economic Policy Research. Research disseminated by CEPR may include views on policy, but the Centre itself takes no institutional policy positions.

The Centre for Economic Policy Research was established in 1983 as an educational charity, to promote independent analysis and public discussion of open economies and the relations among them. It is pluralist and non-partisan, bringing economic research to bear on the analysis of medium- and long-run policy questions.

These Discussion Papers often represent preliminary or incomplete work, circulated to encourage discussion and comment. Citation and use of such a paper should take account of its provisional character.

CEPR Discussion Paper No. 9873

March 2014

ABSTRACT

Optimal Allocations in Round-Robin Tournaments

We study round-robin tournaments with three players whose values of winning are common knowledge. In every stage a pair-wise match is modelled as an all-pay auction. The player who wins in two matches wins the tournament. We characterize the sub-game perfect equilibrium for symmetric (all players have the same value) and asymmetric players (each one is either weak (low value) or strong (high value)) and prove that if the asymmetry between the players' values are relatively weak, each player maximizes his expected payoff if he competes in the first and the last stages of the tournament. Moreover, even when the asymmetry between the players' values are relatively strong, the strong players maximize their expected payoffs if they compete in the first and the last stages. We show that a contest designer who wishes to maximize the length of the tournament such that the winner of the tournament will be decided in the last stage should allocate the stronger players in the last stage. But if he wishes to maximize the players' expected total effort he should not allocate them in the last stage of the tournament.

JEL Classification: D44, D72 and D82
Keywords: all-pay auctions and round-robin tournaments

Alex Krumer
Department of Economics and
Business Management
Ariel University
Ariel 40700
ISRAEL

Reut Megidish
Department of Managing Human
Resources
Sapir Academic College
MP Hof Ashkelon 79165
ISRAEL

Email: alexkr@ariel.ac.il

Email: reutc@bgu.ac.il

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=178123

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=171199

Aner Sela
Department of Economics
Ben-Gurion University of the Negev
Beer--Sheva 84105
ISRAEL

Email: anersela@bgu.ac.il

Submitted 13 February 2014

1   Introduction

Sportive events are commonly organized as round-robin tournaments, two well known examples being professional football and basketball leagues. In the round-robin tournament, every individual player or team competes against all the others and in every stage a player plays a pair-wise match against a different opponent. Sometimes sportive events can also be organized as a combination of a round-robin tournament in the first part of the season and then as an elimination tournament in the second part where in the elimination tournament, players play pair-wise matches and the winner advances to the next round while the loser is eliminated from the competition. Examples of such combinations include US-Basketball, NCAA College Basketball, the FIFA (soccer) World Cup Playoffs and the UEFA Champions' League.

The elimination tournament structure has been widely analyzed in the literature on contests. For example, Rosen (1986) studied an elimination tournament with homogeneous players where the probability of winning a match is a stochastic function of the players' efforts. Gradstein and Konrad (1999) and Harbaugh and Klumpp (2005) studied a rent-seeking contest à la Tullock (with homogenous players). Groh et al. (2012) studied an elimination tournament with four asymmetric players where players are matched in the all-pay auction in each of the stages and they found optimal seedings for different criteria.

In contrast to elimination tournaments, the literature on round-robin tournaments seems to be quite sparse,[1] the reason being the complexity of its analysis. This paper attempts to fill this gap by studying three-stage round-robin tournaments with either symmetric or asymmetric players where in each of the three stages, each of the players competes against a different opponent in the all-pay auction.[2]

We first characterize the sub-game perfect equilibrium of the three-stage round-robin tournament when the players are symmetric, namely, they have the same value of winning the tournament. We also characterize the sub-game perfect equilibrium of the three-stage round-robin tournament when the players are asymmetric, namely, they have either a high value of winning (henceforth referred to as a strong player) or a low value

---

[1]A theoretical paper that deals with this issue is that of Ryvkin and Ortmann (2008) who studied a model of a noisy tournament by which they compared the predictive power of simultaneous, elimination and round-robin tournaments.

[2]Three-stage round-robin tournaments can be found in the real life, for example, the badminton tournament in the Olympic Games, London 2012, was organized in the form of a three-stage round-robin tournament. In addition, three-stage round-robin tournaments are also used in soccer, rugby and even in debates competitions.

of winning (henceforth referred to as a weak player). We focus on the case with one weak and two strong players but it could be shown that the findings for the case of one strong and two weak players are very similar.

We prove that independent of whether the players are symmetric or asymmetric, if the asymmetry between the players' values is relatively weak, then the expected payoff of each player is maximized when he competes in the first and the last stages of the tournament. Moreover, even when the asymmetry between the players' values is relatively strong, the strong players maximize their expected payoff when they compete in the first and the last stages. This result is not straightforward since it is not clear why a player prefers to play in the last stage while there is a positive probability that the winner of the tournament will be decided before the last (third) stage and then there is not any meaning to the match in that stage. We also show that in one of the allocations of players, independent on whether the asymmetry is weak or strong, the weak player has a higher expected payoff than one of the strong players. This result demonstrates the critical effect of the allocation of players on their expected payoffs in the tournament.

We assume that the contest designer wishes to maximize the players' expected total effort in the round-robin tournament. In that case, we explicitly calculate the players' expected total effort for each allocation of players in the asymmetric model, and then by numerical analysis we show that if the asymmetry is relatively low the expected total effort is maximized when the two strong players are matched in the second stage, but, if the asymmetry is relatively high, the expected total effort is maximized when the two strong players are matched in the first stage. In particular, independent of the level of asymmetry, the expected total effort is minimized when the two strong players are matched in the last stage. The intuition behind this result is that the main part of the expected total effort comes from the match between the two strong players, and if they are matched in the last stage, there is a positive probability that the winner will be decided before the last stage and then the total effort in the match of the two strong players will be negligible.

The designer's goal, however, may not necessarily be to maximize the players' expected total effort, but rather to extend the length of the tournament such that the winner of the tournament will be decided as late as possible. In that case, we explicitly calculate the probability that the winner of the tournaments will be decided in the last (third) stage and then by numerical analysis we show that this probability is maximized

when the two strong players are matched in the last stage. The intuition behind this result is that when the strong players are matched in the last stage, the winner of the tournament will be decided before the last stage iff the weak player wins in the first two stages, while in any other allocation of players, the winner of the tournament will be decided before the last stage iff one of the strong players wins in the first two stages. Since the probability that a strong player will win in the first two stages is higher than the probability that the weak player will win in the first two stages, we obtain the result according to which the strong players should be matched in the last stage in order to maximize the probability of a real competition in that stage.

It is important to note that the two designer's goals mentioned above, namely, on the one hand, to maximize the players' expected total effort, and, on the other, to maximize the length of the tournament seem not to conflict. Since if the tournament is longer, the players' expected total effort might be higher. However, we contradict this conjecture by showing that the players' expected total effort is maximized when the two strong players are not matched in the last stage while the probability that the winner of the tournament will be decided in the last stage is obtained when the two strong players are matched in the last stage. Hence, regardless of any allocation of the players, a contest designer cannot simultaneously maximize the expected total effort and the probability that the tournament will be decided in the last stage.

Our paper is related to the statistical literature on the design of various forms of tournaments. The pioneering paper[3] is David (1959) who considered the winning probability of the top player in a four player tournament with a random seeding. This literature assumes that, for each match among players i and j, there is a fixed, exogenously given probability that i beats j. In particular, this probability does not depend on the stage of the tournament where the particular match takes place nor on the identity of the expected opponent at the next stage. In contrast, in our round-robin model, similarly to Groh et al. (2012) who considered an elimination tournament model, each match among two players is an all-pay auction. As a result, winning probabilities in each match become endogenous in that they result from mixed equilibrium strategies, and are positively correlated to win valuations. Moreover, the win probabilities depend on the stage of the tournament where the match takes place, and on the identity of the future expected opponents.

The analysis of our model is also related to the analysis of the best-of-k tournaments (see, Konrad and

---

[3]See also Glenn (1960) and Searles (1963) for early contributions.

Kovenock (2009), Malueg and Yates (2010), Sela (2011) and Krumer (2013)) in which the winner is the one who is first to win $\frac{k+1}{2}$ games.[4] However, since the best-of-k tournaments are two-player contests, our main questions about the optimal allocations of players are not pertinent for this class of contests.

The rest of the paper is organized as follows: Section 2 describes the three-stage round-robin model. Section 3 describes the equilibrium in the one-stage all-pay auction. In Sections 4 and 5 we characterize the sub-game perfect equilibrium in the round-robin tournament with symmetric and asymmetric players respectively. In section 6 we analyze the optimal allocation of players that maximizes the probability that the winner of the tournament will be decided in the last stage. In Section 7 we analyze the optimal allocation of players that maximizes the players' expected total effort. Section 8 concludes.

2   The model

Consider three players (or teams) $i = 1, 2, 3$ who compete in a round-robin all-pay tournament. In each stage t, $t = 1, 2, 3$ there is a different pair-wise match such that each player competes in two different stages. The player who wins two matches wins the tournament and in the case that each player wins only once, each of them wins the tournament with the same probability. If one of the players wins in the first two stages, the winner of the tournament is then decided and the match in the third stage is not played. We model each match among two players as an all-pay auction; both players exert efforts, and the one exerting the higher effort wins. Player $i's$ value of winning the contest is v_i and a player's cost function is $c(x_i) = x_i$, where x_i is his effort.

3   The one-stage all-pay auction

We begin with the analysis of the standard one-stage all-pay auction which plays a key role in our analysis of the round-robin all-pay auction. Consider a one-stage all-pay auction with two players $1, 2$, where the players' values of winning are $v_1 \geq v_2 > 0$. According to Hillman and Riley (1989) and Baye, Kovenock and

---

[4]A classical best-of-three contest is tennis in which the first player to win two sets wins the contest. In certain prominent tennis tournaments for men, including the all four Grand Slam tournaments (the Australian Open, French Open, Wimbledon, and US Open) the first player to win three sets wins the best-of-five contest.

de Vries (1993, 1996), there is always a unique mixed-strategy equilibrium in which the players randomize on the interval $[0, v_2]$ according to their effort cumulative distribution functions which are given by

$$v_1 F_2(x) - x = v_1 - v_2$$

$$v_2 F_1(x) - x = 0$$

Thus, player 1's effort is distributed according to the cumulative distribution function

$$F_1(x) = \frac{x}{v_2}$$

while player 2's effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{v_1 - v_2 + x}{v_1}$$

Given these mixed strategies, player 1's winning probability against player 2 is

$$p_{12} = 1 - \frac{v_2}{2v_1}$$

The players' expected total effort is given by

$$TE = \frac{v_2}{2}(1 + \frac{v_2}{v_1})$$

Using the above analysis of the one-stage all-pay auction we can now turn to analyze the players' equilibrium strategies in the round-robin all-pay auction.

4   The symmetric round-robin tournament

We first assume symmetric players, i.e., the players' values of winning are $v_1 = v_2 = v_3 = v$. In order to analyze the sub-game perfect equilibrium of the round-robin tournament we begin with the last stage of the tournament and go backwards to the previous stages. Figure 1 presents the symmetric round-robin tournament as a tree game. We denote by p_{ij}^* the probability that player i wins against player j in vertex $*$ of the tree game. [**Figure 1 here**].

6

4.1 Stage 3 - player 2 vs. player 3

Players 2 and 3 compete in the last stage only if at least one of them won in the previous stages. Thus, we have the following three scenarios:

1. Assume first that player 2 won the match in the first stage and player 3 won the match in the second stage (vertex A in Figure 1). Then if each of the players wins in stage 3, he also wins the tournament. Thus, following Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996), since $v > 0$, there is always a unique mixed strategy equilibrium in which both players randomize on the interval $[0, v]$ according to their cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$vF_i^{(3)}(x) - x = 0 \quad i = 2, 3 \tag{1}$$

Then, player 2's probability to win in the third stage is

$$p_{23}^A = 0.5$$

and their expected total effort in the third stage is

$$TE^A = v$$

2. Assume now that player 2 won the match in the first stage and player 3 lost the match in the second stage (vertex B in Figure 1). Then, if player 2 wins in this stage, he wins the tournament and his payoff is v, whereas player 3's payoff is zero. But, if player 3 wins in this stage, then each of the players has exactly one win, and then each of the players has an expected payoff of $v/3$. Thus, we obtain that players 2 and 3 randomize on the interval $[0, v/3]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$vF_3^{(3)}(x) + \frac{v}{3}(1 - F_3^{(3)}(x)) - x = \frac{2v}{3} \tag{2}$$
$$\frac{v}{3}F_2^{(3)}(x) - x = 0$$

Then, player 2's probability to win in the third stage is

$$p_{23}^B = 1 - \frac{v}{4v} = 0.75$$

and the players' expected total effort in the third stage is

$$TE^B = \frac{v}{6}(1 + \frac{v}{2v}) = \frac{v}{4}$$

3. Finally, assume that player 2 lost the match in the first stage and player 3 won the match in the second stage (vertex C in Figure 1). Then, similarly to the previous case, we obtain that players 2 and 3 randomize on the interval $[0, v/3]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are now given by

$$vF_2^{(3)}(x) + \frac{v}{3}(1 - F_2^{(3)}(x)) - x = \frac{2v}{3} \qquad (3)$$
$$\frac{v}{3}F_3^{(3)}(x) - x = 0$$

Then, player $2's$ probability to win in the third stage is

$$p_{23}^C = \frac{v}{4v} = 0.25$$

and the players' expected total effort in the third stage is

$$TE^C = \frac{v}{6}(1 + \frac{v}{2v}) = \frac{v}{4}$$

4.2   Stage 2 - player 1 vs. player 3

Based on the results of the match in the first stage, we have two possible scenarios:

1. Assume first that player 1 lost the match in the first stage (vertex D in Figure 1). Then, if player 3 wins in this stage, by (1) his expected payoff in the next stage is zero. If player 3 loses in this stage, by (2) his expected payoff is zero as well. Thus, in such a case, player 3 has no incentive to exert a positive effort and player 1 wins in this stage with a probability of one.[5]

2. Assume now that player 1 won the match in the first stage (vertex E in Figure 1). Then, if he wins again in this stage he also wins the tournament and therefore his payoff is v. The other players' payoffs are

---

[5]It is important to note that when a player has no incentive to exert a positive effort we actually do not have an equilibrium. However, in order to solve this problem, similarly to Groh et al. (2012), we can assume that each player obtains a payment $k > 0$, independent from his performance, and then we consider the limit behavior as $k \to 0$. This assumption does not affect the players' behavior in our model but ensures the equilibrium existence.

then zero. However, if player 1 loses in this stage, his payoff depends on the result of the match between players 2 and 3 in the last stage. If player 3 wins in the last stage, which happens with a probability of 0.75, then by (3) player 3's expected payoff is $2v/3$ and player 1's expected payoff is zero. On the other hand, if player 2 wins in the last stage which happens with a probability of 0.25, each of the players has one win and therefore an expected payoff of $v/3$. In sum, if player 1 loses in this stage, his expected payoff is $v/12$.

Thus, we obtain that players 1 and 3 randomize on the interval $[0, 2v/3]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 1, 3$ which are given by

$$vF_3^{(2)}(x) + \frac{v}{12}(1 - F_3^{(2)}(x)) - x = \frac{v}{3} \qquad (4)$$
$$\frac{2v}{3}F_1^{(2)}(x) - x = 0$$

Then, player 1's probability to win in the second stage is

$$p_{13}^E = 1 - \frac{8}{22} = \frac{7}{11}$$

and the players' expected total effort in the second stage is

$$TE^E = \frac{v}{3}(1 + \frac{8}{11}) = \frac{19v}{33}$$

4.3 Stage 1 - player 1 vs. player 2

If player 1 wins the match in the first stage (vertex F in Figure 1), by (4) his expected payoff in the next stage is $v/3$. But if player 1 loses the match in the first stage, he has an expected payoff of $v/3$ only if he wins in the second stage which happens with a probability of one, and player 2 loses against player 3 in the last stage which happens with a probability of 0.25. Thus, if player 1 loses in the first stage his expected payoff in the next stage is $v/12$.

Now, if player 2 wins the match in the first stage (vertex F in Figure 1), player 1 wins for sure in the second stage and then by (2) player 2's expected payoff is $2v/3$. However, if player 2 loses the match in the first stage, and player 1 wins also in the second stage player 2 has an expected payoff of zero. Furthermore, even if player 1 loses in the second stage, by (3) player 2 has an expected payoff of zero. Thus, we obtain that players 1 and 2 randomize on the interval $[0, v/4]$ according to their effort cumulative distribution functions

$F_i^{(1)}, i = 1, 2$ which are given by

$$\frac{v}{3}F_2^{(1)}(x) + \frac{v}{12}(1 - F_2^{(1)}(x)) - x = \frac{v}{12} \qquad (5)$$

$$\frac{2v}{3}F_1^{(1)}(x) - x = \frac{5v}{12}$$

Then, player 1's probability to win in the first stage is

$$p_{12}^F = \frac{3}{16}$$

and the players' expected total effort in the first stage is

$$TE^F = \frac{v}{8}(1 + \frac{3}{8}) = \frac{11v}{64}$$

By the above analysis we obtain:

Proposition 1 *In the sub-game perfect equilibrium of the round-robin all-pay tournament with three symmetric players, the players' expected payoffs are as follows: player 1's expected payoff is $v/12$, player 2's is $5v/12$, and player 3's is zero.*

By the above analysis we also obtain:

Proposition 2 *In the sub-game perfect equilibrium of the round-robin all-pay tournament with three symmetric players, the players' probabilities to win the tournament are as follows:*

Player 1's probability to win is

$$P_1 = p_{12}^F p_{13}^E + \frac{p_{12}^F p_{31}^E p_{23}^C}{3} + \frac{p_{21}^F p_{13}^D p_{32}^B}{3} = 0.193$$

Player 2's probability to win is

$$P_2 = p_{21}^F p_{13}^D p_{23}^B + p_{21}^F p_{31}^D p_{23}^A + \frac{p_{12}^F p_{31}^E p_{23}^C}{3} + \frac{p_{21}^F p_{13}^D p_{32}^B}{3} = 0.682$$

and player 3's probability to win is

$$P_3 = p_{12}^F p_{31}^E p_{32}^C + p_{21}^F p_{31}^D p_{32}^A + \frac{p_{12}^F p_{31}^E p_{23}^C}{3} + \frac{p_{21}^F p_{13}^D p_{32}^B}{3} = 0.125$$

By Propositions 1 and 2 we can conclude that

Theorem 1 *In the round-robin all-pay tournament with three symmetric players, the player who competes in the first and the last stages has the highest probability to win the tournament as well as the highest expected payoff.*

5   The asymmetric round-robin tournament

We consider now asymmetric players and for simplicity we assume that the players' values of winning are $v_1 = v_2 = v > v_3$; namely we have one weak player (player 3) and two strong players (players 1 and 2). Without loss of generality we also assume that $v_3 = 1$. We say that the symmetry is weak if v is sufficiently close to 1 and the symmetry is strong if v is sufficiently larger than 1. The results of the asymmetric round-robin all-pay tournament depends on the stage in which the two strong players are matched. Below we analyze the results of the tournament for each possible allocation of the players.

5.1   The strong players are matched in the first stage (stage 1: 1 vs. 2, stage 2: 1 vs. 3, stage 3: 2 vs. 3)

Figure 2 presents this tournament as a tree game. [**Figure 2 here**].

Proposition 3 *In the sub-game perfect equilibrium of the round-robin all-pay tournament with one weak and two strong players, if the strong players are matched in the first stage then*

1) if $1 < v \leq 2$, player 1's expected payoff is 1/12, player 2's is $\frac{9-4v}{12}$ and player 3's is zero.

2) if $v > 2$, players 1 and 2's expected payoff is 1/12 and player 3's is zero.

Thus, independent on whether the asymmetry is weak or strong, the expected payoff of the strong player who plays in the first and the last stages (player 2) is higher than or equal to the other players' expected payoffs.

Proof. The proof is obtained by the analysis in Appendix A. ∎

5.2   The strong players are matched in the second stage (stage 1: 1 vs. 3, stage 2: 1 vs. 2, stage 3: 2 vs. 3)

Figure 3 presents this tournament as a tree game. [**Figure 3 here**].

Proposition 4 *In the sub-game perfect equilibrium of the round-robin all-pay tournament with one weak and two strong players, if the strong players are matched in the second stage then*

1) if $1 < v \leq 1.096$, player 1's expected payoff is $\frac{v^2 - 12v + 12}{12}$, player 2's expected payoff is zero, and player 3's expected payoff is $\frac{v^4 - 16v^3 + 44v^2 - 96v + 72}{12v^2}$.

2) if $1.096 < v \leq 1.1$ player 1's expected payoff is $\frac{v^3 - 8v^2 + 24v - 18}{3v^2}$, player 2's expected payoff is zero, and player 3's expected payoff is zero.

3) if $1.1 < v \leq 2$, players 1's expected payoff is $\frac{v^3 - 3v^2 + 24v - 24}{72v - 72}$, player 2's expected payoff is $\frac{v^5 - 15v^4 + 48v^3 - 36v^2}{576v - 576}$ and player 3's expected payoff is zero.

4) if $v > 2$, players 1's expected payoff is $\frac{3v - 1}{12v - 6}$, player 2's expected payoff is $\frac{v^2 - 2v + 1}{12v - 6}$ and player 3's expected payoff is zero.

Thus, if the asymmetry is weak, $1 < v \leq 1.095$, the weak player (player 3) has a higher expected payoff than the both strong players (players 1 and 2).

Proof. The proof is obtained by the analysis in Appendix B. ∎

5.3 The strong players are matched in the third stage (stage 1: 1 vs. 3, stage 2: 2 vs. 3, stage 3: 1 vs. 2)

Figure 4 presents this tournament as a tree game. [**Figure 4 here**].

Proposition 5 *In the sub-game perfect equilibrium of the round-robin all-pay tournament with one weak and two strong players, if the strong players are matched in the third stage then*

1) if $1 < v \leq 1.375$, player 1's expected payoff is $\frac{16v - 11}{12}$, player 2's expected payoff is zero, and player 3's expected payoff is $\frac{1}{12}$.

2) if $v > 1.375$, players 1's expected payoff is $\frac{2v}{3}$, player 2's expected payoff is zero, and player 3's expected payoff is $\frac{1}{12}$.

Thus, independent on whether the asymmetry is weak or strong, the strong player who plays in the first and the last stages (player 1) has a higher expected payoff than the other players and particularly than the

other strong one (player 2). Moreover, independent on whether the asymmetry is weak or strong, the weak

player has a higher expected payoff than one of the strong players.

Proof. The proof is obtained by the analysis in Appendix C. ∎

By a comparing of each player's expected payoffs in the above three cases we obtain

Theorem 2 *In the round-robin all-pay tournament with one weak and two strong players, the strong player*

has the highest expected payoff when he is allocated to play in the first and the last stages.

6 Length of the tournament

In round-robin tournaments there is always a positive possibility that the winner of the tournament will be decided before the last stage. If we assume that the contest designer has a revenue from each match of the tournament, then he might wish to increase the length of tournament, namely, increase the probability that the winner of the tournament will be decided in the last stage or at least as late as possible. Below we analyze the probability that the winner of the tournament is decided before the last stage.

Case A: The strong players are matched in the first stage (stage 1: 1 vs. 2, stage 2: 1 vs. 3, stage 3: 2 vs. 3)

In this case the winner is decided before the last stage iff player 1 wins in the first two stages. If $1 < v \leq 2$, by (21) and (27) the probability that player 1 wins in the first two stages is

$$p_{12}^{F1} p_{13}^{E1} = \left(\frac{16v - 13}{24v - 8}\right)\left(1 - \frac{2v - 6}{v^2 - 12v}\right) = \frac{16v^3 - 237v^2 + 278v - 78}{24v^3 - 296v^2 + 96v}$$

If $v > 2$, by (24) and (30) the probability that player 1 wins in the first two stages is

$$p_{12}^{F2} p_{13}^{E2} = \left(\frac{24v^2 + 2v - 5}{48v^2 + 4v}\right)\left(1 - \frac{1}{4v + 2}\right) = \frac{48v^2 - 8v - 5}{96v^2 + 8v}$$

Case B: The strong players are matched in the second stage (stage 1: 1 vs. 3, stage 2: 1 vs. 2, stage 3: 2 vs. 3)

In this case the winner is decided before the last stage iff player 1 wins in the first two stages. If $1 < v \leq 1.096$ by (54) and (57) the probability that player 1 wins in the first two stages is

$$p_{13}^{F1} p_{12}^{E} = \left(\frac{v^4 - 12v^3 + 8v^2}{8v^3 - 72v^2 + 192v - 144}\right)\left(1 - \frac{6v - 2}{12v - 1}\right) = \frac{6v^5 - 71v^4 + 36v^3 + 8v^2}{96v^4 - 872v^3 + 2376v^2 - 1920v + 144}$$

13

If $1.096 < v \le 1.1$ by (54) and (60) the probability that player 1 wins in the first two stages is

$$p_{13}^{F2}p_{12}^{E} = (1 - \frac{2v^3 - 18v^2 + 48v - 36}{v^4 - 12v^3 + 8v^2})(1 - \frac{6v - 2}{12v - 1}) = \frac{6v^5 - 83v^4 + 142v^3 - 262v^2 + 168v + 36}{12v^5 - 145v^4 + 108v^3 - 8v^2}$$

If $1.1 < v \le 2$, by (54) and (63) the probability that player 1 wins in the first two stages is

$$p_{13}^{F3}p_{12}^{E} = (1 - \frac{v^3 - 3v^2}{48 - 48v})(1 - \frac{6v - 2}{12v - 1}) = \frac{6v^4 - 17v^3 + 285v^2 - 240v - 48}{576v^2 - 624v + 48}$$

and if $v > 2$, by (54) and (66) the probability that player 1 wins in the first two stages is

$$p_{13}^{F4}p_{12}^{E} = (1 - \frac{v - 1}{8v - 4})(1 - \frac{6v - 2}{12v - 1}) = \frac{42v^2 - 11v - 3}{96v^2 - 56v + 4}$$

Case C: The strong players are matched in the third stage (stage 1: 1 vs. 3, stage 2: 2 vs. 3, stage 3: 1 vs. 2)

In this case the winner is decided before the last stage iff player 3 wins in the first two stages. If $1 < v \le 1.375$ by (80) and (86) the probability that player 3 wins in the first two stages is

$$p_{31}^{F1}p_{32}^{E1} = (\frac{11 - 8v}{16v})(1 - \frac{4v}{11}) = \frac{32v^2 - 132v + 121}{176v}$$

If $v > 1.375$, by (83) and (88) the probability that player 3 wins in the first two stages is

$$p_{31}^{F2}p_{32}^{E2} = 0$$

The above analysis is summarized by Figure 5 which shows the probabilities that the winner of the round-robin tournaments is decided after two stages as a function of the level of asymmetry (v) between the players. [**Figure 5 here**].

We can see by Figure 5 that, independent of the level of asymmetry (v), the probability that the winner of the tournament is decided before the last stage is minimized when the two strong players (players 1 and 2) are matched in the last stage. The reason is that if the strong players are matched in the last stage, the winner of the tournament is decided before the last stage iff the weak player (player 3) wins in the first two stages and for any other allocation of players in the tournament, the winner is decided before the last stage iff one of the strong players wins in the first two stages. Thus a contest designer who wishes that the winner of the three-stage round-robin tournament will be decided in the last stage should allocate the two strong players in the last stage of the tournament.

14

7 Total effort

We assume now that the contest designer wishes to maximize the players' expected total effort. Below we analyze the optimal allocation of players for this purpose.

Case A: The strong players are matched in the first stage (stage 1: 1 vs. 2, stage 2: 1 vs. 3, stage 3: 2 vs. 3)

In this case, if $1 < v \leq 2$, by (8), (11), (14), (18), (19), (21), (22),(27) and (28), the players' expected total effort is

$$
\begin{aligned}
TE &= TE^{F1} + p_{12}^{F1}TE^{E1} + p_{21}^{F1}TE^{D} + p_{12}^{F1}p_{31}^{E1}TE^{C1} + p_{21}^{F1}p_{13}^{D}TE^{B} + p_{21}^{F1}p_{31}^{D}TE^{A} \\
&= (\frac{448v^2 - 636v + 221}{288v - 96}) + (\frac{16v - 13}{24v - 8})(\frac{-v^3 + 11v^2 - 12v - 36}{6v^2 - 72v}) \\
&\quad + (\frac{16v - 13}{24v - 8})(\frac{2v - 6}{v^2 - 12v})(\frac{v^2 + 2v}{12}) + (1 - \frac{16v - 13}{24v - 8})(\frac{2v + 1}{12v})
\end{aligned}
$$

But, if $v > 2$, by (8), (11), (17), (18), (19), (24), (25), (30) and (31), the players' expected total effort is

$$
\begin{aligned}
TE &= TE^{F2} + p_{12}^{F2}TE^{E2} + p_{21}^{F2}TE^{D} + p_{12}^{F2}p_{31}^{E2}TE^{C2} + p_{21}^{F2}p_{13}^{D}TE^{B} + p_{21}^{F2}p_{31}^{D}TE^{A} \\
&= (\frac{576v^3 - 192v^2 - 80v + 25}{576v^2 + 48v}) + (\frac{24v^2 + 2v - 5}{48v^2 + 4v})(\frac{v + 1}{6v + 3}) \\
&\quad + (\frac{24v^2 + 2v - 5}{48v^2 + 4v})(\frac{1}{4v + 2})(\frac{v + 2}{3v}) + (1 - \frac{24v^2 + 2v - 5}{48v^2 + 4v})(\frac{2v + 1}{12v})
\end{aligned}
$$

Case B: The strong players are matched in the second stage (stage 1: 1 vs. 3, stage 2: 1 vs. 2, stage 3: 2 vs. 3)

In this case, if $1 < v \leq 1.096$, by (34), (37), (43), (45), (46), (54), (55), (57) and (58), the players'

expected total effort is

$$
\begin{aligned}
TE \;=\;& TE^{F1} + p_{13}^{F1}TE^{E} + p_{31}^{F1}TE^{D1} + p_{13}^{F1}p_{21}^{E}TE^{C} + p_{31}^{F1}p_{12}^{D1}TE^{B1} + p_{31}^{F1}p_{21}^{D1}TE^{A} \\
=\;& (\frac{v^6 - 20v^5 + 76v^4 + 368v^3 - 1448v^2 + 1632v - 576}{-96v^3 + 864v^2 - 2304v + 1728}) \\
& +(\frac{v^4 - 12v^3 + 8v^2}{8v^3 - 72v^2 + 192v - 144})(\frac{72v^2 - 39v + 5}{72v - 6}) \\
& +(1 - \frac{v^4 - 12v^3 + 8v^2}{8v^3 - 72v^2 + 192v - 144})(\frac{v^3 + 11v^2 - 24v + 12}{2v^2}) \\
& +(\frac{v^4 - 12v^3 + 8v^2}{8v^3 - 72v^2 + 192v - 144})(\frac{6v - 2}{12v - 1})(\frac{2v + 1}{12v}) \\
& +(1 - \frac{v^4 - 12v^3 + 8v^2}{8v^3 - 72v^2 + 192v - 144})(1 - \frac{6v - 6}{v^2})(\frac{v^2 + 2v}{12}) \\
& +(1 - \frac{v^4 - 12v^3 + 8v^2}{8v^3 - 72v^2 + 192v - 144})(\frac{6v - 6}{v^2})(\frac{v + 1}{2v})
\end{aligned}
$$

and if $1.096 < v \leq 1.1$, by (34), (37), (43), (45), (46), (54), (55), (60) and (61), the players' expected total effort is

$$
\begin{aligned}
TE \;=\;& TE^{F2} + p_{13}^{F2}TE^{E} + p_{31}^{F2}TE^{D1} + p_{13}^{F2}p_{21}^{E}TE^{C} + p_{31}^{F2}p_{12}^{D1}TE^{B1} + p_{31}^{F2}p_{21}^{D1}TE^{A} \\
=\;& (\frac{v^7 - 17v^6 + 68v^5 + 138v^4 - 1464v^3 + 3456v^2 - 3456v + 1296}{-6v^6 + 72v^5 - 48v^4}) \\
& +(1 - \frac{2v^3 - 18v^2 + 48v - 36}{v^4 - 12v^3 + 8v^2})(\frac{72v^2 - 39v + 5}{72v - 6}) \\
& +(\frac{2v^3 - 18v^2 + 48v - 36}{v^4 - 12v^3 + 8v^2})(\frac{v^3 + 11v^2 - 24v + 12}{2v^2}) \\
& +(1 - \frac{2v^3 - 18v^2 + 48v - 36}{v^4 - 12v^3 + 8v^2})(\frac{6v - 2}{12v - 1})(\frac{2v + 1}{12v}) \\
& +(\frac{2v^3 - 18v^2 + 48v - 36}{v^4 - 12v^3 + 8v^2})(1 - \frac{6v - 6}{v^2})(\frac{v^2 + 2v}{12}) \\
& +(\frac{2v^3 - 18v^2 + 48v - 36}{v^4 - 12v^3 + 8v^2})(\frac{6v - 6}{v^2})(\frac{v + 1}{2v})
\end{aligned}
$$

If $1.1 < v \leq 2$, by (34), (37), (43), (48), (49), (54), (55), (63) and (64), the players' expected total effort is

$$
\begin{aligned}
TE \;=\;& TE^{F3} + p_{13}^{F3}TE^{E} + p_{31}^{F3}TE^{D2} + p_{13}^{F3}p_{21}^{E}TE^{C} + p_{31}^{F3}p_{12}^{D2}TE^{B1} + p_{31}^{F3}p_{21}^{D2}TE^{A} \\
=\;& (\frac{v^6 - 6v^5 - 15v^4 + 96v^3 - 72v^2}{3456v^2 - 6912v + 3456}) + (1 - \frac{v^3 - 3v^2}{48 - 48v})(\frac{72v^2 - 39v + 5}{72v - 6}) \\
& +(\frac{v^3 - 3v^2}{48 - 48v})(\frac{v^4 + 12v^3 - 12v^2}{288v - 288}) + (1 - \frac{v^3 - 3v^2}{48 - 48v})(\frac{6v - 2}{12v - 1})(\frac{2v + 1}{12v}) \\
& +(\frac{v^3 - 3v^2}{48 - 48v})(\frac{v^2}{24v - 24})(\frac{v^2 + 2v}{12}) + (\frac{v^3 - 3v^2}{48 - 48v})(1 - \frac{v^2}{24v - 24})(\frac{v + 1}{2v})
\end{aligned}
$$

and if $v > 2$, by (34), (40), (43), (51), (52), (54), (55), (66) and (67), the players' expected total effort is

$$
\begin{aligned}
TE \;=\;& TE^{F4} + p_{13}^{F4}TE^{E} + p_{31}^{F4}TE^{D3} + p_{13}^{F4}p_{21}^{E}TE^{C} + p_{31}^{F4}p_{12}^{D3}TE^{B2} + p_{31}^{F4}p_{21}^{D3}TE^{A} \\
\;=\;& \frac{5v^2 - 8v + 3}{96v^2 - 96v + 24} + (1 - \frac{v-1}{8v-4})(\frac{72v^2 - 39v + 5}{72v - 6}) \\
& + (\frac{v-1}{8v-4})(\frac{3v^2 - 5v + 2}{12v - 6}) + (1 - \frac{v-1}{8v-4})(\frac{6v-2}{12v-1})(\frac{2v+1}{12v}) \\
& + (\frac{v-1}{8v-4})(\frac{v-1}{4v-2})(\frac{v+2}{3v}) + (\frac{v-1}{8v-4})(1 - \frac{v-1}{4v-2})(\frac{v+1}{2v})
\end{aligned}
$$

Case C: The strong players are matched in the third stage (stage 1: 1 vs. 3, stage 2: 2 vs. 3, stage 3: 1 vs. 2)

In this case, if $1 < v \leq 1.375$, by (70), (73), (76), (77), (78), (80), (81), (86) and (87), the players' expected total effort is

$$
\begin{aligned}
TE \;=\;& TE^{F1} + p_{13}^{F1}TE^{D} + p_{31}^{F1}TE^{E1} + p_{13}^{F1}p_{23}^{D}TE^{A} + p_{13}^{F1}p_{32}^{D}TE^{B} + p_{31}^{F1}p_{23}^{E1}TE^{C} \\
\;=\;& (\frac{-88v + 121}{192v}) + (\frac{-8v + 11}{16v})(\frac{8v^2 + 11v}{33}) \\
& + (1 - \frac{-8v + 11}{16v})\frac{v}{4} + (\frac{-8v + 11}{16v})(\frac{4v}{11})\frac{v}{4}
\end{aligned}
$$

But, if $v > 1.375$, by (70), (73), (76), (77), (78), (83), (84), (88) and (89), the players' expected total effort is

$$
\begin{aligned}
TE \;=\;& TE^{F2} + p_{13}^{F2}TE^{D} + p_{31}^{F2}TE^{E2} + p_{13}^{F2}p_{23}^{D}TE^{A} + p_{13}^{F2}p_{32}^{D}TE^{B} + p_{31}^{F2}p_{23}^{E2}TE^{C} \\
\;=\;& \frac{v}{4}
\end{aligned}
$$

The above analysis is summarized by Figure 6 which shows the expected total effort as a function of the level of asymmetry (v). [**Figure 6 here**].

We can see in Figure 6 that if the asymmetry is relatively low, i.e., $v < 1.6$ the expected total effort is maximized when the two strong players (players 1 and 2) are matched in the second stage, but if the asymmetry is relatively high, i.e., $v > 1.6$ the expected total effort is maximized when the two strong players are matched in the first stage. Moreover, we can see that, independent of the level of asymmetry, the expected total effort is minimized when the two strong players are matched in the last stage. The reason is that most of the expected total effort comes from the match between the strong players and when these players are matched in the last stage there is a positive probability that this match will not be played. Thus,

if a contest designer wishes to maximize the expected total effort he should not allocate the strong players in the last stage of the tournament.

8 Concluding remarks

We analyzed the sub-game perfect equilibrium of the round-robin tournaments with three players. We showed that when the players are symmetric, a player's expected payoff is maximized when he plays in the first and the last stages. We also showed that when the players are asymmetric (one weak and two strong players), each of the strong players maximizes his expected payoff when he plays in the first and the last stages. This result could be also shown for the asymmetric case with one strong and two weak players. We explicitly calculate, on the one hand, the players' expected total effort, and, on the other, the probability that the winner of the tournament will be decided in the last stage. Using these calculations, by numerical analysis we demonstrate that these two possible goals of the contest designer contradict each other since the expected total effort is minimized when the two strong players are matched in the last stage while the probability that the winner of the tournament will be decided in the last stage is maximized when the two strong players are matched in the last stage. It would be interesting although not simple to examine whether or not the results of this model can be generalized for round-robin tournaments with four or a higher number of stages.

9 Appendix A: strong vs. strong in the first stage

As in the symmetric case, in order to analyze the sub-game perfect equilibrium of the round-robin tournament we begin with the last stage of the tournament and go backwards to the previous stages. First, we analyze the case where the strong players (players 1 and 2) are matched in the first stage. The tree game is described by Figure 2.

9.1 Stage 3 - player 2 vs. player 3

We have the following three scenarios:

1) Assume first that player 2 won the match in the first stage and player 3 won the match in the second stage (vertex A in figure 2). Then there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, 1]$ according to their cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$vF_3^{(3)}(x) - x \quad = \quad v - 1 \tag{6}$$

$$F_2^{(3)}(x) - x \quad = \quad 0$$

Then, player 2's probability to win the match in the third stage is

$$p_{23}^A = 1 - \frac{1}{2v} \tag{7}$$

and the players' expected total effort is

$$TE^A = \frac{v + 1}{2v} \tag{8}$$

2) Assume now that player 2 won the match in the first stage and player 3 lost in the second stage (vertex B in Figure 2). Then there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{1}{3}]$ according to their cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$vF_3^{(3)}(x) + \frac{v}{3}(1 - F_3^{(3)}(x)) - x \quad = \quad v - \frac{1}{3} \tag{9}$$

$$\frac{1}{3}F_2^{(3)}(x) - x \quad = \quad 0$$

Then, player 2's probability to win the match in the third stage is

$$p_{23}^B = 1 - \frac{1}{4v} \tag{10}$$

and the players' expected total effort is

$$TE^B = \frac{2v + 1}{12v}. \tag{11}$$

3) Finally, assume that player 2 lost the match in the first stage, and player 3 won in the second stage (vertexes C1-C2 in Figure 2). In this case we have to consider two sub-cases: weak asymmetry, i.e., $1 < v \leq 2$ or strong asymmetry, i.e., $v > 2$. In the case of weak asymmetry (vertex C1 in Figure 2), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{v}{3}]$ according to their

cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$\frac{v}{3} F_3^{(3)}(x) - x = 0 \tag{12}$$

$$F_2^{(3)}(x) + \frac{1}{3}(1 - F_2^{(3)}(x)) - x = 1 - \frac{v}{3}$$

Then, player 3's probability to win the match in the third stage is

$$p_{32}^{C1} = 1 - \frac{v}{4} \tag{13}$$

and the players' expected total effort is

$$TE^{C1} = \frac{v^2 + 2v}{12}. \tag{14}$$

In the case of strong asymmetry, $v > 2$ (vertex C2 in Figure 2), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{2}{3}]$ according to their cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$\frac{v}{3} F_3^{(3)}(x) - x = \frac{v - 2}{3} \tag{15}$$

$$F_2^{(3)}(x) + \frac{1}{3}(1 - F_2^{(3)}(x)) - x = \frac{1}{3}$$

Then, player 3's probability to win the match in the third stage is

$$p_{32}^{C2} = \frac{1}{v} \tag{16}$$

and the players' expected total effort is

$$TE^{C2} = \frac{v + 2}{3v} \tag{17}$$

9.2   Stage 2 - player 1 vs. player 3

We have here two possible scenarios:

1) Assume first that player 1 lost the match in the first stage (vertex D in Figure 2). Then by (6) and (9) the expected payoff of player 3 in the next stage is zero and therefore he has no incentive to exert a positive effort such that player 1 wins with a probability of one. Thus we have

$$p_{13}^D = 1 \tag{18}$$

20

and the players' expected total effort in the second stage is

$$TE^D = 0 \tag{19}$$

2) Assume now that player 1 won the match in the first stage. Then if we assume a weak asymmetry, i.e., $1 < v \leq 2$, (vertex E1 in Figure 2), by (12) and (13) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, 1 - \frac{v}{3}]$ according to their cumulative distribution functions $F_i^{(2)}, i = 1, 3$ which are given by

$$
\begin{aligned}
vF_3^{(2)}(x) + \frac{v^2}{12}(1 - F_3^{(2)}(x)) - x &= \frac{4v - 3}{3} \\
(1 - \frac{v}{3})F_1^{(2)}(x) - x &= 0
\end{aligned}
\tag{20}
$$

Then, player 1's probability to win the match in the second stage is

$$p_{13}^{E1} = 1 - \frac{6 - 2v}{12v - v^2} \tag{21}$$

and the players' expected total effort is

$$TE^{E1} = \frac{-v^3 + 11v^2 - 12v - 36}{6v^2 - 72v}. \tag{22}$$

If, on the other hand, we assume strong asymmetry, i.e., $v > 2$, (vertex E2 in Figure 2), by (15) and (16) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{1}{3}]$ according to their cumulative distribution functions $F_i^{(2)}, i = 1, 3$ which are given by

$$
\begin{aligned}
vF_3^{(2)}(x) + (1 - \frac{1}{v})\frac{v}{3}(1 - F_3^{(2)}(x)) - x &= \frac{3v - 1}{3} \\
\frac{1}{3}F_1^{(2)}(x) - x &= 0
\end{aligned}
\tag{23}
$$

Then, player 1's probability to win the match in the second stage is

$$p_{13}^{E2} = 1 - \frac{1}{4v + 2} \tag{24}$$

and the players' expected total effort is

$$TE^{E2} = \frac{v + 1}{6v + 3} \tag{25}$$

21

9.3 Stage 1 - player 1 vs. player 2

In the case of weak asymmetry, i.e., $1 < v \leq 2$ (vertex F1 in Figure 2), by (9), (10) and (20) there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{16v-13}{12}]$ according to their cumulative distribution functions $F_i^{(1)}, i = 1, 2$ which are given by

$$\frac{4v-3}{3}F_2^{(1)}(x) + \frac{1}{12}(1 - F_2^{(1)}(x)) - x = \frac{1}{12} \tag{26}$$

$$(v - \frac{1}{3})F_1^{(1)}(x) - x = \frac{9 - 4v}{12}$$

Then, player 1's probability to win the match in the first stage is

$$p_{12}^{F1} = \frac{16v - 13}{24v - 8} \tag{27}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F1} = \frac{448v^2 - 636v + 221}{288v - 96} \tag{28}$$

On the other hand, in the case of strong asymmetry, i.e., $v \geq 2$, (vertex F2 in Figure 2), by (10), (15), (23) and (24) there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{12v-5}{12}]$ according to their cumulative distribution functions $F_i^{(1)}, i = 1, 2$ which are given by

$$\frac{3v-1}{3}F_2^{(1)}(x) + \frac{1}{12}(1 - F_2^{(1)}(x)) - x = \frac{1}{12} \tag{29}$$

$$(\frac{3v-1}{3})F_1^{(1)}(x) + (\frac{1}{4v+2})(\frac{v-2}{3})(1 - F_1^{(1)}(x)) - x = \frac{1}{12}$$

Then, player 1's probability to win the match in the first stage is

$$p_{12}^{F2} = \frac{24v^2 + 2v - 5}{48v^2 + 4v} \tag{30}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F2} = \frac{576v^3 - 192v^2 - 80v + 25}{576v^2 + 48v} \tag{31}$$

10 Appendix B: strong vs. strong in the second stage

We analyze here the sub-game perfect equilibrium of the round-robin tournament where the strong players (players 1 and 2) are matched in the second stage. The tree game is described by Figure 3.

10.1 Stage 3 - player 2 vs. player 3

We have the following three scenarios:

1) Assume first that player 3 won the match in the first stage and player 2 won in the second stage (vertex A in Figure 3). Then there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, 1]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$vF_3^{(3)}(x) - x = v - 1 \tag{32}$$

$$F_2^{(3)}(x) - x = 0$$

Then, player 2's probability to win the match in the third stage is

$$p_{23}^A = 1 - \frac{1}{2v} \tag{33}$$

and the players' expected total effort is

$$TE^A = \frac{v+1}{2v} \tag{34}$$

.

2) Assume now that player 3 won the match in the first stage and player 2 lost in the second stage (vertex B1-B2 in Figure 3). Then if the asymmetry is weak, i.e., $1 < v \leq 2$ (vertex B1 in Figure 3), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{v}{3}]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$\frac{v}{3}F_3^{(3)}(x) - x = 0 \tag{35}$$

$$F_2^{(3)}(x) + \frac{1}{3}(1 - F_2^{(3)}(x)) - x = \frac{3-v}{3}$$

Then, player 3's probability to win the match in the third stage is

$$p_{32}^{B1} = 1 - \frac{v}{4} \tag{36}$$

and the players' expected total effort is

$$TE^{B1} = \frac{v^2 + 2v}{12} \tag{37}$$

On the other hand, if the asymmetry is strong, i.e., $v > 2$ (vertex B2 in Figure 3), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{2}{3}]$ according to their effort

cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$\frac{v}{3}F_3^{(3)}(x) - x = \frac{v-2}{3} \tag{38}$$

$$F_2^{(3)}(x) + \frac{1}{3}(1 - F_2^{(3)}(x)) - x = \frac{1}{3}$$

Then, player 3's probability to win the match in the third stage is

$$p_{32}^{B2} = \frac{1}{v} \tag{39}$$

and the players' expected total effort is

$$TE^{B2} = \frac{v+2}{3v} \tag{40}$$

3) Finally, assume that player 3 lost the match in the first stage and player 2 won in the second stage (vertex C in Figure 3). Then, there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{1}{3}]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$vF_3^{(3)}(x) + \frac{v}{3}(1 - F_3^{(3)}(x)) - x = \frac{3v-1}{3} \tag{41}$$

$$\frac{1}{3}F_2^{(3)}(x) - x = 0$$

Then, player 2's probability to win the match in the third stage is

$$p_{23}^C = 1 - \frac{1}{4v} \tag{42}$$

and the players' expected total effort is

$$TE^C = \frac{2v+1}{12v} \tag{43}$$

.

10.2   Stage 2 - player 1 vs. player 2

We have two possible scenarios here:

1) Assume first that player 1 lost the match in the first stage (vertex D1-D2-D3 in Figure 3). By (32), (35) and (36), if the asymmetry is weak, i.e., $1 < v < 1.1$ (vertex D1 in Figure 3), there is a unique mixed

strategy equilibrium in which players 1 and 2 randomize on the interval $[0, v - 1]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 1, 2$ which are given by

$$\frac{v^2}{12} F_2^{(2)} - x = \frac{v^2 - 12v + 12}{12} \tag{44}$$

$$(v - 1) F_1^{(2)}(x) - x = 0$$

Then, player 1's probability to win the match in the second stage is

$$p_{12}^{D1} = 1 - \frac{6v - 6}{v^2} \tag{45}$$

and the players' expected total effort is

$$TE^{D1} = \frac{v^3 + 11v^2 - 24v + 12}{2v^2} \tag{46}$$

If, however, $1.1 < v \leq 2$, (vertex D2 in Figure 3), as previously, there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{v^2}{12}]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 1, 2$ which are given by

$$\frac{v^2}{12} F_2^{(2)} - x = 0 \tag{47}$$

$$(v - 1) F_1^{(2)}(x) - x = \frac{12v - v^2 - 12}{12}$$

Then, player 1's probability to win the match in the second stage is

$$p_{12}^{D2} = \frac{v^2}{24v - 24} \tag{48}$$

and the players' expected total effort is

$$TE^{D2} = \frac{v^4 + 12v^3 - 12v^2}{288v - 288} \tag{49}$$

.

Finally, if $v > 2$, (vertex D3 in Figure 3), by (32), (38) and (39) there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{v-1}{3}]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 1, 2$ which are given by

$$\frac{v - 1}{3} F_2^{(2)} - x = 0 \tag{50}$$

$$(v - 1) F_1^{(2)}(x) + (\frac{v - 2}{3})(1 - F_1^{(2)}(x)) - x = \frac{2v - 2}{3}$$

Then, player 1's probability to win the match in the second stage is

$$p_{12}^{D3} = \frac{v-1}{4v-2} \tag{51}$$

and the players' expected total effort is

$$TE^{D3} = \frac{3v^2 - 5v + 2}{12v - 6} \tag{52}$$

2) Assume now that player 1 won the match in the first stage (vertex E in Figure 3). By (41) and (42), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{3v-1}{3}]$ according to their cumulative distribution functions $F_i^{(2)}, i = 1, 2$ which are given by

$$vF_2^{(2)}(x) + \frac{1}{12}(1 - F_2^{(2)}(x)) - x = \frac{1}{3} \tag{53}$$
$$(\frac{3v-1}{3})F_1^{(2)}(x) - x = 0$$

Then, player 1's probability to win the match in the second stage is

$$p_{12}^E = 1 - \frac{6v-2}{12v-1} \tag{54}$$

and the players' expected total effort is

$$TE^E = \frac{72v^2 - 39v + 5}{72v - 6} \tag{55}$$

10.3   Stage 1 - player 1 vs. player 3

If $1 < v \leq 1.096$ (vertex F1 in Figure 3), by (35), (44), (45), and (53) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{12v-v^2-8}{12}]$ according to their effort cumulative distribution functions $F_i^{(1)}, i = 1, 3$ which are given by

$$\frac{1}{3}F_3^{(1)}(x) + \frac{v^2 - 12v + 12}{12}(1 - F_3^{(1)}(x)) - x = \frac{v^2 - 12v + 12}{12} \tag{56}$$
$$\frac{-v^3 + 9v^2 - 24v + 18}{3v^2}F_1^{(1)}(x) - x = \frac{v^4 - 16v^3 + 44v^2 - 96v + 72}{12v^2}$$

Then, player 1's probability to win the match in the first stage is

$$p_{13}^{F1} = \frac{v^4 - 12v^3 + 8v^2}{8v^3 - 72v^2 + 192v - 144} \tag{57}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F1} = \frac{v^6 - 20v^5 + 76v^4 + 368v^3 - 1448v^2 + 1632v - 576}{-96v^3 + 864v^2 - 2304v + 1728} \tag{58}$$

If $1.096 < v < 1.1$ (vertex F2 in Figure 3), by (35), (44), (45), and (53) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{9v^2 - v^3 - 24v + 18}{3v^2}]$ according to their effort cumulative distribution functions $F_i^{(1)}, i = 1, 3$ which are given by

$$\frac{1}{3}F_3^{(1)}(x) + \frac{v^2 - 12v + 12}{12}(1 - F_3^{(1)}(x)) - x = \frac{v^3 - 8v^2 + 24v - 18}{3v^2} \tag{59}$$
$$\frac{-v^3 + 9v^2 - 24v + 18}{3v^2}F_1^{(1)}(x) - x = 0$$

Then, player 1's probability to win the match in the first stage is

$$p_{13}^{F2} = 1 - \frac{2v^3 - 18v^2 + 48v - 36}{v^4 - 12v^3 + 8v^2} \tag{60}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F2} = \frac{v^7 - 17v^6 + 68v^5 + 138v^4 - 1464v^3 + 3456v^2 - 3456v + 1296}{-6v^6 + 72v^5 - 48v^4} \tag{61}$$

On the other hand, if $1.1 < v \leq 2$, (vertex F3 in Figure 3), by (32), (35), (47), (48), and (53) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{-v^3 + 3v^2}{72v - 72}]$ according to their effort cumulative distribution functions $F_i^{(1)}, i = 1, 3$ which are given by

$$\frac{1}{3}F_3^{(1)}(x) - x = \frac{v^3 - 3v^2 + 24v - 24}{72v - 72} \tag{62}$$
$$\frac{-v^3 + 3v^2}{72v - 72}F_1^{(1)}(x) - x = 0$$

Then, player 1's probability to win the match in the first stage is

$$p_{13}^{F3} = 1 - \frac{v^3 - 3v^2}{-48v + 48} \tag{63}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F3} = \frac{v^6 - 6v^5 - 15v^4 + 96v^3 - 72v^2}{3456v^2 - 6912v + 3456} \tag{64}$$

27

Finally, if $v > 2$, (vertex F4 in Figure 3), by (32), (38), (50), (51), and (53) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{v-1}{12v-6}]$ according to their effort cumulative distribution functions $F_i^{(1)}, i = 1, 3$ which are given by

$$\frac{1}{3}F_3^{(1)}(x) - x = \frac{3v - 1}{12v - 6} \tag{65}$$
$$\frac{v - 1}{12v - 6}F_1^{(1)}(x) - x = 0$$

Then, player 1's probability to win the match in the first stage is

$$p_{13}^{F4} = 1 - \frac{v - 1}{8v - 4} \tag{66}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F4} = \frac{5v^2 - 8v + 3}{96v^2 - 96v + 24} \tag{67}$$

11   Appendix C: strong vs. strong in the third stage

We now analyze the case where the strong players (players 1 and 2) are matched in the last stage. The tree game is described by Figure 4.

11.1   Stage 3 - player 1 vs. player 2

We have the following three scenarios:

1) Assume first that player 1 won the match in the first stage and player 2 won in the second stage (vertex A in Figure 4). Then there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, v]$ according to their cumulative distribution functions $F_i^{(3)}, i = 1, 2$ which are given by

$$vF_2^{(3)}(x) - x = 0 \tag{68}$$
$$vF_1^{(3)}(x) - x = 0$$

Then, player 1's probability to win the match in the third stage is

$$p_{12}^A = 0.5 \tag{69}$$

and the players' expected total effort is

$$TE^A = v \tag{70}$$

2) Assume now that player 1 won the match in the first stage and player 2 lost in the second stage (vertex B in Figure 4). Then there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{v}{3}]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 1, 2$ which are given by

$$vF_2^{(3)}(x) + \frac{v}{3}(1 - F_2^{(3)}(x)) - x = \frac{2v}{3} \tag{71}$$

$$\frac{v}{3}F_1^{(3)}(x) - x = 0$$

Then, player 1's probability to win the match in the third stage is

$$p_{12}^B = 0.75 \tag{72}$$

and the players' expected total effort is

$$TE^B = \frac{v}{4} \tag{73}$$

3) Finally, assume that player 1 lost the match in the first stage and player 2 won in the second stage (vertex C in Figure 4). Then there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{v}{3}]$ according to their cumulative distribution functions $F_i^{(3)}, i = 1, 2$ which are given by

$$\frac{v}{3}F_2^{(3)}(x) - x = 0 \tag{74}$$

$$vF_1^{(3)}(x) + \frac{v}{3}(1 - F_1^{(3)}(x)) - x = \frac{2v}{3}$$

Then, player 2's probability to win the match in the third stage is

$$p_{21}^C = 0.75 \tag{75}$$

and the players' expected total effort is

$$TE^C = \frac{v}{4} \tag{76}$$

11.2 Stage 2 - player 2 vs. player 3

We have here two possible scenarios:

1) Assume first that player 3 lost the match in the first stage (vertex D in Figure 4). Then by (68) and (71) the expected payoff of player 2 in the next stage is zero and therefore he has no incentive to exert a positive effort. In that case, player 3 wins with a probability of one. Thus, we have

$$p_{23}^D = 0 \tag{77}$$

and the players' expected total effort in the second stage is

$$TE^D = 0 \tag{78}$$

2) Assume now that player 3 won the match in the first stage (vertex E1-E2 in Figure 4). If $1 < v \leq 1.375$, (vertex E1 in Figure 4), by (74) and (75) there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{2v}{3}]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 2, 3$ which are given by

$$\frac{2v}{3}F_3^{(2)}(x) - x = 0 \tag{79}$$
$$F_2^{(2)}(x) + \frac{1}{12}(1 - F_2^{(2)}(x)) - x = \frac{-2v + 3}{3}$$

Then, player 2's probability to win the match in the second stage is

$$p_{23}^{E1} = \frac{4v}{11} \tag{80}$$

and the players' expected total effort is

$$TE^{E1} = \frac{8v^2 + 11v}{33} \tag{81}$$

.

If, however, $v > 1.375$, (vertex E2 in figure 4), by (74) there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{11}{12}]$ according to their cumulative distribution functions $F_i^{(2)}, i = 2, 3$ which are given by

$$\frac{2v}{3}F_3^{(2)}(x) - x = \frac{8v - 11}{12} \tag{82}$$
$$F_2^{(2)}(x) + \frac{1}{12}(1 - F_2^{(2)}(x)) - x = \frac{1}{12}$$

Then, player 2's probability to win the match in the second stage is

$$p_{23}^{E2} = 1 - \frac{11}{16v} \tag{83}$$

and the players' expected total effort is

$$TE^{E2} = \frac{88v + 121}{192v} \tag{84}$$

11.3 Stage 1 - player 1 vs. player 3

If $1 < v \leq 1.375$, (vertex F1 in Figure 4), by (71), (72), (77) and (79) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{-8v+11}{12}]$ according to their cumulative distribution functions $F_i^{(1)}, i = 1, 3$ which are given by

$$\frac{2v}{3} F_3^{(1)}(x) - x = \frac{16v - 11}{12} \tag{85}$$

$$(\frac{-2v+3}{3}) F_1^{(1)}(x) + \frac{1}{12}(1 - F_1^{(1)}(x)) - x = \frac{1}{12}$$

Then, player 1's probability to win the match in the first stage is

$$p_{13}^{F1} = 1 - \frac{-8v + 11}{16v} \tag{86}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F1} = \frac{-88v + 121}{192v} \tag{87}$$

On the other hand, if, $v \geq 1.375$, (vertex F2 in Figure 4), by (72), (77) and (82), player 3 has the same expected payoff of $\frac{1}{12}$ in the next stage regardless of whether he wins or loses in this stage, and therefore player 1's probability to win in the first stage is

$$p_{13}^{F2} = 1 \tag{88}$$

and the players' expected total effort in the match (tournament) is

$$TE^{F2} = 0 \tag{89}$$

References

[1] Baye, M., Kovenock, D., de Vries, C.: Rigging the lobbying process. *American Economic Review* 83, 289-294 (1993)

[2] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[3] David, H.: Tournaments and paired comparisons. *Biometrika* 46, 139-149 (1959)

[4] Glenn, W.: A comparison of the effectiveness of tournaments. *Biometrika* 47, 253-262 (1960)

[5] Gradstein, M., Konrad, K.: Orchestrating rent seeking contests. *The Economic Journal* 109, 536-545 (1999)

[6] Groh, C., Moldovanu, B., Sela, A., Sunde, U.: Optimal seedings in elimination tournaments. *Economic Theory* 49, 59-80 (2012)

[7] Harbaugh, N., Klumpp, T.: Early round upsets and championship blowouts. *Economic Inquiry* 43, 316-329 (2005)

[8] Hillman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-40 (1989)

[9] Klumpp, T., Polborn, M.K.: Primaries and the new hampshire effect. *Journal of Public Economics* 90, 1073-1114 (2006)

[10] Konrad, K.: Bidding in hierarchies. *European Economic Review* 48, 1301-1308 (2004)

[11] Konrad, K., Kovenock, D.: Multi-battle contests. *Games and Economic Behavior* 66, 256-274 (2009)

[12] Krumer, A.: Best-of-two contests with psychological effects. *Theory & Decision* 75, 85-100 (2013)

[13] Malueg, D., Yates, A.: Testing contest theory: evidence from best-of three tennis matches. *The Review of Economics and Statistics* 92(3), 689-692 (2010)

[14] Rosen, S.: Prizes and incentives in elimination tournaments. *American Economic Review* 76, 701-715 (1986)

[15] Ryvkin, D., Ortmann, A.: The predictive power of three prominent tournament formats. *Management Science* 54(3), 492-504 (2008)

[16] Searles, D.: On the probability of winning with different tournament procedures. *Journal of the American Statistical Association* 58, 1064-1081 (1963)

[17] Sela, A.: Best-of-three all-pay auctions. *Economics Letters* 112(1), 67-70 (2011)
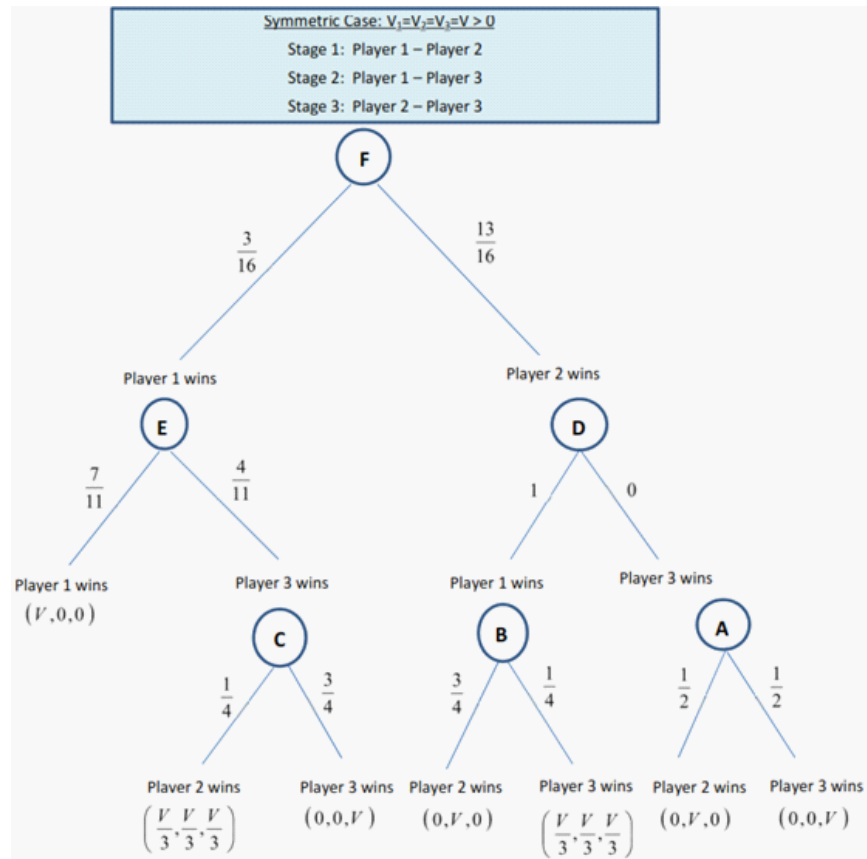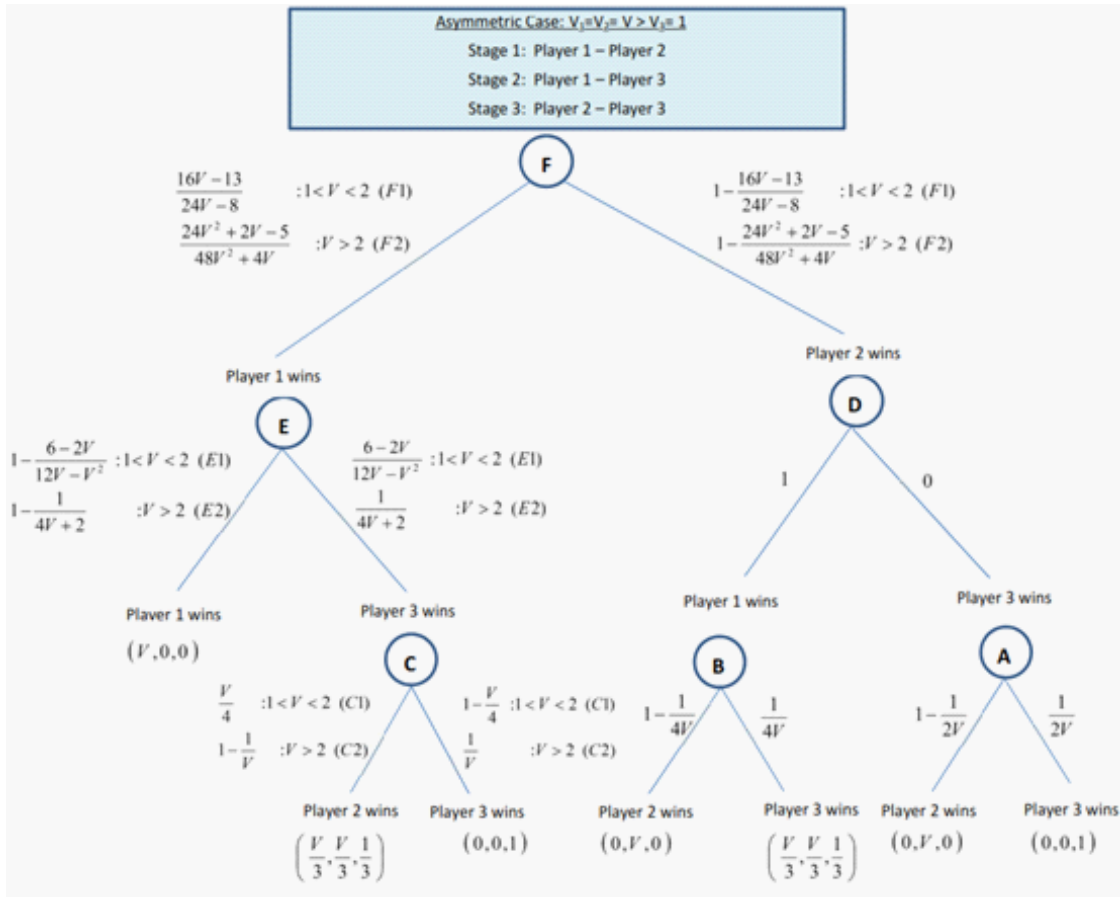
Figure 1: The tree game of the symmetric tournament.

Figure 2: The tree game of the asymmetric tournament where the two strong players are matched in the first stage.
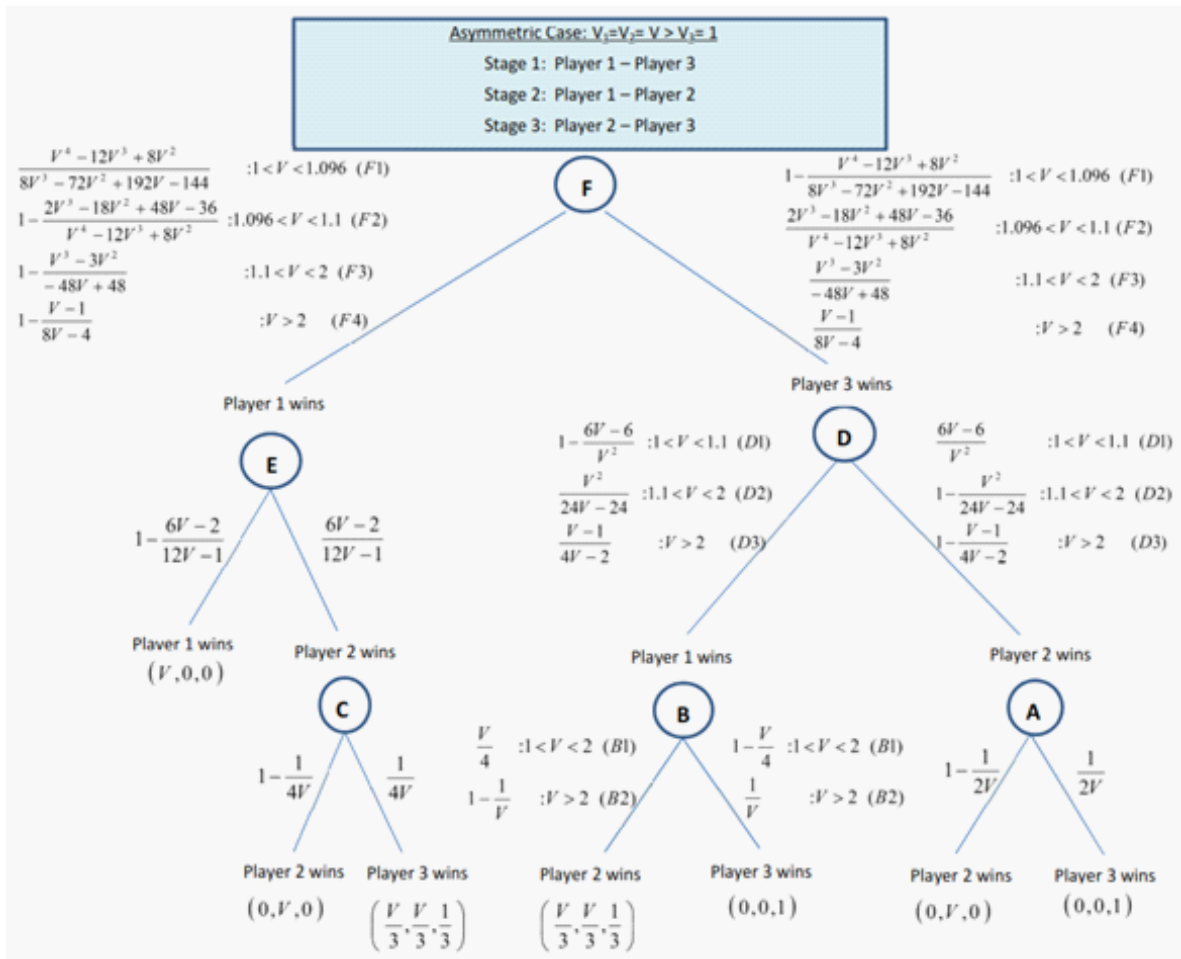
Figure 3: The tree game of the asymmetric tournament where the two strong players are matched in the second stage.
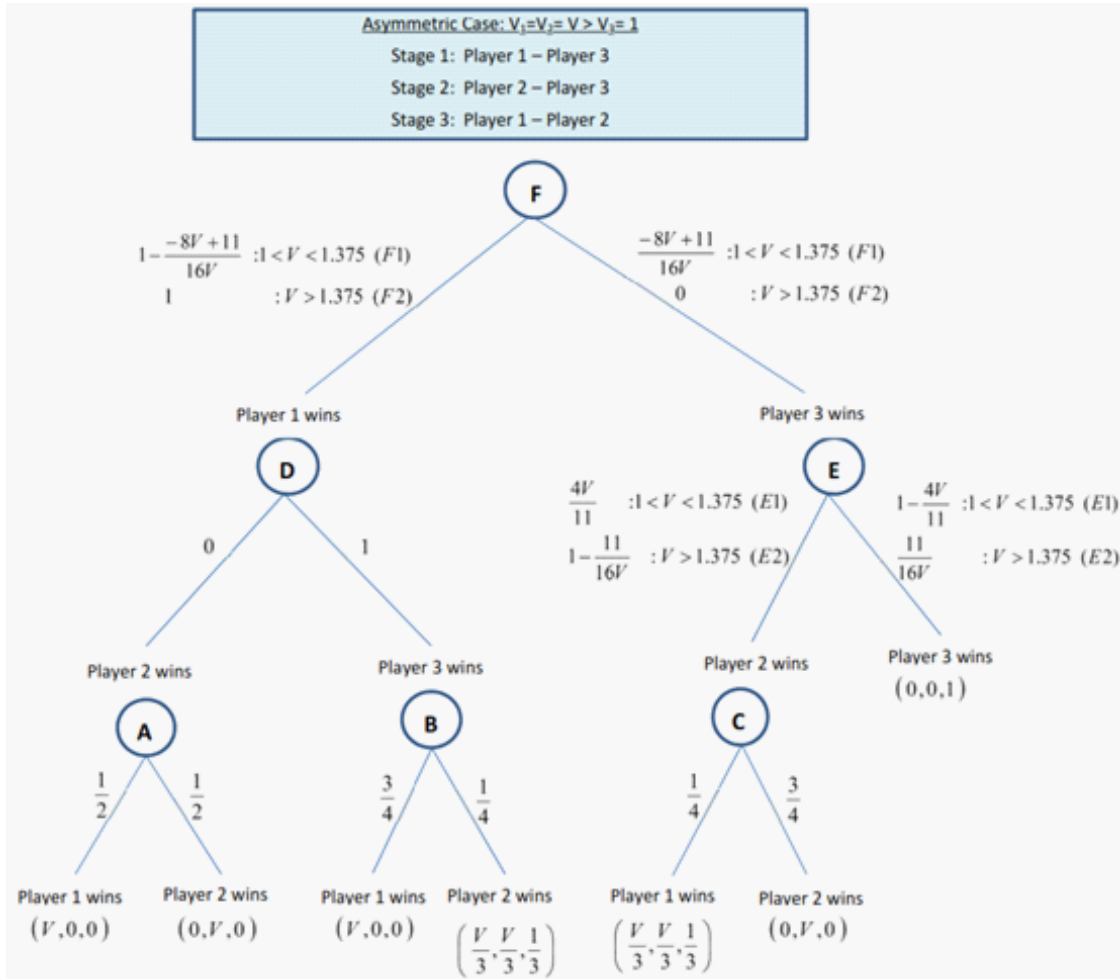
Figure 4: The tree game of the asymmetric tournament where the two strong players are matched in the third stage.
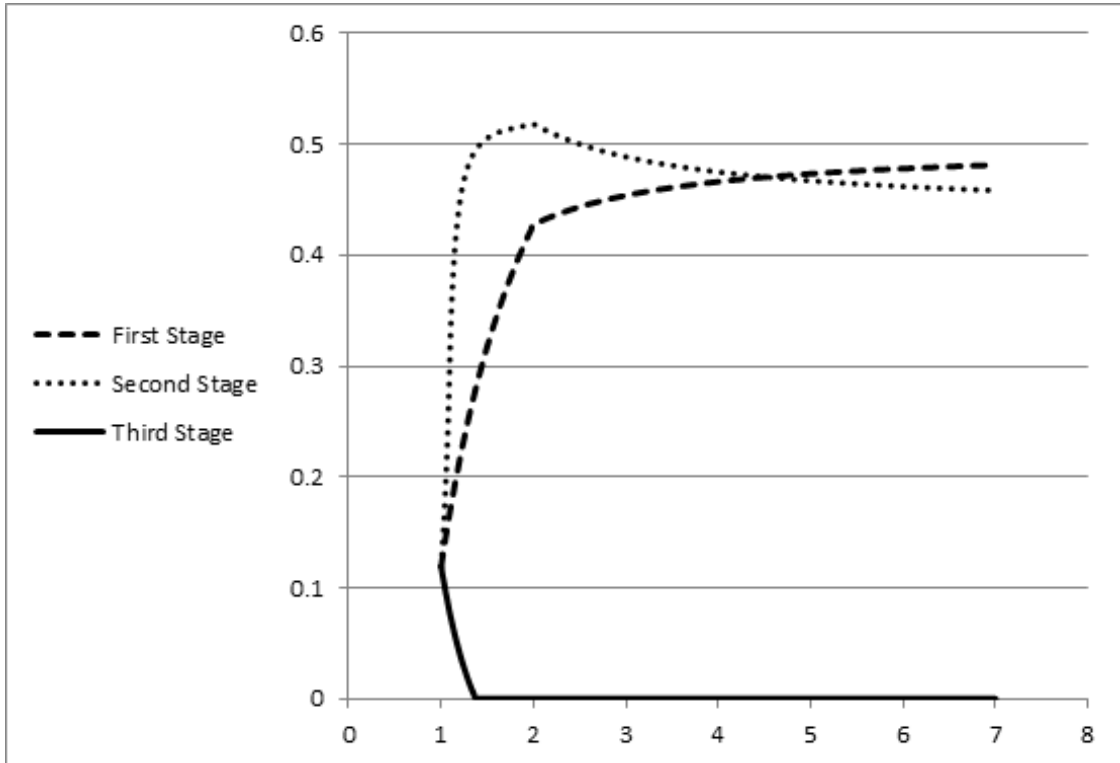
Figure 5: The probability of the tournament to be over after two stages as a function of the level of asymmetry (v) between the players for each asymmetric tournament (strong players are matched either in the first, second or third stage).
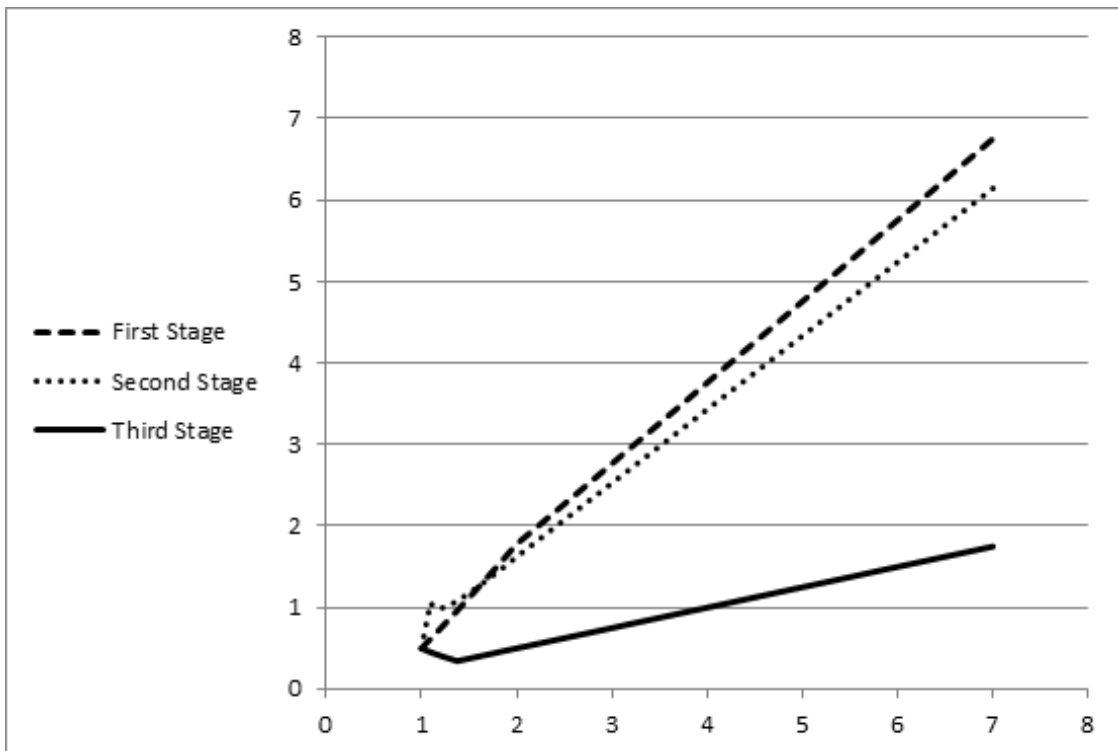
Figure 6: The expected total effort as a function of the level of asymmetry (v) for each asymmetric tournament (the strong players are matched either in the first, second or third stage).