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ABSTRACT

Skewness Risk Premium: Theory and Empirical Evidence*

Using an equilibrium asset and option pricing model in a production economy under jump diffusion, we show theoretically that the aggregated excess market returns can be predicted by the skewness risk premium, which is constructed to be the difference between the physical and the risk-neutral skewness. In an empirical application of the model using more than 20 years of data on S&P500 index options, we find that, in line with theory, risk-averse investors demand risk-compensation for holding stocks when the market skewness risk premium is high. However, when we characterize periods of high and low risk aversion, we show that in line with theory, the relationship only holds when risk aversion is high. In periods of low riskaversion, investors demand lower risk compensation, thus substantially weakening the skewness-risk-premium-return trade off.

JEL Classification: C15 and G12
Keywords: asset pricing, central moments, investor sentiment, option markets, risk aversion and skewness risk premium

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1. Motivation and literature review

Risk compensation theory suggests that systematic negative skewness in asset returns can be considered to be a risk and risk-averse investors want to be compensated for accepting this risk. Hence, expected returns should include a reward for bearing this risk. The first paper that derived a theoretical relation among expected return, variance and skewness is Arditti (1967), where the signs of coefficients for variance and skewness are specified to be positive and negative, respectively. Over time, more and more studies challenge the simple mean-variance asset pricing framework and suggest to include higher moments. Among others, Kraus and Litzenberger (1976) derive a three-moment CAPM and show that systematic skewness is a priced risk factor. Harvey and Siddique (1999, 2000a, 2000b) use conditional skewness to mitigate the shortcomings of mean-variance asset pricing models in explaining cross-sectional variations in expected returns. Their findings suggest that conditional skewness is important and helps explaining ex ante market risk premiums. Other theoretical and empirical studies on the higher-moment CAPM include Friend and Westerfield (1980), Sears and Wei (1985, 1988), Lim (1989), Hwang and Satchell (1999), Dittmar (2002) and, more recently, Chabi-Yo (2008, 2012). Among others, Conrad et al (2012) use options market data to extract estimates of higher moments of stocks` probability density function. They find a significant negative relation between firm`s risk-neutral skewness and subsequent stock returns. In a related study, Chang et al. (2013) show that the market risk premium is a priced risk factor in the cross section of stock returns, which cannot be explained by traditional 4-factor models.

Risk-neutral skewness has long been regarded as a measure of the pronounced volatility smirk observed in options market. This third moment is also mathematically closely linked to and interpreted as a proxy for an observed difference between physical variance and risk-neutral variance. The difference between physical variance and risk-neutral variance, the variance risk premium, has been explored to explain asset prices. (see e.g. Bakshi and Madan (2006), Coval and Shumway (2001), Bakshi and Kapadia (2003), Carr and Wu (2009), Bollerslev, Tauchen and Zhou (2009)). The concept of "skewness risk premium" appears to be new in the asset pricing literature, but has been recently investigated in relation to trading strategies in options market. In a paper by Kozhan, Neuberger and Schneider (2011), the authors discover profits from a trading strategy that directly exploits the skew in implied volatility surface. They attribute the profits to the existence of a “skew risk premium”, which is based on a skew swap that pays the difference between option implied skew and realized skew. In a related paper Ruf (2012) demonstrates how to decompose the price of skewness into realized skewness and a skewness risk premium and shows that depending on strategic situations of arbitrageurs, realized skewness could remain unchanged while the skewness risk premium is changing. The changing skewness risk premium verifies the existence of limits to arbitrage effects in option markets.

Variance and skewness in asset returns represent different types of risks. Using a behavioral paradigm, research in neurology shows that individuals’ choice behavior is sensitive to both, dispersion (variance) and asymmetry (skewness) of outcomes (Symmonds et al (2011)). By scanning subjects with functional magnetic resonance imaging (fMRI), they find that individuals encode variance and skewness separately in the brain, the former being associated with parietal cortex and the latter with prefrontal cortex and ventral striatum. Participants were exposed to choices among a range of orthogonalized risk factors. The authors argue that risk is neither monolithic from a behavioral nor from a neural perspective. Their findings support the argument of dissociable components of risk factors and suggest separable effects of variance and skewness on asset market returns.

We extend the understanding of the impact of skewness on market returns both theoretically and empirically. Theoretically, we build on an equilibrium asset and options pricing model in a production economy derived in Zhang, Zhao and Chang (2012). We theoretically derive and obtain an analytical expression for the relationship between the market equity premium and the skewness risk premium, defined as the difference between physical and risk-neutral skewness. Our paper aims to investigate the properties of skewness risk premium and test its effect on subsequent market excess returns. Our contribution is to provide a theoretical solution for the first time of the relation between market excess return, variance, variance risk premium, skewness and skewness risk premium in an expected utility framework. This paper also completes a series of empirical tests on the relationship between the market risk premium and higher moments of return distributions, physical as well as risk-neutral higher moments. We show that skewness risk premium is economically meaningful and contributes to the market excess return in divergent ways depending on different states of the economy. Recently, Yu and Yuan (2011) find evidence that the risk-return relationship depends on market conditions. Under normal market conditions, in line with risk compensation theory, the stock market’s expected excess return is positively related to the market's conditional variance. However, in times of higher demand for stocks, proxied by all months with a positive realizations of the Baker and Wurgler (2006, 2007) sentiment indicator, the relationship is essentially flat. They argue that sentiment brings higher demand for stocks, pushes up current prices and depresses expected returns. As a result, the return distribution is left skewed in such a regime. Consequently, the perception towards risk of market participants can be assumed to be different and, therefore, we should observe a different level of risk aversion in times when the demand for stocks is high. We control for this effect in our empirical analysis.

The paper proceeds as follows. Section 2 describes the theoretical model. Section 3 discusses the data and section 4 presents the empirical analysis. Section 5 concludes.

2. Jump Diffusion Model

Following the jump-diffusion model in a production economy of Zhang, Zhao and Chang (2012), we assume that the process of the price of an asset S_t (the market portfolio) can be described as

$$(1)\ \frac{dS_t}{S_t} = (r_f + \phi)dt + \sigma dB_t + (e^x - 1)(dN_t - \lambda dt)$$

where r_f is risk-free rate, ϕ represents excess market return, σ denotes volatility, B_t is a standard Brownian motion in $\mathbb{R}$ (and dB_t the increment), N_t is Poisson process with constant intensity λ (and dN_t the increment), $(e^x - 1)$ is the jump size with x following a normal distribution with mean μ_x and variance σ_x^2. We assume that the parameters and initial conditions have sufficient regularity for the solution of (1) to be well defined.

This specification nests many popular models used for option pricing and portfolio allocation applications. Without jumps, $E(dN_t) = \lambda dt = 0$, the model reduces to a standard diffusion model. The drift component of the stock price dynamics increases with the risk-free rate and excess market return, which are associated with the risk-premium for the Brownian motion. However, since pure diffusive model cannot explain the tail-fatness of stock return distribution and cannot explain the volatility smirk phenomena shown in options data (see Andersen et al, 1998; Bakshi et al, 1997; Bates, 2000), the addition of a jump process is of necessity. In our context, the motivation to include jumps is to study how jumps, which are representatives of extreme events, affect the discontinuous behaviors in terms of higher moments of market return distributions, and how jumps are priced in expected market return through its effects on higher moments behaviors, especially through the third moments.

In the model with jumps, the arrival of extreme events is described by the Poisson process, which has $E(dN_t) = \lambda dt$ with arrival intensity $\lambda \geq 0$. The relative jump size of the rare event is $(e^x - 1)$. If x is normally distributed with mean μ_x and variance σ_x^2 , as most literature modeling jump prices suggest, the expected relative jump size could reduce to $E(e^x - 1) = \exp(\mu_x + \sigma_x^2/2) - 1$. Combining the effects of random jump intensity and jump size, the term $\lambda(e^x - 1)dt$ is a compensation for the instantaneous change in expected stock returns introduced by the Poisson process N_t. So, we could call the last term $(e^x - 1)(dN_t - \lambda dt)$ an increment of compensated compound Poisson, which has zero mean to guarantee the expected return to be $\mu \equiv r_f + \phi$ as constructed by Zhang, Zhao and Chang (2012).

Intuitively, the conditional probability at time t of another extreme event before $t + \Delta t$ is approximately $\lambda \Delta t$. Conditioning on the arrival of an extreme event, a negative jump size represents a market crash. The model is therefore able to capture extreme event risk in additional to diffusive risk. Empirical evidence from options market suggests that for investors with a reasonable range of risk aversion, jump risk is compensated more highly than diffusive risk. For example, Bates (2000) regards that investors have differential pricing between diffusive and jump risks and thus have an additional aversion to market crashes; Liu et al (2003) consider an investor with uncertainty aversion towards rare events.

In general, we choose the jump size with a normally distributed component x. The following development of the model will provide explicitly the skewness risk premium in function of the jump size. In the economy, suppose there is a representative investor who has a constant relative risk aversion utility function as

(2) $U(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1 \\ \ln c, & \gamma = 1 \end{cases}$

with $U`(c) > 0$, $U``(c) < 0$ . The coefficient $\gamma = \frac{-cU``(c)}{U`(c)}$ is a measure of the magnitude of relative risk aversion.

Assume the investor has a total wealth W_t at time t. Given the opportunity to invest in the risk-free asset and risky stock, he chooses at each time t to invest a fraction w of his wealth in stock S_t and fraction $(1-w)$ in the risk-free asset. In line with a basic economic setup, a representative investor behaves in order to maximize his expected utility of consumption throughout his lifetime by choosing the fraction w of wealth to investment and the consumption rate c_t at each time t. Mathematically,

(3) $max_{(c_t,w)} E_t \int_t^T \beta(t) U(c_t)\, dt$

Subject to his wealth constraint as

$$\frac{dW_t}{W_t} = \left[r_f + w\phi - w\lambda(e^x - 1) - \frac{c_t}{W_t}\right] dt + w\sigma dB_t + w(e^x - 1) dN_t,$$

where $\beta(t) \geq 0$ $(0 \leq t \leq T)$ is a time preference function.

We note that ϕ represents the risk premium due to investment in risky stocks. In our context, such a risk premium is defined to be the excess market return that is considered to be the compensation for investors bearing both diffusive risk and extreme event risk.

Using Ito's lemma, integration and optimization methods under market clearing conditions, we get to the following propositions.

Proposition1: In equilibrium of the production economy setup (1)(2)(3), market excess return ϕ by definition is equal to the sum of diffusive risk premium ϕ_σ and extreme event risk premium ϕ_J, which are given as follows

(4) $\phi_\sigma = \gamma\sigma^2$

(5) $\phi_J = \lambda E[(1 - e^{-\gamma x})(e^x - 1)]$

(6) $\phi \equiv \mu - r_f \equiv \phi_\sigma + \phi_J =$

$\frac{\gamma}{\tau} Var_t(Y_\tau) + \frac{\gamma}{2\tau}(1-\gamma) Skew_t(Y_\tau) + \frac{\gamma}{12\tau}(2\gamma^2 - 3\gamma + 2) Kurt_t(Y_\tau) - \frac{\gamma}{4\tau}(2\gamma^2 - 3\gamma + 2)[Var_t(Y_\tau)]^2 + \frac{\gamma}{24\tau}(-\gamma^3 + 2\gamma^2 - 2\gamma + 1) Fifth_t(Y_\tau) - \frac{5\gamma}{12\tau}(-\gamma^3 + 2\gamma^2 - 2\gamma + 1)[Var_t(Y_\tau) \times Skew_t(Y_\tau)] + \lambda\gamma \mathrm{E}(\mathrm{o}(\mathrm{x}^6))$

where $Y_\tau = \ln\left(\frac{S_{t+\tau}}{S_t}\right)$; $Var_t(Y_\tau)$, $Skew_t(Y_\tau)$, $Kurt_t(Y_\tau)$, $Fifth_t(Y_\tau)$ *are second-, third-, fourth-, fifth- central moment respectively under the physical measure.*

Proposition2: *In equilibrium of the production economy setup (1)(2)(3), variance risk premium* $VRP_t(Y_\tau)$ *and skewness have the following relation:*

(7) $VRP_t(Y_\tau) \equiv Var_t(Y_\tau) - Var_t^Q(Y_\tau) = \gamma Skew_t(Y_\tau) - \frac{\gamma^2}{2} Kurt_t(Y_\tau) + \frac{3\gamma^2}{2}[Var_t(Y_\tau)]^2 + \frac{1}{6}\lambda\gamma^3\tau \mathrm{E}(\mathrm{o}(\mathrm{x}^5))$

where $Y_\tau = \ln\left(\frac{S_{t+\tau}}{S_\tau}\right)$; $Var_t(Y_\tau)$, $Skew_t(Y_\tau)$, $Kurt_t(Y_\tau)$ *are second-, third-, fourth- central moment respectively under the physical measure;* $Var_t^Q(Y_\tau)$ *is the second moment under the risk-neutral measure.*

Proposition3: *In equilibrium of the production economy setup (1)(2)(3), skewness risk premium* $SRP_t(Y_\tau)$ *and kurtosis have the following relation:*

(8) $SRP_t(Y_\tau) \equiv Skew_t(Y_\tau) - Skew_t^Q(Y_\tau) = \gamma Kurt_t(Y_\tau) - 3\gamma[Var_t(Y_\tau)]^2 - \frac{\gamma^2}{2} Fifth_t(Y_\tau) + 5\gamma^2[Var_t(Y_\tau) \times Skew_t(Y_\tau)] + \frac{1}{6}\lambda\gamma^3\tau \mathrm{E}(\mathrm{o}(\mathrm{x}^6))$

where $Y_\tau = \ln\left(\frac{S_{t+\tau}}{S_\tau}\right)$; $Var_t(Y_\tau)$, $Skew_t(Y_\tau)$, $Kurt_t(Y_\tau)$, $Fifth_t(Y_\tau)$ *are second-, third-, fourth-, fifth- central moment respectively under the physical measure;* $Skew_t^Q(Y_\tau)$ *is the third* *moment under the risk-neutral measure.*

Proposition4: *In equilibrium of the production economy setup (1)(2)(3), market excess return ϕ, variance $Var_t(Y_\tau)$, variance risk premium $VRP_t(Y_\tau)$, skewness $Skew_t(Y_\tau)$ and skewness risk premium $SRP_t(Y_\tau)$ have the following relation:*

(9) $\phi = \frac{\gamma}{\tau} Var_t(Y_\tau) + \frac{-\gamma^3+\gamma^2-1}{6\tau\gamma^2} VRP_t(Y_\tau) + \frac{-2\gamma^3+2\gamma^2+1}{6\tau\gamma} Skew_t(Y_\tau) + \frac{\gamma^3-2\gamma^2+2\gamma-1}{12\tau\gamma} SRP_t(Y_\tau) + \lambda\gamma \mathrm{E}(\mathrm{o}(\mathrm{x}^6))$

where $Y_\tau = \ln\left(\frac{S_{t+\tau}}{S_\tau}\right)$.

Proofs. See Appendix A.

Specifically, we observe from the second proposition that skewness is supposed to have a positive effect on variance risk premium, which verifies the common belief that the negative skewness is in accordance with commonly observed volatility smirk in options market. In the third proposition, the positive coefficient in front of kurtosis suggests that a positive skewness risk premium might well result from a positive kurtosis. The fourth proposition with explicit relations between physical and risk-neutral moments and the relations arching different orders of moments provides for the first time a testable theoretical relation between excess return and skewness risk premium.

We also observe from the fourth proposition that depending on the magnitude of relative risk aversion γ, relations vary. Assume a risk-averse investor who has a constant relative risk aversion coefficient that takes some value in the range of $\gamma \approx 2$. Our theoretical relationships suggest that the coefficient for variance is positive; for variance risk premium is negative; for skewness is negative; and for skewness risk premium is positive. These results all comply with risk-compensation theory. When the representative investor exhibits low risk aversion, for example in case of $\gamma \approx 0.5$, the signs of coefficients for variance and for variance risk premium remain the same as before; but the signs of coefficients for skewness and for skewness risk premium both reverse. Two unchanged signs and two reversed signs in combination is a theoretical reflection of the common sense that investors with low risk aversion demand lower risk compensation, and obviously it is the skewness and skewness risk premium that are much more sensitive to the weakening risk compensation effect. The theoretical predictions can be tested empirically.

To give a more intuitive illustration as to how market crashes affect the skewness risk premium, we take a nonrandom jump size with constant x for simplicity. In such a case, an extreme event is supposed to have a finite definite magnitude of jump size as $E(e^x - 1) = \exp(\mu_x) - 1$, where $\mu_x = x$, $Var(x) = SKew(x) = Kurt(x) = 0$. Based on the pricing kernel constructed in Zhang, Zhao and Chang (2012), $\lambda^Q \equiv \lambda E(e^{-\gamma x}) = \lambda e^{-\gamma x}$, the variance risk premium (VRP) can be written as[1]

$$VRP_t(Y_\tau) \equiv Var_t(Y_\tau) - Var_t^Q(Y_\tau) = \lambda\tau x^2(1 - e^{-\gamma x})$$

As can be seen, the variance risk premium does only depend on the jump risk and not on the diffusion risk. For negative jump size x<0, the variance risk premium is negative. The skewness in both physical and risk-neutral measures can be simplified to

$$Skew_t(Y_\tau) \equiv E_t[Y_\tau - E_tY_\tau]^3 = \lambda\tau[Skew(x) + 3\mu_x Var(x) + \mu_x^3] = \lambda\tau x^3$$
$$Skew_t^Q(Y_\tau) \equiv E_t^Q[Y_\tau - E_tY_\tau]^3 = \lambda^Q\tau\left[Skew^Q(x) + 3\mu_x^Q Var^Q(x) + (\mu_x^Q)^3\right] = \lambda\tau x^3 e^{-\gamma x}$$

And, hence, the skewness risk premium (SRP) is given by

$$SRP_t(Y_\tau) \equiv Skew_t(Y_\tau) - Skew_t^Q(Y_\tau) = \lambda\tau x^3(1 - e^{-\gamma x})$$

We observe that for an extreme event with negative jump size, $x < 0$, the skewness risk premium is supposed to be positive while both the physical and risk-neutral skewness can be negative.

Corollary 1: For a nonrandom negative jump size with $x < 0$*, skewness in both physical and*

[1] Proofs are provided in Appendix A.

risk-neutral measures are negative, while skewness risk premium is positive, namely

$Skew_t(Y_\tau) < 0$

$Skew_t^Q(Y_\tau) < 0$

$SRP_t(Y_\tau) > 0$

where $Y_\tau = \ln\left(\frac{S_{t+\tau}}{S_\tau}\right)$

To the best of our knowledge, the property on skewness risk premium has never been presented in the literature. Even though there is an unanimous agreement that strongly negative risk neutral skewness should be responsible for the observed volatility smirks in options data, empirical actual return skewness is not shown to be equally high and thus risk neutral skewness should be the results of a skew correction. In addition, as pointed out by Polimenis (2006), the third and fourth moments generated by jumps are significant in pricing non-linear payoffs, the question as to which one is most important factor in determining the smirks is still open. Similarly in asset pricing, due to interactions among different orders of moments, it is theoretically hard to distinguish which moments have higher impact. However, through a construction of skewness risk premium, the corollary suggests that skewness risk premium is a much meaningful variable as it might have filtered out statistical interactions among moments and is expected to serve as an important risk component.

3. Data

For the empirical test of the paper, we use the S&P 500 stock index as a broad market portfolio and the 3-month treasury yield as the risk-free interest rate. Options and futures on the S&P 500 index (symbol: SPX) are traded at the Chicago Board Option Exchange (CBOE). The market for S&P index options and futures is the most active index options and futures market in the world. We obtain all risk-neutral volatility and skewness data on a daily basis directly from the exchange. Our data covers the period January 1990 until January 2011.

In a first step, we construct monthly measures of physical moments from daily S&P500 stock returns. On month t, the i-th daily return is given by $p_{t-1+\frac{i}{N}} - p_{t-1+\frac{i-1}{N}}$, where p_h is the natural logarithm of the price observed at time h and N is the number of return observations in a trading month. In order to subsequently calculate central moments, the return is demeaned. The realized central moments of month t under the physical measure are then computed as follows (see Amaya et al. (2012), Andersen et al. (2001, 2003) and Barndorff-Nielsen and Shephard (2002) for details):

$$Variance_t^P = \sum_{i=1}^{N} r_{t,i}^2$$

$$Skewness_t^P = \sqrt{N} \sum_{i=1}^{N} r_{t,i}^3$$

where $r_{t,i}$ is the mean adjusted return on day i and month t, and N is the number of trading days in month t. An appealing characteristic of these measures of realized central moments is that they are essentially model-free. Typically, one refers to these moments as the ex-post central moments under the physical measure. In line with Harvey and Siddique (2000a, 2000b), we consider a skewness, measure, which is not normalized by the standard deviation[2].

In a second step, we derive the risk-neutral counterparts to the physical central moments. Bakshi et al. (2003) derive a model-free measure of risk-neutral variance, skewness and kurtosis based on all options over the complete moneyness range for a particular time to maturity T[3]. They show that the variance and (non-normalized) skewness of the risk-neutral distribution can be computed by

$$Variance_i^Q(T) = e^{rT} V_i(T) - \mu_i^2(T)$$

$$Skewness_i^Q(T) = e^{rT} W_i(T) - 3\mu_i(T) e^{rT} V_i(T) + \mu_i^3(T)$$

[2] One typically normalizes the central moments, which is not appropriate in our case. E.g. normalized skewness can be calculated by $\frac{Skewness}{(Variance)^{3/2}}$.

[3] See Bekkour et al. (2012) and CBOE (2009, 2010) for a discussion of how to implement the method and perform the calculations with actual data.

Where

$$\mu_i(T) = e^{rT} - 1 - \frac{e^{rT}}{2} V_i(T) - \frac{e^{rT}}{6} W_i(T) - \frac{e^{rT}}{24} X_i(T)$$

$$V_i(T) = \int_S^{+\infty} \frac{2\left(1 - \ln\left(\frac{K}{S_i}\right)\right)}{K^2} c_i(T,K)dK + \int_0^S \frac{2\left(1 + \ln\left(\frac{S_i}{K}\right)\right)}{K^2} p_i(T,K)dK$$

$$W_i(T) = \int_S^{+\infty} \frac{6\ln\left(\frac{K}{S_i}\right) - 3(\ln\left(\frac{K}{S_i}\right))^2}{K^2} c_i(T,K)dK - \int_0^S \frac{6\ln\left(\frac{S_i}{K}\right) - 3(\ln\left(\frac{S_i}{K}\right))^2}{K^2} p_i(T,K)dK$$

$$X_i(T) = \int_S^{+\infty} \frac{12\ln\left(\frac{K}{S_i}\right) - 4(\ln\left(\frac{K}{S_i}\right))^3}{K^2} c_i(T,K)dK + \int_0^S \frac{12\ln\left(\frac{S_i}{K}\right) - 4(\ln\left(\frac{S_i}{K}\right))^3}{K^2} p_i(T,K)dK$$

where S_i is the underlying S&P500 index level on day i, K is the exercise price of the option, r is the risk-free interest rate corresponding to the time to maturity (T) of the option and N(.) is the cumulative normal distribution. c and p refer to call and put prices. As a result, one can obtain the risk-neutral moments on a daily basis. Furthermore, in order to obtain the monthly central moments $Variance_t^Q$ and $Skewness_t^Q$ that we use in the subsequent analysis, we calculate an average over the daily risk-neutral moments of the particular month.

As a result, we obtain the risk-neutral counterparts of the realized first and second central moments under the physical measure. Typically, one refers to these moments as the ex-ante central moments under the risk-neutral measure. In a final step, we combine the central moments under both physical and risk-neutral measures and derive the variance risk premium (VRP) and skewness risk premium (SRP), given by,

$$VRP_t = Variance_t^P - Variance_t^Q$$

$$SRP_t = Skewness_t^P - Skewness_t^Q$$

Our final data set for the above empirical test consists of end-of-months observations of all relevant variables.

4. Empirical tests

The main purpose of our empirical analysis is to test the theoretical relationships derived in section 2. Firstly, we discuss the summary statistics for the entire sample of daily as well as monthly data. Secondly, based on the jump diffusion process used in the theoretical derivation of the model, we calculate risk aversion coefficients in order to better understand the theoretical implications of the model. Thirdly, we test the implications of the theoretical model using regression analysis.

Summary statistics

The summary statistics of daily return, monthly excess return and moments are reported in Table 1. The reported values for skewness and kurtosis are non-standardized. The higher moments are oftentimes substantially different between the two markets. The average daily variance in the stock market is significantly lower than the average risk-neutral variance in the options market. Same relation holds in excess return of the monthly data, where physical variance on average (0.288%) is lower than its risk-neutral counterpart (0.404%), resulting in a variance risk premium that is on average negative. This is consistent with our theoretical model as well as previous studies, e.g. Bollerslev, Tauchen and Zhou (2009), who find that option implied volatility is generally higher than realized volatility.

[Table 1]

We obtain a similar, but even more extreme pattern for skewness. Our finding of a negative skewness for S&P500 index returns for the equity as well as the equity option market complies with previous findings. Stock returns are on average left-skewed. Risk-neutral distributions from options data are typically more negatively skewed compared to their physical counterparts. These typical findings in our model are dynamically captured by a key component, the Poisson jump. When there is a negative jump size, which is commonly observed in the stock market, the model generates a risk-neutral skewness that is more negative than its physical counterparts. This would results in a negative skewness risk premium, as has been shown in the corollary. Statistically, we confirm a negative skewness risk premium on average for our dataset.

[Figure 1]

There are also distinct patterns in time series of the two risk premiums over the sample period, as can be seen in Figure 1. Variance risk premium on average remains slightly negative, then peaks to a high positive in crisis periods. Skewness risk premium, on the contrary but intuitively, on average keeps positive, and becomes even more positive during crisis periods. This pattern again is in accordance with our theoretical model that when market crash causes a negative jump size on stock price, it transfers to a positive skewness risk premium.

Risk Aversion

Both our theoretical model and the empirical test suggest that risk aversion is of crucial importance when studying the relationship between asset and option market risk premiums and excess returns. Hence, in the following, we study the parameters of the jump diffusion model, the risk aversion of the representative investor in particular. For expediency, we focus on the special case with constant jump size.

Under jump diffusion, the physical density of daily S&P500 returns r_τ (τ=1/252) is given by

$$p(r_\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Re\left[e^{-ikr_\tau} f_{r_\tau}(k)\right] dk$$

where

$$ln\left(f_{r_\tau}(k)\right) = ik\mu\tau - \frac{1}{2}ik(1-ik)\sigma^2\tau + \lambda\tau\left(e^{ikx} - 1 - ik(e^x - 1)\right)$$

all parameters that characterizing the S&P500 returns r_τ (namely μ, σ, λ, x and τ as defined earlier) are expressed in annual terms and $f_{r_\tau}(k)$ is its characteristic function. The integral can be evaluated numerically, e.g. using Romberg integration methods. The parameters of the model can be obtained by maximizing the log likelihood function $L_{r_\tau}(\mu, \sigma, \lambda, x) = \sum_{n=1}^{N} \ln\left(p_n(r_\tau)\right)$.

The first three central moments in the physical measure are given by

$$E(r_\tau) = \tau\left(\mu - \frac{1}{2}\sigma^2 - \lambda(e^x - 1 - x)\right)$$
$$E\big(r_\tau - E(r_\tau)\big)^2 = \tau(\sigma^2 + \lambda x^2)$$
$$E(r_\tau - E(r_\tau))^3 = \tau\lambda x^3$$

It is apparent from the equations that all central moments are a linear combination of central moments of the diffusion process and the jump process. Jump risk can contribute positively or negatively to the mean, while it always contributes positively to the variance. The skewness of the returns is a result of the jump risk only.

The equity premium is then calculated as $\phi = \mu - r_f$, where r_f is the average risk-free rate over the same period. Finally, the risk aversion coefficient, γ, can be obtained from the following equation

$$\mu - r_f = \gamma\sigma^2 + \lambda(1 - e^{-\gamma x})(e^x - 1)$$

Estimation results are presented in Table 2, Panel A. Over the period 1990-2010, the S&P500 return process exhibits frequent negative jumps of a magnitude of around -3.5%. The combination of jump intensity and negative jump size can explain the negative skewness in the unconditional return distribution. This is consistent with the empirical findings of an implied volatility smirk and a negative variance risk premium. Additionally, negative jumps are consistent with the observed positive skewness risk premium. For our data, negative jumps result in a normalized skewness of returns of -0.65 and jump risk contributes 1/5 to overall volatility of around 18%.

[Table 2]

Given the average 3-months treasury yield r_f over the period (3.69%) as a proxy for the risk-free rate, we obtain 3.78% for the equity premium and, hence, a relative risk aversion coefficient of 1.93, which is in line with estimates obtained in previous studies. 78% of the equity premium is a diffusive risk premium and 22% is a jump risk premium. More importantly, the risk aversion coefficient is in a range, where the theoretical implications of the model comply with risk-compensation theory.

However, there is more and more empirical evidence that risk aversion changes over time. Recently, Yu and Yuan (2011) show that investor sentiment has an influence on the markets' mean-variance tradeoff. They argue and present empirical evidence that in high sentiment periods, more sentiment traders are present in the market and have more impact on stock prices. In those times, the mean-variance relationship is essentially flat. Consequently, the perception towards risk of market participants can be assumed to change and, therefore, we should observe a different level of risk aversion under those market conditions. Given that the theoretical relationship of the variance- or skewness risk premium and the equity premium depends on the risk aversion of the representative investor, time varying risk aversion can be expected to have an impact on our analysis. We construct a set of sentiment index by taking the average of the past six months' Baker and Wurgler end-of-month sentiment index as the current-month index. By doing so, we smooth out some noise in the data. An observation is regarded as in low sentiment regime if the corresponding constructed sentiment index is below zero and as in high sentiment regime if it is above zero. From period 1/2/1990 to 1/28/2011, of the 252 monthly smoothed index observations, 150 observations are below zero, accounting for roughly a bit more than half of the sample.

We hypothesize that periods of high and low sentiment can be associated with different market conditions, where investors exhibit different levels of risk aversion. Hence, we make the physical density and, therefore, the parameters of the model conditional on the two market regimes, e.g.

$$\begin{aligned}
\mu^* &= (1 - D_t^H)\mu_L + D_t^H \mu_H \\
\sigma^* &= (1 - D_t^H)\sigma_L + D_t^H \sigma_H \\
\lambda^* &= (1 - D_t^H)\lambda_L + D_t^H \lambda_H \\
x^* &= (1 - D_t^H)x_L + D_t^H x_H
\end{aligned}$$

where D_t^H is a dummy variable taking the value of 1 in the high sentiment state and 0 otherwise. Again, we calibrate the model on return data by maximizing the log likelihood function $L_{r_\tau}(\mu^*, \sigma^*, \lambda^*, x^*)$ and obtain the parameters $\mu_L, \sigma_L, \lambda_L, x_L, \mu_H, \sigma_H, \lambda_H, x_H$. Results are shown in

Table 2, Panel B.

The estimated parameters obviously show divergent features. Results further suggest that risk aversion in low sentiment regime is relatively high ($\gamma_L = 3.67$); and risk aversion in high sentiment regime is significantly lower ($\gamma_H = 0.22$) (Table 2, Panel C). This finding is consistent with our hypothesis that the average risk aversion is different in the two sentiment regimes[4]. The consistency finds its root in the fact that the sentiment index is linked to economic fundamentals. Other features also reasonably show up. For example, the high sentiment regime is characterized by higher volatile and more negative jump size, than those in the low sentiment regime, resulting in a distribution that is more left skewed in the high sentiment period. Our findings can also be seen as a theoretical motivation of the results obtained in Yu and Yuan (2011). Theoretically, the mean-variance trade-off should be substantially stronger in the high risk aversion (low sentiment) regime compared to the low risk aversion (high sentiment) regime. This is exactly in line with our results and consistent with the main findings in Yu Yuan (2011).

Regression results

We test the theoretical prediction of the model in an empirical application using S&P500 index and index options data. The regression analysis is based on the theoretical model presented in proposition four. In a first step, we regress the monthly excess returns separately on the skewness risk premium. Additionally, we introduce a dummy variable that controls for a two-regime case. Given that the risk aversion is substantially lower in the high sentiment regime, our theoretical model predicts a different impact of the skewness risk premium on the market risk premium. We analyze the following two regression equations:

$$R_{t+1} = a_0 + b_0 SRP_t(R_{t+1}) + \varepsilon_{t+1}$$

$$R_{t+1} = a_0 + b_0 SRP_t(R_{t+1}) + a_1 D_t^H + b_1 D_t^H SRP_t(R_{t+1}) + \varepsilon_{t+1}$$

Where R_{t+1} is our proxy of market excess return ϕ in period t+1, the monthly return on the S&P500 minus the risk-free rate, proxied by the 3-month treasury yield, D_t^H is a dummy

[4] Thus, we could actually use “two-regime sentiment” to represent “different market conditions” or ”time-varying risk aversion”.

variable, taking a value of one in the high sentiment (or low risk aversion) period and zero otherwise and $\mathrm{SRP(t)} \equiv \mathrm{Skewness}_t^P - \mathrm{Skewness}_t^Q$ is the skewness risk premium.

We expect b_0 to be positive since there should be positive risk compensation for bearing a downward market prospect and we expect b_1 to be negative as investors with relatively lower risk aversion preference would weaken a risk-compensation effect.

Results are reported in Table 3. There is indeed a positive tradeoff between skewness risk premium and excess market return either in a general market condition (b_0=1.28) or in a low sentiment regime (b_0=15.12, with t-statistic of 2.59). Such a tradeoff is greatly weakened (b_1=-15.44) during high sentiment period, as sentiment traders are present and influence the level of risk aversion in the market. A lower risk aversion appetite substantially weakens the skewness-risk-premium-return tradeoff. This finding is in line with our theoretical relation of a risk-compensation model.

[Table 3]

It is interesting to find that by distinguishing two market conditions, the impact of skewness risk premium on aggregate excess return becomes divergent. Looking back to the theoretical component of the coefficient in front of skewness risk premium, which is nothing but a relative risk aversion coefficient and a predetermined time horizon, this finding also questions the suitability of a constant relative risk aversion assumption and shows an obvious time-varying characteristic of it.

In order to empirically investigate the effect of time-varying risk aversion, we test other risk premiums in the fourth proposition in regression equations as follows:

$$R_{t+1} = a_0 + b_0 Variance_t^P(R_{t+1}) + a_1 D_t^H + b_1 D_t^H Variance_t^P(R_{t+1}) + \varepsilon_{t+1}$$

$$R_{t+1} = a_0 + b_0 VRP_t(R_{t+1}) + a_1 D_t^H + b_1 D_t^H VRP_t(R_{t+1}) + \varepsilon_{t+1}$$

$$R_{t+1} = a_0 + b_0 Skewness_t^P(R_{t+1}) + a_1 D_t^H + b_1 D_t^H Skewness_t^P(R_{t+1}) + \varepsilon_{t+1}$$

where $\mathrm{R_{t+1}}$, D_t^H, are the same definitions as in the previous regressions. $\mathrm{Variance_t^P}$, $\mathrm{Skewness_t^P}$ are physical variance and physical skewness respectively, and $\mathrm{VRP(t)} \equiv \mathrm{Variance_t^P} - \mathrm{Variance_t^Q}$ is the variance risk premium.

[Table 4]

Estimation results are shown in Table 4. The finding of a significant positive return-variance tradeoff (b_0=4.05, with t-statistic of 2.81) in high risk aversion periods and a significantly weakened tradeoff (b_1=-5.80, with t-statistic of -3.75) in low risk aversion periods, in line with theory, support our view that return-risk tradeoff varies with different risk aversion in the market. The other two risk factors, i.e. variance risk premium and skewness, fail to exhibit expected relations as the t-statistics are not significant.

5. *Conclusion*

Using an equilibrium asset and option pricing model in a production economy under jump diffusion, we theoretically show that the aggregated excess market returns can be predicted by the skewness risk premium, which is constructed to be the difference between the physical and the risk-neutral skewness. In the subsequent empirical test of the model using more than 20 years of options data on the S&P500, we find that, in line with theory, risk-averse investors demand risk-compensation for holding stocks when the market skewness risk premium is high. However, when we characterize periods of high and low risk aversion, we show that the relationship only holds when risk aversion is high. In periods of low risk aversion, investors demand lower risk compensation, thus substantially weakening the skewness-risk-premium-return trade off. Our study also contributes to the literature by studying properties of a skewness risk premium. We show theoretically that the skewness risk premium is essentially captured by the jump risk of stock prices. Negative jump sizes result in a positive skewness risk premium. As in accordance with the well documented negative jumps exhibited in stock market, the observed positive skewness risk premium over the whole sample period verifies the theoretical relation between the two.

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Appendix A

Proof of Proposition1:

Based on the production process in the economy defined in setup formulas (1)(2)(3), the market excess return solves

(A.1)

$$\phi \equiv \phi_\sigma + \phi_J = \gamma\sigma^2 + \lambda E[(1-e^{-\gamma x})(e^x - 1)]$$

Using Taylor expansion on both $(1-e^{-\gamma x})$ and $(e^x - 1)$, we get

(A.2)

$$\begin{aligned}\phi &= \gamma\sigma^2 + \lambda E[(1-e^{-\gamma x})(e^x - 1)]\\ &= \gamma\sigma^2 + \lambda E\left[\gamma x^2 + \frac{1}{2}\gamma(1-\gamma)x^3 + \frac{\gamma}{12}(2\gamma^2 - 3\gamma + 2)x^4 + \frac{\gamma}{24}(-\gamma^3 + 2\gamma^2 - 2\gamma + 1)x^5 \right.\\ &\qquad\left. + \gamma o(x^6))\right]\\ &= \gamma\sigma^2 + \lambda\gamma E(x^2) + \frac{1}{2}\lambda\gamma(1-\gamma)E(x^3) + \frac{\lambda\gamma}{12}(2\gamma^2 - 3\gamma + 2)E(x^4)\\ &\qquad + \frac{\gamma}{24}(-\gamma^3 + 2\gamma^2 - 2\gamma + 1)E(x^5) + \lambda\gamma o(x^6))\end{aligned}$$

Define central moments on jump size component x as follows

(A.3)

$$\mu_x \equiv Ex$$
$$Var(x) \equiv E(x - Ex)^2 \equiv \sigma_x^2$$
$$Skew(x) \equiv E(x - Ex)^3$$
$$Kurt(x) \equiv E(x - Ex)^4$$
$$Fifth(x) \equiv E(x - Ex)^5$$

By employing cumulant generating function, we get relations between moments and central moments as follows

(A.4)

$$E(x^2) = Var(x) + \mu_x^2$$
$$E(x^3) = Skew(x) + 3\mu_x Var(x) + \mu_x^3$$
$$E(x^4) = Kurt(x) + 4\mu_x Skew(x) + 6\mu_x^2 Var(x) + \mu_x^4$$
$$E(x^5) = Fifth(x) + 5\mu_x Kurt(x) + 10\mu_x^2 Skew(x) + 10\mu_x^3 Var(x) + \mu_x^5$$

Next we focus on how to translate the moments on jump size component x into central-moments on returns Y_τ .

Define other conditional central moments on returns $Y_\tau$as follows

(A.5)

$$Var_t(Y_\tau) \equiv E_t[Y_\tau - E_t Y_\tau]^2$$
$$Skew_t(Y_\tau) \equiv E_t[Y_\tau - E_t Y_\tau]^3$$
$$Kurt_t(Y_\tau) \equiv E_t[Y_\tau - E_t Y_\tau]^4$$
$$Fifth_t(Y_\tau) \equiv E_t[Y_\tau - E_t Y_\tau]^5$$

As is in accordance with results in Zhang, Zhao and Chang (2012), after integrating on the jump diffusion model (1), the return process and its conditional expectation can be written in explicit form

(A.6)

$$Y_\tau \equiv \ln\left(\frac{S_{t+\tau}}{S_t}\right) = \left(r_f + \phi - \frac{1}{2}\sigma^2 - \lambda \mathrm{E}(\mathrm{e}^{\mathrm{x}} - 1)\right)\tau + \sigma \mathrm{B}_\tau + \sum_{i=1}^{\mathrm{N}_\tau} x_i$$

$$E_t(Y_\tau) = \left(r_f + \phi - \frac{1}{2}\sigma^2 - \lambda \mathrm{E}(\mathrm{e}^{\mathrm{x}} - 1)\right)\tau$$

$$Y_\tau - E_t(Y_\tau) = \sigma \mathrm{B}_\tau + \sum_{i=1}^{\mathrm{N}_\tau} x_i = \sigma \mathrm{B}_\tau + \left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]$$

Moment-generating functions of a standard Brownian motion, i.e. $g_{B_\tau}(m) = e^{\frac{1}{2}m^2\tau}$ and of a Poisson process, i.e. $g_{N_\tau}(m) = e^{\lambda\tau(\mathrm{e}^{\mathrm{m}}-1)}$, are applied to get the following properties that are needed in order to calculate the conditional central moments on Y_τ.

(A.7)

$$E(B_\tau) = g'_{B_\tau}(m)|_{m=0} = 0$$
$$E(B_\tau^2) = g''_{B_\tau}(m)|_{m=0} = \tau$$
$$E(B_\tau^3) = g'''_{B_\tau}(m)|_{m=0} = 0$$
$$E(B_\tau^4) = g''''_{B_\tau}(m)|_{m=0} = 3\tau^2$$
$$E(B_\tau^5) = g'''''_{B_\tau}(m)|_{m=0} = 0$$
$$E(N_\tau) = g'_{N_\tau}(m)|_{m=0} = \lambda\tau$$
$$E(N_\tau^2) = g''_{N_\tau}(m)|_{m=0} = \lambda^2\tau^2 + \lambda\tau$$
$$E(N_\tau^3) = g'''_{N_\tau}(m)|_{m=0} = \lambda^3\tau^3 + 3\lambda^2\tau^2 + \lambda\tau$$
$$E(N_\tau^4) = g''''_{N_\tau}(m)|_{m=0} = \lambda^4\tau^4 + 6\lambda^3\tau^3 + 7\lambda^2\tau^2 + \lambda\tau$$
$$E(N_\tau^5) = g'''''_{N_\tau}(m)|_{m=0} = \lambda^5\tau^5 + 10\lambda^4\tau^4 + 25\lambda^3\tau^3 + 15\lambda^2\tau^2 + \lambda\tau$$

Replacing (A.5) by (A.6) and by repeatedly using (A.7) and $\mathrm{E}(x - \mu_x) = 0$, we get the relations between central-moments on returns Y_τ and moments on jump size component x as follows

(A.8)

$$Var_t(Y_\tau) \equiv E_t[Y_\tau - E_tY_\tau]^2$$
$$= \sigma^2 E_t(B_\tau^2) + E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^2 = \sigma^2\tau + \lambda\tau E(x^2)$$

$$Skew_t(Y_\tau) \equiv E_t[Y_\tau - E_tY_\tau]^3 = E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^3 = \lambda\tau E(x^3)$$

$$Kurt_t(Y_\tau) \equiv E_t[Y_\tau - E_tY_\tau]^4$$
$$= \sigma^4 E_t(B_\tau^4) + 6\sigma^2 E_t(B_\tau^2)E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^2$$
$$+ E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^4 = \lambda\tau E(x^4) + 3[Var_t(Y_\tau)]^2$$

$Fifth_t(Y_\tau) \equiv E_t[Y_\tau - E_tY_\tau]^5 = 10\sigma^2 E_t(B_\tau^2) \times E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^3 +$
$E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^5 = \lambda\tau E(x^5) + 10[Var_t(Y_\tau) \times Skew_t(Y_\tau)]$

Where the components inside formulas (A.8) are

$$E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right] = 0$$

$$\begin{aligned} &E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^2 \\ &\qquad = \mu_x^2 E_t(\mathrm{N}_\tau - \lambda\tau)^2 + 2E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x \times \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right] \\ &\qquad + E_t\left[\sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^2 = \mu_x^2 E_t(\mathrm{N}_\tau - \lambda\tau)^2 + E_t(\mathrm{N}_\tau)E_t(x_i - \mu_x)^2 \\ &\qquad = \lambda\tau[\mu_x^2 + \sigma_x^2] \end{aligned}$$

$$\begin{aligned} &E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^3 \\ &\qquad = \mu_x^3 E_t(\mathrm{N}_\tau - \lambda\tau)^3 + 3E_t\left[\big((\mathrm{N}_\tau - \lambda\tau)\mu_x\big)^2 \times \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right] \\ &\qquad + 3E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x \times \left(\sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right)^2\right] + E_t\left[\sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^3 \\ &\qquad = \mu_x^3 E_t(\mathrm{N}_\tau - \lambda\tau)^3 + 3E_t\big((\mathrm{N}_\tau - \lambda\tau)\mu_x \mathrm{N}_\tau (x_i - \mu_x)^2\big) + E_t(\mathrm{N}_\tau)E_t(x_i - \mu_x)^3 \\ &\qquad = \lambda\tau[\mu_x^3 + 3\mu_x Var(x) + Skew(x)] \end{aligned}$$

$$\begin{aligned} &E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x + \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^4 \\ &\qquad = \mu_x^4 E_t(\mathrm{N}_\tau - \lambda\tau)^4 + 4E_t\left[\big((\mathrm{N}_\tau - \lambda\tau)\mu_x\big)^3 \times \sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right] \\ &\qquad + 6E_t\left[\big((\mathrm{N}_\tau - \lambda\tau)\mu_x\big)^2 \times \left(\sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right)^2\right] \\ &\qquad + 4E_t\left[(\mathrm{N}_\tau - \lambda\tau)\mu_x \times \left(\sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right)^3\right] + E_t\left[\sum_{i=1}^{\mathrm{N}_\tau}(x_i - \mu_x)\right]^4 \\ &\qquad + 3E_t\big(\mathrm{N}_\tau(\mathrm{N}_\tau - 1)\big)E_t(x_i - \mu_x)^2 E_t(x_i - \mu_x)^2 \\ &\qquad = \mu_x^4 E_t(\mathrm{N}_\tau - \lambda\tau)^4 + 4E_t\big((\mathrm{N}_\tau - \lambda\tau)^3 \mu_x^3 \mathrm{N}_\tau (x_i - \mu_x)\big) \\ &\qquad + 6E_t((\mathrm{N}_\tau - \lambda\tau)^2 \mu_x^2 \mathrm{N}_\tau (x_i - \mu_x)^2) + 4E_t\big((\mathrm{N}_\tau - \lambda\tau)\mu_x \mathrm{N}_\tau (x_i - \mu_x)^3\big) \\ &\qquad + E_t(\mathrm{N}_\tau)E_t(x_i - \mu_x)^4 + 3E_t\big(\mathrm{N}_\tau(\mathrm{N}_\tau - 1)\big)E_t(x_i - \mu_x)^2 E_t(x_i - \mu_x)^2 \\ &\qquad = \lambda\tau[\mu_x^4(3\lambda\tau + 1) + 6\mu_x^2 Var(x)(\lambda\tau + 1) + 4\mu_x Skew(x) + Kurt(x) \\ &\qquad + 3\lambda\tau[Var(x)]^2] \end{aligned}$$

$$
\begin{aligned}
&E_t\left[(\mathrm{N}_\tau-\lambda\tau)\mu_x+\sum_{i=1}^{\mathrm{N}_\tau}(x_i-\mu_x)\right]^5 \\
&\qquad = \mu_x^5 E_t(\mathrm{N}_\tau-\lambda\tau)^5+5E_t\left[\left((\mathrm{N}_\tau-\lambda\tau)\mu_x\right)^4\times\sum_{i=1}^{\mathrm{N}_\tau}(x_i-\mu_x)\right] \\
&\qquad + 10E_t\left[\left((\mathrm{N}_\tau-\lambda\tau)\mu_x\right)^3\times\left(\sum_{i=1}^{\mathrm{N}_\tau}(x_i-\mu_x)\right)^2\right] \\
&\qquad + 10E_t\left[\left((\mathrm{N}_\tau-\lambda\tau)\mu_x\right)^2\times\left(\sum_{i=1}^{\mathrm{N}_\tau}(x_i-\mu_x)\right)^3\right] \\
&\qquad + 5E_t\big((\mathrm{N}_\tau-\lambda\tau)\mu_x\big)E_t\left[\left(\sum_{i=1}^{\mathrm{N}_\tau}(x_i-\mu_x)\right)^4\right. \\
&\qquad \left. + 3E_t\big(\mathrm{N}_\tau(\mathrm{N}_\tau-1)\big)E_t(x_i-\mu_x)^2E_t(x_i-\mu_x)^2\right]+E_t\left[\sum_{i=1}^{\mathrm{N}_\tau}(x_i-\mu_x)\right]^5 \\
&\qquad + 10E_t\big(\mathrm{N}_\tau(\mathrm{N}_\tau-1)\big)E_t(x_i-\mu_x)^2E_t(x_i-\mu_x)^3 \\
&\qquad = \mu_x^5E_t(\mathrm{N}_\tau-\lambda\tau)^5+5E_t((\mathrm{N}_\tau-\lambda\tau)^4\mu_x^4\mathrm{N}_\tau(x_i-\mu_x)) \\
&\qquad + 10E_t((\mathrm{N}_\tau-\lambda\tau)^3\mu_x^3\mathrm{N}_\tau(x_i-\mu_x)^2)+10E_t((\mathrm{N}_\tau-\lambda\tau)^2\mu_x^2\mathrm{N}_\tau(x_i-\mu_x)^3) \\
&\qquad + 5E_t\big((\mathrm{N}_\tau-\lambda\tau)\mu_x\mathrm{N}_\tau(x_i-\mu_x)^4\big) \\
&\qquad + 15E_t((\mathrm{N}_\tau-\lambda\tau)\mu_x)E_t\big(\mathrm{N}_\tau(\mathrm{N}_\tau-1)\big)E_t(x_i-\mu_x)^2E_t(x_i-\mu_x)^2 \\
&\qquad + E_t(\mathrm{N}_\tau)E_t(x_i-\mu_x)^5+10E_t\big(\mathrm{N}_\tau(\mathrm{N}_\tau-1)\big)E_t(x_i-\mu_x)^2E_t(x_i-\mu_x)^3 \\
&\qquad = \lambda\tau[\mu_x^5(10\lambda\tau+1)+10\mu_x^3Var(x)(4\lambda\tau+1)+10\mu_x^2Skew(x)(\lambda\tau+1) \\
&\qquad + 5\mu_xKurt(x)+30\lambda\tau\mu_x[Var(x)]^2+Fifth(x)+10\lambda\tau Var(x)Skew(x)]
\end{aligned}
$$

Inserting formulas (A.8), which give the relations between moments on jump size component x and central-moments on returns Y_τ, into the market excess return formula (A.2), we get

(A.9)

$$
\begin{aligned}
\phi &= \frac{\gamma}{\tau}Var_t(Y_\tau)+\frac{\gamma}{2\tau}(1-\gamma)Skew_t(Y_\tau)+\frac{\gamma}{12\tau}(2\gamma^2-3\gamma+2)Kurt_t(Y_\tau) \\
&\quad -\frac{\gamma}{4\tau}(2\gamma^2-3\gamma+2)[Var_t(Y_\tau)]^2+\frac{\gamma}{24\tau}(-\gamma^3+2\gamma^2-2\gamma+1)Fifth_t(Y_\tau) \\
&\quad -\frac{5\gamma}{12\tau}(-\gamma^3+2\gamma^2-2\gamma+1)[Var_t(Y_\tau)\times Skew_t(Y_\tau)]+\lambda\gamma\mathrm{E}(\mathrm{o}(\mathrm{x}^6))
\end{aligned}
$$

Therefore, we arrive at formula (6).

Proof of Proposition2 &3:

Remember that jump size component x is normally distributed with mean μ_x and variance σ_x^2. By employing cumulant generating function and using the pricing kernel constructed in Zhang, Zhao and Chang (2012) as $\lambda^{\mathrm{Q}}\equiv\lambda E(e^{-\gamma x})$, we are able to write out central moments in risk-neutral measure by Taylor expansion on $e^{-\gamma x}=1-\gamma x+\frac{1}{2}\gamma^2x^2-\frac{1}{6}\gamma^3x^3+o(x^4)$.

(A.10)

$$Var_t^Q(Y_\tau) = \sigma^2\tau + \lambda^Q\tau\left[(\mu_x^Q)^2 + (\sigma_x^Q)^2\right] = \sigma^2\tau + \lambda\tau E(x^2 e^{-\gamma x})$$
$$= \sigma^2\tau + \lambda\tau E(x^2) - \gamma\lambda\tau E(x^3) + \frac{1}{2}\gamma^2\lambda\tau E(x^4) - \frac{1}{6}\gamma^3\lambda\tau E(\mathrm{o}(\mathrm{x}^5))$$
$$Skew_t^Q(Y_\tau) \equiv E_t^Q[Y_\tau - E_t Y_\tau]^3 = \lambda^Q\tau\left[Skew^Q(x) + 3\mu_x^Q Var^Q(x) + (\mu_x^Q)^3\right] = \lambda\tau E(x^3 e^{-\gamma x})$$
$$= \lambda\tau E(x^3) - \gamma\lambda\tau E(x^4) + \frac{1}{2}\gamma^2\lambda\tau E(x^5) - \frac{1}{6}\gamma^3\lambda\tau E(\mathrm{o}(\mathrm{x}^6))$$

Combining with physical central moments, we can write the variance risk premium and the skewness risk premium separately as
(A.11)
$$VRP_t(Y_\tau) \equiv Var_t(Y_\tau) - Var_t^Q(Y_\tau) = \lambda\tau E(x^2) - \lambda\tau E(x^2 e^{-\gamma x})$$
$$SRP_t(Y_\tau) \equiv Skew_t(Y_\tau) - Skew_t^Q(Y_\tau) = \lambda\tau E(x^3) - \lambda\tau E(x^3 e^{-\gamma x})$$

Again by using Taylor expansion on $e^{-\gamma x} = 1 - \gamma x + \frac{1}{2}\gamma^2 x^2 - \frac{1}{6}\gamma^3 x^3 + o(x^4)$, we get
(A.12)
$$VRP_t(Y_\tau) = \lambda\gamma\tau E(x^3) - \frac{1}{2}\lambda\gamma^2\tau E(x^4) + \frac{1}{6}\lambda\gamma^3\tau E(\mathrm{o}(\mathrm{x}^5))$$
$$= \gamma Skew_t(Y_\tau) - \frac{\gamma^2}{2}Kurt_t(Y_\tau) + \frac{3\gamma^2}{2}[Var_t(Y_\tau)]^2 + \frac{1}{6}\lambda\gamma^3\tau\mathrm{E}(\mathrm{o}(\mathrm{x}^5))$$
$$SRP_t(Y_\tau) = \lambda\gamma\tau E(x^4) - \frac{1}{2}\lambda\gamma^2\tau E(x^5) + \frac{1}{6}\lambda\gamma^3\tau E(\mathrm{o}(\mathrm{x}^6))$$
$$= \gamma Kurt_t(Y_\tau) - 3\gamma[Var_t(Y_\tau)]^2 - \frac{\gamma^2}{2}Fifth_t(Y_\tau) + 5\gamma^2[Var_t(Y_\tau) \times Skew_t(Y_\tau)]$$
$$+ \frac{1}{6}\lambda\gamma^3\tau\mathrm{E}(\mathrm{o}(\mathrm{x}^6))$$
Therefore we arrive at formulas (7)(8).

Proof of Proposition4:

Rewrite (A.12) as
(A.13)
$$Kurt_t(Y_\tau) = \frac{-2VRP_t(Y_\tau)}{\gamma^2} + \frac{2Skew_t(Y_\tau)}{\gamma} + 3[Var_t(Y_\tau)]^2$$
$$Fifth_t(Y_\tau) = \frac{-2SRP_t(Y_\tau)}{\gamma^2} + \frac{-4VRP_t(Y_\tau)}{\gamma^3} + \frac{4Skew_t(Y_\tau)}{\gamma^2} + 10[Var_t(Y_\tau) \times Skew_t(Y_\tau)]$$
Substitute (A.13) into formula (6), we get
$$\phi = \frac{\gamma}{\tau}Var_t(Y_\tau) + \frac{-\gamma^3+\gamma^2-1}{6\tau\gamma^2}VRP_t(Y_\tau) + \frac{-2\gamma^3+2\gamma^2+1}{6\tau\gamma}Skew_t(Y_\tau) + \frac{\gamma^3-2\gamma^2+2\gamma-1}{12\tau\gamma}SRP_t(Y_\tau) + \lambda\gamma\mathrm{E}(\mathrm{o}(\mathrm{x}^6))$$

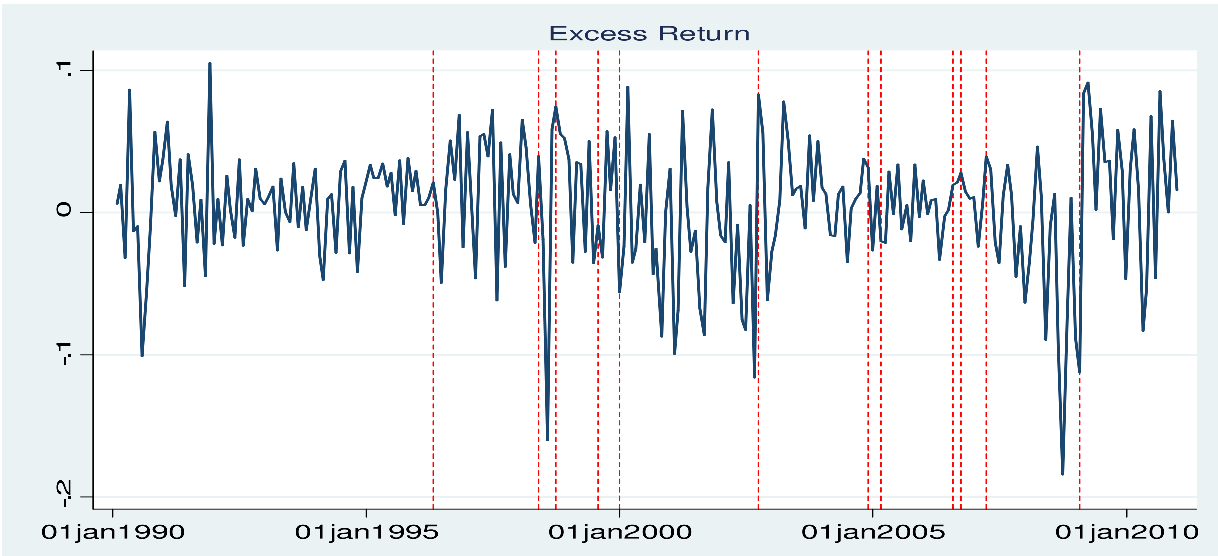
Therefore we arrive at formula (9).

Figure 1: Monthly data time series

This figure presents the monthly return of S&P500 and its corresponding variance- and skewness risk premium, each at a one month horizon. The variance risk premium is physical realized variance minus risk-neutral variance; skewness risk premium is physical realized skewness minus risk-neutral skewness:

$$\mathrm{VRP(t)} \equiv \mathrm{Variance}_t^P - \mathrm{Variance}_t^Q$$
$$\mathrm{SRP(t)} \equiv \mathrm{Skewness}_t^P - \mathrm{Skewness}_t^Q$$

The sample consists of 252 observations from periods 1/31/1990 to 1/28/2011, with 102 observations in high sentiment, low risk aversion periods. The periods dotted in red with horizontal red arrows represent high sentiment periods.



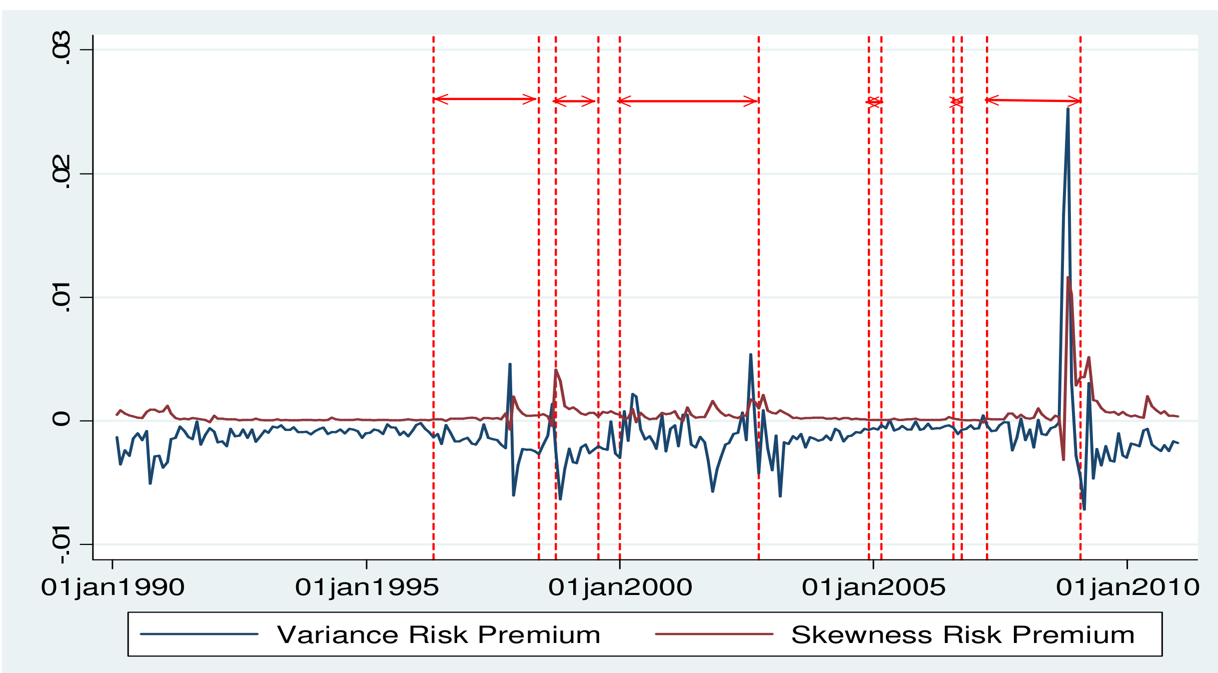


Table 1: Summary statistics

This table presents summary statistics for the sample, which consists of 5302 daily observations (Panel A) and 252 monthly observations (Panel B) from 1/2/1990 to 1/28/2011. Monthly excess return is the sum of daily log-return on the S&P500 minus the risk-free rate, proxied by the 3-month treasury yield. Monthly physical variance is the sum of daily squared logarithm mean-adjusted returns in that month as $\mathrm{Variance}_t^P = \sum_{i=1}^{N} r_{t,i}^2$; Monthly physical skewness is an adjusted sum of daily cubed logarithm mean-adjusted returns in that month as $\mathrm{Skewness}_t^P = \sqrt{N}\sum_{i=1}^{N} r_{t,i}^3$. All risk-neutral moments are derived from daily option prices and averaged over the particular calendar month. The variance risk premium is physical variance minus risk-neutral variance; skewness risk premium is physical skewness minus risk-neutral skewness.

Panel A: Daily Data				
	Mean ($\times 10^3$)	Variance ($\times 10^3$)	Skewness ($\times 10^6$)	Kurtosis ($\times 10^6$)
Returns	0.327	0.137	-0.326	0.223
	Mean	Min	Max	
Risk-Neutral Variance ($\times 10^3$)	0.192	0.034	2.595	
Risk-Neutral Skewness ($\times 10^5$)	-0.568	-21.730	-0.013	

Panel B: Monthly Data				
	Mean ($\times 10^2$)	Variance ($\times 10^2$)	Skewness ($\times 10^4$)	Kurtosis ($\times 10^4$)
Excess Returns	0.411	0.193	-0.667	0.168
	Mean	Min	Max	
Physical Variance ($\times 10^2$)	0.288	0.002	5.735	
Physical Skewness ($\times 10^4$)	-0.321	-39.790	22.985	
Risk-Neutral Variance ($\times 10^2$)	0.404	0.098	3.323	
Risk-Neutral Skewness ($\times 10^3$)	-0.544	-11.37	-0.040	
Variance Risk Premium ($\times 10^2$)	-0.117	-0.717	2.525	
Skewness Risk Premium ($\times 10^3$)	0.512	-3.165	11.651	

Table 2: Maximum likelihood estimation

This table presents parameters estimates when calibrating the model on S&P500 returns. Estimating method is maximum likelihood for a sample from year 1990 to 2010. Parameters in Panels A and B are as follows: μ is average return; σ is average volatility of market return; λ controls for the frequency of jumps; x is a constant jump size. Parameters in Panel B are correspondingly defined as in Panel A, with subscript L denoting low sentiment periods; H denoting high sentiment periods. Panel C reports the relative risk aversion coefficient γ, with its counterparts in the two sub-samples. The calculation is as follows:

$$\mu - r_f = \gamma\sigma^2 + \lambda(1 - e^{-\gamma x})(e^x - 1)$$

where $r_f = 3.69\%$, $r_{f_L} = 3.49\%$ and $r_{f_H} = 3.98\%$.
Significance levels are indicated as ***=1%.

Panel A: Whole Sample

	μ	σ	λ	x
Whole Period	0.0981***	0.1578***	5.5849***	-0.0345***
	(0.0166)	(0.0037)	(1.6693)	(0.0046)

Panel B: Sub-Samples

	$\mu_L(\mu_H)$	$\sigma_L(\sigma_H)$	$\lambda_L(\lambda_H)$	$x_L(x_H)$
Low Sentiment	0.1256***	0.1400***	5.7326***	-0.0292***
	(0.0276)	(0.0013)	(1.7995)	(0.0018)
High Sentiment	0.0489***	0.1880***	3.2051***	-0.0483***
	(0.0136)	(0.0016)	(0.9593)	(0.0024)

Panel C: Risk Aversion

	γ	γ_L	γ_H
Whole Period	1.93		
Low Sentiment		3.67	
High Sentiment			0.22

Table 3: Skewness Risk Premium

This table presents regression results of the impact of skewness risk premiums on excess return. The estimation is as follows (t denotes month t):

$$R_{t+1} = a_0 + b_0 SRP_t(R_{t+1}) + \varepsilon_{t+1}$$
$$R_{t+1} = a_0 + b_0 SRP_t(R_{t+1}) + a_1 D_t^H + b_1 D_t^H SRP_t(R_{t+1}) + \varepsilon_{t+1}$$

where R_{t+1} is a proxy of the market excess return ϕ in period t+1, the monthly return on the S&P500 minus the risk-free rate, proxied by the 3-month T-bond yield, D_t^H is a dummy variable, taking a value of one in the high sentiment (or low risk aversion) period and zero otherwise. $SRP(t) \equiv Skewness_t^P - Skewness_t^Q$ is the skewness risk premium. The sample consists of 252 observations from 1/2/1990 to 1/28/2011.
Significance levels are indicated as ***=1%.

	Skewness Risk Premium			
	a_0	b_0	a_1	b_1
One Regime	0.0035	1.2799		
	(1.13)	(0.53)		
Two Regime	0.0042	15.1217***	-0.0083	-15.4376***
	(1.02)	(2.59)	(-1.34)	(-2.42)

Table 4: Other Risk Premiums, Two Regimes

This table presents regression results of the impacts of other risk premiums on excess return. The estimation is as follows:

$$R_{t+1} = a_0 + b_0 Variance_t^P(R_{t+1}) + a_1 D_t^H + b_1 D_t^H Variance_t^P(R_{t+1}) + \varepsilon_{t+1}$$
$$R_{t+1} = a_0 + b_0 VRP_t(R_{t+1}) + a_1 D_t^H + b_1 D_t^H VRP_t(R_{t+1}) + \varepsilon_{t+1}$$
$$R_{t+1} = a_0 + b_0 Skewness_t^P(R_{t+1}) + a_1 D_t^H + b_1 D_t^H Skewness_t^P(R_{t+1}) + \varepsilon_{t+1}$$

where R_{t+1} is a proxy of the market excess return ϕ in period t+1, the monthly return on the S&P500 minus the risk-free rate, proxied by the 3-month T-bond yield, D_t^H is a dummy variable, taking a value of one in the high sentiment (or low risk aversion) period and zero otherwise. $\mathrm{Variance}_t^P$, $\mathrm{Skewness}_t^P$ are physical variance and physical skewness respectively, and $\mathrm{VRP(t)} \equiv \mathrm{Variance}_t^P - \mathrm{Variance}_t^Q$ is the variance risk premium
The sample consists of 252 observations from 1/2/1990 to 1/28/2011.
Significance levels are indicated as follows:*=10%, **=5%, ***=1%.

	Two Regimes			
	a_0	b_0	a_1	b_1
Variance	0.0023	4.0476***	0.0010	-5.8030***
	(0.53)	(2.81)	(0.15)	(-3.74)
Variance Risk Premium	0.0055	-3.3058	-0.0140*	-1.1717
	(1.07)	(1.14)	(-2.09)	(-0.37)
Skewness	0.0098***	1.6584	-0.0127**	17.6854
	(2.80)	(0.12)	(-2.29)	(1.19)