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Francesco Lippi, EIEF, University of Sassari and CEPR
Stefania Ragni, University of Sassari
Nicholas Trachter, Federal Reserve Bank of Richmond

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Centre for Economic Policy Research
77 Bastwick Street, London EC1V 3PZ, UK
Tel: (44 20) 7183 8801, Fax: (44 20) 7183 8820
Email: cepr@cepr.org, Website: www.cepr.org

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ABSTRACT

State dependent monetary policy*

We study the optimal anticipated monetary policy in a flexible-price economy featuring heterogenous agents and incomplete markets, which give rise to a business cycle. In this setting money policy has distributional effects that depend on the state of the cycle. We parsimoniously characterize the dynamics of the economy and study the optimal regulation of the money supply as a function of the state. The optimal policy prescribes monetary expansions in recessions, when insurance is most needed by cash- poor unproductive agents. To minimize the inflationary effect of these expansions the policy prescribes monetary contractions in good times. Although the optimal money growth rate varies greatly through the business cycle, this policy “echoes” Friedman’s principle in the sense that the expected real return of money approaches the rate of time preference.

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Francesco Lippi
EIEF
Via Sallustiana, 62 – 00187.
00184 Rome
ITALY

Email: f.lippi@uniss.it

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Nicholas Trachter
Federal Reserve Bank of Richmond
Post Office Box 27622
Richmond, VA 23261
USA

Email: nicholas.trachter@rich.frb.org

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Stefania Ragni
University of Sassari
Via Muroni 25
07100 Sassari
ITALY

Email: sragni@uniss.it

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1 Introduction

We study optimal monetary policy in a competitive flexible-price economy where infinity-lived agents are subject to idiosyncratic productivity shocks and money is valued in equilibrium due to anonymity. The state of this economy is described by the wealth distribution, which evolves through time following the history of shocks and determines the value of money and aggregate output. Our objective is to characterize how to optimally regulate the money supply as a function of the state of the economy, what we call an optimal state-dependent monetary policy. A key feature of the setup is that monetary policy affects (and is affected by) the wealth distribution, as in [Wallace \(1984\)](#) or [Berentsen et al. \(2005\)](#) and many other monetary models whose first principles are explicitly spelled out. Although the propagation of such redistributive effects of monetary policy is often “muted” by means of appropriate assumptions for the sake of tractability, as in [Lucas \(1990\)](#), [Shi \(1997\)](#), or [Lagos and Wright \(2005\)](#), in this paper we use an analytically tractable setup that allows us to study the role of systematic monetary policy taking fully into account the dynamics of the wealth distribution. The key assumption to get tractability, following [Scheinkman and Weiss \(1986\)](#), is that we consider the simplest economy with time varying wealth distribution, namely one with two types of agents.¹ We see this as a convenient starting point to study the interactions between the dynamics of the wealth distribution and monetary policy.

We think the question is interesting because it is novel in the theory and because the analysis provides a framework to interpret the large monetary expansions, sometimes observed during deep recessions, with a mechanism that is completely different from the canonical one relying on sticky prices.² The properties of an optimal monetary policy in models where incomplete markets and heterogenous agents allow for a potential redistributive role of monetary policy were first studied by [Levine \(1991\)](#). They have since been explored in a variety of contexts by [Kehoe et al. \(1990\)](#), [Shi \(1999\)](#), [Bhattacharya et al. \(2005\)](#), [Molico \(2006\)](#), [Algan et al. \(2011\)](#), and [Tenreyro and Sterk \(2013\)](#), for example. A common feature of these models is a tension between the benefits of a deflationary policy, one that produces an efficient return on money as under the Friedman’s rule, and the benefits of an expansionary policy, which provides partial insurance to cash-poor agents. A novelty of this paper is that while previous models focused on a constant rule, i.e., seeking the optimal constant rate of monetary expansions, we consider a state-dependent monetary policy, in which the rate of monetary expansion depends on the state of the economy.

¹This assumption is often used in macro theory for tractability, see e.g., [Grossman and Weiss \(1983\)](#), [Alvarez and Lippi \(2013\)](#), or [Golosov and Sargent \(2013\)](#).

²The assumption of flexible prices is useful to emphasize the workings of the redistributive role of monetary policy, and to distinguish it from the better understood mechanism that arises with sticky prices. Both channels are likely relevant in practice.

Our model extends [Scheinkman and Weiss's \(1986\)](#) analysis, which assumes a constant money supply, by letting the government control the money supply through lump-sum transfers.³ We provide a characterization of the price of money and of aggregate production in terms of the policy rule in a competitive equilibrium. We adopt an ex-ante welfare criterion and characterize an efficient monetary policy by solving a Ramsey problem.

Our results cast some light on the interactions between the dynamics of the assets distribution and the optimal anticipated policy. The main policy choice is a tradeoff between the provision of insurance (through monetary expansions) and ensuring an efficient return on savings (through contractions). We show that in our model the importance of the insurance motive varies with the state of the economy, so that a state-dependent monetary policy allows for a significant improvement compared with a constant policy. An expansionary monetary policy turns out to be efficient in recessions, when poor and unproductive economic agents benefit from some wealth redistribution. Surprisingly, in spite of the occasional large monetary expansions in states where the insurance motive is large, the optimal policy neutralizes the inflationary effect of these expansions by contracting the money supply in states where the insurance motive is small. In this way the state-dependent nature of the optimal policy allows for the provision of insurance when mostly needed without severely distorting the return of the asset. Thus, although our setup creates a potentially beneficial role for monetary expansions, the optimal rule prescribes an almost complete undoing of the inflationary effects of those expansions, and it implements a policy that brings the expected return on money as close as possible to the constant rate prescribed by the Friedman rule.

Our research question is related to the one analyzed by [Molico \(2006\)](#), who considers a search model of money with a nondegenerate distribution of cash holdings, showing that mild monetary expansions can be beneficial. In his model randomly matched agents may exchange goods for money. The price paid by the buyer results from bargaining and depends on the amount of money held by each agent upon entering the pairwise meeting. Therefore, the distribution of money is nondegenerate and monetary injections, via lump-sum transfers, can improve the terms of trade of poor buyers. Related results in the context of search models of money and mechanism design are obtained by [Berentsen et al. \(2005\)](#), [Green and Zhou \(2005\)](#), [Deviatov and Wallace \(2012\)](#), [Wallace \(2013\)](#). Our model departs from Molico in a few important ways. First, we restrict attention to an economy with only two types of agents, so that the wealth distribution is analytically tractable, while Molico does not restrict the number of types and as a consequence his model must be solved numerically. Second, and perhaps more importantly, we allow the planner to tie its policy to the distribution

³ As in [Levine \(1991\)](#) we assume that the government does not know which agent is productive, so that the transfers are equal across agents. See [Kehoe et al. \(1990\)](#) for a thorough discussion of this assumption and in particular [Levine \(1991\)](#) for a derivation of the equal-treatment restriction from first principles.

of cash holdings, while Molico considers a constant policy. A key feature of our model is that business cycles, and the magnitude of fluctuations, depend on the tightness of the borrowing constraint. Because the borrowing constraint is tighter in downturns the return on money is high in recessions and low during booms. As a result inflation (the inverse of the return on money) is positively correlated with aggregate activity, thus generating a “Phillips curve’.’ This result relates to [Guerrieri and Lorenzoni \(2009\)](#) and [Guerrieri and Lorenzoni \(2011\)](#), which explore the effects of borrowing constraints on business cycles in a model with liquid assets. Similar to their papers, as the borrowing constraint becomes tighter, economic fluctuations become more severe in our model. The relative simplicity of our setup allows us to investigate the optimal provision of liquidity. Our analysis is also related to [Algan et al. \(2011\)](#), that characterizes the output-inflation tradeoff in a flexible price economy with incomplete markets and persistent wealth inequality among agents. While the setups are similar, the focus is different as monetary policy is treated as an exogenous parameter in these models.

The paper is organized as follows. [Section 2](#) presents the set up of the model. [Section 3](#) sets the ground for the analysis of monetary equilibria. [Section 4](#) characterizes the value of money in equilibrium under a state-dependent rule and compares it with the one with a constant money rule. [Section 5](#) defines an ex ante welfare criterion and studies the best state dependent rule for the supply of liquidity. [Section 6](#) concludes.

2 The model

This section describes the model economy: agents’ preferences, production possibilities, and markets. Two useful benchmarks are presented: the (efficient) allocation with complete markets and the optimal monetary policy with no uncertainty. To finalize, we also argue that value functions and allocations are homogeneous in the exogenous parameters of interest; this is useful as it allows us to reduce the dimensionality of the problem by doing an appropriate normalization.

We consider two types of infinitely lived agents (with a large mass of agents of each type), indexed by $i = 1, 2$, and assume that at each point in time only one type of agent can produce. We further restrict attention to the case where agents of the same type play the same action at every point in time so that we can discuss the model in terms of two representative agents, one of each type. Because there are two agents in this economy, we can solve the model by looking at the problem from the perspective of agent one (i.e., $i = 1$). Let $\mathcal{I}_t = \{0, 1\}$ denote the productive type of the agent at time t . When $\mathcal{I}_t = 1$ the agent can produce and transforms labor into consumption one for one; we label this agent as productive. When

$\mathcal{I}_t = 0$ the agent cannot produce; in this case we label the agent as unproductive. The productivity of labor is state dependent: The duration of productivity spells is random, exponentially distributed, with mean duration $1/\lambda > 0$. Money is distributed at each time t between the two agents so that $m_t^1 + m_t^2 = M_t$. The growth rate of the money supply at time t is μ_t ; then, the money supply follows $M_t = M_0 e^{\int_0^t \mu_j dj}$, with M_0 given. As in [Scheinkman and Weiss \(1986\)](#) and [Levine \(1991\)](#), we let the individual state of an agent be private information, precluding agents from issuing private debt.⁴ A key assumption is that agents face a borrowing constraint restricting their unique savings instrument, money, to be non-negative. Because of the assumption of anonymity, fiscal policy has limited powers in this setup.⁵

Let $\rho > 0$ denote the time discount rate. Each agent chooses consumption c_t , labor supply ℓ_t , and depletion of money balances \dot{m}_t^1 in order to maximize her expected discounted utility,

$$\max_{\{c_t, \ell_t, \dot{m}_t^1\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} (\bar{c} \ln c_t - \mathcal{I}_t \ell_t) dt \right\} \quad (1)$$

subject to the constraints

$$\dot{m}_t^1 \leq (\ell_t + \tau_t - c_t) / \tilde{q}_t \quad \text{if } \mathcal{I}_t = 1 \quad (2)$$

$$\dot{m}_t^1 \leq (\tau_t - c_t) / \tilde{q}_t \quad \text{and } \ell_t = 0 \quad \text{if } \mathcal{I}_t = 0 \quad (3)$$

$$m_t^1 \geq 0, \ell_t \geq 0, c_t \geq 0, m_0^1, \mathcal{I}_0 \text{ given}, \quad (4)$$

where $\bar{c} > 0$ is a parameter governing the marginal utility of consumption, \tilde{q}_t denotes the price of money, i.e., the inverse of the consumption price level, τ_t denotes a government lump-sum transfer to each agent, and where expectations are taken with respect to the productivity process defining \mathcal{I}_t and M_t conditional on time $t = 0$.

A monetary policy with $\mu_t > 0$ is called expansionary, a policy with $\mu_t < 0$ is called contractionary. It is immediate that when the money supply is constant for all t (i.e., $\mu_t = 0 \forall t$) and $\bar{c} = 1$, the economy is the one analyzed by [Scheinkman and Weiss \(1986\)](#). The monetary policy μ_t determines the transfers to the agents τ_t through the government budget constraint,

$$\tilde{q}_t \mu_t M_t = 2\tau_t.$$

The government transfer scheme implies that in the case of a contractionary policy, agents

⁴Having a large mass of agents of each type is important for the argument as it implies that a single agent cannot infer the productive state of a different agent given his own state.

⁵ In the online Appendix we discuss what allocations can be achieved using tax policy under various assumptions about government powers (commitment vs. no commitment), types of available taxes (lump-sum vs. distortionary), and government knowledge about the state (agent's type observable vs. not observable).

must use their money holdings to pay taxes (i.e. $\tau_t < 0$). The “tax solvency” constraint, $m_t^1 \geq 0$, imposes this restriction. Notice that in the continuous time characterization of the model the tax solvency constraint coincides with the borrowing constraint.

Note that the government cannot differentiate transfers across agent types. This follows from the assumption that the identity of the productive type is not known to the government. [Levine \(1991\)](#) shows in a similar setup that, because of anonymity, the best mechanism is linear and resembles monetary policy.

Next we state two important remarks. The first one characterizes a symmetric efficient allocation with complete markets (the proof is standard so we omit it):

Remark 1 *Assume complete markets and an ex-ante equal probability of each productive state. The symmetric efficient allocation prescribes the same constant level of consumption, $c_t = \bar{c}$ for all t .*

Thus, without borrowing constraints, the efficient allocation solves a static problem, and it encodes full insurance: Agents consume a constant amount \bar{c} (since ex-ante agents are equal) and the aggregate output is constant.

The second remark characterizes the optimal monetary policy in the case of no uncertainty. This helps highlighting the essential role of uncertainty in our problem. In particular, consider the case where each agent oscillates deterministically between productive and unproductive cycles of length T . Without loss of generality, for the characterization of the stationary equilibria, let us assume that the economy starts in period $t = 0$ with the agent being productive and holding no money, so that $m_0^1 = 0$. We have

Remark 2 *Consider a deterministic production cycle of length T . The symmetric efficient allocation, $c_t = \bar{c}$ for all t is attained by deflating at the rate of time preference $\mu_t = -\rho$ for all t .*

A proof is available in [Appendix B.1](#). This remark, together with the efficient allocation described in [Remark 1](#), shows that without uncertainty this economy replicates the result of the optimality of the “Friedman rule” in [Townsend \(1980\)](#) and [Bewley \(1980\)](#).

To conclude, notice that, by inspection of the agent problem presented in [equations \(1\) to \(4\)](#) and the evolution of money supply ($M_t = M_0 e^{\int_0^t \mu_j dj}$), it is easily seen that the problem is homogeneous on $\{\lambda, \rho, \mu_t\}$: Allocations (the flows) are homogeneous of degree 0 while prices and values (the stocks) are homogeneous of degree minus 1. This result is a natural consequence of the Poisson rate of changing states λ , the discount rate ρ , and the monetary expansion rate μ_t all being measured with respect to calendar time. This result is useful as it shows that, after normalizing by λ , the model has only three parameters: the normalized

discount rate, ρ/λ , the normalized money growth rate, μ_t/λ , and \bar{c} . This is useful as once we treat μ_t as a policy instrument, the model has only two exogenous parameters: \bar{c} and the normalized discount rate ρ/λ .

3 Characterization of monetary equilibrium

We look for an equilibrium where the price of money depends on the whole history of shocks, as encoded in the current values of the money supply, the distribution of money holdings, and the current state of productivity; that is, we let $\tilde{q}_t = \tilde{q}(M_t, m_t^1, \mathcal{I}_t)$. With a slight abuse of notation this implies $c_t = c(M_t, m_t^1, \mathcal{I}_t)$, $\ell_t = \ell(M_t, m_t^1, \mathcal{I}_t)$, and $\dot{m}_t^1 = \dot{m}^1(M_t, m_t^1, \mathcal{I}_t)$.

As usual the nominal variables are homogenous of degree one in the level of money. With this in mind, we simplify the state space by letting $\hat{q}(z_t, \mathcal{I}_t) = M_t \tilde{q}(M_t, m_t^1, \mathcal{I}_t)$ denote the price of money when the agent holds $z_t \equiv \frac{m_t^1}{M_t}$ share of total money balances—a measure of the wealth distribution—where $z_t \in [0, 1]$. Likewise the consumption and labor supply rules are homogeneous of degree zero in the level of the money supply, $c^p(z_t) = c(M_t, m_t^1, 1)$, $c^u(z_t) = c(M_t, m_t^1, 0)$, and $\ell^p(z_t) = \ell(M_t, m_t^1, 1)$.

Let x_t denote the wealth share in the hands of the unproductive agent. Note that this variable will record discrete jumps every time the identity of the productive type changes: Whenever the identity of the productive type switches, the state x jumps. We allow the planner to choose a Markovian monetary policy μ_t that is a continuous function of the state $x \in [0, 1]$. As a result, notice that x_t summarizes the whole history of the economy and, without loss of generality, $\mu_t = \mu(x_t)$. Given the symmetry of the problem we let $q(x_t)$ denote the price of money in terms of consumption units, which occurs when the unproductive type assets are x_t . That is, if agent 1 holds wealth z_t , $q(x_t) = \hat{q}(z_t, 0)$ if she is unproductive and $q(x_t) = \hat{q}(1 - z_t, 1)$ when she is productive. Next we define a monetary equilibrium.

Definition 1 *Given the continuous policy rule $\mu(x_t)$, the initial level of money supply M_0 , the initial productivity status \mathcal{I}_0 , the an initial distribution of money holdings x_0 , a monetary equilibrium is a price function $\tilde{q}_t = \frac{1}{M_t} q(x_t)$, with $q : [0, 1] \rightarrow \mathbb{R}^+$ and a stochastic process x_t with values in $[0, 1]$ such that, for all t , consumers maximize expected discounted utility (equation (1)) subject to (2), (3) and (4), and the market clearing constraint $c^p(1 - x_t) + c^u(x_t) = \ell(1 - x_t)$ and the government budget constraint are satisfied.*

From now on we omit the time index t to simplify the notation. A straightforward result is that permanent deflations cannot be implemented in equilibrium.⁶ We state this result in the next proposition.

⁶This result relates to [Bewley \(1983\)](#), who showed that in a neoclassical growth model with incomplete markets there is no monetary equilibrium if the interest rate is lower than the discount rate.

Proposition 1 *There is no monetary equilibrium where $\mu(x) < 0$ for all x . Moreover, all monetary equilibria must satisfy $\mu(0) \geq 0$.*

See [Appendix B.2](#) for a proof. The economics of this result is simple. As the length of the unproductive spell cannot be bounded above, there is a nonzero probability that a poor unproductive agent fails to cover her tax obligations. The only way she can fulfill her tax obligations is by keeping half of the money stock and not trading for goods. Because of no trade, money has no value (i.e., $q(x) = 0$ for all x), there is no monetary equilibrium, and the allocation is autarkic. Moreover for any rule, including those that may allow for monetary contractions, the money growth rate cannot be negative at $x = 0$. This is immediate since when $x = 0$ unproductive agents hold no assets and are unable to cover their tax obligations.

Solving the model requires characterizing the marginal value of money given by the Lagrange multipliers for \dot{m} in the problem defined in (1). Let $\tilde{p}(M, m^1)$ and $\tilde{u}(M, m^1)$ denote the un-discounted multipliers associated with the constraints in [equations \(2\)](#) and [\(3\)](#), respectively, so that, e.g., $\tilde{u}(M, m^1)$ measures the marginal value of money for agent 1 when the money supply is M , her wealth share is m^1/M , and she is unproductive. Likewise, $\tilde{p}(M, m^1)$ measures the marginal value for agent 1 when her wealth share is m^1/M and she is productive. Using the homogeneity in the level of money M , we can write $\tilde{u}(M, m^1) = u(z)/M$ and $\tilde{p}(M, m^1) = p(z)/M$.

Combining the first order conditions with respect to ℓ and c^u gives

$$p(z) = q(x) \quad , \quad \frac{\bar{c}}{c^p(z)} = \frac{p(z)}{q(x)} \quad , \quad \text{and} \quad \frac{\bar{c}}{c^u(z)} = \frac{u(z)}{q(x)} \quad , \quad (5)$$

where z is the share of money in the hands of the agent and x is the share of money in the hands of the unproductive agent. These conditions equate marginal costs and benefits of an additional unit of money. The first two equations apply when the agent is productive (i.e. $\mathcal{I} = 1$). The first one states that the marginal benefit of an additional unit of money, $p(z)$, equals the cost of obtaining that unit, i.e. the disutility of work to produce and sell a consumption amount $q(x)$. The second equation states that the marginal cost of the foregone unit of money, which is $p(z)$, equals the marginal benefit, which is given by the product of the price $q(x)$ (consumption per unit of money) times the marginal utility of consumption $\bar{c}/c^p(z)$. Notice that combining these two equations implies that $c^p(z) = \bar{c}$ for all z . Finally, the third equation applies when the agent is unproductive (i.e., $\mathcal{I} = 0$) and states that the marginal cost of the forgone unit of money $u(z)$, equals the benefit that is given by the additional units of consumption that can be bought with it: the product of the price $q(x)$ times the marginal utility of consumption $\bar{c}/c^u(z)$.

The following functions define the evolution of money holdings,

$$\dot{z}^u(z) = h^u(z) \quad \text{and} \quad \dot{z}^p(z) = h^p(z) , \quad (6)$$

where $h^u(z)$ is the change in the share of money holdings of an unproductive agent holding a share z , and $h^p(z)$ is the analogue for a productive agent. It is immediate that $h^u(z) + h^p(1 - z) = 0$. Consider the law of motion for z by type 1 when unproductive,

$$h^u(z) = \mu(x) \left(\frac{1}{2} - z \right) - \frac{c^u(z)}{q(x)} = \mu(x) \left(\frac{1}{2} - z \right) - \frac{\bar{c}}{u(z)} , \quad (7)$$

where we used the budget constraint of the unproductive agent, [equation \(3\)](#), the government budget constraint, and the first order condition in [equation \(5\)](#).

For any $z \in (0, 1)$ the marginal value of money for productive and unproductive agents, $p(z)$, and $u(z)$, solve a system of differential equations, which is the continuous time counterpart of the discrete time Euler equations,

$$(\rho + \mu(x)) p(z) = p'(z) h^p(z) + \lambda(u(z) - p(z)) , \quad (8)$$

$$(\rho + \mu(x)) u(z) = u'(z) h^u(z) + \lambda(p(z) - u(z)) . \quad (9)$$

The derivation is standard so we omit it. To provide some intuition consider the first equation: When the agent is productive and holds a share of money z , the value flow, discounted by the nominal rate $(\rho + \mu(x))$, is equal to the change in the marginal value due to the evolution of her money holdings, $p'(z)h^p(z)$, and to the expectations of the change in value in case the state switches and the agent becomes unproductive: $\lambda(u(z) - p(z))$.

To complete the description of the equilibrium we provide the boundary condition for the marginal value of money $p(z)$ and $u(z)$. The boundary occurs when the unproductive agent has no money. In this case an unproductive agent spends the whole money transfer to finance her consumption, so that $h^u(0) = h^p(1) = 0$.⁷ The budget constraint gives that the consumption of an unproductive agent with no money is $\lim_{z \rightarrow 0} c^u(z) = \tau(0) = q(0)\mu(0)/2$. Using [equation \(5\)](#),

$$\lim_{z \rightarrow 0} u(z) = \frac{2 \bar{c}}{\mu(0)} , \quad (10)$$

with $\lim_{z \rightarrow 0} u(z) = +\infty$ if $\lim_{x \rightarrow 0} \mu(x) = 0$, and where the limit obtains because of Inada conditions. This is an important result in our analysis. An expansionary policy provides an upper bound to the marginal utility of money because the agent enjoys a positive consumption even with no wealth. If there is no money growth when the unproductive agent is poor (i.e.,

⁷We provide a formal proof of this statement in [Section 4](#).

when $\mu(x) \rightarrow 0$ as $x \rightarrow 0$), the agent is not able to consume in poverty and therefore Inada conditions imply that her marginal utility diverges. Also, evaluating [equation \(8\)](#) at $z = 1$ and $x = 0$ gives

$$u(1) = \left(1 + \frac{\rho}{\lambda} + \frac{\mu(0)}{\lambda}\right) p(1) . \quad (11)$$

Notice that $\mu(0)$, the money growth rate when the unproductive agent has zero wealth, appears in both boundaries. An implication is that the choice of the money growth rate simultaneously affects the insurance needs of the unproductive agents and the production incentives of the productive agents.

Using the first order conditions in [equation \(5\)](#) yields an expression for the consumption of an unproductive agent,

$$c^u(z) = \bar{c} \frac{p(1-z)}{u(z)} . \quad (12)$$

This is an important object because unproductive agents need to spend money to consume and therefore monetary policy will affect their consumption behavior. Looking at the consumption level of the unproductive agent when she has no money gives an idea of the insurance role of monetary expansions,

$$\lim_{z \rightarrow 0} c^u(z) = \frac{\mu(0)}{2} p(1) , \quad (13)$$

with $\lim_{z \rightarrow 0} c^u(z) = 0$ if $\lim_{x \rightarrow 0} \mu(x) = 0$, where the limit obtains from the agent's budget constraint. This is important for the welfare analysis that will follow because it shows that monetary transfers provide the unproductive agent with a lower bound to her consumption level. Without transfers, an agent with no money cannot consume.

Another interesting object is the expected real return on money or, equivalently, the expected real interest rate. This object is useful in understanding the workings of the model and illustrates how the market incompleteness affects the economy by generating a risk premium. Let $r(x)$ denote the expected (net) return on money,

$$r(x) \equiv \mathbb{E} \left[\frac{\dot{\tilde{q}}(x)}{\tilde{q}(x)} \middle| x_t = x \right] ,$$

which is just the expected growth rate of the price of money conditional on the unproductive agent money holdings being x . Using that $\tilde{q}(x) = p(1-x)/M$, the expected return can be written as (see [Appendix A](#))

$$r(x) = \rho + \lambda \frac{p(x)}{p(1-x)} \left(\frac{c^u(1-x) - \bar{c}}{c^u(1-x)} \right) . \quad (14)$$

In a complete markets setting, such as the one described in [Remark 2](#), consumption is constant

(at level \bar{c}) and the expected real return equals the time discount, ρ . With incomplete markets the return on money depends on the history of the shocks, as summarized by the wealth distribution x . Recall, by [equation \(5\)](#), that the price of money is determined by the productive agent $q(x) = p(1 - x)$. [Equation \(14\)](#) shows that the expected real rate is proportional to the change in the price of money associated to a switch of the state, i.e., $p(x)/p(1 - x)$, and is proportional to the productive agent's expected consumption growth.⁸

Finally, noting that $1/\tilde{q}(x)$ denotes the price of consumption in units of money, we can compute the expected inflation rate

$$\pi(x) = -\frac{r(x)}{1 + r(x)}. \quad (15)$$

After determining the equilibrium functions $u(z), p(z), c^u(z)$ we will use these formulas to argue that the model displays a “Phillips curve,” a positive correlation between the inflation rate and the output level.

4 The value of money in equilibrium

The previous analysis showed that allocations in a monetary equilibrium are fully characterized by the Lagrange multipliers, $u(z)$ and $p(z)$, that solve the system of Euler equations and associated boundary conditions. This section characterizes the properties of these multipliers, measuring the “value of money” to the productive and unproductive agents, under two rules for the money supply. The first rule is state independent, i.e., equal to a constant non-negative money growth value $\mu \in [0, +\infty)$. The second rule assumes that the money growth rate is a continuous function of the wealth share of the unproductive agent: $\mu(x)$.

Using [equations \(8\)-\(9\)](#) we can define the following system:

$$p'(z) = \frac{(\rho + \lambda + \mu(1 - z))p(z) - \lambda u(z)}{h^p(z)}, \quad (16)$$

$$u'(z) = \frac{(\rho + \lambda + \mu(z))u(z) - \lambda p(z)}{h^u(z)}. \quad (17)$$

The first equation describes the marginal value of money for a productive agent holding z , so that $x = 1 - z$, i.e., monetary policy is a function of the wealth of the unproductive agent. The second equation describes the marginal value for an unproductive agent holding z , so that $x = z$. The solution of this system, together with the boundary condition, fully

⁸Consumption remains constant with probability $1 - \lambda$ per unit of time, or it changes from \bar{c} to $c^u(1 - x)$ with rate λ .

characterizes the value of money in equilibrium.

Next we state a result that is key in characterizing the problem.

Proposition 2 *Assume $\mu(x)$ is continuous in $[0, 1]$ and that the forcing terms on the right hand side of (16)-(17) have no singularities in $(0, 1)$. Then, for any $\bar{c} > 0$, it holds that $h^u(z) < 0$ for all $z \in (0, 1)$ and $\lim_{z \rightarrow 0} h^u(z) = 0$.*

See [Appendix B](#) for a proof.⁹ The economic content of the proposition is that unproductive agents deplete their share of money holdings as long as they remain unproductive. **Proposition 2** allows us to characterize some interesting features of the Lagrange multipliers, the marginal value of money $p(z)$ and $u(z)$, by representing their evolution in the corresponding phase diagram. To this end we define the sets L_p and L_u with elements $(U, P) \in (\mathbb{R}^+)^2$, which evolve as z varies in $[0, 1]$: U is the unproductive agent marginal utility $u(z)$ and P is given by

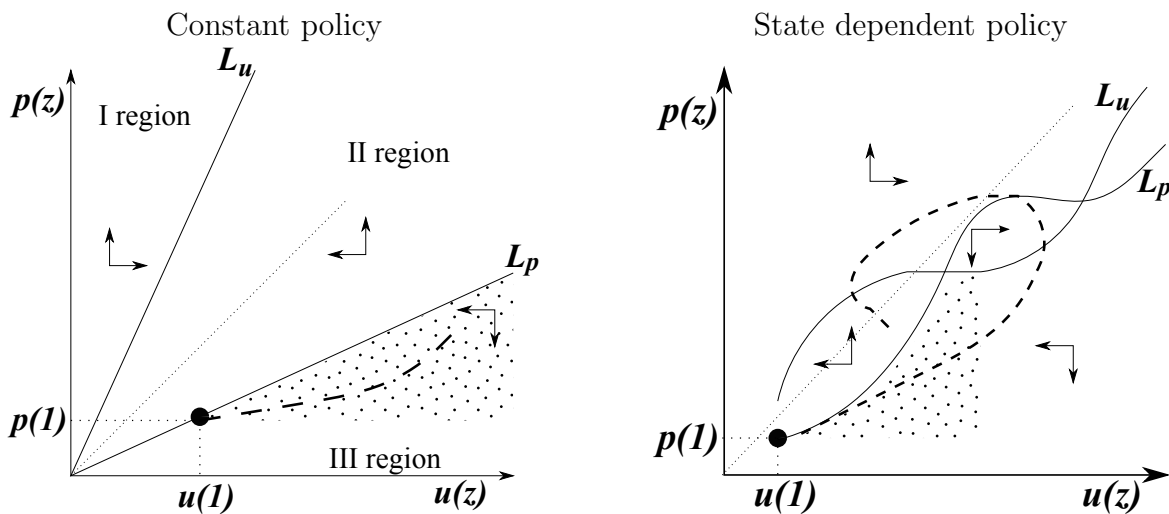
$$L_p : P = \frac{\lambda}{\rho + \lambda + \mu(1-z)}u(z), \quad \text{and} \quad L_u : P = \frac{\rho + \lambda + \mu(z)}{\lambda}u(z),$$

according to z that moves in its domain. These geometrical loci are very useful for investigating some features of the Lagrange multipliers. Indeed, by construction, $p'(z)h^p(z)$ and $u'(z)h^u(z)$ are zero on L_p and L_u , respectively; thus, according to the sign of terms $h^p(z)$ and $h^u(z)$, both L_p and L_u determine different regions in the phase plane (U, P) where the functions $u(z)$, $p(z)$ change behavior according to the sign of their derivative. Two examples are shown in the top panels of [Figure 1](#) where the arrows describe the increasing/decreasing behavior for $u(z)$ and $p(z)$. The direction of the arrows is determined by taking into account both the sign of terms $p'(z)h^p(z)$, $u'(z)h^u(z)$ and the evolution of money holdings established in [Proposition 2](#). The phase diagrams describe the model dynamics: The first case, plotted in the upper-left panel, describes a constant money rule; the second case, describing a state-dependent rule, is plotted in the upper-right panel. The bold dot in each plot corresponds to the boundary condition in [equation \(11\)](#): $u(1) = \frac{\rho + \lambda + \mu(0)}{\lambda}p(1)$, which lies on L_p , the 45-degree line $P = U$ is denoted by dotted line. The dashed curve is a possible path for the solution. Next we exploit the phase diagram to investigate some features of the solution under the different type of policies. In particular, our analysis allows us to provide global results for the constant policy. However, when the policy is allowed to vary with the state, the same results can be obtained only locally when x is sufficiently small (i.e., when the productive agent holds most of the money).

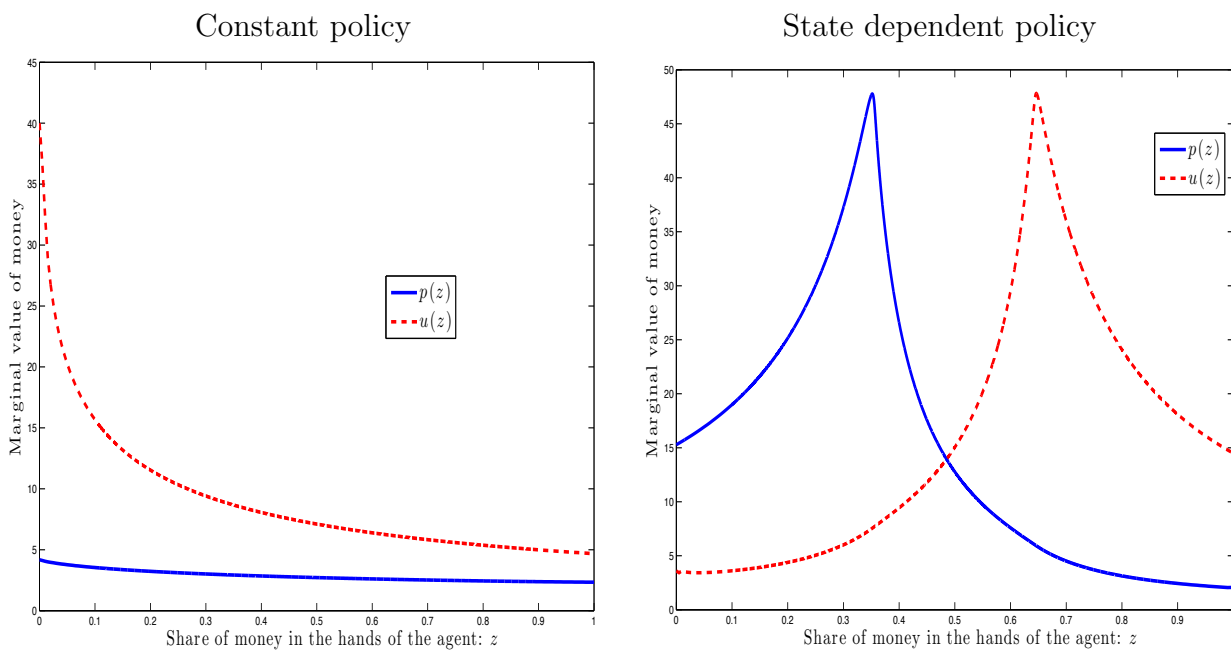
⁹ This result arises from the requirement of continuity and uniform Lipschitz condition in the space $(0, 1)$ for the forcing terms of equations (16)-(17).

Figure 1: The marginal value of money

Phase diagrams



The shadow value of money: Lagrange multipliers



When μ is constant, so that $\mu(x) = \mu \geq 0$ for every value of the aggregate state x , our results extend those provided in [Scheinkman and Weiss \(1986\)](#), which only considered the case where $\mu = 0$. The phase diagram determines three regions where the functions $u(z)$ and $p(z)$ change behavior according to their derivative sign. It is evident that the only positive solutions of the problem must stay in the third region, where both functions are decreasing on the entire $(0,1)$ interval. More precisely, their path develops in the area that is dotted in the figure. The next proposition characterizes the marginal value of money when μ is constant.

Proposition 3 *Under a constant policy $\mu(x) = \mu \geq 0$ we have that for any $\bar{c} > 0$:*

$$(i) \ p(z) < u(z), \quad \text{and} \quad (ii) \ p'(z) < 0, \quad u'(z) < 0 \quad \text{for all } z \in (0,1) .$$

See [Appendix B.4](#) for a proof. The first part of the proposition establishes that when the monetary policy is constant, the value of money for an unproductive agent is higher than the value of money for a productive agent, $u(z) > p(z)$, at all levels of money holdings $z \in (0,1)$. This property, first highlighted by [Scheinkman and Weiss \(1986\)](#), seems intuitive: Because the only difference between productive and unproductive agents holding z is that the productive agent can work, an unproductive agent values more an extra unit of money; as a result, her Lagrange multiplier is higher. Second, the proposition also states that the functions $u(z)$ and $p(z)$ are decreasing in z at all levels of money holdings $z \in (0,1)$. The bottom-left panel in [Figure 1](#) displays these properties and the upper-left panel presents the phase diagram explaining their origin: Since region III is the only admissible region for a solution to satisfy the boundary condition, the curves satisfy the properties listed in [Proposition 3](#).

When the monetary policy depends on the state x , some interesting new features arise. Inspection of the upper-right panel of [Figure 1](#) shows that the equilibrium Lagrange multipliers must reach the boundary condition, which occurs when $x = 0$ and the productive agent money holdings are $z = 1$ (i.e., the bold dot in the figure), in the region that is under L_p and L_u (i.e., the dotted region). As a result, when the productive agent money holdings z are large enough, the solution path develops in the area where both $p(z)$ and $u(z)$ are decreasing functions, as was the case for the constant policy. But crucially this result holds only locally, and in general the functions $p(z)$ and $u(z)$ that solve the problem can take many shapes. The next proposition states this result.

Proposition 4 *Suppose that $\mu(x)$ is continuous and that the following assumptions hold:*

$$(A_1) \ \text{the policy } \mu(x) \ \text{satisfies } \frac{\lambda}{\rho + \lambda + \mu(0)} < \frac{\rho + \lambda + \mu(1)}{\lambda};$$

(A₂) the policy $\mu(x)$ is strictly decreasing at $x = 0$, i.e. $\mu'(x) < 0$ for all $0 \leq x < v$, where v is a suitable positive number.

Then, there is a neighborhood for the productive agent money holdings at $z = 1$, denoted by the interval I_z , where for any $\bar{c} > 0$ the multipliers satisfy

$$(i) \quad p(z) < u(z), \quad \text{and} \quad (ii) \quad p'(z) < 0, \quad u'(z) < 0 \quad \text{for all} \quad z \in I_z .$$

See [Appendix B.5](#) for a proof. The reason for the local similarity of the constant and the state-dependent problem is the following. Start by noting that when the productive agent money holdings are $z = 1$, the unproductive agent money holdings are $x = 0$; in this case, by [Proposition 1](#), $\mu(0) \geq 0$. It follows that by the continuity of $\mu(x)$, for any $z \in I_z$ it is the case that $\mu(1 - z) \geq 0$. This non-negativity constraint on the money growth rate, together with assumptions A_1 and A_2 , produces a “local phase diagram,” and implied behavior of the multipliers, that is similar to the behavior under a constant policy. As shown in the upper-right panel of [Figure 1](#), under the state dependent policy $\mu(x)$ we have that $u(z) > p(z)$ and that both curves are decreasing only when the unproductive agent money holdings are low (i.e., x close to zero) and the productive agent money holdings are high (i.e., z close to one). Assumptions A_1 and A_2 are important as they bound the local behavior of the dynamical system. Assumption A_1 guarantees that in the neighborhood of $z = 1$ the locus L_p lies below the locus L_u , as depicted in the upper-right panel of [Figure 1](#).¹⁰ Assumption A_2 guarantees that in the neighborhood of $z = 1$, the locus L_p is decreasing. While assumptions A_1 and A_2 are useful to analytically characterize the behavior of the economy under a state-dependent policy, they are not imposed in the numerical solution of the optimal policy that is developed below. It is interesting, however, that in all the cases that we analyzed they were always satisfied at an optimum.

Notice that when $z \notin I_z$, money growth $\mu(1 - z)$ can be negative so that $u(z)$ can be below $p(z)$ and one (or both) of them can be increasing. To understand this result it is useful to contrast the state dependent case with the constant case. In the constant case the only difference between productive and unproductive agents is the production opportunity since policy is constant; as shown in [Proposition 3](#) this immediately implies that $u(z) > p(z)$ for all z . But when the monetary rule varies with the state x , a comparative static across productivity states, i.e. comparison of $u(z)$ vs. $p(z)$, also involves a different path for the monetary rule. To make this point clear we make explicit the dependence of the marginal value of money on the money rule μ . That is, let $\{p(z; \mu(x)), u(z; \mu(x))\}$ denote the marginal

¹⁰To see this notice that this assumption can be extended to the inequality $\frac{\lambda}{\rho + \lambda + \mu(1 - z)} \leq \frac{\rho + \lambda + \mu(z)}{\lambda}$ so that L_p lies below L_u . A sufficient condition for satisfying this assumption is $\mu(1) \geq -\rho$.

value of money for an agent with money holdings z and where the money rule is μ . Consider a productive agent holding money z . Under a constant policy her current value is $p(z; \mu)$ and, if the state switches, her value will be $u(z; \mu)$. Under the state dependent rule her current value is $p(z; \mu(1 - z))$ and, if there is a state switch, her value would become $u(z; \mu(z))$. This shows that when the policy is state dependent the value of money across agent types differs not only because of differences in production opportunities, but also because the money rule depends on the wealth distribution. It follows that the restriction $p(z) < u(z)$ needs not to hold over the whole state space under a state-dependent rule. This feature, as we discuss next, allows for a substantial welfare improvement.

To understand why relaxing the properties (i) and (ii) of [Proposition 3](#) is important for our problem, notice that the complete markets allocation features a constant consumption for all agents, at \bar{c} (see [Remark 1](#)). If a money rule existed to implement a first best allocation, it can be seen using [equation \(12\)](#) that the rule would imply

$$u(z) = p(1 - z) \quad \text{for all } z \in [0, 1] \quad (18)$$

or, in words, that the functions $u(z), p(z)$ are symmetric around $z = 1/2$. Obviously this feature cannot be achieved by a constant rule, since in that case the functions $u(z), p(z)$ are decreasing. A state-dependent policy has a chance of getting closer to this symmetric benchmark, as shown by the lower-right panel of [Figure 1](#). However, the next proposition shows that the complete markets allocation cannot be sustained in general (see [Appendix B.6](#) for the proof).

Proposition 5 *Let $\rho/\lambda > \bar{\rho} > 0$. There is no monetary rule $\mu(x)$ that supports the complete markets allocation as a monetary equilibrium.*

5 The optimal supply of liquidity

In this section we define a welfare criterion and explore the optimal supply of liquidity. Let $v(x)$ denote the discounted present value of the sum of utilities of both types of agents, where agents are given the same Pareto weight. This is a function of the money share of unproductive agents, x . The continuous time Bellman equation is

$$\rho v(x) = \bar{c} \left[\ln c^u(x) + \ln \bar{c} - 1 - \frac{c^u(x)}{\bar{c}} \right] + v_x(x) h^u(x) + \lambda (v(1 - x) - v(x)). \quad (19)$$

The flow value $\rho v(x)$ is given by the sum of the period utility for both agents plus the expected change in the value function. The latter occurs because of the evolution of assets

(the change in x) as well as of the possibility that the identity of the productive agent will change. Notice that in this case the state, i.e., the wealth of the unproductive agent, switches from x to $1 - x$.

We consider the problem from an ex-ante perspective, i.e., assuming that at the beginning of time nature assigns the initial productive states and the planner can choose the initial wealth distribution and a policy rule for money growth. We assume that the planner can commit to the optimal policy. Note that because individual types are not observable, and given the symmetry of the environment (and identical Pareto weights), the planner will give the same amount of liquidity to every agent and therefore at the beginning of time $x = \frac{1}{2}$. As a result, the planner chooses the function μ in order to maximize $v(1/2)$.

To evaluate the policy it is useful to define a certainty equivalent compensating variation. Let α denote the consumption equivalent cost of market incompleteness associated with a given policy. That is, α solves the following equation

$$2\bar{c} \ln(\bar{c}(1 - \alpha)) - 2\bar{c} = \rho v(1/2) \quad (20)$$

so that α measures the fraction of the consumption under complete markets that agents would be willing to forgo to eliminate the volatility of consumption due to market incompleteness for a given policy rule μ .¹¹

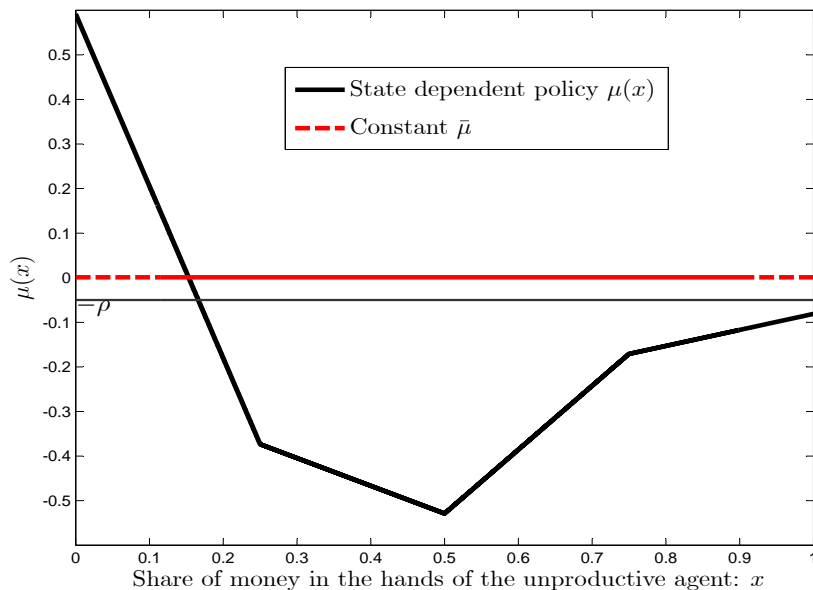
Having established that the complete markets allocation cannot be achieved (**Proposition 5**), we look for the optimal policy $\hat{\mu}(x)$ by searching numerically for the policy that maximizes the ex-ante expected welfare $v(1/2)$. Since the system of differential equations that characterizes the equilibrium allocations cannot be solved in closed form, the numerical approach represents the only means to develop a quantitative analysis. We restrict our attention to monetary policies that are piecewise linear continuous. For simplicity the results discussed in this section are obtained by considering a policy function with five nodes.¹² It is also worth mentioning that neither assumption A_1 or A_2 are imposed in the estimation routine. To aid in the understanding of the policy we also compute an alternative policy, $\bar{\mu}$, which maximizes $v(1/2)$ under the restriction that the policy has to be constant. In all computations reported below, we set the scaling factor $\bar{c} = 1$. **Figure 2** plots both policies obtained for the baseline parametrization of the model where the normalized discount rate ρ/λ equals $1/2$. Given this parameter choice, setting the discount rate to the standard value of 0.05 implies that λ equals $1/10$, so that the average length of a productive state is 10 years.

¹¹Recall that under complete markets $c^u(z) = c^p(z) = \bar{c}$ for all z , and $l^p(z) = 2\bar{c}$ for all z .

¹² Alternatives with more nodes have been explored with little differences in the results. For instance, under our benchmark calibration with $\rho/\lambda = 1/2$, the optimal policy $\hat{\mu}(x)$ shares the same qualitative features under both five and nine nodes. Moreover, the welfare gains of allowing for nine nodes are small.

Qualitatively similar results are obtained for other parameterizations for ρ/λ (see below).

Figure 2: Optimal policy ($\rho/\lambda = 1/2$)



The constant policy $\bar{\mu}$ consists of an expansion of the monetary base of 0.1 percent; $\bar{\mu}$ is positive because the insurance motives outweigh the inflation costs. As shown in Figure 2, the optimal policy $\hat{\mu}(x)$ is very different: It expands the money supply when x is low and it contracts the money supply when x is high; this happens because the optimal policy, through its state-dependent nature, is able to decouple the insurance motives and inflation costs. The welfare cost of market incompleteness α , as defined in equation (20), under the constant policy is 31.7 percent while under the optimal policy is 3.3 percent, which shows that the optimal policy increases welfare substantially.

We now describe the features of the optimal policy and interpret them in terms of the trade-off between insurance motives and the cost of inflation. When the insurance motives are very strong, which happens when $x \approx 0$, the policy prescribes the largest expansion of the money supply. As the insurance motives decrease, which happens as we take x away from zero, the optimal money growth rate falls. Eventually, at $x \approx 0.2$, the cost of inflation (through the production incentives) outweighs the insurance role of money and, as a result, the policy turns to prescribing monetary contractions. When x lies in an intermediate region the policy prescribes the largest monetary contractions, much below the Friedman rule ($-\rho$). These extreme contractions are a reflection of the large expansions that occur when x is low: Because expansions damage the return of money, large (future) monetary contractions undo the detrimental effects of the expansions on production incentives (as agents are forward

looking). Intuitively, the expansions and contractions compensate each other, striking a balance between the insurance provision versus the production incentives.

As x increases toward 1 the level of the monetary contractions decreases, reaching a minimum as x approaches 1. This is a reflection of the provision of insurance that might be needed if a switch of the state occurred: When x is high, the unproductive agent is rich, and her insurance needs are small. But the productive agent is poor and, if a state switch occurs, she will be unproductive and with a high insurance need. Because of the forward looking feature of monetary policy, the optimal policy incorporates this insurance need, and even when x is high, the degree of monetary contractions decreases; this explains the v-shape form of the policy. Nevertheless, because a productive agent can acquire money by working, her insurance motives when poor are lower than those of a poor unproductive agent. This explains why the policy is asymmetric, with larger monetary expansions when the unproductive agent is poor.

Figure 3: Consumption and the return on money under optimal policy rule ($\rho/\lambda = 1/2$)

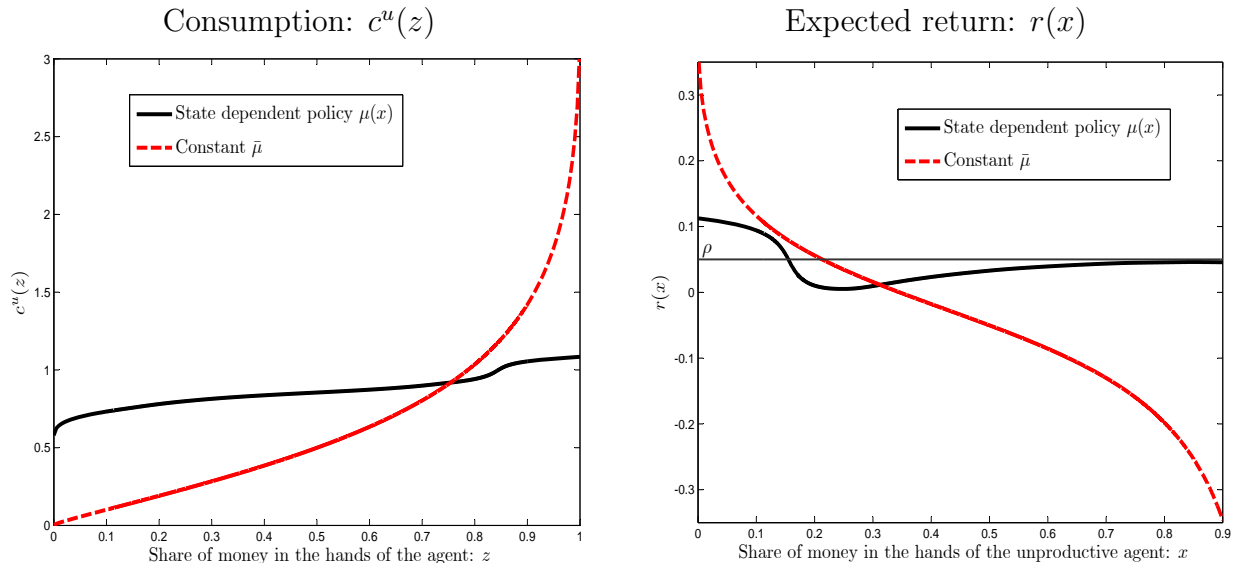


Figure 3 illustrates the profiles of the consumption function and the expected return on money under the optimal policy $\hat{\mu}(x)$ and constant policy $\bar{\mu}$. The left panel of the figure shows that the consumption of the unproductive type under the optimal policy is (i) smoother than under the constant policy, and (ii) closer to the consumption implied by the complete markets allocation (i.e. $c^u(z) = \bar{c} = 1$). The smoother consumption profiles of the state-dependent rule also yields a flatter profile for the expected return on money $r(x)$, as shown in the right panel of the figure. This flatter profile reflects the fact that the consumption of an

unproductive agent is less extreme under $\hat{\mu}(x)$: The smaller expected changes in consumption (hence marginal utilities) associated to a state switch dampen the risk premia and lead to a smoother expected return of the asset. Moreover, because the consumption under the optimal policy is closer to the consumption under complete markets, the expected rate of return is close to the rate of discount ρ , which is the expected rate of return with a complete markets allocation, as shown by [equation \(14\)](#).

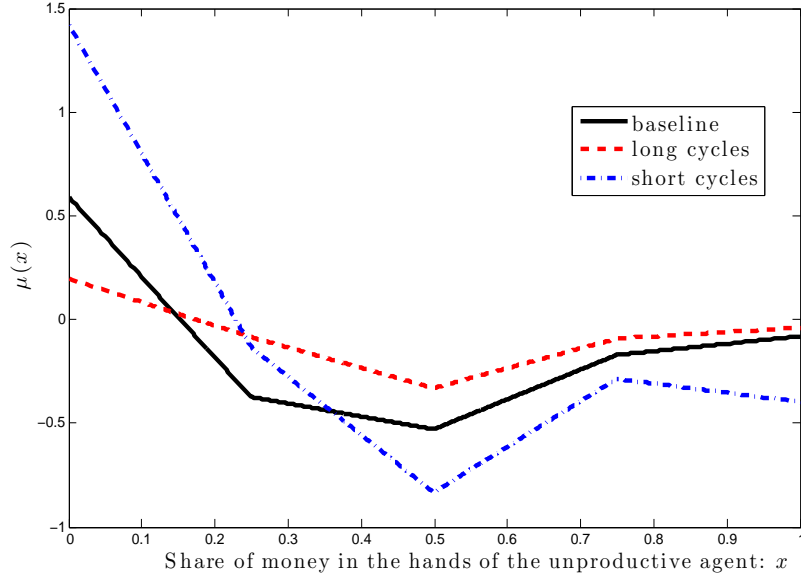
Notice that, because the consumption of the unproductive agent $c^u(x)$ is increasing in x , aggregate production, $c^p(1-x) + c^u(x) = 1 + c^u(x)$, increases with x . Inspection of the right panel of [Figure 3](#) shows that the real return on money is high when x is low. To understand this remember from the discussion of [equation \(14\)](#) that the return on money is proportional to the consumption growth of the productive agent. At $x = 0$ the productive agent is consuming $c = 1$, and expects, in case of a state switch, to consume $c^u(1) > 1$, which explains the high expected return at $x = 0$. An analogous logic explains the low returns when production is high (when x is high). Recall from [equation \(15\)](#) that the expected inflation rate $\pi(x)$ is inversely related to $r(x)$. This implies a positive correlation between the aggregate production and inflation, i.e., the model features a “Phillips curve.”

Recall, as discussed at the end of [Section 4](#), that replicating the first best allocation would require that $u(z) = p(1-z)$. This means that, for any wealth level z , the marginal value of money would have to be equal for both agent’s types; in other words, the marginal value of money has to be symmetric around $z = 1/2$. This symmetry requirement cannot be handled by a constant policy and, as showed in [Proposition 5](#), no state-dependent rule can perfectly achieve this outcome. However when μ is allowed to depend on x , the functions $u(z)$ and $p(z)$ are able, at least partially, to produce the required symmetry. This can be seen in the lower-right panel of [Figure 1](#) where we plot $u(z)$ and $p(z)$ under the optimal money rule $\hat{\mu}(x)$. Moreover, the optimal policy $\hat{\mu}(x)$ satisfies assumptions A_1 and A_2 so that $u(z)$ and $p(z)$ satisfy [Proposition 4](#).

Notice that the state-dependent pattern of the policy rule can be equivalently interpreted in terms of the business cycle. As aggregate production is increasing in x , then the optimal policy $\hat{\mu}(x)$ is such that monetary expansions happen when aggregate production is low and monetary contractions occur when it is high. In other words, the policy is expansionary during recessions and contractionary during expansions.

To conclude this section we explore the effect of changing the average length of the productive cycle, i.e., $1/\lambda$, on the optimal policy. We do so in [Figure 4](#) where we present the policy under three different cases. The first case uses our baseline parameter value for the probability of a state switch, $\lambda = 1/10$, so that the average duration of a productive cycle is 10 years; the second case uses a low probability of a state switch, $\lambda = 1/20$, and

Figure 4: Optimal policy



Note on parameters: $\bar{c} = 1$. Baseline uses $\rho/\lambda = 0.5$, long cycles uses $\rho/\lambda = 1$, short cycles uses $\rho/\lambda = 0.1$.

therefore the economy in this case exhibits long productive cycles (on average, 20 years), while the third one uses a high probability of a state switch, $\lambda = 1/2$, so that the economy displays short cycles (on average, two years). As the figure shows, the qualitative features of the policy are similar for the three parameter configurations: high monetary injections when insurance motives are the highest (when x is low), and high monetary contractions when they are the lowest (when x is close to $1/2$). Even though similar in qualitative terms, they do differ in the level of monetary injections. Monetary expansions and contractions are more extreme the shortest the production cycle. Our intuition is as follows. As the production cycle shortens, the value of money increases so that the production distortion that follows a monetary expansion is milder, and therefore, when x is small, the policy can provide insurance through liquidity expansions at a higher rate. Moreover, as previously discussed, to handle production incentives, the policy prescribes monetary contractions to counterbalance the monetary expansions, which also naturally increase as the cycle becomes shorter. Finally, as previously discussed, the welfare cost α is 3.3 percent under the baseline configuration, while it increases to 11.7 percent when the cycle is long, and it falls to 0.3 percent when it is short. The improvement in welfare as the cycle shortens follows because the insurance motives decrease as the average length of an unproductive spell falls, which, in turn, decreases the degree of the trade-off between insurance motives and production

incentives. Notice that this analysis also applies if we evaluate changes in the discount rate ρ . This follows as, for a given \bar{c} , the unique parameter of the economy is ρ/λ , so that increases in the length of productive cycles are analogous to decreases in the rate of time discount.

6 Concluding remarks

In a variety of monetary models policy has persistent effects on the distribution of wealth (money or other assets). These distributional effects are often muted or kept to a minimum in theoretical models to preserve tractability. This paper began to explore the way these distributional concerns affect monetary policy in a monetary model that allows for a full dynamic analysis of the (two-way) interactions between monetary policy and the wealth distribution.

Two forces shape the design of monetary policy in this setup. As in many other monetary models, efficiency requires distortions are minimized when the return on money equals the rate of time discount (i.e., Friedman’s rule). However, with uncertainty and incomplete markets an expansionary policy can be desirable due to insurance needs. A trade-off arises as expanding the liquidity base dampens the return on the asset, therefore reducing the production incentives. The regulation of liquidity strikes a balance between these two forces. As we discussed in the paper, insurance motives and production incentives depend on the wealth distribution that evolves through the business cycle. The novelty of our paper is that we acknowledge this dependence and we explore how the state-dependent policy balances the costs of anticipated inflation with the needs for insurance along the business cycle. This policy allows for a dramatic improvement in welfare compared with a policy that does not respond to the state. The optimal policy expands the supply of liquidity when the unproductive agents are poor (when the insurance needs are large), and it contracts the liquidity base otherwise to maximize production incentives. The principle underlying this prescription is due to the state-dependent redistributive role of monetary policy, and it differs from the one arising in sticky-price models. Because aggregate production is low when the unproductive group is poor and high when they are rich, the best policy can be interpreted as counter-cyclical. Interestingly, in spite of the policy being far away from contracting the money supply at the rate of preference, the optimal policy “echoes” Friedman’s rule as the expected real return of money approaches the rate of preference.

While the specific predictions of our model are likely to depend on specific modeling assumptions, the paper highlights the potential relevance of a transmission channel of monetary policy that seems reasonable and little explored. Several interesting extensions are left for future work. An important, and classical, assumption for our results is that the planner has

the ability to levy lump-sum taxes, and that agents are not allowed to renege from their obligations. However, as pointed out by [Andolfatto \(2013\)](#), if agents are allowed to voluntarily be subject to taxation, the degree of monetary contractions has to be limited by the voluntary nature of participation. Therefore, it is possible that the incentive-feasible allocation cannot support the optimal policy we constructed in this paper, as this one requires large monetary contractions. An interesting open question is to use the mechanism design toolkit to find the optimal state-dependent policy implementable under voluntary participation. Another interesting extension is to reformulate our model, following [Atkeson and Lucas \(1992\)](#), considering an endowment economy where agents are subject to preference shocks. If only two types of agents are considered, as we did in this paper, the two setups share several similarities and the solution developed for this paper can also be used in this other context. This would be useful to check the robustness of the results. Finally, an extension that seems worth exploring is to extend the model to allow for a larger set of agent types. Although this will come at the cost of losing tractability, as the state space grows with the number of agent types, this model, and the optimal policy, could probably be solved numerically. We conjecture that the main insights of our paper would remain: monetary expansions when unproductive agents are poor counterbalanced by monetary contractions in states where insurance motives are small.

References

- Algan, Y., Challe, E. and Ragot, X. (2011). Incomplete markets and the output-inflation tradeoff, *Economic Theory* **46**(1): 55–84.
- Alvarez, F. and Lippi, F. (2013). Persistent liquidity effect and long run money demand, *American Economic Journal: Macroeconomics* **forthcoming**.
- Andolfatto, D. (2013). Incentive-feasible deflation, *Journal of Monetary Economics* **60**(4): 383–390.
- Ascher, U., Mattheij, R. and Russell, R. (1988). *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Prentice Hall Series in Computational Mathematics.
- Atkeson, A. and Lucas, Robert E, J. (1992). On efficient distribution with private information, *Review of Economic Studies* **59**(3): 427–53.

- Berentsen, A., Camera, G. and Waller, C. (2005). The distribution of money balances and the nonneutrality of money, *International Economic Review* **46**(2): 465–487.
- Bewley, T. (1980). The optimum quantity of money, in J. Kareken and N. Wallace (eds), *Models of monetary economies*, Federal Reserve Bank of Minneapolis, pp. 169–210.
- Bewley, T. (1983). A difficulty with the optimum quantity of money, *Econometrica* **51**(5): 1485–504.
- Bhattacharya, J., Haslag, J. H. and Martin, A. (2005). Heterogeneity, redistribution, and the friedman rule, *International Economic Review* **46**(2): 437–454.
- Deviatov, A. and Wallace, N. (2012). Optimal inflation in a model of inside money, *mimeo*, Dept. of Economics, Penn State.
- Golosov, M. and Sargent, T. J. (2013). Taxation, debt redistribution, *Mimeo*, NYU.
- Green, E. and Zhou, R. (2005). Money as a mechanism in a bewley economy, *International Economic Review* **46**(2): 351–371.
- Grossman, S. and Weiss, L. (1983). A transactions-based model of the monetary transmission mechanism, *American Economic Review* **73**(5): 871–80.
- Guerrieri, V. and Lorenzoni, G. (2009). Liquidity and trading dynamics, *Econometrica* **77**(6): 1751–1790.
- Guerrieri, V. and Lorenzoni, G. (2011). Credit crises, precautionary savings, and the liquidity trap, *NBER Working Papers 17583*, National Bureau of Economic Research, Inc.
- Kehoe, T. J., Levine, D. K. and Woodford, M. (1990). The optimum quantity of money revisited, *Working Papers 404*, Federal Reserve Bank of Minneapolis.
- Lagos, R. and Wright, R. (2005). A unified framework for monetary theory and policy analysis, *Journal of Political Economy* **113**(3): 463–484.
- Levine, D. K. (1991). Asset trading mechanisms and expansionary policy, *Journal of Economic Theory* **54**(1): 148–164.
- Lucas, R. J. (1990). Liquidity and interest rates, *Journal of Economic Theory* **50**(2): 237–264.
- Molico, M. (2006). The distribution of money and prices in search equilibrium, *International Economic Review* **47**(3): 701–722.

- Scheinkman, J. A. and Weiss, L. (1986). Borrowing constraints and aggregate economic activity, *Econometrica* **54**(1): 23–45.
- Shi, S. (1997). A divisible search model of fiat money, *Econometrica* **65**(1): 75–102.
- Shi, S. (1999). Money, capital, and redistributive effects of monetary policies, *Journal of Economic Dynamics and Control* **23**(4): 565–590.
- Tenreyro, S. and Sterk, V. (2013). Durable goods and the transmission mechanism of monetary policy: an alternative framework, *Mimeo*, London School of Economics.
- Townsend, R. M. (1980). Models of money with spatially separated agents, in J. Kareken and N. Wallace (eds), *Models of monetary economies*, Federal Reserve Bank of Minneapolis, pp. 265–304.
- Wallace, N. (1984). Some of the choices for monetary policy, *Quarterly Review* **8**(1).
- Wallace, N. (2013). The optimality of inflation in ‘pure currency’ economies: a conjecture, *Quarterly Journal of Economics* **forthcoming**.

Appendix

A The return on money and inflation

Define the stochastic expected net return on money for a small time interval Δ as

$$r(x)\Delta = \mathbb{E} \left[\frac{\tilde{q}_{t+\Delta}}{\tilde{q}_t} - 1 \middle| x_t = x \right] = \mathbb{E} \left[\frac{q_{t+\Delta}}{q_t} - \Delta\mu_t - 1 \middle| x_t = x \right], \quad (21)$$

where $\tilde{q}_t = q(x_t)/M_t$. Without loss of generality consider the case where at time t agent 1 is productive with money holdings given by z_t^1 , and let z_t^2 denote the money holdings of the unproductive agent at time t . Then, using that $q(x) = p(z)$, we have

$$r(x_t)\Delta = \frac{(1 - \lambda\Delta)p(z_{t+\Delta}^1) + \lambda\Delta p(z_{t+\Delta}^2)}{p(z_t^1)} - 1 - \Delta\mu_t,$$

where we used [equation \(5\)](#). A first order Taylor expansion of $p(z_{t+\Delta}^i)$ gives

$$r(x_t)\Delta = \frac{p'(z_t^1)h^p(z_t^1)\Delta}{p(z_t^1)} + \lambda\Delta \left(\frac{p(z_t^2) + p'(z_t^2)h^p(z_t^2)\Delta}{p(z_t^1)} - \frac{p(z_t^1) + p'(z_t^1)h^p(z_t^1)\Delta}{p(z_t^1)} \right) - \Delta\mu_t. \quad (22)$$

Then, taking the limit as $\Delta \rightarrow 0$ gives that $r(x_t) = \frac{p'(z_t^1)h^p(z_t^1)}{p(z_t^1)} + \lambda \left(\frac{p(z_t^2)}{p(z_t^1)} - 1 \right) - \mu_t$.

We use (8) to get $r(x_t) = \rho + \lambda \left(\frac{p(z_t^2)}{p(z_t^1)} - \frac{u(z_t^1)}{p(z_t^1)} \right)$ or, using that in this proof we assumed that agent 2 was unproductive at time t , i.e. $x_t = z_t^2$,

$$r(x_t) = \rho + \lambda \left(\frac{p(x)}{p(1-x)} - \frac{u(1-x)}{p(1-x)} \right),$$

which yields the expression in the text. Define expected inflation as $\pi(x_t) = \mathbb{E} \left[\frac{1/\tilde{q}_{t+\Delta}}{1/\tilde{q}_t} - 1 \mid x_t = x \right]$ it is easily seen that $\pi(x_t) = -\frac{r(x_t)}{1+r(x_t)}$.

B Proofs

B.1 Proof of Remark 2

Let $i = 1$ be the index for the unproductive agent, and consider her decision problem. The money supply growth is $\mu = \frac{\dot{M}_t}{M_t}$ and let \tilde{u}_t denote the Lagrange multiplier of the money flow constraint in equation (3). The first order condition with respect to c_t gives: $u_t = q_t/c_t$, where we used the homogeneity of degree -1 in the aggregate money supply M_t for both \tilde{u}_t and \tilde{q}_t . The Euler equation for \dot{m}_t^1 gives $\rho = \frac{\dot{u}_t}{u_t} - \frac{\dot{M}_t}{M_t} = \frac{\dot{q}_t}{q_t} - \frac{\dot{c}_t}{c_t} - \mu$ where the last equality uses $u_t = q_t/c_t$. Notice that this is solved by $c_t = \bar{c}$, $\dot{c}_t = \dot{q}_t = 0$, and $\mu = -\rho$.

The constant level of q is pinned down by imposing that total nominal assets at the beginning of a cycle, which in the stationary equilibrium are held by the unproductive agent, are equal the total nominal consumption and tax expenditures over the cycle of length T . Without loss of generality let's consider the first cycle, starting at time 0, where the unproductive agent holds all the money supply; $m_0 = M_0$. Using that $M_t = M_0 e^{\mu t}$, $\mu = -\rho$, and $\tilde{q}_t = \frac{1}{M_t} q$, we have

$$\int_0^T \frac{\bar{c}}{\tilde{q}_t} dt + \int_0^T \rho M_t dt = M_0, \quad \text{which gives} \quad q = \bar{c} \frac{1 - e^{-\rho T}}{\rho e^{-\rho T}} \cong \bar{c} \frac{T}{1 - \rho T}.$$

where the approximation is accurate for small T . It is immediate to verify that this allocation also solves the Euler equation of the productive agent and that her money holdings are never negative.

B.2 Proof of Proposition 1

We first present a proof of the first part of the statement. That is, that there is no monetary equilibrium where $\mu(x) < 0 \forall x$. We then present a proof of the second part of Proposition 1.

A contractionary policy $\mu(x) < 0 \forall x$ requires agents to pay lump sum taxes ($\tau(x) < 0 \forall x$). Consider the case where agent 1 has fraction of money balances z_t , and the current state of the economy is $\mathcal{I}_t = 0$, which means that agent 1 is unproductive. If z_t is low enough, given that $\lambda > 0$ and finite, the agent will fail to comply with the monetary authorities with non-zero probability. On the other hand, consider the case where $z_t = 1$. In this case the agent is able to comply with her tax obligations with certainty, as she can make her consumption profile to be arbitrarily low. This implies that there exists a threshold $\zeta \in (0, 1)$ such that for $z^i \geq \zeta$ the agent is able to cover her lifetime tax needs with probability one. Note that the threshold must be independent of the current state \mathcal{I}_t as with positive probability the states are reversed. In the next claim we characterize this threshold.

Claim 1 *If $\mu_t < 0 \forall t$, for any state of the world \mathcal{I}_t , there is a unique threshold: $\zeta = 1/2$, and a unique ergodic set where $z_t = \frac{1}{2} \forall t$, that ensures tax solvency.*

Proof. We will first prove by contradiction that $\zeta \notin [0, 1/2)$. Then we will show that $\zeta = 1/2$ is enough to cover the lifetime tax obligations. Suppose that $\zeta < 1/2$. Without loss of generality assume that $z_t \in (\zeta, 1/2)$ and agent 1 is unproductive. Conditional on no reversal of the state, it follows that $z_{t+\Delta} < z_t$ for any $\Delta > 0$. Then for a given $\Delta \in \mathbb{R}^+$, $\Pr[z_{t+\Delta} < \zeta] > 0$ and therefore the agent will fail to comply with her tax obligations with positive probability. Then, $\zeta \notin [0, 1/2)$. Consider now the case where $z_t = \zeta = 1/2$. As the agent can decide not to trade she can always keep her share of outstanding money balances z above $1/2$ and therefore for any $\mu \in (0, 1)$ she will be able to cover her tax needs. That $z = 1/2$ is the ergodic set is trivial. If $z < 1/2$ there is a positive probability that an agent fails to pay for her lifetime taxes. An unproductive agent with money holdings $z > 1/2$ is willing to buy goods (and the productive one with $z < 1/2$ willing to take the money) until z reaches $1/2$. ■

Intuitively, given the uncertain duration of the productivity spell, the only value of money holdings that ensures compliance with tax obligations for both types of agents is $z = 1/2$. At this point, for any history of shocks, the identical lump-sum (negative) transfers reduce the money holdings of both agents proportionally, leaving the wealth distribution unaffected. This leads us to

Remark 3 *Let $\mu(x) < 0 \forall x$: In the ergodic set there is no stationary monetary equilibrium and consumption allocations are autarkic.*

The proof of **Remark 3** follows from noting that **Claim 1** implies no trade in the ergodic set. Productive agents have an unsatisfied demand for money and unproductive ones have an unsatisfied demand for consumption goods.

Now we turn to prove that $\mu(0) \geq 0$. Consider the law of motion for the share of money held by type 1 when unproductive, $h^u(z) = \mu(x) \left(\frac{1}{2} - z\right) - \frac{c^u(z)}{q(x)}$. Because $c^u(z) \geq 0$ and $q(x) > 0$ (i.e. prices are positive in a monetary equilibrium) we can bound above the law of motion $h^u(z)$: $h^u(z) \leq \mu(x) \left(\frac{1}{2} - z\right)$. Notice that because the agent is unproductive $z = x$. We apply this result into the previous equation and evaluate it at $x = 0$ to get $h^u(0) \leq \frac{\mu(0)}{2}$, from where it can be seen that $h^u(0) < 0$ if $\mu(0) < 0$ which is inconsistent with the borrowing and tax-solvency constraints. Therefore, $\mu(0) \geq 0$.

B.3 Proof of Proposition 2

The boundary condition (10) yields $\lim_{z \rightarrow 0} h^u(z) = 0$. Moreover, we have $h^u(1/2) = -\frac{\bar{c}}{u(1/2)} < 0$. Under the continuity assumption on $\mu(x)$, $h^u(z)$ is a continuous function; then the inequality can be extended over a suitable neighborhood of the asset $z = \frac{1}{2}$. More precisely, it is possible to consider an interval $(\frac{1}{2} - \delta, \frac{1}{2} + \delta) \subset (0, 1)$ where $h^u(z) < 0$. The further assumption related to the lack of singularities yields that the function $h^u(z)$ cannot nullify within the domain and its sign has to be uniform. It follows that the interval $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ overlaps the whole integration interval and the previous inequality holds for all $z \in (0, 1)$. In this way, the result is completely obtained.

The assumptions in the proposition mean that the solution $(u(z), p(z))$ evolves for $z \in (0, 1)$ in a region of \mathbb{R}_+^2 where the forcing term has no singularity and is continuous, with the classical uniform Lipschitz condition locally satisfied. We recall that this kind of smoothness is a basic assumption which is needed for investigating the solution of the differential problem. As discussed in Ascher et al. (1988), smoothness together with the requirement of solvability of the algebraic system related to the boundary conditions are the crucial issues for establishing the existence of a solution.

B.4 Proof of Proposition 3

The proof follows from the inspection of the phase diagram, which is plotted in upper-left Figure 1. We assume the policy $\mu(x) = \mu \geq 0$ is constant and exploit the geometric loci in $(\mathbb{R}^+)^2$ defined as

$$L_p = \left\{ (U, P) \in (\mathbb{R}^+)^2 \mid P = \frac{\lambda}{\rho + \lambda + \mu} U \right\}, \quad \text{and} \quad L_u = \left\{ (U, P) \in (\mathbb{R}^+)^2 \mid P = \frac{\rho + \lambda + \mu}{\lambda} U \right\}. \quad (23)$$

We first note that $\frac{\lambda}{\rho + \lambda + \mu} < 1 < \frac{\rho + \lambda + \mu}{\lambda}$; this implies that the locus L_p lies under L_u and the dotted line $P = U$ is between them, as it is shown in the figure. Then, according to equations (16)-(17), the terms $p'(z)h^p(z)$ and $u'(z)h^u(z)$ are zero on L_p and L_u , respectively.

Thus, the uniform signs of $h^u(z) < 0$ and $h^p(z) > 0$ allow us to gain an insight about the increasing/decreasing pattern of the Lagrange multipliers $p(z)$ and $u(z)$, as it is shown by the arrows in the figure. It can be seen that region III is the only admissible area for a solution to satisfy the boundary condition $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda}p(1)$, which is on L_p . Indeed the only positive solutions must stay in that region and their path develops in the dotted area. Therefore, both $p(z)$ and $u(z)$ are decreasing functions: it follows that $p'(z) < 0$ and $u'(z)$ everywhere. Moreover, in the same region, we have $p(z) \leq \frac{\lambda}{\rho+\lambda+\mu}u(z) < u(z)$ for all z .

B.5 Proof of Proposition 4

The results in Proposition 4 can be stated by investigating the upper-right plot in Figure 1, where some qualitative features of the solution are represented by exploiting the paths describing two specific different geometric loci in $(\mathbb{R}^+)^2$. We define the following sets

$$L_p = \left\{ (U, P) \in (\mathbb{R}^+)^2 \mid \exists z \in [0, 1] \text{ so that } U = u(z) \text{ solves equations (16)-(17) at the state } z \text{ and } P = \frac{\lambda}{\rho+\lambda+\mu(1-z)}U \right\},$$

and

$$L_u = \left\{ (U, P) \in (\mathbb{R}^+)^2 \mid \exists z \in [0, 1] \text{ so that } U = u(z) \text{ solves equations (16)-(17) at the state } z \text{ and } P = \frac{\rho+\lambda+\mu(z)}{\lambda}U \right\}.$$

They can be considered as a generalization of the geometric loci already defined by (23) in Appendix B.4, where the monetary policy is assumed to be state independent. As a difference with respect to (23), here we consider an explicit dependence on the state z for both U and P in the plane $(\mathbb{R}^+)^2$: the sets are described by the points $\left(u(z), \Gamma_p(z) := \frac{\lambda}{\rho+\lambda+\mu(1-z)}u(z)\right) \in L_p$ and $\left(u(z), \Gamma_u(z) := \frac{\lambda}{\rho+\lambda+\mu(z)}u(z)\right) \in L_u$, as z varies in the domain $[0, 1]$. Again the terms $p'(z)h^p(z)$ and $u'(z)h^u(z)$ in equations (16)-(17) are zero on L_p and L_u , respectively. Then, according to the sign of terms $h^p(z)$ and $h^u(z)$, both L_p and L_u determine different regions in the plane (U, P) where the functions $u(z)$, $p(z)$ change behavior according to the sign of their derivative.

The following features are crucial to draw the loci L_p, L_u , and to characterize the solution of the differential problem assuming A_1 and A_2

(P₁) The loci may intersect in an even number of points, since condition $\Gamma_p(z) = \Gamma_u(z)$ is equivalent to have $\rho^2 + 2\rho\lambda + (\rho + \lambda)(\mu(z) + \mu(1 - z)) + \mu(z)\mu(1 - z) = 0$, which is a symmetric formula with respect to $z = 1/2$.

(P₂) Condition $\frac{\lambda}{\rho+\lambda+\mu(0)} < 1 \leq \frac{\rho+\lambda+\mu(1)}{\lambda}$ in assumption A_1 can be extended in order to have

the inequality $\frac{\lambda}{\rho+\lambda+\mu(1-z)} \leq \frac{\rho+\lambda+\mu(z)}{\lambda}$, satisfied in a suitable neighborhood of $z = 1$. It follows that, in the same neighborhood, the locus L_p is under L_u .

- (P_3) The boundary condition $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda}p(1)$ lies on L_p (denoted by a bold dot in the figures). It has to be reached by the solution in correspondence with z converging to 1.
- (P_4) L_p and L_u determine different regions in the phase plane (U, P) where both functions $u(z)$, $p(z)$ change behavior according to their derivative sign. Since L_p lies under L_u , as shown in the upper-right panel of [Figure 1](#), the local behavior of $u(z)$ and $p(z)$ in a neighborhood of $z = 1$ can be determined as follows. Let the arrows describe the increasing/decreasing patterns for $u(z)$ and $p(z)$, accounting for the signs of $h^u(z) < 0$ and $h^p(z) > 0$ established by [Proposition 2](#). It is evident that $u(z)$ is decreasing when the money holdings of the productive agent is near $z = 1$, since the solution must reach the boundary condition (the bold dot on L_p) according to P_3 .
- (P_5) The increasing/decreasing shape for L_p can be established by noticing that, for any $z \in [0, 1]$, $\Gamma'_p(z) = \frac{\lambda}{\rho+\lambda+\mu(1-z)} \left(\frac{u(z)\mu'(1-z)}{u'(z)(\rho+\lambda+\mu(1-z))} + 1 \right) u'(z)$. We remark that $\mu(1-z) \geq 0$ and $\mu'(1-z) < 0$ near $z = 1$. The first condition is due to the continuity for $\mu(x)$ and $\mu(0) \geq 0$; the second relationship arises from assumption A_2 . In this respect, since $u(z)$ is decreasing when z moves near 1 (see P_4), then $u'(z) < 0$. It follows that $\Gamma'_p(z) < 0$ and the locus L_p decreases in a neighborhood of $z = 1$.

The previous features can be exploited in order to draw the picture in [Figure 1](#), whose inspection lets the proof of [Proposition 4](#) be completed. It is evident that the only solutions that are positive must reach that boundary condition in the dotted region, that is under L_p and L_u (the dashed curve represents a possible path for the solution). Hence there exists a threshold $\bar{z} \in [0, 1]$ such that for all $z \in I_z = [\bar{z}, 1]$ the solution develops in the region where both $p(z)$ and $u(z)$ are decreasing functions, that is equivalent to have $p'(z) < 0$ and $u'(z) < 0$. In the same region we have $p(z) < u(z)$. Indeed, since $\mu(0) \geq 0$ and $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda}p(1)$, the point $(u(1), p(1)) \in L_p$ lies under the 45 degree line and $p(1) < u(1)$. Since the function μ is continuous then the inequality can be extended in I_z . This completes the proof.

B.6 Proof of [Proposition 5](#)

We prove that the complete markets allocation cannot be sustained when $\frac{\rho}{\lambda} > \bar{\rho}$ (i.e. when the expected duration of the cycle is sufficiently long). To this end it suffices to show that consumption is not equal to \bar{c} at all values of the state $x \in [0, 1]$. In particular let us consider $x = 0$. The consumption [equation \(13\)](#) and the continuity of the consumption function give that that $c^u(0) = \mu(0)p(1)/2$. Recall that $\mu(0)$ is finite (since $\mu(x)$ is continuous

over a compact set). The boundary condition in [equation \(11\)](#) can be rewritten as $p(1) = \frac{1}{\frac{\rho}{\lambda} + 1 + \frac{\mu(0)}{\lambda}} u(1)$. Since $u(1)$ is finite in any monetary equilibrium, it follows that $c^u(0)$ can be made arbitrarily small by increasing $\frac{\rho}{\lambda}$, i.e. that $\lim_{\rho/\lambda \rightarrow \infty} c^u(0) = 0$. Therefore there exists a threshold $\bar{\rho} > 0$ such that for all $\rho/\lambda \in (\bar{\rho}, \infty)$ we have that $c^u(0) < \bar{c}$, or that the complete markets allocation cannot be implemented.