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ABSTRACT

Modelling the U.S. sovereign credit rating*

A methodology for generating sovereign credit ratings based on macroeconomic theory is proposed. This is applied to quarterly U.S. data from 1970 to 2011. Over this period the official credit rating of U.S. Treasury securities has been of the highest quality. In contrast, the model-based measure finds that there are two clear instances in which the U.S. sovereign credit rating, if evaluated on the basis of economic fundamentals, should have been downgraded: the first oil crisis of the 1970s and in the aftermath of the Lehman collapse in 2008. This result is robust to several alternative views on the maximum borrowing capacity of the U.S. economy.

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1 Introduction

Credit ratings for both private sector and sovereign debt have come under intense scrutiny since the onset of the 2008 financial crisis. Credit rating agencies (CRAs) have been criticized for failing to identify the amount of risk accumulated by mortgage-backed securities in the United States during the lending boom of the 2000s. Following the recent downgrades of a number of eurozone sovereigns, CRAs have been accused of exacerbating the eurozone debt crisis and contributing to the rise of the cost of borrowing above sustainable levels in several European countries.

Concerns about the adequacy of the operating procedures of the leading CRAs have been recently expressed by the Security and Exchange Commission in a report on the ability of 10 top CRAs to make timely and accurate disclosures, and to allay fears about potential conflicts of interest.[1] The European Commission has recently issued a proposal for stricter rules for CRAs in order to make them more transparent, accountable and to increase competition in the sector. The Commission's proposal stress the role of conflict of interest, political interference and inefficiencies in existing CRAs methodologies. They also suggest the creation of an European ratings foundation in order to counter the influence of U.S.-based CRAs and that individual investors should determine their own independent evaluation of credit ratings (European Commission, 2011).[2] An argument against this proposal is that it would be too costly for individual investors to make their own credit evaluations.

The aim of this paper is to show how it would be possible to provide measures of sovereign credit ratings that are transparent, independent and timely. Transparency refers to the ease of general public to access and to reproduce credit ratings, and to the ability of the public to make its own judgment about their validity. Independence reflects the derivation of sovereign credit ratings that are model-based rather than driven by the subjective evaluation of analysts. The evaluation can be updated systematically using the latest available data, and is timely for this reason. The measure is inexpensive to produce, and can even be automated. Given these properties, we argue that such a procedure can provide benchmark statistics on sovereign credit ratings that could be especially useful not only for individual investors and independent (national and supranational) agencies, but also for policy makers and CRAs.

In advanced economies like the U.S. the repayment of sovereign debt is a constitutional obligation. Thus, a sovereign credit rating should reflect the government's ability to use fiscal policy as an instrument to generate enough savings in the future to meet its outstanding financial obligations. The sovereign credit rating is therefore directly related to this definition of sovereign default probability. The default probability may then be mapped into a credit rating. Our methodology is an adaptation to government debt of Merton's (1974)'s mea-

[1] See Security and Exchange Commission (2011)

[2] Similar reactions from policy makers and commentators were prompted following the Latin American defaults in the early 1980s, the 1997 Asian banking crisis and the default of Enron in 2001.

sure of distance-to-default and default probability. This has three key elements: a forecast of the level of indebtedness over a given time horizon; an estimate of the uncertainty surrounding this forecast; and a measure of the maximum borrowing capacity of the sovereign country. We calculate the time-varying forecasts of the debt-GDP ratio and their volatility using a VAR model based on rolling-window estimation, hereafter a ROVAR model. Rolling-window estimation, favoured by Stock and Watson (2007, 2008) and Orphanides and Wei (2010) among the others, has the advantage of taking account of time-variation in the VAR coefficients and in their volatility over the sample period without taking a specific stance on the source of time-variation. In this way forecasts of the debt-GDP ratio, the uncertainty surrounding them and the implied default probability may be up-dated each period. This procedure is computationally inexpensive as the model can be estimated using standard classical methods. Having determined the default probability profile, this can be mapped into a credit rating using historic information on sovereign default probabilities and credit ratings.

The analysis requires the definition of a debt threshold (debt-GDP limit) beyond which a default event is assumed to occur. We implement two alternative views about the default threshold. The first, based on ad-hoc values, is entirely agnostic about the economic rationale for the debt-GDP limit. Its usefulness is in providing preliminary evidence on the likely values of the model-based sovereign credit rating and on its sensitivity to forecasts of the debt-GDP ratio, macroeconomic volatility and the debt threshold itself. The second employs a real business cycle model with an elastic labour supply and distortionary taxation to derive the debt-GDP limit. This is based on the notion that governments default (either formally or de facto) only when they are not able to meet their financial obligations through using their fiscal instruments. We assess the advantages and disadvantages of four alternative definitions of the debt limit, choosing one that is consistent with the notion of "fiscal limit" recently proposed by Davig, Leeper and Walker (2010). We apply this methodology to the U.S. sovereign credit rating for the period 1969:4 to 2011:2 for both ad-hoc and theory-based debt-GDP limits, constructing quarterly time-series of credit ratings for short-term and long-term U.S. debt.

U.S. Treasury securities have long been considered risk-free assets. Historically, they have received the highest credit-quality rating by all CRAs. Since the latest global financial crisis, however, both prominent economists (see, for example, Buiter (2010)) and market participants have increasingly taken the view that U.S. bonds are no longer risk-free assets. The change in the market sentiment is reflected in the fast-growing trend in the price of credit default swaps (CDS) for U.S. sovereign bonds, an indicator of the market's perception of the U.S. government creditworthiness.

This is illustrated very clearly in Figure 1 which shows the price of U.S. Treasury securities CDSs over the three and half years (from January 2008 to June 2011) before Standard & Poor's downgraded the U.S. sovereign credit rating by one notch from its highest ranking on August 5, 2011.[3] Early in 2008, the 5-year U.S. sovereign CDSs traded below 10 basis points (bps). The price rose substantially in July 2008 when IndyMac Bank collapsed, and rose further in September 2008 when Lehman Brothers declared bankruptcy and AIG attempted to negotiate a bridging loan from the Federal Reserve. CDS prices also increased in early 2009 to just below 100 bps and, after a sharp decline to about 30 bps in the first half of the 2009, again increased steadily. By the end of June 2011 U.S. bonds CDS traded at about 51 bps, twice as much as German sovereign CDS (26 bps) and close to that of Japan CDS (52 bps).[4] Notwithstanding this sharp deterioration of the market's perception of its creditworthiness over this period, U.S. government debt has received the highest quality ranking by all CRAs during this period.[5]

In contrast, the model-based sovereign credit rating derived in this paper appears to be more in line with the market's perceptions about the creditworthiness of the U.S. government. The results suggest that the U.S. sovereign credit rating would have been of the highest quality for most of the last 40 years with the exception of two periods where the credit rating is lower: the oil crisis in the early 1970s and in the aftermath of IndyMac Bank and Lehman Brothers collapse. The extent and duration of these downgrades depend on the level of the debt-GDP limit, but in no case does the credit rating fall to speculative grade.

[3] The data are from Datastream (Thomson Reuters CDS), accessed on July 11, 2012. The sample includes daily observations from January 7, 2008 to June 30, 2011. No data on U.S. sovereign CDS is available prior January 2008. The U.S. macroeconomic data is only available up to 2011:2, therefore defining the end of the CDS sample period.

[4] On July 11, 2012, the 5-year CDSs were traded at about 47 bps, while the price on German CDS was about 32 bps and that of Japan CDS is 59 bps.

[5] In 2011, the main CRAs (Fitch Ratings, Moody's and Standard & Poor's) expressed concern about the medium-term perspectives of the U.S. fiscal outlook and lowered the outlook of the U.S. sovereign debt to negative. All three CRAs also began issuing warnings about a possible (though limited to one or two notches) downgrade of the U.S. sovereign credit rating.

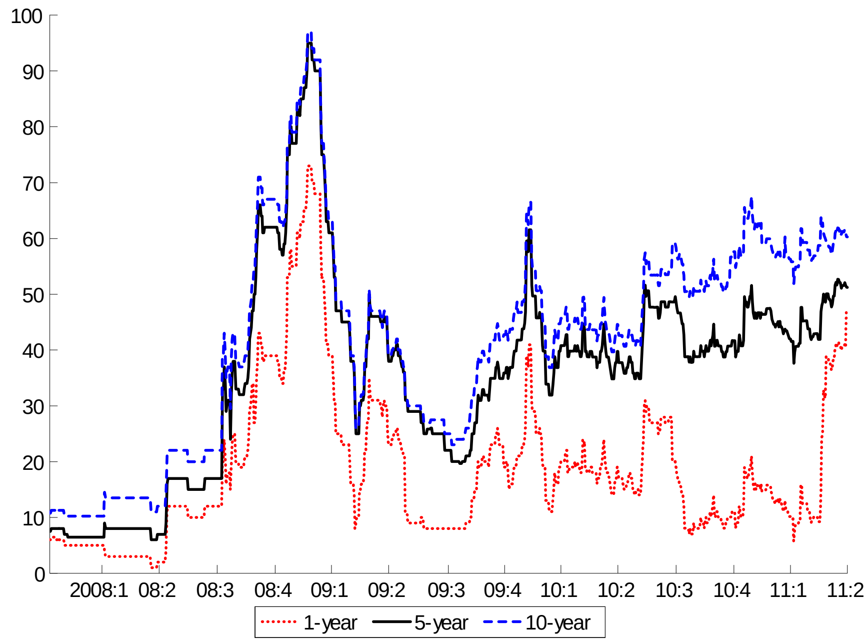

Figure 1: U.S. sovereign CDS, mid rate spread. Source: Datastream (July 2012).

To the best of our knowledge, this paper is the first to attempt to use macroeconomic models to determine sovereign credit ratings. As outlined in sections 2 and 3, there is a large theoretical literature on sovereign default risk, but little attention has been paid to how this translates into a sovereign credit rating. The empirical literature on sovereign credit rating has focused either on determining good predictors of sovereign credit ratings, or how (and to what extent) financial markets and macroeconomic variables react to changes in sovereign credit ratings.

The paper proceeds as follows. Section 2 provides some background information on sovereign credit ratings and the methodologies followed by CRAs. Section 3 discusses how we map the probability of default into credit ratings. Section 4 describes how we adapt Merton's model in order to compute sovereign default probability. Section 5 specifies the ROVAR model used for the derivation of the debt-GDP forecasts and their uncertainty. Section 6 carries out a preliminary analysis of the U.S. sovereign credit rating by combining estimates from the ROVAR with ad-hoc values of the debt-GDP limit. Section 7 assesses various different measures of the debt-GDP limit obtained from a standard dynamic stochastic general equilibrium (DSGE) model and recalculates the U.S. sovereign credit rating based on the notion of fiscal limit of Davig, Leeper and Walker (2010). In Section 8 we summarize our procedure and draw some conclusions about its use. There are also four appendices. Appendix A describes the ROVAR model. Appendix B provides details of the data. Appendix C includes details on the solution of the DSGE model used for the determination of the theory-based default limits. Appendix D describes the algorithm used for computing the distribution of the steady-state values of the theory-based debt-GDP limits

2 Sovereign credit ratings and the CRAs' procedures

A sovereign credit rating is broadly defined as an opinion about the likelihood of default of a government, based on the perceived ability and willingness of the government to meet its financial obligations.[6] A government is able to repay its debt when it can change its policy to generate enough savings to service its obligations. Whether a government wants to repay its liabilities depends on a number of factors that go beyond the ability to generate revenue. These include the existence of binding constraints to enforce debt repayment, sovereign reputation, whether debt holders are either domestic or foreigners, the maturity structure of government debt and the share of government debt held by domestic banks.[7]

The definition of a sovereign credit rating has evolved over time. CRAs currently view a sovereign credit rating as being closely related to a government's ability to meet its financial obligations. The main CRAs also issue separate sovereign recovery ratings for government bonds that reflect the willingness to repay debt. Credit ratings are assigned both to countries and bonds. The former reflects a government's overall financial capacity to pay its financial obligations; the latter measures the quality of securities issued by a country. In addition to evaluating countries and bonds over the long period, CRAs also assign short-term ratings to sovereign issuers and short-term bonds, where short-term is deemed to be a period of time of 12 months or less.

Although they vary across CRAs and have changed considerably over time, sovereign rating methodologies are ultimately based on the individual judgments of rating analysts. For example, no CRA uses a mathematical formula or an economic model to measure the credit rating. Instead, CRAs have sovereign risk units in charge of assigning new credit ratings and of monitoring and reviewing the existing ratings. Ratings are issued following a request for a rating made

[6] This section draws on Gaillard (2012).

[7] The distinction between willingness and ability to pay debt, first highlighted by Eaton and Gersovitz (1981), is also central in the economic literature. The literature on willingness initially looked at the role of sanctions (Sachs, 1984), sovereign reputation (Eaton and Fernandez, 1995) and asymmetric information (Gertler and Rogoff, 1990) for the assessment of sovereign risk. More recent contributions have pointed out the role of debt maturity (Cole and Kehoe, 2000) and secondary markets (Broner, Martin and Ventura, 2010) in determining sovereign default events. There is a vast economic literature on the theoretical determinants of the ability to pay and its empirical measurement. Polito and Wickens (2012) provide an up-to-date survey.

by a sovereign country.[8] The CRA's team of analysts meets with the leading officials of a country, collects data and then formulates a preliminary rating recommendation, which is subsequently ratified (via a voting system) by the members of a rating committee within the CRA. The resulting rating is first notified to the issuer and then disseminated to the public through the media. Fitch Ratings and Standard & Poor's allow the rating issued by the rating committee to be appealed before dissemination, while Moody's does not. Both qualitative and quantitative criteria and variables are employed by the CRAs to determine the credit rating. These vary across CRAs and have changed over time. No information is provided on how each criterion and variable is weighted in the final determination of the overall credit rating.[9]

3 Credit ratings and the probability of default

Credit ratings reflect the probability of default. We require a mapping of the probability of default into a credit rating. This may be derived from historic data made available by the CRAs. Table 1 summarizes the letter-rating scale that we employ as a benchmark for the numerical analysis. This includes 19 categories ranging from Aaa (highest quality and minimum credit risk) to C (lowest quality assigned to bonds or countries already in default). Ratings from Aaa to Baa3 are termed investment grade, whereas ratings below Baa3 are categorized as speculative grade. Short-term ratings are related to long-term ratings and range from Prime, which implies minimum credit risk for obligations of short-term maturity (further distinguished in Prime-1, Prime-2 and Prime-3) to Not Prime.[10]

CRAs provide separate ratings for debt issued in local currency (LC) or in foreign currency (FC). For the purpose of this paper we abstract from this distinction; while this is particularly relevant for emerging economies, it is less important for advanced economies like the U.S. Recent evidence suggests that LC and FC sovereign credit ratings have converged over time with the main CRAs currently regarding governments equally likely to default on both domestic and foreign liabilities (see Gaillard 2011, p24).

[8] Originally CRAs were paid by market investors for issuing credit rating (the so-called "subscriber-pays" model). Since the early 1990s, however, CRAs are paid for their services directly by sovereign countries ("issuer-pays" model).

[9] The economic literature has looked at the determinants and accuracy of sovereign credit ratings, in particular with reference to developing countries. See for example, Eichengreen and Porters (1986) and, more recently, Hill, Brooks and Faff (2010) and Afonso, Gomes and Rother (2011).

[10] Although this rating scale is closely related to that currently employed by Moody's, it also approximates the grading scales employed by other CRAs. The actual rating scale used by Moody's includes 21 categories as the Caa rating is split into Caa1, Caa2 and Caa3. Fitch's scale includes 24 categories from AAA to D, while S&P scale is based on 23 categories from AAA to D.

Investment grade		Speculative grade	
Long-term	Short-term	Long-term	Short-term
Aaa	Prime - 1	Ba1	Not Prime
Aa1	Prime - 1	Ba2	Not Prime
Aa2	Prime - 1	Ba3	Not Prime
Aa3	Prime - 1	B1	Not Prime
A1	Prime - 1	B2	Not Prime
A2	Prime - 1/2	B3	Not Prime
A3	Prime - 1/2	Caa	Not Prime
Baa1	Prime - 2	Ca	Not Prime
Baa2	Prime- 2 or 3	C	Not Prime
Baa3	Prime-3		
Source: www.moodys.com			

Table 1: Sovereign credit rating scale

To map default probabilities into a credit rating we adopt the view that a credit rating reflects exclusively the ability of a government to repay debt as measured through the probability of default. We have noted that there is not a clear-cut definition of sovereign credit rating, but by associating credit ratings with this definition of the probability of default we obtain a rating consistent with the current orientation of the main CRAs.

Table 2 reports the 10-year cumulative default probabilities and the corresponding sovereign credit ratings over the period 1983-2011, as provided by Moody's (2011). Similar tables are provided by Standard & Poor's and Fitch Ratings.[11] The table show a clear negative relation between the sovereign credit rating and the probability of default. For example, a Baa rating implies a probability of default of 2.207 per cent by the end of year 5, while a Ba rating is linked to a higher cumulative default probability (7.435 per cent) over the same period of time.

It is not possible to map these data on sovereign default probabilities directly into the detailed scale of credit ratings reported in table 1 for two reasons. First, from the historic data we are not able to discriminate between either the long-term default probability profiles of Aaa-A ratings or the short-term default probability profiles of Aaa-Baa ratings. Second, the default probability profile is limited to 7 out of the 19 categories in table 1. We therefore use a two-stage

[11] Rather than using historic data, one could specify a new rating scale based on pre-defined cumulative default probabilities. For example, ratings could range from 100 (maximum rating) to 1 (lowest rating) corresponding (one-for-one) to cumulative default probabilities by the end of a specific horison (say 10 year) that instead range from 1 to 100 per cent. The default probability could change over time either linearly or at an increasing rate. The use of a new rating scale would however make more difficult the comparison between the historical credit rating and the model-based measure.

Years	1	2	3	4	5	6	7	8	9	10
Rating					Sovereign					
Aaa	0	0	0	0	0	0	0	0	0	0
Aa	0	0	0	0	0	0	0	0	0	0
A	0	0	0	0	0	0	0	0	0	0
Baa	0	0.476	0.997	1.57	2.207	2.855	2.855	2.855	2.855	2.855
Ba	0.769	1.746	3.433	5.349	7.435	8.949	11.118	13.951	16.416	18.882
B	3.391	7.039	9.204	12.11	15.096	17.986	20.085	21.227	22.735	24.59
Caa-C	23.636	27.727	32.823	32.823	32.823	32.823	32.823	32.823	32.823	32.823
Source: Moody's (2011)										

Table 2: Average cumulative default rates (in percentage), 1983-2010.

linear interpolation to retrieve this missing information. For each year in table 2 we derive the default probability associated with each of the 19 categories in table 1 by interpolating the missing observations.[12] This initial interpolation has the effect of assigning, for each year, nonzero default probabilities for ratings Aaa-Baa3 in year 1, and ratings Aaa-A3 in subsequent years. Having derived a mapping of the probability of default into credit ratings in each of the 10 years, we then derive a quarterly mapping for the whole 10-year period by a further interpolation.[13]

The final four columns of table 3 report the cumulative probability of default by the end of the first, fifth and tenth year, as well as the unweighted average over the whole 10-year period. The 1-year scale is used later to derive the measure of the short-term rating, while the 5-year, 10-year and average scales are used to measure long-term ratings.

The accuracy and interpretation of the default probabilities in table 3 require further comment. The default probabilities in table 2, from which table 3 is obtained, are calculated by forming cohorts of sovereign issuers on the basis of their original ratings and then dividing the number of defaults in the cohort over a specific period of time by the initial size of the cohort. They therefore provide an ex-post measure of the default probability that is not entirely consistent with that required by an ex-ante analysis of the sovereign credit rating. Concerning their reliability, the data in table 3 may understate the default probability profile attached to each credit rating as they exclude the default episodes that occurred between the first and second World Wars, and from the second half of the

[12] We assume that ratings Aa, A, Baa, Ba, B and Caa-C in table 2 correspond respectively to Aa3, A3, Baa3, Ba3, B3 and C in table 1. We also replace the value of 0 for Baa in year 1 of table 2 with 0.476/2, i.e. half of the value in the following year.

[13] This second round of interpolation is carried out assuming that in the first year the default probability at the beginning of the first quarter is 0. We have also replaced the default probabilities at the end of the first year for Aaa ratings from 0.000*e-20 to 0.000499, as the model typically yields nonzero default probability for Aaa ratings.

	Rating		Cumulative default probability			
Category	Long-term	Short-term	1-year	5-year	10-year	average
Investment grade	Aaa	Prime - 1	0.000	0.000	0.000	0.000
	Aa1	Prime - 1	0.026	0.245	0.317	0.208
	Aa2	Prime - 1	0.053	0.490	0.634	0.415
	Aa3	Prime - 1	0.079	0.736	0.952	0.623
	A1	Prime - 1	0.106	0.981	1.269	0.830
	A2	Prime - 1/2	0.132	1.226	1.586	1.038
	A3	Prime - 1/2	0.159	1.471	1.903	1.245
	Baa1	Prime - 2	0.185	1.717	2.221	1.453
	Baa2	Prime- 2 or 3	0.212	1.962	2.538	1.660
	Baa3	Prime-3	0.238	2.207	2.855	1.868
Speculative grade	Ba1	Not Prime	0.415	3.950	8.197	3.942
	Ba2	Not Prime	0.592	5.692	13.540	6.017
	Ba3	Not Prime	0.769	7.435	18.882	8.092
	B1	Not Prime	1.643	9.989	20.785	10.196
	B2	Not Prime	2.517	12.542	22.687	12.299
	B3	Not Prime	3.391	15.096	24.590	14.403
	Caa	Not Prime	10.139	21.005	27.334	19.607
	Ca	Not Prime	16.888	26.914	30.079	24.812
	C	Not Prime	23.636	32.823	32.823	30.016
Source: www.moodys.com (Rating) and authors' calculations						

Table 3: Sovereign credit rating scale and cumulative default probabilities

1970s to the early 1980s. It is also possible that these data may overstate the probability of sovereign default probability as they are based on actual default episodes which include instances of the inability and unwillingness to repay debt.

Nonetheless, in our view, the default probabilities in table 2 represent the best option available for mapping the probability of sovereign default into a sovereign credit rating. CRAs also publish default probabilities associated with corporate credit ratings. These would also suffer of the same limitations as sovereign ratings. Moreover, as sovereign defaults occur less frequently than corporate defaults, corporate default rates cannot be used to map sovereign credit ratings. Rather than use historic data, one could specify a rating scale based on pre-defined cumulative default probabilities (see footnote 10). In addition to being ad-hoc, this procedure would preclude a comparison between the historic credit rating and the model-based measure.

In the following sections we describe and implement a model for computing ex-ante default probabilities over a 10-year forecast period. The model-based measure of the sovereign credit rating is then obtained by comparing the ex-ante default probabilities with the ex-post values in table 3.

4 Measuring the probability of sovereign default

Our conceptual framework for the determination of the probability of sovereign default is adapted for use on government debt from Merton's (1974) model of credit risk. The starting point is the one-period government budget constraint (GBC) which describes the intertemporal dynamics of government debt. Expressed as a proportion of nominal GDP, the GBC can be written as:

$$\frac{d_t}{y_t} + (1+\rho_t)\frac{b_{t-1}}{y_{t-1}} = \frac{b_t}{y_t}, \tag{1}$$

where y_t is real GDP, $\frac{d_t}{y_t}$ is the primary deficit-GDP ratio, ρ_t is the output-adjusted real interest rate on government debt and $\frac{b_t}{y_t}$ is the debt-GDP ratio. The primary deficit $\frac{d_t}{y_t}$ is defined as the difference between government expenditure in goods and services ($\frac{g_t}{y_t}$) plus transfers ($\frac{z_t}{y_t}$), both expressed as a proportion of GDP, and the government revenue-GDP ratio ($\frac{v_t}{y_t}$). The discount rate ρ_t is the nominal interest rate on government bonds (i_t^b) less the inflation rate (π_t) and the growth rate of GDP (γ_t). By expanding the variable $\frac{d_t}{y_t}$ to include seigniorage revenue, equation (1) can also represent the consolidated government budget constraint . Defining $\delta_t = \frac{1}{1+\rho_t}$ as the one period discount factor, the h-period ahead forward solution to the GBC is:

$$\frac{b_t}{y_t} = -E_t \sum_{s=1}^{h} \frac{d_{t+s}/y_{t+s}}{\Pi_{j=1}^{s}\delta_{t+j}} + E_t \frac{b_{t+h}/y_{t+h}}{\Pi_{j=1}^{h}\delta_{t+j}}, \tag{2}$$

where E_t denotes the mathematical expectation conditional on information available at time t. Given a target level of the debt-GDP ratio $\overline{\frac{b_{t+h}}{y_{t+h}}}$, this equation can be written as

$$\frac{b_t}{y_t} + E_t \sum_{s=1}^{h} \frac{d_{t+s}/y_{t+s}}{\Pi_{j=1}^{s}\delta_{t+j}} - E_t \frac{\overline{b_{t+h}/y_{t+h}}}{\Pi_{j=1}^{h}\delta_{t+j}} = E_t \frac{b_{t+h}/y_{t+h}}{\Pi_{j=1}^{h}\delta_{t+j}} - E_t \frac{\overline{b_{t+h}/y_{t+h}}}{\Pi_{j=1}^{h}\delta_{t+j}}.$$

This measures the deviation of the discounted value of the future debt-GDP ratio (equivalent to the sum of the current debt-GDP ratio and the present value of future expected primary surpluses) from the discounted value of the debt-GDP target.[14]

To measure the probability of sovereign default we reinterpret the target debt-GDP ratio as a default threshold. Default is presumed if the expected value of the debt-GDP ratio exceeds this threshold level. The default threshold

[14] This equation is at the basis of numerous short- and medium-term indicators of the fiscal stance proposed in the economic literature and used by policy makers; for example, the tax-gap indicators pioneered by Blanchard (1990) and Blanchard et al. (1990); the S1 and S2 indicators in European Commission (2007), the projections of the fiscal stance in Congressional Budget Office (2010), and model-based measures of the fiscal stance proposed by Polito and Wickens (2011) and (2012).

is interpreted as the market (or an analyst's) expectation of the value of the debt-GDP ratio beyond which the debt cannot be rolled over, thereby bringing about a default event. While a default threshold may be determined by a government's perceived willingness or ability (or both) to meet its financial obligations, for the purposes of measuring the U.S. sovereign credit rating, the default threshold should reflect exclusively the ability of the government to use fiscal instruments to generate enough savings to repay its debt. A default event occurs when the government fails to make scheduled payments (either partially or in full) on the principal or the interest. Typically, this leads to a modification of the terms of the debt contract which either reduces the value of the bonds issued, or extends their maturity, or reduces the interest rate.

To implement this we use the h-period ahead solution of the GBC in equation (2) given by

$$\frac{b_{t+h}}{y_{t+h}} = -\sum_{j=1}^{h} \left[\Pi_{s=1}^{j} \left(1+\rho_{t+s}\right) \frac{d_{t+j}}{y_{t+j}} \right] + \Pi_{s=1}^{h} \left(1+\rho_{t+s}\right) \frac{b_t}{y_t},$$

where the right-hand side is the cumulative saving generated by current and future primary surpluses from t to $t+h$ plus the interest cost of rolling-over the current debt-GDP ratio until period $t+h$.

The probability of sovereign default by period $t+h$ (hazard rate) is the probability of not defaulting prior to year $t+h$ but defaulting in year $t+h$, and hence is given by

$$p_{t,t+h} = p_{t+h} \left(1 - p_{t+h-1}\right) \left(1 - p_{t+h-2}\right) ... \left(1 - p_{t+1}\right), \tag{3}$$

where p_{t+h} denotes the probability of defaulting in period $t+h$ given information up to period t. This is measured by

$$p_{t+h} = \Pr\left(-\sum_{j=0}^{h} \left[\Pi_{s=1}^{j} \left(1+\rho_{t+s}\right) \frac{d_{t+j}}{y_{t+j}} \right] + \Pi_{s=1}^{h} \left(1+\rho_{t+s}\right) \frac{b_t}{y_t} \geq \overline{\frac{b_{t+h}}{y_{t+h}}} | \frac{b_t}{y_t} = \frac{b}{y} \right),$$

where $\Pr(.)$ is assumed to be the normal probability density function. Hence,

$$p_{t+h} = \Pr\left(\frac{b_{t+h}}{y_{t+h}} \geq \overline{\frac{b_{t+h}}{y_{t+h}}} | \frac{b_t}{y_t} = \frac{b}{y} \right). \tag{4}$$

This may be interpreted as measuring the probability of default based on a government's ability or willingness to meet its financial obligations given a default threshold $\overline{\frac{b_{t+h}}{y_{t+h}}}$ that quantifies the amount of debt that a country will be either willing or able to repay at a specific time in the future.[15] In practice,

[15] As the inequality in (4) holds under any monotonic transformation, it can be equivalently written as

$$p_{t,h} = \Pr\left(\ln \frac{b_{t+h}}{y_{t+h}} \geq \ln \overline{\frac{b_{t+h}}{y_{t+h}}} | \ln \frac{b_t}{y_t} = \ln \frac{b}{y} \right).$$

Assuming that the evolution of the debt-GDP ratio follows a geometric Brownian motion, the sovereign default probability can then be expressed in a form equivalent to Merton (1974)-Black and Scholes (1973)'s option model.

market analysts and investors may have in mind a debt-GDP threshold of their own, which may depend upon considerations both about the ability and the willingness of a borrower to service its obligations. In the U.S., as in most advanced countries, honouring sovereign debt is a constitutional obligation.[16] Consequently, for the purposes of measuring the U.S. sovereign credit rating, the default threshold $\overline{\frac{b_{t+h}}{y_{t+h}}}$ must reflect only a government's ability to meet its financial obligations using fiscal policy. This is the definition of default probability that we employ in the empirical analysis in sections 6 and 7.

The debt-GDP ratio at time $t+1$ may be decomposed into

$$\frac{b_{t+1}}{y_{t+1}} = E_t \frac{b_{t+1}}{y_{t+1}} + \xi_{t+1}$$

where $E_t \frac{b_{t+1}}{y_{t+1}}$ is the expectation of the debt-GDP ratio by the end of period $t+1$ conditional on information available in t, and ξ_{t+1} is the corresponding innovation in period $t+1$. The latter may be written as

$$\xi_t = \sigma_t \varepsilon_t,$$

where $\varepsilon_t \sim i.i.d.\,(0,1)$. It then follows that the debt-GDP ratio for period $t+h$ may be written as

$$\begin{aligned} \frac{b_{t+h}}{y_{t+h}} &= E_t \frac{b_{t+h}}{y_{t+h}} + \eta_{t+h} \\ \eta_{t+h} &= \Sigma_{s=1}^{h} \xi_{t+s} \end{aligned}$$

where $V_t(\eta_{t+h}) = \sigma^2_{\eta,t+h} = \Sigma_{s=1}^{h} \sigma^2_{t+s}$ is the conditional variance of the debt-GDP ratio.

The probability of sovereign default in period $t+h$ given information in period t is therefore

$$p_{t+h} = \Pr\left(-DD_{t+h} \leq \eta_{t+h}\right), \quad (5)$$

where

$$DD_{t+h} = \frac{E_t \frac{b_{t+h}}{y_{t+h}} - \overline{\frac{b_{t+h}}{y_{t+h}}}}{\sigma_{\eta,t+h}} \quad (6)$$

is the distance-to-default of sovereign debt, a measure, in terms of standard deviations, of the size of the increase in the expected debt-GDP ratio between t and $t+h$ that would cause default in period $t+h$. The probability of default increases due to a narrowing of $E_t \frac{b_{t+h}}{y_{t+h}} - \overline{\frac{b_{t+h}}{y_{t+h}}}$, to the gap between the expected debt-GDP ratio and the default threshold level, and to increased uncertainty surrounding the forecasts of the debt-GDP ratio (σ_{t+h}). As the base year changes, causing changes in the estimates of the conditional expectation and variance of the debt-GDP ratio, this probability will change over time. From

[16] This has been recently restated by the U.S. Treasury's General Counsel George W. Madison, see The New York Times, July 8 2011.

equation (3), the cumulative default probability (i.e. the probability of default in any period between t and $t+h$) is

$$p^c_{t,t+h} = \sum_{j=1}^{h} p_{t,t+j}, \tag{7}$$

which is calculated assuming a standard cumulative normal distribution. This provides a measure of the cumulative default probability compatible with that used in table 3.

To summarize, equations (5) and (6) show that three fundamental pieces of information are required to measure the probability of sovereign default: a forecast of the debt-GDP ratio over a specific time-horizon, an estimate of the uncertainty surrounding this forecast, and a value for the debt-GDP default threshold. The first two may be determined from a macroeconomic model of the economy. This is described in the next section.

5 Forecasting sovereign debt and volatility

5.1 The model

An economic analysis of the probability of sovereign default and credit ratings requires the specification of a model of the economy in order to forecast the future evolution of the debt-GDP ratio. This could be a structural model of the economy such as a DSGE model or, as in this section, a time series model such as VAR.

In an analysis of the forecasting properties of DSGE models Wickens (2012) has shown that there is little difference in the forecasting properties of DSGE and VAR models. Both forecast well when the economy is growing smoothly but forecast badly when the economy experiences turning points. One reason for poor forecasts from DSGE models is the difficulty of forecasting exogenous variables for which, by definition, there is no theory and so they must be forecast by time series models. The reason why an unrestricted VAR model forecasts as well as a DSGE model is that when the exogenous variables are represented by a VAR, the solution to a DSGE model is a restricted VAR. If these restrictions are correct then the forecasts from a DSGE model will have smaller forecast error variances than those from an unrestricted VAR, but if the restrictions are incorrect then an unrestricted VAR will give less biased forecasts.

There are further advantages to using a VAR. It is theory-free and is easy to implement empirically, both of which are attractive features for market participants seeking to determine their own credit-rating measure. In order to improve the forecasting performance we take account of the uncertainty arising from changes in the processes generating the exogenous variables and any variations in the parameters by estimating the VAR using a rolling-window data period, or ROVAR. Rolling analyses of time-series models are often used in finance to assess a model's stability over time. Stock and Watson (2007, 2008) have used rolling-window estimation to forecast U.S. inflation, while Orphanides and Wei (2010) have demonstrated the effectiveness of rolling-window estimation in capturing time-variation in the parameters and (the volatility of) the residuals in macro-finance models. Recently, Canova and Ferretti (2012) have employed rolling samples to evaluate within a medium-scale DSGE model the impact of policy shocks on the dynamic of inflation. Using a ROVAR it is easy and simple to continually up-date the probability distribution of the debt-GDP ratio on the latest data.

An alternative type of VAR to a ROVAR could also be used. In the monetary policy literature on the determinants of the great acceleration and the great moderation Primiceri (2005) has used a reduced form VAR with time-varying parameters and stochastic volatility (TVP-SV-VAR), and Sims and Zha (2006) have used a VAR with Markow-switching parameters and volatility (MS-VAR). There are, however, two main drawbacks to these VAR models. First, they can only be estimated using Bayesian methods as the analytical form of the likelihood function is not tractable under the assumption that the covariance structure of the shocks follows a stochastic volatility process. Even then, the estimation is generally not straightforward as it depends crucially on the specific assumptions made about the priors; it is also time-intensive, especially for medium- and large-scale models. Second, forecasts and measures of uncertainty from both the TVP-SV-VAR and the MS-VAR are influenced by the particular view taken of the process driving the variance structure of the shocks, whether this is gradual, as in the TVP-SV-VAR, or driven by a Markow-switching process, as in the MS-VAR. An agnostic view on volatility may be preferable. Kapetanios et al. (2012) finds that during the latest financial crisis macroeconomic forecasts obtained from a ROVAR significantly outperform those the TVP-SV-VAR, while being in line with those from the MS-VAR. Forecasts from the three models are instead similar during periods of macroeconomic stability.

Following the VAR literature on time-varying macroeconomic volatility, we specify a ROVAR model with two lags and a constant term. This is estimated using a moving-data window of 40 quarters. At each step of the estimation we derive pseudo out-of-sample forecasts. In essence, we estimate the VAR using data from $t-40$ until date t and then compute $h-$period ahead point forecasts and forecasts errors. These two steps are repeated until t reaches the end of the sample.[17] The vector of variables included in the ROVAR is

$$\mathbf{x}_t = \begin{bmatrix} \gamma_t & \pi_t & \frac{d_t}{y_t} & \frac{b_t}{y_t} & r_t^s & r_t^l & cp_t \end{bmatrix}. \quad (8)$$

The first four variables - the growth rate of real GDP, the inflation rate (GDP deflator), the primary deficit-GDP ratio and the debt-GDP ratio - allow the model to satisfy the GBC, equation (1).[18] The variables r_t^s and r_t^l measure the

[17] Appendix A describes the specification of the ROVAR model and the determination of the recursive forecasts with more details.

[18] This is because there is always an unique nominal rate i_t^b that satisfies (1) which it is not therefore necessary to include explicitly in the model.

Federal funds rate and long-term interest rate on government bonds respectively. These capture the links between the debt-GDP dynamics, monetary policy and the term structure. The last variable, cp_t, is the growth rate of crude oil price, used to proxy changes in commodity prices. This exogenous variable captures the impact of global economic factors on the domestic macroeconomic and the fiscal outlook. Reinhart and Rogoff (2008), pp.6 document that "peaks and troughs in commodity price cycles appear to be leading indicators of peaks and troughs in the capital flow cycle, with troughs typically resulting in multiple defaults".[19] The specification of the variables in $\mathbf{x}_t$ is typical of VAR analyses of fiscal shocks and business cycle fluctuations; see for example Fatas and Mihov (2001), Canzonieri et al. (2002) and Chung and Leeper (2007).

5.2 Data and stylized facts

The data are quarterly for the U.S. from 1969:4 to 2011:2 and are shown in figure 2. Four features of these data are worth bearing in mind when interpreting the empirical analysis of the U.S. sovereign credit rating in the next two sections. The first is the so-called great macroeconomic acceleration and moderation. This is reflected in the higher growth rate of GDP, in the inflation rate and in nominal interest rates in the 1970s and early 1980s relative to the post-1985 period. The second is the great deterioration of the fiscal stance reflected in the upward trend in the debt-GDP ratio since the early 1980s and in the greater amplitude of swings in the primary deficit-GDP ratio starting from the second half of the 1980s.[20] The third is the large fall of the growth rate of GDP and further deterioration of the fiscal stance since the 2007-2009 global financial crisis. The fourth feature is the large spike in oil inflation that occurred in the first quarter of 1974, though the series appear to be stationary throughout the 1970-2011 period.

[19] Full details on the data are in Appendix B.

[20] Like the great moderation, the great deterioration of the fiscal stance is also typical of most of European countries, see Polito and Wickens (2011).

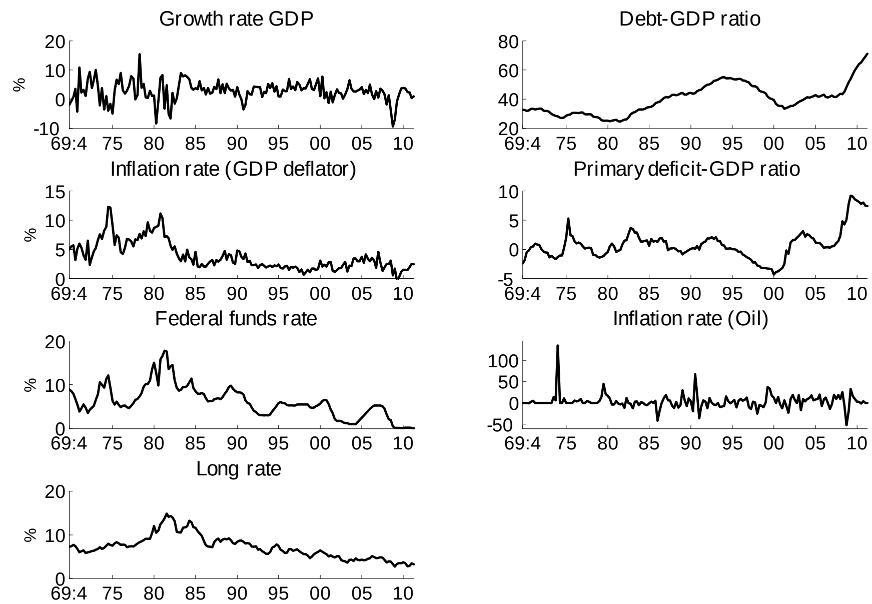

Figure 2: U.S. aggregate macroeconomic and fiscal data, 1969:4-2011:2.

Figure 3 reports estimates of the time-varying volatility of the variables included in the ROVAR, as measured by the standard deviation of the 1-quarter ahead forecast error. Several interesting features can be observed. The most striking is that over the past 40 years there have been large fluctuations not only in the volatility of U.S. macroeconomic variables (output, inflation and interest rates) but also in the volatility of fiscal and global variables. In addition, these fluctuations appear to be highly correlated. For example, the great moderation coincides with a large fall in the volatility of the rate of change in oil prices. The volatility of both output and inflation has steadily (though not sharply) increased since the late 1990s and coincides with a gradual surge in the volatility in the inflation rate in oil prices over the same period. There is also a strong positive correlation between the volatilities of global and fiscal variables. The volatility of the fiscal variables (particularly the deficit-GDP ratio) increased during the 1970s', declined during the 1980s, then increased during the 1990s and 2000s and has picked again during the latest financial crisis. This evidence highlights the significance of fiscal and global factors in determining domestic macroeconomic volatility. It also points to the need to capture time-variation in macroeconomic volatility in the forecasting model.

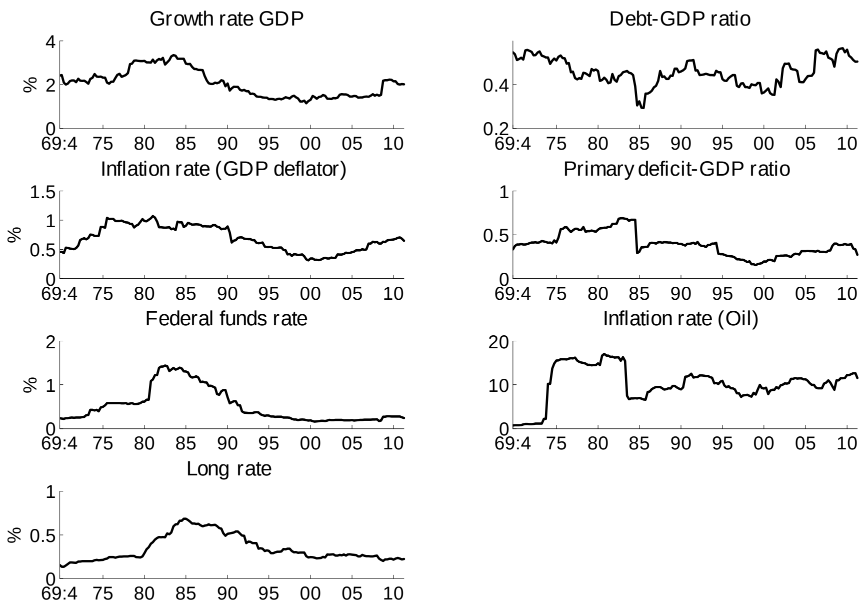

Figure 3: Standard deviation of the 1-quarter ahead forecast error from ROVAR model.

6 Credit ratings with ad-hoc debt limits

We now estimate the probability of default and map this into a credit rating using ad-hoc time-invariant debt-GDP values as proxies for the default threshold. We use these results as a benchmark to assess how the model-based measure of the sovereign credit rating responds to the different perceptions of market analysts and investors of the repayment capacity of the U.S. government. We abstract here from whether this perception is driven by considerations about the ability or the willingness of the U.S. government to meet its financial obligations, or about the impact on the sovereign credit rating of time-varying debt-GDP limits, as these are investigated in the next section.

The sovereign credit rating is determined as follows. First we obtain time-varying debt-GDP forecasts ($E_t \frac{b_{t+h}}{y_{t+h}}$) and forecast-errors standard deviations ($\sigma_{\eta,t+h}$) from the ROVAR for t ranging quarterly from 1969:4 to 2011:2 and over a 10-year horizon, i.e. $h = 1, ..., 40$. We then use these to compute the probability of default in each quarter of the forecasting horizon. As the ad-hoc debt-GDP threshold does not change over time, effectively, we compute the probability of default in period $t+h$ from

$$p_{t+h} = \Pr\left(\frac{E_t \frac{b_{t+h}}{y_{t+h}} - \overline{\frac{b}{y}}}{\sigma_{\eta,t+h}} \leq \eta_{t+h}\right). \quad (9)$$

We have simulated the model over a wide range of debt-GDP limits, from 100 to 500 per cent, but for illustrative purposes report only the results obtained for debt-GDP limits equal to 150, 200 and 300 per cent which are sufficient to highlight the main results.[21] The corresponding hazard rates can then be calculated using equation (3) and converted into cumulative default probabilities using equation (7). Finally, the model-based measure of the credit rating is derived by mapping this probability into an initial sovereign credit rating using table 3. As this may be too sensitive to swings in debt-GDP forecasts and uncertainty, we later revise this initial rating to provide a smoother rating which we call the final rating.

6.1 Initial credit rating

Figure 4 plots the initial model-based sovereign credit rating against the historic U.S. rating as reported in Moody's (2011). Short- and long-term sovereign credit ratings are obtained by comparing the cumulative default probability profile derived from the model over horizons of 1 (top panel), 5 (middle panel) and 10 years (bottom panel) with those corresponding to columns 4, 5 and 6 in table 3. The short-term credit rating derived from the model is Prime 1 for the whole sample period, regardless of the default threshold. This coincides with the actual rating, indicating that the model attaches a zero probability to a U.S. default within 1-year over the whole sample period. For a 5-year horizon the initial rating is lowered from Aaa to C in the first two quarters of 1974 regardless of the debt-GDP limit. A downgrade from Aaa to C also occurs following the bankruptcy of Lehman Brothers in September 2008, though its duration reduces as the debt-GDP limit increases.[22] Using a 10-year horizon the model gives the same downgrade in early 1974 as for the 5-year horizon, but the downgrade to C is longer-lasting (for five consecutive quarters, starting from 2008:4) during the latest financial crisis for any of the debt-GDP limits considered. The model also lowers the credit rating in 2008:2 to C, Ba3 and Aa1 if the debt-GDP limits are, respectively, 150, 200 and 300.[23]

[21] Sovereign credit ratings obtained from other values of the debt-GDP limit within the 100-500 per cent range can be made available upon request.

[22] This downgrade lasts four quarters (from 2008:4 to 2009:3) when $\overline{\frac{b}{y}} = 150$ per cent, three quarters (2009:1-2009:3) when $\overline{\frac{b}{y}} = 200$ per cent and two quarters (2009:1-2009:2) with $\overline{\frac{b}{y}} = 300$ per cent.

[23] With $\overline{\frac{b}{y}} = 200$ per cent, further downgrades from Aaa to Aa1 occur in 1987:1, 1992:2, 1993:1, 1997:2 and 1997:3. With the lowest debt-GDP threshold, $\overline{\frac{b}{y}} = 150$ per cent, the model yields further downgrades from Aaa in 1970:1(to Aa1), 1987:1 (to Aa1), in the first

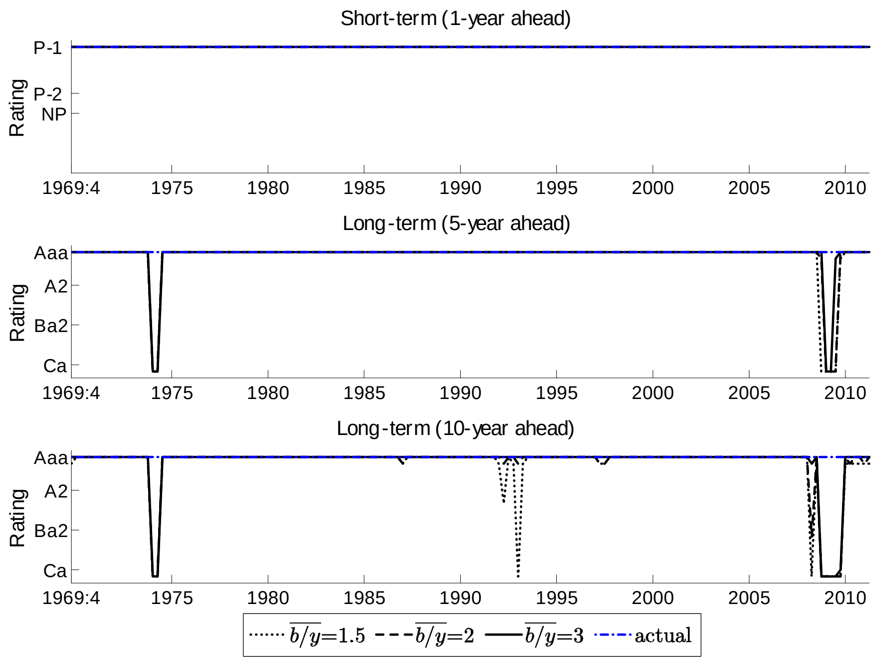

Figure 4: Initial model-based U.S. sovereign credit rating (short and long term) under ad-hoc debt-GDP limit, 1969:4-2011:2. Actual corresponds to Moody's (2011) historical credit rating.

6.2 Final credit rating

This initial sovereign credit rating is very sensitive to fluctuations in the debt-GDP forecast and may be very volatile especially during periods of high macroeconomic instability. This is in contrast with the operating procedures of CRAs, which typically revise sovereign credit ratings twice at year rather than on a quarterly basis and place a sovereign credit rating under review in the short term, perhaps before changing it later.[24] Sovereign credit ratings, at least for industrialized countries, tend to be altered gradually rather than suddenly.[25] We therefore provide a final sovereign credit rating, which is less volatile than the initial credit rating, by imposing the constraint that quarterly changes in the sovereign credit rating may occur only gradually. This is calculated each

two quarters of 1992 (to Aa1 and Baa1 respectively), during 1992:4-1993:2 (Aa1, C and Aa1 respectively), in 1997:2 (to Aa1). In addition, the credit rating remains at Aa1 from 2010:1 onward.

[24]This is referred to "creditwatch" by S&P, "watchlist" by Moody's and "ratingwatch" by Fitch. Possible rating changes can be either upgrade, or downgrade, or with uncertain direction uncertain. Sovereign credit ratings also can be changed without being first under review.

[25]See Sovereign Bond Rating Histories, in Appendix III of Moody's (2011).

quarter from the initial rating as follows: in the first period of the sample the final credit rating is set equal to the initial credit rating; if the new initial credit rating (from the second period onwards) is the same as the previous quarter's initial rating, the new final rating is set equal to the previous quarter's final rating; if the new initial credit rating is higher (lower) than the previous period's initial rating then the new final credit rating is upgraded (downgraded) by one notch.

The final credit ratings are reported in figure 5. We observe no change in the final short-term credit rating compared to the initial measure as this is Prime-1 throughout. With a 5-year horizon, the model would have downgraded U.S. sovereign debt from Aaa to Aa1 in 1974:1 and to Aa2 in 1974:2, regardless of the debt-GDP limit, and it would have restored the Aaa credit rating gradually over the next two quarters. The final rating further downgrades U.S. debt during the latest financial crisis. The starting date and the magnitude of the downgrade vary with the debt-GDP limit but, unlike the initial rating, in no case is the U.S. credit rating lowered below speculative grade.[26]

Using a 10-year horizon has the effect of deepening the reduction and duration of the downgrade during the latest financial crisis. The lowest value reached is A2 in 2009:4 regardless of the debt-GDP limit. With the lowest debt-GDP limit the credit rating is lowered to A1 in 2008:1 but is restored to Aaa in the following quarter. It then reduces to A2 in 2009:4 and reaches A1 in 2010:4 where it remains until 2011:2. The same pattern is found with the intermediate and high debt-GDP limits, except that a Aaa rating is restored in 2011:2 and 2011:1. With a 150 per cent debt-GDP limit, minor downgrades occur in the mid 1980s and during the 1990s; with the lowest debt-GDP limit, the credit rating is lowered to Aa1 in 1987:1, 1992:1, 1992:3 1997:2 and 1997:3, and is Aa2 in 1992:2; and with the intermediate debt-GDP limit the credit rating is Aa1 in 1987:1, 1992:2, 1993:1, 1997:1 and 1997:2.

[26] With the lowest and intermediate debt-GDP limits, the credit rating is gradually lowered from Aaa in 2008:3 to A1 in 2009:3 and recovers the Aaa status by 2010:3. With the 300 per cent debt-GDP limit, the credit rating is gradually lowered from Aaa in 2008:4 to Aa2 in 2009:2 and the Aaa rating is restored by 2009:4.

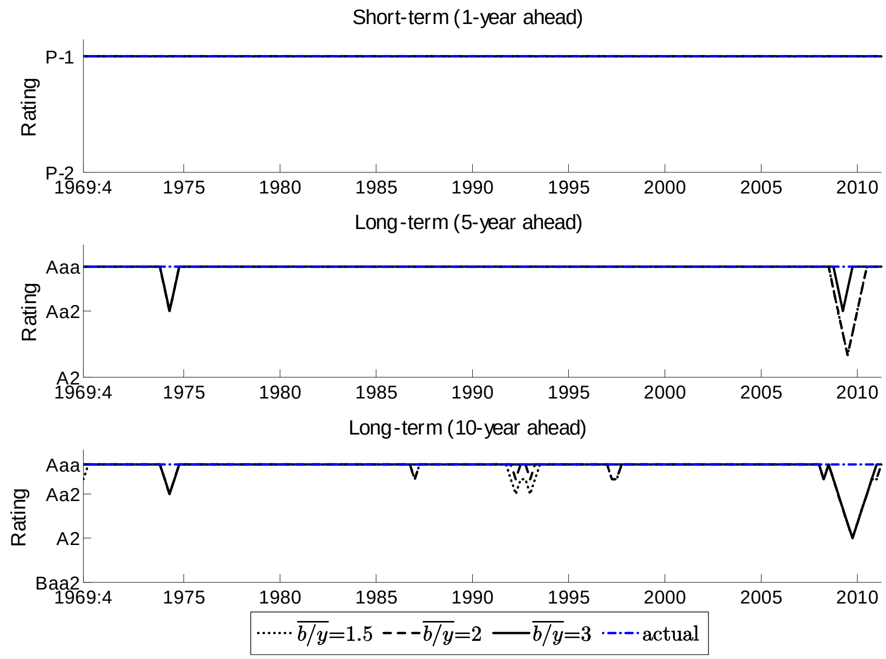

Figure 5: Final model-based U.S. sovereign credit rating (short and long term) under ad-hoc debt-GDP limit, 1969:4-2011:2. Actual corresponds to Moody's (2011) historical credit rating.

To summarize, the model-based final sovereign credit rating suggests that there are two clear episodes over the past 40 years in which U.S. debt should have been downgraded: the first coincides with the oil crisis starting in October 1973; the second coincides with the period after the collapse of IndyMac Bank and Lehman Brothers. These findings are generally robust to the specification of the debt-GDP limit within the range 150-300 per cent. The extent and duration of the downgrade varies with the debt-GDP limit and the time horizon. A mild downgrade would have occurred following the earlier 1990s recession and in the latest 1980s and 1990s. Clearly, the model-based measure of the U.S. sovereign credit rating is related to key events of the U.S. macroeconomic history, with downgrades typically being associated with episodes of economic recessions, swings in commodity prices and large increase in government spending. This is in contrast with the credit ratings issued by the main CRAs. As noted in the introduction, the CRAs have given U.S. debt the highest credit rating throughout the whole 1969:4-2011:2 period. Standard & Poor only lowered this by one notch in its rating on August 5, 2011. This was about 12 quarters after the collapse of IndyMac Bank and Lehman Brothers.

6.3 Sensitivity analysis

We now assess how sensitive the model-based measure of the sovereign credit rating is to alternative mappings of the probability of default into credit ratings, and whether downgrades are driven by changes in the debt-GDP forecast or volatility or both.

An alternative to calculating the probability of default at a particular time horizon is to use the average cumulative probability of default over the whole 10-year horizon. The results, shown in figure 6, are very similar to assuming the medium-term horizon that were reported in figure 5. The credit rating is lower than Aaa in the early 1974 and during the latest global financial crisis, and minor downgrades occur between these two periods as the debt-GDP limit falls.

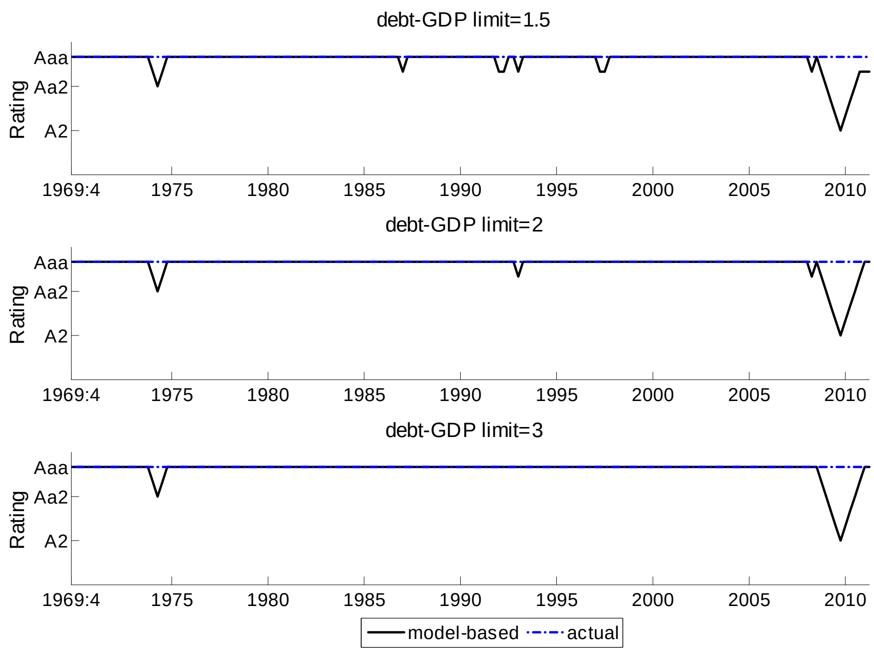

Figure 6. Final model-based U.S. sovereign credit rating using average cumulative default probabilities over 10 year, 1969:4-2011:2. Actual corresponds to Moody's (2011) historical credit rating.

To assess whether the results in figures 4-6 are driven by swings in either the debt-GDP forecasts or volatility, we have recalculated the sovereign credit rating assuming that the standard deviation of the forecast in equation (9) is constant over the forecasting horizon and equal to its first-period value. We therefore compute the probability of default using

$$p_{t,h} = \Pr\left(-\frac{E_t \frac{b_{t+h}}{y_{t+h}} - \overline{\frac{b}{y}}}{\sigma_t} \leq \eta_{t+h} \right). \tag{10}$$

The final credit ratings using the average cumulative default probability over the whole 10-year horizon that is implied by (10) are shown in figure 7. These can be compared directly with the measure of the credit rating reported in figure 6. Differences are due to the contribution of uncertainty in the determination of the sovereign credit rating. We find that under (10) the lower U.S. sovereign credit rating in the 1980s and 1990s would not have occurred. This suggests that these downgrades were influenced by a temporary increase in uncertainty and not just a deterioration in the forecasted debt-GDP ratio.

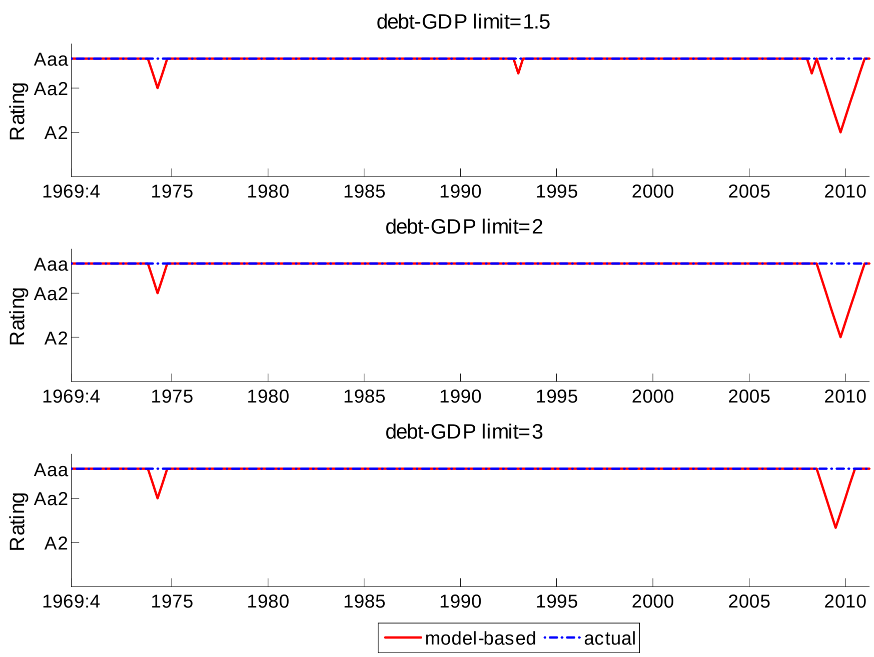

Figure 7. Final model-based sovereign credit rating using average cumulative default probabilities over 10 year with fixed uncertainty, 1969:4 -2011:2. Actual corresponds to Moody's (2011) historical credit rating.

7 Credit ratings with theory-based debt limits

7.1 The debt-GDP limit

The previous analysis of sovereign credit ratings is based on an ad-hoc choice of the debt-GDP limit. We now consider a limit derived from the intertemporal government budget constraint (IGBC) embedded in a microfounded DSGE model of the economy. The debt-GDP limit may be obtained either directly from the steady-state value of the IGBC, or by maximizing this with respect to tax rates, or by minimizing the steady-state IGBC with respect to expenditure.[27] All these notions of the maximum borrowing capacity compare the outstanding liabilities of the government with the market expectation of future saving that the government can generate by either anticipated or unanticipated fiscal policy changes. Default occurs whenever the present value of this expected saving is lower than the existing debt-GDP ratio.

The starting point for the derivation of a debt-GDP limit is the one-period GBC in equation (1). The solution to an infinite horizon DSGE model constrains the government from rolling over its liabilities forever and implies that the current period debt-GDP ratio must satisfy the IGBC every period. Denoting this level of the debt-GDP ratio as $\frac{b_t}{y_t}^{IGBC}$, the resulting equilibrium condition is

$$\frac{b_t}{y_t}^{IGBC} = -E_t \sum_{j=1}^{\infty} \frac{\frac{g_{t+j}}{y_{t+j}} + \frac{z_{t+j}}{y_{t+j}} - \frac{v_{t+j}}{y_{t+j}},}{\prod_{s=1}^{j} \left(1+\rho_{t+s}\right)}, \tag{11}$$

where g_t is government expenditures on goods and services, z_t is transfers, v_t is government revenues and ρ_t is the discount rate defined earlier. The right-hand side of equation (11) is the market's expectation of the present value of current and future primary surpluses given information up to time t, and is a measure of a government's borrowing capacity based on the market's anticipation of the future evolution of fiscal policy.

Other measures of the debt-GDP limit exploit the fact that unanticipated changes in fiscal policy may increase a government's ability to raise further debt. Unanticipated changes in policy are unpredictable: however market participants can determine the maximum impact of any surprise change in government expenditure and tax policy if these were to occur. This logic is at the basis of three other definitions of the debt-GDP limit.

The first is obtained by adapting Aiyagari (1994)'s natural debt limit (NDL) for government policy. In a DSGE model, the NDL corresponds to the representative household's wealth and comes from the requirement that it must be

[27] There are several alternatives to using the IGBC to measure the debt-GDP limit of a government. The IGBC approach is related to Sachs (1989)'s concept of debt Laffer curve based on the notion of debt overhang, a situation where high debt levels (relative to the repayment capacity of a country) result in efficiency losses due, for example, to rescheduling negotiations of debt obligations, changes in trade patterns, or crowding out of investment. The quantification of these costs is however not straightforward. Cole and Kehoe (2000)'s model of liquidity crisis can be used to derive a debt-GDP threshold below which governments are always solvent and default never occurs. However, default above this threshold is undetermined as it is subject to whether a country can still avoid a liquidity crisis. There is also an extensive empirical literature that quantifies "safe" debt-GDP thresholds. These however vary from country to country, change over time, and crucially depend on the default history of a country (see, for example, Reinhart, Rogoff and Savastano (2003) for emerging markets and Reinhart and Rogoff (2011) for advanced economies).

feasible for the household to repay its state contingent debt in every possible state by eliminating its expenditures. When applied to fiscal policy, the NDL is obtained by setting to zero all sources of government expenditure in (11). The implied debt-GDP limit, denoted by $\frac{b_t}{y_t}^{NDL}$, then corresponds to the expected present value of current and future tax revenue and is given by

$$\frac{b_t}{y_t}^{NDL} = E_t \sum_{j=1}^{\infty} \frac{\frac{v_{t+j}}{y_{t+j}}}{\prod_{s=1}^{j} \left(1 + \rho_{t+s}\right)}. \tag{12}$$

This debt-GDP limit precludes a government from being able to finance higher debt levels from unanticipated reductions in expenditure; unanticipated increases in tax revenues or changes in monetary policy must occur in order to increase the debt-GDP limit.

The second has been suggested recently by Davig, Leeper and Walker (2010, 2011). The starting point is the notion that distortionary taxation implies that there exists a set of tax rates that maximizes tax revenues. For this to happen taxes must satisfy a Laffer effect. The debt-GDP limit then emerges as the expected present value of future primary deficits under the assumption that tax revenue is maximized in each period and there will be no unanticipated changes in the conduct of government expenditure policy. Davig, Leeper and Walker (2010) refer to this as the fiscal limit (FL). This is given by

$$\frac{b_t}{y_t}^{FL} = E_t \sum_{j=1}^{\infty} \frac{\frac{d_{t+j}}{y_{t+j}}^{FL}}{\prod_{s=1}^{j} \left(1 + \rho_{t+s}\right)}, \tag{13}$$

where $\left\{ \frac{d_{t+j}}{y_{t+j}}^{FL} = \frac{g_{t+j}}{y_{t+j}} + \frac{z_{t+j}}{y_{t+j}} - \frac{v_{t+j}^{\max}}{y_{t+j}} \right\}_{j=0}^{\infty}$ denotes the sequence of primary deficit achievable when government tax revenues are maximized. It disregards the potential impact of unanticipated changes in government expenditure. Thus Davig, Leeper and Walker's (2010) fiscal limit occurs when the government no longer has the ability to increase its borrowing capacity by increasing tax revenues. Nonetheless, it could still either change its expenditure policy or use monetary policy, or both.

A further possibility is to maximize tax revenues whilst setting government expenditure and transfers equal to zero in (13). This gives the weakest of the debt-GDP limits and is denoted by (WL):

$$\frac{b_t}{y_t}^{WL} = E_t \sum_{j=1}^{\infty} \frac{\frac{v_{t+j}^{\max}}{y_{t+j}}}{\prod_{s=1}^{j} \left(1 + \rho_{t+s}\right)}. \tag{14}$$

When the limit in (14) is reached, a government can no longer use unanticipated fiscal policy to finance additional debt and would need to resort to monetary policy.

Implementing these four measures of the debt-GDP limit presents a number of problems. For example, computation of the debt-GDP limit in equation (11) may be problematic when using historic data from countries that have run persistent deficits in the past as the debt-GDP limit implied by the projection of future deficits may be negative. This point is illustrated in the next section. The debt-GDP limits based on (12) and (14) are always positive and the maximization of tax revenue in (13) is also likely to produce a positive the debt-GDP limit. The fiscal changes required to achieve the debt-GDP limits in (12), (13) and (14) may not, however, be politically feasible.[28]

We implement the debt-GDP limits in equations (11), (12), (13) and (14) by including distortionary taxation in an otherwise standard real business cycle model with an elastic labour supply. Davig, Leeper and Walker (2010) have used a similar model to compute the debt-GDP limit for the U.S. economy, but extended to include nominal rigidities. They assume that labour and capital income are taxed at the same rate and do not consider revenue generated by the taxation of consumption. This model is closely related to the neoclassical growth model employed by Trabandt and Uhlig (2011) to estimate Laffer curves for the U.S. and for individual EU countries. We depart from Davig, Leeper and Walker (2010) in that we consider separately taxation on consumption, labour and capital income. We also depart from Trabandt and Uhlig (2011) because we derive debt-GDP limits rather than Laffer hills alone. Another difference is that we use the model to derive a quarterly time series of debt-GDP limits over the past 40 years by allowing taxes, expenditures and technological progress in the U.S. to change and to have time-varying volatility.

7.2 The model

The model is defined in real terms. We assume that households derive utility from consumption c_t and leisure $1 - n_t$, and want to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - n_t\right),$$

where E_0 denotes a mathematical expectation conditioned on time 0 information; $\beta \in (0,1)$ is the household discount factor; $u\left(.\right)$ is a twice continuously differentiable, increasing, strictly concave utility function; and n_t denotes labour. Households face a sequence of budget constraints

$$\begin{aligned} (1+\tau_t^c)c_t + k_t + b_t &= (1-\tau_t^n)w_t n_t + [1 + \left(r_t^k - \delta\right)\left(1-\tau_t^k\right)]k_{t-1} \\ &\quad + \left(1 + r_t^b\right) b_{t-1} + z_t, \end{aligned}$$

where k_t, b_t, w_t, r_t^k, r_t^b, z_t, δ, τ_t^c, τ_t^n and τ_t^k denote physical capital, government bonds, wages, the gross real rate of return on capital, the real rate of return on bonds, government transfers, the rate of physical depreciation of capital, a tax

[28] Davig, Leeper and Walker (2010) discuss the use of logistic functions to compute feasible debt-GDP limits. But this is beyond the scope of our paper.

rate on consumption, a tax rate on labour income and a tax rate on net income from capital (as in Prescott (2002)). Output is generated by a labor-augmenting Cobb-Douglas production function

$$y_t = k_t^{\alpha} \left(A_t n_t\right)^{1-\alpha},$$

where A_t denotes technological progress and α is the income share of capital. The stock of physical capital evolves according to

$$k_t - k_{t-1} = x_t - \delta k_{t-1},$$

where x_t denotes investment. The government budget constraint (GBC) is

$$g_t + \left(1 + r_t^b\right) b_{t-1} + z_t = v_t + b_t$$

where g_t is government expenditure on goods and services and $v_t = \tau_t^c c_t + \tau_t^n w_t n_t + \tau_t^k \left(r_t^k - \delta\right) k_{t-1}$ is total tax revenue. As we treat the U.S. as a closed economy, the national income identity is

$$y_t = c_t + g_t + x_t.$$

The model does not include a seigniorage tax from inflation for two main reasons. First and foremost, from an investor's point of view, the borrowing capacity of a country (as reflected in its debt-GDP limit and consequently in its credit rating) should be evaluated by taking account of the potential ability of the country to repay its liabilities using tax policy rather than inflation. Indeed, the policy mix of strategies proposed by the IMF to reduce the large debt-GDP ratios accumulated by advanced economies since the 2008 global financial crisis explicitly exclude inflation as a policy instrument for repaying debt, see Coratelli (2010). Second, historically, the contribution of inflation to the U.S. budget deficit has been negligible, see King (1995) and Woodford (1996). The real rate r_t^b captures the impact of inflation on payments to bondholders. This too has made little contribution to the U.S. budget deficit since the 1970s, see Hall and Sargent (2011). A government using inflation to generate revenue is de facto defaulting on its obligations in real terms. The fiscal limits in equations (11), (12), (13) and (14) are all implicitly based on the notion that investors assess the creditworthiness of a government exclusively by looking at the saving that it can generate through its fiscal policy.

Like Davig, Leeper and Walker (2010, 2011) and the baseline analysis of Bi (2011), the model does not incorporate a default risk premium in r_t^b. In principle, as the probability of default approaches close to unity - but not before - it may be expected that purchasers of government debt would demand a risk premium.[29] As this would increase the cost of borrowing close to default, it would also lower the debt-GDP limit. In practice, because according to our model the

[29] To illustrate, in order for the expected pay-off when the probability of default next period is p to be the same as when the probability is zero, the risk premium ξ must satisfy $1 + r^b = (1 + r^b + \xi)(1 - p)$ giving a risk premium of $\xi = \frac{p(1+r^b)}{1-p}$, where r^b is the risk-free return.

probability of sovereign default by the U.S. is close to zero, this adjustment is unnecessary. Although for the U.S. we may ignore such considerations of risk in the theoretical model of the debt-GDP limit, the debt-GDP ratio and the interest rate forecasted from the VAR are not constrained in this way. The use of ad-hoc debt-GDP limits in section 6 is, of course, free from this problem.

In stationary equilibrium, the steady-state debt-GDP ratio implied by (11) is:

$$\frac{b}{y}^{IGBCSS} = \frac{1}{\rho}\left(\frac{v}{y} - \frac{g}{y} - \frac{z}{y}\right), \tag{15}$$

where $\rho = r^b - \gamma$, with $r^b = i^b - \pi$, is the output-adjusted real rate of return on government bonds. The maximum debt-GDP ratio sustainable under distortionary taxation is obtained at the peak of the Laffer hill associated with tax rates on labour and capital income alone. The consumption tax rate is held at its current level since in a standard real business cycle model the tax revenue generated by the consumption tax increases monotonically with the consumption tax rate, implying that there is no interior maximum.[30] For this reason Davig, Leeper and Walker (2010) exclude consumption taxation from their model on the debt limit; and Trabandt and Uhlig (2011) look only at Laffer curves generated by taxation of income from labor and capital.

Appendix C shows that assuming

$$u\left(c_t, 1-n_t\right) = \log c_t + \psi \log\left(1-n_t\right),$$

the first-order conditions for households and firms imply that the steady-state value of tax revenues as a proportion of GDP (the average tax rate) can be computed as:

$$\frac{v}{y} = \tau^c \chi\left(\frac{1}{\varphi k} - 1\right) + \tau^n\left(1-\alpha\right) + \tau^k \alpha\left[1 - \delta\left(\frac{\beta^{-1}-1}{1-\tau^k} + \delta\right)^{-1}\right], \tag{16}$$

where

$$\chi = \frac{(1-\tau^n)}{\psi(1+\tau^C)}\left(1-\alpha\right) \tag{17}$$

$$k = \frac{\mu + (1+\tau^c)g}{[(1+\tau^c)\Omega + \mu\varphi]} \tag{18}$$

$$\mu = \frac{1}{\psi}\left(1-\tau^n\right)\left(1-\alpha\right)A^{1-\alpha}\varphi^{-\alpha} \tag{19}$$

$$\Omega = \left(A\varphi\right)^{1-\alpha} - \delta \tag{20}$$

$$\varphi = \left[\frac{\beta^{-1} - 1 + \delta\left(1-\tau^k\right)}{\alpha A^{1-\alpha}\left(1-\tau^k\right)}\right]^{\frac{1}{1-\alpha}}. \tag{21}$$

[30] Similarly, inflation has no interior maximum when introduced in a general equilibrium model using a cash-in-advance constraint on consumption expenditure. For further discussion on this see Chiari, Christiano and Kehoe (1996).

Substituting equation (16) into (15) gives the steady-state debt-GDP limit implied by the IGBC in (11). To implement a time-varying version of this debt-GDP limit, we employ time-varying values of the steady-state levels of the three tax rates τ^c, τ^n and τ^k, and the expenditure processes for $\frac{g}{y}$ and $\frac{z}{y}$. These steady-state values are estimated by their rolling-window means and hence form a time-series. Further detail is given in appendix D (see steps 2 - 4 in particular).

The steady-state debt-GDP limit corresponding to (12) is derived as

$$\frac{b}{y}^{NDLSS} = \frac{1}{\rho}\frac{v}{y} \tag{22}$$

where $\frac{v}{y}$ is still calculated from (16). The time-varying version of this debt-GDP limit is calculated using rolling-window means of the three tax rates τ^c, τ^n and τ^k. The steady-state debt-GDP limit corresponding to (13) is

$$\frac{b}{y}^{FLSS} = \frac{1}{\rho}\left(\frac{v^{\max}}{y} - \frac{g}{y} - \frac{z}{y}\right) \tag{23}$$

where

$$\frac{v^{\max}}{y} = \tau^c\chi\left(\frac{1}{\varphi k} - 1\right) + \tau^{n,\max}(1-\alpha) + \tau^{k,\max}\alpha\left[1 - \delta\left(\frac{\beta^{-1}-1}{1-\tau^k} + \delta\right)^{-1}\right]$$

is the maximum steady-state revenue-GDP ratio obtained by the government through distortionary taxation. This is implemented by using the time-varying steady state value of τ^c and the time-varying maximum levels of $\tau^{n,\max}$ and $\tau^{k,\max}$. The steady-state equivalent of equation (14) is obtained by setting $\frac{g}{y} = \frac{z}{y} = 0$ in (23) to give

$$\frac{b}{y}^{WLSS} = \frac{1}{\rho}\frac{v^{\max}}{y}. \tag{24}$$

We note that since the debt-GDP limits are based on the steady-state solution, they would be unaffected if the model above is extended to include nominal (price and wages) rigidities as well as imperfect competition in the production sector. These changes would impact on the short-term dynamics of the endogenous variables, but not on the determinants of the steady-state debt-GDP limits in equation (15), (22), (23) and (24).

7.3 Evolution of debt-GDP limits

Quarterly time series of debt-GDP limits for the period 1969:4-2011:2 are obtained through numerical simulation of the model in section 6.2. These data are generated from steady-state values of the ratios of government expenditure to GDP and transfers to GDP, technological progress and actual tax rates using 40-period rolling window of sample means. Time-varying volatility is introduced by simulating the stochastic processes of the ratios of government expenditure to GDP and transfers to GDP and technological progress from AR(1) processes driven by shocks with time-varying volatility using a 40-period rolling window of sample standard deviations derived from the original data. All other parameters are calibrated on an annual basis using standard values for the U.S. The Markov Chain Monte Carlo algorithm used for the simulation is described in Appendix D. As a result, in each period, we obtain a distribution of the debt-GDP limit.

The differences between the four measures of the steady state debt-GDP limits depend on the evolution of their main components. Rolling window means of the ratios of government expenditure to GDP, transfers to GDP, tax revenues to GDP and the maximum revenue to GDP are shown in figure 8 for the sample period 1969:4-2011:2. While government transfers show a slight upward trend, government expenditures and both measures of the tax revenues have remained relatively stable. Higher government transfers lower the debt-GDP limits obtained from (11) and (13), but do not affect those from (12) and (14). As government expenditures, actual and maximum tax revenues are relatively stable over this sample period, we would expect the debt-GDP limits from (11) and (13) to decline over time, and those from (12) and (14) to be relatively stable.

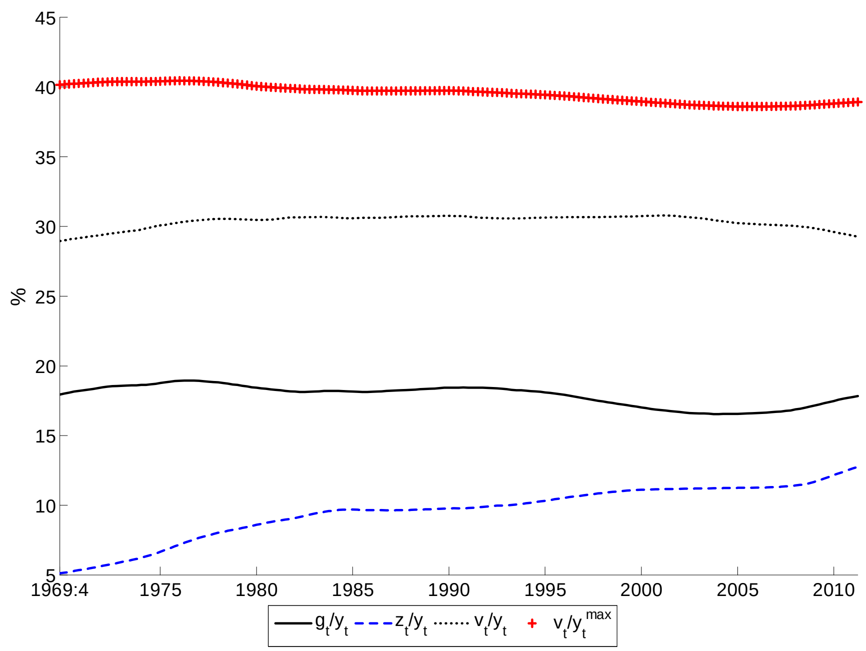

Figure 8: Rolling window (10-year) means of government expenditure-GDP ratio $\frac{g_t}{y_t}$, transfers-GDP ratio $\frac{z_t}{y_t}$, actual revenue-GDP ratio $\frac{v_t}{y_t}$ and maximum revenue-GDP ratio $\frac{v_t}{y_t}^{\max}$.

The steady-state values of the four debt-GDP limits are shown in figure 9 for the period 1969:4-2011:2. The four measures give a very different picture of the likely evolution of the borrowing capacity of the U.S. economy. The highest debt-GDP limit is given by $\frac{b}{y}^{WLSS}$ in which the debt limit is on average 7.5 times U.S. GDP. The next highest is $\frac{b}{y}^{NDLSS}$ on average 5.7 times the U.S. GDP. Both $\frac{b}{y}^{WLSS}$ and $\frac{b}{y}^{NDLSS}$ are relatively stable over the sample period. This is consistent with the relative stability of the actual and maximum ratios of tax revenue to GDP observed in figure 8. In contrast, the other two debt-GDP limits, $\frac{b}{y}^{FLSS}$ and $\frac{b}{y}^{IGBCSS}$ are lower and decline over time due to the upward trend in the ratio of government transfers to GDP.

These results highlight several practical issues in measuring the debt-GDP limit from the macroeconomic theory. The average debt-GDP limit obtained from the IGBC is only 0.56 and becomes negative in the last 10 quarters of the sample. The empirical results in section 5, based on the ad-hoc default thresholds, suggest that this definition of the debt-GDP limit would imply a U.S. sovereign credit rating of speculative grade (or possibly C) throughout most of the sample period and therefore is unsuitable. Davig, Leeper and Walker (2010) argue that the fiscal limit derived from (13) may overstate the true debt-GDP limit since governments would face increasing political pressures as they increase taxes in order to reach the peak of the Laffer hill.[31] This suggests that the debt-GDP limits implied by (12) and (14) would also be implausible (according to the empirical results based on ad-hoc limits, they are likely to yield an Aaa rating throughout the whole sample period). Choosing a satisfactory theoretically-base debt-GDP limit is, therefore, problematic. Despite the misgivings of Davig, Leeper and Walker, it seems that the best choice from the four we have considered is the debt-GDP limit given by equation (13). Consequently, we adopt equation (23) and $\frac{b}{y}^{FLSS}$ for our numerical analysis.

[31] It is also true, however, that the same debt-GDP limit can be achieved through an appropriate fiscal policy mix including government expenditure cuts as well as increasing taxes.

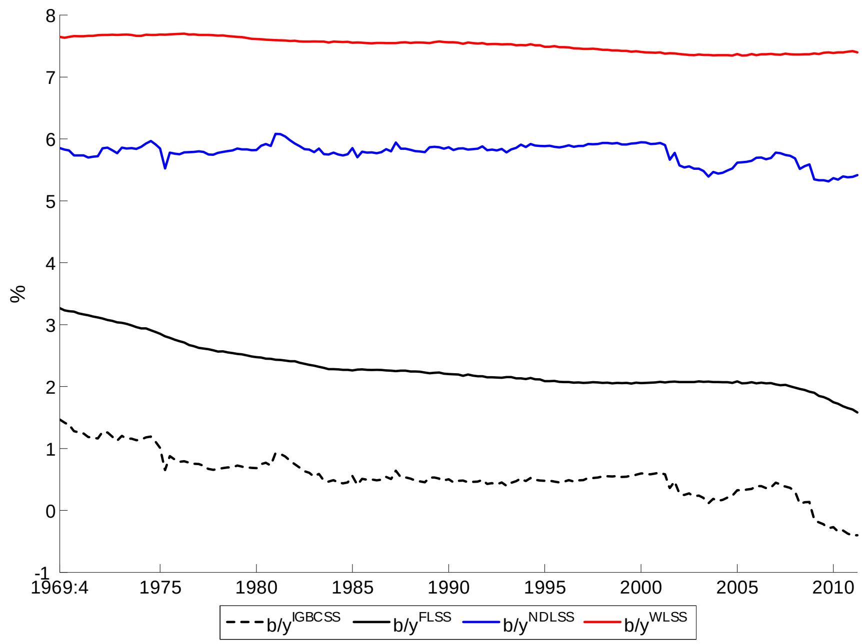

Figure 9: Evolution of the debt-GDP limit $\frac{b}{y}^{IGBCSS}$, $\frac{b}{y}^{FLSS}$, $\frac{b}{y}^{NDLSS}$ and $\frac{b}{y}^{WLSS}$ in the U.S. from 1969:4 to 2011:2.

The probability density functions (PDFs) of the fiscal limit in equation (23) at six different points in the sample are shown in figure 10. We find that over time the PDFs shifts to the left, thereby reducing the average debt-GDP limit. The largest shifts occurred during the 1970s and in the latest financial crisis. Changes in dispersion, reflecting time-varying volatility in the shocks, are also evident, with the greatest dispersion occurring in the 1970s and since the year 2007.

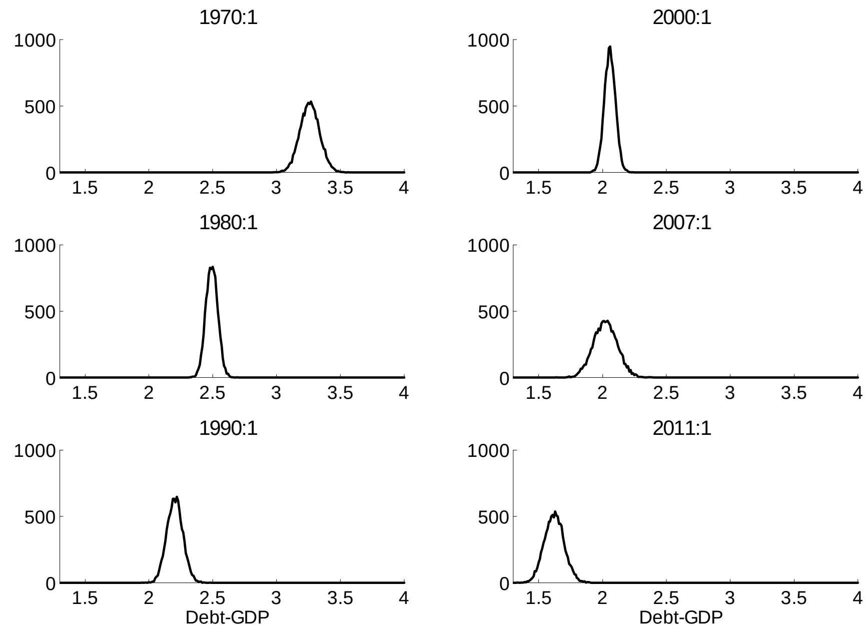

Figure 10: State-dependent probability density functions (PDFs) of the fiscal limit $\frac{b}{y}^{FLSS}$ at different points of the 1969:4-2011:2 sample period.

The corresponding cumulative density functions (CDFs) are shown in figure 11. These measure the probability of sovereign default (i.e. the probability that the debt-GDP ratio is above the fiscal limit in equation (23)) over different values of the debt-GDP ratio. The state of the economy at the time has a clear impact on this probability. For example, throughout the whole period 1969:4-2007:1 and with a debt-GDP of 150%, the probability of default at every point has been zero, but by 2011 it reached about 5%.[32]

[32] Using a microfounded model with fixed parameters, time-invariant volatilities, and calibrated using average data for 8 advanced countries, Bi (2011) finds that the average fiscal limit is close to the 150 per cent debt-GDP threshold we compute for the U.S. by the end of the sample period.

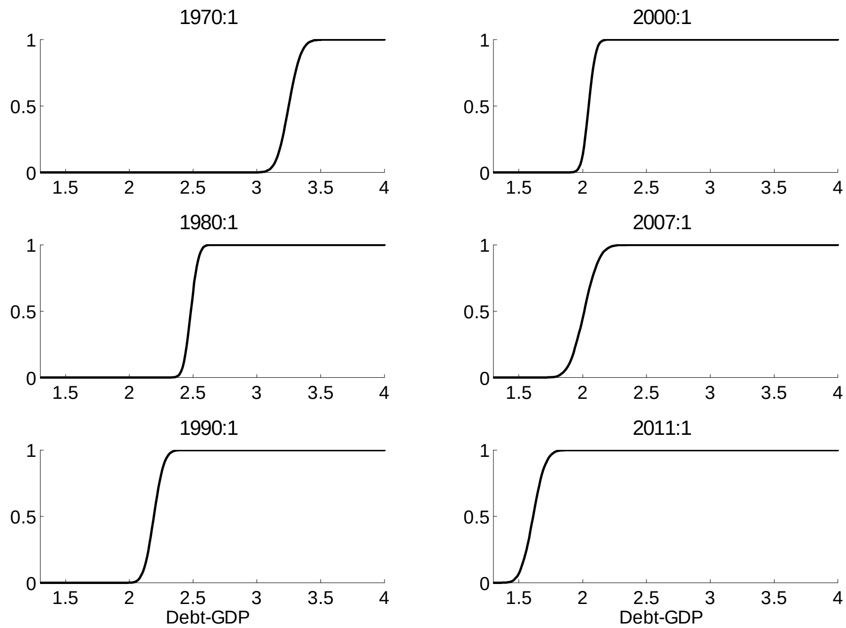

Figure 11: State-dependent cumulative density functions (CDFs) of the fiscal limit $\frac{b}{y}^{FLSS}$ at different points of the 1969:4-2011:2 sample period.

The uncertainty surrounding the fiscal limit in equation (23) can be captured by confidence bands. These may be calculated each quarter of the sample period from the 5 and 95 per cent quantiles of the distribution of the debt-GDP limit. The top panel in figure 12 shows the debt-GDP limit $\frac{b}{y}^{FLSS}$ with 95 per cent confidence bands. The bottom panel plots the time-varying volatility (standard deviation) of $\frac{b}{y}^{FLSS}$, thus showing how the uncertainty surrounding the debt-GDP threshold of the U.S. economy has changed over time. Our estimates show that the volatility of $\frac{b}{y}^{FLSS}$ tend to increase not only during recessions (for instance 1981-1982, the early 1990s, and late 2000s) but also during periods of significant tax reforms (for instance during the Reagan's tax cuts in 1981 and the increase in military spending of 1986 and the early 2000s). Based on these estimates, the fiscal limit in the U.S. display quantitatively significant time-varying volatility. The increase in uncertainty surrounding the borrowing capacity of the U.S. economy during the latest financial crisis is unprecedented.

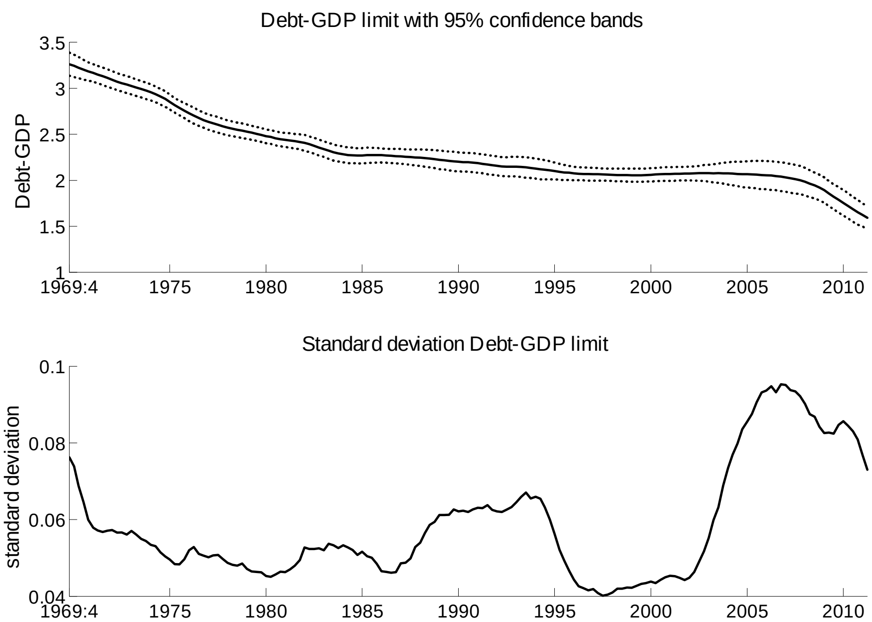

Figure 12: Confidence bands (top panel) and standard deviation (bottom panel) of the debt-GDP limit $\frac{b}{y}^{FLSS}$.

7.4 Credit rating

We now recalculate the U.S. sovereign credit rating over the 1970:-2011:2 period based on the theoretical debt-GDP limit in equation (23) using

$$p_{t+h} = \Pr\left(-\frac{E_t \frac{b_{t+h}}{y_{t+h}} - \frac{b_t}{y_t}^{FLSS}}{\sigma_{\eta,t+h}} \leq \eta_{t+h}\right). \tag{25}$$

We estimate the probability of default using the mean of the distribution of $\frac{b}{y}^{FLSS}$ and compute 95% confidence bands from the 5 and 95 per cent percentiles of the distribution to allow for uncertainty about the debt-GDP limit and the central debt-GDP forecast. The results are reported in figures 13 and 14. Figure 13 is based on a mapping from default probabilities at fixed time horizons into credit rating (as in figure 5). Figure 14 employs the mapping from the average default probability over the 10-year horizon into credit rating (as in figure 6).

Both mappings give credit ratings similar to those based on ad-hoc debt-GDP limits. Over a short time horizon (1–year ahead) a Prime-1 credit rating is found throughout the whole sample period. Over a longer term horizon, the credit rating is lower than Aaa only in the early 1970s and during the latest financial crisis. Allowing for uncertainty in the debt-GDP limit does not change the credit rating significantly. Increasing the debt-GDP limit to the upper 95% bound leaves the credit rating unaltered if already Aaa. Reducing the limit to the lower 5% bound causes only minor revisions to the credit rating.

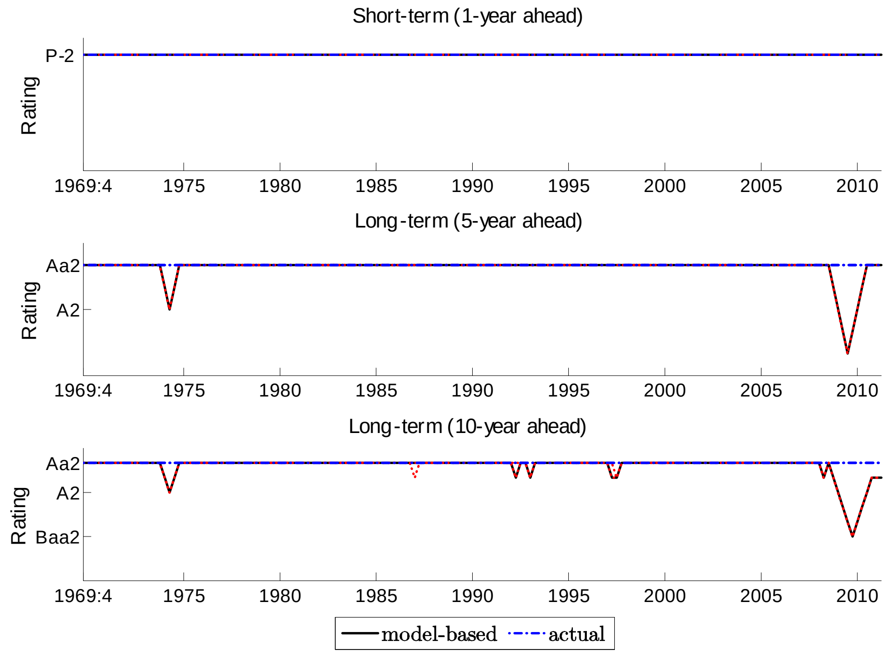

Figure 13: Final model-based U.S. sovereign credit ratings (short and long term) under the debt-GDP limit $\frac{b}{y}^{FLSS}$, 1969:1-2011:2. Actual corresponds to Moody's (2011) historical credit rating. Dotted lines denote confidence bands at 95 per cent.

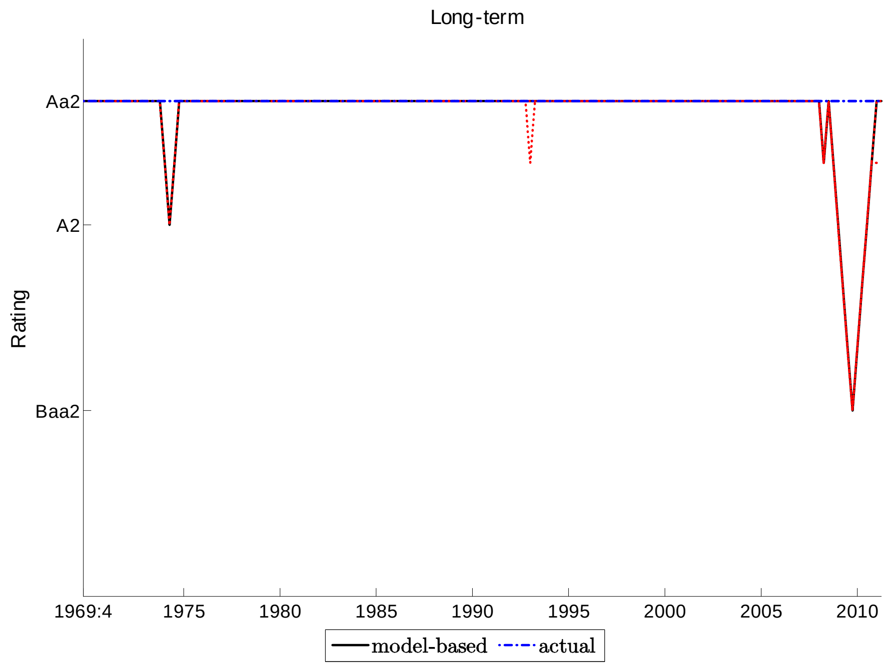

Figure 14: Final model-based U.S. sovereign credit ratings (10-year average) under the $\frac{b}{y}^{FLSS}$, 1969:1:2011:2. Actual corresponds to Moody's (2011) historical credit rating. Dotted lines denote confidence bands at 95 per cent. Dotted lines denote confidence bands at 95 per cent.

8 Summary and conclusions

8.1 Methodology

We have proposed a simple and easily automated way of generating sovereign credit ratings which we have illustrated using quarterly U.S. data from 1970-2011. This requires a forecast of the debt-GDP ratio and its forecast-error variance, together with a debt-GDP limit beyond which default is assumed to occur. We then estimate the probability that the forecast of the debt-GDP ratio will exceed the debt-GDP limit both in any period and by a particular future date.

The forecasts can be derived from either a structural or a reduced-form economic model. We have chosen to use a VAR as it has been found to forecast at least as well as a typical DSGE model, and is much easier to implement. The VAR we have used has two desirable features: it includes variables that make it consistent with the government budget constraint, the key determinant of debt, and it allows for time-variation both in the parameters and in volatility. We have suggested that a simple way to estimate such a VAR, that reproduces features usually captured by more complex estimation methods, is by rolling-window estimation (i.e. a ROVAR).

How to determine the debt-GDP limit is more problematic. We have considered two solutions. The simplest is to use ad-hoc limits. A more complicated, and potentially more controversial, alternative is to derive this from the macroeconomic theory. We investigated four alternative views of the debt-GDP limit, based on either anticipated or unanticipated changes in future fiscal policy. The former implies that the debt-GDP limits must satisfy the IGBC of the economy. The latter yields three weaker definitions of the debt-GDP limits, depending whether government expenditure is reduced to the minimum (NDL), or tax revenue is increased to the maximum (FL) or both (WL). Each of these is implemented empirically from the simulation of a DSGE model calibrated to the U.S. economy.

Having determined the debt-GDP limit we then calculate the probability of default and mapped this into a credit rating using Moody's historic records on sovereign default probability and credit ratings.

8.2 Main findings

The model-based measure of the credit rating proposed in this paper is affected by a number of factors: in particular, the lower the debt limit, the longer the time horizon and the greater is macroeconomic uncertainty as measured by the volatility of macroeconomic shocks, the higher is the probability of default and thus the likelihood that the model will determine a reduction of the credit rating.

The empirical study of the U.S. sovereign credit rating covers the period from 1969:4 to 2011:2. During this time, which includes two major oil crisis, six macroeconomic recessions, and significant domestic fiscal reforms, U.S. Treasury securities have received the highest credit-quality rating by all CRAs.

The model-based measure of the sovereign credit rating however suggest that there are at least two clear episodes over the past 40 years in which U.S. debt should have been downgraded: the first coincides with the oil crisis starting in October 1973; the second with the period after the collapse of IndyMac Bank and Lehman Brothers. These findings are generally robust to the specification of the debt-GDP limit, whether ah-hoc or theoretical. While the downgrade in the early 1970s is of two notches and lasts for two to three quarters, the extent and duration of the downgrade predicted by the model during the latest financial crisis varies with the debt-GDP limit and the time horizon, though in no case the credit rating reaches speculative grade.

The empirical analysis of the historic evolution of the theoretical debt-GDP limits in the U.S. offers useful insights on the evolution of the U.S. government borrowing capacity. Economic crisis tend to coincide with temporary and, in times, large increases in government expenditure, in turn shifting the distribution of the debt-GDP limit: when these shifts are particularly significant, like in the early 1970s and the late 2000s, the model-based measure of the sovereign credit rating falls. A temporary loss of the highest credit-quality rating is not the only worry for U.S. policy makers: the estimated evolution of the fiscal limit

(FL) suggests that the maximum borrowing capacity of the U.S. economy has been significantly eroded over time, mainly as a result of the steady increase in government transfers as a proportion to GDP over the past 40 years. Stabilisation of this trend can prevent further deterioration of the U.S. borrowing capacity: failure in achieving this will imply that the U.S. will not be able to maintain is Aaa rating indefinitely.

8.3 Assessment

One of the motivations of this paper was to show how it is possible to derive sovereign ratings in a simple, transparent and relative inexpensive manner. The results are very encouraging for those who wish to calculate their own credit ratings on a frequent basis.

Nonetheless, the procedure is mechanical and does not take into account several potentially important considerations which cause the markets to be either more or less fearful of sovereign default. Principal among these is the underlying political context. It may, for example, be politically infeasible to attain the maximum debt-GDP limit due to the degree of fiscal austerity entailed. It should also be remembered that sovereign default is an instrument of economic policy and that it may be optimal to default before reaching the debt-GDP limit.

Our analysis of fiscal limits has excluded the use of higher inflation to erode the real burden of government debt. This is because we assumed a political context in which high rates of inflation are ruled out. As a result, the sustainable debt-GDP limits we have derived are much lower than would be possible if inflation were used as a policy instrument.

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A ROVAR model

The specification of the ROVAR model can be represented as:

$$\begin{aligned}\mathbf{x}_t &= \mathbf{a}_{0t} + \sum_{p=1}^{2} \mathbf{A}_{pt}\mathbf{x}_{t-p} + \mathbf{C}_{0t}\boldsymbol{\epsilon}_t \\ & \boldsymbol{\epsilon}_t \tilde{} N(\mathbf{0}, \mathbf{I}).\end{aligned}$$

where $\mathbf{x}_t$ is a 7×1 vector of variables described in (8); $\mathbf{a}_{0t}$ is a 7×1 vector of time-varying coefficients; $\mathbf{A}_{pt}$ and $\mathbf{C}_{0t}$ are 7×7 matrices of time-varying coefficients; and $\boldsymbol{\epsilon}_t$ is a 7×1 vector of random variables satisfying $\boldsymbol{\epsilon}_t \tilde{} N(\mathbf{0}, \mathbf{I})$. For forecasting purposes it is convenient to formulate the ROVAR model in its companion form

$$\mathbf{X}_t = \mathbf{A}_t\mathbf{X}_{t-1} + \mathbf{C}_t\mathbf{w}_t$$

where

$$\mathbf{X}_t = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{x}_{t-1} \\ \mathbf{1}_n \end{bmatrix}, \mathbf{A}_t = \begin{bmatrix} \mathbf{A}_{1t} & \mathbf{A}_{2t} & \mathbf{a}_{0t} \\ \mathbf{I}_7 & \mathbf{O}_7 & \mathbf{0}_7 \\ \mathbf{O}_7 & \mathbf{I}_7 & \mathbf{0}_7 \end{bmatrix}, \mathbf{C}_t = \begin{bmatrix} \mathbf{C}_{0t} & \mathbf{O}_7 & \mathbf{0}_7 \\ \mathbf{O}_7 & \mathbf{O}_7 & \mathbf{0}_7 \\ \mathbf{O}_7 & \mathbf{O}_7 & \mathbf{0}_7 \end{bmatrix}, \mathbf{w}_t = \begin{bmatrix} \boldsymbol{\epsilon}_t \\ \mathbf{0}_7 \\ \mathbf{0}_7 \end{bmatrix}$$

with $\mathbf{1}_7$ and $\mathbf{0}_7$ denoting 7×1 vectors of ones and zeros respectively; and $\mathbf{I}_7$ and $\mathbf{O}_7$ denoting respectively an identity matrix and a matrix of zeros of order 7.

The time-varying (optimal) h-periods ahead forecasts are then computed using

$$E_t\mathbf{X}_{t+h} = \mathbf{A}_t^h\mathbf{X}_t$$

where E_t denotes expectation based on information available up to the current period t. The corresponding time-varying h-periods ahead forecast error is

$$\mathbf{X}_{t+h} - E_t\mathbf{X}_{t+h} = \sum_{j=1}^{h-1} \mathbf{A}_t^j\mathbf{C}_t\mathbf{w}_{t-j+h}$$

thus implying that the time-varying covariance matrix of the h-periods ahead forecast error is given by

$$E\left(\mathbf{X}_{t+h}-E_t\mathbf{X}_{t+h}\right)\left(\mathbf{X}_{t+h}-E_t\mathbf{X}_{t+h}\right)'=\sum_{j=1}^{h-1}\mathbf{A}_t^j\mathbf{C}_t\mathbf{C}_t'\mathbf{A}_t^j=\mathbf{\Sigma}_{t,h},$$

where $\mathbf{\Sigma}_{t,h}$ can be computed using the recursive formulae

$$\begin{aligned}\mathbf{\Sigma}_{t,1} &= \mathbf{C}_t\mathbf{C}_t' \\ \mathbf{\Sigma}_{t,h} &= \mathbf{C}_t\mathbf{C}_t'+\mathbf{A}_t\mathbf{\Sigma}_{t,h-1}\mathbf{A}_t', \text{ for } h\geq 2.\end{aligned}$$

B Data

The data are quarterly for the U.S. for the period 1960:1 to 2011:2. They are taken from three sources: the National Income and Product Accounts (NIPA) Tables held by the Bureau of Economic Analysis, last revised on August 26, 2011; from the International Financial Statistics (IFS) of the IMF; and from the OECD Economic Outlook, as available from Datastream in August 2011.

The data taken from NIPA are GDP deflator (DGDP), table 1.1.4. line 1; nominal GDP (NGDP) Table 1.1.5., line 1; NGDP; real GDP (RGDP), table 1.1.6., line 1; government current receipts (CURREC), table 3.1., line 1; government current expenditures (CUREXP), table 3.1., line 15; government transfers receipts, (TRAREC), table 3.1., line 11; government interest receipts (INTREC), table 3.1., line 9; government transfers payments, (TRAEXP), table 3.1., line 19; government interest payments (INTEXP), table 3.1. line 22. These are used to construct the first three variables in the vector $\mathbf{x}_t$ in (8) as follows. The growth rate of real GDP (γ_t) is calculated as the change of the natural logarithm of RGDP times 400. The inflation rate (π_t) is the change of the natural logarithm of DGDP times 400. To compute the deficit-GDP ratio ($\frac{d_t}{y_t}$) we use the following data: government current receipts (NIPA, table 3.1, line 1), interest receipts (NIPA, table 3.1, line 9), transfers receipts (NIPA, table 3.1, line 11), current expenditures (NIPA, table 3.1, line 15), interest payments (NIPA, table 3.1, line 22), transfers payments (NIPA, table 3.1, line 19). The primary deficit is computed by subtracting current receipts net of interest receipts (CURREC-INTREC) from net expenditures (CUREXP-INTEXP). This is then scaled by NGDP and multiplied by 100.

The three components of the defict-GDP ratio ($\frac{d_t}{y_t}=\frac{g_t}{y_t}+\frac{z_t}{y_t}-\frac{v_t}{y_t}$) are measured as follows. Net expenditure in goods and services ($\frac{g_t}{y_t}$) is computed as CUREXP-TRAEXP-INTEXP; net transfers ($\frac{z_t}{y_t}$) are measures as TRAEXP-TRAREC; and revenue ($\frac{v_t}{y_t}$) is computed net of transfers and interest receipts using CURREC-TRAREC-INTREC.

The debt-GDP ratio ($\frac{b_t}{y_t}$) and the global factor variable (cp_t) are taken from the OECD Economic Outlook using government net financial liabilities as a percentage of GDP and the growth rate of crude oil price respectively. The short rate (r_t^s) and the long rate (r_t^l) are taken from the IFS using the Federal funds rate and the 10-year bond yield respectively.

C Model solution

Maximizing inter-temporal utility subject to the household budget constraint using the method of Lagrange multipliers gives the following first-order conditions for consumption, labour, capital and bonds for $t \geq 0$

$$\begin{aligned}
c_t &: \beta^t u_{c,t} - \lambda_t(1+\tau_t^c) = 0 \\
n_t &: \beta^t u_{n,t} + \lambda_t(1-\tau_t^n)w_t = 0 \\
k_t &: \lambda_{t+1}\left[1+\left(r_{t+1}^k-\delta\right)\left(1-\tau_{t+1}^k\right)\right] - \lambda_t = 0 \\
b_t &: \lambda_{t+1}\left(1+r_{t+1}^b\right) - \lambda_t = 0,
\end{aligned}$$

where λ_t denotes the Lagrange multiplier attached to the household budget constraint. The first-order conditions for the firm's problem are, for $t \geq 0$,

$$\begin{aligned}
k_t &: r_t^k = \alpha k_t^{\alpha-1}\left(A_t n_t\right)^{1-\alpha} \\
n_t &: w_t = (1-\alpha)\, A_t k_t^{\alpha}\left(A_t n_t\right)^{-\alpha}.
\end{aligned}$$

Assuming the utility function $u(c_t, 1-n_t) = \log c_t + \psi \log(1-n_t)$, the Euler equation for capital, the intratemporal condition between consumption and labour, and the no-arbitrage condition for $t \geq 0$ are

$$\begin{aligned}
\frac{(1+\tau_{t+1}^c)c_{t+1}}{(1+\tau_t^c)c_t} &= \beta\left\{1+\left[\alpha k_{t+1}^{\alpha-1}\left(A_{t+1}n_{t+1}\right)^{1-\alpha}-\delta\right]\left(1-\tau_{t+1}^k\right)\right\} \\
\psi\frac{c_t}{1-n_t} &= \frac{(1-\tau_t^n)}{(1+\tau_t^c)}(1-\alpha)\,A_t k_t^{\alpha}\left(A_t n_t\right)^{-\alpha} \\
r_{t+1}^b &= \left(r_{t+1}^k-\delta\right)\left(1-\tau_{t+1}^k\right).
\end{aligned}$$

The stationary equilibrium value of capital is given by equation (18), while the corresponding solutions for consumption and labour are:

$$\begin{aligned}
c &= \Omega k - g \\
n &= \varphi k
\end{aligned}$$

where μ, Ω and φ defined in equations (19), (20) and (21) respectively. The stationary equilibrium values of all other variables can then be retrieved from

$$\begin{aligned}
y &= k^{\alpha}\left(An\right)^{1-\alpha} \\
r^k &= \alpha k^{\alpha-1}\left(An\right)^{1-\alpha} \\
w &= A\left(1-\alpha\right)k^{\alpha}\left(An\right)^{-\alpha} \\
r^b &= \left[\alpha k^{\alpha-1}\left(An\right)^{1-\alpha}-\delta\right]\left(1-\tau^k\right).
\end{aligned}$$

The stationary equilibrium values of the capital-output ratio, the labour productivity, the before tax wage and the consumption-output ratio are therefore

$$\begin{aligned}
\frac{k}{y} &= \left[\frac{\beta^{-1}-1}{\alpha\left(1-\tau^{k}\right)}+\frac{\delta}{\alpha}\right]^{-1} \\
\frac{y}{n} &= A\left[\frac{\beta^{-1}-1}{\alpha\left(1-\tau^{k}\right)}+\frac{\delta}{\alpha}\right]^{-\frac{\alpha}{1-\alpha}} \\
w &= (1-\alpha)\,A\left[\frac{\beta^{-1}-1}{\alpha\left(1-\tau^{k}\right)}+\frac{\delta}{\alpha}\right]^{-\frac{\alpha}{1-\alpha}} \\
\frac{c}{y} &= \chi\left(\frac{1}{\varphi k}-1\right),
\end{aligned}$$

where χ is defined in equation (17). The stationary solution of the tax-GDP ratio is

$$\frac{v}{y}=\tau^{c}\frac{c}{y}+\tau^{n}w\frac{n}{y}+\tau^{k}\left(r^{k}-\delta\right)\frac{k}{y}$$

Eliminating the macroeconomic variables on the right-hand side gives equation (16) which expresses the average tax rate as a function of the marginal tax rates.

The $\frac{b}{y}^{FLSS}$ and $\frac{b}{y}^{WLSS}$ are obtained by maximising the average tax rate with respect to the labour and capital taxes. If the capital stock is held fixed then an analytic solution can be obtained that gives an interior maximum. But when the capital stock is not fixed, as here, the debt-GDP limit must be obtained by numerical maximisation. This solution, which uses MCMC simulation, is described appendix D.

D Markov Chain Monte Carlo algorithm

The Markov Chain Monte Carlo simulation includes the following steps:

1. Estimate the time-varying volatility of technology shocks. From the log transformation of the labour-augmenting Cobb-Douglas production, we derive a time-series for the logarithm of technological progress using

$$\ln A_t=\frac{1}{1-\alpha}\left[\ln y_t-\alpha\ln k_t-(1-\alpha)\ln n_t\right]$$

over the period 1960:1-2011:2. This uses real GDP data described in appendix B, together with data on private sector employment and private nonresidential gross fixed capital from the OECD Economic Outlook (Datastream, August 2011) and assumes a capital share of output of 0.3. We then measure the rolling-window (40 quarters) standard deviation of the derived series for $\ln A_t$ which is used as proxy for the time-varying volatility of technological progress. Given the size of the rolling window, this series ranges from 1969:4 to 2011:2.

2. Estimate the time-varying mean and volatility of $\frac{g_t}{y_t}$ and $\frac{z_t}{y_t}$. These are derived by calculating rolling window (40 periods) means and standard deviations for the time series for government expenditure-GDP and transfers-GDP described in Appendix B. As data are available from 1960:1 onward, these time-varying means and standard deviations range from 1969:4 to 2011:2.

3. Estimate time-varying actual tax rates on consumption, capital and labour. We follow the methodology employed by Fernandez-Villaverde et al. (2012) and calculate time series for these three tax rates over the period 1969:4-2011:2.

4. Estimate Laffer hills. We simulate numerically the steady-state solution to the model over the period 1969:4-2011:2 using rolling-window mean values of $\frac{g_t}{y_t}$ (from 2) and τ^c (from 3) and allowing in each quarter τ^n and τ^k to range from 0.01 to 0.99 (step = 0.01). We then use grid search to find the combination of τ^n and τ^k that maximizes the revenue-GDP ratio in each quarter. This yields a series of $\tau^{n,\max}$ and $\tau^{k,\max}$ that correspond with the pick of the Laffer hill at each quarter of the sample period. The simulation is carried out using β=0.95, α=0.3224, δ=0.012, ψ=0.6, and $\overline{A} = 1$.

5. Stochastic simulation of the shocks. We assume that the natural logarithm of $\frac{g_t}{y_t}$, $\frac{z_t}{y_t}$ and A_t follows an AR(1) process with time-varying volatility (taken from 1 and 2) and time-varying mean (for $\frac{g_t}{y_t}$ and $\frac{z_t}{y_t}$) taken from 2. The mean of the technological progress is normalized to 1. Thus we specify
$$\ln s_t = (1 - \rho^s) \ln \overline{s}_t + \rho^s \ln s_{t-1} + \epsilon_t^s,\ \epsilon_t^s \tilde{} N\left(0, \sigma_s^2\right)$$
with $s = \frac{g_t}{y_t}$, $\frac{z_t}{y_t}$ and A_t. We simulate these AR(1) process 200 times at each quarter over the 1969:4-2011:2 period using a constant mean reversion coefficient $\rho^s = 0.553$.

6. Compute time-varying steady states. Using the tax rates from either 3 or 4, we calculate the steady-state solution to the model and the implied consumption path, for each of the 200 values of $\frac{g_t}{y_t}$ and A_t simulated from 5.

7. Compute time-varying debt-GDP limit. Using the simulated values of $\frac{v_t^{\max}}{y_t}$, $\frac{v_t}{y_t}$, $\frac{g_t}{y_t}$ and $\frac{z_t}{y_t}$ we calculate the present value of the maximum deficit obtained by setting expenditure to zero and generating the maximum tax revenue. Discounting is carried out using the stochastic discount factor generated by the consumption series simulated in 6.

8. Posterior means and distribution of the debt-GDP limits. We repeat steps 5-7 10000 times to get the posterior mean and distribution of each of the debt-GDP limits.