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ABSTRACT

Common-Value All-Pay Auctions with Asymmetric Information*

We study two-player common-value all-pay auctions (contests) with asymmetric information under the assumption that one of the players has an information advantage over his opponent. We characterize the unique equilibrium in these contests, and examine the role of information in determining the players' expected efforts, probabilities of winning, and expected payoffs. In particular, we show that the players always have the same probability of winning the contest, and that their expected efforts are the same, but their expected payoffs are different. It is also shown that budget constraints may have an unanticipated effect on the players' expected payoffs, i.e., a player's information advantage may turn into a payoff disadvantage.

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Keywords: all-pay auctions, asymmetric information and information advantage

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1 Introduction

All-pay auctions are used in diverse areas of economics, such as lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, and biological wars of attrition. In the all-pay auction each player submits a bid (effort) and the player who submits the highest bid wins the contest, but, independently of success, all players bear the cost of their bids. All-pay auctions have been studied both under a complete information framework where each player's type (the value of winning the contest or ability) is common knowledge (see, e.g., Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993, 1996), Che and Gale (1998), and Siegel (2009)), and under an incomplete information framework where each player's type is private information and only the distribution from which the players' types is drawn is common knowledge (see, for example, Amann and Leininger (1996), Moldovanu and Sela (2001, 2006), and Moldovanu et al. (2010)). In most of the literature on all-pay auctions with incomplete information it is assumed that the players' types are independent. However, in several competitive environments the players' types may not be independent (see Milgrom and Weber 1982).[1] Krishna and Morgan (1997) analyzed the equilibrium strategies of the all-pay auction with interdependent types in the Harsanyi-type formulation of Bayesian games. They assumed that the players' types are affiliated and symmetrically distributed. A generalization of their work to a model where players' types are asymmetrically distributed is usually not tractable.

We study the value of information in a contest with ex-ante asymmetrically informed players by considering a two-player common-value all-pay auction with asymmetric information, where the value of winning is the same for all players in the same state of nature, but the information about which state of nature was realized is different. This model captures situations in which winning a contest would be of similar benefit to each contestant, but the precise value of winning, which depends on several random parameters, may be unknown. In our framework, the information a player has about the value of winning is described by a partition of the space of states of nature, which is assumed to be finite. This partition representation is equivalent to the more common Harsanyi-type formulation of Bayesian games (see Jackson (1993) and Vohra (1999)) but it is more suitable for expressing the information advantage some players may have over others,

[1] Examples of auctions with intedependent values include treasury bill auctions, spectrum auctions, and oil and gas leases.

which will figure prominently in our model.

In our (two-player) model of asymmetric information, we assume that information sets of each player are connected with respect to the value of winning the contest (see Einy et al. (2001, 2002) and Forges and Orzach (2011)). This means that if a player's information partition does not enable him to distinguish between two possible values of winning, then he also cannot distinguish between all intermediate values. Connectness seems plausible in environments where the information of a player allows him to put upper and lower bounds on the actual value of winning, without ruling out any outcome within these bounds. We additionally assume that one player has an information advantage over the other, which means that his information partition is finer than that of his opponent. It can be shown that without loss of generality, we can assume that one player is completely informed about the state of nature, while the other player is completely uninformed.

We establish the uniqueness of equilibrium in mixed strategies in this class of contests, and provide its complete characterization. In equilibrium, the expected payoff of the uninformed player is zero, while the expected payoff of the informed payer is positive. Our results also show that although the players have asymmetric strategies that yield different expected payoffs, the expected efforts of both players are the same. Moreover, the probability of each player to win the contest in equilibrium is the same. Hence, we find that asymmetry of information between the players does not result in different expected efforts or different chances to win the contest, but it does affect the allocation of payoffs between the players.

We then examine how the relation between players' information sets affect their expected total effort. We find that maximizing the total effort calls for narrowing the information gap between the players. Specifically, if there are three players (a,b and c) where a has an information advantage over b who has an information advantage over c, then the expected total effort in the contest between a and c is necessarily lower than in the contest between b and c. In other words, when the players' information are similar to each other their total effort is larger.

Finally, we assume that players face budget constraints, which implies that there are caps on the bids that the players are able to place. A budget constraint changes the players' equilibrium behavior compared to the same contests without budget constraints. This was shown, among others, by Che and Gale (1998) and Gavious, Moldovanu and Sela (2003) in the standard all-pay auction under complete and incomplete information.[2] This observation is also valid in common-value all-pay auctions where the budget constraint may drastically change the relation between the players' expected payoffs. Furthermore, we show that the budget constraint may imply the unusual result according to which the player with the information advantage will have a lower expected payoff than his opponent. In other words, an information advantage may turn into a payoff disadvantage.

Several researchers used the same framework as ours to analyze common-value second-price auctions (see Einy et al. (2001, 2002), Forges and Orzach (2011), and Abraham et al. (2012)) and common-value first-price auctions (see Malueg and Orzach (2009, 2012)) but not to analyze asymmetric all-pay auctions. Without budget constraints, in the common-value all-pay auctions, as well as in the common-value first-price and second-price auctions, the player with an information advantage has a higher expected payoff than his opponent. However, in common-value all-pay auctions the players' bids (efforts) as well as their chances of winning are the same despite the asymmetry of information.

Although we analyze two-player (common-value all-pay) auctions, our results can be generalized to any number of players as long as the players' information partitions can be ranked, namely, in all pairwise comparisons one player will have an information advantage over the other. In such a case, as well as in the complete information all-pay auction (see Baye et al. 1996), there will be an equilibrium in which only the two most informed players participate, and the rest stay out of the contest (or, alternatively, place bids of zero).

The paper is organized as follows. In Section 2 we present the model. In Section 3 we give a numerical example that demonstrates how to find the equilibrium in our model. Section 4 is divided into three subsections: In Section 4.1 we characterize the equilibrium and prove its uniqueness. In Section 4.2 we analyze the players' expected effort, their probabilities of winning, and their expected payoffs. In Section 4.3 we examine the effect of information on the players' total effort. In section 5 we study the model with budget-constrained players. Section 6 concludes. The proof of Proposition 1 is provided in the appendix.

[2] Che and Gale (1998) and Gavious, Moldovanu and Sela (2003) deal with all-pay auctions with bid caps (budget constraints that the contest designer imposes on the contestants).

2 The model

Consider the set $\mathcal{N} = \{1, 2, ..., N\}$ of $N \geq 2$ players who compete in an all-pay auction where the player with the highest effort (output) wins the contest, but all the players bear the cost of their effort. The uncertainty in our model is described by a finite set Ω of states of nature, and a probability distribution p over Ω which can be interpreted as the common prior belief about the realized state of nature (w.l.o.g. $p(\omega) > 0$ for every $\omega \in \Omega$). A function $v : \Omega \to \mathbb{R}_+$ represents the common value of winning the contest, i.e., if $\omega \in \Omega$ is realized then the value of winning is $v(\omega)$ for every player.

The private information of each player $n \in \mathcal{N}$ is described by a partition Π_n of Ω. We assume that each Π_n is *connected* with respect to the common value function v, i.e., for every element $\pi_n \in \Pi_n$, if $\omega_1, \omega_2 \in \pi_n$ and $\omega \in \Omega$ satisfy $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$, then $\omega \in \pi_n$.[3]

A common-value all-pay auction starts when nature chooses a state ω form Ω according to the distribution p. Each player $n \in \mathcal{N}$ is informed of the element $\pi_n(\omega)$ of Π_n which contains ω. Thus, $\pi_n(\omega)$ constitutes the information set of player n at ω, and then he chooses an effort $x_n \in \mathbb{R}_+$. The players will typically have different information partitions, and thus are ex-ante asymmetric.

The utility (payoff) of player $n \in \mathcal{N}$ is given by the function $u_n : \Omega \times \mathbb{R}_+^N \to \mathbb{R}$ as follows:

$$u_n(\omega, x) = \begin{cases} \frac{1}{m(x)} v(\omega) - x_n, & \text{if} \quad x_n = \max\{x_k\}_{k \in \mathcal{N}}, \\ -x_n, & \text{if} \quad x_n < \max\{x_k\}_{k \in \mathcal{N}}, \end{cases}$$

where $m(x)$ denotes the number of players who exert the highest effort, namely, $m(x) = |n \in N : x_n = \max\{x_k\}_{k \in \mathcal{N}}|$. A common-value all-pay auction with differential information is fully described by and identified with the collection $G = (N, (\Omega, p), \{u_n\}_{n \in \mathcal{N}}, \{\Pi_n\}_{n \in \mathcal{N}})$.

In all-pay auctions, there is usually no equilibrium in pure strategies. Thus our attention will be given to mixed strategy equilibria. A mixed strategy of player $n \in \mathcal{N}$ is a function $F_n : \Omega \times \mathbb{R}_+ \to [0, 1]$, such that for every $\omega \in \Omega$, $F_n(, \cdot)$ is a cumulative distribution function (c.d.f.) on $\mathbb{R}_+$, and for all $x \in \mathbb{R}_+$, $F_n(\cdot, x)$ is a Π_n-measurable function (that is, $F_n(\cdot, x)$ is constant on every element of Π_n). If player n plays a pure

[3] It is worth noting that our analysis remains valid if Ω is an infinite set of states of nature provided the partitions are finite. To see this simply replace Ω with a finite Ω', which is the coarsest partition of Ω that refines all $\{\Pi_n\}_{n \in \mathcal{N}}$, and, for each $\pi \in \Omega'$, let the value of winning at π, $v(\pi)$, be equal to the conditional expectation $E(v(\cdot) \mid \pi)$.

strategy given π_n, i.e., if the distribution represented by $F_n(\pi_n, \cdot)$ is supported on some $y \in \mathbb{R}_+$, we will identify between $F_n(\pi_n, \cdot)$ and y wherever appropriate.

Given a mixed strategy profile $F = (F_1, ..., F_N)$, denote by $E_n(F)$ the expected payoff of player n when players use that strategy profile, i.e.,

$$E_n(F) \equiv E(\int_0^\infty ... \int_0^\infty u_n(\cdot, (x_1, ..., x_N))dF_1(\cdot, x_1), ..., dF_N(\cdot, x_N)).$$

For $\pi_n \in \Pi_n$, $E_n(F \mid \pi_n)$ will denote the conditional expected payoff of player n given his information set π_n, i.e.,

$$E_n(F \mid \pi_n) \equiv E(\left[\int_0^\infty ... \int_0^\infty u_n(\cdot, (x_1, ..., x_n))dF_1(\cdot, x_1), ..., dF_i(\cdot, x_i), ..., dF_N(\cdot, x_N)\right] \mid \pi_n).$$

An N-tuple of mixed strategies $F^* = (F_1^*, ..., F_N^*)$ constitutes a Bayesian equilibrium in the common-value all-pay auction G if for every player n, and every mixed strategy F_n of that player, the following inequality holds:

$$E_n(F^*) \geq E_n(F_1^*, ..., F_n, ..., F_N^*).$$

3 An Example

We begin with a simple example to illustrate the players' behavior in our model. Consider a common-value all-pay auction with two players. Assume that there are three states of nature such that in state ω_i the value of winning is $v(\omega_i) = i$ with probability of $p_i = \frac{1}{3}$, for $i = 1, 2, 3$. Player 1 knows *only* the prior distribution p, and hence he has the trivial information partition, $\Pi_1 = \{\{\omega_1, \omega_2, \omega_3\}\}$, while player 2 is completely informed of the value of winning, hence $\Pi_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ partitions Ω into singletons.

It can be easily verified that the corresponding common-value all-pay auction does not have an equilibrium in pure strategies. However, there does exist a mixed strategy equilibrium. In this equilibrium, player 1's mixed strategy F_1^* is a state-independent c.d.f. given by

$$F_1^*(x) = \begin{cases} 0 & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{x}{2} + \frac{1}{6}, & \text{if } \frac{1}{3} < x \leq 1, \\ \frac{x}{3} + \frac{1}{3} & \text{if } 1 < x \leq 2, \\ 1, & \text{if } 2 < x. \end{cases}$$

Player 2's mixed strategy F_2^* does depend on the state of nature (of which he is informed):

$$F_2^*(\omega_1, x) = \begin{cases} 0 & \text{if } x < 0, \\ 3x, & \text{if } 0 \leq x \leq \frac{1}{3}, \\ 1 & \text{if } x > \frac{1}{3}, \end{cases}$$

$$F_2^*(\omega_2, x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{3}, \\ \frac{3}{2}x - \frac{1}{2}, & \text{if } \frac{1}{3} < x \leq 1, \\ 1, & \text{if } x > 1, \end{cases}$$

$$F_2^*(\omega_3, x) = \begin{cases} 0, & \text{if } x < 1, \\ x - 1, & \text{if } 1 \leq x \leq 2, \\ 1, & \text{if } x > 2. \end{cases}$$

In order to see that the above strategies are in equilibrium, note that, given player 2's mixed strategy F_2^*, player 1's expected payoff if he exerts effort $x \in [1, 2]$ is

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 \cdot (x - 1) - x = 0$$

When $x \in [\frac{1}{3}, 1]$,

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 \cdot (\frac{3}{2}x - \frac{1}{2}) - x = 0,$$

and when $x \in [0, \frac{1}{3}]$,

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 \cdot (3x) - x = 0.$$

As any effort above 2 would result in a negative expected payoff, $[1, 2]$ is the set of player 1's pure strategy best responses to to F_2^*, and in particular his mixed strategy F_1^* is a best response to F_2^* as it results in an expected payoff of zero.

Now, fix payer 1's mixed strategy F_1^*, and assume that ω_3 is the realized state of nature. If player 2 exerts effort $x \in [1, 2]$, then his conditional expected payoff is

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot (\frac{x}{3} + \frac{1}{3}) - x = 1.$$

If he exerts $x \in [\frac{1}{3}, 1)$ or $x \in [0, \frac{1}{3}]$, his expected payoff is, correspondingly,

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot (\frac{x}{2} + \frac{1}{6}) - x = \frac{x}{2} + \frac{1}{2} < 1,$$

or

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot x - x = 2x < 1,$$

and thus, conditional on the realization of ω_3, $[1, 2]$ is the set of player 2's pure strategy best responses to F_1^*. In particular, conditional on ω_3, $F_2^*(\omega_3, \cdot)$ is a mixed strategy best response to F_1^*.

If ω_2 is the realized state, by exerting $x \in [\frac{1}{3}, 1]$ player 2 obtains the expected payoff

$$E_2(\{\omega_2\}, F_1^*, x) = 2 \cdot (\frac{x}{2} + \frac{1}{6}) - x = \frac{1}{3}.$$

As before, it can be seen that all effort levels outside $[\frac{1}{3}, 1]$ lead to a lower expected payoff, and thus conditional on ω_2, $F_2^*(\omega_2, \cdot)$ is a mixed strategy best response to F_1^*.

If ω_1 is the realized state, by exerting $x \in [0, \frac{1}{3}]$ player 2, in expectation, obtains

$$E_2(\{\omega_1\}, x) = 1 \cdot x - x = 0,$$

while effort levels outside $[0, \frac{1}{3}]$ lead to negative expected payoffs. Thus, also conditional on ω_1, $F_2^*(\omega_1, \cdot)$ is a mixed strategy best response to F_1^*. We conclude that F_2^* is a best response of player 2 also w.r.t. the unconditional expected payoff. Hence, the pair (F_1^*, F_2^*) is a mixed strategy equilibrium. The expected payoff of player 2 is then

$$E_2(F_1^*, F_2^*) = \frac{1}{3}(E_2(\{\omega_1\}, F_1^*, F_2^*) + E_2(\{\omega_2\}, F_1^*, F_2^*) + E_2(\{\omega_3\}, F_1^*, F_2^*)) = \frac{4}{9}.$$

In the next section, we characterize the players' mixed-strategy equilibrium in a general two-player common-value all-pay auction, and prove its uniqueness.

4 Results

4.1 Equilibrium analysis

We will consider first the case of two players, where player 2 has an information advantage over player 1 (i.e., information partition Π_2 of player 1 is finer than Π_1). Without loss of generality, we assume that $\Pi_1 = \{\Omega\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$.[4] That is, player 1 has no information on the realized state of nature (other than the common prior distribution p) and thus has the trivial information partition, while player 2 knows the realized state precisely, and thus his information partition is the finest one possible.

For each state of nature $\omega_i \in \Omega$, denote $v_i = v(\omega_i)$ and $p_i = p(\omega_i) > 0$. Assume that the possible values are strictly ranked: $0 < v_1 < v_2 < ... < v_n$. In what follows, we describe a mixed strategy equilibrium (F_1^*, F_2^*) of the all-pay auction.

Let $x_0 \equiv 0$, and

$$x_i \equiv \sum_{j=1}^{i} p_j v_j \tag{1}$$

for each $i = 1, ..., n$. Thus, $x_0 < x_1 < ... < x_n$. Consider a function F_1^* on $\mathbb{R}_+$ given by

$$F_1^*(x) = \begin{cases} 0 & \text{if } x < 0, \\ . & . \\ \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right], & \text{if } x \in [x_{i-1}, x_i], \qquad i = 1, ..., n \\ . & . \\ 1, & \text{if } x > x_n. \end{cases} \tag{2}$$

[4] In the general case of Π_2 being finer than Π_1 note the following. Given $\pi_1 \in \Pi_1$, the event π_1 is common knowledge at any $\omega \in \pi_1$. Thus, the equilibrium analysis can be carried out separately for each $\pi_1 \in \Pi_1$, as the auction G conditional on the occurence of π_1 can be viewed as a distinct common-value all-pay auction G', where the set of states of nature is $\Omega' = \pi_1$ and the conditional distribution $p(\cdot \mid \pi_1)$ serves the common prior distribution p'. In G', player 1 has the trivial information partition, $\Pi_1' = \{\pi_1\}$. Furthermore, since the mixed strategies of both players are constant on every $\pi_2 \subset \pi_1$, $\pi_2 \in \Pi_2$, there will be no payoff distinction between G' and its variant G'', where the set of states of nature $\Omega'' = \Pi_2 \mid_{\pi_1}$ consists of those elements of Π_2 that are subsets of π_1 (i.e., all states of nature in Ω' that are contained in the same element of π_1 are lumped into one state). By definition, the information partition of player 2 in G'' will be the finest possible, consisting of singletons in $\Pi_2 \mid_{\pi_1}$.

It is easy to see that $F_1^*(x)$ is well defined, strictly increasing, and continuous. Moreover, $F_1^*(x_0) = 0$ and $F_1^*(x_n) = 1$. Thus, $F_1^*(x)$ is a c.d.f. of a continuous probability distribution supported on the interval $[x_0, x_n]$. (Such a distribution is obtained by assigning probability p_i to each interval $[x_{i-1}, x_i]$, randomly choosing an interval, and then selecting a point w.r.t. the uniform distribution on the chosen interval). Being that the function F_1^*, is state-independent, it can be viewed as a mixed strategy of the uninformed player 1.

Note next that

$$\begin{aligned} E_2(\{\omega_i\}, F_1^*, x) &= v_i F_1^*(x) - x & (3) \\ &= v_i F_1^*(x_{i-1}) - x_{i-1} = E_2(\{\omega_i\}, F_1^*, x_{i-1}) & (4) \end{aligned}$$

for every $x \in [x_{i-1}, x_i]$, and $i = 1, ..., n$. Thus, given that ω_i was realized, the informed player 2 is indifferent between all efforts in the interval $[x_{i-1}, x_i]$, provided that his rival acts according to F_1^*. Since the slopes of the function $v_i F_1^*(x) - x$ are positive when $x < x_{i-1}$ and negative when $x > x_{i-1}$, the set of player 2's pure strategy best responses is the interval $[x_{i-1}, x_i]$.

Now, for each $i = 1, ..., n$, consider a function $F_2^*(\omega_i, x)$ on $\mathbb{R}_+$ given by

$$F_2^*(\omega_i, x) = \begin{cases} 0 & \text{if } x < x_{i-1}, \\ \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i}, & \text{if } x \in [x_{i-1}, x_i], \\ 1 & \text{if } x > x_i, \end{cases} \tag{5}$$

Note that $F_2^*(\omega_i, x)$ is well defined, strictly increasing, continuous, $F_2^*(\omega_i, x_{i-1}) = 0$ and $F_2^*(\omega_i, x_i) = 1$. Thus, $F_2^*(\omega_i, x)$ is a c.d.f. of a probability distribution supported on $[x_{i-1}, x_i]$, and in particular F_2^* constitutes a mixed strategy of player 2. Moreover,

$$E_1(x, F_2^*) = \sum_{j=1}^{i-1} p_j v_j + p_i v_i F_2^*(\omega_i, x) - x = 0 \tag{6}$$

for every $x \in [x_{i-1}, x_i]$. Thus, player 1 is (in expectation) indifferent between all efforts in $[x_0, x_n]$ (and is obviously worse off when efforts are outside $[x_0, x_n]$) provided his rival 2 acts according to F_2^*.

We conclude that (F_1^*, F_2^*) is a mixed strategy equilibrium. It turns out that it is the only one:

Proposition 1 *The mixed strategy equilibrium (F_1^*, F_2^*) described above is the unique equilibrium in G.*

Proof. See Appendix. ■

Thus far we have assumed that there are only two players. This entails no loss of generality in the following sense. Suppose that there are $N > 2$ players, such that the players' information endowments are ranked as follows: player 2 has an information advantage over player 1, and player 1 has an information advantage over (or the same information endowment as) players $3, ..., N$. Let (F_1^*, F_2^*) be the unique equilibrium in the contest between 1 and 2 (which exists by Proposition 1 and footnote 4). We claim that in the contest between $1, 2, ..., N$, strategy profile $(F_1^*, F_2^*, 0, 0, ...0)$ constitutes a Bayesian equilibrium. That is, all but the two players with the best information submit bids of zero which means that they are effectively staying out of contest, while players 1 and 2 behave as if they were engaged in a two-player contest. This will ensure that any N-player contest in which information endowments are ranked possesses a reduction to the two-player case.

In order to see that $(F_1^*, F_2^*, 0, 0, ...0)$ is a Bayesian equilibrium, note first that players 1 and 2 have no incentive to unilaterally deviate from their strategies in $(F_1^*, F_2^*, 0, 0, ...0)$, since their payoffs are identical to those in a two-player contest where such deviations are not profitable in expectation. Note next that if any of the remaining players (say, player 3) had a profitable deviation F_3 from bid 0, we would have had

$$E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > E_3(F_1^*, F_2^*, 0, 0, ..., 0) = 0,$$

and hence

$$E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > 0.$$

Since player 1 has an information advantage over (or the same information as) player 3, F_3 is also a Bayesian strategy of player 1. As F_1^* is 1's best response to $(F_1^*, F_2^*, 0, 0, ...0)$, it follows that

$$\begin{aligned} E_1(F_1^*, F_2^*, 0, 0, ...0) &\geq E_1(F_3, F_2^*, 0, 0, ..., 0) \\ &= E_3(0, F_2^*, F_3, 0, ..., 0) \\ &\geq E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > 0. \end{aligned}$$

Thus

$$E_1(F_1^*, F_2^*, 0, 0, ...0) > 0,$$

and in particular $E_1(F_1^*, F_2^*) > 0$ in the two-player contest between 1 and 2. However, it follows from (6), Proposition 1, and Footnote 4 that the expected payoff to player 1 in the unique equilibrium is zero, a contradiction. We conclude that players $3, .., N$ cannot unilaterally deviate from bid 0 and make profit, and hence that $(F_1^*, F_2^*, 0, 0, ...0)$ is a Bayesian equilibrium of the N-player contest, as claimed.

In the next section, using the characterization of equilibrium in Proposition 1, we study the effect of information on players' payoffs, efforts and probabilities of winning.

4.2 Expected payoffs and efforts

We have shown that the equilibrium strategies in a two-payer common-value all-pay auction are determined uniquely. The expected equilibrium payoff of player 1 is zero: it follows from (6) that

$$E_1(F_1^*, F_2^*) = 0. \tag{7}$$

It follows from (3)-(4) that player 2's expected payoff is

$$\begin{aligned} E_2(F_1^*, F_2^*) &= \sum_{i=1}^{n} p_i(v_i F_1(x_{i-1}) - x_{i-1}) \qquad (8) \\ &= \sum_{i=1}^{n} p_i(\sum_{j=1}^{i-1} p_j(v_i - v_j)). \end{aligned}$$

The equilibrium strategies F_1^*, F_2^* of the two players are quite different. Among other distinctions, F_2^* is state-dependent, while F_1^* is not. However, both players have the same *ex-ante* distribution of the effort they make. Indeed, for every $i = 1, 2, ..., n$, and every $x \in [x_{i-1}, x_i]$ (where x_i is given by (1)), the ex-ante probability $F_2(x)$ that player 2 will exert an effort that is smaller than or equal to x according to his strategy F_2^* is given by

$$\begin{aligned} F_2(x) &= \sum_{j=1}^{i-1} p_j + p_i F_2^*(\omega_i, x) = \sum_{j=1}^{i-1} p_j + p_i \cdot \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i} \\ &= \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] = F_1^*(x). \end{aligned}$$

Thus, the ex-ante distribution of equilibrium effort is identical for both players. This fact leads to the following proposition.

Proposition 2 *In the unique equilibrium (F_1^*, F_2^*) of every two-player common-value all-pay auction:*

(i) each player has probability $\frac{1}{2}$ to win;

and

(ii) both players exert the same expected effort

$$\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_j + \frac{1}{2} p_i v_i \right).$$

Proof. It was shown above that the players have ex-ante identical (and, obviously, independent) distributions of efforts, and hence, as claimed in (i), each wins the contest with the same probability. It also follows that the expected efforts of both players are equal. Calculating the expected payoff for player 1 (using (2)) leads to the formula claimed in (ii):

$$EE_1 = \int_{x_0}^{x_n} x dF_1^* (x) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \frac{x}{v_i} dx = \sum_{i=1}^{n} \frac{x_i^2 - x_{i-1}^2}{2v_i} = \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_j + \frac{1}{2} p_i v_i \right).$$

Q.E.D. ■

According to Proposition 2, the asymmetry in information does not affect the ratio of the two players' expected efforts, as the expected efforts are equal. However, the asymmetric information does affect the players' expected *total* effort. In the next section we will examine the effect of asymmetry in information on the expected total effort and the expected total payoff.

4.3 Comparative results

We just showed that the expected payoff of player 1, over whom player 2 has an information advantage, is zero in equilibrium. We will now examine how the *extent* of information advantage affects the expected payoff of player 2. Assume, as before, that $\Pi_1 = \{\Omega\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$. Also consider an additional player 2' with an "intermediate" connected information partition Π_2', which is a strict coarsening of Π_2 and a strict refinement of Π_1. Then we have the following comparative result.

Proposition 3 *In a two-player common-value all-pay auction, the expected payoff of player 2 (when he competes against player 1) is higher than the expected payoff of player 2' (when he competes against player 1).*

Proof. By (8), the expected payoff of player 2, when he competes against player 1, is given in equilibrium by

$$E_2 = \sum_{i=1}^{n} p_i (\sum_{k=1}^{i-1} p_k (v_i - v_k)).$$

Regarding player 2', assume first that Π_2' is different from Π_2 only in that player 2' cannot distinguish between the states ω_j and ω_{j+1}, for some $1 \le j < n$. Thus, $\Pi_2' = \{\{\omega_1\}, \{\omega_2\}, ...\{\omega_{j-1}\}, \{\omega_j, \omega_{j+1}\}, \{\omega_{j+2}\}, ...\{\omega_n\}\}$. The auction in which player 2' competes against player 1 is amenable to our previous analysis, but with a minor modification: the set of states of nature must be redefined as $\Omega' = (\Omega \backslash \{\omega_j, \omega_{j+1}\}) \cup \{\omega_{j,j+1}\}$, where the new state $\omega_{j,j+1}$ is the amalgamation of ω_j and ω_{j+1}, occurring with probability $p_{j,j+1} = p_j + p_{j+1}$ and having the common value of $v_{j,j+1} = \frac{p_j}{p_j + p_{j+1}} v_j + \frac{p_{j+1}}{p_j + p_{j+1}} v_{j+1}$. In this modified contest (payoff-equivalent to the original), player 1 has the trivial information, while player 2' has the finest possible information partition. Applying (8) to this contest, the expected payoff of player 2 is equilibrium is given by

$$\begin{aligned} E_2' &= \sum_{i=1}^{j-1} p_i (\sum_{k=1}^{i-1} p_j (v_i - v_j)) \\ &+ p_{j,j+1} \sum_{k=1}^{j-1} p_k (v_{j,j+1} - v_k) \\ &+ \sum_{i=j+2}^{n} p_i (\sum_{k \le i-1, k \ne j, k \ne j+1} p_k (v_i - v_k) + p_{j,j+1} (v_i - v_{j,j+1})). \end{aligned}$$

Then we have

$$E_2 - E_2' = p_j p_{j+1} (v_{j+1} - v_j) + \sum_{i=j+2}^{n} p_i (\sum_{k=j}^{j+1} p_k (v_i - v_k) - p_{j,j+1} (v_i - v_{j,j+1})).$$

Since $p_j p_{j+1} (v_{j+1} - v_j) > 0$ and $\sum_{i=j+2}^{n} p_i (\sum_{k=j}^{j+1} p_k (v_i - v_k) - p_{j,j+1} (v_i - v_{j,j+1})) = 0$ we obtain that $E_2 - E_2' > 0$.

We have thus shown that player 2' obtains in expectation less than player 2 (when competing against 1 in a two-player auction) if Π_2' is a connected partition which is a strict coarsening of Π_2 with $|\Pi_2'| = |\Pi_2| - 1$. Inductively, the claim can be extended to *any* connected partition Π_2' with $|\Pi_2'| < n$. *Q.E.D.* ∎

The next result shows that there is an opposite relation between the players' total expected payoff and their total expected effort (bid).

Proposition 4 *In a two-player common-value all-pay auction, the expected total effort when player 2 competes against player 1 is lower than the expected total effort when player 2' competes against player 1.*

Proof. In every common-value all-pay auction, the relation between the players' expected total effort and their expected total payoff is

$$\text{Expected total effort} = \text{Expected reward} - \text{Expected total payoff}$$

Since the expected payoff of player 1 when he competes against player 2 or against 2' is zero (see (7)), in both auctions

$$\text{Expected total effort} = \text{Expected reward} - \text{Expected payoff of player 2 (or, 2')}$$

By Proposition 3, the expected payoff of player 2 is higher than that of player 2' (when competing against player 1). On the other hand, both contests clearly have the same expected reward, $E(v)$. Thus, the expected total effort when player 2 competes against player 1 is lower than when player 2' competes against 1. $Q.E.D.$

■

The above propositions demonstrate that increasing asymmetry between players in a two-player common-value all-pay auction has a positive effect on the expected payoff of the player with an information advantage, and a negative effect on the expected total effort.

5 Budget Constraints

We have thus far assumed that players submit any bids they wish without any constraint. In this section we will show that having (even identical) budget constraints can change our results in a significant way. We shall assume, as in Section 4, that there are two players, and that $\Pi_1 = \{\Omega\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$, i.e., player 1 has no information on the realized states of nature (other than the common prior distribution p), while player 2 knows the realized state precisely. As before, for each state of nature $\omega_i \in \Omega$, denote $v_i = v(\omega_i)$ and $p_i = p(\omega_i)$. Each player can submit any bid that is lower than or equal to a given budget constraint $d > 0$. The following example (the same one as in Section 3 but with budget constrained players) demonstrates the effect of the budget constraint on the players' equilibrium strategies, and, in particular, shows that having a budget constraint may imply a higher expected payoff to player 1 (the uninformed player) compared to player 2 (the informed player).

Example 1 *Assume that* $n = 3$*, and that in state* ω_i *the value of winning is* $v(\omega_i) = i$ *with probability of* $p_i = \frac{1}{3}, i = 1, 2, 3$*. Assume also that the players have the same budget constraint,* $d = \frac{5}{6}$*. Then the contest possesses the following pure strategy Bayesian equilibrium: player 1's bid is independent of the state of nature,* $x_1^* \equiv \frac{5}{6}$*, and player 2's state-dependent bid is given by*

$$x_2^*(\omega) = \begin{cases} 0, & \text{if } \omega = \omega_1 \\ \frac{5}{6}, & \text{if } \omega \neq \omega_1 \end{cases} .$$

The expected payoff of player 1 is then

$$E_1 = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot \frac{1}{2} \cdot 3 - \frac{5}{6} = \frac{1}{3},$$

and the expected payoff of play player 2 is

$$E_2 = \frac{1}{3} \cdot (\frac{1}{2} \cdot 2 - \frac{5}{6}) + \frac{1}{3} \cdot (\frac{1}{2} \cdot 3 - \frac{5}{6}) = \frac{5}{18}.$$

Thus, the expected payoff of the uninformed player (player 1) is higher than that of the informed player (player 2).

In Example 1, we showed that the uninformed player's expected payoff could be higher than that of the informed player. In the following we describe some sufficient conditions under which this unusual result is obtained.

Proposition 5 *Consider a two-player common-value all-pay auction with a budget constraint* d *for both players. Suppose that there exists* $1 \leq j \leq n - 1$ *such that*

(i)

$$\frac{1}{2} \sum_{m=j+1}^{n} p_j v_j \geq d, \tag{9}$$

(ii)

$$\frac{v_j}{2} \leq d < \frac{v_{j+1}}{2}, \tag{10}$$

and

(iii)

$$\sum_{m=1}^{j} p_m (v_m - d) \geq 0. \tag{11}$$

Then, there exists a pure strategy Bayesian equilibrium in which the expected payoff of the uninformed player (player 1) is higher than that of the informed player (player 2).

Proof. Given the conditions (9) and (10), the contest possesses a pure strategy Bayesian equilibrium (x_1^*, x_2^*), in which

$$x_1^* \equiv d, \tag{12}$$

and

$$x_2^*(\omega_k) = \begin{cases} 0, & \text{for } k = 1, ..., j, \\ d, & \text{for } k = j+1, ..., n \end{cases}. \tag{13}$$

If the players use the strategies given by (12) and (13), the expected payoff of player 2 is

$$E_2\left(x_1^*, x_2^*\right) = \sum_{m=j+1}^{n} p_m\left(\frac{1}{2}v_m - d\right), \tag{14}$$

and the expected payoff of player 1 is then

$$E_1\left(x_1^*, x_2^*\right) = \sum_{m=1}^{j} p_m(v_m - d) + \sum_{m=j+1}^{n} p_m\left(\frac{1}{2}v_m - d\right). \tag{15}$$

To check that (x_1^*, x_2^*) constitutes an equilibrium, we do the following. If player 1 submits a bid of $x_1 = \varepsilon < d$, then by (9)

$$E_1\left(x_1, x_2^*\right) = \sum_{m=1}^{j} p_m v_m - \varepsilon < E_1\left(x_1^*, x_2^*\right)$$

Thus x_1^* is player 1's (unique) best response to x_2^*.

If player 2 unilaterally deviates from x_2^* to a strategy x_2 with $0 < x_2(\omega_k) = \varepsilon \le d$ for some $1 \le k \le j$, then

$$\begin{aligned} E_2\left(\{\omega_k\}, x_1^*, x_2\right) &\le \max\left(-\varepsilon, \frac{v_k}{2} - d\right) \\ &\le 0 = E_2\left(\{\omega_k\}, x_1^*, x_2^*\right), \end{aligned}$$

where the second inequality is implied by (10). But, also by (10), if player 2 unilaterally deviates from x_2^* to a strategy x_2 with $0 \le x_2(\omega_k) = \varepsilon < d$ for some $j+1 \le k \le n$, then

$$\begin{aligned} E_2\left(\{\omega_k\}, x_1^*, x_2\right) &= -\varepsilon \\ &\le 0 \le \frac{v_k}{2} - d = E_2\left(\{\omega_k\}, x_1^*, x_2^*\right), \end{aligned}$$

By taking expectation over ω_k, it follows that

$$E_2\left(x_1^*, x_2\right) \le E_2\left(x_1^*, x_2^*\right)$$

for *any* pure strategy x_2 of player 2 that obeys his budget constraint, and thus x_2^* is player 2's best response to x_1^*.

We can conclude now that (x_1^*, x_2^*) is an equilibrium. By comparing the players' expected payoffs given by (14) and (15), we obtain that the expected payoff of player 1 is higher than that of player 2 if and only if (11) holds. Q.E.D. ■

The existence of a pure strategy equilibrium as established in Proposition 5 is based on assumptions (9) and (10). Furthermore, if the budget constraint d is below $\frac{v_1}{2}$, both players would make a bid equal to the bid cap in all the states of nature in a pure strategy equilibrium. But our model with identical budget constraints may also have a mixed strategy equilibrium, and for a sufficiently large cap d the equilibrium would in fact be unique and identical to the one in Proposition 1.

6 Concluding remarks

In models with asymmetric information, differences in players' information usually result in different equilibrium strategies, probabilities of winning, and expected payoffs. In this model we show that even when the players' information can be ranked, with one player having an information advantage over his opponent, the players' expected efforts as well as their probabilities of winning the contest are the same. The difference in information only manifested itself in the different expected payoffs. We also show that the highest expected total effort is obtained when the difference in the players' information is as small as possible. Thus, a contest designer who wishes to maximizes the players' expected total effort has an incentive to reduce the information difference between the players. But, if players face budget constraints, the information advantage might become a disadvantage: the player with the information advantage may have a lower expected payoff than his opponent. This unusual result implies that by imposing bid caps, the contest designer can control the relation between the players' expected payoffs.

We established our results under the assumptions that information sets of each player are connected with respect to the value of winning the contest and that the different information endowments can be ranked. These assumptions are found to be sufficient for the existence of a unique equilibrium. When information cannot be ranked, however, the existence of an equilibrium remains an open problem.

It would be interesting to examine whether the results in this paper can, at least partially, carry over to other contest forms with common values and asymmetric information. In particular, the question of how asymmetric information is reflected in the relation between the players' expected efforts and in their probabilities of winning seems worthy of further attention.

6.1 Proof of Proposition 1

Fix an equilibrium (F_1, F_2) in the auction G. We will prove that $(F_1, F_2) = (F_1^*, F_2^*)$.

In what follows, for $k = 1, 2$ and $\omega \in \Omega$, $F_k(\omega, \cdot)$ will be treated either as a *probability distribution* on $\mathbb{R}_+$, or as the corresponding *c.d.f.*, depending on the context. Also, as F_1 is state-independent, $F_1(\omega, \cdot)$ will be shortened to $F_1(\cdot)$, whenever convenient.

Notice that $F_k(\cdot, \{c\}) \equiv 0$ for any effort $c > 0$ and $k = 1, 2$. Indeed, if $F_k(\omega, \{c\}) > 0$ for some k and ω, then $F_m(\omega', (c - \varepsilon, c]) = 0$ for the other player m and every $\omega' \in \Omega$, and some sufficiently small $\varepsilon > 0$. But then k would be strictly better off by shifting the probability from c to $c - \frac{\varepsilon}{2}$, a contradiction to F_k being an equilibrium strategy. Thus, $F_1(\cdot), F_2(\omega, \cdot)$ are non-atomic on $(0, \infty)$ for every $\omega \in \Omega$. Notice also that there is no interval $(a, b) \subset (0, \infty)$ on which in some state of nature *only one* player places positive probability according to his equilibrium strategy. Indeed, otherwise there would exist $a' > a$ such that only one player places positive probability on (a', b), and it would then be profitable for that player to deviate (in at least one state of nature, if this is the informed player 2) by shifting positive probability from (a', b) to a'.

Suppose now that there is a bounded interval $(a, b) \subset (0, \infty)$ such that $F_1((a, b)) = 0$ (and thus $F_2(\omega, (a, b)) = 0$ for every $\omega \in \Omega$, by the previous paragraph), but $F_1([0, a]) > 0$ and $F_1([b, \infty)) > 0$. By extending this interval if necessary, it can also be assumed that (a, b) is maximal with respect to this property, i.e., that $F_1([\max(a - \varepsilon, 0), a]) > 0$ and $F_1([b, b + \varepsilon]) > 0$ for every small enough $\varepsilon > 0$. However, using the fact that $F_2(\omega, \cdot)$ is non-atomic on $(0, \infty)$ for every $\omega \in \Omega$, the expected payoff of player 1 at $\frac{a+b}{2}$ is strictly bigger than his payoff for any effort in $[b, b + \varepsilon]$, if $\varepsilon > 0$ is small enough. This contradicts the assumption that $F_1([b, b+\varepsilon]) > 0$. This contradiction shows that there exists no interval (a, b) as above, meaning that $F_1(\cdot)$ must have *full support* on some closed interval. Denote this interval[5] by $[c, d]$. Notice also that, for every $\omega \in \Omega$, $F_2(\omega, \cdot)$ must be supported on the interval $[c, d]$ (though there need not be full support), since otherwise there would be an interval where only player 2 places positive probability, and this was ruled out.

Note next that $c = 0$. Indeed, if $c > 0$ then $F_2(\cdot, \{c\}) \equiv 0$, and thus player 1 has a negative expected payoff for efforts in $[c, c+\varepsilon]$ for all small enough $\varepsilon > 0$ (because with efforts in $[c, c+\varepsilon]$ he loses the contest almost for sure while expending positive effort of at least c). He would then profitably deviate from F_1 by shifting the probability from $[c, c+\varepsilon]$ to effort 0. Thus, indeed, $c = 0$. Note also that the interval $[0, d]$ is non-degenerate, i.e., $0 < d$, since otherwise the equilibrium strategies would prescribe the constant effort 0, and it is clear that each player would have a profitable unilateral deviation to some $\varepsilon > 0$.

Given i, $i = 1, ..., n$, we will now show that $F_2(\omega_i, \cdot)$ has full support on a (possibly degenerate) *subinterval* of $[0, d]$. Indeed, if not, there would exist an open subinterval $(a, b) \subset [0, d]$ such that $F_2(\omega_i, (a, b)) = 0$, but $F_2(\omega_i, [0, a]) > 0$ and $F_2(\omega_i, [b, d]) > 0$. Since $F_1((a, b)) > 0$, there must be $j \neq i$ such that $F_2(\omega_j, (a, b)) = 0 > 0$. Assume that $i < j$ (the opposite case is treated similarly). Then there are $x \in [b, d]$ and $y \in (a, b)$ such that

$$\begin{aligned} v_i F^1(x) - x &= E_2\left(\{\omega_i\}, F^1, x\right) && (16) \\ &\geq E_2\left(\{\omega_i\}, F^1, y\right) = v_i F^1(y) - y && (17) \end{aligned}$$

and

$$\begin{aligned} v_j F^1(x) - x &= E_2\left(\{\omega_j\}, F^1, x\right) && (18) \\ &\leq E_2\left(\{\omega_j\}, F^1, y\right) = v_j F^1(y) - y. && (19) \end{aligned}$$

But $x > y$, and therefore

$$(v_j - v_i)F^1(x) > (v_j - v_i)F^1(y) \tag{20}$$

since $v_i < v_j$ and the c.d.f. F^1 is strictly increasing on $[0, d]$. Adding (20) to the inequality in (16)-(17) contradicts the inequality obtained in (18)-(19), and therefore no such (a, b) exists. Consequently, each

[5] The interval must be bounded as no efforts above v_n will be made in equilibrium, due to the associated negative payoff.

$F_2(\omega_i,\cdot)$ has full support on some subinterval[6] $[a_i,b_i]$ of $[0,d]$. Moreover, if $i<j$ then $[a_i,b_i]$ lies below $[a_j,b_j]$ (barring boundary points), since otherwise it would have been possible to find $x>y$, where $x\in[a_i,b_i]$ and $y\in[a_j,b_j]$, such that inequalities (16)-(17) and (18)-(19) hold. As above, this would lead to a contradiction via (20).

Thus, the intervals $\{[a_i,b_i]\}_{i=1}^n$ are disjoint (barring boundary points), and "ordered" according to the index i on the interval $[0,d]$. Moreover, $\cup_{i=1}^n[a_i,b_i]=[0,d]$, since otherwise there would be a "gap" (a,b) on which only player 1 places positive probability, which is impossible as we have seen earlier. It follows that there are points $0=x_0\le x_1\le ...<x_n\equiv d$ such that $[a_i,b_i]=[x_{i-1},x_i]$ for every $i=1,2,...,n$, i.e., $F^1(\cdot)$ has full support on $[0,x_n]$, and, for $i=1,...,n$, $F_2(\omega_i,\cdot)$ has full support on $[x_{i-1},x_i]$. Denote by i_0 the smallest integer with $x_{i_0}>0$.[7]

Since $F^1(\cdot)$ has full support on $[0,x_n]$ and $F_2(\omega,\cdot)$ has no atoms (except possibly at 0), player 1 is indifferent between any two efforts in $(0,x_n]$. Thus, the following equality must hold for every $i=i_0,...,n$ and every positive $x\in[x_{i-1},x_i]$:

$$\sum_{j=1}^{i-1}p_jv_j+p_iv_iF_2(\omega_i,x)-x=E_1(x,F_2)=\lim_{y\searrow 0}E_1(y,F_2)\equiv e_1\ge 0.$$

In particular,

$$F_2(\omega_i,x)=\frac{x-\sum_{j=1}^{i-1}p_jv_j+e_1}{p_iv_i} \tag{21}$$

for every $i=i_0,...,n$ and every positive $x\in[x_{i-1},x_i]$. Since $F_2(\omega_i,\cdot)$ is supported on $[x_{i-1},x_i]$, we have $F_2(\omega_i,x_i)=1$, and thus

$$x_i=\sum_{j=1}^{i}p_jv_j-e_1 \tag{22}$$

for every $i=i_0,...,n$.

Since, for $i=i_0,...,n$, $F_2(\omega_i,\cdot)$ has full support on $[x_{i-1},x_i]$ and $F_1(\cdot)$ has no atoms (except, possibly, at 0), player 2 is indifferent between all positive efforts in $[x_{i-1},x_i]$. Thus, the following equality must hold for every positive $x\in[x_{i-1},x_i]$:

$$\begin{aligned} v_iF_1(x)-x &= E_2(\{\omega_i\},F_1,x)\\ &= E_2(\{\omega_i\},F_1,x_i)=v_iF_1(x_i)-x_i. \end{aligned}$$

[6] All these subintervals are either non-degenerate (of positive length), or $\{0\}$, as only the latter can be an atom of $F_2(\omega_i,\cdot)$.

[7] Since each interval $[x_{i-1},x_i]$ is either non-degenerate or $\{0\}$, $0=x_0=...=x_{i_0-1}<x_{i_0}<...<x_n$.

In particular,

$$F_1(x) = \frac{x}{v_i} + F_1(x_i) - \frac{x_i}{v_i},$$

and using the fact that $F_1(x_n) = 1$ and (22), we obtain

$$F_1(x) = \frac{x + e_1}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] \tag{23}$$

for every $i = i_0, ..., n$, and every positive $x \in [x_{i-1}, x_i]$.

If $e_1 > 0$, it follows from (23) that $F_1(\cdot)$ has an atom at effort 0. Then, obviously $F_2(\omega_i, \cdot)$ cannot have an atom at 0, for any i, since otherwise each player would have a profitable unilateral deviation that shifts the probability from zero to an effort slightly above zero. In particular, all intervals $\{[x_{i-1}, x_i]\}_{i=1}^{n}$ are non-degenerate, i.e., $i_0 = 1$. But then, by (21), $F_2(\omega_1, \cdot)$ has an atom at 0, a contradiction. We conclude that $e_1 = 0$.

If $i_0 > 1$, $x_{i_0-1} = 0$, and thus (21) should hold for $i = i_0$ and any sufficiently small x. But then, if $x < p_1 v_1$

$$F_2(\omega_{i_0}, x) = \frac{x - \sum_{j=1}^{i_0-1} p_j v_j}{p_{i_0} v_{i_0}} \leq \frac{x - p_1 v_1}{p_{i_0} v_{i_0}} < 0,$$

and thus $F_2(\omega_{i_0}, x)$ is not a c.d.f., a contradiction. Consequently, $i_0 = 1$.

It now follows from (22), (21), and (23) that

$$x_i = \sum_{j=1}^{i} p_j v_j$$

for every $i = 1, ..., n$, that

$$F_1(x) = \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right]$$

for every $i = 1, ..., n$ and every $x \in [x_{i-1}, x_i]$, and that

$$F_2(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i}$$

for every $i = 1, ..., n$ and positive $x \in [x_{i-1}, x_i]$. Thus, (F_1, F_2) coincides with (F_1^*, F_2^*) as described in (2) and (5).

Q.E.D.

References

[1] Abraham, I., Athey, S., Babaiof, M., Grubb, M.: Peaches, Lemons, and Cookies: Designing auction markets with dispersed information. Working paper, Harvard University (2012)

[2] Amman, E., Leininger,W.: Asymmetric all-pay auctions with incomplete information: the two-player case. *Games and Economic Behavior* 14, 1-18 (1996)

[3] Baye, M. R., Kovenock, D., de Vries, C. G.: "Rigging the lobbying process: an application of the all-pay auction." *American Economic Review* 83, 289-294 (1993)

[4] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[5] Che, Y-K., Gale, I.: Caps on political lobbying. *American Economic Review* 88(3), 643-651 (1998)

[6] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Dominant strategies, superior information, and winner's curse in second-price auctions. *International Journal of Game Theory* 30, 405-419 (2001)

[7] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Dominance solvability of second-price auctions with differential information. *Journal of Mathematical Economics* 37, 247-258 (2002)

[8] Forges, F., Orzach, R.: Core-stable rings in second price auctions with common values. *Journal of Mathematical Economics* 47, 760-767 (2011)

[9] Gavious, A., Moldovanu, B., Sela, A.: Bid costs and endogenous bid caps. *Rand Journal of Economics* 33(4), 709-722 (2003)

[10] Hillman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-40 (1989)

[11] Hillman, A., Samet, D.: Dissipation of contestable rents by small numbers of contenders. *Public Choice* 54(1), 63-82 (1987)

[12] Jackson, M.: Bayesian implementation. *Econometrica* 59, 461-477 (1993)

[13] Krishna, V., Morgan, J.: An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory* 72(2), 343-362 (1997)

[14] Malueg, D., Orzach, R.: Revenue comparison in common-value auctions: two examples. *Economics Letters* 105, 177-180 (2009)

[15] Malueg, D., Orzach, R.: Equilibrium and revenue in a family of common-value first-price auctions with differential information. *International Journal of Game Theory* 41(2), 219-254 (2012)

[16] Milgrom, P., Weber, R.: A theory of auctions and competitive bidding. *Econometrica* 50(5), 1089-1122 (1982)

[17] Moldovanu, B., Sela, A.: The optimal allocation of prizes in contests. *American Economic Review* 91(3), 542-558 (2001)

[18] Moldovanu, B., Sela, A.: Contest architecture. *Journal of Economic Theory* 126(1), 70-97 (2006)

[19] Moldovanu, B., Sela, A., Shi, X.: Carrots and sticks: prizes and punishments in contests. *Economic Inquiry* 50(2), 453-462 (2012)

[20] Siegel, R.: All-pay contests. *Econometrica* 77 (1), 71-92 (2009)

[21] Vohra, R.: Incomplete information, incentive compatibility and the core. *Journal of Economic Theory* 54, 429-447 (1999)