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Claudio Michelacci and Josep Pijoan-Mas

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**Claudio Michelacci, Centre for Monetary and Financial Studies and CEPR**
**Josep Pijoan-Mas, Centre for Monetary and Financial Studies and CEPR**

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Centre for Economic Policy Research
77 Bastwick Street, London EC1V 3PZ, UK
Tel: (44 20) 7183 8801, Fax: (44 20) 7183 8820
Email: cepr@cepr.org, Website: www.cepr.org

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# ABSTRACT

## Labor Supply with Job Assignment under Balanced Growth*

We consider a competitive equilibrium growth model where technological progress is embodied into new jobs that are assigned to workers of different skills. In every period workers decide whether to actively participate in the labor market and if so how many hours to work on the job. Balanced growth requires that the job technology is complementary with the worker's total labor input in the job, which is jointly determined by his skill and his working hours. Since lower skilled workers can supply longer hours, we show that the equilibrium features positive assortative matching (higher skilled workers are assigned to better jobs) only if differences in consumption are small relative to differences in worker skills. When the pace of technological progress accelerates, wage inequality increases and workers participate less often in the labor market but supply longer hours on the job. This mechanism can explain why, as male wage inequality has increased in the US, labor force participation of male workers of different skills has fallen while their working hours have increased.

JEL Classification: E24, G31 and J31
Keywords: job heterogeneity, participation, wage inequality and working hours

Claudio Michelacci
CEMFI
Casado del Alisal 5
28014 Madrid
SPAIN

Email: c.michelacci@cemfi.es

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Josep Pijoan-Mas
CEMFI
Casado del Alisal 5
28014 Madrid,
SPAIN

Email: pijoan@cemfi.es

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=160822

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## 1 Introduction

The idea that new technologies come embodied into a limited supply of new capital vintages dates back at least to Solow (1960). If the production technology requires each worker to be assigned to a specific capital unit, technological progress also leads to heterogeneity in jobs. In the words of Akerlof (1981), good jobs become a scarce resource, which the economy should assign to workers with potentially different skills. This assignment friction has been widely studied, see Sattinger (1993) for a literature review. But existing assignment models have typically abstracted away from labor supply decisions either at the intensive margin (how many hours to work on the job) or at the extensive margin (whether to actively participate in the labor market). This is an interesting issue because, in assignment models, standard income and substitution effects in labor supply lead to a non-trivial allocation problem between the number of hours worked in the job, which determines the output each job produces, and labor force participation, which determines the number and quality of operating jobs. Income and substitution effects also play a non trivial role in determining whether the equilibrium features positive assortative matching—i.e. whether higher skilled workers are assigned to better jobs. This is because the amount of labor input supplied by a worker on the job is determined by his skill as well as by his working hours. So a low skilled worker can supply greater working hours to compensate for his lower skill level, which implies that standard conditions for assortative matching based on capital-skill complementarity (Becker 1973) are directly affected by labor supply.

To study labor supply in an assignment model, we consider a simple neoclassical growth model with perfectly competitive labor markets and vintage capital as in Jovanovic (1998). Technological progress is embodied into new jobs, which are slowly created over time. Hence in equilibrium there is dispersion in job technologies. Workers differ in skills and they can be employed in at most one job. This leads to a simple assignment problem in the spirit of Becker (1973) and Sattinger (1975). But in our framework labor supply is endogenous because in every period each worker decides whether to actively participate in the labor market, which involves a fixed utility cost, and how many hours to work in the job he is assigned to. To guarantee the existence of a balanced growth path, we assume log preferences in consumption (so that in the long run income and substitution effects cancel out) and a production technology in the job that features unitary elasticity of substitution between the job technology and worker's total labor input, which is jointly determined by the worker's skill and his working hours. In equilibrium, the model endogenously generates inequality in jobs, wages, and labor supply, but all workers of the same skill consume the same amount—which is a natural implication of the permanent income hypothesis. Subject to the assignment friction, the competitive equilibrium is efficient and its allocation coincides with the solution chosen by a social planner who gives (potentially) different Pareto weights to workers of different skills.

When labor supply is exogenous, complementarity between the job technology and worker skill ensures that the equilibrium features positive assortative matching (see for instance Becker (1973)). But in our framework the amount of labor input supplied by a worker in a job is function both of his skill and his working hours. Since working hours depend positively on the job technology (due to the substitution effect) and negatively on the worker's wealth (due to the income effect), the total labor input supplied by a poor low skilled workers assigned to a high technology job could be higher than the analogous amount supplied by a wealthy high skilled worker in the same job. This could make profitable assigning a low skilled worker to a high technology job. We show that positive assortative matching requires that workers consumption differences are small relative to their skill differences. This ensures that a low skilled worker assigned to a high technology job faces a small substitution effect relative to the income effect, which in turn guarantees that his total labor input in the job is smaller than the analogous amount supplied by a high skilled worker in the same job. In the social planner problem consumption differences just reflect differences in Pareto weights. But in the decentralized economy, consumption differences are determined in equilibrium through wages and reflect differences both in workers initial wealth and in their skills. In the absence of differences in initial wealth, the condition for positive assortative matching requires that workers skill differences are large enough compared to differences in job technologies. If this is not the case, positive assortative matching still arises in equilibrium if low skilled workers enjoy a sufficiently large amount of non-labor income.

In the model technological progress is embodied into a limited supply of new jobs. So when technological progress accelerates, newly created jobs become more technologically advanced than old jobs that embody relatively more obsolete technologies. As jobs technological differences widen, working hours in high technology jobs increase, while more technologically obsolete jobs are scrapped earlier. This makes aggregate hours per worker increase, while labor force participation falls. The assignment friction is essential for generating opposite movements in the intensive and extensive margins of labor supply. When workers can operate any amount of capital units, so that the assignment friction is absent, faster technological progress affect both margins symmetrically, so hours per worker and labor force participation never move in opposite directions.

In principle, this mechanism can explain why in the US since the 70's, as wage inequality has increased, labor force participation of male workers has fallen while hours per employed worker have increased. To study the quantitative relevance of the mechanism, we parametrize the model to account for differences in employment rates, hours per worker, labor income and consumption across educational groups in the 1970's. The calibrated model implies that, in the 70's, 70 percent of the hourly wage premium between college graduates and high school dropouts was due to skill differences, while the remaining 30 percent was due to differences in job technologies. We then follow Greenwood and Yorokoglu (1997), Greenwood, Hercowitz, and Krusell (1997) and Violante (2002) in arguing that the speed of technological progress embodied in new jobs has increased over the period 1970-2000. Since the supply of workers of different educational groups has changed over the period, we also match changes in their relative proportion in the US male population. Overall, the model accounts for one third of the observed rise in labor income inequality across educational groups. The model generates a fall in 8 percentage points in the participation rate and an increase of 1.3 hours worked per week by an average employed worker. This is in line with the data, which also shows an 8 percent fall in the aggregate participation rate and an increase of 1.5 weekly hours. Finally, the model accounts well for observed variation across educational groups. In particular the fact that highly educated workers have experienced a larger increase in hours per worker and a less pronounced fall in participation rates.

Our findings are related to Elsby and Shapiro (2012) who argue that the fall in productivity growth in the US since the 70's has caused a decrease in the return to labor market experience, which can explain why male employment rates for different skill groups have fallen. Our model provides a novel alternative mechanism whereby changes in the long run rate of growth affect labor supply in models with balanced growth preferences. According to our model, employment rates have fallen because of an acceleration in technological progress which has exacerbated technological differences across jobs.

The remaining of the article is organized as follows. In Section 2 we set up an economy with identical workers, which we solve in Section 3. In Section 4 we allow for heterogeneous workers and analyze conditions for an assortative matching equilibrium to exist. In Section 5 we discuss how to decentralize this equilibrium allocation. Section 6 discusses our quantitative results. Section 7 concludes.

## 2 The economy with homogenous workers

We start characterizing an economy where all workers are homogenous. This is useful to analyze the key trade-off between labor force participation and working longer hours in an economy with assignment frictions. In Section 4 we extend the model by allowing workers to have different skills.

### 2.1 Worker preferences

The economy is in continuous time and it is populated by a measure one of identical infinitely lived workers. The consumption good is the numeraire. The time-$t$ instantaneous utility of a worker consuming $\tilde{c}_t$ units and working $n_t$ hours in the period is

$$u(\tilde{c}_t, n_t) = \ln \tilde{c}_t - v(n_t)$$

where $v(n_t)$ is the worker's individual disutility of working which is equal to

$$v(n) = \begin{cases} \lambda_0 + \lambda_1 \frac{n^{1+\eta}}{1+\eta} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{1}$$

where $\lambda_0 > 0$ measures the fixed cost of going to work, $\lambda_1 > 0$ governs the magnitude of the variable component and $\eta \geq 0$ regulates the Frisch elasticity. Log preferences in consumption are needed to guarantee the existence of a balanced growth path with constant aggregate labor supply. To produce output a worker has to be matched with a machine. We assume that, at a time, workers can not use more than one machine and that a machine can not be matched with more than one worker. This is the key friction of our economy, which arises because workers and machines are indivisible. Since a machine identifies a job, thereafter we will use the two terms interchangeably. A machine of quality $k$ when matched with a worker who supplies $n$ hours of work produces an output level given by the homogenous of degree one function $f(k, n) = k^{\alpha} n^{1-\alpha}$, with $\alpha \in (0, 1)$. We restrict the production function to be Cobb-Douglas to guarantee the existence of a balanced growth path with constant working hours and participation rate.

### 2.2 Machine qualities

As in Jovanovic (1998), at every instant in time $t$, $m < \infty$ new machines of quality $e^{qt}$ become available, with $q > 0$ measuring the speed of embodied technical change. Machines are in excess supply because the number of potential workers is fixed to one while new machines of relatively better quality become continuously available. There is heterogeneity in the quality of available machines and at each point in time we rank machines by their age $\tau$. Let $\tau^*$ denote the critical age such that all machines older than $\tau^*$ are scrapped. Then the age distribution of operating machines is uniform with support $[0, \tau^*]$, which implies a probability density equal to $1/\tau^*$. Let $p$ denote the aggregate *participation rate*, i.e. the fraction of workers who actively participate to the labour market. We focus the analysis on a balanced growth path equilibrium, where the aggregate participation rate $p$ and the critical age threshold $\tau^*$ are constant over time. For simplicity, we assume $p$ to be in the open interval $(0, 1)$. Since every worker is paired with a machine, the number of machines in operation must be equal to the number of employed workers

$$\int_0^{\tau^*} m \, d\tau = p$$

which implies that $\tau^* = \frac{p}{m}$. We start assuming, but later relax in Section 4, that $m = 1$. So we have $\tau^* = p$, which means that the age distribution of operating machines has support over the interval $[0, p]$ and density $1/p$. Since the economy will be growing at rate $\alpha q$, we will denote by $c_t = \tilde{c}_t e^{-\alpha q t}$ detrended consumption.

Let $\tilde{k}_t^\tau = e^{q(t-\tau)}$ denote the quality of a machine of age $\tau$ at time $t$. Then, the quality of the worst machine in operation at time $t$ can be expressed as $\tilde{k}_t^* \equiv \tilde{k}_t^{\tau^*} = e^{q(t-\tau^*)}$. In the rest of the paper we will work with detrended machine qualities:

$$k^\tau = \tilde{k}_t^\tau e^{-qt} = e^{-q\tau} \tag{2}$$

This implies that the detrended quality of the best machine in operation is equal to $k^0 = 1$, while the worse machine quality in operation is $k^* \equiv k^{\tau^*} = e^{-qp}$.

## 3 Solving the economy with homogenous workers

We focus on the social planner problem and we postpone the discussion on how to decentralize the social planner allocation to Section 5. As a term of reference, we start characterizing the economy where workers can operate any amount of machines, so that no assignment friction is present. We then study the equilibrium properties of our economy. The analysis focuses on the effects of technological progress on labor supply under balanced growth, which is the topic of the quantitative exercise in Section 6.

### 3.1 The frictionless economy

The assignment friction involves two types of constraints. The first is that capital labor ratios are not equalized across jobs, the second is that not all capital available in the economy can be used in production. This leads to two possible alternative ways of characterizing the corresponding economy with no assignment frictions. In the first characterization we remove both constraints at the same time by assuming that all workers use the same amount of capital and that all capital is used in production. In the second characterization we remove just the first constraint by assuming that capital labor ratios are equalized across jobs, but we keep the number of machines used in production equal to the participation rate $p$.

Aggregate production is obtained by combining aggregate capital $K$ with aggregate labor $L$ according to the Cobb-Douglas production function $K^{\alpha}L^{1-\alpha}$. When all capital units can be used in production, case $j = 1$, detrended capital is equal to

$$K_{f1} = \int_0^{\infty} k^{\tau} d\tau = \int_0^{\infty} e^{-q\tau} d\tau = \frac{1}{q}$$

while in the second case $j = 2$ capital is equal to

$$K_{f2} = \int_0^{p} k^{\tau} d\tau = \int_0^{p} e^{-q\tau} d\tau = \frac{1 - e^{-qp}}{q} \tag{3}$$

which imposes the constraint that the number of machines used in production is equal to the aggregate participation rate. In either case the aggregate supply of labour is equal to $L = np$, where $n$ denotes average hours worked by workers who are actively participating in the labor market in a period. Clearly (detrended) consumption should be equal to (detrended) output:

$$c = K_{fj}^{\alpha} (np)^{1-\alpha}, \ \forall j = 1, 2. \tag{4}$$

The social planner cares equally for all workers and chooses to give them the same level of (detrended) consumption $c$. His problem is intrinsically static and it amounts to maximizing the sum of the instantaneous utility of workers:

$$\max_{c,p,n} \{\log c - pv(n)\} \tag{5}$$

subject to the aggregate resource constraint in (4). By focusing on the case $j = 1$ and taking the first order conditions with respect to $n$ and $p$ we obtain the two conditions:

$$\frac{1-\alpha}{n} = v'(n) \tag{6}$$

$$\frac{1-\alpha}{p} = v(n) \tag{7}$$

which immediately implies that $n$ and $p$ are independent of $q$. This follows from consumption log-preferences that make income and substitution effects cancel out exactly.

When we focus on the case $j = 2$ and we maximize with respect to $n$, we still find condition (6), which is again independent of $q$. The first order condition for $p$ now changes slightly and it reads as follows:

$$\alpha \frac{k^*}{K_{f2}} + \frac{1-\alpha}{p} = v(n) \tag{8}$$

where $k^* = e^{-qp}$ is the value of the (detrended) marginal unit of capital quality used in production. This condition differs from (7), because by employing an additional worker the aggregate capital stock increases by the value of the marginal machine in operation. Given (3) the $\frac{k^*}{K_{f2}}$ ratio can be expressed as equal to

$$\frac{k^*}{K_{f2}} = \frac{q}{e^{qp} - 1},$$

that is decreasing in $q$ for given $p$ (see Appendix), which means that an increase in $q$ decreases the value of the marginal worker. Overall we can conclude that the left hand side of (8) is decreasing in both $p$ and $q$. Hence an increase in $q$ has to lead to an increase in $p$ to keep (8) satisfied. In either case $j = 1$ or $j = 2$, we then have:

**Proposition 1** *In the absence of assignment frictions, when the pace of technological progress accelerates (q goes up), labor force participation and average hours worked on the job never move in opposite directions.*

### 3.2 The social planner problem

In our economy the social planner equalizes (detrended) consumption $c$ across all workers, as in (5). In every period, the planner also chooses how many workers should actively participate in the labor market, and if so the machine they should be matched with as well as their working hours in the match. As workers are homogenous, the exact identity of workers is irrelevant and thereby indeterminate. The planner's problem is intrinsically static and consists in maximizing the sum of workers' instantaneous utilities

$$\max_{c,p,n^\tau} \left\{ \log c - \int_0^p v(n^\tau)\, d\tau \right\} \tag{9}$$

subject to the constraint that (detrended) consumption is equal to (detrended) output:

$$c = \int_0^p f\left(e^{-q\tau}, n^\tau\right) d\tau \tag{10}$$

Here $n^{\tau}$ denotes the working hours of a worker matched with a machine of age $\tau$.

Let $\mu$ denote the Lagrange multiplier of the resource constraint in (10). Then, by maximizing (9) with respect to $c$ we immediately obtain that $\mu$ is equal to the marginal utility of consumption: $\mu = \frac{1}{c}$. By writing the first order conditions with respect to $p$ and $n^{\tau}$ we obtain:

$$v\left(n^{*}\right) = \mu f\left(k^{*}, n^{*}\right) \tag{11}$$

$$v'\left(n^{\tau}\right) = \mu f_{2}\left(k^{\tau}, n^{\tau}\right) \tag{12}$$

where $n^{*} \equiv n^{\tau^{*}}$ denotes hours worked in the marginal job. Equation (11) implicitly determines the participation rate by equating the disutility of sending the marginal individual to work to the value of output in the marginal job. Equation (12) determines working hours in machines of age $\tau$, $n^{\tau}$, by equating the marginal value of hours in production to the marginal disutility of a working hour. This condition determines $n^{\tau}$ as a function of machine age $\tau$ and the marginal value of income $\mu$:

$$n^{\tau} = \psi\left(\tau, \mu\right) = \left(\frac{1-\alpha}{\lambda_{1}}\right)^{\frac{1}{\eta+\alpha}} e^{-\frac{\alpha q}{\eta+\alpha}\tau} \mu^{\frac{1}{\eta+\alpha}} \tag{13}$$

This implies that hours are increasing in machine quality so decreasing in machine age, $\psi_{1} < 0$, which characterizes the substitution effect. Hours are also increasing in the marginal value of income $\mu$, $\psi_{2} > 0$, which characterizes the income effect. The amount of hours in the marginal job $k^{*}$ can be characterized by evaluating (12) at $n^{\tau} = n^{*}$ and then dividing the resulting expression side by side by (11). After rearranging this yields

$$\frac{n^{*} v'\left(n^{*}\right)}{v\left(n^{*}\right)} = 1 - \alpha \tag{14}$$

which determines $n^{*}$ just as a function of preferences and the output elasticity to labor, which is constant under a Cobb-Douglas production function. Overall we have proved that hours worked have the following properties:

**Lemma 1** *Hours worked are decreasing in the age of the machine the worker is matched with and increasing in the marginal utility of consumption. Hours worked in the marginal machine $n^{*}$ depend just on preferences and the output elasticity to labor.*

### 3.3 An increase in the speed of embodied technical change

Clearly an acceleration in the pace of technological progress (an increase in $q$) leads to an increase in welfare. But, when focusing on detrended quantities, the increase in $q$ is equivalent to an increase in the depreciation rate of capital. To see this notice that after detrending, the quality of a newly created machine is always equal to one, $k^0 = 1$, while the quality of a machine of any age $\tau$, $k^\tau = e^{-q\tau}$, falls with $q$. This makes detrended output and consumption $c$ fall, while differences in job technologies, as measured by the ratio between the quality of a newly created machine and a marginal machine, equal to $1/k^* = e^{qp}$, increase. Also notice that the fall in detrended consumption together with (13) evaluated at $\tau = 0$ implies that hours worked in newly created machines $n^0$ increase. From (14) it instead follows that hours worked $n^*$ in the marginal machine $k^*$ remain unchanged when $q$ increases. Overall these considerations lead to the following Lemma;

**Lemma 2** *When technological progress accelerates ($q$ goes up): (a) Detrended consumption $c$ falls; (b) The ratio of the quality between the top and the marginal machine, $\frac{1}{k^*}$ increases; (c) Hours worked in newly created machines $n^0$ increase; (d) Hours worked $n^*$ in the marginal machine $n^*$ remain unchanged.*

**Proof of Lemma 2.** Only points (a) and (b) were not formally proved by the considerations above. To prove (a), we first totally differentiate (10) to obtain,

$$\left(1 + \frac{1}{c^2}\int_0^p f_2\psi_2 d\tau\right) dc = f\left(k^*, n^*\right) dp - \left[\int_0^p \tau\left(e^{-q\tau} f_1 + \frac{\alpha}{\eta+\alpha} f_2\psi\right) d\tau\right] dq \tag{15}$$

Similarly by taking logs in (11), and then totally differentiating after remembering that (14) implies that $n^*$ is independent of $q$, we obtain:

$$\frac{dc}{c} = -\alpha\left(pdq + qdp\right). \tag{16}$$

After solving for $dp$ in (16) we obtain:

$$dp = -\frac{1}{\alpha qc} dc - \frac{p}{q} dq$$

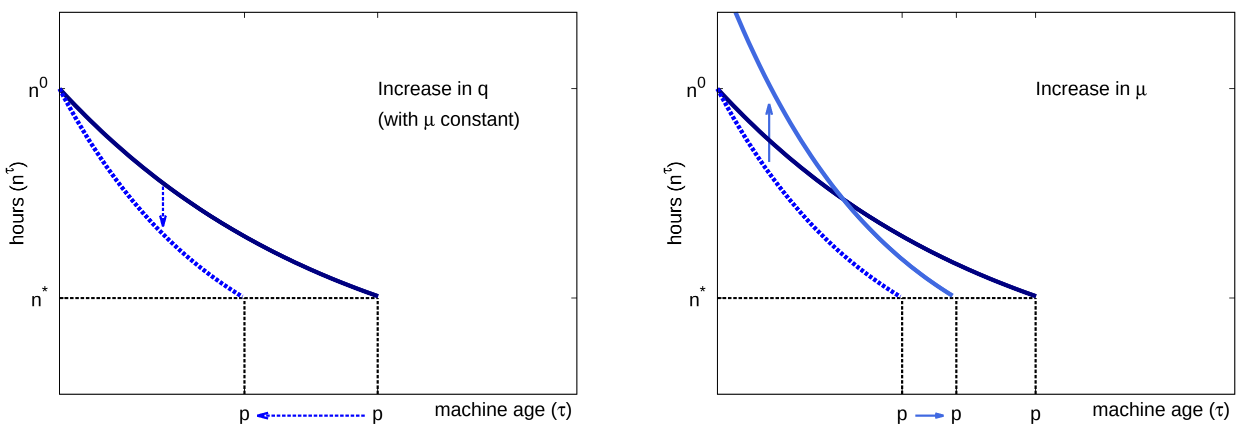
which substituted into (15), and after some rearranging, leads to:

$$\frac{dc}{dq} = -\frac{f\left(k^*, n^*\right)\alpha cp + \alpha qc\int_0^p \tau\left(e^{-q\tau} f_1 + \frac{\alpha}{\eta+\alpha} f_2\psi\right) d\tau}{\alpha qc + \frac{\alpha q}{c}\int_0^p f_2\psi_2 d\tau + f\left(k^*, n^*\right)} < 0$$

To prove (b) just notice that (11), for given $n^*$, implies that when $\mu$ goes up (which happens when $c$ falls), $k^*$ falls. ■

Figure 1 provides a graphical representation of the response of working hours to the change in $q$. The solid line in the left panel corresponds to the profile of working hours in machines of different ages $\tau$—i.e. to the function $n^\tau$ defined in (13). Hours are maximum at $\tau = 0$ and fall exponentially with age until reaching the age of the marginal machine $\tau^*$, which is equal to $p$. From (13) and (14) it follows that an increase in $q$, while holding consumption constant, leads to a fall in hours in all machines except in new machines $\tau = 0$ and in the marginal machine $\tau^*$. The right panel characterizes the total effects of

Figure 1: Hours worked and machine age




the increase in $q$ on working hours by incorporating the effects of the fall in consumption $c$, which leads to an upward movement of $n^\tau$.

Lemma 2 implies that when technological change accelerates ($q$ goes up) technological differences across machines widen, so working hours in high technology jobs increase, while old more technologically obsolete jobs are scrapped earlier. As a result average hours per employed worker, which are equal to

$$\overline{n} = \int_0^p \psi(\tau, \mu) \frac{1}{p} \, d\tau \tag{17}$$

increase, while the participation rate $p$ falls. This is the content of the following Proposition:

**Proposition 2** *When technological progress accelerates ($q$ increases) the participation rate $p$ falls, while average hours per employed worker $\overline{n}$ increase.*

**Proof of Proposition 2.** The resource constraint in (10) can be written as

$$c = \left(\mu \frac{1-\alpha}{\lambda_1}\right)^{\frac{1-\alpha}{\eta+\alpha}} \int_0^p e^{-\frac{\alpha q(1+\eta)}{\eta+\alpha}\tau} d\tau$$

After solving the integral, remembering that $\mu = 1/c$, and some rearranging we obtain

$$c^{\frac{1+\eta}{\eta+\alpha}} = \left(\frac{1-\alpha}{\lambda_1}\right)^{\frac{1-\alpha}{\eta+\alpha}} \frac{\eta+\alpha}{\alpha q\,(1+\eta)} \left[1 - e^{-\frac{\alpha(1+\eta)}{\eta+\alpha}qp}\right] \tag{18}$$

By rewriting (11) and then solving for consumption we obtain

$$c = e^{-\alpha pq} \frac{(n^*)^{1-\alpha}}{v\,(n^*)} \tag{19}$$

which can be used to replace $c$ in (18) to yield

$$\left[\frac{(n^*)^{1-\alpha}}{v\,(n^*)}\right]^{\frac{1+\eta}{\eta+\alpha}} = \left(\frac{1-\alpha}{\lambda_1}\right)^{\frac{1-\alpha}{\eta+\alpha}} \frac{\eta+\alpha}{\alpha\,(1+\eta)} \cdot \frac{e^{\frac{\alpha(1+\eta)}{\eta+\alpha}qp} - 1}{q}$$

The left hand side is independent of $q$ by point (c) in Lemma 2. The right hand side is increasing both in $p$ and in $q$ since the Appendix shows that the function $\frac{e^{\gamma_0 x}-1}{x}$ is increasing in $x$ when $\gamma_0 > 0$. This proves that $\frac{dp}{dq} < 0$.

Integrating (17) after using (13) yields

$$\overline{n} = \frac{\eta+\alpha}{\alpha} \left(\frac{1-\alpha}{\lambda_1 c}\right)^{\frac{1}{\eta+\alpha}} \frac{1 - e^{-\frac{\alpha}{\eta+\alpha}qp}}{qp}$$

After using (19) to replace consumption we finally obtain

$$\overline{n} = \frac{\eta+\alpha}{\alpha} \left[\frac{(1-\alpha)v\,(n^*)}{\lambda_1\,(n^*)^{1-\alpha}}\right]^{\frac{1}{\eta+\alpha}} \frac{e^{\frac{\alpha}{\eta+\alpha}qp} - 1}{qp}$$

which implies that average hours per worker $\overline{n}$ are increasing in $qp$, due again to the properties of the function $\frac{e^{\gamma_0 x}-1}{x}$. This concludes the proof, since point (b) in Lemma 2 states that $k^* = e^{-qp}$ is decreasing in $q$. ■

By comparing this result with Proposition 1, we conclude that the assignment friction is essential for generating opposite movements in the intensive and extensive margins of labor supply in response to an increase in $q$. With assignment frictions, as technology differences widen, working longer hours in new technologically advanced jobs is more valuable, while old jobs are more technologically obsolete and are scrapped earlier.

## 4 The model with worker heterogeneity

We now extend the model to allow workers to have different skills and consumption levels. After characterizing the economy we discuss under which conditions the equilibrium features positive assortative matching. We then solve the social planner problem and characterize optimal choices for consumption, working hours, and labor market participation.

### 4.1 Assumptions

There are $N$ types of workers with skill level $h_i > h_{i+1} > 0$ for $i = 1, 2, \ldots, N-1$. The mass of type $i$ workers is $z_i \in (0,1)$ and $\sum_{i=1}^{N} z_i = 1$. We assume that a worker with human capital $h_i$ working $n$ hours supplies

$$e = h_i^{1-\theta} n^{\theta} \tag{20}$$

efficiency units of labour, which are combined with capital quality $k$ to produce output according to:

$$f(k, h_i, n) = k^{\alpha} \left(h_i^{1-\theta} n^{\theta}\right)^{1-\alpha}$$

This specification allows the existence of a balanced growth path with constant growth. To allow consumption levels to vary by worker's type, we assume that the social planner gives different Pareto weights $\nu_i$ to workers of different type. We impose $\nu_i \geq \nu_{i+1}$ for all $i = 1, 2, \ldots, N-1$ with $\sum_{i=1}^{N} \nu_i z_i = 1$. Strict equality implies equality of consumption across skill types. Strict inequality means that more skilled workers enjoy higher consumption, which is the empirically relevant case. This also justifies why we disregard the case $\nu_i < \nu_{i+1}$. To improve the quantitative fit of the model in Section 6 we now leave unrestricted the value of the mass of newly created machines $m$ and we allow capital to depreciate at a constant rate $\delta$, as in Hornstein, Krusell, and Violante (2007). This implies that, the (detrended) quality of a machine of age $\tau$ is now equal to $k^{\tau} = e^{-\frac{q+\delta}{m}\tau}$.

### 4.2 Assortative matching

Let $p_i$ denote the participation rate of workers of type $i$. Let $c_i$ denote their (detrended) consumption and let $n_i^{\tau}$ denote their working hours when matched with a machine of age $\tau$. Finally let $\mu$ denote the Lagrange multiplier of the aggregate resource constraint, which measures the marginal value of income to the social planner. Then the utility value of matching a machine of age $\tau$ with a worker of type $i$ can be expressed as equal to:

$$\widetilde{V}_i(\tau) = \max(0, V_i(\tau)) \tag{21}$$

where $V_i(\tau)$ measures the value in consumption units of actively participating in the labor market (i.e. choosing positive working hours):

$$V_i(\tau) = \max_{n>0} \left\{ f\left(e^{-\frac{q+\delta}{m}\tau}, h_i, n\right) - \frac{\nu_i}{\mu} v(n) \right\}. \tag{22}$$

This is equal to the difference between the value of the income the worker produces in the job and the disutility cost of working to the social planner, measured in consumption units. The zero value in (21) simply reflects the option value of staying out of the labor market. By solving for $n$ in (22) we immediately obtain that a worker of type $i$ in a job of age $\tau$ works

$$n_i^\tau = \left[\frac{(1-\alpha)\,\theta h_i^{(1-\alpha)(1-\theta)}}{\lambda_1 \nu_i}\right]^{\frac{A}{1+\eta}} e^{-\frac{\alpha q A}{1+\eta}\tau} \mu^{\frac{A}{1+\eta}} \tag{23}$$

hours, where $A = \frac{(1+\eta)}{1+\eta-(1-\alpha)\theta} > 1$. By susbstituting this expression for $n_i^\tau$ into (22), and after remembering (1) we obtain that

$$V_i(\tau) = \overline{V}_i e^{-\frac{(q+\delta)A}{m}\tau} - \frac{\nu_i}{\mu}\lambda_0 \tag{24}$$

where the sequence of constants $\overline{V}_i$'s are given by

$$\overline{V}_i = B\left[\mu \frac{h_i^{\frac{(1+\eta)(1-\theta)}{\theta}}}{\nu_i}\right]^{A-1} \tag{25}$$

with $B = \left[\frac{\theta(1-\alpha)}{\lambda_1}\right]^{A-1} \frac{1}{A}$. For given participation rates $p_i$'s, it is optimal to assign higher skilled workers to higher quality machines, if and only if higher skilled workers are relatively more valuable in new than in old machines, which is equivalent to having $\frac{\partial[V_i(\tau)-V_{i+1}(\tau)]}{\partial\tau} \leq 0$. Given (24) and (25), this means that positive assortative matching requires that the sequence of constants $\overline{V}_i$ is decreasing in $i$, which happens if and only if the following condition holds:

$$\frac{h_i}{h_{i+1}} \geq \left(\frac{\nu_i}{\nu_{i+1}}\right)^{\frac{\theta}{(1-\theta)(1+\eta)}}, \quad \forall i < N. \tag{A1}$$

This immediately leads to the following Proposition:

**Proposition 3** *The equilibrium of the model with heterogenous workers skills features positive assortative matching if and only if condition A1 holds true.*

Condition A1 states that human capital differences are large relative to Pareto weights. The condition is more likely to hold when $\theta$ is low, which implies that hours matter less for the total labor input supplied in the job or when $\eta$ is large, which means that substitutions effects have small effects on working hours in the job.

### 4.3 The planner problem

We now write the social planner problem under Assumption A1, so that the equilibrium features positive assortative matching. The mass of machines assigned to workers of type $i$ is given by $p_i z_i$. Let define $\tau_0^* = p_0 z_0 = 0$. Then the minimal age of a machine operated by workers of type $i \geq 1$ is $\tau_{i-1}^*$ while the maximal age is $\tau_i^*$ where

$$\tau_i^* = \frac{\sum_{j=0}^{i} p_j z_j}{m} = \tau_{i-1}^* + \frac{p_i z_i}{m} \tag{26}$$

The aggregate resource constraint implies that total consumption expenditures are equal to aggregate output $Y$:

$$\sum_{i=1}^{N} z_i c_i = Y \equiv \sum_{i=1}^{N} \int_{\tau_{i-1}^*}^{\tau_i^*} f\left(e^{-\frac{q+\delta}{m}\tau}, h_i, n_i^\tau\right) m\, d\tau. \tag{27}$$

The social planner then solves the problem

$$\max_{c_i, p_i, n_i^\tau} \sum_{i=1}^{N} \nu_i \left[ z_i \log c_i - \int_{\tau_{i-1}^*}^{\tau_i^*} v\left(n_i^\tau\right) m\, d\tau \right] \tag{28}$$

subject to the aggregate resource constraint in (27).

### 4.4 Solving the model with worker heterogeneity

We now characterize choices for consumption $c_i$, participation rates $p_i$ and hours worked $n_i^\tau$ in the problem (28). The first order conditions for consumption $c_i$ leads to

$$c_i = \frac{\nu_i}{\mu} \tag{29}$$

which implies that the relative consumption of different worker types is equal to their relative Pareto weights. Notice that after multiplying the left and right hand side by $z_i$, adding up over all $i$'s and after using (27) and after remembering that $\sum_{i=1}^{N} \nu_i z_i = 1$ we also have that the Lagrange multiplier of the aggregate resource constraint in (27) satisfies

$$\mu = \frac{1}{Y}.$$

To write the first order condition with respect to $p_i$, notice that (26) implies that $\frac{d\tau_j^*}{dp_i} = \frac{z_i}{m}$, $\forall j \geq i$ and zero otherwise. This is because, as the participation of type $i$ workers increases, all workers of lower type, $j > i$, are displaced to marginally older machines, while workers of higher types, $j < i$, are left unaffected. Let start assuming for simplicity that $p_i \in (0, 1)$, $\forall i$. Then the first order condition with respect to $p_N$ immediately leads to

$$V_N\left(\tau_N^*\right) = 0, \tag{30}$$

which means that the worst quality machine operated by the lowest skill workers has zero value to the social planner. The analogous condition for $p_i$ $i < N$ can be expressed as

$$V_i\left(\tau_i^*\right) - \sum_{j=i+1}^{N} \left[V_j\left(\tau_{j-1}^*\right) - V_j\left(\tau_j^*\right)\right] = 0, \tag{31}$$

which emphasizes that assigning a machine to a worker of type $i$ has an opportunity cost, because the same machine can not be operated by other workers. So when we employ one more worker of type $i$, this worker would operate a machine of age $\tau_i^*$ that has value $V_i\left(\tau_i^*\right)$ to the social planner. But since this machine was already operated by a type $i+1$ worker, the net increase in social value is smaller than $V_i\left(\tau_i^*\right)$. This fall in value is measured by the second term in (31), which takes into account that, as the mass of type $i$ workers used in production increase, all employed workers of type $j > i$ are displaced to marginally older machines. Condition (31) can be solved recursively using (30) and starting from $i = N - 1$ to obtain

$$V_i\left(\tau_i^*\right) = V_{i+1}\left(\tau_i^*\right), \quad \forall i < N \tag{32}$$

which simply says that at the critical age threshold $\tau_i^*$ the planner is indifferent between using a type $i$ or a type $i+1$ worker. This again emphasizes the opportunity cost of assigning a machine to a type $i$ worker rather than a type $i+1$ worker.

Finally the first order condition for working hours $n_i^\tau$ immediately leads to (23), which determines working hours of workers of type $i$ as a function of machine age $\tau$ and the marginal value of income $\mu$. Exactly as in the one-type model, hours worked decrease with $\tau$, while they are increasing in $\mu$. For given $\tau$ and $\mu$, hours worked are increasing in the worker's skill $h_i$ and decreasing in the worker's Pareto weight $\nu_i$, which, given (29) determine worker's consumption. Since higher skilled workers have both higher skill $h_i$ and higher Pareto weight $\nu_i$, it is generally unclear whether working hours are increasing in workers skills. By evaluating (23) for workers of different type $i$, we can characterize the conditions under which $n_i^\tau$ decrease with $i$:

**Lemma 3** *For given marginal value of income $\mu$ and machine quality $k^\tau$, working hours are increasing in the skill type of workers if and only if*

$$\frac{h_i}{h_{i+1}} > \left(\frac{\nu_i}{\nu_{i+1}}\right)^{\frac{1}{(1-\alpha)(1-\theta)}} \quad \forall i < N \tag{33}$$

Basically this condition says that working hours are increasing in the skill type of workers when the skill advantage of better workers is large relative to their consumption premium, which simply means that the substitution effect dominates the income effect. Of course, if (33) holds, also output in a job is increasing in skill type but the converse is not necessarily true. For output to be increasing in the skill type of a worker it has to be that the efficiency units of labor as defined in (20) are increasing in the skill type. After using (23), we obtain that a worker of type $i$ in a machine of age $\tau$ produces an amount of output equal to

$$f\left(e^{-\frac{q+\delta}{m}\tau}, h_i, n_i^\tau\right) = \left[\frac{\mu\theta\left(1-\alpha\right)}{\lambda_1 \nu_i}\right]^{A-1} h_i^{(1-\theta)(1-\alpha)A} e^{-\frac{\alpha A(q+\delta)}{m}\tau} \tag{34}$$

which immediately leads to the following Lemma:

**Lemma 4** *For given machine age $\tau$, output is increasing in the skill type of workers if and only if condition A1 holds true.*

This means that the condition to guarantee that output in a job is increasing in the skill type of workers is also the condition to guarantee that positive assortative matching is an equilibrium.

A particular case arises when Pareto weights are independent of worker skills, $\nu_i = \nu$ $\forall i$. In this case, consumption is equalized across workers, see (29), and, since A1 is satisfied, the equilibrium features positive assortative matching. By Lemma 3 we also have that working hours are increasing in the skill type of workers. Finally, since the value of labor market participation is higher for higher skilled workers at all machine ages, $V_i(\tau) > V_j(\tau)$, $\forall j > i, \forall \tau$, condition (32) can never hold as an equality, leading to corner solutions in participation rates. In particular, there will be an $i^*$ such that $p_i = 1, \forall i < i^*$ and $p_i = 0, \forall i > i^*$, which implies that higher skilled workers participate more in the labor market.

## 5 Decentralization

We now discuss how the social planner allocation can be decentralized through prices. The labor market is characterized by a wage function $w_{it}(n)$, that specifies the (detrended) income paid to workers of type $i$ when supplying $n > 0$ hours in a job, and by an assignment function $\varphi_{it}(\tau)$ that specifies the probability density at which a worker of type $i$ actively participating in the labor market in the period is assigned to a machine of age $\tau$. Given the wage function and the assignment function, firms can freely choose their demand for working hours while workers choose their labor supply, i.e. whether to actively participate in the labor market and how many hours to supply in the job. Stable assignment requires that no firm should find optimal to deviate and hire a worker of a type different from that prescribed by the assignment function $\varphi_{it}(\tau)$. We conjecture and later verify that the equilibrium features positive assortative matching.

### 5.1 Representative households

All workers of the same type $i$ are endowed with the same initial level of wealth. Workers are infinitely lived, they can freely borrow and save at the equilibrium interest rate, and there is no aggregate uncertainty. Therefore workers can achieve perfect consumption smoothing, which guarantees the existence of a representative household for workers of the same type $i$. The representative household will give the same consumption level to all workers of the same type and ensure that the present value of the disutility cost of working is equalized across all workers in the household. The representative household of type $i$ chooses, in each period $t$, the probability $p_{it}$ with which each of its members goes to work, and if so how many hours to supply in the job he is assigned to $n_{it}^{\tau}$. This yields labor income

$$W_{it} = p_{it} \int_{R^+} e^{\alpha q t} w_{it}\left(n_{it}^{\tau}\right) \varphi_{it}(\tau)\, d\tau \tag{35}$$

to the household, where $\varphi_{it}(\tau)$ is the probability density that a worker of type $i$ actively participating in the labor market in the period is assigned to a machine of age $\tau \geq 0$, while $e^{\alpha q t} w_{it}\left(n_{it}^{\tau}\right)$ denotes the total labour income earned by a type $i$ worker when assigned to a machine of age $\tau$. Notice that household size is normalized to one. The household starts with wealth $\tilde{b}_{i,0}$, can freely borrow and save at the market interest rate $r_t$, discount utility flows at rate $\rho > 0$, and, in each period, chooses each household member consumption $\tilde{c}_{it}$ and household's total assets $\tilde{b}_{it}$. This leads to the following problem:

$$\max_{\tilde{c}_{it},\tilde{b}_{it},p_{it},n_{it}^{\tau}} \int_0^{\infty} e^{-\rho t} \left[ \log \tilde{c}_{it} - p_{it} \int_{R^+} v\left(n_{it}^{\tau}\right) \varphi_{it}\left(\tau\right) d\tau \right] dt$$

subject to the sequence of budget constraints:

$$\dot{\tilde{b}}_{it} = W_{it} - \tilde{c}_{it} + r_t \tilde{b}_{it}, \tag{36}$$

where a dot denotes a time derivative. By solving the problem we obtain the standard Euler equation for consumption:

$$\frac{\dot{c}_{it}}{c_{it}} = r_t - \rho - \alpha q \tag{37}$$

where $c_{it} = e^{-\alpha q t} \tilde{c}_{it}$ is detrended consumption. This condition can be used to integrate forward (36), which, together with the transversality condition, yields:

$$\int_0^{\infty} e^{-\overline{r}_t} c_{it} dt = b_{i,0} + \int_0^{\infty} e^{-\overline{r}_t} W_{it} dt \tag{38}$$

where $\overline{r}_t \equiv \int_0^t \left(r_u - \alpha q\right) du$ is the horizon $t$, relevant interest rate for detrended quantities.

The first order condition for $n_{it}^{\tau}$ reads as

$$c_{it} v'\left(n_{it}^{\tau}\right) = w_{it}' \tag{39}$$

Finally, we can write the first order condition for labor market participation $p_{it}$, which for simplicity we assume holds as an equality, $p_{it} \in (0, 1)$.[1] This yields

$$c_{it} v\left(n_{it}^{\tau}\right) = w_{it}\left(n_{it}^{\tau}\right). \tag{40}$$

[1] If instead participation rates are at a corner, $p_i \in \{0, 1\}$, we should have that

$$\left[w_{it}\left(n_{it}^{\tau}\right) - c_{it} v\left(n_{it}^{\tau}\right)\right]\left(1 - 2p_i\right) \leq 0$$

which says that the value of participating in the labor market is negative if $p_i = 0$, while it is positive if $p_i = 1$.

## 5.2 Firms

A firm with a machine of age $\tau$ (detrended quality $k^{\tau} = e^{-\frac{q+\delta}{m}\tau}$) matched with a worker of type $i$ chooses hours to maximize $e^{\alpha qt}\pi_{it}(\tau)$ where

$$\pi_{it}(\tau) = \max_{n_{it}^{\tau}} f\left(k^{\tau}, h_i, n_{it}^{\tau}\right) - w_{it}\left(n_{it}^{\tau}\right) \tag{41}$$

denotes detrended profits. By maximizing we then obtain

$$f_3\left(k^{\tau}, h_i, n_{it}^{\tau}\right) = w_{it}'\left(n_{it}^{\tau}\right), \tag{42}$$

which implicitly defines the demand of a firm with capital quality $k^{\tau}$ for the hours of a type $i$ worker.

## 5.3 Matching

In a balanced growth path equilibrium with positive assortative matching there are machine age thresholds $\tau_i^*$ that satisfy (26) as in the social planner problem, with $\tau_0^* = 0$. This implies that the assignment function $\varphi_{it}$ is independent of time and that, for given $i$, $\varphi_i(\tau)$ is equal to zero for any $\tau$ outside the interval $\left[\tau_{i-1}^*, \tau_i^*\right]$. So we have

$$\varphi_i(\tau) = \begin{cases} \frac{1}{\tau_i^* - \tau_{i-1}^*}, & \text{if } \tau \in \left[\tau_{i-1}^*, \tau_i^*\right] \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

which integrates to one over the support $[\tau_{i-1}^*, \tau_i^*]$ of the probability density of workers of type $i$.

## 5.4 Free entry and stable assignment

Since there is an excess supply of machines, it must be that at the critical technological gap $\tau_N^*$, a firm makes zero profits:

$$\pi_{Nt}\left(\tau_N^*\right) = 0 \tag{44}$$

Moreover a stable matching between workers and firms require that $\forall\tau \in \left[\tau_{i-1}^*, \tau_i^*\right]$ a firm prefers to hire workers of type $i$ rather than any other worker type

$$\pi_{it}(\tau) = \max_{j} \pi_{jt}(\tau) \geq 0, \quad \forall\tau \in \left[\tau_{i-1}^*, \tau_i^*\right], \ \forall i \geq 1 \tag{45}$$

This guarantees that no firm finds optimal to deviate and hire a worker of a type different from that prescribed by the assignment function $\varphi_{it}(\tau)$. The positive constraint simply states that a firm should at least break even. Notice that, at the age threshold $\tau_i^*$, (45) should hold both for $i$ and $i+1$. So it must be that

$$\pi_{it}(\tau_i^*) = \pi_{i+1,t}(\tau_i^*), \quad \forall i < N, \tag{46}$$

which means that at the marginal machine, higher skill workers capture all rents from their higher human capital.

### 5.5 Financial markets

At time zero each worker of type $i$ owns shares $s_{i0}$ of the aggregate portfolio of firms. Type $i$ household detrended wealth at time $t$ is equal to $b_{it} = e^{-\alpha q t}\tilde{b}_{it} = s_{it}\mathbf{p}_t$ where $\mathbf{p}_t$ denotes the time-$t$ (detrended) price of equity shares which is equal to

$$\mathbf{p}_t = \int_t^\infty e^{-\bar{r}_s}\Pi_s ds \tag{47}$$

with

$$\Pi_t = \sum_{i=1}^N z_i p_{it} \int_{\tau_{i-1}^*}^{\tau_i^*} \pi_{it}(\tau)\varphi_i(\tau)\, d\tau$$

denoting the (detrended) aggregate profits which are rebated back to households in each period. Since there are no borrowing constraints, in each period $s_{it}$ can be negative, but since the total supply of shares has to add up to one, $\sum_i s_{it}z_i = 1$, it must be that

$$\sum_{i=1}^N z_i b_{it} = \mathbf{p}_t \tag{48}$$

The financial return is given by the sum of dividend payments and capital gains:

$$r_t = \alpha q + \frac{\dot{\mathbf{p}}_t}{\mathbf{p}_t} + \frac{\Pi_t}{\mathbf{p}_t}. \tag{49}$$

### 5.6 Balanced growth path equilibrium

In a balanced growth path equilibrium de-trended consumption $c_i$, assets $b_i$, stock market value $\mathbf{p}$, wages schedules $w_i(n)$, and profits $\pi_i(\tau)$ as well as the assignment function $\varphi_i(\tau)$, participation rates $p_i$, working hours $n_i^\tau$, and the interest rate $r$ remain all constant through time. After defining the tuple

$$x = [c_i, b_i, \mathbf{p}, w_i(n), \pi_i(\tau), \varphi_i(\tau), p_i, n_i^\tau, r]$$

we can then state the following definition for a balanced growth equilibrium:

**Definition** A balanced growth equilibrium is a tuple $x$ such that (i) workers solve their optimization problem, so that (36)-(40) hold; (ii) firms maximize profits, so (42) holds; (iii) the conditions for free entry and stable assignment (44)-(45) are satisfied; (iv) the labor market and the capital market clear, so (26) and (43), and (47)-(49) are satisfied.

### 5.7 Decentralized equilibrium

We conjecture that, if $\forall i\ p_i \in (0, 1)$, the equilibrium features the wage function

$$w_i(n) = \begin{cases} c_i\lambda_0 + c_i\lambda_1 \frac{n^{1+\eta}}{1+\eta}, & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{50}$$

which implies the key property that (39) and (40) in the household problem hold as an identity.[2] As in Prescott, Rogerson, and Wallenius (2009), this means that, in every period, households are just indifferent about whether to participate in the labor market and about how many hours to supply in the job. In equilibrium the aggregate use of labor is determined by firms demand for labor. By comparing (22) with (41), we also have that, under (50), the value of a job to the social planner is equal to the firm's private value, $V_i(\tau) = \pi_i(\tau)$, $\forall \tau \geq 0$ and $i \geq 1$, if and only if type $i$ households consume the same in the two economies. Given (29), this requires that the initial assets of type $i$ households, $b_i(0)$, are such that the consumption $c_i$ that solves (38) is equal to a fraction $\nu_i$ of aggregate output $Y$:

$$c_i = \nu_i Y, \quad \forall i \geq 1. \tag{51}$$

When (51) holds, it can be easily checked that the equilibrium conditions of the decentralized economy are identical to the conditions that characterize the solution of the social planner problem. For example by comparing (30) and (32) with (44) and (46), we immediately see that the critical age thresholds $\tau_i^*$'s are equal, while by comparing (23) with (42) we obtain the same working hours decisions $n_i^\tau$. To analyze under which conditions

[2] If some $p_i$'s are at a corner (either equal zero or one), then the fixed terms in the wage compensation schedule $w_i(n)$ in (50), call it $a_{0i}$, will have to be modified slightly. Generally the $a_{0i}$'s are pinned down by the conditions (44) and (46) leading to $a_{0i} < c_i\lambda_0$ if $p_i = 0$, to $a_{0i} > c_i\lambda_0$ if $p_i = 1$, and to $a_{0i} = c_i\lambda_0$ if $p_i \in (0, 1)$.

the decentralized equilibrium features positive assortative matching we can use (45) and apply the same logic that allowed us to prove Proposition 3. All this immediately leads to the following Proposition:

**Proposition 4** *If the initial assets of type i household, $b_i(0)$ are such that*

$$\frac{h_i}{h_{i+1}} \geq \left(\frac{c_i}{c_{i+1}}\right)^{\frac{\theta}{(1-\theta)(1+\eta)}}, \quad \forall i < N, \tag{A2}$$

*then the equilibrium of the decentralized economy features positive assortative matching and its allocation solves the social planner problem in (28) with the set of Pareto weights $\nu_i$ which satisfy (51).*

For a given vector of model parameters, there is a unique set of Pareto weights $\nu_i$ such that the social planner problem and the decentralized equilibrium yield the same allocation. So, if we change some model parameters, then the set of Pareto weights should also change to have that the new decentralized equilibrium still coincides with the social planner allocation. This also clarifies the difference between Assumption A1 and A2: in the decentralized equilibrium, consumption differences are an equilibrium outcome, while in the social planner problem they just reflect differences in Pareto weights, which are taken as given. Generally A2 requires that skill differences are large relative to consumption differences. A2 is more likely to hold when technological differences across jobs are small: due to the capital skill complementarity induced by the assignment friction, small differences in technologies compress the return to skill and thereby reduce differences in consumption. It is also easy to prove that A2 is more likely to hold when the share of non labor income on total income of low skilled workers is greater, which in the model can only be because of their initial wealth.

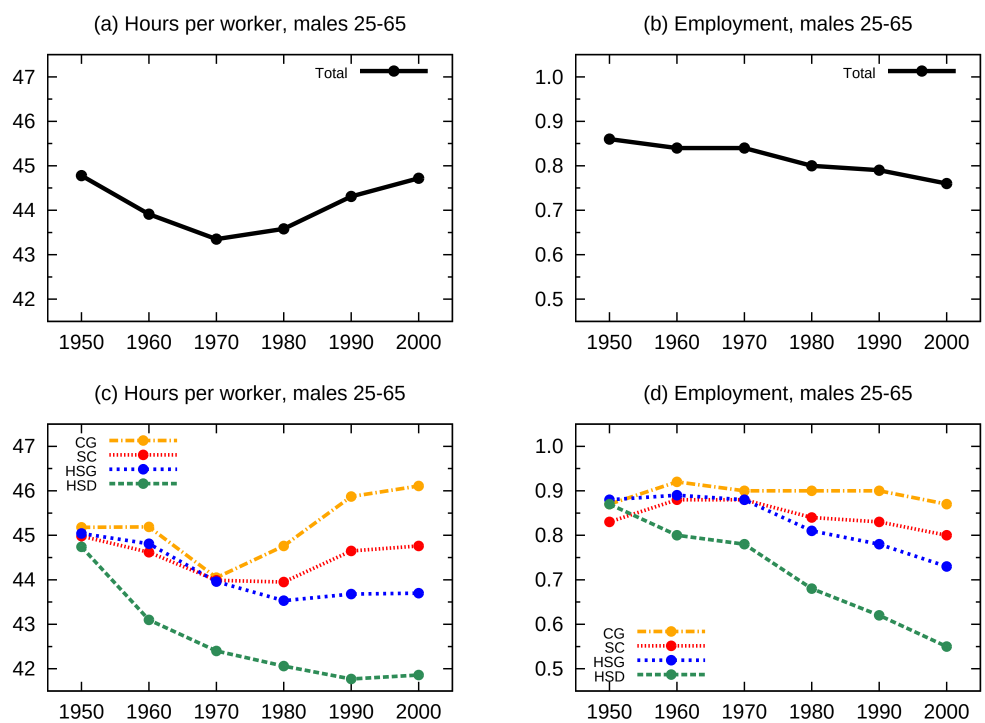
## 6 A quantitative exercise

Our theory states that an increase in the speed of embodied technical change rises wage inequality, rises hours per employed worker, and diminishes participation. According to Greenwood and Yorokoglu (1997) and Greenwood, Hercowitz, and Krusell (1997) investment specific technological progress has actually accelerated since the 1970, and it has been argued —see for instance Violante (2002)— that this is the cause of the increase in US wage inequality documented among other by Katz and Autor (1999) and Heathcote, Perri, and Violante (2010). In this Section we want to quantify the potential role of the observed change in the pace of investment specific technological progress in accounting for some important changes in male labor supply observed in the US over the 1970-2000 period.

### 6.1 Some facts

Over the period, male labor supply has indeed changed substantially in the US. As documented by Juhn (1992), Aaronson, Fallick, Figura, Pingle, and Wascher (2006) and Michelacci and Pijoan-Mas (2012) the participation rate of US male workers has fallen substantially while average hours worked per employed worker have increased. Figure 2 documents these facts using the 1 percent sample of the decennial Census, as provided by the Integrated Public Use Microdata Series (IPUMS) at the University of Minnesota (www.ipums.org). We focus the analysis on a sample of male workers aged between 25 and 64 years old. Panels (a) and (b) describe the evolution of average hours per employed worker and of the employment rate, respectively. It is also well known that these changes have varied depending on the skill level of workers, here identified using four different educational levels. As shown in Panels (c) and (d), high skilled workers have experienced a larger increase in hours per worker and a smaller fall in employment rates.

Figure 2: Hours and Participation




Over the period the composition by skill group of the US male labor force has also changed. For example, as documented in Table 1 the share of workers with a college degree has increased substantially, from 15% to 27%,, while the share of high school dropouts in the US male working age population has fallen from 43% to 14%. We will take these compositional changes into account to evaluate the quantitative performance of the model.

Table 1: Population share by education groups

| | 1970 | 2000 | $\Delta_{00-70}$ |
|---|---|---|---|
| College graduates | 15 | 27 | +12 |
| Some college | 11 | 28 | +17 |
| High school graduates | 31 | 31 | 0 |
| High school dropouts | 43 | 14 | -29 |

Note. All statistics are computed using the sample of male workers of age 25-65 from US Census.

### 6.2 Calibration in the 70's

To match the evidence in Section 6.1, we consider a version of the decentralized economy studied in Section 5.[3] We assume there are 4 worker types, $N = 4$, corresponding to workers with 4 different educational levels: college graduates, workers with some college education but no college degree, high school graduates, and high school dropouts. Calibrating this version of the model involves setting 21 parameters: 8 are set directly, 3 are set using one normalization and two add-up constraints, while the remaining 10 are set by requiring that the model equilibrium matches some moments from the data. Table 2 summarizes the resulting parameter values and the corresponding calibration targets.

As standard in the literature we set the annual discount rate $\rho$ to 4%, the Frisch labor elasticity parameter $\eta$ to 2 and the depreciation rate $\delta$ to 6%, which is taken from Nadiri and Prucha (1996). As in Greenwood, Hercowitz, and Krusell (1997) and Hornstein, Krusell, and Violante (2007) we set the rate of growth of capital-embodied technical change, $q$ to 2% before the 70's and to 4.5% in the late 90's. The value for $m$ is chosen to match the value for the average age of private fixed assets reported by the Bureau of Economic Analysis, which in the mid 60's was equal to 11.5 years, see Table 2.10 at http://www.bea.gov/national/FA2004/. The age of the oldest machine in the model

[3] Alternatively we could have used a version of the social planner problem studied in Section 4. But in this model, for given Pareto weights, a change in technological progress would have no effects on relative consumption by skill groups, which as shown in Table 4 would be highly counterfactual.

Table 2: Parameter values and calibration targets, 1970

| **Model parameter** | | **Calibration target** | |
|---|---|---|---|
| **Symbol** | **Value** | **Statistic** | **Value** |
| Preferences | | | |
| $\rho$ | 0.04 | – | |
| $\eta$ | 2 | – | |
| $\lambda_0$ | 0.65 | average employment to population ratio | 0.84 |
| $\lambda_1$ | 6.88 | average hours per employed person | 43.3 |
| Technology | | | |
| $\delta$ | 0.06 | – | |
| $q$ | 0.02 | rate of fall of price of investment goods | 0.02 |
| $m$ | 0.03652 | average age of fixed assets (in years) | 11.5 |
| $\alpha$ | 0.45 | capital share | 0.33 |
| $\theta$ | 0.64 | income ratio between group 4 and 1, $\overline{w}_4/\overline{w}_1$ | 0.54 |
| Population | | | |
| $z_1$ | 0.15 | add-up constraint $\sum z_i = 1$ | 0.15 |
| $z_2$ | 0.11 | population share of group 2, $z_2$ | 0.11 |
| $z_3$ | 0.31 | population share of group 3, $z_3$ | 0.31 |
| $z_4$ | 0.43 | population share of group 4, $z_4$ | 0.43 |
| $h_1$ | 1 | normalization | 1 |
| $h_2$ | 0.83 | consumption for group 2 relative to group 1, $c_2/c_1$ | 0.84 |
| $h_3$ | 0.75 | consumption for group 3 relative to group 1, $c_3/c_1$ | 0.76 |
| $h_4$ | 0.64 | consumption for group 4 relative to group 1, $c_4/c_1$ | 0.68 |
| $s_{10}$ | 1.01 | add-up constraint, $\sum s_{i0}z_i = 1$ | |
| $s_{20}$ | 0.92 | participation rate for group 2, $p_2$ | 0.88 |
| $s_{30}$ | 0.90 | participation rate for group 3, $p_3$ | 0.88 |
| $s_{40}$ | 1.09 | participation rate for group 4, $p_4$ | 0.78 |

Note. Group 1 refers to college graduates, group 2 refers to high school graduates with some college education, group 3 refers to high school graduates and group 4 to high school dropouts. All statistics are computed using the sample of male workers of age 25-65. Population shares, employment rates, hours per worker and income differences are from the 1970 US Census. Consumption level are from 1980 CEX.

economy is equal to $p/m$ and the distribution of ages is uniform. Hence the average machine age in the economy is $\frac{p}{2m}$. Since the aggregate employment rate $p$ will be a calibration target (see below), $m$ can then be chosen directly by matching the average age of capital assets in the US economy. The shares $z_i$ of workers of different skill types are taken to match the values in the US in the 70's as reported in Table 1.

The values for $\lambda_0$ and $\lambda_1$ are set to match the aggregate average male employment rate and hours per employed male worker in the US in 1970, which are equal to 0.84 and 43.3 weekly hours, respectively. Hours in the model are calibrated to 43.3/112, where 112 corresponds to the amount of non-sleeping hours in a week available to the worker (7 days a week times 16 hours a day). Hours in the model are then multiplied back by 112 to report the results in tables.

We normalize $h_1$ to one. The remaining three values for $h_i$, together with the values for $s_{i0}$ and those for $\alpha$ and $\theta$ are chosen to match the employment rate for the three educational types (the fourth is then matched since the aggregate participation rate is a target), the consumption level of workers of different educational level relative to the consumption level of workers with a college degree (which in the model correspond to $c_i/c_1$ for $i = 1, 2, 3$), the labor share in GDP, and the average labor income per employed worker of the lowest skilled relative to the highest skilled workers. In the model the average labor income per employed worker of type $i$ is defined as equal to

$$\overline{w}_i = \int_{\tau_{i-1}^*}^{\tau_i^*} c_i v\left(n_i^\tau\right) \varphi_i\left(\tau\right) d\tau$$

while the labor share is calculated as equal to

$$\text{Labor share} = \frac{\sum_{i=1}^{4} z_i p_i \overline{w}_i}{Y}$$

Relative consumption comes from CEX in 1980, which is the first wave available. Average labor incomes by skill group are calculated using the Census in 1970. This implies that the sampling period for the consumption and the income data are slightly different.

Column 1 in Table 3 presents the data for employment rates and hours worked in 1970. These numbers are the same used in Figure 3. In Column 3 we report the employment rates and hours per worker predicted by our model in 1970. The average employment rate, the average hours per worker and the employment rates by education groups are calibration targets and thereby are matched perfectly. In the calibration we do not target, hours per employed worker by educational group, which in the model corresponds to $\overline{n}_i \equiv$

$\int_{\tau_{i-1}^*}^{\tau_i^*} n_i^\tau \varphi_i(\tau)\, d\tau$. Still the model rightly predicts that better educated workers work longer hours, although the model slightly over-predicts differences by educational group. Column 1 and 3 in Table 4 report the values for relative labor income and relative consumption of the different skill groups in the data and in the model. Relative consumption patterns are matched by construction and so is the labor income of high school dropouts relative to college graduates. But the relative labor income of the two other educational groups was not targeted, still the model matches their value quite nicely.

We also use the calibrated model to measure how much of the wage returns to education is due to worker skill differences and how much is due to job differences. To do so, we solve the calibrated model for a very large value of $m$ set equal to $10^3$, which *de facto* makes all jobs identical. We find that 70% of the hourly wage ratio between college workers and high school dropouts is still present in the high $m$ economy. This implies that, in 1970, 70% of the college premium was due to differences in worker skills with the remaining 30% being due to job differences. Finally, we checked that the condition A2 for positive assortative matching is satisfied in the calibrated economy.

Table 3: Labor supply

| | Data | | Model | | | |
|---|---|---|---|---|---|---|
| Statistic | 1970 | $\Delta_{00-70}$ | 1970 | $\Delta q$ | $\Delta z_i$ | both |
| Participation rate | 0.84 | -0.08 | 0.84 | -0.08 | 0.00 | -0.08 |
| College graduates | 0.90 | -0.03 | 0.90 | -0.06 | -0.05 | -0.12 |
| Some college | 0.88 | -0.08 | 0.88 | -0.07 | -0.03 | -0.09 |
| High school graduates | 0.88 | -0.15 | 0.88 | -0.07 | -0.01 | -0.09 |
| High school dropouts | 0.78 | -0.23 | 0.78 | -0.09 | -0.04 | -0.14 |
| Hours per worker | 43.3 | +1.5 | 43.3 | +1.2 | +0.1 | +1.3 |
| College graduates | 44.1 | +2.5 | 47.4 | +2.0 | 0.0 | +2.0 |
| Some college | 44.0 | +1.0 | 46.7 | +1.8 | -2.2 | -0.8 |
| High school graduates | 44.0 | -0.2 | 44.2 | +1.3 | -3.3 | -2.7 |
| High school dropouts | 42.3 | -0.5 | 40.0 | +0.4 | -1.8 | -1.7 |

### 6.3 Calibration in the 00's

We now increase the value of $q$ from 2% to 4.5%. In Column 4 of Table 3 we report the implied changes to the economy. The model predicts a fall in the aggregate participation rate of 8 percentage points, as in the data. The model also predicts an increase of 1.2 weekly hours, which is slightly smaller than the observed increase of 1.5 hours. The model also matches well the observed variation by skill groups: for the more highly educated it predicts a greater increase in hours per worker and a lower fall in participation.

In Column 4 of Table 4 we report the implications for relative labor income and consumption. The increase in $q$ generates a small increase in labor income inequality, with the labor income of high school dropouts relative to college graduates falling from 0.54 to 0.49, which is one third of the fall observed in the data. The consumption of high school dropouts relative to college graduates falls from 0.68 to 0.66. This small increase in income and consumption inequality is due to the fact that low skilled workers receive a substantial amount of non labor income: in our calibrated economy, only 58 percent of the permanent income of high school dropouts comes from their labor, while this figure is 74 percent for college educated workers. In the model this is due to a relative high value for $s_4$. But in the data, the high share of non labor income for high school dropouts is mainly due to government transfers, see Budría, Díaz-Giménez, Quadrini, and Ríos-Rull (2002).

Table 4: Labor income and consumption

| | Data | | Model | | | |
|---|---|---|---|---|---|---|
| Statistic | 1980 | 2000 | 1980 | $\Delta q$ | $\Delta z_i$ | both |
| Average labor income | | | | | | |
| College graduates | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| Some college | 0.79 | 0.66 | 0.81 | 0.79 | 0.82 | 0.79 |
| High school graduates | 0.69 | 0.54 | 0.71 | 0.68 | 0.73 | 0.70 |
| High school dropouts | 0.54 | 0.39 | 0.54 | 0.49 | 0.56 | 0.50 |
| Average consumption | | | | | | |
| College graduates | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| Some college | 0.84 | 0.77 | 0.84 | 0.83 | 0.85 | 0.84 |
| High school graduates | 0.76 | 0.68 | 0.76 | 0.74 | 0.78 | 0.76 |
| High school dropouts | 0.68 | 0.54 | 0.68 | 0.66 | 0.71 | 0.70 |

We now replace the value of the population shares $z_i$'s in 1970 with their value in 2000, as reported in Table 1. Changes in population shares matter because, due to the assignment frictions, they affect directly the quality of jobs that are available to skill groups. When we just change the composition of the labor force, the aggregate participation and hours per worker are hardly affected, see Column 5 in Table 3. Still the participation and the hours per worker of each educational group fall. This is because with a very much larger share of high skilled workers in the economy, every skill group end up working with machines of average lower quality. Finally, Column 6 of Tables 3 and 4 report the overall effect of changing both $q$ and the population shares $z_i$'s. The aggregate participation rate and hours per worker behave as in the economy where only $q$ changes, while the changes in the lower skill group are substantially more pronounced, which again reflects that, with a larger supply of high skilled worker, the quality of jobs they have access to worsens substantially.

## 7 Conclusions

We have studied labor supply decisions in an assignment model with balanced growth. In the model, technological progress is embodied into new jobs which are slowly created over time. Hence there is dispersion in job technologies. Workers differ in skills and they can be employed in at most one job. This leads to a simple assignment problem in the spirit of Becker (1973) and Sattinger (1975). But in our framework labor supply is endogenous because in every period each worker decides whether to actively participate in the labor market, and how many hours to work in the job he is assigned to. Since lower skilled workers can supply longer hours, we have shown that the equilibrium features positive assortative matching (higher skilled workers are assigned to better jobs) only if differences in consumption are small relative to differences in workers skills, which guarantees that low skilled workers do not compensate their lower skill level with much greater working hours. In equilibrium, the model endogenously generates inequality in jobs, wages, and labor supply, but all workers of the same skill consume the same amount. When the pace of technological progress accelerates, differences in job technologies widen, wage inequality increases and workers participate less often in the labor market but supply longer hours on the job. We have shown quantitatively that this mechanism can explain why, as male wage inequality has increased in the US, labor force participation of male workers of different skills has fallen while their working hours have increased. The model also matches well the observed variation by skill groups.

Our analysis could be extended along several dimensions. In particular, in our model skill differences are perfectly observable, constant over time, and exogenously given. This simplifies the analysis, but it neglects some important features of the labor market, such as worker types learning, as in Eeckhout and Weng (2011) and Groes, Kircher, and Manovskii (2010), or human capital accumulation as in Eeckhout and Jovanovic (2002), Imai and Keane (2004), and Michelacci and Pijoan-Mas (2012). Introducing dynamic elements into the analysis would make the return to labor supply intertemporal, which would affect the incentive to participate in the labor market, working hours decisions and the value of being matched with a specific machine. Additionally, in our model machines and workers are combined in a fixed proportion which is exogenously given. As in Eeckhout and Kircher (2011), it would be interesting to have a richer theory of the firm where not only the skill level but also the number of workers matched with each machine is endogenously determined.

## A Proof that $\frac{\mathbf{e^{\gamma_0 q}-1}}{\mathbf{q}}$ is increasing in q when $\gamma_0 > 0$ and $q \geq 0$

Let $\gamma_0 > 0$. The derivative of the function

$$z(q) = \frac{e^{\gamma_0 q} - 1}{q} \tag{52}$$

has the same sign as

$$g(q) = \gamma_0 e^{\gamma_0 q} q - e^{\gamma_0 q} + 1.$$

which is positive for $q \geq 0$ since $g(0) = 0$ and $g(q)$ is increasing in $q$ for all $q > 0$ which follows from

$$g'(q) = \gamma_0^2 e^{-\gamma_0 q} q > 0.$$

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