# DISCUSSION PAPER SERIES

No. 9705

## METHODS FOR MEASURING EXPECTATIONS AND UNCERTAINTY IN MARKOV-SWITCHING MODELS

Francesco Bianchi

*INTERNATIONAL MACROECONOMICS*

**Centre for Economic Policy Research**

**www.cepr.org**

**Available online at:** **www.cepr.org/pubs/dps/DP9705.php**

*ISSN 0265-8003*

# METHODS FOR MEASURING EXPECTATIONS AND UNCERTAINTY IN MARKOV-SWITCHING MODELS

**Francesco Bianchi, Duke University and CEPR**

Discussion Paper No. 9705
October 2013

Centre for Economic Policy Research
77 Bastwick Street, London EC1V 3PZ, UK
Tel: (44 20) 7183 8801, Fax: (44 20) 7183 8820
Email: cepr@cepr.org, Website: www.cepr.org

CEPR Discussion Paper No. 9705

October 2013

# ABSTRACT

## Methods for Measuring Expectations and Uncertainty in Markov-Switching Models

I develop a toolbox to analyze the properties of multivariate Markov-switching models. I first derive analytical formulas for the evolution of first and second moments, taking into account the possibility of regime changes. The formulas are then used to characterize the evolution of expectations and uncertainty, the propagation of the shocks, the contribution of the shocks to the overall volatility, and the welfare implications of regime changes in general equilibrium models. Then, I show how the methods can be used to capture the link between uncertainty and the state of the economy. Finally, I generalize Campbell's VAR implementation of Campbell and Shiller's present value decomposition to allow for parameter instability. The applications reveal the importance of taking into account the effects of regime changes on agents' expectations, welfare, and uncertainty. All results are derived analytically, do not require numerical integration, and are therefore suitable for structural estimation.

JEL Classification: C11, C32, E31, E52 and G12
Keywords: Bayesian methods, DSGE, impulse responses, Markov-switching, uncertainty, VAR and welfare

Francesco Bianchi
Department of Economics
Duke University
213 Social Sciences Buillding
419 Chapel Drive, Box 90097
Durham, NC 27708-0097

Email: francesco.bianchi@duke.edu

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=173067

Submitted 16 October 2013

# Methods for Measuring Expectations and Uncertainty in Markov-Switching Models*

Francesco Bianchi
Duke University and CEPR

October 2013

**Abstract**

I develop a toolbox to analyze the properties of multivariate Markov-switching models. I first derive analytical formulas for the evolution of first and second moments, taking into account the possibility of regime changes. The formulas are then used to characterize the evolution of expectations and uncertainty, the propagation of the shocks, the contribution of the shocks to the overall volatility, and the welfare implications of regime changes in general equilibrium models. Then, I show how the methods can be used to capture the link between uncertainty and the state of the economy. Finally, I generalize Campbell's VAR implementation of Campbell and Shiller's present value decomposition to allow for parameter instability. The applications reveal the importance of taking into account the effects of regime changes on agents' expectations, welfare, and uncertainty. All results are derived analytically, do not require numerical integration, and are therefore suitable for structural estimation.

**Keywords:** Markov-switching, VAR, DSGE, Bayesian methods, Expectations, Uncertainty, Impulse responses, Welfare, Moments.

**JEL codes:** C32, C11, E31, E52, G12

## 1 Introduction

Since the seminal contribution of Hamilton (1989), Markov-switching models have become a popular tool to allow for parameter instability. In recent years, the univariate framework proposed by Hamilton (1989) has been extended to the multivariate case. Sims and Zha (2006) have used a Markov-switching vector autoregression (MS-VAR) to investigate the possibility of structural breaks in the conduct of monetary policy, while Sims, Waggoner, and Zha (2008) have outlined the methods for inference in this class of models. Furthermore, a growing literature has moved in the direction of modeling parameter instability in dynamic stochastic general equilibrium (DSGE) models using Markov-switching processes.

*I would like to thank Luigi Bocola, Frank Diebold, Leonardo Melosi, Juan Rubio-Ramirez, Frank Schorfheide, Allan Timmermann, and all seminar participants at the University of Pennsylvania and the NBER Summer Institute for useful comments and suggestions. Yang Yu provided excellent research assistance. Correspondence: Francesco Bianchi, francesco.bianchi@duke.edu.

While the methods to estimate multivariate Markov-switching models are by now quite well understood, regimes are often studied in isolation and the profession is still missing a framework to systematically analyze the properties of these models. This paper aims to fill this gap. I first derive a toolbox that can be used to characterize agents' expectations, model dynamics, and uncertainty in multivariate Markov-switching models. I then present a wide range of applications meant to highlight the importance of taking into account the possibility of regime changes when characterizing agents' uncertainty, the link between the macroeconomy and uncertainty, and the welfare consequences of uncertainty.

In the first part of the paper, I derive analytical laws of motion for the first and second moments of the endogenous variables. These are then combined to obtain the evolution of the covariance and auto-covariance matrices. Means and variances derived in this way take into account all sources of uncertainty, including the possibility of regime changes. I then state the conditions under which the moments converge to finite values. Specifically, borrowing from the engineering literature, I make use of the concept of mean square stability. A process is mean square stable if its first and second moments converge to finite values as the time horizon goes to infinity. It is then straightforward to derive ergodic values for the first and second moments and, consequently, for volatilities. Mean square stability seems a desirable condition to impose on a statistical process when thinking about economic applications. First, it implies that agents' expectations and uncertainty converge as the time horizon increases. Second, under the assumption of ergodicity of the Markov chain, a Markov-switching (MS) model is mean-square stable if and only if it is asymptotically covariance stationary.

I make use of these results to emphasize how MS models can be a powerful tool to characterize the evolution of agents' expectations and uncertainty. I consider a MS-DSGE model that allows for heteroskedasticity and changes in monetary policy. Once it is linearized and solved, the model returns a multivariate Markov-switching model of the kind studied by Sims and Zha (2006). As a first application, I show how to characterize the historical evolution of agents' expectations and uncertainty. At each point in time I compute the expected values and the volatilities for each of the endogenous variables at different horizons: $\mathbb{E}_t(Z_{t+s})$ and $sd_t(Z_{t+s}) = \sqrt{\mathbb{V}_t(Z_{t+s})}$. Expectations and uncertainty computed in this way reflect all sources of uncertainty faced by an agent in the model. Specifically, they take into account the possibility of regime changes, uncertainty around the state of the economy, uncertainty about the regime in place, and the possibility of Gaussian shocks. Therefore, they provide an accurate characterization of agents' expectations and uncertainty, based on the estimates for the model parameters and the regime probabilities.

The same formulas can be easily adapted to compute impulse responses, taking into account the possibility of regime changes. When working with models with parameter instability, two different sets of results might be of interest. First, it might be useful to understand how shocks propagate under a specific regime. In this case, the evolution of the variables of interest can be computed assuming that a specific regime is in place over the relevant horizon. However, in many other situations it might be important to take into account the possibility of regime changes. For example, a policy maker might be interested in the propagation of a shock, taking into account uncertainty about the underlying state of the economy. Alternatively, a practitioner could find it important to control for uncertainty about the future conduct of fiscal and monetary policies. In all of these cases, an impulse response can be obtained shocking the economy and then using the law of motion for first moments to project the shock into the future. The resulting impulse response automatically integrates over all possible regime paths.

A similar argument holds for uncertainty. When taking into account the possibility of regime changes measures of uncertainty can change substantially and surprising results can arise. For example, in the context of the MS-DSGE model described above, if a very volatile regime is in place today, uncertainty becomes hump shaped with respect to the time horizon. In other words, agents can be more uncertain about the short run than the long run. This is because two competing forces are at play. On the one hand, events that are further into the future are naturally harder to predict. On the other hand, in the long run the probability of still being in the high volatility regime declines. This latter mechanism also determines a decline in the upper bound for uncertainty with respect to the case in which the possibility of regime changes is ruled out: When agents are in a very volatile regime combination, they are aware that eventually the economy will move to more favorable outcomes.

In other contexts, the upper bound for uncertainty can also increase as a result of regime changes. This is because regime changes can be regarded as shocks themselves. An increase in volatility is more likely to occur when regime changes also affect the conditional steady states of the model, i.e., the values to which the state variables converge if a regime is in place for a prolonged period of time. The conditional steady states are not necessarily reached by the model, given that convergence can be slow when compared to the regime persistences. Nevertheless, they generally determine important swings in the model dynamics. This additional source of volatility cannot be detected if uncertainty is computed conditioning on a specific regime. Therefore, if an economist is interested in characterizing the effective level of uncertainty implied by an MS model, it is important to take into account the possibility of regime changes.

The same logic applies if the goal is to understand the sources of uncertainty. Some shocks might be very important under a specific regime, but much less under another one. If regime changes are ruled out when computing the variance decomposition, the importance of a specific shock might be dramatically overstated. This is because in a model subject to regime changes, it is not only the size and the contemporaneous impact of a shock that matter. A regime might be characterized by very large shocks, but such shocks may occur very infrequently or only for a very short period of time. Alternatively, it might be systematically followed by an offsetting regime that strongly mitigates the propagation of the shocks. In both cases, the overall contribution of the shocks associated with such a regime is going to be very small.

Correctly characterizing the level of uncertainty is extremely important when conducting welfare analysis in a general equilibrium model. This is because measures of medium- and long-run uncertainty change substantially when taking into account the possibility of regime changes. As a result, the importance of the regime that is in place at a particular point in time is substantially reduced. If welfare were computed assuming a regime in place for a prolonged period of time, the results could be completely misleading. In other words, it is not enough to account for the size and the contemporaneous impact of the shocks when evaluating the welfare implications of a regime. The results derived in this paper can be used to address these issues in a systematic way. Following Rotemberg and Woodford (1999), Woodford (2003), and Gali (2008), I use a period welfare loss function that depends on expected quadratic deviations of inflation and the output gap from their respective steady states. For each initial regime, these squared deviations need to be computed integrating over all possible regime paths. Under the assumption of mean square stability, this can be done in one step by computing the discounted present value of the expected second moments as implied by the corresponding law of motion. It is worth pointing out that this way of calculating welfare takes into account uncertainty around the regime that is in place today, the current state of the economy, and the possibility of regime changes. In the long run, the second moments converge to their ergodic steady states, while the first moments converge to zero. Therefore at long horizons, welfare is determined by the ergodic variance. This is in line with standard results in the literature about welfare calculations in new-Keynesian models.

Markov-switching models can also generate interesting dynamics between uncertainty and the endogenous variables. To make this point, I simulate a bivariate MS-VAR with no Gaussian shocks. In this context, the only source of variation is represented by swings in the constant. What emerges is a model in which a variable can experience a sharp drop anticipated by a sudden increase in uncertainty. At the same time, in an MS model uncertainty moves in response to the state of the economy. This is not the case in a model with fixed coefficients or in which the only source of parameter instability is due to heteroskedasticity or to shifts in the constant. The intuition for this result stems from the fact that in an MS-VAR model there is uncertainty for the way the lagged values are projected into the future because the autoregressive component is subject to changes. If the only source of parameter instability is represented by changes in the constant or in the volatility of the shocks, there is no uncertainty about the propagation mechanism, but only about the magnitude and the direction of the innovations. The ability of an MS model to generate non-trivial connections between the dynamics of the endogenous variables and the level of uncertainty represents a promising feature to study the link between uncertainty and real activity. This is intriguing in light of the attention that uncertainty has recently received in the profession following the seminal contribution of Bloom (2009).

The possibility of using a well-defined concept of stability should not be undervalued. When working with models with parameter instability, the problem of how to impose stability for the model-implied forecasts arises quite often. In the time-varying VAR literature, for example, practitioners have followed two alternative approaches. In the first approach, anticipated utility, forecasts are performed disregarding the possibility of parameter changes, simply using the set of parameters that are in place at a particular point in time. It is easy to see that this approach automatically disregards a very important source of uncertainty: parameter instability. In the second approach, a sequence for the parameter values is simulated into the future. Then stability is checked at each point of the sequence. If at any point, the set of parameters turns out to be unstable when taken in isolation, the entire draw is disregarded. It is clear that this approach is not completely satisfactory because it does not allow for the possibility that over a certain period of time the variables are not on a stable path. In MS models, instead, mean square stability allows for the possibility that some of the regimes are unstable when taken in isolation, as long as first and second moments remain stable. This seems a very desirable property when working with economic and financial series. For example, when dealing with macroeconomic data we might want to allow for the possibility of recurrent episodes of hyperinflation followed by a painful disinflation. Similarly, in finance we often observe bubbles and crashes. As an illustrative example, I show how under the assumption of mean square stability it is possible to extend the Campbell's (1991) VAR implementation of Campbell and Shiller's (1988) present value decomposition

This discussion should also make apparent another advantage of using Markov-switching processes to model parameter instability. All results of this paper are obtained analytically, implying that numerical integration is not required. This represents a significant advantage of Markov-switching models and opens the door to the possibility of including the moments in an estimation exercise. In a model with smoothly time-varying parameters the computational burden implied by numerical integration is substantially larger, implying that anticipated utility is in many cases the only feasible approach to model agents' expectations inside the model or to augment the list of observables in an estimation exercise.

Timmermann (2000) derives the moments for a range of univariate Markov-switching models. Three main differences between the two contributions should be pointed out. First, I derive results for multivariate Markov-switching models, while Timmermann (2000) works with univariate processes. Second, I derive formulas for the dynamic evolution of the objects of interest conditional on a particular information set, while Timmermann (2000) only presents results for the unconditional moments. Finally, the focus of the two papers is quite different, given that I illustrate a series of applications in the context of the macroeconomic and finance literature. From this point of view, this paper complements the work of Sims and Zha (2006) and Sims, Waggoner, and Zha (2008), who illustrate estimations techniques for MS-VAR; Schorfheide (2005), Liu, Waggoner, and Zha (2011), Bianchi (2013), and Davig and Doh (2013) who explain how to estimate MS-DSGE models; Fernandez-Villaverde, Guerron-Quintana, and Rubio-Ramirez (2010), Fernandez-Villaverde, Rubio-Ramirez, Guerron-Quintana, and Uribe (2011), and Justiniano and Primiceri (2008), who work with models in which parameters change smoothly over time; and Ang and Bekaert (2002) and Pesaran, Pettenuzzo, and Timmermann (2006) that use MS models in forecasting financial series.

The techniques developed here can be easily combined with the framework developed by Bianchi and Melosi (2012b) to characterize the evolution of expectations and uncertainty in general equilibrium models in which agents have to learn the stochastic properties of regimes. The central insight of Bianchi and Melosi (2012b) consists of recognizing that the evolution of agents' beliefs can be captured by defining a set of *regimes* that are characterized by the degree of agents' pessimism, optimism, and uncertainty about future equilibrium outcomes. Once this structure has been imposed, the model can be recast as a Markov-switching dynamic stochastic general equilibrium model with perfect information and the resulting equilibrium can be analyzed with the toolbox developed in this paper. For example, Bianchi and Melosi (2013a) apply the methods developed here to characterize the progressive increase in uncertainty that stems from a deterioration of policy makers' reputation for fiscal virtue.

The remainder of the paper is organized as follows. Section 2 presents the class of models considered in this paper and derives the laws of motion for first moments, second moments, and covariance and auto-covariance matrices. Section 3 introduces the notion of mean square stability, states conditions under which it holds, and derives conditional and ergodic steady states. Section 4 presents a series of applications using an MS-DSGE model. Section 5 illustrates how the methods can be used to characterize the relation between the endogenous variables and uncertainty and to extend Campbell's (1991) VAR implementation of Campbell and Shiller's (1988) present value decomposition. Section 6 concludes.

## 2 The Class of Models

Consider the following multivariate Markov-switching model:

$$Z_t = c_{\xi_t} + A_{\xi_t} Z_{t-1} + V_{\xi_t} \varepsilon_t \tag{1}$$

$$V_{\xi_t} = R_{\xi_t} \Sigma_{\xi_t}, \ \varepsilon_t \sim N(0, I_k) \tag{2}$$

where $Z_t$ is an $n \times 1$ vector of variables, $c_{\xi_t}$ is an $n \times 1$ vector of constants, $A_{\xi_t}$ is an $n \times n$ matrix of coefficients, $R_{\xi_t}$ is an $n \times k$ matrix of coefficients mapping the $k \times 1$ vector of structural shocks $\varepsilon_t$ into the $n$ endogenous variables, and the matrix $\Sigma_{\xi_t}$ is a diagonal matrix capturing the size of the different shocks.[1] The hidden Markov process $\xi_t$ controls the regime that is in place at time $t$, assumes values from 1 to $m$, evolves according to the transition matrix $H$, and is assumed to be ergodic and independent from $\varepsilon_t$. The elements of the transition matrix $H$ are defined as $h_{ji} = P(\xi_{t+1} = j | \xi_t = i)$. Therefore the columns of $H$ sum to one.

Changes in coefficients and changes in volatilities are assumed to follow the same chain. This is just to make the exposition simpler, as it is straightforward to allow for two independent chains. It might be useful to think of $\xi_t$ as a composite hidden variable, $\xi_t = (\xi_t^\Phi, \xi_t^\Sigma)$, where $\Phi_{\xi_t} = \{c_{\xi_t}, A_{\xi_t}, R_{\xi_t}\}$ and $\xi_t^\Phi$ and $\xi_t^\Sigma$ summarize the regimes in place for the structural parameters and the stochastic volatilities, respectively. Furthermore, the assumption of one lag should not be considered restrictive, given that a VAR with more than a lag can always be recast in this way using the companion form. Finally, the assumption of independence between $\xi_t$ and $\varepsilon_t$ is less restrictive than what it might seem: If we are interested in modeling a regime change triggered by a shock, it is enough to discretize the shock and include it in $c_{\xi_t}$. For example, Bianchi and Melosi (2013b) use this approach to model zero-lower-bound episodes triggered by a demand shock.

The vector $Z_t$ can be observable or not. In the former case, we have a standard MS-VAR, while in the latter case the MS model (1)-(2) will be combined with an observation equation mapping the vector $Z_t$ into a set of observable variables (as in DSGE or factor models). Furthermore, while in a Markov-switching VAR the number of structural shocks $k$ is generally equal to the number of endogenous variables $n$, I allow for the possibility that $k < n$ to accommodate the case in which the law of motion (1) arises as the output of a Markov-switching DSGE model of the kind studied by Schorfheide (2005), Davig and Leeper (2007), Farmer, Waggoner, and Zha (2009), Bianchi (2013), and Liu, Waggoner, and

[1] The matrix $\Sigma_{\xi_t}$ could also be assumed to be non diagonal. However, the idea here is that all of the contemporaneous relations between the shocks are captured by the matrix $R_{\xi_t}$.

Zha (2011). In such models, the coefficients describing the law of motion will generally depend on a set of underlying deep parameters $\theta$ , the regime in place at time $t$, and the probability of moving across regimes:

$$Z_t = c(\xi_t, \theta, H) + A(\xi_t, \theta, H) Z_{t-1} + R(\xi_t, \theta, H) \Sigma(\xi_t) \varepsilon_t$$

Notice that in this kind of model the regime-dependent law of motion is affected by the transition matrix $H$. While this feature poses some challenges when estimating the model, it does not affect the derivation of the results shown below. Therefore, without loss of generality we will refer to the law of motion given by (1)-(2).

In what follows, I will derive laws of motion for first and second moments for (1)-(2). I will then state the conditions for stability of these processes following Costa, Fragoso, and Marques (2004) (CFM). I will then use these results to construct a toolbox that can be used to characterize agents' expectations and uncertainty, the welfare implications of regime changes, and the output of a multivariate MS model. The results presented here for the law of motion of the moments will not coincide with the ones obtained by CFM. This is because of two reasons. First, CFM do not model the constants and the Gaussian innovations separately. Second, they study models in which the regimes are known one period in advance. This implies that when agents form expectations, they do not face uncertainty about the regime that will prevail in the next period.

### 2.1 Law of motion for first moments

Consider the MS model (1)-(2) and suppose that we are interested in $\mathbb{E}_0(Z_t) = \mathbb{E}(Z_t|\mathbb{I}_0)$ with $\mathbb{I}_0$ being the information set available at time 0. Define the $mn \times 1$ column vector $q_t$ as:

$$q_t = \left[q_t^{1\prime}, ..., q_t^{m\prime}\right]'$$

where $q_t^i = \mathbb{E}_0\left(Z_t 1_{\xi_t=i}\right) = \mathbb{E}\left(Z_t 1_{\xi_t=i}|\mathbb{I}_0\right)$ where $1_{\xi_t=i}$ is an indicator variable that is one when regime $i$ is in place. Note that:

$$q_t^i = \mathbb{E}_0\left(Z_t 1_{\xi_t=i}\right) = \mathbb{E}_0\left(Z_t|\xi_t = i\right)\pi_t^i$$

where $\pi_t^i = P_0(\xi_t = i) = P(\xi_t = i|\mathbb{I}_0)$. Therefore we can express $\mu_t = \mathbb{E}_0(Z_t)$ as:

$$\mu_t = \mathbb{E}_0(Z_t) = \sum_{i=1}^m q_t^i = wq_t$$

where the matrix $w = [I_n, ..., I_n]$ is obtained placing side by side $m$ $n$-dimensional identity matrices. Then we can derive the following proposition:[2]

**Proposition 1** *Consider a Markov-switching model whose law of motion can be described by (1)-(2) and define $q_t^i = \mathbb{E}_0\left(Z_t 1_{\xi_t=i}\right)$ for $i = 1...m$, then:*

$$q_t^j = \sum_{i=1}^m \left[c_j \pi_{t-1}^i + A_j q_{t-1}^i\right] h_{ji} \text{ or } q_t^j = c_j \pi_t^j + \sum_{i=1}^m A_j q_{t-1}^i h_{ji}.$$

[2]See Appendices A and B for a proof and an illustrative example.

Using this result, we can write the law of motion of $q_t$ as:

$$q_t = C\pi_t + \Omega q_{t-1} \tag{3}$$

$$\pi_t = H\pi_{t-1} \tag{4}$$

with $\pi_t = \left[\pi_t^1, ..., \pi_t^m\right]'$, $\Omega = bdiag\left(A_1, ..., A_m\right)\left(H \otimes I_n\right)$, and $C = bdiag\left(c_1, ..., c_m\right)$, where $\otimes$ represents the Kronecker product and *bdiag* is a matrix operator that takes a sequence of matrices and use them to construct a block diagonal matrix. Finally, if we define the column vector $\widetilde{q}_t = \left[q_t', \pi_t'\right]'$, the laws of motion (3) and (4) can be written in a compact form:

$$\widetilde{q}_t = \widetilde{\Omega}\widetilde{q}_{t-1} \tag{5}$$

where

$$\widetilde{\Omega} = \begin{bmatrix} \Omega & CH \\ & H \end{bmatrix}$$

Section 4 will illustrate how the previous formulas can be used to compute a series of objects of interest such as impulse responses and expectations at different time horizons. For now, notice that it is straightforward to compute expectations conditional on the information available at a particular point in time. Suppose we are interested in $\mu_{t+s|t} = \mathbb{E}_t\left(Z_{t+s}\right)$, i.e. the expected value for the vector $Z_{t+s}$ conditional on the information set available at time $t$. If we define:

$$q_{t+s|t} = \left[q_{t+s|t}^{1\prime}, ..., q_{t+s|t}^{m\prime}\right]'$$

where $q_{t+s|t}^i = \mathbb{E}_t\left(Z_{t+s}1_{\xi_{t+s}=i}\right)$, we have $\mathbb{E}_t\left(Z_{t+s}\right) = wq_{t+s|t}$.

If we define $\widetilde{q}_{t|t} = \left[q_{t|t}', \pi_{t|t}'\right]'$, where $\pi_{t|t}$ is a column vector whose *i-th* element coincides with $\pi_{t|t}^i$, where $\pi_{t|t}^i = P_t\left(\xi_t = i\right)$ represents the probability of being in regime $i$ at time $t$ conditional on the information set available at time $t$, we can compute the conditional expectations in one step:

$$\mu_{t+s|t} = \mathbb{E}_t\left(Z_{t+s}\right) = \widetilde{w}\widetilde{\Omega}^s\widetilde{q}_{t|t}$$

where $\widetilde{w} = \left[w, 0_{n\times m}\right]$. Note that this expected value reflects the possibility of regime changes and uncertainty around the regime in place at time $t$ and the state vector $Z_t$. If $Z_t$ is observable at time $t$ we have $q_{t|t}^i = \mathbb{E}_t\left(Z_t 1_{\xi_t=i}\right) = Z_t\pi_{t|t}^i$. Furthermore, if the regime is observable $\pi_{t|t}^i$ will be 0 or 1. In general, $\pi_{t|t}^i$ will be the outcome of a filtering problem.

## 2.2 Law of motion for second moments

I will now derive the law of motion for the second moments. Before proceeding, let me define the vectorization operator $\varphi\left(X\right)$ that takes the matrix $X$ as an input and returns a column vector stacking the columns of the matrix $X$ on top of one another. We will also make use of the following result: $\varphi\left(X_1X_2X_3\right) = \left(X_3' \otimes X_1\right)\varphi\left(X_2\right)$.

Define the vector $n^2m \times 1$ column vector $Q_t$ as:

$$Q_t = \left[Q_t^{1\prime}, ..., Q_t^{m\prime}\right]'$$

where the $n^2 \times 1$ vector $Q_t^i$ is given by $Q_t^i = \varphi\left[\mathbb{E}_0\left(Z_t Z_t' 1_{\xi_t=i}\right)\right]$. This implies that we can compute the vectorized matrix of second moments $M_t = \varphi\left[\mathbb{E}_0\left(Z_t Z_t'\right)\right]$ as:

$$M_t = \varphi\left[\mathbb{E}_0\left(Z_t Z_t'\right)\right] = \textstyle\sum_{i=1}^m Q_t^i = W Q_t$$

where the matrix $W = [I_{n^2}, ..., I_{n^2}]$ is obtained placing side by side $m$ $n^2$-dimensional identity matrices. We can then state the following proposition:[3]

**Proposition 2** *Consider a Markov-switching model whose law of motion can be described by (1)-(2) and define* $Q_t^i = \varphi\left[\mathbb{E}_0\left(Z_t Z_t' 1_{\xi_t=i}\right)\right]$ *for* $i = 1...m$*, then:*

$$\begin{aligned} Q_t^j &= \textstyle\sum_{i=1}^m \left[\begin{array}{c} \left(c_j \otimes c_j\right)\pi_{t-1}^i + \left(A_j \otimes A_j\right) Q_{t-1}^i + \left(V_j \otimes V_j\right) K_{t-1}^i \\ + \left[\left(A_j \otimes c_j\right) + \left(c_j \otimes A_j\right)\right] q_{t-1}^i \end{array}\right] h_{ji} \\ &\quad or \\ Q_t^j &= \left(c_j \otimes c_j\right)\pi_t^j + \left(V_j \otimes V_j\right) K_t^j + \textstyle\sum_{i=1}^m \left[\left(A_j \otimes A_j\right) Q_{t-1}^i + \left[\left(A_j \otimes c_j\right) + \left(c_j \otimes A_j\right)\right] q_{t-1}^i\right] h_{ji} \end{aligned}$$

*where* $q_t^i = \mathbb{E}_0\left[Z_t 1_{\xi_t=i}\right]$, $\pi_t^i = P_0\left(\xi_t = i\right)$, *and* $K_t^i = \varphi\left[\mathbb{E}_0\left[I_k 1_{\xi_t=i}\right]\right] = \varphi\left[I_k\right] * \pi_t^i$.

Using matrix algebra we obtain:

$$Q_t = \Xi Q_{t-1} + \widehat{DAC} q_{t-1} + \widehat{VV} K_t + \widehat{cc}\pi_t \quad (6)$$

$$q_t = C\pi_t + \Omega q_{t-1},\ K_t = L_K K_{t-1},\ \pi_t = H\pi_{t-1}. \quad (7)$$

Therefore, if we define

$$\widetilde{\Xi} = \left[\begin{array}{cc|cc} \Xi & \widehat{VV} L_K & \widehat{DAC} & \widehat{cc} H \\ & L_K & & \\ \hline & & \Omega & CH \\ & & & H \end{array}\right], \widetilde{Q}_t = \left[\begin{array}{c} Q_t \\ K_t \\ q_t \\ \pi_t \end{array}\right]$$

where $\widehat{cc}_j = \left(c_j \otimes c_j\right)$, $\widehat{AA}_j = \left(A_j \otimes A_j\right)$, $\widehat{VV}_j = \left(V_j \otimes V_j\right)$, and

$$\begin{aligned} \widehat{cc} &= bdiag(\widehat{cc}_1, ..., \widehat{cc}_m),\ K_t = \pi_t \otimes \varphi\left[I_k\right],\ L_K = \left(H \otimes I_{k^2}\right) \\ \Xi &= bdiag(\widehat{AA}_1, ..., \widehat{AA}_m)(H \otimes I_{n^2}),\ \widehat{VV} = bdiag(\widehat{VV}_1, ..., \widehat{VV}_m) \\ \widehat{DAC}_j &= \left(A_j \otimes c_j\right) + \left(c_j \otimes A_j\right),\ \widehat{DAC} = bdiag(\widehat{DAC}_1, ..., \widehat{DAC}_m)(H \otimes I_n) \end{aligned}$$

[3]See Appendices A and B for a proof and an illustrative example.

we can express the law of motion for the second moments as:

$$\widetilde{Q}_t = \widetilde{\Xi}\widetilde{Q}_{t-1}. \tag{8}$$

Section 4 will illustrate how the formulas can be used to characterize the evolution of uncertainty. For now, notice that it is straightforward to compute the evolution of second moments conditional on the information available at a particular point in time. Suppose we are interested in $\mathbb{E}_t\left(Z_{t+s}Z'_{t+s}\right)$, i.e. the second moment of the vector $Z_{t+s}$ conditional on the information available at time $t$. If we define:

$$Q_{t+s|t} = \left[Q^{1\prime}_{t+s|t}, ..., Q^{m\prime}_{t+s|t}\right]'$$

where $Q^i_{t+s|t} = \varphi\left[\mathbb{E}_t\left(Z_{t+s}Z'_{t+s}1_{\xi_{t+s}=i}\right)\right]$, we obtain $\varphi\left[\mathbb{E}_t\left(Z_{t+s}Z'_{t+s}\right)\right] = WQ_{t+s|t}$.

If we define $\widetilde{Q}_{t|t} = [Q'_{t|t}, K'_{t|t}, q'_{t|t}, \pi'_{t|t}]'$, where $Q^i_{t|t} = \varphi\left[\mathbb{E}_t\left(Z_tZ'_t1_{\xi_t=i}\right)\right]$ and $K_{t|t} = \pi_{t|t} \otimes \varphi\left[I_k\right]$, we can compute the second moments conditional on the information available at time $t$ in one step:

$$M_{t+s|t} = \varphi\left[\mathbb{E}_t\left(Z_{t+s}Z'_{t+s}\right)\right] = \widetilde{W}\widetilde{\Xi}^s\widetilde{Q}_{t|t} \tag{9}$$

where $\widetilde{W} = \left[W, 0_{n^2\times m(k^2+n+1)}\right]$. Note that the result reflects the possibility of regime changes, the presence of Gaussian shocks, and uncertainty around the regime in place at time $t$ and the state vector $Z_t$. Finally, if $Z_t$ is observable at time $t$ we have $Q^i_{t|t} = \varphi\left[Z_tZ'_t\right]\pi^i_{t|t}$.

### 2.3 Variance

With the first and second moments at hand, it is then possible to compute the variance implied by the Markov-switching model, $\mathbb{V}_0\left(Z_t\right)$:

$$\varphi\left[\mathbb{V}_0\left(Z_t\right)\right] = M_t - \varphi\left[\mu_t\mu'_t\right] \tag{10}$$

In many practical applications, we will be interested in the variance $s$ periods ahead conditional on the information available at time $t$:

$$\varphi\left[\mathbb{V}_t\left(Z_{t+s}\right)\right] = M_{t+s|t} - \varphi\left[\mu_{t+s|t}\mu'_{t+s|t}\right] \tag{11}$$

It is worth pointing out that in an MS model the variance will change in response to changes in the initial values for the endogenous variables, $Z_t$, and for the regime probabilities, $\pi_t$. Instead, in a fixed coefficient VAR the initial values of the endogenous variables $Z_t$ do not affect the variance. This is because in an MS-VAR the way the endogenous variables evolve changes over time through the effect of a multiplicative term: $A_{\xi_t}$. Appendix C presents a simple example for a univariate MS model, showing that if $A_{\xi_t}$ is not subject to regime changes, then the variance does not depend on the initial values for the endogenous variables. Appendix D shows how to write a law of motion for the squared first moments, $\mu^2_t = \varphi\left[\mu_t\mu'_t\right]$, introducing the variable $\widetilde{q}q_t = \varphi\left[\widetilde{q}_t\widetilde{q}'_t\right]$.

### 2.4 Autocovariance

The autocovariance matrix requires some additional work, but it can be useful to characterize the evolution of autocorrelation. By definition $cov_0(Z_{t+s}, Z_t) = \mathbb{E}_0(Z_{t+s}Z_t') - \mathbb{E}_0(Z_{t+s})\mathbb{E}_0(Z_t')$. We have already derived formulas for $\mathbb{E}_0(Z_{t+s})\mathbb{E}_0(Z_t')$. Then, to compute $\mathbb{E}_0(Z_{t+s}Z_t')$, we will introduce some additional notation. Define:

$$\begin{aligned} Q_{t,t+s}^{i,j} &= \varphi\left[\mathbb{E}_0\left(Z_{t+s}Z_t' 1_{\xi_t=i} 1_{\xi_{t+s}=j}\right)\right] \\ Q_{t,t+s}^{\cdot,j} &= \varphi\left[\mathbb{E}_0\left(Z_{t+s}Z_t' 1_{\xi_{t+s}=j}\right)\right] \\ q_{t,t+s}^{\cdot,j} &= \mathbb{E}_0\left(Z_t 1_{\xi_{t+s}=j}\right) \end{aligned}$$

and note that $q_{t,t}^{\cdot,j} = \mathbb{E}_0\left(Z_t 1_{\xi_t=j}\right) = q_t^j$ and $Q_{t,t}^{\cdot,j} = \varphi\left[\mathbb{E}_0\left(Z_t Z_t' 1_{\xi_t=j}\right)\right] = Q_t^j$. If we define $M_{t,t+s} = \varphi\left[\mathbb{E}_0\left(Z_{t+s}Z_t'\right)\right]$, we have:

$$\varphi\left[\mathbb{E}_0\left(Z_{t+s}Z_t'\right)\right] = \sum_{j=1}^m Q_{t,t+s}^{\cdot,j}.$$

Then, we can derive the following proposition:[4]

**Proposition 3** *Consider a Markov-switching model whose law of motion can be described by (1)-(2) and define* $Q_{t,t+s}^{\cdot,j} = \varphi\left[\mathbb{E}_0\left(Z_{t+s}Z_t' 1_{s_{t+s}=j}\right)\right]$ *and* $q_{t,t+s}^{\cdot,j} = \mathbb{E}_0\left(Z_t 1_{\xi_{t+s}=j}\right)$ *for* $j = 1...m$*, then:*

$$Q_{t,t+s}^{\cdot,j} = \widehat{Ic}_j q_{t,t+s}^{\cdot,j} + \sum_{i=1}^m \widehat{IA}_j h_{ji} Q_{t,t+s-1}^{\cdot,i}$$

*where* $\widehat{Ic}_j = (I_n \otimes c_j)$ *and* $\widehat{IA}_j = (I_n \otimes A_j)$*.*

Then, using the fact that $Q_{t,t} = Q_t$ and $q_{t,t} = q_t$, we have:

$$\begin{bmatrix} Q_{t,t+s} \\ q_{t,t+s} \end{bmatrix} = \widetilde{\widetilde{\Xi}}_{1,l} \begin{bmatrix} Q_{t,t+s-1} \\ q_{t,t+s-1} \end{bmatrix} = \widetilde{\widetilde{\Xi}}_{1,l}^s \begin{bmatrix} Q_t \\ q_t \end{bmatrix}, \text{ with } \widetilde{\widetilde{\Xi}}_{1,l} \equiv \begin{bmatrix} \Xi_{1,l} & \widehat{Ic}\,(H \otimes I_n) \\ & (H \otimes I_n) \end{bmatrix} \quad (12)$$

where $\widehat{IA}_j = (I_n \otimes A_j)$, $\widehat{Ic}_j = (I_n \otimes c_j)$, $\Xi_{1,l} = bdiag(\widehat{IA}_1, ..., \widehat{IA}_m)(H \otimes I_{n^2})$, $\widehat{Ic} = bdiag(\widehat{Ic}_1, ..., \widehat{Ic}_m)$, $Q_{t,t+s} = \left[\left(Q_{t,t+s}^{\cdot,1}\right)', ..., \left(Q_{t,t+s}^{\cdot,m}\right)'\right]'$, and $q_{t,t+s} = \left[\left(q_{t,t+s}^{\cdot,1}\right)', ..., \left(q_{t,t+s}^{\cdot,m}\right)'\right]'$. Analogous formulas can be derived for $\mathbb{E}_0\left(Z_t Z_{t+s}'\right)$.

# 3 Mean Square Stability and Steady States

So far we have illustrated how to characterize the evolution of first moments, second moments, and covariance matrices in an MS model. In many applications, it would be useful to know if these objects converge to finite values. For example, we might find it desirable that as the horizon goes to infinity, uncertainty stabilizes. In order to establish whether this is the case, I will borrow the concept of mean

[4] See Appendices A and B for a proof and an illustrative example.

square stability (MSS) from the engineering literature. Once convergence of first and second moments has been established, I will derive formulas for the steady states of expectations and uncertainty.

### 3.1 Mean square stability

Mean square stability is defined as follows:

**Definition 1** *An n-dimensional process $Z_t$ is mean square stable if and only if there exists an n-vector $\overline{\mu}$ and an $n^2$-vector $\overline{M}$ such that:*

*1)* $\lim_{t\to\infty} \mathbb{E}_0 [Z_t] = \overline{\mu}$

*2)* $\lim_{t\to\infty} \mathbb{E}_0 [Z_t Z_t'] = \overline{M}$

*for any initial $Z_0$ and $\xi_0$.*

MSS requires that first and second moments converge as the time horizon goes to $\infty$. MSS has been used by Svensson and Williams (2007) to study optimal monetary policy in an uncertain environment and by Farmer, Waggoner, and Zha (2009) to derive conditions for uniqueness of a solution in DSGE models subject to regime changes. Farmer, Waggoner, and Zha (2009) argue that MSS is an appealing stability concept when dealing with Markov-switching general equilibrium models. Using MSS in place of bounded stability allows for the possibility that one or more regimes are unstable, as long as this regime is not too persistent. This seems a very desirable property, especially when thinking about asset prices or macroeconomic models. For example, MSS allows for the possibility of a bubble regime during which stock prices keep increasing for a prolonged period of time or hyperinflationary episodes during which inflation follows an unstable path. Finally, under the assumptions that the Markov-switching process $\xi_t$ is ergodic and that the innovation process $\varepsilon_t$ is asymptotically covariance stationary, Costa, Fragoso, and Marques (2004) show that a multivariate Markov-switching model as the one described by (1)-(2) is mean-square stable if and only if it is asymptotically covariance stationary. Both conditions hold for the models studied in this paper and are usually verified in economic models.

Costa, Fragoso, and Marques (2004) show that in order to establish MSS of a process such as the one described by (1)-(2), it is enough to check MSS stability of the correspondent homogeneous process: $Z_t = A_{\xi_t} Z_{t-1}$. In this case, formulas (5) and (8) simplify substantially: $q_t = \Omega q_{t-1}$ and $Q_t = \Xi Q_{t-1}$. Let $r_\sigma (X)$ be the operator that given a square matrix $X$ computes its largest eigenvalue. We then have:

**Proposition 4** *A Markov-switching process whose law of motion can be described by (1)-(2) is mean square stable if and only if $r_\sigma (\Xi) < 1$.*

### 3.2 Steady states

When mean square stability holds, it is possible to compute analytically the unconditional first and second moments that can be used to obtain the steady-state values for the levels and the volatilities of the endogenous variables taking into account the possibility of regime changes. This is done in Subsection 3.2.1. When the single regimes taken in isolation are stationary, it is also possible to compute steady-state values for first and second moments conditional on a specific regime being in place for a prolonged period of time. This is done in Subsection 3.2.2.

#### 3.2.1 Ergodic steady states

Recall that the law of motion for first and second moments is entirely summarized by (8). Notice that $K_t$ and $\pi_t$ evolve independently from the endogenous variables, given that they only depend on the Markov-switching process. Therefore, we have $\pi_t \to \overline{\pi}$ and $K_t \to \overline{K} = \overline{\pi} \otimes \varphi[I_k]$, where $\overline{\pi}$ is the $m \times 1$ vector containing the ergodic probabilities of the $m$ regimes that can be obtained computing the normalized right eigenvector of the transition matrix $H$ associated with the unit eigenvalue. Furthermore, the law of motion for the first moments does not depend on the law of motion for the second moments. Therefore, under the assumption of MSS, we obtain $\overline{q} = (I_{nm} - \Omega)^{-1} C\overline{\pi}$, implying that the ergodic steady state for the first moments can be easily computed as

$$\overline{\mu} = \mathbb{E}(Z_t) = w\overline{q} = w(I_{nm} - \Omega)^{-1} C\overline{\pi} \tag{13}$$

For the second moments, using (6), we have $Q_t \to \overline{Q}$ where

$$\overline{Q} = (I_{mn^2} - \Xi)^{-1} \left(\widehat{VV}\overline{K} + \widehat{cc}\overline{\pi} + \widehat{DAC}\overline{q}\right)$$

and the ergodic steady state for the vectorized second moments can be computed as

$$\overline{M} = W\overline{Q} = W(I_{mn^2} - \Xi)^{-1} \left(\widehat{VV}\overline{K} + \widehat{cc}\overline{\pi} + \widehat{DAC}\overline{q}\right) \tag{14}$$

With these formulas at hand, it is then straightforward to compute ergodic values for variances and correlations. For example, the ergodic (vectorized) covariance matrix of the endogenous variables is given by:

$$\varphi[\mathbb{V}(Z_t)] = \overline{M} - \varphi\left[\overline{\mu\mu}'\right]. \tag{15}$$

In a model with no Markov-switching constants, these formulas simplify to:

$$\overline{q} = 0,\ \overline{Q} = (I_{mn^2} - \Xi)^{-1} \widehat{VV}\overline{K},\ \varphi[\mathbb{V}(Z)] = W\overline{Q}.$$

#### 3.2.2 Conditional steady states

For all regimes that are stationary when taken in isolation, we can easily compute conditional steady states. When conditioning on a single regime, formulas (5) and (8) simplify to:

$$\begin{bmatrix} Q_{t|i} \\ q_{t|i} \end{bmatrix} = \begin{bmatrix} \widehat{VV}_i \varphi(I_k) + \widehat{cc}_i \\ c_i \end{bmatrix} + \left[\begin{array}{c|c} \widehat{AA}_i & \widehat{DAC}_i \\ \hline & A_i \end{array}\right] \begin{bmatrix} Q_{t-1|i} \\ q_{t-1|i} \end{bmatrix} \tag{16}$$

where $q_{t|i} = \mathbb{E}[Z_t|\xi_t = i\ \forall\ t]$ and $Q_{t|i} = \mathbb{E}[Z_t Z_t'|\xi_t = i\ \forall\ t]$. Notice that (16) implies $q_{t|i} = c_i + A_i q_{t-1|i}$.

Therefore the conditional steady states can be computed as:

$$\mathbb{E}_i(Z_t) = \overline{\mu}_i = \overline{q}_i = (I_n - A_i)^{-1} c_i \tag{17}$$

$$\overline{M}_i = \overline{Q}_i = \left(I_{n^2} - \widehat{AA}_i\right)^{-1} \left(\widehat{VV}_i \varphi(I_k) + \widehat{cc}_i + \widehat{DAC}_i \overline{q}_i\right) \tag{18}$$

$$\varphi\left[\mathbb{V}_i\left(Z_t\right)\right] = \overline{M}_i - \varphi\left[\overline{\mu}_i\overline{\mu}_i'\right]. \tag{19}$$

# 4 Applications for a Microfounded Model

The second part of the paper is entirely dedicated to showing how the results derived above can be used to analyze the features of a Markov-switching model. In this section, I will illustrate a series of applications using a Markov-switching DSGE model. The solution of the model assumes the form of an MS-VAR with no constant. In Section 5, I will consider two additional applications for the case of an MS-VAR with MS conditional steady states. In the interest of brevity, I am not reporting results for persistences and autocorrelations. These are available upon request.

## 4.1 A new-Keynesian model with regime changes

In what follows, I describe a small new-Keynesian model of the kind used by Clarida, Gali, and Gertler (2000), Woodford (2003), Lubik and Schorfheide (2004), and Gali (2008), in which the behavior of the monetary authority and the volatility of the exogenous shocks are subject to regime changes. For the sake of brevity, I will present the linearized version of the model. The model is not meant to provide a full characterization of the US economy. Instead, it has been chosen because of its relative simplicity and in order to illustrate a wide range of applications of the methods developed in the paper. Please refer to Bianchi (2013) for a model that includes capital and investment and is explicitly derived from agents' optimizations problems. Finally, it is important to emphasize that the methods illustrated here can be applied to *any* model whose solution can be expressed as an MS-VAR.

The private sector can be described by a system of two equations:

$$\Delta p_t = \beta\mathbb{E}_t(\Delta p_{t+1}) + \kappa(y_t - z_t) \tag{20}$$
$$y_t = \mathbb{E}_t(y_{t+1}) - \tau^{-1}(R_t - \mathbb{E}_t(\Delta p_{t+1})) + d_t \tag{21}$$

where $\Delta p_t$ represents inflation, $y_t$ is the output gap, and $R_t$ is the nominal interest rate. The variables are expressed in deviations from a deterministic steady state that does not depend on regime changes.[5] Inflation dynamics are described by the expectational Phillips curve (20) with slope $\kappa$. This relation can be derived assuming a quadratic adjustment cost or Calvo pricing. Equation (21) can be derived starting from an intertemporal Euler equation describing the households' optimal choice of consumption and bond holdings. The parameter $\tau^{-1} > 0$ can be interpreted as intertemporal substitution elasticity and $0 < \beta = 1/\left(1 + r^*\right) < 1$ is the households' discount factor, where $r^*$ is the steady-state real interest rate. The process $z_t$ can be interpreted as a supply/mark-up shock, while the process $d_t$ summarizes changes in preferences and other demand side disturbances. The two shocks evolve according to:

$$d_t = \rho_d d_{t-1} + \sigma_{d,\xi_t^{vo}}\epsilon_{d,t},\ \epsilon_{d,t} \sim N\left(0,1\right) \tag{22}$$
$$z_t = \rho_z z_{t-1} + \sigma_{z,\xi_t^{vo}}\epsilon_{z,t},\ \epsilon_{z,t} \sim N\left(0,1\right) \tag{23}$$

[5] See Schorfheide (2005), Liu, Waggoner, and Zha (2011), and Bianchi, Ilut, and Schneider (2012) for models in which regime changes affect the steady state.

The central bank responds to the output gap and deviations of inflation from its target level $\Delta p^*$ adjusting the monetary policy interest rate. Unanticipated deviation from the systematic component of the monetary policy rule are captured by $\epsilon_{R,t}$:

$$R_t = \rho_{R,\xi_t^{mp}} R_{t-1} + (1 - \rho_{R,\xi_t^{mp}})(\psi_{\Delta p,\xi_t^{mp}} \Delta p_t + \psi_{y,\xi_t^{mp}} y_t) + \sigma_{R,\xi_t^{vo}} \epsilon_{R,t}, \; \epsilon_{R,t} \sim N(0,1) \tag{24}$$

where $\xi_t^{mp}$ is an unobserved state variable capturing the monetary policy regime that is in place at time $t$ and evolves according to a two state Markov chain with transition matrix $H^{mp}$. The hidden variable $\xi_t^{vo}$ allows for changes in the volatility of the exogenous disturbances and evolves according to an independent two-state Markov chain with transition matrix $H^{vo}$.[6] Agents in the model know the probability of moving across regimes and they use this information when forming expectations.

If we define the matrix $\Sigma_{\xi_t^{vo}} \equiv diag\left(\sigma^2_{R,\xi_t^{vo}}, \sigma^2_{d,\xi_t^{vo}}, \sigma^2_{z,\xi_t^{vo}}\right)$ and the DSGE state vector $Z_t$ as:

$$Z_t = [y_t, \Delta p_t, R_t, d_t, z_t, \mathbb{E}_t(y_{t+1}), \mathbb{E}_t(\Delta p_{t+1})]'$$

we can rewrite the system of equations (20)-(24) in a more compact form:

$$\Gamma_{0,\xi_t^{mp}} Z_t = \Gamma_{1,\xi_t^{mp}} Z_{t-1} + \Psi_{\xi_t^{mp}} \Sigma_{\xi_t^{vo}} \epsilon_t + \Pi \eta_t \tag{25}$$

with $\eta_t$ a vector containing the expectations errors. The model can be solved with any of the solution methods developed to solve MS-DSGE models: Davig and Leeper (2007), Farmer, Waggoner, and Zha (2009), Cho (2012), and Foerster, Rubio-Ramirez, Waggoner, and Zha (2010). I make use of Farmer, Waggoner, and Zha (2009). When a solution exists, it can be characterized as a regime-switching vector-autoregression, of the kind studied by Hamilton (1989), Chib (1996), and Sims and Zha (2006):

$$Z_t = A_{\xi_t^{mp}} Z_{t-1} + R_{\xi_t^{mp}} \Sigma_{\xi_t^{vo}} \epsilon_t \tag{26}$$

Therefore, the model solution assumes the same form of the MS process described in (1), with the only notable difference being that there is no MS constant. It is worth emphasizing that the law of motion of the DSGE states depends on the structural parameters, the regime in place, and the probability of moving across regimes. This is because the model is solved under the assumption that while agents can observe the regime they are in, they are uncertain about the regime that will prevail in the future. While this assumption makes the estimation of the model challenging, it will not prevent us from applying the methods described in this paper.

## 4.2 Parameter values and regime probabilities

Once the law of motion (26) is combined with a system of observation equations, the model can be estimated with the methods described in Bianchi (2013). The time series are extracted from the Global Insight database. The output gap is measured as the percentage deviations of real per capita GDP from

[6] Here and later on, *mp* and *vo* stand, respectively, for monetary policy and volatilities.

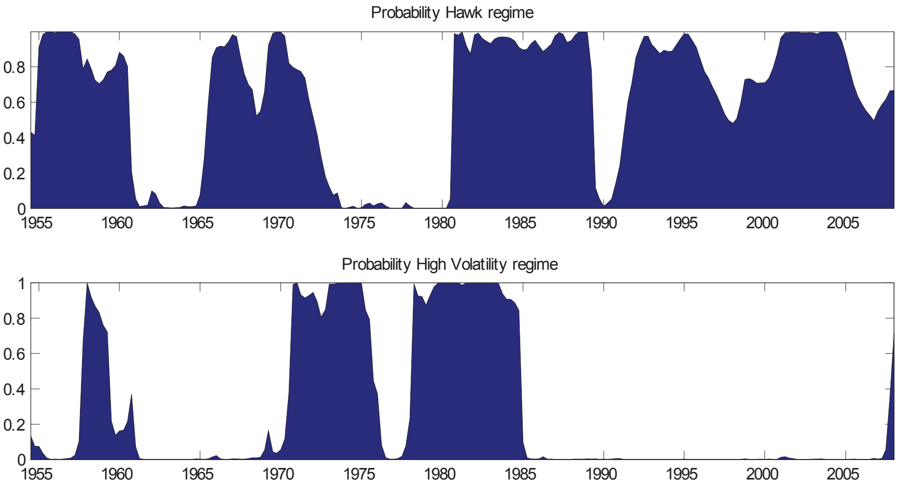


Figure 1: Posterior mode smoothed probabilities for the *Hawk regime* and the *High volatility regime*.

| Parameter | $\xi^{mp}=1$ | $\xi^{mp}=2$ | Parameter | $\xi^{vo}=1$ | $\xi^{vo}=2$ |
|---|---|---|---|---|---|
| $\psi_\pi$ | 2.0528 | 0.5907 | $\sigma_R$ | 0.3211 | 0.0741 |
| $\psi_y$ | 0.2744 | 0.3824 | $\sigma_d$ | 0.3522 | 0.1483 |
| $\rho_R$ | 0.7530 | 0.7881 | $\sigma_z$ | 1.8538 | 0.5842 |
| $\tau$ | 2.8744 | | $\sigma_\pi$ | 0.2929 | |
| $\kappa$ | 0.0257 | | $H_{11}^{mp}$ | 0.9186 | |
| $\rho_d$ | 0.8404 | | $H_{22}^{mp}$ | 0.9211 | |
| $\rho_z$ | 0.9071 | | $H_{11}^{vo}$ | 0.8948 | |
| $\beta$ | 0.9952 | | $H_{22}^{vo}$ | 0.9556 | |
| $100r^*$ | 0.4812 | | $100\Delta p^*$ | 0.7811 | |

Table 1: Posterior mode estimates of the model parameters.

a trend obtained with the HP filter. Inflation is the annualized quarterly percentage change of CPI (Urban, all items). The nominal interest rate is the average federal funds rate in percentage points.

Table 1 reports the posterior mode estimates for the parameters of the model. These are the values that will be used to obtain all the results that follow. Concerning the parameters of the Taylor rule, we find that under Regime 1 ($\xi_t^{mp}=1$) the federal funds rate reacts strongly to deviations of inflation from its target, while the output gap does not seem to be a major concern. The opposite occurs under Regime 2, under which the response to inflation is substantially weaker. The degree of interest rate smoothing turns out to be similar across regimes. In what follows, I will refer to Regime 1 as the *Hawk* regime, while Regime 2 will be the *Dove* regime.

Figure 1 shows the smoothed probabilities assigned to $\xi_t^{mp}=1$ (top panel) and $\xi_t^{vo}=1$ (lower panel). The *Hawk* regime prevails over the early years, during the second half of the '60s and for a large part of the second half of the sample. Over the period 1961-1965 and during the '70s the *Dove* regime was dominant, while over the second half of the sample its probability is close to one only during the '91 recession. The timing of regime changes roughly coincides with the narrative evidence that suggests the Fed was passive in the '70s and very aggressive against inflation after the appointment of

Paul Volcker in August '79. As for the stochastic volatilities, it emerges that Regime 1, characterized by large volatilities for all shocks, prevails for a long period that goes from the early '70s to 1985, with a break between the two oil crises. I will refer to Regime 1 as the *high volatility regime*. Finally, it is worth pointing out that all regimes have relatively low persistence, except for the *low volatility regime*.

### 4.3 Historical evolution of expectations and uncertainty

Given that agents are rational and aware of regime changes, the possibility of moving across regimes should have a significant impact on expectations and uncertainty. Therefore, given the parameter estimates and the probabilities assigned to different regimes, a researcher might be interested in characterizing the historical evolution of expectations and uncertainty at different horizons. In other words, the researcher could be interested in computing $\mathbb{E}_t (Z_{t+s})$ and $\mathbb{V}_t (Z_{t+s})$ for $s > 0$ from the point of view of an agent living in the economy described by equations (20)-(24).

The first step consists of defining the composite regime $\xi_t = [\xi_t^{mp}, \xi_t^{vo}]$ and the corresponding transition matrix $H = H^{mp} \otimes H^{vo}$. We then have a total of four regimes corresponding to all possible combinations of $\xi_t^{mp}$ and $\xi_t^{vo}$. Then, given the smoothed estimates for the DSGE states and the regime probabilities, we can construct $q_{t|t}$ and $Q_{t|t}$:

$$q_{t|t} = \left[q_{t|t}^{1\prime}, ..., q_{t|t}^{4\prime}\right]', \; Q_{t|t} = \left[Q_{t|t}^{1\prime}, ..., Q_{t|t}^{4\prime}\right]'$$

where $q_{t|t}^i = Z_t \pi_{t|t}^i$ and $Q_{t|t}^i = \varphi \left[Z_t Z_t'\right] \pi_{t|t}^i$ for $i = 1, 2, 3, 4.$[7] Finally, we can use (5) and (8) to compute $\mu_{t+s|t} = w q_{t+s|t}$ and $\varphi \left[\mathbb{V}_t (Z_{t+s})\right] = M_{t+s|t} - \varphi \left[\mu_{t+s|t} \mu_{t+s|t}'\right]$ with $M_{t+s|t} = W Q_{t+s|t}$. Notice that in this case, the formulas simplify because we do not have an MS constant:

$$\begin{bmatrix} Q_{t+s|t} \\ K_{t+s|t} \\ q_{t+s|t} \end{bmatrix} = \left[\begin{array}{c|c} \begin{matrix} \Xi & \widehat{VV} L_K \\ & L_K \end{matrix} & \\ \hline & \Omega \end{array}\right]^s \begin{bmatrix} Q_{t|t} \\ K_{t|t} \\ q_{t|t} \end{bmatrix}.$$

Figure 2 reports the evolution of model-implied expectations at each point in time. For a variable $X_t$ and a horizon $s$, expectations are measured by $\mathbb{E}_t (X_{t+s})$. The time horizon goes from one quarter, light blue area, to five years, dark red area. In the long run all macroeconomic variables return to the unique deterministic steady state. However, this requires an extended period of time: After five years the expected values for the variables of interest are often different from zero. Not surprisingly, persistent deviations can be detected during the Great Inflation of the '70s.

Figure 3 reports the evolution of uncertainty. For a variable $X_t$ and a horizon $s$, uncertainty is measured by its standard deviation: $sd_t (X_{t+s}) = \left[\mathbb{V}_t (X_{t+s})\right]^{.5}$. The horizons are still from one quarter to five years. It is clear that the volatility regime that is in place heavily affects the evolution of uncertainty. To see this, notice that even at a five-year horizon uncertainty is above the ergodic steady

[7] Here and thereafter $X^T = \{X_t\}_{t=1}^T$. In what follows, we treat the DSGE state vector as observable using the smoothed series $Z^T$. Notice that the econometrician uses the entire sample to form the smoothed estimates and infer the state of the economy a time $t$. However, expectations and uncertainty are computed from the point of view of the agent in the model that has only the information set available at time $t$. This is why I use the notation $q_{t|t}$ and $Q_{t|t}$ instead of $q_{t|T}$ and $Q_{t|T}$.

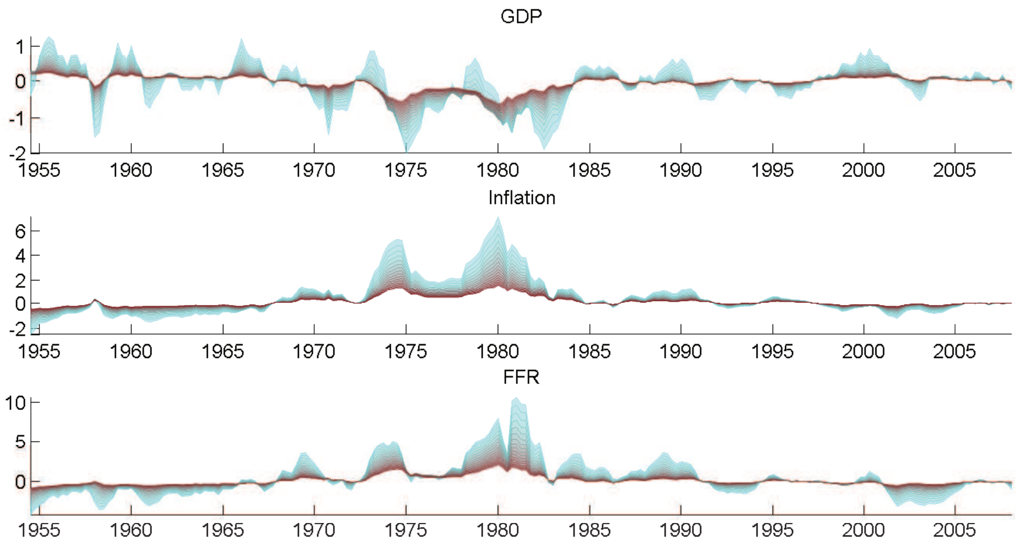


Figure 2: Historical evolution of the expected values for output, inflation, and FFR for horizons going from 1 quarter to 5 years. The light blue areas correspond to the short horizons.

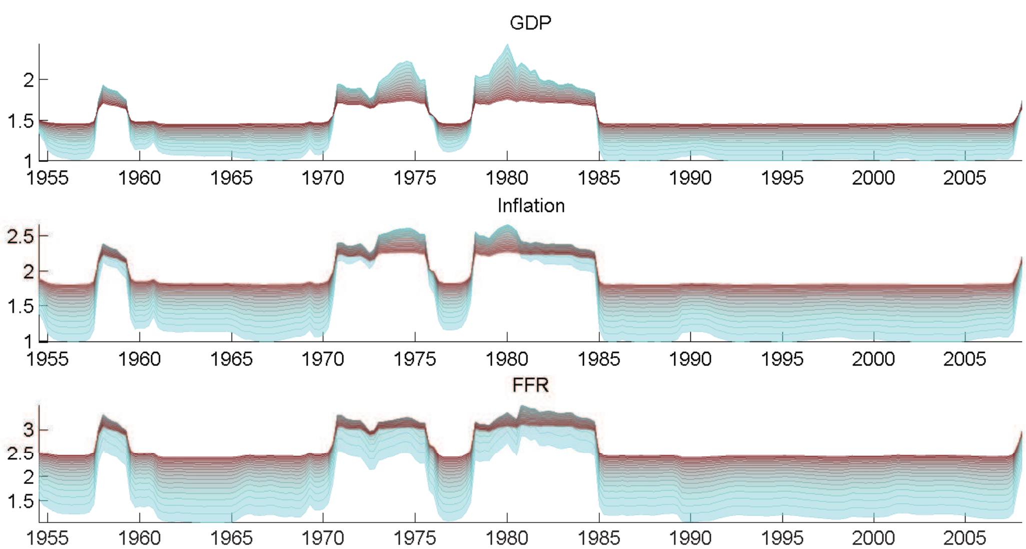


Figure 3: Historical evolution of uncertainty for output, inflation, and FFR for horizons going from 1 quarter to 5 years. The light blue areas correspond to the short horizons.

state when the high volatility regime is in place. Furthermore, the fact that agents are aware of regime changes determines an hump shape in uncertainty when the high volatility regime is in place: Short run uncertainty can be larger than long run uncertainty. As carefully explained in subsection 4.5, this is because two competing forces affect uncertainty as the horizon increases. On the one hand, events further into the future are harder to predict. On the other hand, as the time horizon increases, the probability of still being in the high volatility regime declines. Finally, the monetary policy regime does not seem to be very important. As it will be emphasized in subsections 4.5 and 4.7, this result stems from the fact that agents are aware of regime changes.

It is worth pointing out that the possibility of analytically characterizing uncertainty in a model with stochastic volatility and regime changes in policy makers' behavior represents a useful tool, especially in light of the recent attention given to uncertainty following the seminal contribution of Bloom (2009). When modeling parameter instability with smoothly time-varying coefficients as in Primiceri (2005) and Cogley and Sargent (2006), a researcher has only two options when trying to characterize expectations and uncertainty. She can decide to ignore parameter instability and use anticipated utility, or she can decide to use numerical integration (Bianchi, Mumtaz, and Surico (2009)). The first approach is often used in the learning literature or when agents' expectations are used as additional observables in an estimation exercise. The second approach is generally used ex post to carefully characterize expectations for a given set of estimated parameters. What cannot be easily done is to use numerical integration to compute agents' expectations while at the same time estimating the model, given that this would lead to an unsustainable computational burden. The results presented here suggest that when parameter instability is modeled with an MS process, agents' uncertainty and expectations can be easily computed and used as observables in an estimation algorithm. For example, the observation equation could be augmented with a measure of inflation expectations as in Del Negro and Eusepi (2011) or long term interest rates as in De Graeve, Emiris, and Wouters (2009) while allowing for regime changes.[8]

## 4.4 Impulse responses

An important and useful application of the methods outlined above is given by impulse responses. In a model with Markov-switching changes a researcher might be interested in both *conditional* and *unconditional* impulse responses. The former are computed assuming that over the relevant horizon a specific regime path will prevail; the latter are derived taking into account the possibility of regime changes. In what follows, I will show that the two approaches can lead to very different results.

When conditioning on a specific regime path $\xi^S$, impulse responses to the shock $x$ over the time horizon $S$ can be computed recursively:

$$\begin{aligned} Z_{1|1,\xi^S} &= c_{\xi_1} + A_{\xi_1}\overline{\mu}_{\xi_1} + R_{\xi_1}\Sigma_{\xi_1}\varepsilon^x \\ Z_{s|1,\xi^S} &= c_{\xi_s} + A_{\xi_s}Z_{s-1|1,\xi^S} \text{ for } 1 < s \leq S \end{aligned}$$

[8]Notice that the alternative approach of expanding the state space to recursively model the evolution of inflation expectations would lead to a *substantial* increase of the time required to solve the MS-DSGE model.

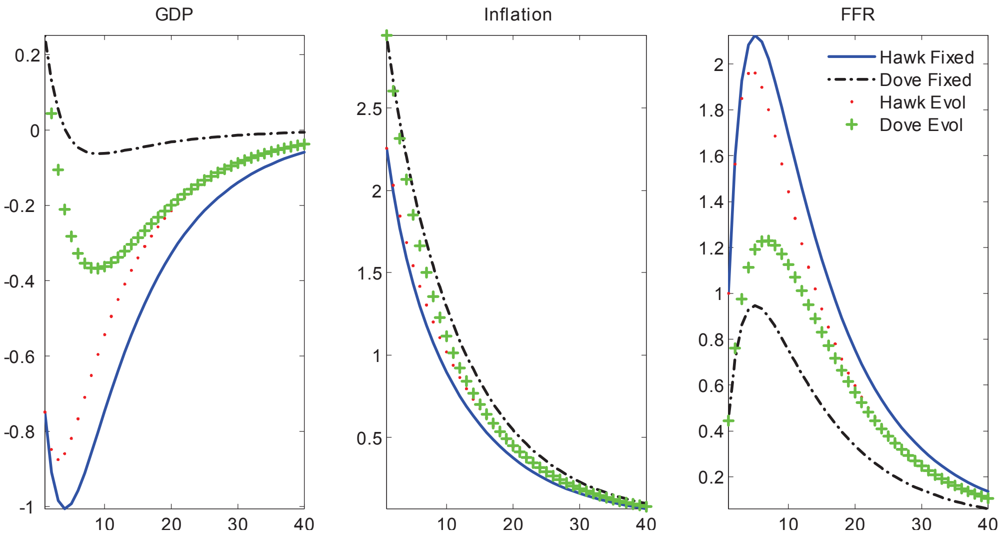


Figure 4: Impulse responses to an adverse supply shock. The blue solid line and the dashed black line assume that a particular regime is in place over the entire sample. The red dotted line and plus green line take into account the possibility of regime changes. The initial shock is equal to minus two standard deviations under the high volatility regime.

where $\varepsilon^x$ is a $k \times 1$ vector with all elements equal to zero, except for the $x$-$th$ element that is equal to the size of the shocks in terms of standard deviations. The initial value $\overline{\mu}_{\xi_1}$ is the conditional steady state associated with the regime in place at the time of the shock. This can be replaced with the ergodic steady state. I choose the conditional steady state because in practice it is often useful to understand how the economy behaves once it has spent a significant amount of time in a specific regime. Furthermore, in a model that allows for a constant, the initial regime change becomes a shock itself if the ergodic steady state is chosen as the starting endogenous variable. This point will be illustrated in Section 5.

To compute impulse responses taking into account the possibility of regime changes, we can use Equation (5). Once again, it can be assumed that the economy has spent a significant amount of time under the regime in place at the time of the shock:

$$\widetilde{q}_{1|1,\xi_1} = \left[ \left( e_{\xi_1} \otimes \left( c_{\xi_1} + A_{\xi_1} \overline{\mu}_{\xi_1} + R_{\xi_1} \Sigma_{\xi_1} \varepsilon^x \right) \right)', e'_{\xi_1} \right]'$$

$$\mathbb{E}_{1,\xi_1}(Z_s) = \mu_{s|1,\xi_1} = \widetilde{w} \widetilde{\Omega}^{s-1} \widetilde{q}_{s|1,\xi_1} \text{ for } 1 \leq s \leq S$$

where $\mathbb{E}_{1,\xi_1}(Z_s)$ represents the expected value of the variable $s-1$ periods ahead. The regime in place at the time of the shock is captured by $e_{\xi_1}$, a column vector with all elements equal to zero except for the one at position $\xi_1$. Notice that even in this case we can consider a different starting value, for example, the ergodic steady state $\overline{\mu}$. In that case, if the model allows for an MS constant, the regime in place at time $t$ would become a shock itself, since it would determine a shift with respect to the ergodic steady state $\overline{\mu}$. Alternatively, $e_{\xi_1}$ can be replaced with the probability vector $\pi_{t|t}$. For example, the probabilities could be obtained by filtering the data and would reflect the uncertainty around the regime prevailing at time $t$. Notice that this approach would be appropriate if an economist were interested in studying the impact of a specific intervention without fully knowing the current economic environment. Finally, in the case of the MS-DSGE model described above, the constant drops out and the conditional steady state coincides with the ergodic steady state. In Section 5, I will present results for a model in which the conditional steady states do not coincide with the ergodic steady state.

Figure 4 reports the responses to an adverse supply shock. The solid blue line assumes that the *Hawk* regime prevails over the entire sample, while the dashed black line corresponds to the case in which the *Dove* regime is in place. The red dotted line and the green plus line instead take into account the possibility of regime changes. The former assumes that the *Hawk* regime is in place at the time of the shock, while the latter conditions on the *Dove* regime being in place at the time of the shock. By construction, the impulse responses do not differ on impact. However, the dynamics that follow are substantially different if conditioning or not to a specific regime being in place over the relevant horizon. The behavior of the Federal Reserve differs substantially across the two regimes. Under the *Hawk* regime, the Fed is willing to accept a recession in order to fight inflation. The FFR reacts strongly on impact and keeps rising for one year. On the contrary, under the *Dove* regime the response of the policy rate is much weaker because the Fed tries to keep the output gap around zero, at the cost of higher inflation. Note that on impact the economy experiences a boom: The increase in expected inflation determines a decline in the real interest rate that boosts the economy in the short run. When taking into account the possibility of regime changes the impulse responses become very similar after three or four years. This is because the regimes are not very persistent. Therefore, before the shock is completely reabsorbed there is a high probability that regime changes will occur.

The choice between conditional and unconditional impulse responses clearly depends on the goal of the analysis. Conditioning on a particular regime highlights the regime-specific features. It is generally useful when trying to understand the properties of the model and when trying to infer how the economy will behave if a specific regime will be in place over the relevant horizon. On the other hand, an agent that does not have control over regime changes, such as a market operator, might be more interested in a path that reflects all sources of uncertainty. In that case, the unconditional responses are more appropriate. If an economist is also uncertain about which regime is currently prevailing, she can decide to control for such uncertainty by assigning some probability to each of the regimes and allowing for the possibility of regime changes.

### 4.5 Evolution of uncertainty

Different regimes tend to have very different implications for the evolution of uncertainty. It is therefore very informative to understand how uncertainty evolves across different regimes. In Subsection 4.3, I showed how to characterize the historical evolution of uncertainty taking into account regime uncertainty. A similar exercise can be conducted conditional on an initial regime. Once again, I assume that the economy has spent a significant amount of time under a specific regime and I therefore use the conditional steady state for the regime of interest. I further assume that no Gaussian shocks have occurred in the economy, implying that the initial level of uncertainty only reflects the possibility of future shocks.

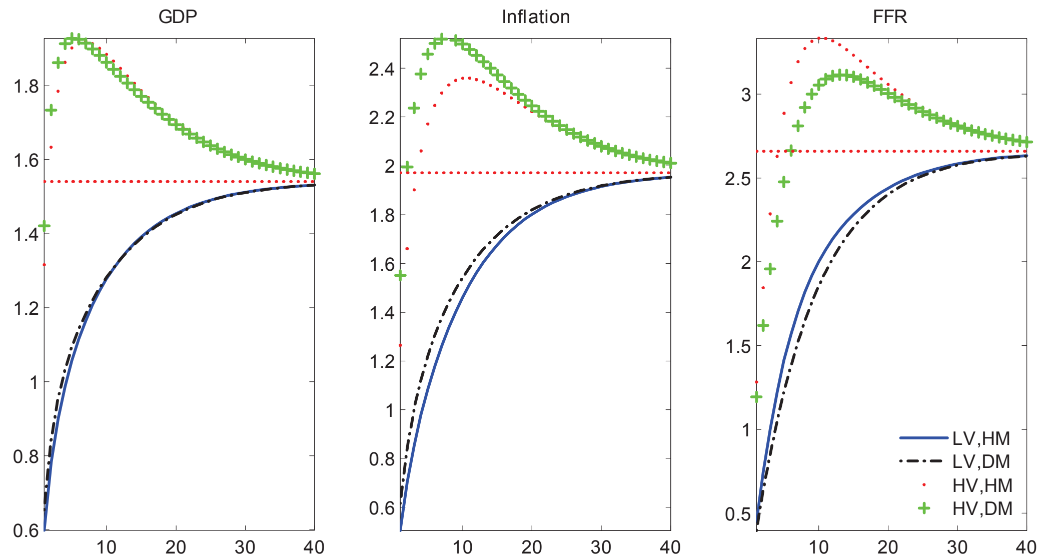


Figure 5: For each possible regime combination, the three panels report the evolution of uncertainty at different horizons. The measures of uncertainty take into account the possibility of regime changes. The horizontal line represents the ergodic steady state for uncertainty.

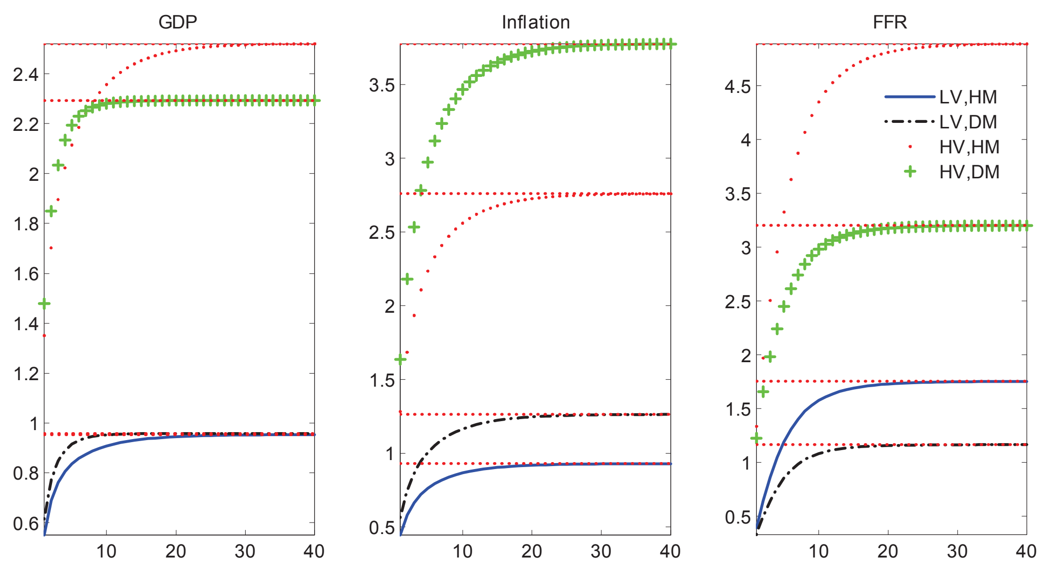


Figure 6: For each possible regime combination, the three panels report the evolution of uncertainty at different horizons. The measures of uncertainty are computed under the assumption that a particular regime combination is in place over the entire horizon. The horizontal lines represent the conditional steady states for uncertainty.

Conditional on an initial regime $\xi_1$, the evolution of uncertainty taking into account the possibility of regime changes can be computed using (5) and (8):

$$M_{s|1,\xi_1} = \varphi\left[\mathbb{E}_{1,\xi_1}\left(Z_s Z_s'\right)\right] = \widetilde{W}\widetilde{Q}_{1|1,\xi_1} = \widetilde{W}\widetilde{\Xi}^{s-1}\widetilde{Q}_{1|1,\xi_1}$$

$$\mathbb{E}_{1,\xi_1}\left(Z_s\right) = \mu_{s|1,\xi_1} = \widetilde{w}\widetilde{\Omega}^{s-1}\widetilde{q}_{1|1,\xi_1}$$

$$\varphi\left[\mathbb{V}_{1,\xi_1}\left(Z_s\right)\right] = M_{s|1,\xi_1} - \varphi\left[\mu_{s|1,\xi_1}\mu_{s|1,\xi_1}'\right] \tag{27}$$

for $1 \leq s \leq S$. The initial values $\widetilde{q}_{1|1,i}$ and $\widetilde{Q}_{1|1,\xi_1}$ are defined as follows

$$\widetilde{q}_{1|1,\xi_1} = \left[e_{\xi_1}' \otimes \overline{\mu}_{\xi_1}', e_{\xi_1}'\right]' \text{ and } \widetilde{Q}_{1|1,\xi_1} = \left[Q_{1|1,\xi_1}', K_{1|1,\xi_1}', \widetilde{q}_{1|1,\xi_1}'\right]'$$

where $Q_{1|1,\xi_1} = e_{\xi_1} \otimes \overline{M}_{\xi_1}$ and $K_{1|1,\xi_1} = e_{\xi_1} \otimes \varphi\left[I_k\right]$. As the time horizon $s$ increases, uncertainty measures will converge to their corresponding ergodic values, independently of the regime from which the economy started. The ergodic value for uncertainty can be derived in one step using (15). Finally, for the starting value, the conditional steady state $\mu_{s|1,\xi_1}$ could be replaced with the ergodic steady state, $\overline{\mu}$. In this case, the regime occurring at time one would become a shock itself. This is because volatility moves in response to a change in the probabilities assigned to different regimes.

Figure 5 reports the evolution of uncertainty for each of the possible starting regime combinations. When evaluating these measures of uncertainty, there are three effects that should be taken into account. First, events that are more distant in the future tend to be more uncertain because shocks cumulate and propagate over time. In fact, uncertainty is zero at time $t$, when agents can observe the variables of interest (we do not report time $t$ in the graphs). Second, different regimes have different implications for the magnitude of the shocks (high volatility vs. low volatility) and the way these shocks propagate through the economy (*Hawk* vs. *Dove*). Third, over time the initial regime becomes irrelevant, while the relative regimes' frequencies start to matter more as the regime probabilities converge to their ergodic probabilities. These three aspects are clearly reflected in the results. Notice that uncertainty tends to always increase in the beginning, as agents tend to forecast variables further into the future. However, when the low volatility regime is in place, uncertainty always remains below its ergodic value for all three variables. Instead, when the high volatility regime is in place uncertainty becomes hump-shaped, crosses the long-term ergodic value, and converges to it from above. It is interesting to note that the peak of uncertainty occurs fairly soon, around the two-year horizon. This is because uncertainty results from the combined effect of the relatively long horizon and the high probability of the high volatility regime. As the horizon increases, the probability of the high volatility regime keeps declining to finally converge to its ergodic value.

In many empirical papers, we are interested in evaluating the importance of policy makers' behavior vis-a-vis changes in the volatility of the exogenous disturbances. It is obvious that policy makers' behavior does not seem to have a dramatic impact on uncertainty across all horizons when the low volatility regime is in place, while it determines a visible reduction in uncertainty when the high volatility regime is in place. The importance of the monetary policy regime tends to disappear as the horizon

| | | *MP* | | | *DEM* | | | *SUP* | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Variable | *Std* | *LV* | *HV* | *TOT* | *LV* | *HV* | *TOT* | *LV* | *HV* | *TOT* |
| GDP | 1.54 | 0.32 | 2.88 | 3.20 | 21.38 | 50.46 | 71.84 | 4.92 | 20.03 | 24.95 |
| Inflation | 1.97 | 0.03 | 0.23 | 0.26 | 3.34 | 7.88 | 11.21 | 17.47 | 71.06 | 88.53 |
| FFR | 2.66 | 1.03 | 9.13 | 10.16 | 13.61 | 32.12 | 45.74 | 8.70 | 35.41 | 44.11 |

Table 2: For each variable, the table reports the ergodic standard deviation and the ergodic variance decomposition. For each shock, the total contribution is reported together with the contributions of the low and high volatility shocks.

increases because the probabilities of the different regimes converge to their ergodic values. While a more hawkish monetary policy leads to a reduction in the volatility of inflation, it also leads to an increase in the volatility of the monetary policy interest rate.

We might also be interested in characterizing the evolution of uncertainty conditional on a specific path for the regimes.[9] The law of motion for first and second moments can then be obtained using a simplified version of (8):

$$\left[\begin{array}{c} Q_{s|1,\xi^S} \\ q_{s|1,\xi^S} \end{array}\right] = \left[\begin{array}{c} \widehat{VV}_{\xi_s}\varphi\left(I_k\right) + \widehat{cc}_{\xi_s} \\ c_{\xi_s} \end{array}\right] + \left[\begin{array}{c|c} \widehat{AA}_{\xi_s} & \widehat{DAC}_{\xi_s} \\ \hline & A_{\xi_s} \end{array}\right] \left[\begin{array}{c} Q_{s-1|1,\xi^S} \\ q_{s-1|1,\xi^S} \end{array}\right], \; s>0 \qquad (28)$$

where the initial values are obtained as described above. Notice that the measures of uncertainty computed in this way do not reflect any regime uncertainty, given that we are conditioning on a specific path. Furthermore, if the same regime is assumed to be in place over the entire period, the horizon is long enough, and the regime is stationary when taken in isolation, then uncertainty will converge to its corresponding conditional steady state. For each of the stationary regimes, this can be obtained by taking the square root of (19).

Figure 6 reports this alternative measure of uncertainty. Now, regime uncertainty has been removed with the result that events further in the future are always more uncertain than events close in time. When comparing the two sets of results, it is clear that the monetary policy regime turns out to be much more important when disregarding the possibility of regime changes, especially for inflation volatility. When the low volatility regime is in place, the conduct of monetary policy is basically irrelevant for output volatility, but not for inflation and the FFR. A more hawkish monetary policy always implies an important reduction in inflation volatility and an increase in interest rate volatility. When the high volatility regime is in place, it also determines a visible increase in output volatility. Finally, comparing Figures 5 and 6, it is important to point out the upper bound for uncertainty is greatly reduced when the possibility of regime changes is taken into account. This is because even when the economy is under the most volatile regime, agents form expectations taking into account the possibility of moving to more favorable outcomes. These patterns will be important when considering the welfare implications of regime changes in subsection 4.7.

[9] If we were interested in computing expectations and uncertainty assuming a certain probability for different regimes being in place in the future, (5) and (8) could be easily modified to reflect a specific path for the regime probabilities.

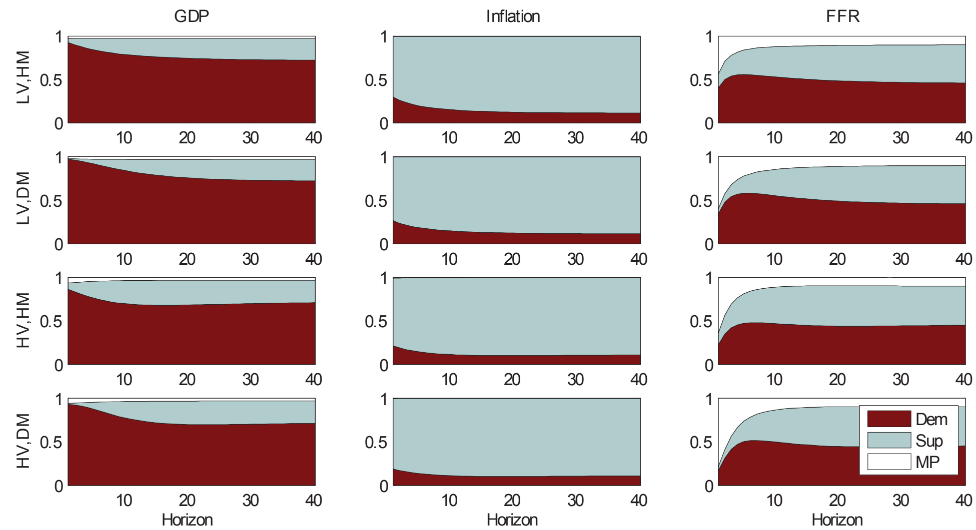


Figure 7: Variance decomposition taking into account the possibility of regime changes and conditioning on a specific *starting* regime. In the long run, the variance decomposition converges to the values reported in Table 2 under the column labeled $TOT$.

## 4.6 Variance decomposition

The previous section has shown how to characterize the dynamic evolution of uncertainty and its ergodic counterpart. However, in many economic applications it is also very important to be able to detect which shocks play a key role in determining such uncertainty. A variance decomposition is a simple way to summarize this information. This can be computed taking into account the possibility of regime changes or conditioning on a particular regime path.

In order to define the contribution of the *k-th* shock, occurring under the $\xi_t$-*th* regime, it is useful to define the $n \times k$ matrix $V_{\xi_t,k} = R_{\xi_t}\Sigma_{\xi_t,k}$, where $\Sigma_{\xi_t,k}$ is obtained restricting all coefficients of the diagonal matrix $\Sigma_{\xi_t}$ to zero except the *k-th* diagonal element. Suppose that the goal is to obtain a dynamic variance decomposition that takes into account the possibility of regime changes, only conditioning on an initial regime $\xi_1$. We can proceed in two steps. First, we can use (27) to derive the evolution of the variance associated with the shock of interest. Then, the contribution of the shock is obtained by dividing the variance associated with the matrix $V_{\xi_t,k}$ by the overall variance derived using the full matrix $V_{\xi_t} = R_{\xi_t}\Sigma_{\xi_t}$. For short horizons, the importance of the shocks will vary across regimes. However, as the time horizon goes to infinity, the importance of the initial regime declines and the variance decomposition converges to its ergodic counterpart. Notice that this can be obtained in one step using (15) to compute the ergodic variance associated with $V_{\xi_t,k}$ and dividing it by the overall ergodic variance implied by $V_{\xi_t}$.

Figure 7 reports the dynamic variance decomposition, while Table 2 illustrates the contribution of the shocks to the ergodic volatility. While for short horizons some differences across regimes can be detected, the convergence to the asymptotic values occurs quite fast, reflecting the fact that the regimes are not very persistent. For each shock, Table 2 reports the overall contribution (see the column labeled

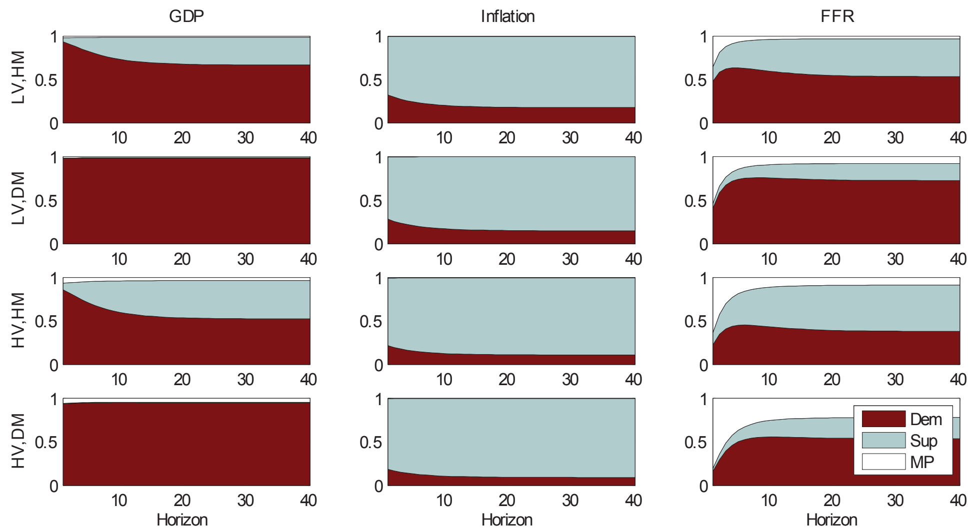


Figure 8: Variance decomposition conditional on a specific regime being in place over the entire horizon. In the long run, the variance decomposition converges to the values reported in Table 3.

| | *Hawk* | | | | *Dove* | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Variable | *Std* | $MP$ | $DEM$ | $SUP$ | *Std* | $MP$ | $DEM$ | $SUP$ | *Volatility* |
| GDP | 2.52 | 3.49 | 52.57 | 43.94 | 2.29 | 4.53 | 94.86 | 0.61 | |
| Inflation | 2.76 | 0.38 | 10.99 | 88.62 | 3.78 | 0.22 | 9.11 | 90.67 | *High* |
| FFR | 4.89 | 8.77 | 37.96 | 53.28 | 3.20 | 21.95 | 53.36 | 24.69 | |
| GDP | 0.95 | 1.17 | 66.56 | 32.27 | 0.96 | 1.25 | 98.39 | 0.37 | |
| Inflation | 0.93 | 0.16 | 17.59 | 82.25 | 1.26 | 0.09 | 14.75 | 85.16 | *Low* |
| FFR | 1.75 | 3.27 | 53.32 | 43.41 | 1.17 | 7.92 | 72.59 | 19.48 | |

Table 3: For each variable, the table reports the standard deviation and the variance decomposition conditioning on each of the possible regime combinations.

$TOT$) and the relative contribution of the shocks that occur under the high and low volatility regimes. The results show that supply shocks are very important for inflation volatility, while GDP volatility is mostly explained by demand shocks. The shocks that occur under the high volatility regime explain a larger fraction of the overall volatility. Notice this does *not* have to be the case: If the high volatility regime has a much lower persistence than the low volatility regime, then the result would be reverted. In other words, it is not only the size of the shocks that matter, but also the frequency with which a specific regime occurs. Furthermore, in some applications the sequence of the regimes might be important as well. For example, when a policy makers' behavior is always triggered by a specific state of the world, the overall volatility is going to be affected. The methods illustrated in this paper take these aspects into account.

It might also be interesting to assess what the contributions of the shocks would be conditional on a specific regime path or if the economy were to spend a prolonged period of time in a specific regime. In order to do this, it is enough to replace $V_{\xi_t}$ with $V_{\xi_t,k}$ and compute the evolution for first and second moments described by the law of motion (28). We can then compute the shock contribution at different horizons, taking the ratio between the variance implied by the single shock and the total variance. If all regime combinations are stationary when taken in isolation, we can also compute the long-run variance decomposition conditional on a specific regime. In this case, we can use (19) to compute the conditional steady-state value of the variance when only the *k-th* shock is active and divide it by the overall conditional steady-state variance, still obtained using (19).

The results, reported in Figure 8, assume that a single regime is in place over the entire horizon. Fixing the sequence of regimes has important effects on the contribution of the shocks. The most notable difference with respect to Figure 7 is obtained when imposing the *Dove* regime over the entire sample. In this case, no matter the volatility regime in place, GDP is substantially unaffected by supply shocks at each horizon. As the horizon increases, the variance decomposition stabilizes around the long-run values obtained using (19). The long-run variance decomposition is reported in Table 3. As pointed out before, the monetary policy regime plays an important role for the volatility of inflation, while the impact on output volatility is much more modest (see the column labeled *Std*). The table confirms that when the central bank behaves according to the *Hawk* regime, the contribution of the supply shocks to the overall volatility of output is quite large. On the other hand, when the *Dove* regime is in place, the effect of the shock is almost totally absorbed by inflation and the contribution of supply shocks to output volatility is close to zero, in line with what is illustrated in Figure 4. This is because under this scenario agents disregard the possibility of moving to the *Hawk* regime.

Summarizing, in this section I have illustrated two alternative ways to characterize the sources of uncertainty in an MS model. If the researcher is interested in describing the properties of the different regimes, it might be useful to condition on a specific regime being in place for a prolonged period of time. If instead the goal is to capture the effective level of uncertainty, it is more appropriate to take into account regime changes. In general, the two approaches might return quite different results, and it is therefore important to understand what the goal of the analysis is. Finally, it is important to keep in mind that if the model also includes changes in the constant, such changes would represent an additional source of volatility. This implies that even when restricting all Gaussian shocks to zero, there will still be uncertainty deriving from the Markov-switching innovations in the constant. Section 5 illustrates this point.

## 4.7 Welfare calculations

In this section, I show how the formulas presented in this paper can be used to correctly characterize the welfare implications of the different regimes. Following Rotemberg and Woodford (1999), Woodford (2003), and Gali (2008), the period welfare loss is obtained by taking a log-quadratic approximation of the representative household's utility function:[10]

$$\mathbb{L}_t = \sum_{s=0}^{\infty} \beta^s \left[ \mathbb{E}_t \left( \Delta p_{t+s}^2 \right) + (\kappa/\upsilon) \, \mathbb{E}_t \left( y_{t+s}^2 \right) \right] \tag{29}$$

[10] Recall that regime changes do not affect the unique deterministic steady state, but only the way shocks propagate around it.

where $\upsilon$ is the elasticity of substitution between two differentiated goods. The output gap enters the welfare function because it reflects the difference between the marginal rate of substitution and the marginal product of labor, which is a measure of the economy's aggregate inefficiency. Inflation deviations from its steady-state level reduce welfare by raising price dispersion. The elasticity of substitution between two differentiated goods $\upsilon$ raises the weight of inflation fluctuations relative to the output gap because it amplifies the welfare losses associated with any given price dispersion. Nominal rigidities, whose size is inversely related to the slope of the New Keynesian Phillips curve $\kappa$, raise the degree of price dispersion resulting from any given deviation from the steady-state inflation rate. In what follows, I fix the value of $\upsilon$ to 6, a value in line with what is used in the literature. However, given the estimated low value for the slope of the Phillips curve $\kappa$, the results are robust to different values for this parameter.

In order to compute the welfare loss taking into account the possibility of regime changes, recall that, for a variable $x_{t+s}$, the evolution of second moments is pinned down by:

$$\mathbb{E}_t \left[x_{t+s}^2\right] = e_x M_{t+s|t} = e_x \widetilde{W} \widetilde{\Xi}^s \widetilde{Q}_{t|t}$$

Therefore, the welfare loss (29) becomes (see Appendix E):

$$\mathbb{L}_t = \left(e_{\Delta p} + (\kappa/\varepsilon)\, e_y\right) \widetilde{W} \left(I - \beta\widetilde{\Xi}\right)^{-1} \widetilde{Q}_{t|t} \tag{30}$$

where the column vectors $e_{\Delta p}$ and $e_y$ select the appropriate elements. It is worth pointing out that this way of calculating welfare takes into account uncertainty around the regime that is in place, the current state of the economy, and the possibility of regime changes. In the long run, the second moments converge to their ergodic steady states, while the first moments converge to zero. Therefore at long horizons, welfare is determined by the model ergodic variance. This is in line with standard results in the literature about welfare calculations in new-Keynesian models. In fact, if we were only interested in computing welfare conditional on a specific initial regime, the law of motion of the second moments would coincide with the law of motion of the variance at each point in time, given that the first moments are in this case always zero. Finally, if the goal is to compute an unconditional measure of welfare, the different regimes can be weighed according to their ergodic probabilities. This is the approach taken by Bianchi and Melosi (2012a) to analyze the welfare implications of central bank reputation and transparency.

The first row of Table 4 reports the (rescaled) welfare losses conditional on a specific regime combination being in place at time $t$ and assuming that the economy starts from the ergodic steady state. In order to obtain such a measure of welfare loss, it is enough to replace $\widetilde{Q}_{t|t}$ in (30) with $\widetilde{Q}_{1|1,\xi_1}$ as defined in Subsection 4.5 and assume $\widetilde{q}_{1|1,\xi_1} = 0$. Given the large weight assigned to the squared deviations of inflation in the loss function and the results shown in Subsection 4.5, it is not surprising that the worst case scenario corresponds to the case in which the high volatility regime is combined with dovish monetary policy. To facilitate the interpretation of the results, for each alternative regime combination, the second row of the table reports the percentage change in the welfare loss with respect to this worst

| | $DM, HV$ | $DM, LV$ | $HM, HV$ | $HM, LV$ |
|---|---|---|---|---|
| Benchmark: Loss*100 | 0.5551 | 0.5051 | 0.5466 | 0.5025 |
| Benchmark: % change | – | −9.01 | −1.54 | −9.48 |
| Anticipated: Loss*100 | 1.8630 | 0.2118 | 1.0315 | 0.1188 |
| Anticipated: % change | – | −88.63 | −44.63 | −93.63 |

Table 4: For each regime combination, the table reports the rescaled welfare loss and the percentage change with respect to the worst case scenario in which volatility is high and monetary policy is dovish. The first two rows refer to the benchmark case in which agents take into account the possibility of regime changes, the third and fourth rows assume that agents disregard the possibility of regime changes. This corresponds to the anticipated utility assumption.

case scenario. The results confirm that a change in the volatility of the exogenous shocks has a large impact on welfare, with a reduction in welfare loss of more than 9%. On the other hand, moving from the *Dove* regime to the *Hawk* regime is much less effective, with a reduction in welfare loss of 1.54%. The third and fourth rows of Table 4 repeat the same exercise under the anticipated utility assumption according to which agents disregard the possibility of regime changes. Notice that in this second case regime changes turn out to be substantially more important and monetary policy plays a key role for agents' welfare when the high volatility regime is in place.

Figure 9 reports the historical evolution of welfare losses based on the posterior mode estimates. In both panels, the solid blue line measures the effective loss, given that it takes into account both the state of the economy and the regime probabilities at each point in time. The dashed red line in the first panel reports the evolution of the welfare loss that is implied by the regime probabilities, assuming that the economy starts from the ergodic steady states. This can be considered a useful benchmark to capture the relative contribution of the regime combination with respect to the state of the economy. Two lessons can be learned. First, the actual state of the economy played a very large role in determining the extent of the welfare losses in correspondence to the two run-ups in inflation that occurred in the mid and late '70s. Second, the losses implied by the regime combination closely track the occurrence of the high volatility regime.

The second panel of Figure 9 compares the benchmark historical welfare losses (solid blue line) with the alternative measure of welfare in which agents are assumed to disregard regime changes (red dashed line). This second case corresponds to the assumption of anticipated utility and clearly leads to overstating the welfare consequences of the regime in place at each point in time and the effects of changes in monetary policy. Both sets of results are consistent with the evidence presented in Table 4. Finally, it is worth emphasizing that the results presented in this subsection are perfectly in line with the evolution of uncertainty under the two different assumptions presented in Figures 5 and 6.

Summarizing, in this subsection I have shown how welfare can be characterized in models in which agents are aware of regime changes. This is not a matter of secondary importance, because, as shown in subsection 4.5, measures of medium- and long-run uncertainty change substantially when taking into account the possibility of regime changes. In this context, the contribution of the different regimes to the overall volatility does not depend only on the size of the shocks or on policy makers' behavior, but

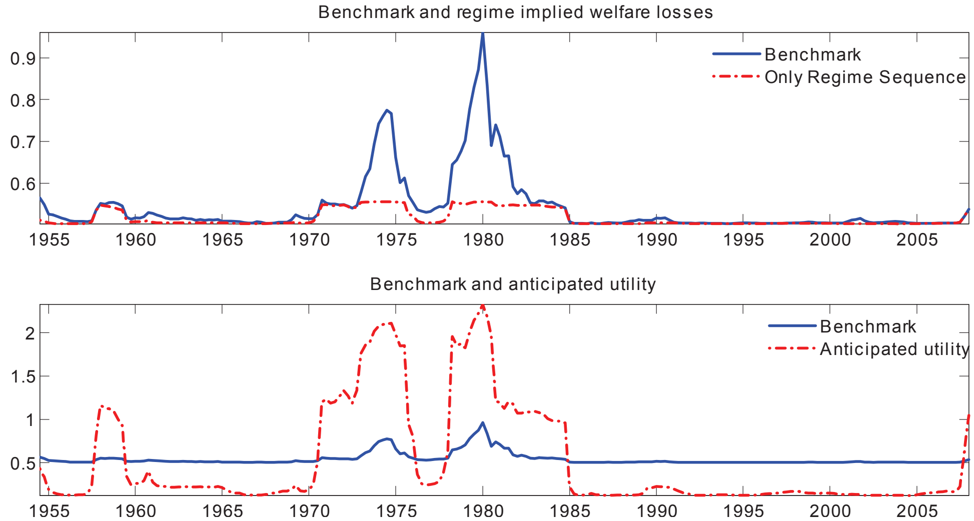


Figure 9: Historical welfare losses. In both panels, the solid blue line plots the benchmark historical welfare losses that take into account the possibility of regime changes and the state of the economy. In the first panel, the benchmark case is compared with an alternative measure that reflects the welfare losses implied by the regimes themselves (red dashed line). In the second panel, the benchmark case is compared with an alternative measure that assumes that agents disregard the possibility of regime changes (red dashed line). This second case corresponds to the assumption of anticipated utility.

also on the shocks' frequency and persistence. As a result, the importance of the regime that is in place at a particular point in time is substantially reduced when taking into account the possibility of regime changes. If welfare were computed assuming a regime in place for a prolonged period of time, then more substantial differences would arise, but this would be completely misleading. In other words, it is not enough to account for the size and the contemporaneous impact of the shocks when evaluating welfare because agents are likely to take into account the possibility of moving across regimes.

# 5 Applications for Uncertainty and Asset Pricing

In this last section, I describe two additional applications. The first application shows how MS models can generate interesting interactions between uncertainty and the endogenous variables of the model, while the second one describes how to use the methods of this paper to generalize Campbell's (1991) VAR implementation of Campbell and Shiller's (1988) present value decomposition.[11]

## 5.1 Regime changes and uncertainty

Since the seminal contribution by Bloom (2009), the need for modeling the link between real activity and uncertainty has become more and more acknowledged in the profession. As highlighted in Section 4.5, the methods described in this paper allow for a convenient characterization of agents' uncertainty

[11] See Ang and Timmermann (2012) for a comprehensive review of the use of Markov-switching models in finance. See Bansal, Tauchen, and Zhou (2004) for a multivariate MS term structure model.

|  | $c_\xi$ | $A_\xi$ | | $H_{\xi\xi}$ | $V_\xi$, Sim. 1 | $V_\xi$, Sim. 2 | |
|---|---|---|---|---|---|---|---|
| $\xi = 1$ | .1 | 0.9 | $-.1$ | .90 | 0 | .1 | 0 |
|  | .2 | .1 | .7 |  |  | $-.05$ | .3 |
| $\xi = 2$ | .3 | 0.5 | $-.1$ | .95 | 0 | .05 | 0 |
|  | .1 | .1 | .9 |  |  | .02 | .1 |

Table 5: Parameter values for an MS-VAR with switches in all parameters. Two calibrations are considered: with and without Gaussian shocks.

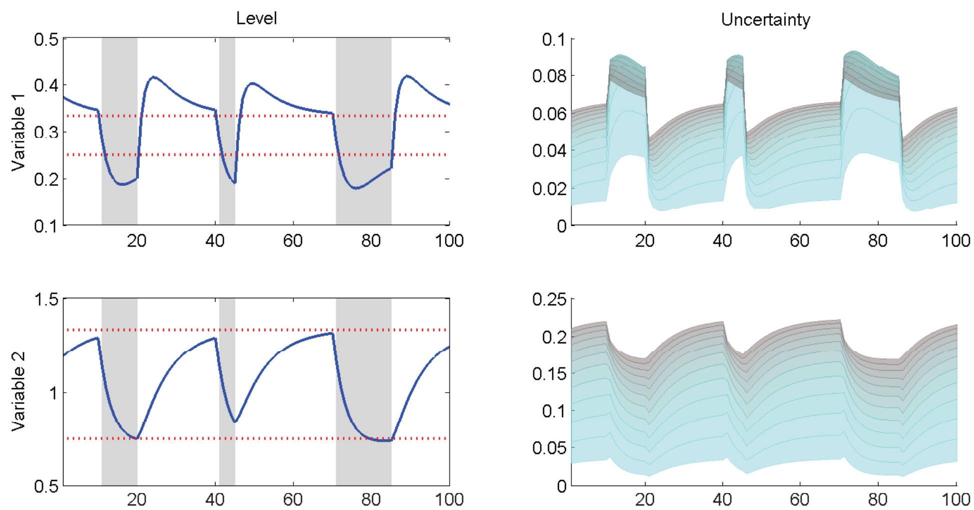


Figure 10: Evolution of levels and standard deviations for a MS-VAR with changes in the constant $c_\xi$ and the autoregressive matrix $A_\xi$. The values for $A_\xi$, $c_\xi$, and the diagonal elements of the transition matrix $H$ are reported in Table 5. The matrices $V_\xi$ are restricted to zero in this simulation (Sim 1). The gray areas mark the occurrence of Regime 1, the least persistent regime.

that takes into account the possibility of changes in the structure of the economy. In this section, I extend the analysis to show how an MS-VAR can generate interesting interactions between the levels and the volatility of the variables of interest. Specifically, I consider a bivariate Markov-switching VAR with one lag in which the constant is also moving over time. The parameter values are reported in Table 5. I consider two calibrations. In the first one, all Gaussian shocks are set to zero, imposing the constraint $V_{\xi_t} = 0$, while in the second calibration a lower triangular structure is assumed.

Figure 10 simulates the model under the first calibration in which the matrices $V_{\xi_t}$ are restricted to zero. The left panel reports the evolution of the two variables. The horizontal dashed red lines report the conditional steady states computed according to (17), while the gray areas mark the occurrence of Regime 1, the less persistent of the two regimes. The right panel reports the corresponding evolution of uncertainty, measured by $sd_t\left(X_{t+s}\right) = \sqrt{\mathbb{V}_t\left(X_{t+s}\right)}$, where $\mathbb{V}_t\left(X_{t+s}\right)$ is computed using (11). The time horizon runs from 1 to 12 periods, with the light blue areas denoting the short run. The simulation considers three occurrences of Regime 1. The three deviations last ten, five, and fifteen periods, respectively. Given that the expected duration of Regime 1 is ten periods, its first occurrence is representative of the typical realization of this regime, whereas the second and third realizations are unusually short

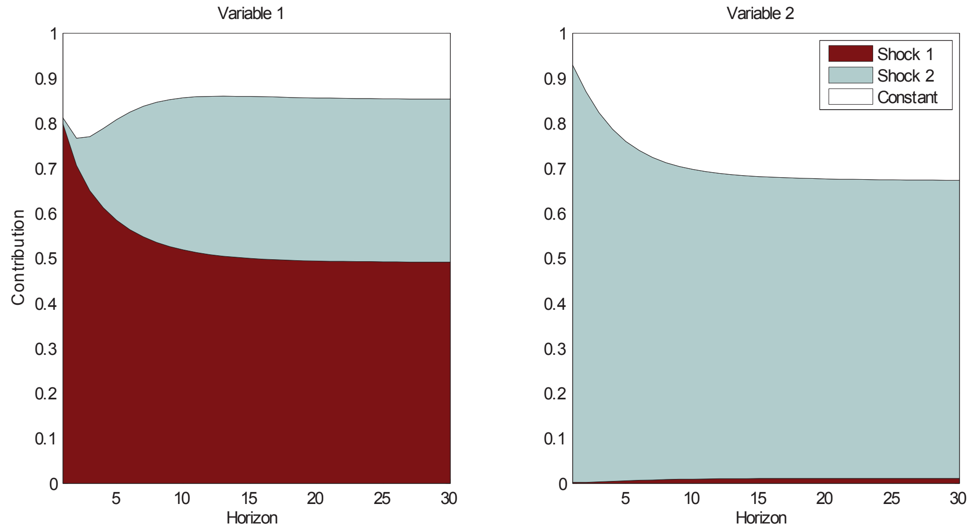


Figure 11: Dynamic variance decomposition for an MS-VAR in which all parameters change. The values for the parameters are reported in Table 5 (Sim 2).

and unusually long, respectively.

Several properties of the model are worth commenting on. First, as a result of the switches in the constant, uncertainty is still positive for both variables, even if all Gaussian shocks have been restricted to zero. This illustrates that the possibility of movements in the constant generates uncertainty. Second, whenever Regime 1 occurs, we observe a quite sharp drop in the first variable associated with an increase in its volatility. Notice that uncertainty moves faster than the variable itself. Uncertainty increases sharply at each horizon and then starts declining as more time is spent under Regime 1. It is also possible to detect a hump shape in the evolution of uncertainty with respect to the time horizon, with medium-run uncertainty larger than short- and long-run uncertainty. Similarly, when the system moves back to Regime 2, the decline in uncertainty occurs before the variable reaches its peak. These two facts determine a strong negative correlation between the level and the *one-period-lagged* measures of uncertainty: from $-.95$ to $-.85$ depending on the time horizon for uncertainty. The ability of an MS model to generate this kind of dynamics is intriguing in light of the documented link between uncertainty and real activity. Third, the behavior of the two variables shows an interesting asymmetry. The first variable overshoots before approaching its conditional steady states, while the second variable tends to bounce between them. Furthermore, the first variable generally does not reach the conditional steady states, especially when the less persistent Regime 1 occurs. This result highlights the importance of simulating an MS model. Only by studying the typical behavior of the model is it possible to uncover the interaction between the regime persistences and the statistical properties of the regimes themselves. Finally, it is worth emphasizing that uncertainty keeps moving even when no regime changes occur. This is because in an MS model the evolution of uncertainty depends on the starting value $Z_t$. As pointed out in Subsection 2.3, this is not the case in a model with fixed coefficients.

Figure 11 further elaborates on the role played by an MS constant in determining uncertainty. We now allow for both Gaussian and Markov-switching shocks (Sim. 2 in Table 5). For each variable, the

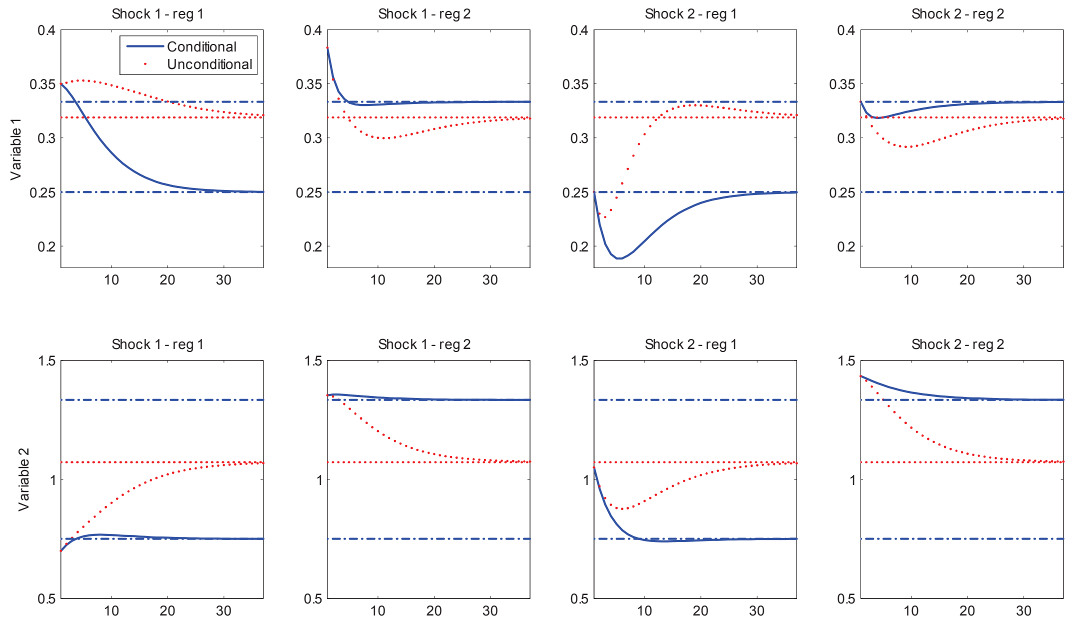


Figure 12: Impulse responses based on a two-variable MS-VAR. The starting point is always given by one of the two conditional steady states (dashed blue horizontal lines). Then, two impulse responses are considered: Unconditional and conditional on the same regime being in place over the entire horizon. In the former case, the economy converges to the ergodic mean represented by the dotted red horizontal line.

figure reports the dynamic variance decomposition assuming that the economy starts from the ergodic steady state. The contribution of the Gaussian shocks to the overall uncertainty is obtained using (11) twice. First, set the constants to zero and replace $V_{\xi_t}$ with the matrix $V_{\xi_t,k}$ obtained excluding all but the *k-th* shock. Then consider the law of motion for the overall variance. The ratio between these two quantities is used to compute the shock contributions. The contribution of the MS constant is obtained using a similar procedure, with the only difference being that the constants are unrestricted, while all the Gaussian shocks are set to zero. The figure highlights the fact that even when Gaussian shocks are allowed for, changes in the constant can explain a sizeable fraction of the overall volatility at all horizons.

Figure 12 reports impulse responses for this second calibration. Impulse responses are computed assuming that the economy starts from one of the two conditional steady states.[12] In each row, the first two panels refer to the first shock, while the remaining two consider the second shock. For each of the four pairs of panels, the first panel assumes that initially Regime 1 is prevailing, while the second panel assumes that Regime 2 is prevailing. In each case, the dotted red line shows the evolution of the economy conditional on the starting regime prevailing, while the solid blue line represents the impulse response taking into account the possibility of regime changes.

The calibration is chosen to illustrate how all sorts of outcomes can arise. For example, under Regime 1, following Shock 1, the first variable, associated with the lowest conditional steady state, can

[12]This is possible because the two regimes are both stable when taken in isolation.

increase to the point of overshooting with respect to the largest conditional steady state. The dynamics that follow are then substantially different depending on whether the regime that was in place at the time of the shock is assumed to be in place over the entire horizon. As for the second variable, the shock implies only very small fluctuations when assuming that the regime will stay constant, while when taking into account the possibility of regime changes, we observe significant swings. This does not have to always be the case, as under some circumstances the shock brings the endogenous variables closer to the ergodic steady state, as is the case for the second variable in response to Shock 2 under Regime 1.

### 5.2 Present value decomposition

In their seminal contribution Campbell and Shiller (1988) propose a loglinear approximation to study movements in the dividend price ratio as a result of changes in forecasts about future cash flows and future discount rates. Campbell (1991) uses this framework and a VAR to decompose market returns into cash-flow news and discount-rate news. Since then, the idea has spread beyond the boundaries of the asset pricing literature. For example, building on Campbell and Shiller's (1988) methods, Gourinchas and Rey (2007) show that current trade imbalances must be offset by future improvements in trade surpluses, or excess returns on the net foreign portfolio, or both.

At the same time, it is often the case that the relations between the variables of interest are not stable over time. For example, in financial markets we can observe a smooth and prolonged increase in stock prices followed by a sudden crash. Alternatively, when dealing with macroeconomic data, we might want to allow for the possibility of prolonged periods of persistent and high inflation followed by a painful disinflation. Agents are likely to keep the possibility of these scenarios in mind when forming expectations. Therefore, it would seem desirable to be able to extend Campbell's (1991) VAR implementation of Campbell and Shiller's (1988) present value decomposition to analyze economic environments subject to parameter instability. In this section, I show how this can be done for the case in which the law of motion of the state variables is described by a Markov-switching VAR. I derive my results in the context of the "Bad Beta, Good Beta" asset pricing model that has been introduced by Campbell and Vuolteenaho (2004). However, the approach shown here can be applied to any model in which the present value of a variable depends on agents' expectations about future outcomes that can be modeled using an MS-VAR.

Campbell and Vuolteenaho (2004) point out that returns on the market portfolio can be split into two components. An unexpected change in excess returns can be determined by news about future cash flows or by a change in the discount rate that investors apply to these cash flows. While a fall in expected cash flows is simply bad news, an increase in discount rates implies at least an improvement in future investment opportunities. This means that the single capital asset pricing model (CAPM) beta can be decomposed into two sub-betas: one reflecting the covariance with news about future cash flows (bad beta), the other linked to news about discount rates (good beta). The previous argument suggests that given two assets with the same CAPM beta, the one with the highest cash-flow beta should have a larger return. Using an intertemporal CAPM along the lines of the one proposed by Merton (1973), it can be shown that the price of risk for the discount-rate beta should equal the variance of the market return, while the price of risk for the cash-flow beta should be $\gamma$ times greater, where $\gamma$ is the investor's coefficient of relative risk aversion. Campbell and Vuolteenaho (2004) show that the Bad Beta, Good Beta ICAPM is able to explain the cross section of asset returns.

Campbell and Vuolteenaho (2004) use a VAR to model the expectation formation mechanism and to derive the news. Using the loglinear approximation for returns introduced by Campbell and Shiller (1988), unexpected excess returns can be approximated by:

$$r_{t+1} - \mathbb{E}_t r_{t+1} = \underbrace{(\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j}}_{N_{CF,t+1}} - \underbrace{(\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}}_{N_{DR,t+1}} \tag{31}$$

where $r_{t+1}$ is a log stock market excess return, $d_{t+1}$ is the log dividend paid by the stock, $\Delta$ denotes a one period change, and $\rho < 1$ is the discount coefficient. $N_{CF,t+1}$ and $N_{DR,t+1}$ represent news about future market cash flows and news about future market discount returns, respectively. Following Campbell (1991), agents' expectations are modeled by a VAR: $Z_{t+1} = c + AZ_t + u_{t+1}$, where $Z_t$ is a vector of state variables with the excess return ordered first and $u_{t+1} = V\varepsilon_{t+1}$, $\varepsilon_{t+1} \sim N(0, I)$ is the vector of reduced-form residuals. The two types of news can then be obtained according to the following transformation of the residuals:

$$r_{t+1} - \mathbb{E}_t r_{t+1} = e_1' u_{t+1} \tag{32}$$

$$N_{CF,t+1} = \left(e_1' + e_1'\lambda\right) u_{t+1} \tag{33}$$

$$N_{DR,t+1} = e_1' \lambda u_{t+1} \tag{34}$$

where $\lambda = \rho A (I - \rho A)^{-1}$ and $e_1' = [1, 0, ..., 0]'$.

Suppose that the law of motion of the financial variables is instead given by an MS-VAR:

$$Z_{t+1} = c_{\xi_{t+1}} + A_{\xi_{t+1}} Z_t + u_{t+1}$$

where $u_{t+1} = R_{\xi_{t+1}} \Sigma_{\xi_{t+1}} \varepsilon_{t+1}$, $\varepsilon_{t+1} \sim N(0, I)$ is the vector of reduced-form residuals, and the unobserved state $\xi_t$ can take a finite number of values and evolves according to the transition matrix $H$. Appendix F shows that when MSS holds, given a sequence of probabilities $\pi^T$ or a posterior draw for the regime sequence $\xi^T$, it is straightforward and computationally efficient to compute the entire sequences of discount rate news and cash-flow news in one step:

$$N_{DR}^T = e_1' w \left[\lambda^q v^{q,T} + \lambda^\pi v^{\pi,T}\right] \tag{35}$$

$$N_{CF}^T = e_1' w \left[(I_r + \lambda^q) v^{q,T} + \lambda^\pi v^{\pi,T}\right] \tag{36}$$

$$u^T = e_1' w v^{q,T} \tag{37}$$

where $\lambda^q = (I_{nm} - \rho\Omega)^{-1} \rho\Omega$, $\lambda^\pi = (I_{nm} - \rho\Omega)^{-1} \rho C H (I_r - \rho H)^{-1}$, $v_t^q = q_{t+1|t+1} - q_{t+1|t}$, and $v_{t+1}^\pi = \pi_{t+1|t+1} - \pi_{t+1|t}$. It is worth emphasizing that now news has two components. The first one is represented by the standard Gaussian innovation, while the second component derives from the revision in beliefs about the regime that is in place: $v^{\pi}_{t+1} = \pi_{t+1|t+1} - \pi_{t+1|t}$. For a given Gaussian innovation, the change in beliefs determines a change in the way the shocks are mapped into the future. When the two regimes coincide, formulas (35)-(37) collapse to (32)-(34). So the above formulas can be treated as a generalization of the ones used in Campbell and Vuolteenaho (2004).

It is worth pointing out that the approach described above can model situations in which not all $m$ regimes are stable. This is because in order to be able to compute the news we only need the discounted expectations to be stable. Mean square stability guarantees stability for first and second moments. Notice that this is in fact more than what is actually necessary for two reasons. First, the VAR implementation does not require the variance to be stable, but only that agents' expectations converge. Second, even if first moments are not stable, discounted first moments might be. However, it might be argued that imposing MSS is still desirable, given that it implies that agents' uncertainty converges to a finite value no matter the regime that is in place today.

Summarizing, the MS approach seems very appealing when thinking about modeling agents' expectations formation mechanism because it allows for temporary deviations from the stationarity assumption. Instead, this assumption is assumed to hold at each point in time when computing the news in a fixed coefficient framework. At the same time, the MS approach retains the convenience of having analytical expressions for the news. In other words, numerical integration is not necessary. This is a key ingredient if an economist were interested in jointly estimating the MS-VAR and the news series using GMM (Hansen (1982)). I regard this as a promising area for future research.

# 6 Conclusions

In this paper, I develop a toolbox for multivariate Markov-switching models. The building blocks are represented by the laws of motion for the first and second moments of the endogenous variables. Once these have been derived, they can be used to characterize the evolution of expectations and uncertainty, taking into account the possibility of regime changes. If mean square stability holds, these objects converge to finite values as the time horizon goes to infinity. In this case, it is possible to derive analytical formulas for the ergodic steady state and uncertainty.

The results can then be used to derive a series of objects of interest such as the historical evolution of expectations and uncertainty, impulse responses, the dynamic evolution of uncertainty, and variance decompositions. In the context of a general equilibrium model in which agents are aware of regime changes, these formulas provide the building blocks to derive the welfare implications of regime changes and to describe the historical evolution of welfare losses. Furthermore, the methods can be used to derive the joint evolution of the state variables and uncertainty. Finally, under the assumption of mean square stability, it is possible to extend Campbell's (1991) VAR implementation of Campbell and Shiller's (1988) present value decomposition to the case in which the law of motion of the state variables is described by a Markov-switching VAR.

Along the way, I have highlighted why this toolbox will be very useful for economists. First, all results are derived analytically, implying that agents' expectations and uncertainty can be computed repeatedly without requiring numerical integration. Therefore, the measures of expectations and volatilities can be easily included in an estimation exercise. Second, the toolbox allows the relationship between the levels and the volatilities of the endogenous variables to be characterized in a very parsimonious way. Third, the toolbox can be used to construct forecasts that take into account the possibility of changes in the macroeconomic environment or that allow for temporary deviations from the assumption of stationarity.

## References


Ang, A., and G. Bekaert (2002): "Regime Switches in Interest Rates," *Journal of Business & Economic Statistics*, 20(2), 163–82.

Ang, A., and A. Timmermann (2012): "Regime Changes and Financial Markets," *Annual Review of Financial Economics*, 4(1), 313–337.

Bansal, R., G. Tauchen, and H. Zhou (2004): "Regime Shifts, Risk Premiums in the Term Structure, and the Business Cycle," *Journal of Business & Economic Statistics*, 22, 396–409.

Bianchi, F. (2013): "Regime Switches, Agents' Beliefs, and Post-World War II U.S. Macroeconomic Dynamics," *Review of Economic Studies*, 80(2), 463–490.

Bianchi, F., C. Ilut, and M. Schneider (2012): "Business Cycles and Asset Prices: The Role of Volatility Shocks under Ambiguity Aversion," Duke University-Stanford University, working paper.

Bianchi, F., and L. Melosi (2012a): "Constrained Discretion and Central Bank Transparency," working paper.

——— (2012b): "Modeling the Evolution of Public Expectations and Uncertainty in General Equilibrium," Duke University, working paper.

——— (2013a): "Dormant Shocks and Fiscal Virtue," 2013 NBER Macroeconomics Annual, forthcoming.

——— (2013b): "Escaping the Great Recession," working paper.

Bianchi, F., H. Mumtaz, and P. Surico (2009): "The Great Moderation of the Term Structure of U.K. Interest Rates," *Journal of Monetary Economics*, 56, 856–871.

Bloom, N. (2009): "The Impact of Uncertainty Shocks," *Econometrica*, 77, 623–685.

Campbell, J. Y. (1991): "A Variance Decomposition for Stock Returns," *Economic Journal*, 101(405), 157–179.

Campbell, J. Y., and R. J. Shiller (1988): "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies*, 1, 195–228.

Campbell, J. Y., and T. Vuolteenaho (2004): "Bad Beta, Good Beta," *American Economic Review*, 94, 1249–1275.

Chib, S. (1996): "Calculating Posterior Distributions and Model Estimates in Markov Mixture Models," *Journal of Econometrics*, 75, 79–97.

Cho, S. (2012): "Characterizing Markov-Switching Rational Expectations Models," working paper.

Clarida, R., J. Gali, and M. Gertler (2000): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115, 147–180.

Cogley, T., and T. J. Sargent (2006): "Drifts and Volatilities: Monetary Policies and Outcomes in the Post WWII U.S.," *Review of Economic Dynamics*, 8, 262–302.

Costa, O., M. Fragoso, and R. Marques (2004): *Discrete-Time Markov Jump Linear Systems*. Springer, New York.

Davig, T., and T. Doh (2013): "Monetary Policy Regime Shifts and Inflation Persistence," The Review of Economics and Statistics, forthcoming.

Davig, T., and E. M. Leeper (2007): "Generalizing the Taylor Principle," *American Economic Review*, 97(3), 607–635.

De Graeve, F., M. Emiris, and R. Wouters (2009): "A Structural Decomposition of the US Yield Curve," *Journal of Monetary Economics*, 56(4), 545–559.

Del Negro, M., and S. Eusepi (2011): "Fitting Observed Inflation Expectations," *Journal of Economic Dynamics and Control*, 35(12), 2105–2131.

Farmer, R. E. A., D. Waggoner, and T. Zha (2009): "Understanding Markov-Switching Rational Expectations Models," *Journal of Economic Theory*, 144, 1849–1867.

Fernandez-Villaverde, J., P. Guerron-Quintana, and J. F. Rubio-Ramirez (2010): "Fortune or Virtue: Time-Variant Volatilities Versus Parameter Drifting in U.S. Data," NBER Working Papers 15928, National Bureau of Economic Research, Inc.

Fernandez-Villaverde, J., J. F. Rubio-Ramirez, P. Guerron-Quintana, and M. Uribe (2011): "Risk Matters. The Real Effects of Volatility Shocks," *American Economic Review*, 101(3), 2530–2561.

Foerster, A., J. F. Rubio-Ramirez, D. Waggoner, and T. Zha (2010): "Perturbation Methods for Markov-Switching Models," working paper.

Gali, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*. Princeton University Press.

Gourinchas, P.-O., and H. Rey (2007): "International Financial Adjustment," *Journal of Political Economy*, 115(4), 665–703.

Hamilton, J. D. (1989): "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle," *Econometrica*, 57, 357–384.

Hansen, L. P. (1982): "Large Sample Properties of Generalized Method of Moments Estimators," *Econometrica*, 50, 1029–1054.

Justiniano, A., and G. Primiceri (2008): "The Time Varying Volatility of Macroeconomic Fluctuations," *American Economic Review*, 98(3), 604–41.

Liu, Z., D. Waggoner, and T. Zha (2011): "Sources of the Great Moderation: A Regime-Switching DSGE Approach," *Quantitative Economics*, 2(2), 251–301.

Lubik, T., and F. Schorfheide (2004): "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *American Economic Review*, 94(1), 190–217.

Merton, R. (1973): "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41(4), 867–887.

Pesaran, M. H., D. Pettenuzzo, and A. Timmermann (2006): "Forecasting Time Series Subject to Multiple Structural Breaks," *Review of Economic Studies*, 73, 1057–1084.

Primiceri, G. (2005): "Time Varying Structural Vector Autoregressions and Monetary Policy," *Review of Economic Studies*, 72(3), 821–852.

Rotemberg, J. J., and M. Woodford (1999): "The Cyclical Behavior of Prices and Costs," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and M. Woodford, vol. 1 of *Handbook of Macroeconomics*, chap. 16, pp. 1051–1135. Elsevier.

Schorfheide, F. (2005): "Learning and Monetary Policy Shifts," *Review of Economic Dynamics*, 8(2), 392–419.

Sims, C. A., D. F. Waggoner, and T. Zha (2008): "Methods for Inference in Large Multiple-Equation Markov-Switching Models," *Journal of Econometrics*, 146, 113–144.

Sims, C. A., and T. Zha (2006): "Were There Regime Switches in US Monetary Policy?," *American Economic Review*, 91(1), 54–81.

Svensson, L. E. O., and N. Williams (2007): "Monetary Policy with Model Uncertainty: Distribution Forecast Targeting," Princeton University, mimeo.

Timmermann, A. (2000): "Moments of Markov Switching Models," *Journal of Econometrics*, 96(1), 75–111.

Woodford, M. (2003): *Interest and prices: Foundations of a theory of monetary policy*. Princeton University Press, Princeton, New Jersey.

# A Proofs

In this appendix I report the proofs for Propositions 1-3. Please, refer to Costa, Fragoso, and Marques (2004) for a proof of Proposition 4.

## A.1 Proposition 1

Consider a Markov-switching model whose law of motion can be described by (1) and define $q_t^i = \mathbb{E}_0\left(Z_t 1_{\xi_t=i}\right)$ for $i = 1...m$, then:

$$q_t^j = \sum_{i=1}^m \left[c_j \pi_{t-1}^i + A_j q_{t-1}^i\right] h_{ji}$$

**Proof.** By definition, $q_t^j = \mathbb{E}_0\left[Z_t 1_{\xi_t=j}\right]$. Then:

$$\begin{aligned} q_t^j &= \mathbb{E}_0\left[\left(c_{\xi_t} + A_{\xi_t} Z_{t-1}\right) 1_{\xi_t=j}\right] \\ &= \sum_{i=1}^m \mathbb{E}_0\left[\left(c_{\xi_t} + A_{\xi_t} Z_{t-1}\right) | \xi_t = j, \xi_{t-1} = i\right] P_0\left(\xi_t = j\right) P_0\left(\xi_{t-1} = i | \xi_t = j\right) \\ &= \sum_{i=1}^m \mathbb{E}_0\left[\left(c_j + A_j Z_{t-1}\right) | \xi_{t-1} = i\right] P_0\left(\xi_t = j | \xi_{t-1} = i\right) P_0\left(\xi_{t-1} = i\right) \\ &= \sum_{i=1}^m \left[c_j \pi_{t-1}^i + A_j q_{t-1}^i\right] h_{ji} \end{aligned}$$

where we have used the fact that $h_{ji} = P_0\left(\xi_t = j | \xi_{t-1} = i\right)$ and the definition of $q_{t-1}^i$. ■

## A.2 Proposition 2

Consider a Markov-switching model whose law of motion can be described by (1) and define $Q_t^i = \varphi\left[\mathbb{E}_0\left(Z_t Z_t' 1_{\xi_t=i}\right)\right]$ for $i = 1...m$, then:

$$Q_t^j = \sum_{i=1}^m \begin{bmatrix} \left(c_j \otimes c_j\right) \pi_{t-1}^i + \left(A_j \otimes A_j\right) Q_{t-1}^i + \left(V_j \otimes V_j\right) K_{t-1}^i \\ + \left[\left(A_j \otimes c_j\right) + \left(c_j \otimes A_j\right)\right] q_{t-1}^i \end{bmatrix} h_{ji}$$

where $Q_t^i = \varphi\left[\mathbb{E}_0\left[Z_t Z_t' 1_{\xi_t=i}\right]\right], q_t^i = \mathbb{E}_0\left[Z_t 1_{\xi_t=i}\right], \pi_t^i = P_0\left(\xi_t = i\right),$ and $K_t^i = \varphi\left[\mathbb{E}_0\left[I_k 1_{\xi_t=i}\right]\right] = \varphi\left[I_k\right] * \pi_t^i$.

**Proof.** By definition, $Q_t^j = \varphi\left[\mathbb{E}_0\left[Z_t Z_t' 1_{\xi_t=j}\right]\right]$. Then:

$$\begin{aligned} Q_t^j &= \varphi\left[\sum_{i=1}^m \mathbb{E}_0\left[\left(c_j + A_j Z_{t-1} + V_j \varepsilon_t\right)\left(c_j + A_j Z_{t-1} + V_j \varepsilon_t\right)' 1_{\xi_t=j} 1_{\xi_{t-1}=i}\right]\right] \\ &= \varphi\left[\sum_{i=1}^m \mathbb{E}_0\left[\left(c_j c_j' + A_j Z_{t-1} Z_{t-1}' A_j' + V_j \varepsilon_t \varepsilon_t' V_j' + c_j Z_{t-1}' A_j' + A_j Z_{t-1} c_j'\right) 1_{\xi_{t-1}=i}\right] h_{ji}\right] \\ &= \sum_{i=1}^m \mathbb{E}_0\left[\begin{pmatrix} \left(c_j \otimes c_j\right) + \left(A_j \otimes A_j\right)\varphi\left(Z_{t-1} Z_{t-1}'\right) + \left(V_j \otimes V_j\right)\varphi\left(\varepsilon_t \varepsilon_t'\right) \\ + \left[\left(A_j \otimes c_j\right) + \left(c_j \otimes A_j\right)\right]\varphi\left(Z_{t-1}\right) \end{pmatrix} 1_{\xi_{t-1}=i}\right] h_{ji} \\ &= \sum_{i=1}^m \begin{bmatrix} \left(c_j \otimes c_j\right) \pi_{t-1}^i + \left(A_j \otimes A_j\right) Q_{t-1}^i + \left(V_j \otimes V_j\right) K_{t-1}^i \\ + \left[\left(A_j \otimes c_j\right) + \left(c_j \otimes A_j\right)\right] q_{t-1}^i \end{bmatrix} h_{ji} \end{aligned}$$

where we have used the fact that $h_{ji} = P_0\left(\xi_t = j|\xi_{t-1} = i\right)$ and the definitions of $Q^i_{t-1}$, $q^i_{t-1}$, and $K^i_{t-1}$. ■

### A.3 Proposition 3

Consider a Markov-switching model whose law of motion can be described by (1) and define $Q^{\cdot,j}_{t,t+s} = \varphi\left[\mathbb{E}_0\left(Z_{t+s}Z'_t 1_{s_{t+s}=j}\right)\right]$ and $q^{\cdot,j}_{t,t+s} = \mathbb{E}_0\left(Z_t 1_{\xi_{t+s}=j}\right)$ for $j = 1...m$, then:

$$Q^{\cdot,j}_{t,t+s} = \widehat{Ic}_j q^{\cdot,j}_{t,t+s} + \textstyle\sum_{i=1}^m \widehat{IA}_j h_{ji} Q^{\cdot,i}_{t,t+s-1}$$

where $\widehat{Ic}_j = (I_n \otimes c_j)$ and $\widehat{IA}_j = (I_n \otimes A_j)$.
**Proof.** By definition, $Q^{\cdot,j}_{t,t+s} = \varphi\left[\mathbb{E}_0\left(Z_{t+s}Z'_t 1_{\xi_{t+s}=j}\right)\right]$. Then:

$$\begin{aligned}
Q^{\cdot,j}_{t,t+s} &= \varphi\left[\mathbb{E}_0\left[\left(c_{\xi_{t+s}} + A_{\xi_{t+s}} Z_{t+s-1}\right) Z'_t 1_{\xi_{t+s}=j}\right]\right] \\
&= \varphi\left[\textstyle\sum_{i=1}^m \mathbb{E}_0\left[\left(c_j Z'_t + A_j Z_{t+s-1} Z'_t\right) 1_{\xi_{t+s}=j} 1_{\xi_{t+s-1}=i}\right]\right] \\
&= \varphi\left[\textstyle\sum_{i=1}^m \mathbb{E}_0\left[\left(c_j Z'_t + A_j Z_{t+s-1} Z'_t\right) 1_{\xi_{t+s-1}=i}\right] h_{ji}\right] \\
&= \textstyle\sum_{i=1}^m \varphi\left[c_j \mathbb{E}_0\left[Z'_t 1_{\xi_{t+s-1}=i}\right] h_{ji}\right] + \textstyle\sum_{i=1}^m \varphi\left[A_j \mathbb{E}_0\left[Z_{t+s-1} Z'_t 1_{\xi_{t+s-1}=i}\right] h_{ji}\right] \\
&= \widehat{Ic}_j \textstyle\sum_{i=1}^m q^{\cdot,i}_{t,t+s-1} h_{ji} + \textstyle\sum_{i=1}^m \widehat{IA}_j Q^{\cdot,i}_{t,t+s-1} h_{ji}
\end{aligned}$$

where we have used the fact that $h_{ji} = P_0\left(\xi_t = j|\xi_{t-1} = i\right)$ and the definitions of $Q^{\cdot,i}_{t,t+s-1}$ and $q^{\cdot,i}_{t,t+s-1}$. Finally, we can use the fact that $q^{\cdot,j}_{t,t+s} = \sum_{i=1}^m q^{\cdot,i}_{t,t+s-1} h_{ji}$. ■

## B Illustrative Examples

In this appendix I provide some illustrative examples. Consider the model described in (1) with $m = 2$ and the transition matrix

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

where $h_{ji} = P_0\left[\xi_{t+1} = j|\xi_t = i\right]$. This implies that $h_{ji} = 1 - h_{ii}$ for $i = 1, 2$, $j \neq i$. In what follows we will assume that $Z_t$ is observable at time $t$.

### B.1 Law of motion of first moments

Define the expectation conditional on the information available at time $t$ as:

$$q^i_{t|t} = \mathbb{E}_t\left(Z_t 1_{\xi_t=i}\right) = Z_t \pi^i_{t|t} \text{ for } i = 1, 2$$

where we have used the fact that $Z_t$ is observable at time $t$. Notice that:

$$Z_t = Z_t \pi^1_{t|t} + Z_t \pi^2_{t|t} = q^1_{t|t} + q^2_{t|t}$$

For this simple model we can explicitly compute $\mathbb{E}_t(Z_{t+1})$:

$$\begin{aligned}
\mathbb{E}_t(Z_{t+1}) &= q^1_{t+1|t} + q^2_{t+1|t} \\
&= (c_1 + A_1 Z_t)\,\pi^1_{t+1|t} + (c_2 + A_2 Z_t)\,\pi^2_{t+1|t} \\
&= (c_1 + A_1 Z_t)\left(h_{11}\pi^1_{t|t} + h_{12}\pi^2_{t|t}\right) + (c_2 + A_2 Z_t)\left(h_{21}\pi^1_{t|t} + h_{22}\pi^2_{t|t}\right) \\
&= \left(c_1 h_{11}\pi^1_{t|t} + A_1 h_{11} q^1_{t|t}\right) + \left(c_1 h_{12}\pi^2_{t|t} + A_1 h_{12} q^2_{t|t}\right) \\
&\quad + \left(c_2 h_{21}\pi^1_{t|t} + A_2 h_{21} q^1_{t|t}\right) + \left(c_2 h_{22}\pi^2_{t|t} + A_2 h_{22} q^2_{t|t}\right)
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
q^i_{t+1|t} &= \mathbb{E}_t\left(Z_{t+1} 1_{\xi_{t+1}=i}\right) = (c_i + A_i Z_t)\,\pi^i_{t+1|t} \\
&= (c_i + A_i Z_t)\left(h_{i1}\pi^i_{t|t} + h_{i2}\pi^i_{t|t}\right)
\end{aligned}$$

Now consider using (5):

$$\underbrace{\begin{bmatrix} q_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix}}_{\widetilde{q}_{t+1|t}} = \underbrace{\begin{bmatrix} \Omega & CH \\ & H \end{bmatrix}}_{\widetilde{\Omega}} \underbrace{\begin{bmatrix} q_{t|t} \\ \pi_{t|t} \end{bmatrix}}_{\widetilde{q}_{t|t}}$$

$$\begin{bmatrix} q^1_{t+1|t} \\ q^2_{t+1|t} \\ \pi^1_{t+1|t} \\ \pi^2_{t+1|t} \end{bmatrix} = \begin{bmatrix} h_{11}A_1 & h_{12}A_1 & c_1 h_{11} & c_1 h_{12} \\ h_{21}A_2 & h_{22}A_2 & c_2 h_{21} & c_2 h_{22} \\ & & h_{11} & h_{12} \\ & & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} q^1_{t|t} \\ q^2_{t|t} \\ \pi^1_{t|t} \\ \pi^2_{t|t} \end{bmatrix}$$

We then obtain:

$$\begin{bmatrix} q^1_{t+1|t} \\ q^2_{t+1|t} \\ \pi^1_{t+1|t} \\ \pi^2_{t+1|t} \end{bmatrix} = \begin{bmatrix} h_{11}A_1 q^1_{t|t} + h_{12}A_1 q^2_{t|t} + c_1 h_{11}\pi^1_{t|t} + c_1 h_{12}\pi^2_{t|t} \\ h_{21}A_2 q^1_{t|t} + h_{22}A_2 q^2_{t|t} + c_2 h_{21}\pi^1_{t|t} + c_2 h_{22}\pi^2_{t|t} \\ h_{11}\pi^1_{t|t} + h_{12}\pi^2_{t|t} \\ h_{21}\pi^1_{t|t} + h_{22}\pi^2_{t|t} \end{bmatrix}$$

implying that we can compute the one-step-ahead first moment in one step:

$$\mu_{t+1|t} = \mathbb{E}_t(Z_{t+1}) = \widetilde{q}^1_{t+1|t} + \widetilde{q}^2_{t+1|t} = \widetilde{w}\widetilde{\Omega}\widetilde{q}_{t|t}$$

Similarly, all first moments at different horizons can be computed in one step:

$$\mu_{t+s|t} = \mathbb{E}_t(Z_{t+s}) = \widetilde{w}\widetilde{q}_{t+s|t} = \widetilde{w}\widetilde{\Omega}^s\widetilde{q}_{t|t}$$

### B.2 Law of motion of second moments

Define the second moment conditional on the information available at time $t$ as:

$$Q^i_{t|t} = \varphi\left[\mathbb{E}_t\left(Z_t Z_t' 1_{\xi_t=i}\right)\right] = \varphi\left[Z_t Z_t'\right]\pi^i_{t|t} \text{ for } i = 1, 2$$

where $\pi^i_{t|t}$ is the probability of being in regime $i$ at time $t$ given the information available at time $t$. Notice that:

$$\varphi\left(Z_t Z_t'\right) = \varphi\left(Z_t Z_t'\right)\left[\pi_{t,1|t} + \pi_{t,2|t}\right] = Q^1_{t|t} + Q^2_{t|t}$$

Given that we only have two regimes, we can explicitly derive the one-step-ahead second moment:

$$\begin{aligned}
\varphi\left[\mathbb{E}_t\left(Z_{t+1}Z'_{t+1}\right)\right] &= Q^1_{t+1|t} + Q^2_{t+1|t} \\
&= \varphi\left[\mathbb{E}_t\left(Z_{t+1}Z'_{t+1}|\xi_{t+1}=1\right)\right]\pi^1_{t+1|t} + \varphi\left[\mathbb{E}_t\left(Z_{t+1}Z'_{t+1}|\xi_{t+1}=2\right)\right]\pi^2_{t+1|t} \\
&= \varphi\left[\begin{array}{c} \mathbb{E}_t\left(\left(c_1 + A_1 Z_t + V_1\varepsilon_{t+1}\right)\left(c_1 + A_1 Z_t + V_1\varepsilon_{t+1}\right)'\right)\pi^1_{t+1|t} \\ +\mathbb{E}_t\left(\left(c_2 + A_2 Z_t + V_2\varepsilon_{t+1}\right)\left(c_2 + A_2 Z_t + V_2\varepsilon_{t+1}\right)'\right)\pi^2_{t+1|t} \end{array}\right].
\end{aligned}$$

Using the fact that $\pi^1_{t+1|t} = h_{11}\pi^1_t + h_{12}\pi^2_t$ and taking expectations we have:

$$\begin{aligned}
\varphi\left[\mathbb{E}_t\left(Z_{t+1}Z'_{t+1}\right)\right] &= \varphi\left[\begin{array}{c} \left(c_1c_1' + A_1Z_tZ_t'A_1' + V_1V_1' + c_1Z_t'A_1' + A_1Z_tc_1'\right)\left(h_{11}\pi^1_t + h_{12}\pi^2_t\right) \\ +\left(c_2c_2' + A_2Z_tZ_t'A_2' + V_2V_2' + c_2Z_t'A_2' + A_2Z_tc_2'\right)\left(h_{21}\pi^1_t + h_{22}\pi^2_t\right) \end{array}\right] \\
&= \left[\begin{array}{c} \left(\widehat{AA}_1 h_{11} Q^1_{t|t} + \widehat{VV}_1 h_{11} K^1_t + \widehat{DAC}_1 q^1_{t|t} + \widehat{cc}_1 h_{11}\right) \\ + \left(\widehat{AA}_2 h_{21} Q^1_{t|t} + \widehat{VV}_2 h_{21} K^1_t + \widehat{DAC}_2 q^1_{t|t} + \widehat{cc}_2 h_{21}\right) \\ + \left(\widehat{AA}_1 h_{12} Q^2_{t|t} + \widehat{VV}_1 h_{12} K^2_t + \widehat{DAC}_1 q^2_{t|t} + \widehat{cc}_1 h_{12}\right) \\ + \left(\widehat{AA}_2 h_{22} Q^2_{t|t} + \widehat{VV}_2 h_{22} K^2_t + \widehat{DAC}_2 q^2_{t|t} + \widehat{cc}_2 h_{22}\right) \end{array}\right]
\end{aligned}$$

It is straightforward to verify that if we define $\widetilde{Q}_{t|t} = [Q'_{t|t}, K'_{t|t}, q'_{t|t}, \pi'_{t|t}]'$ we can compute the one-step-ahead conditional second moment simply using (8):

$$M_{t+1|t} = \varphi\left(\mathbb{E}_t\left(Z_{t+1}Z'_{t+1}\right)\right) = Q^1_{t+1|t} + Q^2_{t+1|t} = \widetilde{W}\widetilde{\Xi}\widetilde{Q}_{t|t}$$

To see this, recall that (8) implies:

$$Q_{t+1|t} = \Xi Q_{t|t} + \widehat{VV} L_K K_{t|t} + \widehat{DAC} q_{t|t} + \widehat{cc} H \pi_{t|t}$$

with

$$\Xi = \begin{bmatrix} h_{11}\widehat{AA}_1 & h_{12}\widehat{AA}_1 \\ h_{21}\widehat{AA}_2 & h_{22}\widehat{AA}_2 \end{bmatrix}, \ \widehat{VV}L_K = \begin{bmatrix} h_{11}\widehat{VV}_1 & h_{12}\widehat{VV}_1 \\ h_{21}\widehat{VV}_2 & h_{22}\widehat{VV}_2 \end{bmatrix}$$

$$\widehat{DAC} = \begin{bmatrix} h_{11}\widehat{DAC}_1 & h_{12}\widehat{DAC}_1 \\ h_{21}\widehat{DAC}_2 & h_{22}\widehat{DAC}_2 \end{bmatrix}, \ \widehat{cc}H = \begin{bmatrix} h_{11}\widehat{cc}_1 & h_{12}\widehat{cc}_1 \\ h_{21}\widehat{cc}_2 & h_{22}\widehat{cc}_2 \end{bmatrix}$$

Then:

$$\begin{bmatrix} Q^1_{t+1|t} \\ Q^2_{t+1|t} \end{bmatrix} = \begin{bmatrix} h_{11}\widehat{AA}_1 Q^1_{t|t} + h_{12}\widehat{AA}_1 Q^2_{t|t} \\ h_{21}\widehat{AA}_2 Q^1_{t|t} + h_{22}\widehat{AA}_2 Q^2_{t|t} \end{bmatrix} + \begin{bmatrix} h_{11}\widehat{VV}_1 K^1_{t|t} + h_{12}\widehat{VV}_1 K^2_{t|t} \\ h_{21}\widehat{VV}_2 K^1_{t|t} + h_{22}\widehat{VV}_2 K^2_{t|t} \end{bmatrix}$$
$$+ \begin{bmatrix} h_{11}\widehat{DAC}_1 q^1_{t|t} + h_{12}\widehat{DAC}_1 q^2_{t|t} \\ h_{21}\widehat{DAC}_2 q^1_{t|t} + h_{22}\widehat{DAC}_2 q^2_{t|t} \end{bmatrix} + \begin{bmatrix} h_{11}\widehat{cc}_1 \pi^1_{t|t} + h_{12}\widehat{cc}_1 \pi^2_{t|t} \\ h_{21}\widehat{cc}_2 \pi^1_{t|t} + h_{22}\widehat{cc}_2 \pi^2_{t|t} \end{bmatrix}$$

Similarly, all second moments at different horizons can be computed in one step:

$$M_{t+s|t} = \varphi\left(\mathbb{E}_t\left(Z_{t+s}Z'_{t+s}\right)\right) = \widetilde{W}\widetilde{Q}_{t+s|t} = \widetilde{W}\widetilde{\Xi}^s\widetilde{Q}_{t|t}$$

### B.3 Law of motion of autocovariance

We have

$$\begin{aligned}
\varphi\left[\mathbb{E}_t\left[Z_{t+1}Z'_t\right]\right] &= Q^{\cdot,1}_{t,t+1|t} + Q^{\cdot,2}_{t,t+1|t} \\
&= \varphi\left[\left(A_1Z_tZ'_t + c_1Z'_t\right)\pi^1_{t+1|t} + \left(A_2Z_tZ'_t + c_2Z'_t\right)\pi^2_{t+1|t}\right] \\
&= \varphi\begin{bmatrix} \left(A_1Z_tZ'_t + c_1Z'_t\right)\left(h_{11}\pi^1_{t|t} + h_{12}\pi^2_{t|t}\right) \\ +\left(A_2Z_tZ'_t + c_2Z'_t\right)\left(h_{22}\pi^2_{t|t} + h_{21}\pi^1_{t|t}\right) \end{bmatrix} \\
&= \varphi\begin{bmatrix} h_{11}\pi^1_{t|t}\left(A_1Z_tZ'_t + c_1Z'_t\right) + h_{12}\pi^2_{t|t}\left(A_1Z_tZ'_t + c_1Z'_t\right) \\ +h_{21}\pi^1_{t|t}\left(A_2Z_tZ'_t + c_2Z'_t\right) + h_{22}\pi^2_{t|t}\left(A_2Z_tZ'_t + c_2Z'_t\right) \end{bmatrix} \\
&= \begin{bmatrix} h_{11}A_1Q^1_{t|t} + h_{11}c_1q^1_{t|t} + h_{12}A_1Q^2_{t|t} + h_{12}c_1q^2_{t|t} \\ +h_{21}A_2Q^1_{t|t} + h_{21}c_2q^1_{t|t} + h_{22}A_2Q^2_{t|t} + h_{22}c_2q^2_{t|t} \end{bmatrix}
\end{aligned}$$

It is straightforward to verify that the same result would have been obtained using (12) and replacing $Q_t$ and $q_t$ with $Q_{t|t}$ and $q_{t|t}$, respectively.

## C Uncertainty and MS models

In this appendix I show that in an MS model the initial values for the state variables affect uncertainty as long as the autoregressive component is MS. Consider a univariate MS process with two regimes:

$$z_t = c_{\xi_t} + a_{\xi_t}z_{t-1} + \sigma_{\xi_t}\varepsilon_t$$

Then

$$
\begin{aligned}
\mathbb{E}_t\left(z_{t+1}\right) &= \left(c_1+a_1 z_t\right)\pi^1_{t+1|t}+\left(c_2+a_2 z_t\right)\pi^2_{t+1|t} \\
\left[\mathbb{E}_t\left(z_{t+1}\right)\right]^2 &= \left(c_1+a_1 z_t\right)^2\left(\pi^1_{t+1|t}\right)^2+\left(c_2+a_2 z_t\right)^2\left(\pi^2_{t+1|t}\right)^2 \\
&\quad +2\left(c_1+a_1 z_t\right)\left(c_2+a_2 z_t\right)\pi^1_{t+1|t}\pi^2_{t+1|t} \\
\mathbb{E}_t\left(z^2_{t+1}\right) &= \left(c_1^2+a_1^2 z_t^2+\sigma_1^2+2c_1 a_1 z_t\right)\pi^1_{t+1|t} \\
&\quad +\left(c_2^2+a_2^2 z_t^2+\sigma_2^2+2c_2 a_2 z_t\right)\pi^2_{t+1|t}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
V_t\left(z_{t+1}\right) &= \left(\begin{array}{c} c_1^2+a_1^2 z_t^2+2c_1 a_1 z_t \\ +c_2^2+a_2^2 z_t^2+2c_2 a_2 z_t \\ -2\left(c_1 c_2+a_1 a_2 z_t^2+c_1 a_2 z_t+c_2 a_1 z_t\right)\end{array}\right)\pi^1_{t+1|t}\left(1-\pi^1_{t+1|t}\right) \\
&\quad +\sigma_1^2\pi^1_{t+1|t}+\sigma_2^2\left(1-\pi^1_{t+1|t}\right)
\end{aligned}
$$

- Assume $c_1 = c = c_2$:

$$
\begin{aligned}
V_t\left(z_{t+1}\right) &= \left(a_1^2+a_2^2-2a_1 a_2\right) z_t^2\pi^1_{t+1|t}\left(1-\pi^1_{t+1|t}\right) \\
&\quad +\sigma_1^2\pi^1_{t+1|t}+\sigma_2^2\left(1-\pi^1_{t+1|t}\right)
\end{aligned}
$$

  $\Longrightarrow$ the variance depends on $z_t^2$.

- Assume $a_1 = a = a_2$:

$$
\begin{aligned}
V_t\left(z_{t+1}\right) &= \left(c_1^2+c_2^2-2c_1 c_2\right)\pi^1_{t+1|t}\left(1-\pi^1_{t+1|t}\right) \\
&\quad +\sigma_1^2\pi^1_{t+1|t}+\sigma_2^2\left(1-\pi^1_{t+1|t}\right)
\end{aligned}
$$

  $\Longrightarrow$ the variance does not depend on $z_t^2$ or $z_t$. Therefore, in this case the initial state variables do not affect uncertainty.

- Assume $c_1 = c = c_2$ and $a_1 = a = a_2$:

$$
V_t\left(z_{t+1}\right)=\sigma_1^2\pi^1_{t+1|t}+\sigma_2^2\left(1-\pi^1_{t+1|t}\right)
$$

- Finally, in a model with fixed coefficients $V_t\left(z_{t+1}\right)=\sigma^2$ and the variance does not depend on the initial state vector.

## D Squared First Moments

The squared first moments can generally be computed starting from the output of the law of motion for the first moments. However, in some situations it might be more convenient to use explicit formulas. In this appendix I derive such formulas. Recall that:

$$\mathbb{E}_0\left(Z_t\right)=\sum_{i=1}^{m} q_t^i=\widetilde{w}\widetilde{q}_t$$

If we define $\widetilde{q}q_t=\varphi\left(\widetilde{q}_t\widetilde{q}_t'\right)$ and $\widetilde{q}q_{t,t+s}=\varphi\left(\widetilde{q}_{t+s}\widetilde{q}_t'\right)$, then:

$$\widetilde{q}q_{t+s}=\varphi\left(\widetilde{q}_{t+s}\widetilde{q}_{t+s}'\right)=\varphi\left(\widetilde{\Omega}^s\widetilde{q}_t\widetilde{q}_t'\widetilde{\Omega}^{s\prime}\right)=\widehat{\widetilde{\Omega}\widetilde{\Omega}}^s\,\widetilde{q}q_t$$

$$\widetilde{q}q_{t,t+s}=\varphi\left(\widetilde{q}_{t+s}\widetilde{q}_t'\right)=\varphi\left(\widetilde{\Omega}^s\widetilde{q}_t\widetilde{q}_t'\right)=\widehat{I\widetilde{\Omega}}^s\,\widetilde{q}q_t$$

where $\widehat{\widetilde{\Omega}\widetilde{\Omega}}=\widetilde{\Omega}\otimes\widetilde{\Omega}$ and $\widehat{I\widetilde{\Omega}}=I_{(n+1)m}\otimes\widetilde{\Omega}$.

Therefore the squared first moments can be computed as:

$$\mu_{t+s}^2=\varphi\left[\mathbb{E}_0\left(Z_{t+s}\right)\mathbb{E}_0\left(Z_{t+s}\right)'\right]=\varphi\left[\left(\widetilde{w}\widetilde{q}_{t+s}\right)\left(\widetilde{w}\widetilde{q}_{t+s}\right)'\right]=\widehat{\widetilde{w}\widetilde{w}}\widetilde{q}q_{t+s}=\widehat{\widetilde{w}\widetilde{w}}\widehat{\widetilde{\Omega}\widetilde{\Omega}}^s\,\widetilde{q}q_t$$

$$\mu_{t,t+s}^2=\varphi\left[\mathbb{E}_0\left(Z_{t+s}\right)\mathbb{E}_0\left(Z_t\right)'\right]=\varphi\left[\left(\widetilde{w}\widetilde{q}_{t+s}\right)\left(\widetilde{w}\widetilde{q}_t\right)'\right]=\widehat{\widetilde{w}\widetilde{w}}\widetilde{q}q_{t,t+s}=\widehat{\widetilde{w}\widetilde{w}}\widehat{I\widetilde{\Omega}}^s\,\widetilde{q}q_t$$

where $\widehat{\widetilde{w}\widetilde{w}}=\left(\widetilde{w}\otimes\widetilde{w}\right)$. Conditional on the information at time $t$ we have:

$$\widetilde{q}q_{t+s|t}=\widehat{\widetilde{\Omega}\widetilde{\Omega}}^s\,\widetilde{q}q_{t|t}\Rightarrow\mu_{t+s|t}^2=\varphi\left[\mathbb{E}_t\left(Z_{t+s}\right)\mathbb{E}_t\left(Z_{t+s}\right)'\right]=\widehat{\widetilde{w}\widetilde{w}}\widehat{\widetilde{\Omega}\widetilde{\Omega}}^s\,\widetilde{q}q_{t|t}$$

$$\widetilde{q}q_{t,t+s|t}=\widehat{I\widetilde{\Omega}}^s\,\widetilde{q}q_t\Rightarrow\mu_{t,t+s}^2=\varphi\left[\mathbb{E}_t\left(Z_{t+s}\right)Z_t'\right]=\widehat{\widetilde{w}\widetilde{w}}\widehat{I\widetilde{\Omega}}^s\,\widetilde{q}q_t$$

Therefore, the definition for the variance (10) and (11) can be expressed as

$$\begin{aligned}\varphi\left[\mathbb{V}_0\left(Z_t\right)\right] &= M_t-\mu_t^2 &\quad(38)\\ \varphi\left[\mathbb{V}_t\left(Z_{t+s}\right)\right] &= M_{t+s|t}-\mu_{t+s|t}^2. &\quad(39)\end{aligned}$$

## E Welfare Calculations

In this appendix, I report an extended derivation of the welfare calculations of Section 4.7. Following Rotemberg and Woodford (1999), Woodford (2003), and Gali (2008), the period welfare loss is obtained by taking a log-quadratic approximation of the representative household's utility function:

$$\mathbb{L}_t=\sum_{s=0}^{\infty}\beta^s\left[\mathbb{E}_t\left(\widehat{\pi}_{t+s}^2\right)+\left(\kappa/\upsilon\right)\mathbb{E}_t\left(\widehat{y}_{t+s}^2\right)\right]\quad(40)$$

where $\upsilon$ is the elasticity of substitution between two differentiated goods and $\kappa$ is the slope of the Phillips curve. In order to compute the welfare loss, taking into account the possibility of regime changes, recall that for a variable $\widehat{x}_t$ the evolution of second moments is pinned down by:

$$\mathbb{E}_t\left[\widehat{x}_{t+s}^2\right] = e_x M_{t+s|t} = e_x \widetilde{W}\widetilde{\Xi}^s \widetilde{Q}_{t|t}$$

Therefore each element of (40) can be computed as:

$$\begin{aligned}
\textstyle\sum_{s=0}^{\infty}\beta^s \mathbb{E}_t\left[\widehat{x}_{t+s}^2\right] &= \textstyle\sum_{s=0}^{\infty}\beta^s e_x \widetilde{W}\widetilde{\Xi}^s \widetilde{Q}_{t|t} \\
&= e_x\widetilde{W}\textstyle\sum_{s=0}^{\infty}\left(\beta\widetilde{\Xi}\right)^s \widetilde{Q}_{t|t} \\
&= e_x\widetilde{W}\left(I-\beta\widetilde{\Xi}\right)^{-1}\widetilde{Q}_{t|t}
\end{aligned}$$

Then:

$$\begin{aligned}
\mathbb{L}_t &= e_\pi\widetilde{W}\left(I-\beta\widetilde{\Xi}\right)^{-1}\widetilde{Q}_{t|t} + (\kappa/\varepsilon)\, e_y\widetilde{W}\left(I-\beta\widetilde{\Xi}\right)^{-1}\widetilde{Q}_{t|t} \\
&= \left(e_\pi + (\kappa/\varepsilon)\, e_y\right)\widetilde{W}\left(I-\beta\widetilde{\Xi}\right)^{-1}\widetilde{Q}_{t|t}
\end{aligned}$$

# F News in Markov Switching Models

Consider an MS-VAR:

$$Z_t = c_{\xi_t} + A_{\xi_t} Z_{t-1} + R_{\xi_t}\Sigma_{\xi_t}\varepsilon_t$$

where $Z_t$ is a column vector containing $n$ variables observable at time $t$ and $\xi_t = 1, ..., m$, with $m$ the number of regimes, evolves following the transition matrix $H$.

Define the column vectors $q_t$ and $\pi_t$:

$$q_t = \left[q_t^{1\prime}, ..., q_t^{m\prime}\right]', q_t^i = \mathbb{E}_0\left(Z_t 1_{\xi_t = i}\right), \pi_t = \left[\pi_t^1, ..., \pi_t^m\right]',$$

where $\pi_t^i = P_0(\xi_t = i)$ and $1_{\xi_t = i}$ is an indicator variable that is equal to one when regime $i$ is in place and zero otherwise. The law of motion for $\widetilde{q}_t = [q_t', \pi_t']'$ is then given by

$$\underbrace{\begin{bmatrix} q_t \\ \pi_t \end{bmatrix}}_{\widetilde{q}_t} = \underbrace{\begin{bmatrix} \Omega & CH \\ & H \end{bmatrix}}_{\widetilde{\Omega}} \begin{bmatrix} q_{t-1} \\ \pi_{t-1} \end{bmatrix} \tag{41}$$

where $\pi_t = [\pi_{1,t}, ..., \pi_{m,t}]'$, $\Omega = bdiag\,(A_1, ..., A_m)\, H$, and $C = bdiag\,(c_1, ..., c_m)$. Recall that:

$$\mathbb{E}_0(Z_t) = \sum_{i=1}^{m} q_t^i = wq_t,\;\; w = \left[\underbrace{I_n, ..., I_n}_{m}\right]$$

To compute the news, define:

$$\begin{aligned} q^i_{t+s|t} &= \mathbb{E}_t\left(Z_{t+s}1_{\xi_{t+s}=i}\right) = \mathbb{E}\left(Z_{t+s}1_{\xi_{t+s}=i}|\mathbb{I}_t\right) \\ e_1' &= [1,0,0,0]',\ mn = m*n \end{aligned}$$

where $\mathbb{I}_t$ contains all the information that agents have at time $t$, including the probability of being in one of the $m$ regimes. Note that $q^i_{t|t} = Z_t\pi^i_t$.

Now, consider the formula for the discount rate news:

$$N_{DR,t+1} = \left(\mathbb{E}_{t+1} - \mathbb{E}_t\right)\sum_{j=1}^{\infty}\rho^j r_{t+1+j}$$

The first term is:

$$\begin{aligned} \mathbb{E}_{t+1}\sum_{j=1}^{\infty}\rho^j r_{t+1+j} &= \sum_{j=1}^{\infty}\rho^j e_1' w q_{t+1+j|t+1} \\ &= e_1' w\left[\rho q_{t+2|t+1} + \rho^2 q_{t+3|t+1} + \rho^3 q_{t+4|t+1} + ....\right] \\ &= e_1' w\left(I_r - \rho\Omega\right)^{-1}\left[\rho\Omega q_{t+1|t+1} + \rho CH\left(I_r - \rho H\right)^{-1}\pi_{t+1|t+1}\right] \end{aligned}$$

The second term is:

$$\begin{aligned} \mathbb{E}_t\sum_{j=1}^{\infty}\rho^j r_{t+1+j} &= \sum_{j=1}^{\infty}\rho^j e_1' w q_{t+1+j|t} \\ &= e_1' w\left(I_r - \rho\Omega\right)^{-1}\left[\rho\Omega q_{t+1|t} + \rho CH\left(I_r - \rho H\right)^{-1}\pi_{t+1|t}\right] \end{aligned}$$

The formulas can also be derived using (5). For example the second term is:

$$\begin{aligned} \mathbb{E}_t\sum_{j=1}^{\infty}\rho^j r_{t+1+j} &= e_1'\widetilde{w}\sum_{j=1}^{\infty}\rho^j\widetilde{q}_{t+1+j|t} = e_1'\widetilde{w}\sum_{j=1}^{\infty}\rho^j\widetilde{\Omega}^j\widetilde{q}_{t+1|t} \\ &= e_1'\widetilde{w}\left(I - \rho\widetilde{\Omega}\right)^{-1}\rho\widetilde{\Omega}\widetilde{q}_{t+1|t} \\ &= e_1' w\left(I_{mn} - \rho\Omega\right)^{-1}\left[\rho\Omega q_{t+1|t} + \rho CH\left(I_m - \rho H\right)^{-1}\pi_{t+1|t}\right] \end{aligned}$$

where we have used the fact that $I_m + \left(I_m - \rho H\right)^{-1}\rho H = \left(I_m - \rho H\right)^{-1}$ and

$$\begin{aligned} \left(I - \rho\widetilde{\Omega}\right)^{-1}\rho\widetilde{\Omega} &= \begin{bmatrix} \left(I_{mn} - \rho\Omega\right)^{-1} & \left(I_{mn} - \rho\Omega\right)^{-1}\rho CH\left(I_m - \rho H\right)^{-1} \\ & \left(I_m - \rho H\right)^{-1} \end{bmatrix}^{-1}\begin{bmatrix} \rho\Omega & \rho CH \\ & \rho H \end{bmatrix} \\ &= \begin{bmatrix} \left(I_m - \rho\Omega\right)^{-1}\rho\Omega & \left(I_{mn} - \rho\Omega\right)^{-1}\rho CH\left(I_m - \rho H\right)^{-1} \\ & \left(I_m - \rho H\right)^{-1}\rho H \end{bmatrix}^{-1} \end{aligned}$$

Therefore:

$$\begin{aligned} N_{DR,t+1} &= e_1' w \left[\lambda^q v_{t+1}^q + \lambda^\pi v_{t+1}^\pi\right] \\ \lambda^q &= (I_r - \rho\Omega)^{-1} \rho\Omega \\ \lambda^\pi &= (I_r - \rho\Omega)^{-1} \rho CH (I_r - \rho H)^{-1} \end{aligned}$$

Then, we can easily compute the residuals:

$$\begin{aligned} u_{t+1} &= Z_{t+1} - \mathbb{E}_t Z_{t+1} \\ e_1' u_{t+1} &= r_{t+1} - \mathbb{E}_t (r_{t+1}) \end{aligned}$$

and the news about future cash flows can be obtained as:

$$N_{CF,t+1} = e_1' u_{t+1} + N_{DR,t+1}$$

Note that given a sequence of probabilities or a draw for the MS states and a set of parameters, it is easy and computationally efficient to compute the entire sequences $v^{q,T}$, $v^{\pi,T}$, and $u^T$:

$$\begin{aligned} N_{DR}^T &= e_1' w \left[\lambda^q v^{q,T} + \lambda^\pi v^{\pi,T}\right] \\ N_{CF}^T &= e_1' w \left[(I_r + \lambda^q) v^{q,T} + \lambda^\pi v^{\pi,T}\right] \\ u^T &= e_1' w v^{q,T} \end{aligned}$$

## G Some Useful Results

Size of the matrices and vectors:

$$\underset{([n^2+k^2+n+1]m \times [n^2+k^2+n+1]m)}{\widetilde{\widetilde{\Xi}}} = \left[\begin{array}{cc|cc} \underset{(n^2m \times n^2m)}{\Xi} & \underset{(n^2m \times k^2m)}{\widehat{VV} L_K} & \underset{(n^2m \times nm)}{\widehat{DAC}} & \underset{(n^2m \times m)}{CH} \\ & \underset{(k^2m \times k^2m)}{L_K} & & \\ \hline & & \underset{(nm \times nm)}{\Omega} & \underset{(nm \times m)}{CH} \\ & & & \underset{(m \times m)}{H} \end{array}\right], \widetilde{Q}_t = \begin{bmatrix} \underset{(n^2m \times 1)}{Q_t} \\ \underset{(k^2m \times 1)}{K_t} \\ \underset{(nm \times 1)}{q_t} \\ \underset{(m \times 1)}{\pi_t} \end{bmatrix}$$

Kronecker product and constants:

$$\varphi\left(c_j c_j'\right) = c_j \otimes c_j$$