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## **GAMES ON NETWORKS**

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## ABSTRACT

### Games on Networks\*

We provide an overview and synthesis of the literatures analyzing games where players are connected via a network structure. We study, in particular, the impact of the structure of the network on individuals' behaviors. We focus on the game theoretic modeling, but also include some discussion of analyses of peer effects, as well as applications to diffusion, employment, crime, industrial organization, and education.

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# 1 Introduction

Social networks are important in many facets of our lives. Most decisions that people make, from which products to buy to whom to vote for, are influenced by the choices of their friends and acquaintances. For example, the decision of an individual to whether buy or not a new product, attend a meeting, commit a crime, find a job is often influenced by the choices of his or her friends and acquaintances (be they social or professional). The emerging empirical evidence on these issues motivates the theoretical study of network effects.

Here, we provide an overview of literatures on the analysis of the interaction of individuals who are connected via a network and whose behaviors are influenced by those around them.<sup>1</sup> Such interactions are natural ones to model using game theory, as the payoffs that an individual receives from various choices depends on the behaviors of his or her neighbors.

Of course, one can view these settings as special cases of game theory more generally, and so results from the general literature directly apply: for example, existence of various forms of equilibria can be deduced from standard results. The interest, therefore is instead in whether there is anything we can deduce that holds systematically regarding how play in a game depends on the network structure of interactions. For example, if individuals only wish to buy a new product if a sufficient fraction of their friends do, can we say something about how segregation patterns in the network of friendships affects the purchase of the product? Can we say anything about who is the most influential individual in a network where people look to their peers in choosing an effort level in education? Thus, our discussion of the literature focuses on investigations relating network characteristics to behavior.

The main challenge that faced in studying strategic interaction in social settings is the inherent complexity of networks. Without focusing in on specific structures in terms of the

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<sup>1</sup>As game theoretic models of network formation have been surveyed extensively elsewhere (see, for example, Jackson (2003, 2004, 2005b, 2008a, 2011), Goyal (2007), De Martí Beltran and Zenou (2011)), we concentrate here on games on networks. We also do not survey the literature on communication structures in cooperative games. This is another large literature, following Myerson (1977) that has been surveyed elsewhere (see Slikker and van den Nouweland (2001) for the graph-based literature, and Jackson (2005a, 2008a) for the allocation rule literature and allocation rules). Finally, there is also a large literature on agent-based models including networked interactions (e.g., see Tesfatsion (2006) for some references) that is already surveyed elsewhere, and not included here. Exploiting the growing capabilities of computers, agent-based methods have been used to analyze dynamic systems of interacting agents in cases where the network is fixed (see Wilhite, 2006) as well as where the relationships are endogenous (Vriend, 2006).

games, it is hard to draw any conclusions. The literature has primarily taken three approaches to this challenge, and form the basis for our discussion. One involves looking at games of strategic complements and strategic substitutes, where the interaction in payoffs between players satisfies some natural and useful monotonicity properties. With strategic complementarities, a player’s incentives to take an action (or a “higher” action) are increasing in the number of his or her friends who take the (higher) action; and with strategic substitutes the opposite incentives are in place. We show that the monotonicity properties of these structured interactions have allowed the literature to deduce a number of results related to equilibrium behavior as well as dynamics, and how those depend on network structure. A second approach relies on looking at quite tractable “linear-quadratic” setting where agents choose a continuous level of activity. That simple parametric specification permits an explicit solution for equilibrium behavior as a function of a network, and thus leads to useful comparative statics and other results that are useful in empirical work. A third approach considers settings with an uncertain pattern of interactions, where players make choices (such as learning a language) without being certain about with whom they will interact. The uncertainty actually simplifies the problem since behavior depends on anticipated rates of interaction, rather than complex realizations of interactions. Together all of these various approaches and models make a number of predictions about behavior, relating levels of actions to network density, relating players’ behaviors to their position in the network, and relating behavior to things like the degree distribution and cost of taking given actions. The theory thus makes predictions both about how a player’s behavior relates to his/her position in a network, as well as what overall behavior patterns to expect as a function of the network structure.

## 2 Background Definitions

We begin with a class of canonical and widely applicable games; specifically, games where there is a fixed and given network of interactions. Links indicate which players’ strategies affect which others’ payoffs. In particular, a given player’s payoff depends only on the play of his or her neighbors. Of course, this results in indirect network effects since there may be chains of influence.

We provide some basic definitions of games on networks.<sup>2</sup>

## 2.1 Players and Networks

We consider a finite set of players  $N = \{1, \dots, n\}$  who are connected in a network.

A *network* (or *graph*) is a pair  $(N, \mathbf{g})$ , where  $\mathbf{g}$  is a network on the set of nodes  $N$ .

We abuse notation and let  $\mathbf{g}$  denote the two standard ways in which networks are represented: by their adjacency matrices as well as by listing the pairs of nodes that are connected. Thus,  $\mathbf{g}$  will sometimes be an  $n \times n$  adjacency matrix, with entry  $g_{ij}$  denoting whether  $i$  is linked to  $j$  and can also include the intensity of that relationship. At other times  $\mathbf{g}$  denotes the set of all relationships that are present, and so we use notation  $ij \in \mathbf{g}$  to indicate that  $i$  is linked to  $j$ .

A network is *undirected* if  $\mathbf{g}$  is symmetric so that it is required that  $g_{ij} = g_{ji}$ , and is *directed* otherwise.

A relationship between two nodes  $i$  and  $j$ , represented by  $ij \in \mathbf{g}$ , is referred to as a link. Links are also referred to as edges or ties in various parts of the literature; and sometimes also directed links, directed edges, or arcs in the specific case of a directed network. Shorthand notations for the network obtained by adding or deleting a link  $ij$  to or from an existing network  $\mathbf{g}$  are  $\mathbf{g} + ij$  and  $\mathbf{g} - ij$ , respectively.

A *walk* in a network  $(N, \mathbf{g})$  refers to a sequence of nodes,  $i_1, i_2, i_3, \dots, i_{K-1}, i_K$  such that  $i_k i_{k+1} \in \mathbf{g}$  for each  $k$  from 1 to  $K - 1$ .<sup>3</sup> The *length* of the walk is the number of links in it, or  $K - 1$ .

A *path* in a network  $(N, \mathbf{g})$  is a walk in  $(N, \mathbf{g})$ ,  $i_1, i_2, i_3, \dots, i_{K-1}, i_K$ , such that all the nodes are distinct.

A *cycle* in a network  $(N, \mathbf{g})$  is a walk in  $(N, \mathbf{g})$ ,  $i_1, i_2, i_3, \dots, i_{K-1}, i_K$ , such that  $i_1 = i_K$ .

A network  $(N, \mathbf{g})$  is *connected* if there is a path in  $(N, \mathbf{g})$  between every pair of nodes  $i$  and  $j$ .<sup>4</sup>

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<sup>2</sup>We provide terse definitions here. For a reader new to these definitions, Chapter 2 in Jackson (2008a) provides more discussion and background.

<sup>3</sup>Standard definitions of walks, paths, and cycles specify them as sets of nodes together with sets of links. The definitions here simplify notation, and for the purposes of this chapter the difference is inconsequential.

<sup>4</sup>Each of these definitions has an analog for directed networks, simply viewing the pairs as directed links and then having the name directed walk, directed path, and directed cycle. In defining connectedness for a directed network one often uses a strong definition requiring a directed path from each node to every other

A *component* of a network  $(N, \mathbf{g})$  is a subnetwork  $(N', \mathbf{g}')$  (so  $N' \subset N$  and  $\mathbf{g}' \subset \mathbf{g}$ ) such that

- there is a path in  $\mathbf{g}'$  from every node  $i \in N'$  to every other node  $j \in N', j \neq i$ ,
- $i \in N'$  and  $ij \in \mathbf{g}$  implies  $j \in N'$  and  $ij \in \mathbf{g}'$ .

Thus, a component of a network is a maximal connected subgraph with all adjacent links, so that and there is no way of expanding the set of nodes in the subgraph and still having it be connected.

The *distance* between two nodes in the same component of a network is the length of a shortest path (also known as a *geodesic*) between them.

The *neighbors* of a node  $i$  in a network  $(N, \mathbf{g})$  are denoted by  $N_i(\mathbf{g})$ . As we predominantly discuss settings where  $N$  is fixed, we omit dependence on the set of nodes  $N$ , and so write  $N_i(\mathbf{g})$  rather than  $N_i(N, \mathbf{g})$ . We have:

$$N_i(\mathbf{g}) = \{j | ij \in \mathbf{g}\}$$

The *degree* of a node  $i$  in a network  $(N, \mathbf{g})$  is the number of neighbors that  $i$  has in the network, so that  $d_i(\mathbf{g}) = |N_i(\mathbf{g})|$ .<sup>5</sup>

The  $k$ th power  $\mathbf{g}^k = \mathbf{g} \times \overset{(k \text{ times})}{\dots} \mathbf{g}$  of the adjacency matrix  $\mathbf{g}$  keeps track of indirect connections in  $\mathbf{g}$ . More precisely, the coefficient  $g_{ij}^{[k]}$  in the  $(i, j)$  cell of  $\mathbf{g}^k$  gives the number of walks of length  $k$  in  $\mathbf{g}$  between  $i$  and  $j$ .

An *independent set* relative to a graph  $(N, \mathbf{g})$  is a subset of nodes  $A \subset N$  for which no two nodes are adjacent (i.e., linked).  $A$  is a maximal independent set if there does not exist another independent, set  $A' \neq A$ , such that  $A \subset A' \subset N$ . A *dominating set* relative a graph  $(N, \mathbf{g})$  is a subset of nodes  $A \subset N$  such that every node in  $A$  is connected to every other node in  $A$  via a path that involves only nodes in  $A$ , and every node not in  $A$  is linked to at least one member of  $A$ . For example, the central node in a star forms a dominating set and also a maximal independent set, while each peripheral node is an independent set and the set of all peripheral nodes is a maximal independent set. Any set including the central node and some peripherhal nodes is a dominating set, but not an independent set.

Let  $G(N)$  be the set of networks on the nodes  $N$  under consideration, which will often be the set of simple (nondirected) networks unless otherwise stated.

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node.

<sup>5</sup>Unless otherwise stated, let us suppose that  $g_{ii} = 0$ , so that nodes are not linked to themselves.

## 2.2 Games on Networks

Players in  $N$  have action spaces  $A_i$ . Let  $A = A_1 \times \cdots \times A_n$ . In most of our discussion, the action spaces are finite sets or subsets of a Euclidean space.

Player  $i$ 's payoff function is denoted  $u_i : A \times G(N) \rightarrow \mathbb{R}$ .

Unless otherwise indicated when we say *equilibrium*, we refer to a pure strategy Nash equilibrium: a profile of actions  $\mathbf{a} \in A = A_1 \times \cdots \times A_n$ , such that

$$u_i(a_i, \mathbf{a}_{-i}, \mathbf{g}) \geq u_i(a'_i, \mathbf{a}_{-i}, \mathbf{g})$$

for all  $a'_i \in A_i$ .

A given player's payoff depends on other players' behaviors, but only on those to whom the player is (directly) linked in the network. In fact, without loss of generality the network can be taken to indicate the payoff interactions in the society. More formally,  $i$ 's payoff depends only on  $a_i$  and  $\{a_j\}_{j \in N_i(\mathbf{g})}$ , so that for any  $i$ ,  $a_i$ , and  $\mathbf{g}$ :

$$u_i(a_i, \mathbf{a}_{-i}, \mathbf{g}) = u_i(a_i, \mathbf{a}'_{-i}, \mathbf{g}),$$

whenever  $\mathbf{a}_j = \mathbf{a}'_j$  for all  $j \in N_i(\mathbf{g})$ .

To fix ideas, let us consider a couple of examples.

**EXAMPLE 1** *The Majority Game.*<sup>6</sup>

Players' action spaces are  $A_i = \{0, 1\}$ . This covers applications where a player can choose to either do something or not to, for instance, buying a product, attending a party, and so forth. The payoff to a player from taking action 1 compared to action 0 depends on the number of neighbors who choose action 1, so that

$$\text{sign} (u_i(1, a_{N_i(\mathbf{g})}) - u_i(0, a_{N_i(\mathbf{g})})) = \text{sign} \left( \sum_{j \in N_i(\mathbf{g})} a_j - \sum_{j \in N_i(\mathbf{g})} (1 - a_j) \right).$$

In particular, if more than one half of  $i$ 's neighbors choose action 1, then it is best for player  $i$  to choose 1, and if fewer than one half of  $i$ 's neighbors choose action 1 then it is best for player  $i$  to choose action 0. There are clearly multiple equilibria in this game. For example, all players taking action 0 (or 1) is an equilibrium. Figure 1 displays another equilibrium of the majority game.

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<sup>6</sup>This game has been studied extensively in physics (see, e.g. Conway's "game of life") and in various agent-based models that followed, such as the "voter model" (see e.g. Clifford and Sudbury, 1973, and Holley and Liggett, 1975)



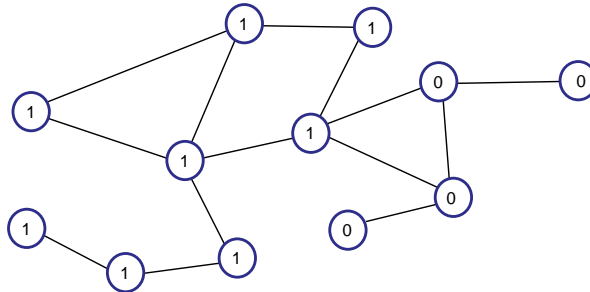


Figure 1: An Equilibrium in the Majority Game

**EXAMPLE 2** “Best-Shot” Public Goods Games

Another canonical example of a game on a network is based on what are known as “best-shot” public goods games (see Hirshleifer, 1983). For instance, the action might be learning how to do something, where that information is easily communicated; or buying a book or other product that is easily lent from one player to another. Taking the action 1 is costly and a player would prefer that a neighbor take the action than having to do it himself or herself; but, taking the action and paying the cost is better than having nobody take the action.

$$u_i(\mathbf{a}, \mathbf{g}) = \begin{cases} 1 - c & \text{if } a_i = 1, \\ 1 & \text{if } a_i = 0, a_j = 1 \text{ for some } j \in N_i(\mathbf{g}) \\ 0 & \text{if } a_i = 0, a_j = 0 \text{ for all } j \in N_i(\mathbf{g}), \end{cases}$$

where  $1 > c > 0$ .

There are many possible equilibria in the best-shot public goods game and Figure 2 displays one of them. Interestingly, the equilibria in this game correspond exactly to having

the set of players who choose action 1 form a maximal independent set of nodes in the network; that is, a maximal set of nodes that have no links to each other in the network. In Figure 2, it is clear that nobody wants to deviate from their Nash equilibrium actions. Take, for example, the central player who chooses action 1. His/her utility is  $1 - c$ . Since all his/her neighbors choose action 0, deviating by choosing action 0 would give him/her a utility of  $0 < 1 - c$ . Similarly, for each player who chooses action 0, his/her utility is 1 since at least one of his/her neighbors choose action 1. Choosing action 1 would give him/her  $1 - c < 1$ .

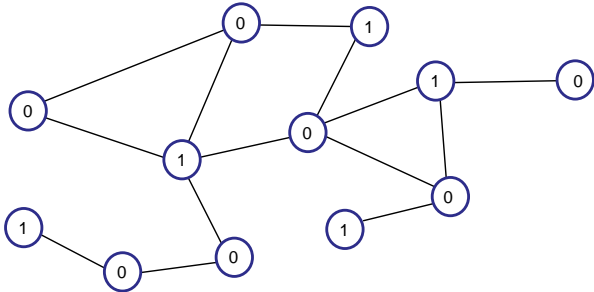


Figure 2: An equilibrium in the best-shot public good game

### 3 Strategic Complements and Strategic Substitutes

Although there are many forms that networked games can take, there are two prominent and broadly encompassing classes of games. In fact, the two previous examples are typical members of these two classes of games. The distinction between these types of games relates

to the nature of the strategic interaction. To see the nature of the distinction, let us take the actions in the games to be well-ordered, such as a subset of the real line (or more generally a lattice, as detailed shortly). The first class of examples, of which coordination games are the canonical example, are games of *strategic complements*. In games of strategic complements, an increase in the actions of other players leads a given player's higher actions to have relatively higher payoffs compared to that player's lower actions. Examples of such games include things like the adoption of a technology, human capital decisions, criminal efforts, smoking behaviors, etc. Games of *strategic substitutes* are such that the opposite is true: an increase in other players' actions leads to relatively lower payoffs to higher actions of a given player. Applications of strategic substitutes include, for example, local public good provision and information gathering.

### 3.1 Defining Strategic Complements and Substitutes

Let us take  $A_i$  (the action space) to be a complete lattice with an associated partial order  $\geq_i$ , for each  $i$ .<sup>7</sup> Then it is easy to see that  $A$  is also complete lattice, if we define  $\mathbf{a} \geq \mathbf{a}'$  if and only if  $a_i \geq a'_i$  for every  $i$ , and where for any  $S \subset A$  we define  $\inf(S) = (\inf_i\{a_i : \mathbf{a} \in S\})_i$  and  $\sup(S) = (\sup_i\{a_i : \mathbf{a} \in S\})_i$ .

A game exhibits *strategic complements* if it exhibits *increasing differences*; that is, for all  $i$ ,  $a_i \geq_i a'_i$  and  $a_{-i} \geq_{-i} a'_{-i}$ :

$$u_i(a_i, a_{-i}, \mathbf{g}) - u_i(a'_i, a_{-i}, \mathbf{g}) \geq u_i(a_i, a'_{-i}, \mathbf{g}) - u_i(a'_i, a'_{-i}, \mathbf{g}).$$

A game exhibits *strategic substitutes* if it exhibits *decreasing differences*; that is, for all  $i, j$ , with  $i \neq j$ ,  $a_i \geq_i a'_i$  and  $a_{-i} \geq_{-i} a'_{-i}$ :

$$u_i(a_i, a_{-i}, \mathbf{g}) - u_i(a'_i, a_{-i}, \mathbf{g}) \leq u_i(a_i, a'_{-i}, \mathbf{g}) - u_i(a'_i, a'_{-i}, \mathbf{g}).$$

These notions are said to apply strictly if the inequalities above are strict whenever  $a_i >_i a'_i$  and  $a_{-i} \geq_{-i} a'_{-i}$  with  $a_j >_j a'_j$  for  $j \in N_i(\mathbf{g})$ .

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<sup>7</sup>Let  $\geq_i$  be a partial order on a (nonempty) set  $A_i$  (so  $\geq_i$  is reflexive, transitive and antisymmetric).  $(A_i, \geq_i)$  is a lattice if any two elements  $a_i$  and  $a'_i$  have a least upper bound (supremum) (so,  $\sup_i(a_i, a'_i) \geq a_i$  and  $\sup_i(a_i, a'_i) \geq a'_i$ ), and a greatest lower bound (infimum) (so,  $\inf_i(a_i, a'_i) \leq a_i$  and  $\inf_i(a_i, a'_i) \leq a'_i$ ), in the set. A lattice  $(A_i, \geq_i)$  is *complete* if every nonempty subset of  $A_i$  has a supremum and an infimum in  $A_i$ .

The majority game (Example 1) is one of strategic complements while the best-shot public goods game (Example 2) is one of strategic substitutes.

## 3.2 Existence of Equilibrium

### 3.2.1 Games of Strategic Complements

Beyond capturing many applications, games of strategic complements are well-behaved in a variety of ways. Not only do equilibria generally exist, but they form a lattice so that they are well-ordered and there are easy algorithms for finding the maximal and minimal equilibria.

**THEOREM 1** *Consider a game of strategic complements such that:*

- *for every player  $i$ , and specification of strategies of the other players,  $a_{-i} \in A_{-i}$ , player  $i$  has a nonempty set of best responses  $BR_i(a_{-i})$  that is a closed sublattice of the complete lattice  $A_i$ ,<sup>8</sup> and*
- *for every player  $i$ , if  $a'_{-i} \geq a_{-i}$ , then  $\sup_i BR_i(a'_{-i}) \geq_i \sup_i BR_i(a_{-i})$  and  $\inf_i BR_i(a'_{-i}) \geq_i \inf_i BR_i(a_{-i})$ .*

*An equilibrium exists and the set of equilibria form a (nonempty) complete lattice.<sup>9</sup>*

Variations on this theorem can be found in Topkis (1979) and Zhou (1994), and for arbitrary sets of players in Acemoglu and Jackson (2011).

In games of strategic complements such that the set of actions is finite, or compact and payoffs are continuous, the conditions of the theorem apply and there exists an equilibrium. Note that the equilibria exist in pure strategies, directly in terms of the actions  $A$  without requiring any additional randomizations. The same is not true games of strategic substitutes.

Finding maximal and minimal equilibria in a game of strategic complements is then quite easy. Let us describe an algorithm for the case where  $A$  is finite. Begin with all players playing the maximal action  $a^0 = \bar{a}$ . Let  $a_i^1 = \sup_i(BR_i(a_{-i}^0))$  for each  $i$  and, iteratively,

<sup>8</sup>Closure requires that  $\sup_i(BR_i(a_{-i})) \in BR_i(a_{-i})$  and  $\inf_i(BR_i(a_{-i})) \in BR_i(a_{-i})$ .

<sup>9</sup>The set of equilibria is not necessarily a sublattice of  $A$  (see Topkis, 1979, and Zhou, 1994). That is, the sup in  $A$  of a set of equilibria may not be an equilibrium, and so sup and inf have to be restricted to the set of equilibria to ensure that the set is a complete lattice, but the same partial order can be used.

let  $a_i^k = \sup_i(BR_i(a_{-i}^{k-1}))$ . It follows that a point such that  $a^k = a^{k-1}$  is the maximal equilibrium point, and given the finite set of strategies this must occur in a finite number of iterations. Analogously, starting from the minimal action and iterating upward, one can find the minimal equilibrium point.<sup>10</sup>

This also means that dynamics that iterate on best response dynamics will generally converge to equilibrium points in such games (e.g., see Milgrom and Roberts, 1990).

### 3.2.2 Games of Strategic Substitutes and other Games on Networks

Moving beyond games of strategic complements, existence of equilibria and the structure of the set are no longer so nicely behaved.

Existence of equilibria can be guaranteed along standard lines: for instance equilibria exist if  $A_i$  is a nonempty, compact, and convex subset of a Euclidean space and  $u_i$  is continuous and quasi-concave for every  $i$ . This covers the canonical case where  $A_i$  are the mixed strategies associated with an underlying finite set of pure actions and  $u_i$  is the expected payoff and hence quasi-concave. Nonetheless, this means that pure strategy equilibria may not exist unless the game has some specific structure (and we discuss some such cases below).

In addition, with the lack of lattice structure, best responses are no longer so nicely ordered and equilibria in many network games can be more difficult to find.<sup>11</sup>

Nonetheless, some games of strategic substitutes on networks still have many important applications and are tractable in some cases. For example, consider the best-shot public goods game discussed above (example 2). As we showed above, best-shot public goods games on a network always have pure strategy equilibria, and in fact those equilibria are the situations where the players who take action 1 form a maximal independent set.

Finding all of the maximal independent sets is computationally intensive, but finding one such set is easy. Here is an algorithm that finds an equilibrium.<sup>12</sup> At a given step  $k$ ,

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<sup>10</sup>Calvó-Armengol and Jackson (2004, 2007) use this technique to calculate the maximal equilibrium in their dynamic game with strategic complementarities. They propose an important application of this game by looking at labor-market networks and showing that investing in human capital depends on having access to job information.

<sup>11</sup>For results on the complexity of finding equilibria in games on networks beyond the strategic complements and strategic substitutes cases see, for example, Kearns, Littman and Singh (2001), Kakade, Kearns and Ortiz (2004), Daskalakis, Goldberg and Papadimitriou (2009), and Papadimitriou and Roughgarden (2008).

<sup>12</sup>This is from Jackson (2008a, pp. 304-306), based on an obvious algorithm for finding a maximal inde-

the algorithm lists a set of the providers of the public good (the independent set of nodes),  $P_k$ , and a set of non-providers of the public good (who will not be in the eventual maximal independent set of nodes),  $NP_k$ , where the eventual maximal independent set will be the final  $P_k$ . In terms of finding an equilibrium to the best-shot game, the final  $P_k$  is the list of players who take action 1, and the final  $NP_k$  is the set of players who take action 0.

Step 1: Pick some node  $i$  and let  $P_1 = \{i\}$  and  $NP_1 = N_i(\mathbf{g})$ .

Step  $k$ : Iterate by picking one of the players  $j$  who is not yet assigned to sets  $P_{k-1}$  or  $NP_{k-1}$ .

Let  $P_k = P_{k-1} \cup \{j\}$  and  $NP_k = NP_{k-1} \cup N_j(\mathbf{g})$ .<sup>13</sup>

End: Stop when  $P_k \cup NP_k = N$ .

More generally, one might ask the question of whether it is possible to find the “best” equilibrium in the best-shot game. Given that in every equilibrium all players get a payoff of either 1 or  $1 - c$ , minimizing the number of players who pay the cost  $c$  would be one metric via which to rank equilibria. As discussed by Dall’Asta, Pin and Ramezanpour (2011), finding such equilibria can be difficult but finding them (approximately) through an intuitive class of mechanisms that tradeoff accuracy against speed is possible.

There are other games of strategic substitutes where at least some equilibria are also easy to find.

### EXAMPLE 3 A “Weakest-Link” Public Goods Game

Another local-public goods example of a game on a network is based on what are known as “weakest-link” public goods games (see Hirshleifer, 1983).

Here each player chooses some level of public good contribution (so  $A_i \subset \mathbb{R}_+$ ) and the payoff to a player is the *minimum* action taken by any player in his or her neighborhood (in contrast to the maximum, as in the best-shot game). We have:

$$u_i(a_i, a_{N_i(\mathbf{g})}) = \min_{j \in N_i(\mathbf{g}) \cup \{i\}} \{a_j\} - c(a_i)$$

where  $c$  is an increasing, convex and differentiable cost function.

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pendent set.

<sup>13</sup>Note that this is well-defined, since no neighbors of  $j$  can be in  $P_{k-1}$  as otherwise  $j$  would have been in  $NP_{k-1}$ .

In the case where there is a smallest  $a^*$  such that  $c'(a^*) \geq 1$ , and each player has at least one neighbor in the network  $g$ , any profile of actions where every player chooses the same contribution  $a_i = a^*$  is an equilibrium of this game. Note that in a network in which every player has at least one neighbor, everyone playing  $a_i = 0$  is also an equilibrium, and so the game will have multiple equilibria when it is nondegenerate.

### 3.2.3 Games with Strategic Substitutes, Continuous Action Spaces and Linear Best-Replies

We have seen that equilibria in games with strategic substitutes are difficult to characterize and multiple equilibria rather than unique equilibrium are the rule. If, however, the best-reply functions are linear, then some further results can be obtained, both in terms of characterization of equilibria and comparative statics. Bramoullé and Kranton (2007a) and Bramoullé, Kranton, and D'Amours (2012) study these type of games. In their models, players experiment to obtain new information and benefit from their neighbors' experimentation. Each player  $i$  selects an action  $a_i \geq 0$ , and obtains a payoff  $u_i(\mathbf{a}, \mathbf{g})$  that depends on the action profile  $\mathbf{a}$  and on the underlying network  $\mathbf{g}$ , in the following way:

$$u_i(\mathbf{a}, \mathbf{g}) = v \left( a_i + \phi \sum_{j=1}^n g_{ij} a_j \right) - c a_i \quad (1)$$

where  $v(\cdot)$  is an increasing, differentiable and strictly concave function on  $\mathbb{R}_+$  and  $c > 0$  is the constant marginal cost of own action such that  $v'(0) > c > v'(x)$  for some  $x > 0$ . As in Bramoullé and Kranton (2007a), consider the case when  $\phi = 1$ . This is clearly a game with (strict) strategic substitutes since

$$\frac{\partial u_i(\mathbf{a}, \mathbf{g})}{\partial a_i a_j} = v'' \left( a_i + \sum_{j=1}^n g_{ij} a_j \right) < 0$$

Denote by  $a^*$  the action level of a player who experiments by him/herself, i.e.  $a^* = v'^{-1}(c)$ . Then, the best-reply function, for each individual  $i$ ,  $i$ 's best response to  $\mathbf{a}_{-i}$  is linear and given by:

$$a_i^* = \begin{cases} a^* - \sum_{j=1}^n g_{ij} a_j & \text{if } a^* > \sum_{j=1}^n g_{ij} a_j \\ 0 & \text{if } a^* \leq \sum_{j=1}^n g_{ij} a_j \end{cases}$$

We can distinguish between two types of equilibria. An action profile  $\mathbf{a}$  is *specialized* if players actions are such that  $a_i = 0$  or  $a_i = a^*$  for every  $i$ . A player for which  $a_i = a^*$  is a

“specialist.” An action profile  $\mathbf{a}$  is *distributed* when all players choose a positive action less than the individually optimal action level:  $0 < a_i < a^*, \forall i \in N$ . *Hybrid* equilibria are other than these extremes.

Because actions are strategic substitutes, maximal independent sets are a natural notion in this model (see Section 3.2.2 above). Indeed, in equilibrium, no two specialists can be linked. Hence, specialized equilibria are characterized by this structural property of a network, i.e. the specialists are equal to a maximal independent set of the network. A result of Bramoullé and Kranton (2007a) can be stated as follows:

**PROPOSITION 1** *A specialized profile is a Nash equilibrium of the above game if and only if its set of specialists is a maximal independent set of the structure  $\mathbf{g}$ . Since for every  $\mathbf{g}$  there exists a maximal independent set, there always exists a specialized Nash equilibrium.*

Figure 3 illustrates this proposition for a star-shaped network with 4 players. Observe that, in a star network, there are two maximal independent sets: the one that only includes the central player and the one that includes all peripheral players. As a result, using Proposition 1, there are two specialized equilibria: (i) the center is a specialist and provides action  $a^*$  while peripheral players choose actions of 0 (Figure 3, left panel); (ii) the center chooses action 0 while peripheral players are specialists and exert effort  $a^*$  each (Figure 3, right panel).

Bramoullé, Kranton, and D’Amours (2012) provide more results on these types of games. Denote by  $\mu_0(\mathbf{g})$ , the lowest eigenvalue of the adjacency matrix  $\mathbf{g}$ . They show that if  $\phi < -1/\mu_0(\mathbf{g})$ , there exists a unique Nash equilibrium of this game while, when  $\phi > -1/\mu_0(\mathbf{g})$ , there always exists a corner equilibrium, i.e. for some player  $i$ ,  $a_i^* = 0$ . In terms of comparative statics, they also show that any increase in  $\phi$  or any additional link to the network leads to an equilibrium with lower total actions. Thus, while some players may increase their actions, the decreases dominate.

### 3.3 Two-Action Games on Networks

The class of (complete information) games on networks where players can choose one of two actions, so that  $A_i = \{0, 1\}$  for all  $i$ , is an important class of games in terms of its applications, one that has been widely studied, and one that allows us to see some general insights. It includes coordination games, and generally all sort of games where players choose



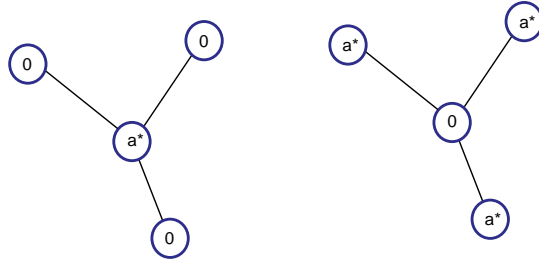


Figure 3: Equilibria in a local public good game

whether to do something (adopt a new technology, participate in something, provide a public good effort) or not. These were called *graphical games* by Kearns, Littman and Singh (2001) and Kakade, Kearns, and Ortiz (2004) who studied the complexity of finding equilibria in such games.

In particular, let us concentrate on a class of such games that are referred to as “semi-anonymous” by Jackson (2008a). These games are not fully anonymous because players are connected in a network and only care about their neighbors’ behaviors, but players care about their neighbors equally. That is, they care only about how many of their neighbors take action 1 and how many take action 0, but not which particular neighbors choose 1 versus 0. Thus, such games are anonymous except to the extent of the interaction patterns governed by the network. In addition, in keeping with the semi-anonymity, we can consider payoff functions that do not depend on a player’s identity, but only on how many neighbors that he or she has.

Such games then have some nice properties and best responses are easily described.

Letting  $d$  be the a player’s degree, in the case of strategic complements there is a threshold  $t(d)$  such that if more than  $t(d)$  neighbors choose action 1 then the player prefers to choose action 1, while if fewer than  $t(d)$  neighbors choose 1 then the player prefers to choose 0. It is possible to have situations where an individual is exactly indifferent at the threshold, although that would not occur for generic specifications of payoff functions. Analogously, for the case of strategic substitutes, there is also a threshold, but the best response of the player is reversed, so that he or she prefers to take action 0 if more than  $t(d)$  neighbors take action 1, and prefers action 1 if fewer than  $t(d)$  neighbors take action 1.

The majority game (example 1) is a game of strategic complements where the threshold is  $d/2$ , whereas the best-shot public good game (example 2) is a game of strategic substitutes where the threshold is anything between 0 and 1.

### 3.3.1 Changes in Behaviors as the Network Varies

The threshold expression of the two-action semi-anonymous games on networks allows us to easily deduce a few useful comparative statics.

For example, as discussed Galeotti, Goyal, Jackson, Vega-Redondo and Yariv (2010),<sup>14</sup> it is easy to see that in games of complements where the threshold is nonincreasing in degree, adding links will lead to (weakly) higher actions as players will have higher numbers of neighbors taking action 1.

**PROPOSITION 2** *Consider a semi-anonymous two-action game of strategic complements on a network  $(N, \mathbf{g})$  and such that the threshold for taking action 1 is nonincreasing as a function of degree (so that  $t(d+1) \leq t(d)$  for each  $d$ ). If  $\mathbf{g}'$  is obtained by adding links to the network  $\mathbf{g}$  (so that  $\mathbf{g} \subset \mathbf{g}'$ ), then for any equilibrium  $\mathbf{a}$  under  $\mathbf{g}$ , there exists an equilibrium  $\mathbf{a}'$  under  $\mathbf{g}'$  such that  $\mathbf{a}' \geq \mathbf{a}$ , so that all players play at least as high an action under  $\mathbf{a}'$  as under  $\mathbf{a}$ .*

The case of strategic substitutes does not lead to the same clear-cut conclusions since adding links can change the structure of payoffs in unpredictable ways. Decreasing actions for some players can lead to increasing actions for others in the case of strategic substitutes, and so changing network structure leads to more complex changes in behavior (see Figure 2 in Galeotti et al., 2010, for an illustration of this and Jackson, 2008a, for additional examples and results).

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<sup>14</sup>See the working paper version, or Jackson (2008a) for details.

The comparative statics in equilibrium behavior become clearer in games with incomplete information, as detailed in Section 5 below.

### 3.3.2 Coordination Games

Perhaps one of the most important and extensively studied classes of games of strategic complements on networks are coordination games. It is an important class of games because of its many applications: including the choice of a language, a technology, whether to buy a new product, adopt a particular behavior, and so forth; when there are complementarities in actions between friends or acquaintances.

The majority game is an example of this class, but more generally the threshold for wanting to match one’s neighbors may differ from fifty percent, depending both on the payoff to matching versus failing to match one’s neighbors. A standard representation of such a game would be as follows (normalizing the payoff to action 0 to zero and then keeping track of the difference in payoffs):

	1	0
1	$(b, b)$	$(-c, 0)$
0	$(0, -c)$	$(0, 0)$

where  $b$  and  $c$  are both strictly positive. Thus, coordinating on action 1 is overall better for society, but involves some risk from miscoordination  $-c$ . A player has to choose either 0 or 1 and then matches against each neighbor. Here there is a threshold fraction  $q = \frac{c}{c+b}$  such that action 1 is a best response for a given player if and only if at least a fraction  $q$  of the player’s neighbors choose 1. This fraction is the same for all players independent of their degrees. Thus, this is a game of strategic complements where the threshold in terms of numbers of neighbors of degree  $d$  is simply  $qd$ .

Here the Pareto efficient equilibrium (and sometimes referred to as the payoff dominant equilibrium) is for all players to play action 1. However, reaching this play will depend on what players expect their neighbors to play. In setting where  $q > 1/2$  (so  $c > b$ ), then action 0 is said to be the “risk dominant” action (as named by Harsanyi and Selten, 1988), in the sense that if a player had a uniform prior over other players’ plays - so an equal chance of each other player playing 0 or 1, then action 0 would be the best expected payoff maximizing choice for the player. In a case where  $q < 1/2$ , then action 1 is both risk dominant and payoff dominant (Pareto efficient).

We know from Theorem 1 that the set of equilibria form a lattice, and here the maximum equilibrium is where all players choose action 1 while the minimum equilibrium is where all players choose action 0.

What else can we deduce about the set of equilibria? Do there exist equilibria where some players choose 1 and others choose 0? What happens if we start with some small initial seed of players choosing 1 and others choosing 0, and then iterate on best replies? Morris (2000) provides some answers to these questions.<sup>15</sup>

If we let  $S$  be the set of players who play action 1 in an equilibrium, then it is clear that each player in  $S$  must have at least a fraction  $q$  of his/her neighbors in the set  $S$ , and also each player outside of  $S$  must a fraction of no more than  $q$  of his/her neighbors in  $S$ , or equivalently has a fraction of at least  $1 - q$  of his/her neighbors outside of  $S$ .

To capture this, given  $1 \geq r \geq 0$ , Morris (2000) defined the set of nodes  $S$  to be *r-cohesive* with respect to a network  $(N, \mathbf{g})$  if each node in  $S$  has at least a fraction  $r$  of its neighbors in  $S$ . That is,  $S$  is *r-cohesive* relative to  $\mathbf{g}$  if

$$\min_{i \in S} \frac{|N_i(\mathbf{g}) \cap S|}{d_i(\mathbf{g})} \geq r, \quad (2)$$

where  $0/0$  is set to 1.

The *cohesiveness* of a given set  $S$  relative to a network  $(N, \mathbf{g})$  is then the maximum  $r$  such that  $S$  is *r-cohesive*. The following proposition of Morris (2000) follows directly:

**PROPOSITION 3** *Consider a network  $(N, \mathbf{g})$  and a coordination game as described above. There exists an equilibrium where action 1 is played by  $S \subset N$  and action 0 by  $N \setminus S$  if and only if  $S$  is  $q$ -cohesive and such that its complement  $N \setminus S$  is  $(1 - q)$ -cohesive.*

Cohesiveness provides enough of a “separation” in a network for different behaviors to exist on different parts of a network.

With this proposition as background, it is then easy to see how behavior might spread in a network. In particular, start from a network  $(N, \mathbf{g})$  with all players choosing action 0. Next, “infect” a set of players by switching them to play action 1 (and they can never switch back). Next, let players (other than the initially infected) best respond to the current actions of their neighbors, switching players to action 1 if their payoffs are at least as good

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<sup>15</sup>The analysis here follows Jackson’s (2008a) adaptation of Morris’s results to a finite population setting. See also Ellison (1993) and Blume (1993) for some related analyses.

with action 1 as with action 0 against the actions of the other players. Repeat this process starting from the new actions, and stop at a stage where no new players change to action 1. If there is some set of players  $S$  whose initial infection leads to all players taking action 1 under the best response dynamics, then we say that there is a *contagion from  $S$* .

Define a set  $S$  to be *uniformly no more than  $r$ -cohesive* if there is no nonempty subset of  $S$  that is more than  $r$ -cohesive. We then can deduce the following proposition.

**PROPOSITION 4** *Consider a network  $(N, \mathbf{g})$  and a coordination game as described above. Contagion from the set  $S$  occurs if and only if its complement is uniformly no more than  $(1 - q)$ -cohesive.*

The proof of this proposition is straightforward: If the complement of  $S$  has a subset  $S'$  that is more than  $(1 - q)$ -cohesive, then  $S'$  will all play 0 under the process above, at every step. Thus, it is necessary for the complement to be uniformly no more than  $(1 - q)$ -cohesive in order to have contagion to all remaining players outside of  $S$ . To see that this condition is also sufficient, note that if the complement is uniformly no more than  $(1 - q)$ -cohesive, then the complement is no more than  $(1 - q)$ -cohesive. This means that there must be at least one player in the complement who has at least a fraction of  $q$  of his or her neighbors in  $S$ . So, at the first step, at least one player changes strategies. Subsequently, at each step, the set of players who have not yet changed strategies is no more than  $(1 - q)$ -cohesive, and so some player must have at least  $q$  neighbors who are playing 1 and will change. Thus, as long as some players have not yet changed, the process will have new players changing, and so every player must eventually change.

The cohesion conditions used in the above results can be difficult to check in a network, but the concept is still quite useful in terms of outlining what is necessary to maintain different behaviors in a society: one needs a sufficient schism between (at least) two groups. This is closely related to homophily in the network, whereby a group's members tend to be relatively more connected to each other than with outsiders (see e.g. McPherson, Smith-Lovin and Cook, 2001; Currarini, Jackson, and Pin, 2009, 2010; Golub and Jackson, 2010, 2012a,b,c; Jackson, 2008b; Jackson and Lopez-Pintado 2011; Bramoullé, Currarini, Jackson, Pin, and Rogers, 2012; Bramoullé, Kranton and D'Amours, 2012).<sup>16</sup>

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<sup>16</sup>Chwe (1999, 2000) studies related classes of games, but instead where players care not only about the behavior of their own neighborhood, but of the collective, with an application to collective actions like

### 3.3.3 Stochastically Stable Play in Coordination Games on Networks

While analyses of equilibria in complete information games on networks provide some insights and intuitions, they often lack the randomness (and heterogeneity) that are pervasive in many applications. In fact, in some cases adding randomness can substantially refine the multiplicity of equilibria that exist in the complete information noiseless setting. To see how this works, let us visit some of the literature on stochastic stability in coordination games.

Let us consider the following variation on settings considered by Kandori, Mailath and Rob (1993) and Young (1993). Start with  $(N, \mathbf{g})$  being the complete network, so that each player plays with all others and chooses strategy either 1 or 0 in the coordination game from Section 3.3.2. Players play the game repeatedly over time. However, they are myopic: in each period a player chooses a best reply to the play of the others in the previous period. So far, this is similar to our analysis of the previous section except for the complete network. Indeed, here there are two obvious trajectories of society: all playing 1 and all playing 0 in every period. In fact, it is clear that if the starting proportion is sufficiently different from  $q$  (at least  $1/(n-1)$  above or below suffices), then one of these two states is reached after the first period, and even away from that, if  $q$  is not too close to  $1/2$  (at least  $1/n$  above or below suffices), then one of the states is reached after at most two periods.<sup>17</sup>

However, to refine the predictions of the system, let us add slight perturbations to the choice of actions: in each period, a player's choice is reversed (1 to 0 or 0 to 1) with a slight probability  $\varepsilon > 0$ , and this is done independently across players. Players still best reply to what was played in the previous period, but sometimes their chosen action is changed by some exogenous perturbation. This dynamic process is now a Markov chain that is irreducible and aperiodic. This process thus has very nice ergodic properties that are easy

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participating in a large protest or revolution. His motivation is similar, in looking at the network structure and investigating where it is possible to sustain a collective action when players can only be sure of their neighbors' play and worry about a failed collective action: so they will only participate if they are sure that some quota is exceeded. Granovetter (1978) and Schelling (1978) also provide dynamic models to analyze such issues, but avoiding a clear linkage of outcomes to the social network structure in society. In particular, a snowball effect might be generated only if there are initially enough activists who are willing to participate, independently of others decisions. Once these activists start the process.

<sup>17</sup>Some starting configurations that have a fraction of players sufficiently close to  $q$  playing 1, when  $q$  is sufficiently close to  $1/2$ , can cycle endlessly, where two approximately equal-sized groups continue to switch back and forth between 1 and 0, each miscoordinating with the other group over time.

to deduce from standard results in Markov theory. In fact, without even relying on much of the mathematics, it is easy to see some basic properties of this process:

- Any configuration of play is possible in any period since there is at least an  $\varepsilon$  probability of each player playing either action in every period, given that the perturbations are independent across players.
- If  $\varepsilon$  is sufficiently small then once the population has sufficiently more or fewer than a fraction of  $q$  playing 1, then the next period with a high probability, all players play 1 or all players play 0.
- Thus, the process will continue to exhibit all possible plays infinitely often over time, but, as  $\varepsilon$  tends to 0, it will spend most of its time with all players choosing the same action.

The important insight that emerged from Kandori, Mailath, and Rob (1993) and Young (1993) is that if  $q$  is not too close to  $1/2$  (more than  $1/(n-1)$  away), then, as  $\varepsilon$  tends to 0:

- if  $q > 1/2 + 1/(n-1)$ , then the fraction of periods where all players play action 0 tends to one, and
- if  $q < 1/2 + 1/(n-1)$ , then the fraction of periods where all players play action 1 tends to one.

Thus, if  $q$  is not too close to  $1/2$ , then stochastic stability picks out the risk dominant action: the action that is better for a player when others play uniformly at random.

The intuition behind this result is relatively straightforward. Considering that most of the time all players choose the same action, the important determinant of the system is how long it takes to transition from an extreme where all play 0 to the other extreme where all play 1, and vice versa. Here  $q$  enters. If we start with all playing 0, then we need roughly  $qn$  “perturbations” of the best replies within a single period before the system transitions to all playing 1, and, in the reverse direction, at least  $(1-q)n$  perturbations are needed. For small  $\varepsilon$ , the probabilities of this happening are effectively on the order of  $\varepsilon^{qn}$  and  $\varepsilon^{(1-q)n}$ , respectively. As  $\varepsilon$  tends to 0, if  $q < 1/2$  then the first becomes infinitely more likely than the second, and if  $q > 1/2$  then the reverse is true.

Thus, introducing very slight trembles to a coordination game can end up making very stark predictions in terms of the ergodic play of a society. What are some issues with this? One is that as  $\varepsilon$  tends to 0, it could take an enormous amount of time to reach a transition. For example, suppose that  $q = 1/3$  and we start with all players choosing 0. Even though we know that in the “long run,” for small  $\varepsilon$  the process spends most of its time with all players choosing 1, it could take a very long time before we leave the starting state of everybody playing 0. In a sense, the “short run” could last a very long time in a large society, as many very unlikely simultaneous trembles could be needed to transition.<sup>18</sup>

The more interesting case in terms of many applications is when one turns to a setting of less than complete networks. A first work in this direction was Ellison (1993) who pointed out that this could have important implications for the speed of transitions from one state to another. In particular, as pointed out by Ellison (1993), for example, if players are connected in a “circle” so that each player has two adjacent neighbors (so,  $i$ ’s neighbors in the network are  $i - 1$  and  $i + 1$ , with the wrap-around convention that  $n$  is connected to 1), then the transition can happen much more quickly. Instead of waiting for  $n/3$  perturbations when  $q = 1/3$ , if two adjacent players are perturbed, then that is enough to spread behavior 1 to the whole society; something which is much more likely to happen. Thus, network structures significantly affect the times to transition in the network.

The network structure can also have profound effects on the long-run distribution of play, as shown by Jackson and Watts (2002a,b).

Jackson and Watts (2002b) propose a model in which the interaction pattern is an arbitrary network  $\mathbf{g}$ . In each period  $t$ , a player  $i$  chooses an action  $a_i^t \in \{1, 0\}$  and then receives a payoff equals to

$$u_i(\mathbf{a}^t, \mathbf{g}) = \sum_{j \neq i} g_{ij} v_i(a_i^t, a_j^t)$$

where  $v_i(a_i^t, a_j^t)$  is a payoff that depends on the actions chosen. The dynamic process is described as follows. In each period one player is chosen at random (with equal probability across players) to update his/her strategy. A player updates his/her strategy myopically, best responding to what the other players with whom he/she interacts did in the previous period. There is also a probability  $1 > \varepsilon > 0$  that a player trembles, and chooses a strategy

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<sup>18</sup>Another issue is raised by Bergin and Lipman (1996) who point out that this depends on the error structure being the same  $\varepsilon$  regardless of circumstances. If the probability of trembles is enriched arbitrarily, then the relative probability of various transitions can be made arbitrary and the selection is lost.



that he/she did not intend to. Thus, with probability  $1 - \varepsilon$ , the strategy chosen is  $a_i^t = \arg \max_{a_i} u_i(a_i, a_{-i}^{t-1}, \mathbf{g})$  and with probability  $\varepsilon$  the strategy is  $a_i^t \neq \arg \max_{a_i} u_i(a_i, a_{-i}^{t-1}, \mathbf{g})$ . The probabilities of trembles are identical and independent across players, strategies, and periods. Again, this is a well-defined Markov chain where the state is the vector of actions  $\mathbf{a}^t$ , that are played in period  $t$ . Since this process is aperiodic and irreducible, the Markov chain has a unique stationary distribution, denoted  $\mu^\varepsilon(\mathbf{a})$ .

If we consider the *complete network* where all players are connected to each other so that each player has  $n - 1$  links, then the analysis is as above from the models of Kandori, Mailath and Rob (1993) and Young (1993)<sup>19</sup> and the unique stochastically stable state is the risk-dominant equilibrium (provided that  $q$  is not too close to  $1/2$ ).

However, instead let us consider a *star network* where player 1 is the at the center of the star, connected to every other player but where other players are only connected to player 1. Jackson and Watts (2002b) show that, in this case, there are two stochastically stable states: one where all players play 1 and the other where all players play 0; regardless of  $q$ . Note that now peripheral players  $i > n$  care only about what player 1 is playing, and they update to play whatever 1 played last period when called on to update. Player 1, in contrast, cares about what all the players are doing. Thus one tremble by player 1 can lead from a network where all play 1 to one where all play 0. Thus starting from either equilibrium of all play 1 or all play 0, we need only one tremble to have updating lead naturally to the other equilibrium. As the relative number of trembles is the important factor in determining the set of stochastically stable states, both of these states are stochastically stable.

The important difference from the complete network setting is that regardless of the starting condition, if the central player changes actions, that can change the subsequent play of all other players. Thus, all playing 1 and all playing 0 are both more “easily” destabilized, and the long-run distribution does not place all weight on just one equilibrium.<sup>20</sup>

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<sup>19</sup>The Jackson and Watts process involves changes of players’ actions one at a time, which is easier to handle when the network is endogenized (see Section 6.1), but differs from the Kandori, Mailath and Rob (1993) and Young (1993) where decisions and trembles are simultaneous. That difference is largely inconsequential when applied to a fixed network.

<sup>20</sup>The results on this do depend on the perturbation structure. As shown by Blume (1993) and Young (1998), if one uses a structure in which errors are exponentially proportional to the payoff loss of making the error, then the center makes infinitely fewer errors in moving away from 1 than away from 0 when  $q < 1/2$  (and vice versa when  $q > 1/2$ ), which can then restore the original conclusion that the risk-dominant play is

This shows that network structure can influence the play of a society, even in terms of selecting among (strict) Nash equilibria. Exactly what emerges depends on a number of details and how the long-run play depends on the structure of the network can be quite complex. Nonetheless, there are some clean results that emerge when one further enriches the structure to allow players also to choose the network along with their play, as discussed in Section 6.

## 4 A Model with Continuous Actions, Quadratic Payoffs, and Strategic Complementarities

Although models with finite numbers of actions capture many applications, there are others that are well-approximated by continuous actions; for instance in which players have choices of how much time or effort to exert in an activity like education or crime. In this section, we analyze a simple model of a game with strategic complements where the utility function is linear-quadratic. An advantage of this formulation is that it allows for an easy characterization of equilibrium as a function of network structure.

### 4.1 The Benchmark Quadratic Model

Consider a game in which players decide how much time or effort to exert in some activity, denoted  $a_i \in \mathbb{R}_+$ . The payoff to player  $i$  as a function of the action profile and network,  $u_i(\mathbf{a}, \mathbf{g})$ , is given by

$$u_i(\mathbf{a}, \mathbf{g}) = \alpha a_i - \frac{1}{2}a_i^2 + \phi \sum_{j=1}^n g_{ij}a_i a_j, \quad (3)$$

where  $\alpha, \phi > 0$ . In this formulation, players are ex ante homogeneous in terms of observable characteristics (i.e. they all have the same  $\alpha, \phi$ ) and their heterogeneity stems entirely from their position in the network. The first two terms of (3),  $\alpha a_i - \frac{1}{2}a_i^2$ , give the benefits and costs of providing the action level  $a_i$ . The last term of this utility function,  $\phi \sum_{j=1}^n g_{ij}a_i a_j$ , reflects the strategic complementarity between friends' and acquaintances' actions and own action. This peer effect depends on the different locations of players in the network  $\mathbf{g}$ . When

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obtained. However, that then relies on some errors by individual players becoming infinitely more likely than others, which seems implausible if errors can result from things like errors in calculation or some unmodeled preference shock, and so forth.

$i$  and  $j$  are directly linked, i.e.  $g_{ij} = 1$ , the cross derivative is  $\phi > 0$  and reflects *strategic complementarity* in efforts. When  $i$  and  $j$  are not direct friends, i.e.  $g_{ij} = 0$ , this cross derivative is zero.

Ballester, Calvó-Armengol, and Zenou (2006) have analyzed this game, determining its unique Nash equilibrium in pure strategies (when it exists, provided that  $\phi$  is not too large). The first-order necessary condition for each player  $i$ 's choice of action to maximize his or her payoff is :

$$\frac{\partial u_i(\mathbf{a}, \mathbf{g})}{\partial a_i} = \alpha - a_i + \phi \sum_{j=1}^n g_{ij} a_j = 0,$$

which leads to:

$$a_i^* = \alpha + \phi \sum_{j=1}^n g_{ij} a_j^*. \quad (4)$$

In matrix form:

$$\mathbf{a}^* = \alpha \mathbf{1} + \phi \mathbf{g} \mathbf{a}^*$$

where  $\mathbf{1}$  is the column vector of 1 and  $\mathbf{g}$  is the adjacency matrix. Solving this leads to:

$$\mathbf{a}^* = \alpha (\mathbf{I} - \phi \mathbf{g})^{-1} \mathbf{1} \quad (5)$$

where  $\mathbf{I}$  is the identity matrix (and provide the inverse of  $(\mathbf{I} - \phi \mathbf{g})$  is well-defined).

#### 4.1.1 Katz-Bonacich Network Centrality and Strategic Behavior

The equilibrium action profile of this quadratic model is related to a centrality measure. Let

$$\mathbf{M}(\mathbf{g}, \phi) = (\mathbf{I} - \phi \mathbf{g})^{-1} = \sum_{k=0}^{+\infty} \phi^k \mathbf{g}^k$$

**Definition 1 (Katz, 1953; Bonacich, 1987)** Given  $\mathbf{w} \in \mathbb{R}_+^n$  and  $\phi \geq 0$ , the vector of weighted Katz-Bonacich centralities for a network  $\mathbf{g}$  relative to  $\phi$  is:

$$\mathbf{b}_{\mathbf{w}}(\mathbf{g}, \phi) = \mathbf{M}(\mathbf{g}, \phi) \mathbf{w} = (\mathbf{I} - \phi \mathbf{g})^{-1} \mathbf{w} = \sum_{p=0}^{+\infty} \phi^p \mathbf{g}^p \mathbf{w}. \quad (6)$$

In particular, when  $\mathbf{w} = \mathbf{1}$ , the *unweighted* Katz-Bonacich centrality of node  $i$  is  $b_{1,i}(\mathbf{g}, \phi) = \sum_{j=1}^n M_{ij}(\mathbf{g}, \phi)$ , and counts the *total* number of walks in  $\mathbf{g}$  starting from  $i$ , discounted exponentially by  $\phi$ . It is the sum of all loops  $M_{ii}(\mathbf{g}, \phi)$  from  $i$  to  $i$  itself, and of all the outer walks  $\sum_{j \neq i} M_{ij}(\mathbf{g}, \phi)$  from  $i$  to every other player  $j \neq i$ , that is:

$$b_{1,i}(\mathbf{g}, \phi) = M_{ii}(\mathbf{g}, \phi) + \sum_{j \neq i} M_{ij}(\mathbf{g}, \phi).$$

By definition,  $M_{ii}(\mathbf{g}, \phi) \geq 1$ , and thus  $b_i(\mathbf{g}, \phi) \geq 1$ , with equality when  $\phi = 0$ .

#### 4.1.2 Nash equilibrium

We now characterize the Nash equilibrium of the game where players choose their effort level  $a_i \geq 0$  simultaneously. Let  $\mu_1(\mathbf{g})$  denote the spectral radius of  $\mathbf{g}$ .

**PROPOSITION 5** *If  $\phi\mu_1(\mathbf{g}) < 1$ , the game with payoffs (3) has a unique (and interior) Nash equilibrium in pure strategies given by:*

$$\mathbf{a}^* = \alpha \mathbf{b}_1(\mathbf{g}, \phi). \quad (7)$$

This result shows that Katz-Bonacich centrality embodies the feedback calculations that underlie equilibrium behavior when utility functions are linear-quadratic. In (3), the local payoff interdependence is restricted to direct network contacts. In equilibrium, however, this local payoff interdependence spreads indirectly through the network. The condition  $\phi\mu_1(\mathbf{g}) < 1$  stipulates that local complementarities must be small enough compared to own concavity, which prevents excessive feedback which can lead to the absence of a finite equilibrium solution.

There are different ways of proving Proposition 5. Ballester, Calvó-Armengol, and Zenou (2006) show that the condition  $\phi\mu_1(\mathbf{g}) < 1$  ensures that the matrix  $\mathbf{I} - \phi\mathbf{g}$  is invertible by Theorem III of Debreu and Herstein (1953, p. 601). Then, using (5), it is straightforward to see that the optimal efforts are equal given by Katz-Bonacich centrality. Finally, they rule out corner solutions to establish uniqueness. Since  $\frac{\partial u_i(\mathbf{a}, \mathbf{g})}{\partial a_i} |_{\mathbf{a}=\mathbf{0}} = \alpha > 0$ , then  $\mathbf{a} = \mathbf{0}$  cannot be a Nash Equilibrium. It is also clear from the first-order condition that a deviation to positive effort is profitable if only some subgroup  $S \subset N$  of players provides zero effort.<sup>21</sup> As a result, the Nash equilibrium is unique and each  $a_i^*$  is interior.

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<sup>21</sup>From Theorem I of Debreu and Herstein (1953, p. 600),  $\phi\mu_1(\mathbf{g}) < 1$  also guarantees that  $\mathbf{I} - \phi\mathbf{g}_S$  is invertible for  $\mathbf{g}_S \subset \mathbf{g}$ , where  $\mathbf{g}_S$  is the adjacency matrix for the subgroup  $S \subset N$  of players.

Another simple way of proving Proposition 5 is to observe that this game is a potential game (as defined by Monderer and Shapley, 1996)<sup>22</sup> with potential function:<sup>23</sup>

$$\begin{aligned} P(\mathbf{a}, \mathbf{g}, \phi) &= \sum_{i=1}^n u_i(\mathbf{a}, \mathbf{g}) - \frac{\phi}{2} \sum_{i=1}^n \sum_{j=1}^n g_{ij} a_i a_j \\ &= \sum_{i=1}^n \alpha a_i - \frac{1}{2} \sum_{i=1}^n a_i^2 + \frac{\phi}{2} \sum_{i=1}^n \sum_{j=1}^n g_{ij} a_i a_j, \end{aligned}$$

or in matrix form:

$$\begin{aligned} P(\mathbf{a}, \mathbf{g}, \phi) &= \alpha \mathbf{a}^\top \mathbf{1} - \frac{1}{2} \mathbf{a}^\top \mathbf{a} + \mathbf{a}^\top \frac{\phi}{2} \mathbf{g} \mathbf{a} \\ &= \alpha \mathbf{a}^\top \mathbf{1} - \frac{1}{2} \mathbf{a}^\top (\mathbf{I} - \phi \mathbf{g}) \mathbf{a}. \end{aligned}$$

It is well-known (see e.g., Monderer and Shapley, 1996) that solutions of the program  $\max_{\mathbf{a}} P(\mathbf{a}, \mathbf{g}, \phi)$  are a subset of the set of Nash equilibria. This program has a unique interior solution if the potential function  $P(\mathbf{a}, \mathbf{g}, \phi)$  is strictly concave on the relevant domain. The Hessian matrix of  $P(\mathbf{a}, \mathbf{g}, \phi)$  is easily computed to be  $-(\mathbf{I} - \phi \mathbf{g})$ . The matrix  $\mathbf{I} - \phi \mathbf{g}$  is positive definite if for all non-zero  $\mathbf{a}$

$$\mathbf{a}^\top (\mathbf{I} - \phi \mathbf{g}) \mathbf{a} > 0 \Leftrightarrow \phi < \left( \frac{\mathbf{a}^\top \mathbf{g} \mathbf{a}}{\mathbf{a}^\top \mathbf{a}} \right)^{-1}.$$

By the Rayleigh-Ritz theorem, we have  $\mu_1(\mathbf{g}) = \sup_{\mathbf{a} \neq \mathbf{0}} \left( \frac{\mathbf{a}^\top \mathbf{g} \mathbf{a}}{\mathbf{a}^\top \mathbf{a}} \right)$ . Thus a necessary and sufficient condition for having a strict concave potential is that  $\phi \mu_1(\mathbf{g}) < 1$ , as stated in Proposition 5.

To illustrate the results and to describe a *social multiplier*, let us consider the case of a dyad, i.e.  $n = 2$ . In that case, the utility (3) can be written as:

$$u_i(\mathbf{a}, \mathbf{g}) = \alpha x_i - \frac{1}{2} a_i^2 + \phi a_i a_j$$

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<sup>22</sup>A game is a potential game if there is a function  $P : X \rightarrow \mathbb{R}$  such that, for each  $i \in N$ , for each  $x_{-i} \in X_{-i}$ , and for each  $x_i, z_i \in X_i$ ,

$$u_i(x_i, x_{-i}) - u_i(z_i, x_{-i}) = P(x_i, x_{-i}) - P(z_i, x_{-i})$$

<sup>23</sup>Here the potential  $P(\mathbf{x}, \mathbf{g}, \phi)$  is constructed by taking the sum of all utilities, a sum that is corrected by a term which takes into account the network externalities exerted by each player  $i$ .

If there were no social interactions, the unique Nash equilibrium would be:

$$a_1^* = a_2^* = \alpha,$$

while, for a dyad where the two individuals are linked to each other (i.e.  $g_{12} = g_{21} = 1$ ), the unique Nash equilibrium is given by (if  $\phi < 1$ ):<sup>24</sup>

$$a_1^* = a_2^* = \frac{\alpha}{1 - \phi}.$$

In the dyad, complementarities lead to an effort level above the equilibrium value for an isolated player ( $a_1^* = a_2^* = \alpha$ ). The factor  $1/(1 - \phi) > 1$  is often referred to as a *social multiplier*.<sup>25,26</sup>

In terms of comparative statics, it is clear that, by standard strategic-complementarity arguments, increasing the number of links of any player raises his/her effort, and for this specification of utility functions also increases his/her payoff.

### 4.1.3 Welfare

It is obvious that the Nash equilibrium in such a game is inefficient, as there are positive externalities in efforts. There is *too little effort* at the Nash equilibrium as compared to the social optimum outcome, because each individual ignores the positive impact of her effort on others. As a result, there are benefits from subsidizing effort, as we now detail.

To analyze welfare, we first relate the equilibrium utility of player  $i$  to Katz-Bonacich

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<sup>24</sup>It is straightforward to check that:

$$\mathbf{b}_1(\mathbf{g}, \phi) = (\mathbf{I} - \phi\mathbf{g})^{-1} \mathbf{1} = \begin{pmatrix} 1/(1 - \phi) \\ 1/(1 - \phi) \end{pmatrix}$$

<sup>25</sup>See, for instance, Glaeser, Sacerdote and Scheinkman (2003), and references therein.

<sup>26</sup>Belhaj and Deroian (2011a) have extended the previous model by considering two substitute activities, so that player  $i$  chooses both  $a_{1,i}$  and  $a_{2,i}$  such that  $a_{1,i} + a_{2,i} = 1$ . The payoff is:

$$u_i(\mathbf{a}_1, \mathbf{a}_2, \mathbf{g}) = \alpha_1 a_{1,i} - \frac{1}{2} a_{1,i}^2 + \alpha_2 a_{2,i} - \frac{1}{2} a_{2,i}^2 + \sum_{j=1}^n g_{ij} (\phi_1 a_{1,i} a_{1,j} + \phi_2 a_{2,i} a_{2,j}).$$

The model incorporates local synergies and both lower and upper bounds on efforts, which facilitates the analysis of equilibrium. Belhaj and Deroian (2011a) study both interior and corner solutions, and provide comparative statics with respect to activity cost.

centrality:

$$\begin{aligned} u_i(\mathbf{a}^*, \mathbf{g}) &= \alpha a_i^* - \frac{1}{2} a_i^{*2} + \phi \sum_{j=1}^n g_{ij} a_i^* a_j^* \\ &= a_i^* \left( \alpha + \phi \sum_{j=1}^n g_{ij} a_j^* \right) - \frac{1}{2} a_i^{*2}. \end{aligned}$$

By (4) where  $a_i^* = \alpha + \phi \sum_{j=1}^n g_{ij} a_j^*$ :

$$u_i(\mathbf{a}^*, \mathbf{g}) = a_i^{*2} - \frac{1}{2} a_i^{*2} = \frac{1}{2} a_i^{*2} = \frac{1}{2} [b_{1,i}(\mathbf{g}, \phi)]^2. \quad (8)$$

The total equilibrium welfare (i.e. the sum of all equilibrium utilities) is then:

$$\mathcal{W}(\mathbf{a}^*, \mathbf{g}) = \mathbf{u}(\mathbf{a}^*, \mathbf{g}) \cdot \mathbf{1} = \frac{1}{2} \mathbf{b}_1^\top(\mathbf{g}, \phi) \mathbf{b}_1(\mathbf{g}, \phi). \quad (9)$$

Following Helsley and Zenou (2011) and Ballester, Calvó-Armengol, and Zenou (2011), let us show that the Nash equilibrium (5) is not efficient. For that, the planner chooses  $a_1, \dots, a_n$  that maximize total welfare, that is:

$$\max_{\mathbf{a}} \mathcal{W}(\mathbf{a}, \mathbf{g}) = \max_{a_1, \dots, a_n} \sum_{i=1}^n u_i(\mathbf{a}, \mathbf{g}) = \max_{a_1, \dots, a_n} \left\{ \sum_{i=1}^n \left[ \alpha a_i - \frac{1}{2} a_i^2 \right] + \phi \sum_{i=1}^n \sum_{j=1}^n g_{ij} a_i a_j \right\}.$$

The first-order conditions are that for each  $i = 1, \dots, n$ :

$$\alpha - a_i + \phi \sum_j g_{ij} a_j + \phi \sum_j g_{ji} a_j = 0,$$

which implies that (since  $g_{ij} = g_{ji}$ ):

$$a_i^O = \alpha + 2\phi \sum_j g_{ij} a_j^O, \quad (10)$$

where the superscript  $O$  refers to the “social optimum”. In matrix form,

$$\mathbf{a}^O = \alpha (\mathbf{I} - 2\phi \mathbf{g})^{-1} \mathbf{1} = \alpha \mathbf{b}_1(\mathbf{g}, 2\phi). \quad (11)$$

Examination of (7) and (11) shows that the two solutions differ and that the Nash equilibrium effort of each individual  $i$  is inefficiently low.

Analogously to the derivation of (9), we see that

$$\mathcal{W}(\mathbf{a}^O, \mathbf{g}) = \frac{1}{2} \mathbf{b}_1^\top(\mathbf{g}, 2\phi) \mathbf{b}_1(\mathbf{g}, 2\phi). \quad (12)$$

Here a Pigouvian *subsidy* can lead individuals to choose the socially optimal effort levels. Let us derive this optimal subsidy that can restore the first-best outcome (10).

The timing is as follows. First, the government announces the per-effort subsidy  $s_i(\mathbf{a}) \geq 0$  for each individual  $i = 1, \dots, n$ . Then each individual  $i$  chooses effort  $a_i$  to maximize his or her payoff accounting for the subsidy. Because of the planner's subsidy, individual  $i$ 's utility is now given by:

$$u_i(\mathbf{a}, \mathbf{g}; s_i) = [\alpha + s_i] a_i - \frac{1}{2} a_i^2 + \phi \sum_{j=1}^n g_{ij} a_i a_j \quad (13)$$

Helsley and Zenou (2011) and Ballester, Calvó-Armengol, and Zenou (2011) show the following result:

**PROPOSITION 6** *Assume that  $2\phi\mu_1(\mathbf{g}) < 1$ . If subsidies that satisfy  $s_i = \phi \sum_j g_{ij} a_j^O$  are in place, then the first-best efforts form a Nash equilibrium. In equilibrium, this per-effort subsidy is equal to:  $s_i = \frac{1}{2} [b_{1,i}(\mathbf{g}, 2\phi) - \alpha]$ . Thus, the planner gives a higher per-effort subsidy to more central players in the network.*

This proposition does not address the financing of the subsidy. Here since the anticipated total cost of the subsidies is calculable ex ante (knowing the network and players' preferences), the planner can raise the value of the subsidies by any tax scheme that is independent of  $\mathbf{a}$ .

## 4.2 The model with global congestion

In some applications, in addition to local complementarities, players might experience global competitive effects. When global competition or congestion matters (e.g., see our discussion of applications of this model to crime, Cournot competition, etc.), we can modify the utility function (3) to allow for global competitive effects. One such formulation is:

$$u_i(\mathbf{a}, \mathbf{g}) = \alpha a_i - \frac{1}{2} a_i^2 + \phi a_i \sum_{j=1}^n g_{ij} a_j - \gamma a_i \sum_{j=1}^n a_j \quad (14)$$

where global congestion is captured by the factor  $-\gamma \sum_{j=1}^n a_j$  that multiplies player  $i$ 's action. Compared to the previous case without global substitutes where no player had interest in providing zero effort (see (4)), it is now possible that corner solutions arise at the Nash



equilibrium. Ballester, Calvó-Armengol, and Zenou (2006) show that, if  $\phi\mu_1(\mathbf{g}) < 1 + \gamma$ , then there exists a unique interior Nash equilibrium given by:

$$\mathbf{a}^* = \frac{\alpha}{1 + \gamma [1 + b_1(\mathbf{g}, \phi/(1 + \gamma))]} \mathbf{b}_1(\mathbf{g}, \phi/(1 + \gamma)), \quad (15)$$

where  $b_1(\mathbf{g}, \phi/(1 + \gamma)) = \mathbf{1}^\top \mathbf{M}(\mathbf{g}, \phi, \gamma) \mathbf{1}$  (where  $\mathbf{M}(\mathbf{g}, \phi, \gamma) = [(1 + \gamma) \mathbf{I} - \phi \mathbf{g}]^{-1}$ ) is the sum of unweighted Bonacich centralities of all players, i.e.,

$$b_1(\mathbf{g}, \phi/(1 + \gamma)) = b_{1,1}(\mathbf{g}, \phi/(1 + \gamma)) + \dots + b_{n,1}(\mathbf{g}, \phi/(1 + \gamma)).$$

Thus, the model is easily adapted to include such effects, provided they also fit into a linear-quadratic form.

In terms of comparative statics, the standard complementarity argument for the benchmark model, which implies that equilibrium efforts increase (on a player by player basis) in any component with in which links are added, does not apply here because of the competition effect. However, the following result regarding total aggregate effort can be shown:

**PROPOSITION 7** *Let  $\mathbf{g}$  and  $\mathbf{g}'$  be symmetric and such that  $\mathbf{g}' \geq \mathbf{g}$  and  $\mathbf{g}' \neq \mathbf{g}$ . If  $\phi\mu_1(\mathbf{g}) < 1$  and  $\phi'\mu_1(\mathbf{g}') < 1$ , then  $\sum_i a^*(\mathbf{g}')_i > \sum_i a^*(\mathbf{g})_i$ .*

Proposition 7 shows that an increase in network relationships (in a partial order sense) increases aggregate activity. This result is due to the fact that direct connections are the source of local complementarities. As a result, players obtain more local complementarities in  $\mathbf{g}'$  than in  $\mathbf{g}$ , and equilibrium aggregate activity is thus higher in  $\mathbf{g}'$  than in  $\mathbf{g}$ . Symmetry of the adjacency is not really needed here: Using the Farkas' Lemma, Belhaj and Deroïan (2011b) have shown that, even with asymmetric  $\mathbf{g}$  and  $\mathbf{g}'$ , Proposition 7 holds. They show that if the transposed system admits a positive solution, then any perturbation of the linear system that enhances complementarities leads to an increase of average effort.

### 4.3 The model with ex ante heterogeneity

So far, players were only heterogeneous due to their positions in the network. Let us now extend the model to allow for players who are also heterogeneous in terms of characteristics; e.g., age, race, gender, education, etc., which is important when one would like to bring the

model to the data and to test it. One way to introduce some aspects of heterogeneity while still maintaining tractability is to allow the utility function of player  $i$  to be given by:

$$u_i(\mathbf{a}, \mathbf{g}) = \alpha_i a_i - \frac{1}{2} a_i^2 - \gamma \sum_{j=1}^n a_i a_j + \phi \sum_{j=1}^n g_{ij} a_i a_j, \quad (16)$$

where  $\alpha_i$  depends on the characteristics of player  $i$ . In many empirical applications (see e.g. Calvó-Armengol, Patacchini, and Zenou 2009, for education, or Patacchini and Zenou, 2012, for crime),  $\alpha_i$  could also include the average characteristics of individual  $i$ 's neighbors, i.e. the average level of parental education of  $i$ 's friends, etc., which are referred to as contextual effects. In particular, assume that

$$\alpha_i = \sum_{h=1}^H \beta_h x_i^h + \frac{1}{d_i(\mathbf{g})} \sum_{h=1}^H \sum_{j=1}^n \gamma_h g_{ij} x_j^h \quad (17)$$

where  $(x_i^h)_h$  is a set of  $H$  variables accounting for individual characteristics and  $\beta_h, \gamma_h$  are parameters.

Let  $b_{\alpha}(\mathbf{g}, \phi/(1+\gamma)) = \boldsymbol{\alpha}^{\top} \mathbf{M}(\mathbf{g}, \phi, \gamma) \boldsymbol{\alpha}$  be the weighted sum of weighted Bonacich centralities of all players, i.e.  $b_{\alpha} = \alpha_1 b_{\alpha,1} + \dots + \alpha_n b_{\alpha,n}$ . Calvó-Armengol, Patacchini and Zenou (2009) show the following.

#### PROPOSITION 8

- (i) If  $\boldsymbol{\alpha} = \alpha \mathbf{1}$ , then the game has a unique Nash equilibrium in pure strategies if and only if  $\phi \mu_1(\mathbf{g}) < 1 + \gamma$ . This equilibrium  $\mathbf{a}^*$  is interior and described by (15).
- (ii) If  $\boldsymbol{\alpha} \neq \alpha \mathbf{1}$ , then let  $\bar{\alpha} = \max \{\alpha_i \mid i \in N\} > \underline{\alpha} = \min \{\alpha_i \mid i \in N\} > 0$ . In this case, if  $\phi \mu_1(\mathbf{g}) + n\gamma(\bar{\alpha}/\underline{\alpha} - 1) < 1 + \gamma$ , then this game has a unique Nash equilibrium in pure strategies and it is interior and described by:

$$\mathbf{a}^* = \frac{1}{1+\gamma} \left[ \mathbf{b}_{\alpha}(\mathbf{g}, \phi/(1+\gamma)) - \frac{\gamma b_{\alpha}(\mathbf{g}, \phi/(1+\gamma))}{1+\gamma+\gamma b_{\mathbf{1}}(\mathbf{g}, \phi/(1+\gamma))} \mathbf{b}_{\mathbf{1}}(\mathbf{g}, \phi/(1+\gamma)) \right]. \quad (18)$$

## 4.4 Some Applications of the Quadratic Model

Part of the usefulness of the quadratic model is that it can provide explicit relationships between network structure and behavior, and thus can make sharp predictions in context. Let us examine some specific contexts where it generates further results.

#### 4.4.1 Crime

Criminal activity is, to some extent, a group phenomenon, and the crime and delinquency are related to positions in social networks (see e.g. Sutherland, 1947, Sarnecki, 2001 and Warr, 2002). Indeed, delinquents often have friends who have committed offenses, and social ties are a means of influence to commit crimes. In fact, the *structure* of social networks matters in explaining an individual's delinquent behavior: in adolescents' friendship networks, Haynie (2001), Patacchini and Zenou (2008), Calvó-Armengol, Patacchini and Zenou (2005) find that individual Katz-Bonacich centrality together with the density of friendship links affect the delinquency-peer association. This suggests that the properties of friendship networks should be taken into account to better understand peer influence on delinquent behavior and to craft delinquency-reducing policies.

Glaeser, Sacerdote and Scheinkman (1996) were among the first to model criminal social interactions. Their model clearly and intuitively shows how criminal interconnections act as a social multiplier on aggregate crime. They consider, however, only a specific network structure where criminals are located on a circle. Following Calvó-Armengol and Zenou (2004) and Ballester, Calvó-Armengol, and Zenou (2010), we examine a model that can encompass any social network. Let us reinterpret the local-complementarity model with global congestion described in Section 4 in terms of criminal activities. Let  $a_i$  be the criminal effort level of delinquent  $i$ . Following Becker (1968), assume that delinquents trade off the costs and benefits of delinquent activities. The expected delinquency gains to delinquent  $i$  are:

$$u_i(\mathbf{a}, \mathbf{g}) = \underbrace{y_i(\mathbf{a})}_{\text{benefits}} - \underbrace{p_i(\mathbf{a}, \mathbf{g})}_{\text{prob.caught}} \underbrace{f}_{\text{fine}}, \quad (19)$$

where

$$\begin{cases} y_i(\mathbf{a}) = \alpha'_i a_i - \frac{1}{2} a_i^2 - \gamma \sum_{j=i}^n a_i a_j \\ p_i(\mathbf{a}, \mathbf{g}) = p_0 a_i \max \left\{ 1 - \phi' \sum_{j=1}^n g_{ij} a_j, 0 \right\} \end{cases}$$

The proceeds  $y_i(\mathbf{a})$  include the *global* payoff interdependence. The expected cost of criminal activity,  $p_i(\mathbf{a}, \mathbf{g})f$ , is positively related to  $a_i$  as the apprehension probability increases with one's involvement in delinquency. The crucial assumption is that delinquents' activity has complementarities with their friends' criminal activity, but a criminal also faces global competition as well as increased expected costs as he or she increases activity.

By direct substitution:

$$u_i(\mathbf{a}, \mathbf{g}) = (\alpha'_i - p_0 f) a_i - \frac{1}{2} a_i^2 - \gamma \sum_{j=i}^n a_i a_j + p_0 f \phi' \sum_{j=1}^n g_{ij} a_i a_j. \quad (20)$$

It should be clear that this utility (20) and (14) are equivalent if  $\alpha_i = \alpha'_i - p_0 f > 0$  and  $\phi = p_0 f \phi'$ . Thus, we can apply Proposition 8 (ii): if  $p_0 f \phi' \mu_1(\mathbf{g}) + n\gamma(\bar{\alpha}/\underline{\alpha} - 1) < 1 + \gamma$ , then there exists a unique Nash equilibrium:

$$\mathbf{a}^* = \frac{1}{1 + \gamma} \left[ \mathbf{b}_\alpha(\mathbf{g}, p_0 f \phi' / (1 + \gamma)) - \frac{\gamma b_\alpha(\mathbf{g}, p_0 f \phi' / (1 + \gamma))}{1 + \gamma + \gamma b_1(\mathbf{g}, p_0 f \phi' / (1 + \gamma))} \mathbf{b}_1(\mathbf{g}, p_0 f \phi' / (1 + \gamma)) \right].$$

An interesting aspect of a network analysis of criminal activity is that it allows us to derive a *key player* policy: If we could choose to push one player's activity to 0 and remove all his/her existing links, which player's removal would lead to the highest overall criminal activity reduction when examining the resulting impact on others' behaviors?<sup>27</sup> Given that delinquent removal has both a direct and an indirect effect on the group outcome, the choice of the key player results from considering both effects. In particular, the key player need not necessarily be the one with the highest criminal activity (the one with the highest Bonacich centrality measure). Formally, the planner's problem is :

$$\max_i \{a^*(\mathbf{g}) - a_{-i}^*(\mathbf{g}_{-i})\},$$

where  $a^* = \mathbf{a}^{*\top} \mathbf{1}$ . The program above is equivalent to:

$$\min_i \{a_{-i}^*(\mathbf{g}_{-i})\} \quad (21)$$

Following Ballester, Calvó-Armengol, and Zenou (2006, 2010), define a network centrality measure  $d_\alpha(\mathbf{g}, \phi)$  that corresponds to this program. Recall that  $b_\alpha(\mathbf{g}, \phi) = \sum_{i=1}^n b_\alpha(\mathbf{g}, \phi)_i$ .

## Definition 2

(i) If  $\alpha = \alpha \mathbf{1}$ , then the intercentrality measure of player  $i$  is:

$$d_1(\mathbf{g}, \phi)_i = b_1(\mathbf{g}, \phi) - b_1(\mathbf{g}_{-i}, \phi) = \frac{[b_1(\mathbf{g}, \phi)_i]^2}{M_{ii}(\mathbf{g}, \phi)}. \quad (22)$$

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<sup>27</sup>See Dell (2011) for related analyses of spillover effects and reactions of criminals to enforcement with respect to Mexican drug cartels.

(ii) If  $\alpha \neq \alpha \mathbf{1}$ , then the intercentrality measure of player  $i$  is:

$$d_{\alpha}(g, \phi)_i = b_{\alpha}(\mathbf{g}, \phi) - b_{\alpha}(\mathbf{g}_{-i}, \phi) = \frac{b_{\alpha}(\mathbf{g}, \phi)_i \sum_{j=1}^n M_{ji}(\mathbf{g}, \phi)}{M_{ii}(\mathbf{g}, \phi)}. \quad (23)$$

The intercentrality measure  $d_{\alpha}(\mathbf{g}, \phi)_i$  of delinquent  $i$  is the sum of  $i$ 's centrality measures in  $\mathbf{g}$ , and  $i$ 's contribution to the centrality measure of every other delinquent  $j \neq i$  also in  $\mathbf{g}$ . The following result establishes that intercentrality captures the two dimensions of the removal of a delinquent from a network: the direct effect on criminal activity and the indirect effect on others' criminal activities.

**PROPOSITION 9** *A player  $i^*$  is the key player who solves (21) if and only if  $i^*$  has the highest intercentrality in  $g$ :  $d_{\alpha}(\mathbf{g}, \phi)_{i^*} \geq d_{\alpha}(\mathbf{g}, \phi)_i$ , for all  $i = 1, \dots, n$ .*

The key player policy is such the planner perturbs the network by removing a delinquent and all other delinquents are allowed to change their effort after the removal but the network is not "rewired", and so is a *short-term policy*.<sup>28</sup>

The individual Nash equilibrium efforts are proportional to the Katz-Bonacich centrality network measures, while the key player is the delinquent with the highest intercentrality measure. This difference is not surprising, as the equilibrium index derives from individual considerations, while the intercentrality measure solves the planner's optimality collective concerns. The measure  $d_{\alpha}(\mathbf{g}, \phi)$  goes beyond the measure  $\mathbf{b}_{\alpha}(\mathbf{g}, \phi)$  by keeping track of the cross-contributions that arise.

#### 4.4.2 Education

The influence of peers on education outcomes has been studied extensively (see e.g. Evans et al., 1992; Sacerdote, 2001; Zimmerman, 2003; for a survey, see Sacerdote, 2011).<sup>29</sup> Following

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<sup>28</sup>Liu, Patacchini, Zenou and Lee (2012) develop a dynamic network formation model where, once a delinquent is removed from the network, the remaining delinquents can form new links. They do not have a formula like the intercentrality measure here, but test the model empirically. The invariant and dynamic models often lead to the same key player in the AddHealth data.

<sup>29</sup>For recent models of human capital investment, social mobility and networks, see Calvó-Armengol and Jackson (2009), Jackson (2007), and Bervoets, Calvó-Armengol and Zenou (2012). The papers study the intergenerational relationship between parents' and offsprings' educational outcomes. They show that a positive correlation emerges between their education status, *without any direct interaction*, because of the overlap in the surroundings that influence their education decisions.

Calvó-Armengol, Patacchini and Zenou (2009), let us reinterpret the benchmark model of Section 4 in terms of education:  $a_i$  is an educational effort. Applying Proposition 5, if  $\phi\mu_1(\mathbf{g}) < 1$ , the equilibrium effort levels are  $a_i^* = b_1(\mathbf{g}, \phi)_i$ .

In practice, we are not interested in the effort per se, but in the outcome of effort; i.e. the educational achievement obtained by each student. Let the educational achievement of student  $i$  be denoted by  $\beta_i^*$  and described by

$$\beta_i^* = \alpha_i + a_i^*$$

where  $\alpha_i$  is again a parameter that depends on the characteristics of student  $i$ . Thus,

$$\beta_i^* = \alpha_i + b_1(\mathbf{g}, \phi)_i.$$

Calvó-Armengol, Patacchini and Zenou (2009) estimate this last equation using the AddHealth data. Estimating such an equation is useful because it allows the authors to disentangle between the effects of own characteristics  $\alpha_i$  from a student's location in the network on the educational outcome (i.e. the grades of the student).<sup>30</sup> In terms of magnitude, they find that a standard deviation increase in the Katz-Bonacich centrality of a student increases his or her performance in school by about seven percent of one standard deviation.

Again, one can ask what an optimal subsidy would be in this setting. Ballester, Calvó-Armengol, and Zenou (2011) determine the optimal per-effort subsidy for each student in order to maximize total output (i.e. the sum of all students' efforts). They take the planner's objective to be to choose  $\mathbf{s}$  to maximize

$$a^* = \sum_i a_i^* = \sum_i b_{\alpha+\mathbf{s}}(\mathbf{g}, \phi)_i = \mathbf{1}^\top \mathbf{M}(\alpha + \mathbf{s})$$

subject to a budget constraint:

$$\sum_i s_i a_i^* \leq B \quad \text{and} \quad s_i \geq 0, \forall i,$$

where  $B > 0$  is the budget that the government has for this policy.

Assume that  $\alpha_1 < \dots < \alpha_n$ . Ballester, Calvó-Armengol, and Zenou (2011) demonstrate the following result.

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<sup>30</sup>To evaluate the effect of Katz-Bonacich centrality, they first estimate  $a_i^* = \alpha_i + \phi \sum_{j=1}^n g_{ij} a_j^*$ . This leads to an estimated value of  $\phi$ , denoted by  $\hat{\phi}$ , for each network of friends in the AddHealth data. Remember that  $\hat{\phi}$  not only captures the intensity of complementarities in the utility function (3) but also the decay factor in the Katz-Bonacich centrality, i.e. how much weight to put on distant walks in the network.

**PROPOSITION 10** *Assume  $\phi\mu_1(\mathbf{g}) < 1$  and  $\alpha_n < \sqrt{\frac{4B+b_w\alpha}{b_1}}$ . There exists a unique solution and it is interior and described by*

$$\mathbf{s}^* = \frac{1}{2} \left[ \sqrt{\frac{4B + b_w\alpha}{b_1}} \mathbf{1} - \boldsymbol{\alpha} \right]. \quad (24)$$

The optimal subsidy  $\mathbf{s}^*$  for each student depends on the heterogeneity  $\alpha_i$  and on overall network characteristics (i.e.,  $b_1$  and  $b_w\alpha$ ), but not on the individual position in the network. Ballester Calvó-Armengol, and Zenou (2011) show that

$$\frac{\partial s_i^*}{\partial \alpha_i} \begin{matrix} \geq \\ \leq \end{matrix} 0 \Leftrightarrow b_1 \begin{matrix} \leq \\ \geq \end{matrix} \frac{(b_{\alpha,i})^2}{4(4B + b_w\alpha)},$$

implying that the planner does not always give higher per-effort subsidies to either the more or less advantaged students in terms of background characteristics.

#### 4.4.3 Industrial organization

Another application of the model is to collaboration between firms. Collaboration takes a variety of forms which includes creation and sharing of knowledge about markets and technologies (via joint research activities), setting market standards and sharing facilities (such as distribution channels). Recent studies (see for instance Hagedoorn, 2002) suggest that joint ventures seem to have become less popular while non-equity contractual forms of R&D partnerships, such as joint R&D pacts and joint development agreements, have become more prevalent modes of inter-firm collaborations. Little is known, however, about how the structure of these collaboration networks affects the profits of firms. The linear-quadratic model of Section 4 can be applied to this question.<sup>31</sup>

Following, König, Liu and Zenou (2012), we study competition in quantities a la Cournot between  $n$  firms with homogeneous products and price, given the following linear inverse market demand for each firm  $i$ :

$$p_i = \theta_i - \sum_{j \in N} q_j,$$

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<sup>31</sup>There is a growing literature on industrial organization and networks, much of which has focused on network formation. See, e.g., Manshadi and Johari (2009) for Bertrand competition; Goyal and Moraga (2001), Goyal and Joshi (2003), Deroian and Gannon (2006), Goyal, Konovalov and Moraga-Gonzalez (2008), and Westbrook (2010) for Cournot competition and R&D networks; Candogan, Bimpikis and Ozdaglar (2011) and Bloch and Quérou (2011) for monopoly pricing.

where  $\theta_i > 0$  and  $q_j$  is the quantity produced by firm  $j$ . Assume throughout that  $\theta_i$  is assumed to be large enough for all  $i = 1, \dots, n$  so that price and quantities are always strictly positive. The marginal cost of each firm  $i$  is:

$$c_i(\mathbf{g}) = \phi_0 - \phi \left[ \sum_{j=1}^n g_{ij} q_j \right], \quad (25)$$

where  $\phi_0 > 0$  represents the firm's marginal cost when it has no links and  $\phi > 0$  influences the cost reduction induced by each link formed by a firm. The marginal cost of each firm  $i$  is a decreasing function of the quantities produced by all firms  $j \neq i$  that have a direct link with firm  $i$ . Assume that  $\phi_0$  is large enough so that  $c_i(g) \geq 0, \forall i \in N, \forall \mathbf{g}$ . The profit function of firm  $i$  in a network  $\mathbf{g}$  is thus:

$$\pi_i(\mathbf{q}, \mathbf{g}) = p q_i - c_i(\mathbf{g}) q_i = \alpha_i q_i - q_i^2 - \sum_{j \neq i} q_i q_j + \phi \sum_{j=1}^n g_{ij} q_i q_j \quad (26)$$

where  $\alpha_i \equiv \theta_i - \phi_0$ . We can apply Proposition 8 (ii) to obtain the following result. Let  $\bar{\alpha} > \underline{\alpha} > 0$ . If  $\phi \mu_1(\mathbf{g}) + n(\bar{\alpha}/\underline{\alpha} - 1)/3 < 1$ , then this game has a unique Nash equilibrium in pure strategies  $\mathbf{q}^*$ , which is interior and given by:

$$\mathbf{q}^* = \mathbf{b}_a(\mathbf{g}, \phi) - \frac{b_a(\mathbf{g}, \phi)}{1 + b_1(\mathbf{g}, \phi)} \mathbf{b}_1(\mathbf{g}, \phi). \quad (27)$$

This characterizes the equilibrium quantities produced by firms as a function of their position in the network (again, as measured by their Katz-Bonacich centrality).

Proposition 7 then provides comparative statics for the total activity in the industry. Overall industry output increases when the network of collaboration links expands, irrespective of the network geometry and the number of additional links.<sup>32</sup>

#### 4.4.4 Cities

Helsley and Zenou (2011) use the quadratic model to investigate the interaction between the social space (i.e. the network) and the geographical space (i.e. the city). For that, they consider a city which consists of two locations, a *center*, where all social interactions occur,

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<sup>32</sup>For the case of a linear inverse demand curve, this generalizes the findings in Goyal and Moraga-González (2001) and Goyal and Joshi (2003), where monotonicity of industry output is established for the case of regular collaboration networks, where each firm forms the same number of bilateral agreements. For such regular networks, links are added as multiples of  $n$ , as all firms' connections are increased simultaneously.



and a *periphery*. Players are located in either the center or the periphery. The distance between the center and the periphery is normalized to one.

Let  $l_i \in \{0, 1\}$  represent the location of player  $i$ , defined as his/her distance from the interaction center. Players derive utility from a numeraire good  $z_i$  and interactions with others according to the utility function:

$$u_i(\mathbf{a}, \mathbf{g}) = z_i + \alpha a_i - \frac{1}{2} a_i^2 + \phi \sum_{j=1}^n g_{ij} a_i a_j \quad (28)$$

where  $a_i$  (effort) is the number of visits that player  $i$  makes to the center. Thus, utility depends on the visit choice of player  $i$ , the visit choices of other players and on player  $i$ 's position in the social network  $\mathbf{g}$ . Players located in the periphery must travel to the center to interact with others. Letting  $I$  represent income and  $\tau$  represent marginal transport cost, budget balance implies that expenditure on the numeraire is  $z_i = I - \tau l_i a_i$ . Using this expression to substitute for  $z_i$  in (28), we obtain:

$$u_i(\mathbf{a}, \mathbf{g}) = I + \alpha_i a_i - \frac{1}{2} a_i^2 + \phi \sum_{j=1}^n g_{ij} a_i a_j, \quad (29)$$

where  $\alpha_i = \alpha - \tau l_i > 0$ . Consider the case where location  $l_i$  of each player  $i$  is fixed. Then, using Proposition 8 (ii) for  $\gamma = 0$ , if  $\phi \mu_1(\mathbf{g}) < 1$ , there is a unique (and interior) Nash equilibrium given by:  $\mathbf{a}^* = \mathbf{b}_\alpha(\mathbf{g}, \phi)$ . The Nash equilibrium number of visits  $a_i^*$  thus depends on position in the social network and geographic location. This implies that a player who is more central in the social network makes more visits to the interaction center in equilibrium, as he or she has more to gain from interacting with others and so exerts higher interaction effort for any vector of geographic locations.

Helsley and Zenou (2011) extend this model to allow players to choose between locating in the center and the periphery. They assume that because the center has more economic activity, there is an exogenous cost differential  $C > 0$  associated with locating at the center. This cost differential might arise from congestion effects or reflect a difference in location rent from competition among other activities for center locations. They show that, in equilibrium, players who are most central in the social network locate at the interaction center, while players who are less central in the social network locate in the periphery. This expresses the salient relationship between position in the social network and geographic location. If interaction involves costly transportation, then players who occupy more central positions in the social network have the most to gain from locating at the interaction center.

Applying the logic of Proposition 6, it is again clear that the Nash equilibrium outcome is inefficient and there will be too little social interaction. The first-best outcome can be restored if the planner subsidizes  $i$ 's activity with the optimal subsidy  $\phi \sum_j g_{ij} a_j$ . In this case, it is optimal for the planner to give higher subsidies to more central players in the social network.

#### 4.4.5 Conformity and Conspicuous Effects

Let us consider one last application of the quadratic model, with some modifications. In that model, it is the *sum* of friends' activities that impact a player's utility from increasing his or her action (3). This is clearly not always the right model, as it might be some other function of friends' activities that matters. Patacchini and Zenou (2012) alter the model so that it is the *average effort level* of friends that affects a player's marginal utility of own action.

Let  $\hat{g}_{ij} = g_{ij}/d_i(\mathbf{g})$ , and set  $\hat{g}_{ii} = 0$ . By construction,  $0 \leq \hat{g}_{ij} \leq 1$ .

Let  $\bar{a}_i$  the average effort of individual  $i$ 's friends: given by:

$$\bar{a}_i(\mathbf{g}) = \frac{1}{d_i(\mathbf{g})} \sum_{j=1}^n g_{ij} a_j = \sum_{j=1}^n \hat{g}_{ij} a_j. \quad (30)$$

Player  $i$ 's payoff is described by:

$$u_i(\mathbf{a}, \mathbf{g}) = \alpha a_i - \frac{1}{2} a_i^2 - \frac{\delta}{2} (a_i - \bar{a}_i(\mathbf{g}))^2 \quad (31)$$

with  $\delta \geq 0$ . The last term  $\delta (a_i - \bar{a}_i)^2$  is a standard manner of capturing conformity.<sup>33</sup> Each player wants to minimize the distance between his or her action and the average action of his or her reference group, where  $\delta$  is a parameter describing the taste for conformity.

First-order conditions imply that:

$$a_i^* = \frac{\alpha}{1 + \delta} + \frac{\delta}{(1 + \delta)} \sum_{j=1}^n \hat{g}_{ij} a_j^*$$

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<sup>33</sup>See, for instance, Kandel and Lazear (1992), where peer pressure arises when individuals deviate from a well-established group norm, e.g., individuals are penalized for working less than the group norm; Berman (2000), where praying is more satisfying the more average participants there are; and Akerlof (1980, 1997), Bernheim (1994), where deviations from an average action imply a loss of reputation and status. Calvó-Armengol and Jackson (2010) model explicit peer pressure (as a strategy in the game).

or in matrix form

$$\mathbf{a}^* = \frac{\alpha}{1 + \delta} \left( \mathbf{I} - \frac{\delta}{(1 + \delta)} \widehat{\mathbf{g}} \right)^{-1} \mathbf{1} = \frac{\alpha}{1 + \delta} \mathbf{b}_1(\widehat{\mathbf{g}}, \delta / (1 + \delta))$$

Using the AddHealth data, Liu, Patacchini and Zenou (2012) test which model (local average or local aggregate ) does a better job of matching behaviors with regards to effort in education, sport, and screen activities (e.g., video games) for adolescents in the United States. They find that peer effects are not significant for screen activities. For sports activities, they find that students are mostly influenced by the aggregate activity of their friends; while for education they show that both the aggregate performance of friends as well as conformity matter, although the magnitude of the effect is higher for the latter.<sup>34</sup>

Finally, we can provide a more general function for how own action interacts with the average of a player's friends actions. For that, we can adapt the models proposed by Clark and Oswald (1998) to include a network. Consider the following utility function:

$$u_i(\mathbf{a}, \mathbf{g}) = \alpha a_i - \frac{1}{2} a_i^2 + \phi v(a_i - \bar{a}_i(\mathbf{g}))$$

where  $\bar{a}_i$  is defined by (30),  $v(\cdot)$  is an increasing function of  $a_i - \bar{a}_i$  and  $v'' < 0$ . First-order conditions imply that if there is an interior equilibrium then it must satisfy:

$$a_i = \alpha + \phi v'(a_i - \bar{a}_i(\mathbf{g}))$$

and also that

$$\frac{\partial a_i}{\partial \bar{a}_i} = \frac{v''(a_i - \bar{a}_i(\mathbf{g}))}{-1 + \phi v''(a_i - \bar{a}_i(\mathbf{g}))}.$$

If  $v'' < 0$ , then a rise in others' efforts leads to an increase in own effort. This is due to the fact that if other individuals set a high  $\bar{a}_i(\mathbf{g})$ , that reduces  $a_i - \bar{a}_i(\mathbf{g})$ , and this increases the marginal benefit from effort  $a_i$  for those with such a comparison-concave utility. If  $v'' > 0$ , then a rise in others' efforts leads to a decrease in own effort.

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<sup>34</sup>Ghiglini and Goyal (2010) propose an alternative model with *conspicuous effects* so that individuals are happier the higher is their effort compared to that of their peers (direct links) and derive negative utility if they effort is below that of their peers. They also compare the local aggregate and the local average model in the context of a pure exchange economy where individuals trade in markets and are influenced by their neighbors. They found that with *aggregate* comparisons, networks matter even if all people have same wealth. With *average* comparisons networks are irrelevant when individuals have the same wealth. The two models are, however, similar if there is heterogeneity in wealth.

To illustrate this model, consider an example in which:

$$u_i(\mathbf{a}, \mathbf{g}) = \alpha a_i - \frac{1}{2}a_i^2 + \phi \frac{a_i}{\bar{a}_i(\mathbf{g})}, \quad (32)$$

and where  $\phi$  can be positive or negative. If  $\phi > 0$ , then there is a conspicuous effect since individuals increase their utility by having higher effort than their peers. If  $\phi < 0$ , then it becomes costly for individuals to differentiate themselves from their peers. First-order conditions imply that if there is an interior equilibrium then:

$$a_i^* = \alpha + \frac{\phi}{\bar{a}_i^*(\mathbf{g})}.$$

An advantage of (32) is that the characterization of the Nash equilibrium is easy.

## 5 Network Games with Incomplete Information

While settings with a fixed network are widely applicable, there are also many applications where players choose actions without fully knowing with whom they will interact. For example, learning a language, investing in education, investing in a software program, and so forth. These can be better modeled using the machinery of incomplete information games.

It is also important to point out that this class of games also helps in tractability relative to complete information games. The results on games on networks that we have outlined so far primarily rely on either games of strategic complements or in cases with a continuum of actions and quadratic payoff functions. The analysis of other games, and in particular of games of strategic substitutes, even with just two actions, is difficult in the context of complete information, but becomes much more tractable in the context of incomplete information, as we detail below.

### 5.1 Incomplete Information and contagion effects

The following discussion builds on the models of Jackson Yariv (2005, 2007, 2011), Jackson and Rogers (2007b), Sundararajan (2007), and Galeotti et al. (2010), primarily incorporating aspects of Jackson and Yariv (2007) and Galeotti et al. (2010).

### 5.1.1 A Model of Network Games with Incomplete Information

Instead of a fixed network, players are now unsure about the network that will be in place in the future, but have some idea of the number of interactions that they will have. To fix ideas, think of choosing whether to adopt a new software program that is only useful in interactions with other players who adopt the software as well, but without being sure of with whom one will interact in the future.

In particular, the set of players  $N$  is fixed, but the network  $(N, \mathbf{g})$  is unknown when players choose their actions. A player  $i$  knows his or her own degree  $d_i$ , when choosing an action, but does not yet know the realized network.

Players choose actions in  $\{0, 1\}$ , and we normalize the payoff to choosing 0 to be 0, and so effectively consider the difference in payoffs between choosing action 0 and 1. Player  $i$  has a cost of choosing action 1, denoted  $c_i$ . Player  $i$ 's payoff from action 1 when  $i$  has  $d_i$  neighbors and expects them each independently to choose 1 with a probability  $x$  is

$$v(d_i, x) - c_i,$$

and so action 1 is a best response for player  $i$  if and only if  $c_i \leq v(d_i, x)$ .

It is easy to see how this incorporates some of the games we considered earlier. For instance, in the case of a best-shot public goods game of Example 2.<sup>35</sup>

$$v(d_i, x) = (1 - x)^{d_i}$$

In the case of our coordination game from Section 3.3.2, the payoff is

$$v(d_i, x) = \sum_{m=0}^{d_i} B_{d_i}(m, x) [mb - (d_i - m)c]$$

where  $B_{d_i}(m, x)$  is the binomial probability of having exactly  $m$  neighbors out of  $d_i$  play action 1 when they independently choose action 1 with probability  $x$ .

The uncertainty about the network affects a player's beliefs about the play of his or her neighbors.

In particular, let us consider a case where a player's probability distribution over the types of neighbors that he or she faces is governed by beliefs about the distribution of other players

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<sup>35</sup>We have normalized the payoff to action 0 to 0, so this is the difference between action 1 and action 0. If no other player chooses action 1, then the difference in overall payoff is  $1 - c_i$  and otherwise it is  $-c_i$ .

types. To keep things simple, let us examine a case where  $c_i$  is described by an atomless distribution  $F$ , independently across players, and independently of a player's degree.

Let a player's beliefs about the degree of each of his or her neighbors be described by a degree distribution  $\tilde{P}(d)$ , independently across neighbors.<sup>36</sup> One example, is one in which players are matched uniformly at random, and players have degrees distributed according to some  $P$ , in which case the probability of matching to another player of degree  $d$  is  $P(d)d/E_P[d] = \tilde{P}(d)$ . So for instance, a player is twice as likely to be matched with someone who has two interactions as someone who has one. Galeotti et al. (2010) discuss the more general case where the distribution of neighbors' degrees can depend on own degree.

With this structure, we can then consider (*symmetric*) *Bayesian equilibria*: players choose an action based on their type  $d_i, c_i$ , and so let the strategy be denoted by  $\sigma(d_i, c_i) \in \{0, 1\}$ . Given the atomless distribution of costs, it is enough to consider pure strategy equilibria, as at most one (probability zero) cost type is ever indifferent for any given  $d_i$ .

A simple equation is then sufficient to characterize equilibria. Again, letting  $x$  be the probability that a randomly chosen neighbor chooses action 1, a player of type  $d, c$  plays action 1 if and only if (up to ties)

$$v(d, x) \geq c.$$

Thus,  $F(v(d, x))$  is the probability that a random (best-responding) neighbor of degree  $d$  chooses the action 1. A characterization of equilibrium is then that

$$x = \Phi(x) = \sum_d \tilde{P}(d)F(v(d, x)). \quad (33)$$

In cases where  $F$  and  $v$  are continuous, existence of equilibrium follows directly from the fact that the right hand side is a continuous function mapping from  $[0, 1]$  into  $[0, 1]$ .

It is easy to keep track of equilibria directly in terms of  $x$  since that ties down behaviors of all types of players (up to sets of measure 0).

### 5.1.2 Monotonicity of Equilibria

The nice aspect of the equilibria in the incomplete information setting, is that behavior can now be nicely ordered, depending on the properties of  $v$ . In many applications,  $v$  is either

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<sup>36</sup>When fixing a size of a society, only certain configurations of degrees are possible, and so in general there are some interdependencies in the degrees of any player's neighbors. This is thus a limiting case.

increasing in  $x$  (strict strategic complements) or decreasing in  $x$  (strict strategic substitutes). It also tends to be either increasing or decreasing in  $d$ , although there are other cases of interest (e.g., as in a game where the player cares precisely about the average payoff from play with neighbors).

The strategic substitute or complement nature of a game imply that strategies of players can be represented (up to indifferences) by threshold functions  $\tau(d_i, c_i)$ , so that  $i$  plays 1 if and only if  $x$  exceeds  $\tau$  in the case of complements or is below it in the case of substitutes.

The monotonicity in degree affects how the thresholds behave as a function of  $d$ , as seen in the following proposition, of which variations appear in Jackson and Yariv (2005, 2007) and Galeotti et al (2010).

**PROPOSITION 11** *If  $v$  is increasing in  $d$  for every  $x$ , then there exists a symmetric pure strategy Bayesian equilibrium such that equilibrium strategies are increasing in the sense that higher degree players have higher (or at least as high) probabilities of choosing action 1 compared to lower degree players, and correspondingly higher degree players have lower thresholds given the same  $c_i$ . Similarly, if  $v$  is decreasing in  $d$  for every  $x$ , then the reverse conclusion holds.*

These observations are useful in understanding dynamics of equilibria and comparative statics of equilibria in response to various changes in the primitives of the environment.

### 5.1.3 A Dynamic Best Reply Process

Let us consider a more general version of the contagion/diffusion process that we discussed in Section 3.3.2. At time  $t = 0$ , a fraction  $x^0$  of the population is exogenously and randomly assigned the action 1, and the rest of the population is assigned the action 0. At each time  $t > 0$ , each player<sup>37</sup> best responds to the distribution of players choosing the action 1 in period  $t - 1$  under the distributions described above. We work with expectations, or a mean-field approximation of the system.<sup>38</sup>

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<sup>37</sup>In contrast to our earlier discussion, in this process every player adjusts, including those assigned to action 1 at the outset.

<sup>38</sup>A mean-field version of a model is a deterministic approximation of the statistical system where interactions take place at their expected rates. See Vega-Redondo (2007) or Jackson (2008a) for some discussion of these techniques in network analysis.

In particular, let  $x^t$  denote the expected probability that a player's neighbor will choose action 1 at time  $t$  as a function of  $x^{t-1}$ :

$$x^t = \Phi(x^{t-1}) \equiv \sum_d \tilde{P}(d) F(v(d, x^{t-1})).$$

Equilibria are again fixed points, but now we view this as a dynamic. In a game of strategic complements, convergence of behavior from any starting point to some equilibrium is monotone, either upwards or downwards, and any player switches behaviors at most once. In a game of strategic substitutes, convergence is not ensured from some starting points in some cases, but there are still things that we can deduce about the equilibria.

We can see that there can exist multiple equilibria, as pictured in Figure 4.

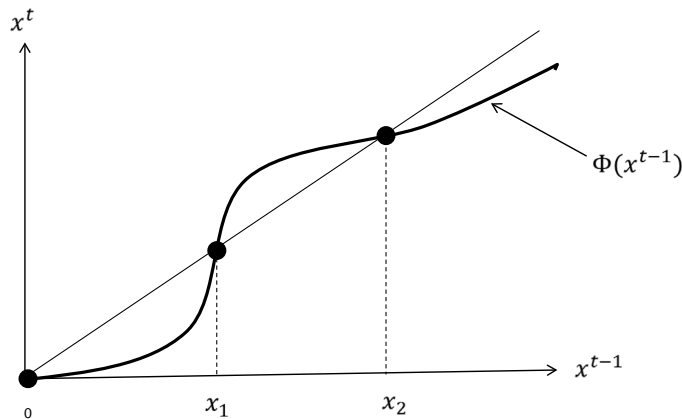


Figure 4: Best response curve and equilibria

Equilibria are points where the curve  $\Phi$  intersects the 45 degree line. In games of strategic substitutes,  $\Phi$  is a decreasing function, and so the equilibrium is unique. In contrast, in games of strategic complements,  $\Phi$  is increasing and can have multiple equilibria.<sup>39</sup>

<sup>39</sup>As pointed out in Jackson and Yariv (2007), some of these equilibria are robust to small perturbations



Galeotti et al. (2010) and Jackson and Yariv (2005, 2007) provide comparative statics in equilibria, as a function of various changes in networks and payoffs. In particular, it is easy to see that as one shifts a continuous  $\Phi$  upwards or downwards, equilibria shift in predictable ways. In particular, as  $\Phi$  is shifted upwards (so  $\bar{\Phi}(x) > \Phi(x)$  for all  $x$ , the greatest equilibrium under  $\bar{\Phi}$  moves shifts upward (unless it was already 1), as pictured in Figure 5:<sup>40</sup> To understand what leads to such shifts, let us examine  $\Phi$  as it is defined in (33).

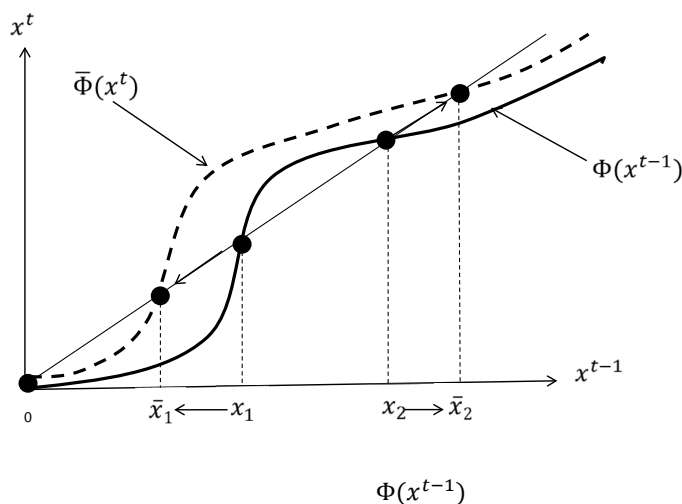


Figure 5: The effects of shifting  $\Phi$

First, let us consider changes in the relative costs of the actions; for instance, an increase and others are not, depending on the shape of  $\Phi$  and how it intersects the 45 degree line. In short, stable equilibria are ones where the curve  $\Phi$  cuts through the 45 degree line from the left. In cases where the intersection is a single point, such equilibria are stable in that the dynamics from a slight perturbation return naturally to the original equilibrium. In contrast, if  $\Phi$  cuts from the right or below the 45 degree line, then the equilibrium is unstable. For example, in Figure 4, the equilibrium corresponding to  $x_1$  is unstable while those corresponding to 0 and  $x_2$  are stable.

<sup>40</sup>One can deduce more about the shifts in equilibria: the stable equilibria (weakly) decrease from local shifts, and the unstable ones weakly increase from such local shifts of  $\Phi$ . See Jackson and Yariv (2005, 2007) for details.

in the cost of adopting action 1. This can be captured by a strict first-order stochastic dominance (FOSD) shift of the cost distribution from  $F$  to  $\bar{F}$  (so  $\bar{F}(y) \leq F(y)$  for each  $y$  with strict inequalities for some  $y$ 's in the relevant range). It then follows that

$$\bar{\Phi}(x) = \sum_d \tilde{P}(d)\bar{F}(v(d, x)) \leq \sum_d \tilde{P}(d)F(v(d, x)) = \Phi(x)$$

for every  $x$ . Thus, an increase in costs shifts the curve downwards, and so the greatest equilibrium shifts downward (as does any stable equilibrium for small changes).<sup>41</sup> This has a simple explanation: increasing the cost of choosing action 1 raises the thresholds for adopting that action and lowers the adoption.

One can also deduce how changes in network structure, here captured via the degree distribution, affect equilibrium, based on arguments developed by Jackson and Rogers (2007b) and Jackson and Yariv (2005, 2007), and also studied extensively in Galeotti et al. (2010).

For example, consider an increase in the anticipated degree of each random neighbor that a player has in the sense of first-order stochastic dominance. That is, suppose  $\tilde{P}'$  FOSD  $\tilde{P}$ , so that  $\sum_{d \leq d'} \tilde{P}'(d) \leq \sum_{d \leq d'} \tilde{P}(d)$  for each  $d'$ . Couple this with  $v(d, x)$  being non-decreasing in  $d$ , so that individuals who have more interactions are at least as likely to adopt action 1 at the same level of behavior per neighbor. For instance, this holds if it is the number of neighbors adopting action 1 that matters, or even if it is just the fraction that matters. Then, it follows from the definition of first-order stochastic dominance that

$$\Phi'(x) = \sum_d \tilde{P}'(d)F(v(d, x)) \geq \sum_d \tilde{P}(d)F(v(d, x)) = \Phi(x),$$

and, so under  $P'$ , the greatest equilibrium increases. If we reverse things so that  $v(d, x)$  is non-increasing in  $d$ , then the conclusion is reversed.

Again, this has an intuition related to the best replies. If  $v(d, x)$  is increasing in  $d$ , then higher degree individuals are more likely to adopt a behavior for any given anticipated level of activity of neighbors. Thus, starting from the maximal equilibrium, as we increase the anticipated degrees of each player's neighbors, we increase the activity generated for any given  $x$ . Regardless of how players respond to this (strategic substitutes or complements), this shifts the whole curve upwards and the new equilibrium higher.<sup>42</sup>

<sup>41</sup>One can also deduce similar statements for the least and other equilibria.

<sup>42</sup>In the case of strategic complements, all players end up with (weakly) higher activity, while in the case of strategic substitutes, some players' actions may increase and others decrease, but the overall effect from the shift in  $\phi$  has to be upwards.

This sort of analysis can also be extended to other changes in anticipated network structure, such as mean preserving spreads. For example, Jackson and Yariv (2007) show that if  $F(v(d, x))$  is non-decreasing and convex in  $d$ , then mean preserving spreads in the degree distributions lead to nicely ordered  $\Phi$ s. For example, power, Poisson, and regular degree distributions with identical means can be ordered in terms of mean-preserving spreads and then lead to respective  $\Phi^{power}$ ,  $\Phi^{Poisson}$ , and  $\Phi^{regular}$  such that

$$\Phi^{power}(x) \geq \Phi^{Poisson}(x) \geq \Phi^{regular}(x)$$

for all  $x$ , which leads to a corresponding ranking of maximal equilibria.<sup>43</sup>

Finally, using this sort of approach, one can study the time series of adoption, since  $x^t - x^{t-1} = \Phi(x^{t-1}) - x^{t-1}$  is described by  $\Phi(x) - x$ . Thus by studying the shape of this curve as a function of  $x$ , one can deduce things about classic diffusion patterns (e.g., “S-Shaped” rates of adoption), as shown by Jackson and Yariv (2007) as follows:

**PROPOSITION 12** *Let  $F(v(d, x))$  be twice continuously differentiable and increasing in  $x$  for all  $d$ . If  $F(v(d, x))$  is strictly concave (convex) in  $x$  for all  $d$ , then there exists  $x^* \in [0, 1]$  such that  $\Phi(x) - x$  is increasing (decreasing) up to  $x^*$  and then decreasing (increasing) past  $x^*$  (whenever  $\Phi(x) \neq 0, 1$ ).*

The idea is that as  $x$  increase, the concavity of  $F(v(d, x))$  initially leads high feedback effects of changes in  $x$  leading to greater changes in  $\Phi(x)$ , but then it eventually slows down.

## 5.2 Incomplete information about Payoffs

The previous analysis was centered on uncertainty about the interaction patterns. Instead, one might also consider situations where the network of interactions is fixed and known, but there is some uncertainty about payoffs.

Consider the payoff function from (3). De Martí Beltran and Zenou (2012) analyze the quadratic game when the parameter  $\alpha$  is common to but unknown by all players. Players know, however, the value of the synergy parameter  $\phi$ .<sup>44</sup> Let  $\alpha$  take two possible values:

<sup>43</sup>The welfare effects of changes in equilibria depend on changes in the overall population utilities, which require a bit more structure to discern. See Galeotti et al. (2010) and Jackson and Yariv (2005, 2007) for discussion.

<sup>44</sup>We can also analyze a game where  $\phi$  is unknown. The results are similar.

$\alpha_H > \alpha_L > 0$ . All individuals share a common prior, which is  $\Pr(\alpha = \alpha_H) = p \in (0, 1)$ . Each player receives a private signal,  $s_i \in \{h, l\}$ , such that  $\Pr(s_i = h | \alpha_H) = \Pr(s_i = l | \alpha_L) = q \geq 1/2$ .

There is no communication between players. Player  $i$  chooses an action  $a_i(s_i) \geq 0$  as a function of the signal  $s_i \in \{h, l\}$ . Let  $\underline{a}_i = a_i(l)$  and  $\bar{a}_i = a_i(h)$ . The expected utility of player  $i$  is thus equal to:

$$E_i[u_i(\mathbf{a}, \mathbf{g}) | s_i] = E_i[\alpha | s_i] a_i - \frac{1}{2} a_i^2 + \phi \sum_{j=1}^n g_{ij} a_i E_i[a_j | s_i].$$

Let  $\hat{\alpha}_L = E_i[\alpha | s_i = l]$  and  $\hat{\alpha}_H = E_i[\alpha | s_i = h]$ . The best-reply functions are thus:

$$\begin{aligned} \underline{a}_i^* &= \hat{\alpha}_L + \phi \sum_{j=1}^n g_{ij} [\Pr(s_j = l | s_i = l) \underline{a}_j^* + \Pr(s_j = h | s_i = l) \bar{a}_j^*] \\ \bar{a}_i^* &= \hat{\alpha}_H + \phi \sum_{j=1}^n g_{ij} [\Pr(s_j = l | s_i = h) \underline{a}_j^* + \Pr(s_j = h | s_i = h) \bar{a}_j^*] \end{aligned}$$

De Martí Beltran and Zenou (2012) show that if  $\phi \mu_1(\mathbf{g}) < 1$ , then there exists a unique Bayesian-Nash equilibrium of this game where equilibrium efforts are given by:

$$\begin{aligned} \mathbf{a}^* &= \hat{\alpha} \mathbf{b}_1(\mathbf{g}, \phi) - \left( \frac{1 - \gamma_H}{2 - \gamma_H - \gamma_L} \right) (\hat{\alpha}_H - \hat{\alpha}_L) \mathbf{b}_1(\mathbf{g}, (\gamma_H + \gamma_L - 1) \phi) \\ \bar{\mathbf{a}}^* &= \hat{\alpha} \mathbf{b}_1(\mathbf{g}, \phi) + \left( \frac{1 - \gamma_H}{2 - \gamma_H - \gamma_L} \right) (\hat{\alpha}_H - \hat{\alpha}_L) \mathbf{b}_1(\mathbf{g}, (\gamma_H + \gamma_L - 1) \phi) \end{aligned}$$

where

$$\hat{\alpha} = \frac{(1 - \gamma_H) \hat{\alpha}_L + (1 - \gamma_L) \hat{\alpha}_H}{2 - \gamma_H - \gamma_L}.$$

Here, the equilibrium efforts become a combination of two Katz-Bonacich centralities,  $\mathbf{b}_1(\mathbf{g}, \phi)$  and  $\mathbf{b}_1(\mathbf{g}, (\gamma_H + \gamma_L - 1) \phi)$ , reflecting the uncertainty between the two values that the parameter could take. Thus, the model can be adapted to incorporate some uncertainty, and that ends up mitigating the effort levels.

### 5.3 Incomplete information with communication in networks

When players desire to coordinate their behaviors, and they face some uncertainty, it is natural for them to communicate. This has interesting implications for who talks to whom in a network setting.

For example, Hagenback and Koessler (2011) develop an interesting model in which players have to take actions and care about how those actions relate to three things: (i)

some private parameter, *(ii)* a state of nature common to all players, and *(iii)* other players' actions. The private information that players' have about the state of nature motivates communication. Players would like to convey information to others, but their private biases may mean that revealing information to others could move those players' actions in the wrong directions. The tension is due to the fact that a player wishes to have other players' choose actions similar to his or her action, which involves matching the state of nature; but those players have different desires in terms of how close they wish to be to the state of nature, based on their private parameters.<sup>45</sup>

This is a challenging setting to analyze, as even settings with two players are complex (e.g., Crawford and Sobel, 1982). Nonetheless, Hagenback and Koessler (2011) find some very interesting features of equilibria. They find that there are multiple equilibria, in terms of which players reveal their information to which others, and that players can only talk to people whose private biases are similar enough to their own. However, communication patterns between any two players depend not only on the private biases of the two players, but also on the relative private biases of the other players with whom each of them communicates, as that affects how information translates into actions. This leads to the multiplicity of equilibria.

There are a variety of other related models, including Calvó-Armengol and de Martí Beltran (2009), Galeotti, Ghiglino, Squantini (2011), Acemoglu et al. (2011), Calvó-Armengol, de Martí Beltran and Prat (2012). Also related is a literature on updating and learning originating from DeGroot (1974), with some earlier roots. That literature examines whether behavior converges over time, how quickly it converges, whether a consensus opinion or behavior is reached, and what the limit is (e.g., Bala and Goyal, 1998; DeMarzo, Vayanos and Zwiebel, 2003; Golub and Jackson, 2010, 2012a,b,c; see Jackson 2008a, 2011 for more background). The DeGroot model can be interpreted as one where players wish to behavior in ways similar to their neighbors, and they myopically best respond.

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<sup>45</sup>This is also related to Chwe (2000), who analyzes a game in which players want to coordinate their binary decisions and guess the action of others based on the local information that players communicate to their neighbors in the network about their preferences.

## 6 Choosing both actions and links

An important aspect of strategic behavior in network settings that has been looked at but is far from being completely understood is the coevolution of networks and behavior. Much of the literature that we have discussed to this point focused primarily on the influence of a network on behavior. On the other side, there is also a large literature that we have not discussed here that analyzes network formation taking the payoffs from forming links as a given.<sup>46</sup> It is clear, however, that there is feedback: People adjust their behaviors based on that of their friends and they choose their friends based on behaviors. Kandel (1978) provides interesting evidence suggesting that both effects are present in friendship networks, so that over time players adjust actions to match that of their friends and are more likely to maintain and form new friendships with other individuals who act similarly to themselves. Bringing the formation and behavior literatures together is thus important. In this section, we therefore study models where players choose both actions and links. In particular, we would like to see (i) how this changes the conclusions one gets from analyzing the choices separately, and (ii) whether this produces some interesting patterns/dynamics.

### 6.1 Coordination games

Let us return to consider the widely applicable class of coordination games, such as those examined in Sections 3.3.2 and 3.3.3.

We slightly re-normalize the game structure so that payoffs from any pairwise interaction are given by

	1	0
1	$(b, b)$	$(z, y)$
0	$(y, z)$	$(x, x)$

where  $x > z$  and  $b > y$ . This allows for a coordination game with arbitrary payoffs.<sup>47</sup>

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<sup>46</sup>For example, see Jackson and Wolinsky (1996), Dutta and Mutuswami (1997), Bala and Goyal (2000), Dutta and Jackson (2000), and the overview in Jackson (2008a). Those models can allow for the equilibrium of a game on a network to generate payoffs as a function of the network, but do not explicitly include such an analysis.

<sup>47</sup>In the previous analysis, it was fine to normalize payoffs of one of the actions to 0. Here, since there are also costs of links to consider, absolute payoffs of actions matter (not just relative differences), and so we keep track of the value all possible combinations of actions.

The threshold fraction  $q = \frac{x-z}{x-z+b-y}$  is such that action 1 is a best response for a given player if and only if at least a fraction  $q$  of the player's neighbors choose 1. Here, the Pareto efficient equilibrium (payoff dominant equilibrium) depends on whether  $x$  or  $b$  is larger. The risk dominant equilibrium is governed by the size of  $q$ . If  $q > 1/2$  (so  $x - z > b - y$ ), then action 0 is risk dominant, and if  $q < 1/2$ , then action 1 is both risk dominant and payoff dominant (Pareto efficient).

As we saw in Section 3.3.3, in an evenly matched society, the stochastically stable equilibrium was the risk dominant profile of actions, as it takes more trembles to move away from the risk dominant profile than to it. We also saw that the network structure of interactions can matter: for instance, in a star network all playing 0 and all playing 1 are both stochastically stable.

An important question emerges as to how this all depends on choice of partners. A basic intuition arises that it is better to choose partners who are playing an action that leads to a higher payoff. However, this is complicated by history: if one has many friends choosing a low payoff action, how easy is it to change partners and get to better play?

Ely (2002)<sup>48</sup> provides an analysis of this by allowing players to choose both actions and neighborhoods (i.e. with whom they want to interact by choosing a location). In other words, when a revision opportunity arises, a player simultaneously chooses a strategy and location in order to maximize his/her expected payoff. In the model of Ely (2002), if some player randomly moves to an unoccupied location and plays the efficient strategy, then other players would like to move to that location and play the efficient strategy rather than staying at a location where they play the inefficient strategy. As a result, Ely shows that the limit distribution (with small trembles) places probability one on the efficient outcome, so that risk-dominance ceases to play a role in determining long-run play.

This result depends on the locational aspect of the interaction patterns, and caring about average play rather than numbers of interactions. In particular, in changing locations, players can sever all old ties, form new ties, and switch technologies simultaneously, and the number of new versus old ties is not important to the players.

Jackson and Watts (2002b) instead propose a model which is not a location one, but rather one where players choose their interaction patterns on an individual-by-individual basis. In other words, they model the interaction pattern as a network where individuals

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<sup>48</sup>See also Mailath, Samuelson, and Shaked (2001).

periodically have the discretion to add or sever links to other players.

In each period  $t$ , a player  $i$  chooses an action  $a_i^t \in \{1, 0\}$  and then receives a payoff of

$$u_i(\mathbf{a}^t, \mathbf{g}^t) = \sum_{j \neq i} g_{ij}^t [v_i(a_i^t, a_j^t) - k(d_i(\mathbf{g}^t))]$$

where  $v_i(a_i^t, a_j^t)$  is a payoff that depends on the actions chosen, the network  $\mathbf{g}^t$ , and a cost  $k(d_i(\mathbf{g}^t))$  of having  $d_i(\mathbf{g}^t)$  links.

The full version of the Jackson and Watts (2002b) dynamic process is as follows.

- Period  $t$  begins with network  $\mathbf{g}^{t-1}$  in place.
- One link  $ij_t$  is chosen at random with probability  $p_{ij}$ . If the link is not in  $\mathbf{g}^{t-1}$  then it is added if both players weakly benefit from adding it and at least one strictly benefits, under the myopic assumption that the rest of the network stays fixed and actions will be  $\mathbf{a}^{t-1}$ . If the link is in  $\mathbf{g}^{t-1}$  then it is deleted if either player strictly benefits from deleting it, under the myopic assumption that the rest of the network stays fixed and actions will be  $\mathbf{a}^{t-1}$ . With a probability  $\gamma$ ,  $1 > \gamma > 0$ , the decision to add or delete the link is reversed. This results in a new network  $\mathbf{g}^t$ .
- Next, one player  $i$  is randomly chosen with some probability  $q_i$ . That player chooses  $a_i^t$  to maximize  $u_i(a_i^t, \mathbf{a}_{-i}^{t-1}, \mathbf{g}^t)$ . With a probability  $\varepsilon$ ,  $1 > \varepsilon > 0$ , the action of player  $i$  is reversed. This results in a new action profile  $\mathbf{a}^t$ .<sup>49</sup>

The probabilities of trembles are identical and independent across players, strategies, and periods. This is well-defined Markov chain where the state is the vector of actions  $\mathbf{a}^t$ , that are played in period  $t$ . Since this process is aperiodic and irreducible, the Markov chain has a unique stationary distribution, denoted  $\mu^{\gamma, \varepsilon}(\mathbf{g}, \mathbf{a})$ .

Jackson and Watts (2002b) analyze a variety of different cost structures, but let us just consider one of those and refer the reader to the paper for other details.

Consider a cost of link structure  $k(d)$  that is equal to  $kd$  for  $d \leq D$  and infinite if  $d > D$ . So, players have a constant marginal cost of links up to some level degree  $D$ , and then the cost of maintaining additional costs is arbitrarily large, so that they effectively have a capacity constraint on friendships.

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<sup>49</sup>Disconnected players choose a best response to the last period distribution of play of the others.



Stochastic stability now involves looking at the limit of the process  $\mu^{\gamma, \varepsilon}(\mathbf{g}, \mathbf{a})$  as  $\gamma$  and  $\varepsilon$  both go to zero (at similar rates, so take  $\gamma = f\varepsilon$  for some constant  $f$ ). Jackson and Watts (2002b) show:<sup>50</sup>

**PROPOSITION 13** *Let  $D$  be even and such that  $n > D > 1$ . Also, let  $1 - 2/D > q > 2/D$  and  $q \neq 1/2$ .*

- (i) *If  $x - k > 0$  and  $b - k < 0$ , then the set of stochastically stable states involve all players playing 0; and, analogously, if  $x - k < 0$  and  $b - k > 0$ , then the set of stochastically stable states involve all players playing 1. There can be a variety of network configurations in stochastically stable states.*
- (ii) *If  $y - k > 0$  and  $z - k > 0$ , then in any stochastically stable state each player has  $D$  links and plays the risk dominant action.*
- (iii) *If  $y - k < 0$  and/or  $z - k < 0$ , and  $x - k > 0$  and  $b - k > 0$ , then in stochastically stable states involve all players playing 0 as well as all players playing 1. There can be a variety of network configurations in stochastically stable states.*

Although the proof of the proposition is quite involved, some of the intuitions are relatively easy to see. In case (i), there is only one of the actions that can lead to a positive payoff and so links can only exist in the long run between players playing the action leading to a positive payoff, and indeed that turns out to be stochastically stable as if players randomly start playing the action that leads to a positive payoff then they end up linking to each other, and so only a couple of trembles can start growing a network of people playing the action that leads to positive payoffs. In case (ii), both actions are lead to positive payoffs (up to the capacity constraint) regardless of what ones neighbors do. Here, standard risk dominance arguments take over as players form the full number of  $D$  links, and then trembles in actions play the leading role in determining the outcome. Case (iii) is perhaps the most subtle and interesting one. It addresses the situation where at least one of the actions leads to sufficiently poor payoffs from miscoordination, that it is better to sever a link to a player with whom one is not coordinating when playing that action, than to maintain that link. In

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<sup>50</sup>Jackson and Watts (2002b) provide more analysis of this model, and Goyal and Vega-Redondo (2005) find similar results for a model with unilateral link formation. There is also an analysis by Droste, Gilles and Johnson (2000) for the case of geographic link costs.

this situation, trembles in actions can lead to changes in network structure, which then aids in changes in behavior. For example, suppose that all players are initially playing the risk dominant action. A tremble can lead a player who changes action to be disconnected from the network. With two such trembles, two disconnected players can then form a component playing the other action, which continues to grow as trembles accumulate. This process moves symmetrically between all playing the risk dominant action, and the other case (and each intermediate equilibrium with some mixture of play is destabilized by a single tremble).

There are also questions as to the relative rates at which actions change compared to network structure, and that can affect the overall convergence to equilibrium, as one sees in Ehrhardt, Marsili, and Vega-Redondo (2006), as well as Holme and Newman (2006) and Gross and Blasius (2008). There are also analyses of equilibrium play in games when players are matched in heterogeneous roles (e.g., firms and workers, husbands and wives, etc.) as in Jackson and Watts (2010).

While the details can be complicated, the main message is that endogenizing the interaction structure between players can have a profound impact on the way in which play evolves in a society. Thus, the endogeneity of relationships cannot be ignored when trying to make robust predictions about behaviors in a society, despite the complexity that endogenous relationships introduce.

## 6.2 Network Formation in Quadratic Games

In addition to the models discussed above, there are other models that have combined action and link choices. For example, Bloch and Dutta (2009) proposed a model with both link intensities and communication.

We expose two models, one static (Cabrales, Calvo-Armengol and Zenou, 2011) and one dynamic (König, Testone and Zenou, 2010, 2011), which both use the quadratic model from Section 4.

Consider a simultaneous move game of network formation (or social interactions) and investment.  $T = \{1, \dots, t\}$  is a finite set of types for the players. We let  $n$  be a multiple of  $t$ :  $n = mt$  for some integer  $m \geq 1$ , and there is the same number of players of each type. The case where  $n = t$  is referred to as *the baseline game* and the case where  $n = mt$  as *the  $m$ -replica* of this baseline game. Let  $\tau(i) \in T$  be player  $i$ 's type.

Let  $c > 0$ . Player  $i$ 's payoff is described by:

$$u_i(\mathbf{a}, \mathbf{s}) = \alpha_{\tau(i)} a_i + \phi \sum_{j=1, j \neq i}^n g_{ij}(\mathbf{s}) a_j a_i - \frac{1}{2} c a_i^2 - \frac{1}{2} s_i^2 \quad (34)$$

where  $a_i \geq 0$  is the action (productive investment) taken by player  $i$  while  $s_i \geq 0$  is the socialization effort of player  $i$ , with  $\mathbf{s} = (s_1, \dots, s_n)$ . The returns to the investment are the sum of a private component and a synergistic component. The private returns are heterogeneous across players and depend on their type.

The network of interactions  $g_{ij}(\mathbf{s})$  between players  $i$  and  $j$  are determined by

$$g_{ij}(\mathbf{s}) = \rho(\mathbf{s}) s_i s_j \quad (35)$$

where

$$\rho(\mathbf{s}) = \begin{cases} 1 / \sum_{j=1}^n s_j, & \text{if } \mathbf{s} \neq \mathbf{0} \\ 0, & \text{if } \mathbf{s} = \mathbf{0} \end{cases} \quad (36)$$

so that  $g_i(\mathbf{s}) = s_i$ . That is, players decide upon their total interaction intensity.

This sort of approach to network formation appears in the models of Curarrini, Jackson and Pin (2009, 2010) and has also been used by Golub and Levne (2011), among others. An important aspect is that the synergistic effort  $\mathbf{s}$  is *generic* within a community – a scalar decision. Network formation is not the result of an earmarked socialization process. Using (35) and (36), one can see that the probability of forming a link,  $g_{ij}(\mathbf{s})$ , is equal to  $s_i s_j / \sum_{j=1}^n s_j$ . This means that the more time two players  $i$  and  $j$  spend socializing, the more likely they form a link together.

Let

$$\phi(\boldsymbol{\alpha}) = \phi \frac{\sum_{\tau=1}^t \alpha_{\tau}^2}{\sum_{\tau=1}^t \alpha_{\tau}}. \quad (37)$$

Cabrales, Calvó-Armengol, and Zenou (2011) demonstrate the following result:

**PROPOSITION 14** *Suppose that  $2(c/3)^{3/2} > \phi(\boldsymbol{\alpha}) > 0$ . Then, there exists an  $m^*$  such that for all  $m$ -replicas with  $m \geq m^*$ , there are two (stable) interior pure strategy Nash equilibria. These pure strategy Nash equilibria are such that for all players  $i$  of type  $\tau$ , the strategies  $(s_i, a_i)$  converge to  $(s_{\tau(i)}^*, a_{\tau(i)}^*)$  as  $m$  goes to infinity, where  $s_{\tau(i)}^* = \alpha_{\tau(i)} s$ ,  $a_{\tau(i)}^* = \alpha_{\tau(i)} a$ , and  $(s, a)$  are positive solutions to*

$$\begin{cases} s = \phi(\boldsymbol{\alpha}) a^2 \\ a [c - \phi(\boldsymbol{\alpha}) s] = 1 \end{cases} \quad (38)$$

When  $\phi(\boldsymbol{\alpha})$  is small enough compared to the infra-marginal cost for a productive investment, the system of two equations (38) with two unknowns has exactly two positive solutions. As  $m$  gets large, each such solution gets arbitrarily close to a pure strategy Nash equilibrium of the corresponding  $m$ -replica game. The multiplicity reflects complementarities in socialization and in actions.

The approximated equilibria characterized in Proposition 14 display important features. First, the level of socialization per unit of productive investment is the same for all players, that is,  $s_i^*/a_i^* = s_j^*/a_j^*$ , for all  $i, j$ . This is equivalent to having a marginal rate of substitution of socialization versus action uniform across all players. Second, differences in actions reflect differences in idiosyncratic traits. More precisely,  $a_i^*/a_j^* = \alpha_{\tau(i)}/\alpha_{\tau(j)}$ , for all  $i, j$ . Third, in the presence of synergies, actions are all scaled up (compared to the case without synergies) by a multiplier. This multiplier, which is homogeneous across all players, is a decreasing function of the cost  $c$ , and an increasing function of the second-order average type  $\phi(\boldsymbol{\alpha})$ .

Figure 6 plots equations (38).

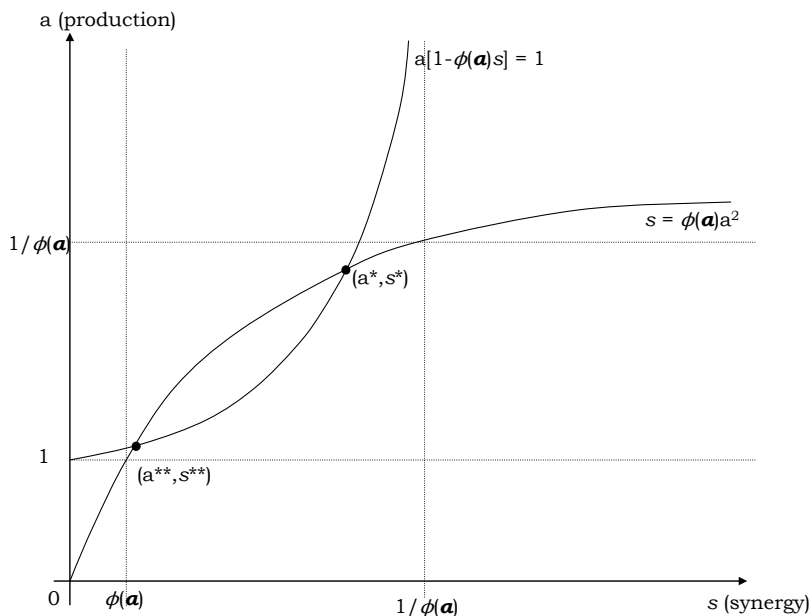


Figure 6: Multiple equilibria when players choose actions and socialization

From this figure, it is clear that the system (38) need not always have a positive solution. The upper bound on  $\phi(\boldsymbol{\alpha})$  in Proposition 14 is a necessary and sufficient condition for the two graphs to cross in the positive orthant of the space  $(s, a)$ . When  $\phi(\boldsymbol{\alpha})$  is too large, the synergistic multiplier operates too intensively and there is no intersection: there is a feedback where increases in socialization lead to increase actions and vice versa..

The equilibrium actions can be ranked component-wise, and the equilibrium payoffs can be Pareto-ranked accordingly. There is a Pareto-superior approximate equilibrium  $((\mathbf{s}^*, \mathbf{a}^*)$  in Figure 6) and a Pareto-inferior approximate equilibrium  $((\mathbf{s}^{**}, \mathbf{a}^{**})$  in Figure 6) while the socially efficient outcome lies in between the two equilibria.<sup>51</sup> Formally,  $(\mathbf{s}^*, \mathbf{a}^*) \geq (\mathbf{s}^O, \mathbf{a}^O) \geq (\mathbf{s}^{**}, \mathbf{a}^{**})$  and  $\mathbf{u}(\mathbf{s}^O, \mathbf{a}^O) \geq \mathbf{u}(\mathbf{s}^*, \mathbf{a}^*) \geq \mathbf{u}(\mathbf{s}^{**}, \mathbf{a}^{**})$ .

Another model based on the quadratic model where players choose with whom they want to interact is that of König, Tessone and Zenou (2011). At each period of time, players play a two-stage game: in the first stage, players play their equilibrium actions in the quadratic game, while in the second stage a randomly chosen player can update his/her linking strategy by creating a new link as a best response to the current network. Links do not last forever but have an exponentially distributed life time. A critical assumption is that the most valuable links (i.e. the ones with the highest Bonacich centrality) decay at a lower rate than those that are less valuable. As in the literature on evolutionary models described in Section 6.1, the authors introduce some noise to this model. There is, however, a possibility of error, captured by the stochastic term in the utility function. The authors then analyze the limit of the invariant distribution, the stochastically stable networks, as the noise vanishes.

The network generated by this dynamic process is a nested split graph. The authors also show that the stochastically stable network is a nested split graph. These graphs, which are relatively well-known in the applied mathematics literature (see, in particular, Mahadev and Peled, 1995), have a simple structure that make them tractable. In order to define nested split graphs, we consider the degree partition of a graph.

**Definition 3** *Let  $(N, \mathbf{g})$  be a graph whose distinct positive degrees are  $d_{(1)} < d_{(2)} < \dots < d_{(k)}$ , and let  $d_0 = 0$  (even if no player with degree 0 exists in  $\mathbf{g}$ ). Further, define  $\mathcal{D}_i = \{j \in N : d_j = d_{(i)}\}$  for  $i = 0, \dots, k$ . Then the set-valued vector  $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_k)$  is the degree partition of  $\mathbf{g}$ .*

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<sup>51</sup>The superscript  $O$  refers to the “social optimum” outcome.

**Definition 4** A graph  $(N, \mathbf{g})$  is a nested split graph if it satisfies the following. Let  $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_k)$  be its degree partition. Then the nodes  $N$  can be partitioned into independent sets  $\mathcal{D}_i$ ,  $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$  and a “dominating” set  $\bigcup_{i=\lfloor \frac{k}{2} \rfloor + 1}^k \mathcal{D}_i$  in the graph  $(N \setminus \mathcal{D}_0, \mathbf{g}')$ , where  $\mathbf{g}'$  is the corresponding subgraph.<sup>52</sup> Moreover, the neighborhoods of the nodes are nested: for each node  $j \in \mathcal{D}_i$ ,  $i = 1, \dots, k$ ,

$$N_j(\mathbf{g}) = \begin{cases} \bigcup_{j=1}^i \mathcal{D}_{k+1-j} & \text{if } i = 1, \dots, \lfloor \frac{k}{2} \rfloor, \\ \bigcup_{j=1}^i \mathcal{D}_{k+1-j} \setminus \{j\} & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \dots, k. \end{cases} \quad (39)$$

Figure 7 (left) illustrates the degree partition  $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_6)$  and the nested neighborhood structure of a nested split graph. A line between  $\mathcal{D}_i$  and  $\mathcal{D}_j$  indicates that every node in  $\mathcal{D}_i$  is linked to every node in  $\mathcal{D}_j$  for any  $i, j = 1, \dots, 6$ . The nodes in the dominating set included in the solid frame induce a clique while the nodes in the independent sets that are included in the dashed frame induce an empty subgraph. Figure 7 (right) shows the stepwise adjacency matrix corresponding to the nested split graph with  $n = 10$  nodes shown on the left hand side. If we let the nodes indexed by the order of the rows in the adjacency matrix, then it is easily seen that for example  $\mathcal{D}_6 = \{1, 2 \in \mathcal{N} : d_1 = d_2 = d_{(6)} = 9\}$  and  $\mathcal{D}_1 = \{9, 10 \in N : d_9 = d_{10} = d_{(1)} = 2\}$ . From the stepwise property of the adjacency matrix it follows that a connected nested split graph contains at least one spanning star, that is, there is at least one player that is connected to all other players.

Let us give some more intuition of the result that the stochastically stable network is a nested split graph. In this model, because of complementarities, players always want to link to others who are more central since this leads to higher actions (as actions are proportional to centrality) and higher actions raise payoffs more. Similarly, links decay to those with lower centrality as these players have lower actions and hence lower payoff effects. Notice moreover that, once a player loses a link, he/she becomes less central and this makes it more likely that another link decays. Thus link gains and losses are self reinforcing. This intuition suggests that if the probability of adding links is large then the process should approximate complete network while if it is small then the process should approximate the star network. The key insight of this model is that, for intermediate values of this probability, the network is a nested split graph.

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<sup>52</sup> $\lceil x \rceil$  denotes the smallest integer larger or equal than  $x$  (the ceiling of  $x$ ). Similarly,  $\lfloor x \rfloor$  denotes the largest integer smaller or equal than  $x$  (the floor of  $x$ ).

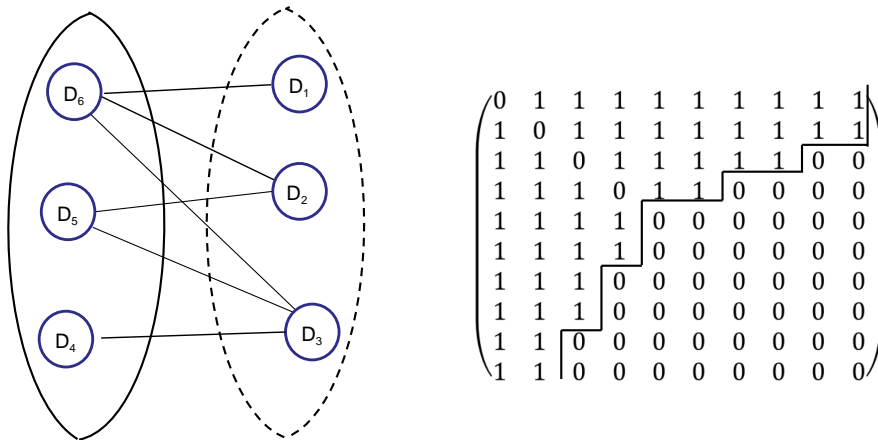


Figure 7: Example of a nested-split graph. On the left: the  $D_i$ 's are collections of nodes with given degrees and links between them indicate that all nodes in one set are connected to all nodes in the other set. Nodes in a  $D_i$  on the left group are all linked to each other, while nodes in the right group are all independent. On the right: the associated adjacency matrix for this example with 10 nodes.

König, Tessone and Zenou (2011) also explicitly characterize the degree distribution  $P(d)$  in the stochastically stable networks. Instead of relying on a mean-field approximation of the degree distribution and related measures as some dynamic network formation models do, because of the nature of nested split graphs, the authors are able to derive explicit solutions for network statistics of the stochastically stable networks (by computing the adjacency matrix). They find that, as rates at which linking opportunities arrive and links decay, a sharp transition takes place in the network density. This transition entails a crossover from highly centralized networks when the linking opportunities are rare and the link decay is high to highly decentralized networks when many linking opportunities arrive and only few links are removed.

### 6.3 Games with strategic substitutes

We now turn to discuss network formation in the context of games with strategic substitutes. Let us extend the model of Bramoullé and Kranton (2007a) exposed in Section 3.2.3 to introduce link formation. Remember that in a game with no link formation, there were specialized equilibria in which the set of players exerting full effort forms a maximal independent and vice versa.

Galeotti and Goyal (2010) consider a variation in which the utility function of player  $i$  is given by:

$$u_i(\mathbf{a}, \mathbf{g}) = v \left( a_i + \phi \sum_{j=1}^n g_{ij} a_j \right) - c a_i - k d_i(\mathbf{g}), \quad (40)$$

where  $v(\cdot)$  is a strictly increasing and strictly concave function on  $\mathbb{R}_+$  and  $c > 0$  is the constant marginal cost of own action such that  $v'(0) > c > v'(x)$  for some  $x$ , and  $k > 0$  is the cost of linking with an other player.

Consider *directed networks*, in which a player can unilaterally form a link with another player without his/her consent and pays the cost of forming this link. As in Section 3.2.3, let  $a^{**}$  be the effort level of an individual who experiments by him/herself, i.e.  $v'(a^{**}) = c$ . Assume  $k < c a^{**}$  so that it is cheaper to link to another player who exerts effort than to exert effort directly. This ensures that some players form links with others. Every (strict) equilibrium of this game has a core-periphery architecture so that the players in the core acquire information personally, while the peripheral players acquire no information personally but form links and get all their information from the core players. Galeotti and Goyal (2010) also show that, under some conditions, the socially optimal outcome is a star network in which the hub chooses some positive level of effort while all other players choose 0.<sup>53</sup>

López-Pintado (2008) presents an interesting dynamic network formation model with strategic substitutabilities. She assumes that players make a binary decision, whether or not to exert effort, rather than exerting a continuous effort. In each period, a player, uniformly selected at random from the population, updates his/her strategy and chooses a myopic best response. In other words, taking as given the behavior of others, the player chooses the action that maximizes his/her current benefits. López-Pintado (2008) shows that this

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<sup>53</sup>The equilibrium and socially optimal networks are similar to those obtained by König, Testone and Zenou (2011) in the context of a dynamic network formation model with strategic complementarities (see Section 6.2) since nested-split graphs have clearly a core-periphery structure.



dynamic process converges to a Nash equilibrium of the static model. She then studies a mean-field version of the myopic best response dynamics and considers the asymptotic properties of the equilibria in (general) random networks when the network size becomes large. In this model, López-Pintado shows that the dynamics converge to a unique, globally stable fraction of free-riders. She also demonstrates that the higher is the degree of players in a homogeneous network, the higher is the proportion of players free-riding and the proportion of links pointing to a free-rider. Under additional conditions on the degree distribution, she shows that the proportion of links pointing to a free-rider increases under a first-order stochastic dominance (FOSD) shift of the degree distribution and decreases under a mean preserving spread (MPS) of the degree distribution. These results suggest that there tends to be more free-riding in denser or more equal networks.

## 7 Repeated Games and Network Structure

Several papers have provided a theoretical analysis of a repeated games in network settings. This includes Raub and Weesie (1990), Ali and Miller (2009, 2012), Lippert and Spagnolo (2011), Mihm, Toth and Lang (2009), Jackson, Rodriguez-Barraquer, Tan (2011), and Fainmesser (2012).

There are several ideas that emerge from this literature. One relates to network structure and information travel. The basic idea is that to enforce individual cooperation in prisoner's dilemma sorts of games, other players have to be able to react to a given player's deviations. Raub and Weesie (1990) and Ali and Miller (2009) show how completely connected networks shorten the travel time of contagion of bad behavior which can quicken punishment for deviations. Players only observe their own interactions, and so punishments travel through the network only through contagious behavior, and the challenge to enforcing individual cooperation is how long it takes for someone's bad behavior to come to reach their neighbors through contagion.

A second idea relates to heterogeneity. Haag and Lagunoff (2006) show how heterogeneity can also favor cliques. They allow for preference differences that can preclude cooperative behavior within a repeated game, and so partitioning society into homogeneous subgroups can enable cooperative behavior that might not be feasible otherwise.

A third idea relates to robustness of networks, which favor networks that look like "quilts"

of small cliques pasted together. Jackson, Rodriguez-Barraquer, and Tan (2012) examine self-enforcing favor exchange, where players can periodically do favors for their friends but at a cost. In order to ensure that a player performs a favor when called upon, the player fears losing several friendships and thus many future favors. There are many such networks that can enforce favor exchange, by threatening to sever links to players who fail to perform favors when they are asked to. However, as Jackson, Rodriguez-Barraquer, and Tan (2012) show, only very special networks are “robust”. Robustness requires two things: a form of renegotiation so that the punishments are really credible, and immunity to large contagions - the only people who end up losing links are friends of the player failing to do a favor. These networks consist of (many) small cliques of players, where if a player fails to perform a favor then a particular clique disintegrates, but that does not lead to further contagion. Other networks experience further contagions where society loses more of its links.

The many settings of repeated interactions between friends makes understanding repeated games on networks essential. This literature is still at its beginning and many open questions remain.

## 8 Concluding Remarks and Further Areas of Research

The settings where social network structure has profound implications for human behavior are quite numerous. Thus, it is not surprising that the literature that relates to this subject is enormous. We have focused mainly on the central theoretical underpinnings of games played on networks.

Networks are inherently complex, and so much of the progress that has been made required some focus on specific game structures. There are three main ways in which progress has been made. One involves looking at games of strategic complements and strategic substitutes, where the interaction in payoffs between players satisfies some natural and useful monotonicity properties. That monotonicity provides a basis for understanding how patterns of behaviors relate to network structure. A second approach relies on looking at a simple parametric specification of a game in terms of a quadratic form that permits an explicit solution for equilibrium behavior as a function of a network. A third approach considers settings where the specific pattern of interactions is uncertain, in which case equilibrium can be expressed nicely as a function of the number of interactions that players expect to have.

These models make a number of predictions about behavior, relating levels of actions to network density, relating players' behaviors to their position in the network, and relating behavior to things like the degree distribution and cost of taking given actions. Thus, we end up both with predictions about how a specific player's behavior relates to his/her position in a network, as well as what overall behavior patterns to expect as a function of the network structure. While much progress has been made, the innumerable applications and importance of the subject cry out for additional study.

Let us close with a brief discussion of various areas of research that are closely related, but we have not covered due to space constraints.

## 8.1 Bargaining and Exchange on Networks

An important area in terms of applications to economics concerns bargaining and exchange on networks. Many business relationships are not centralized, but take place through networks of interactions. A standard modeling technique in the literature that has modeled this is to have only pairs of players connected in the network can trade. The key questions addressed by this literature are: How does the network structure influence market outcomes? How does a player's position in the network determine his/her bargaining power and the local prices he/she faces? Who trades with whom and on what terms? Are trades efficient? If players can choose their links, do the efficient networks form? How does this depend on the bargaining protocol and the costs of links?

The literature on this subject includes the early experimental investigations of Cook and Emerson (1978), and what followed in the literature on "exchange theory."<sup>54</sup> That literature documents how positions in a network influence terms of trade, testing theories of power and brokerage.

The theoretical analyses of this subject later emerged using models of trade based on tools including auction theory, noncooperative analyses of bargaining games, and use of the core. Much of that literature assumes that buyers have unit demands and sellers have unit supply, and that these are pre-identified. For example, Kranton and Minehart (2001) considered a setting where prices are determined by an ascending-bid auction mechanism. They showed that the unique equilibrium in weakly undominated strategies leads to an efficient allocation

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<sup>54</sup>Early writings on exchanges in networks include Homans (1958, 1961), Blau (1964), and eventually included explicit consideration of social network structure as in Emerson (1962, 1976).

of the goods. They found that the network formation was also efficient in a setting where buyers pay the cost of connection. Jackson (2003) shows that this result does not generalize, but is special to buyers bearing the entire connection cost, and that cost being born before buyers know their valuations for the good. He shows that both under-connection (in cases where some players see insufficient shares of the gains from trade) and over-connection (as players try to improve their bargaining position) are possible sources of inefficiency.

The literature that followed explored a variety of different exchange mechanisms and settings in terms of valuations. Calvó-Armengol (2003) explores the power of a position in a network, in a context of pairwise sequential bargaining with neighbor players. Polanski (2007) assumes that a maximum number of pairs of linked players are selected to bargain every period. Corominas-Bosch (2004) considered an alternating offer bargaining game. One of the most general analyses is by Elliott (2011) who allows for general valuations among pairs of players and then characterizes the core allocations. Using that as a basis, he then documents the inefficiencies that arise in network formation, showing how under-connection and over-connection depend on how costs of links are allocated. Elliott also documents the size of the potential inefficiencies, and shows that small changes in one part of a network can have cascading effects. Manea (2011) and Abreu and Manea (2012) provide a non-cooperative models of decentralized bargaining in networks when players might not be ex ante designated as buyers or sellers, but where any pair may transact. Condorelli and Galeotti (2012) provide a look at a setting where there is incomplete information about potential valuations and possibilities of resale.

Related models include those studying oligopoly games that are played by networked firms as in Nava (2009), who analyzes when it is that competitive outcomes are reached, and Lever (2011) who examines network formation. Blume et al (2009) examine the role of middlemen in determining profits and efficient trade.

There are also studies, such as Kakade, Kearns and Ortiz (2004) and Kakade et al. (2005), that examine exchange on random graph-generated network structures. One finding is that if the network is sufficiently symmetric so that buyers have similar numbers of connections, as well as sellers, then there are low levels of price dispersion, while sufficient asymmetry allows for more price dispersion.

The literature to date provides some foundations for understanding how networks influence terms of trade. Still largely missing are empirical tests of some of the theory with field

data. Some experimental evidence (e.g., Charness, Corominas-Bosch, and Frechette, 2005) validates some of the bargaining predictions in lab settings.

## 8.2 Risk-sharing networks

Another important application where networks play a prominent role in shaping behavior is in terms of risk-sharing. In much of the developing world, people face severe income fluctuations due to weather shocks, diseases affecting crops and livestock, and other factors. These fluctuations are costly because households are poor and lack access to formal insurance markets. Informal risk-sharing arrangements, which help cope with this risk through transfers and gifts, are therefore widespread.<sup>55,56</sup>

Again, this is an area where simple game theoretic models can add substantial insight. Some studies to date include Bramoullé and Kranton (2007b), Bloch, Genicot and Ray (2008), Ambrus, Mobius and Szeidl (2010), Belhaj and Deroian (2012), and Jackson, Rodriguez-Barraquer, and Tan (2012). This is an area where rich repeated game studies should help deepen our understanding of the relationship between networks of interactions and the (in)-efficiency of risk-sharing.

## 8.3 Dynamic games and network structure

Most of our discussion has focused on static games or else simple variations on dynamics with perturbations. Beyond those, and the repeated games mentioned above, another important area where networks can have profound implications is in terms of the dynamics of patterns that emerge. Just as an example, Calvó-Armengol and Jackson (2004) study the dynamics of labor markets: the evolution over time of employment statuses of  $n$  workers connected by a network of relationships where individuals exchange job information only between direct neighbors.<sup>57</sup> Accounting for the network of interactions can help explain things like

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<sup>55</sup>Rosenzweig (1988) and Udry (1994) document that the majority of transfers take place between neighbors and relatives (see also Townsend, 1994). Other empirical work confirms this finding with more detailed social network data (Dercon and De Weerd, 2006; Fafchamps and Gubert, 2007; Fafchamps and Lund, 2003).

<sup>56</sup>Although this is clearly vital in the developing world, substantial sharing of risks is also prevalent in the developed world.

<sup>57</sup>See also the model of Calvó-Armengol, Verdier and Zenou (2007) who study the importance of weak ties in crime and employment.

the duration of unemployment of each worker. A worker whose friends are unemployed has more difficulty in hearing about job openings and this leads to increased spells of unemployment, as well as decreased incentives to invest in human capital. The complementarities in that setting, coupled with the network structure, helps provide an understanding of various empirical observations of employment. Calvó-Armengol and Jackson (2004) also show that education subsidies and other labor market regulation policies display local increasing returns due to the network structure. Subsidies and programs are more tightly clustered with network structure in mind can then be more effective.

That is just an example of an area where the coupling of game theoretic analysis and network structure can have important implications for dynamics and for policy. This is an important and still under-studied area.

## 8.4 More Empirical Applications based on Theory

Part of the reason that accounting for networks in studying behavior is so essential, is that understanding the relationship can shape optimal policy. For example, in Section 4.4.1, we discussed the idea of key player in crime, i.e. the criminal who once removed generates the highest reduction in crime. Liu et al. (2012) examined this idea using data on adolescent youths in the US (AddHealth) and show that, indeed, a key player policy reduces crime more than a random-targeted policy. In other words, targeting nodes or players in a network can have snow-ball effects because of the interactions between players in the network. This is the idea of the social multiplier developed in Section 4.1.

Other examples of areas where models of games on networks help inform policy and learn from field work include financial networks, diffusion of innovations, and political interactions. For example, financial markets can be considered as a network where links are transactions of dependencies between firms or other organizations (Leitner, 2005; Cohen-Cole, Patacchini and Zenou 2011, Elliott, Golub and Jackson (2012)). Network analyses in financial settings can enhance our understanding of the interactions and optimal regulation and policy. It is also clear that networks influence adoption of technologies. There is indeed empirical evidence of social learning (e.g., Conley and Udry, 2010). Theory (e.g., Section 5.1.3) tells us that the adoption of a new technology is related to network structure. In a recent paper, Banerjee, Chandrasekhar, Duflo and Jackson (2012) study the adoption of a microfinance program in villages in rural India. They find that participation to the program is higher

in villages when the first set of people to be informed are more important in a network sense in that they have higher eigenvector centrality (which is a measure related to Katz-Bonacich centrality). As a last example, there is evidence that personal connections amongst politicians have a significant impact on the voting behavior of U.S. politicians (Cohen and Malloy, 2010). Here there is a need for both theory and additional empirical work, that helps us understand how the network of interactions between politicians shapes legislation and a host of other governmental outcomes.

As the theory of games on networks continues to develop, the interaction with field work will continue to become richer and more valuable.

## 8.5 Lab and Field Experiments

While our knowledge of peer effects is growing, the complexities involved mean that there is still much to be learned. Given the enormous difficulty of identifying social effects in the field, essential tools in this area of research are laboratory and field experiments, where one can control and directly measure how players' behaviors relate to network structure.

Experiments have been used to study strategic network formation (e.g., Callander and Plott, 2005; Pantz and Zeigelmeyer, 2003; Falk and Kosfeld, 2003; Goeree, Riedl and Ule, 2009; and Charness and Jackson 2007), learning in network settings, (e.g., Choi, Gale and Kariv, 2012; Celen, Kariv and Schotter, 2004; Möbius, Phan and Sziedl, 2010; Chandrasekhar, Larreguy, and Xandri, 2011) as well as games played on networks (see Kosfeld, 2004, and Jackson and Yariv, 2011, for additional background).

For instance, Goeree et al. (2007) and Leider et al. (2009) find that play in games is related to social distance in a network, with players being more cooperative or generous to those who are direct friends or close in a social distance sense compared to those who are more distant in the network. Kearns et al. (2009) find that a society's ability to reach a consensus action in a game of complementarity depends on network structure.

Moreover, experiments can be very useful in directly testing some of the theoretical predictions given above. For example, Charness, Feri, Melendez-Jimenez and Sutter (2011) test games of networks by looking at two important factors: *(i)* whether actions are either strategic substitutes or strategic complements, and *(ii)* whether subjects have complete or incomplete information about the structure of a random network. They find that subjects conform to the theoretical predictions of the Galeotti et al. (2010) model that we exposed

in Section 5.1. They also find some interesting selections of equilibria that suggest that additional theory would be useful.

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