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ABSTRACT

Multi-Stage Sequential All-Pay Auctions*

We study multi-stage sequential all-pay contests (auctions) where heterogeneous contestants are privately informed about a parameter (ability) that affects their cost of effort. We characterize the sub-game perfect equilibrium of these multi-stage sequential all-pay contests and analyze the effect of the number of contestants, their types, and their order on the expected highest effort.

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1 Introduction

Contests are one of the most common economic interactions and are taking place, for example, in job markets, politics, R&D, and sports. A contest is defined as an activity in which players exert effort in order to win a prize (or several prizes). Many real-life contests are sequential in nature. For instance, in many sport contests (e.g., athletics and gymnastics) the contestants perform one after the other. Likewise, in contests in the labor market several job candidates compete for the same job and arrive one by one. Even R&D tournaments can sometimes be sequential when one firm develops a product to compete with an existing product of another firm. The outcomes of such contests are obviously affected by the number of the players, their abilities and their order. This paper analyzes a sequential all-pay contest with incomplete information and a general number of players. We address the following questions. If the designer of the contest wishes to maximize the players' expected highest effort, is it always better to have more players? Is it always better to have a contest among strong players than among weak players? If the players are ex-ante asymmetric, who should be first, the stronger or the weaker player? How should the designer order the contestants?

Most existing studies on all-pay contests (auctions) deal with simultaneous all-pay contests where each player submits a bid (effort) and the player who submits the highest bid wins the contest, but, independently of success, all players bear the cost of their bids.[1] We, on the other hand, study multi-stage sequential all-pay contests under incomplete information where the ability of each contestant is his private information and contestants submit their bids one after the other.[2] In Segev and Sela (2011) we studied sequential all-pay contests with two contestants who compete for a prize of size 1. Contestant 1 (the first mover)

[1] All-pay auctions have been studied either under complete information (see, for example, Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993, 1996), Che and Gale (1998) and Siegel (2009)) or under incomplete information (see, for example, Hillman and Riley (1989), Amman and Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006) and Moldovanu et al. (2010)).

[2] Sequential all-pay auctions under complete information vave been studied in the literature. For example, Leininger (1991) modeled a patent race between an incumbent and an entrant as a sequential asymmetric all-pay auction under complete information, and Konrad and Leininger (2007) characterized the equilibrium of the all-pay auction under complete information in which a group of players choose their effort 'early' and the other group of players choose their effort 'late'.

makes an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then makes an effort in the second period. Contestant 2 wins the contest if his effort is larger than or equal to the effort of contestant 1; otherwise, contestant 1 wins. Here, we generalize this model and study a sequential all-pay auction with $n \geq 2$ contestants, who compete for a prize of size 1, where in each period of the contest, $1 \leq j \leq n$, a new contestant joins and chooses an effort. Contestant j, $j = 1, ..., n$ observes the efforts of all contestants in the previous $j-1$ periods and then exerts an effort in period j. Contestant j wins if his effort is larger than or equal to the efforts of all the contestants in the $j-1$ previous periods and strictly larger than the efforts of all the contestants in the following $n-j$ periods. We assume throughout that the contest designer's goal is to maximize the expected highest effort which indeed is the case in many real-life contests. This is especially true in sport competitions where the designer wishes to see as many records broken as possible as well as in R&D contests in which the goal of the society is that the product being developed will have the highest quality.

We first show that the expected highest effort in the multi-stage sequential all-pay contest is not necessarily monotonic in the number of contestants (stages). Thus, if the designer adds new contestants the expected highest effort might decrease. This implies that the designer can sometimes increase the expected highest effort by excluding some contestants from participation. This result holds regardless of whether the contestants are ex-ante symmetric or ex-ante asymmetric. The intuition is that each contestant when deciding what effort to exert takes into account his probability of winning which is determined by the identity and the number of contestants that perform after him. Therefore, as we formally show, the contest designer may sometimes increase the expected highest effort by reducing the number of stages or, alternatively, the number of active contestants since then the contestants in the early stages will exert higher efforts. Moreover, similarly to the result of Moldovanu and Sela (2006) in the simultaneous all-pay auction with n players under incomplete information, we show that in the multi-stage sequential all-pay auction under incomplete information the optimal expected highest effort might be obtained for any number of contestants $2 \leq k \leq n$. Thus, the expected highest effort is neither increasing nor decreasing in the number of contestants.

In our sequential model we say that contestant A is stronger than contestant B if contestant A's ability distribution first-order stochastically dominates contestant B's ability distribution. In that case, we obtain another interesting result that does not hold in the simultaneous all-pay auction under incomplete information. This result shows that the expected highest effort in a contest among n weak players might be higher than the expected highest effort in a contest among n strong players. Therefore, a contest designer who wishes to maximize the expected highest effort might want to choose a pool of contestants who are not too strong but also not too weak.

In order to address the question of how to order asymmetric players over the stages according to their ability distributions, we focus here on a family of distribution functions of the contestants' abilities for which we are able to explicitly derive the contestants' equilibrium efforts. We show that in the two-stage sequential all-pay auction the stronger contestant should always be allocated to the first stage and the weaker to the second. However, if the number of contestants is larger than two, the optimal order of the players cannot be determined and depends on the exact distribution of the contestants' abilities. On the other hand, we show that if one of the contestants is substantially stronger than the others, he should never be allocated to the last stage. Interestingly, in several other contest forms, it is shown that the strongest contestant should be excluded in order to increase the competitive balance and therefore the contestants' expected efforts (see Baye, Kovenock and de Vries 1993). In our sequential model, however, we show that the designer can never achieve a strictly higher expected highest effort by excluding the strongest contestant.

1.1 Related literature

It is well known that in order to maximize the players' expected efforts it might be profitable to exclude the players with the lower abilities. Fullerton and McAfee (1999) showed in a two-stage model that the optimal research tournament requires competing firms to participate in an all-pay auction with entry fees, while only a subset of the most competitive firms engage in innovation activities. Fu and Lu (2009) studied a multi-stage simultaneous elimination Tullock contest, and showed that the optimal contest eliminates one contestant at each stage until the final, and the winner of the final takes the entire prize sum. In our sequential model, however, some players should be excluded but not necessarily those with the lowest abilities.

Furthermore, from the literature on contests we know that the expected effort in any two-player contest does not solely depend on the ability (or strength) of the respective players, but also on their relative ability. While Baye et al. (1993), as was mentioned previously, looked for the optimal set of contestants in an all-pay auction, and found that it is sometimes beneficial to exclude the strongest player, Clark and Riis (1998) showed that, similarly to our sequential model, it does not pay-off to exclude the strongest player from a simultaneous moves contest if there are several prizes.

The question of how to allocate the players according to their types (abilities) has been extensively studied by several researchers. Rosen (1986) considered an elimination tournament with homogeneous players where the probability of winning a match is a stochastic function of the players' efforts. His main result is that rewards in later stages must be higher than rewards in earlier stages in order to sustain a non-decreasing effort along the tournament. He also considered an example with four players who can be either "strong" or "weak" and found (numerically) that a random seeding yields a higher total effort than the seeding where strong players meet weak players in the semifinals. Groh et al. (2012) studied an elimination all-pay auction with heterogenous players whose ability is common knowledge. For tournaments with four players, they found optimal seedings for several criteria. While these authors focused on which types of players should be in the final, Amegashie (1999) determined the optimal number of finalists in a two-stage contest à la Tullock with homogenous players.

In a related paper referred to earlier, Moldovanu and Sela (2006) studied a two-stage all-pay auction under incomplete information and examined how the players are split among several sub-contests whose winners compete against each other (while other players are eliminated). They showed that if the designer maximizes the expected total effort, the optimal architecture is a single grand static contest but if he maximizes the expected highest effort and if there are sufficient competitors, it is optimal to split the competitors into two divisions, and then hold a final among the two divisional winners. Amegashie (2000) studied a two-stage contests and compared between pooling (players compete against all others in each stage) and grouping (players are divided into groups). He showed that the former generates a higher rent-dissipation rate.

The rest of the paper is organized as follows: Section 2 characterizes the subgame perfect equilibrium of our multi-stage sequential all-pay auction. Section 3 analyzes the effect of the number of players and their types on the expected highest effort, while Section 4 analyzes the effect of the players' allocation over the stages on the expected highest effort. Section 5 concludes. Some of the proofs are in the Appendix.

2 The model

We consider a multi-stage sequential all-pay auction with $n \geq 2$ contestants who compete for a prize of size 1. Contestants arrive one by one. In period j, $1 \leq j \leq n$, contestant j observes the efforts of contestants $1, 2, ..., j-1$ in the previous periods and then exerts an effort x_j. The winner is the contestant who exerts the highest effort. We break ties in favor of later contestants. Thus, contestant j wins the contest if his effort is larger than or equal to the efforts of all the contestants in the previous periods and if his effort is strictly larger than the efforts of all the contestants in the following periods. Formally, contestant j wins iff $x_j \geq x_i$ for all $i < j$ and $x_j > x_i$ for all $i > j$. Contestant j's cost of effort x_j is given by $\frac{x_j}{a_j}$ where a_j is a parameter that describes his ability (a higher a_j causes a lower cost and therefore a higher ability) and is private information to contestant j.[3] It is common knowledge among the contestants that contestant j's ability a_j is drawn from a distribution F_j with a continuous and positive density function f_j and a support $[0, 1]$. Our aim is to characterize the perfect Bayesian equilibrium effort strategies of the contestants.

We start the analysis by describing contestant n's effort strategy. Contestant n exerts an effort that is equal to the highest effort of all the previous contestants as long as his type a_n is larger than or equal to this highest effort; otherwise he stays out of the contest. The equilibrium effort of contestant n is then given by

$$b_n(a_n; b_1, ..., b_{n-1}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_n < \max_{j<n} b_j \\ \max_{j<n} b_j & \text{if} \quad \max_{j<n} b_j \leq a_n \leq 1 \end{cases}$$

[3] An equivalent interpretation is that a_j is player's j valuation for the prize and his cost is equal to his bid.

where $b_j, j < n$ is the effort of contestant j. Given this equilibrium strategy, contestant $n-1$'s maximization problem becomes

$$\max_b \left\{ F_n(b) - \frac{b}{a_{n-1}} \right\}$$

$$s.t \quad b \geq \max_{j=1,\ldots,n-2} b_j$$

Let $\gamma_{n-1} = \max_{j=1,\ldots,n-2} b_j$. By deriving the above maximization problem w.r.t. b we get that if F_n is concave then contestant $n-1$'s equilibrium effort strategy is given by

$$b_{n-1}(a_{n-1}; b_1, \ldots, b_{n-2}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_{n-1} < \frac{\gamma_{n-1}}{F_n(\gamma_{n-1})} \\ \gamma_{n-1} & \text{if} \quad \frac{\gamma_{n-1}}{F_n(\gamma_{n-1})} \leq a_{n-1} < \min\left\{ \frac{1}{f_n(\gamma_{n-1})}, 1 \right\} \\ (f_n)^{-1}\left(\frac{1}{a_{n-1}}\right) & \text{if} \quad \min\left\{ \frac{1}{f_n(\gamma_{n-1})}, 1 \right\} \leq a_{n-1} \leq 1 \end{cases}$$

Note that since $\frac{\gamma_{n-1}}{F_n(\gamma_{n-1})} > \gamma_{n-1}$, some types of contestant $n-1$ with higher abilities than the maximum effort of contestants $1, \ldots, n-2$ choose to stay out of the contest and do not exert an effort. The reason is simply that if such a contestant exerts an effort of γ_{n-1} (the lowest effort that provides him a positive probability of winning) then his expected payoff is $F_n(\gamma_{n-1}) - \frac{\gamma_{n-1}}{a_{n-1}}$ which is non-negative only when $a_{n-1} \geq \frac{\gamma_{n-1}}{F_n(\gamma_{n-1})}$. Moreover, if $\frac{\gamma_{n-1}}{F_n(\gamma_{n-1})} > \frac{1}{f_n(\gamma_{n-1})}$ contestant $n-1$ does not exert an effort of γ_{n-1} and we have

$$b_{n-1}(a_{n-1}; b_1, \ldots, b_{n-2}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_{n-1} < \frac{\gamma_{n-1}}{F_n(\gamma_{n-1})} \\ (f_n)^{-1}\left(\frac{1}{a_{n-1}}\right) & \text{if} \quad \frac{\gamma_{n-1}}{F_n(\gamma_{n-1})} \leq a_{n-1} \leq 1 \end{cases}$$

Finally, as long as $\frac{\gamma_{n-1}}{F_n(\gamma_{n-1})} \leq a_{n-1} < \min\left\{ \frac{1}{f_n(\gamma_{n-1})}, 1 \right\}$ the constraint in the maximization problem of contestant $n-1$ is binding and then this contestant exerts an effort that is equal to γ_{n-1}, the highest effort of the contestants in the previous stages. When $a_{n-1} \geq \min\left\{ \frac{1}{f_n(\gamma_{n-1})}, 1 \right\}$, the constraint in the maximization problem of contestant $n-1$ is not binding and then his equilibrium effort is obtained by solving the following first-order condition (F.O.C.):

$$a_{n-1} f_n(b_{n-1}(s)) b'_{n-1}(s) - b'_{n-1}(s) = 0$$

The second-order condition (S.O.C.) is then

$$a_{n-1} f'_n(b_{n-1}(s))(b'_{n-1}(s))^2 + a_{n-1} f_n(b_{n-1}(s)) b''_{n-1}(s) - b''_{n-1}(s) = a_{n-1} f'_n(b_{n-1}(s))(b'_{n-1}(s))^2$$

If F_n is not concave then the S.O.C. does not hold. In particular, if F_n is convex then $F_n(b) - \frac{b}{a_{n-1}}$ is negative for all b. Therefore $b_{n-1}(a_{n-1}) = 0$. Thus, in the rest of this paper we assume that F_n is concave. Contestant $n-2$'s maximization problem is then

$$\max_b \left\{ F_{n-1}\left(\frac{b}{F_n(b)}\right) F_n(b) - \frac{b}{a_{n-2}} \right\}$$
$$s.t \;\; b \geq \max_{j=1,\dots,n-3} b_j$$

Now, let $\gamma_{n-2} = \max_{j=1,\dots,n-3} b_j$. By deriving the above maximization problem w.r.t. b we get that if $G_{n-2}(x) = F_{n-1}\left(\frac{x}{F_n(x)}\right) F_n(x)$ is concave, contestant $n-2$'s equilibrium effort strategy is given by

$$b_{n-2}(a_{n-2}; b_1, \dots, b_{n-3}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_{n-2} < \frac{\gamma_{n-2}}{F_{n-1}\left(\frac{\gamma_{n-2}}{F_n(\gamma_{n-2})}\right) F_n(\gamma_{n-2})} \\ \gamma_{n-2} & \text{if} \quad \frac{\gamma_{n-1}}{F_{n-1}\left(\frac{\gamma_{n-2}}{F_n(\gamma_{n-2})}\right) F_n(\gamma_{n-2})} \leq a_{n-2} < \bar{a}_{n-2} \\ h(a_{n-2}) & \text{if} \quad \bar{a}_{n-2} \leq a_{n-2} \leq 1 \end{cases}$$

where $h(a_{n-2})$ is implicitly defined as the solution of the following F.O.C.

$$f_{n-1}\left(\frac{b}{F_n(b)}\right)\left(1 - \frac{b f_n(b)}{F_n(b)}\right) + F_{n-1}\left(\frac{b}{F_n(b)}\right) f_n(b) - \frac{1}{a_{n-2}} = 0$$

and $\bar{a}_{n-2}$ is the minimum between 1 and the type a_2 for which $h(a_{n-2}) = \gamma_{n-2}$.

At this stage, it is clear that we cannot derive a closed-form solution to the maximization problem of contestant j for $j \leq n-2$. In the next subsection, we therefore restrict attention to a specific family of concave distribution functions ($F_i(x) = x^{c_i}, 0 < c_i < 1$) for which we are able to explicitly calculate the subgame perfect equilibrium effort of each contestant. This will allow us to derive some general (negative) results on multi-stage sequential all pay auctions.

2.1 A special case $F_i(x) = x^{c_i}$

We now assume that the distributions of contestants' abilities are $F_i(x) = x^{c_i}$ for $i = 1, \dots, n$ where $0 < c_i < 1$. Let $d_j = \Pi_{l=j+1}^{n}(1 - c_l)$ for $j = 1, \dots, n-1$ and $d_n = 1$. Moreover, let $\gamma_j = \max_{i<j} b_i(a_i; b_1, \dots, b_{i-1})$ for $j = 1, \dots, n$. Then we have the following perfect Bayesian equilibrium.

Proposition 1 *In the multi-stage sequential all-pay auction when $F_i(x) = x^{c_i}$, $i = 1, ..., n$, $0 < c_i < 1$, the perfect Bayesian equilibrium effort strategies are given by*

$$b_n(a_n; b_1, ..., b_{n-1}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_n < \gamma_n \\ \gamma_n & \text{if} \quad \gamma_n \leq a_n \leq 1 \end{cases} \tag{1}$$

for $j = 2, ..., n-1$

$$b_j(a_j; b_1, ..., b_{j-1}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_j < \gamma_j^{d_j} \\ \gamma_j & \text{if} \quad \gamma_j^{d_j} \leq a_j < \min\left\{\frac{1}{(1-d_j)}\gamma_j^{d_j}, 1\right\} \\ ((1-d_j)\,a_j)^{\frac{1}{d_j}} & \text{if} \quad \min\left\{\frac{1}{(1-d_j)}\gamma_j^{d_j}, 1\right\} \leq a_j \leq 1 \end{cases} \tag{2}$$

and,

$$b_1(a_1) = ((1-d_1)\,a_1)^{\frac{1}{d_1}} \text{ for all } 0 \leq a_1 \leq 1 \tag{3}$$

Proof. *In the Appendix.* ■

Note that $\gamma_j^{d_j} < \min\left\{\frac{1}{(1-d_j)}\gamma_j^{d_j}, 1\right\}$ since $\gamma_j < 1$ (no contestant ever exerts an effort higher than 1) and $d_j < 1$. In this case, we observe that a contestant's effort in stage j, $1 \leq j \leq n-1$ is not affected by the order of contestants in the following stages, only by their identity. This is true since the only relevant parameter for this contestant is $d_j = \Pi_{l=j+1}^{n}(1-c_l)$. Furthermore, since $b_1(a_1) > 0$ for all $0 < a_1 \leq 1$ and $\gamma_j^{d_j} < 1$ for all $j = 2, ..., n$ we can conclude that in a multi–stage all-pay auction where $F_i(x) = x^{c_i}, 0 < c_i < 1, i = 1, ..., n$ there are types of contestants who exert positive efforts in all the n stages of the sequential contest.

3 The optimal number of contestants

We assume now that the contest designer wishes to maximize the expected highest effort. Therefore, given a realization of the contestants' abilities, we define the highest effort as

$$HE(a_1, ..a_n) = \max\{b_1(a_1), b_2(a_2; b_1), ..., b_n(a_n; b_1, ..., b_{n-1})\}$$

The expected highest effort is then given by

$$HE = \int_0^1 .... \int_0^1 HE(a_1, ..a_n)\, f_n(a_n)\, da_n.....f_1(a_1)\, da_1$$

We first prove an interesting and rather counter-intuitive result which shows that the expected highest effort is non-monotonic in the number of contestants n. Therefore, adding more contestants is not always profitable when the goal is to maximize the expected highest effort.

Proposition 2 *The expected highest effort in a multi-stage sequential all-pay auction, regardless of whether the contestants are ex-ante symmetric or asymmetric, is not necessarily monotonic in the number of contestants n.*

Proof. *Consider a multi-stage sequential all-pay auction with j symmetric contestants. Denote by $HE_{n=j}(c)$ the expected highest effort in the contest with j, $j = 2, 3, 4$ contestants and a distribution function $F(x) = x^c, 0 < c \leq 1$. The following figure presents the expected highest efforts $HE_{n=j}(c), j = 2, 3, 4$ as a function of the parameter c.*

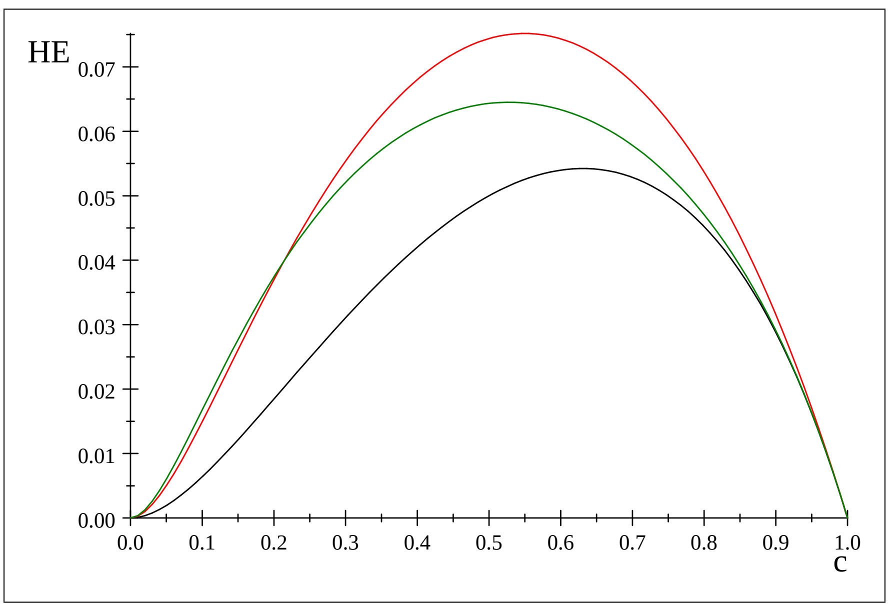

Figure 1: The expected highest effort as a function of c. For $n = 2$ the curve is in black, for $n = 3$ it is in red and for $n = 4$ it is in green.

For example, when $c = \frac{1}{2}$ we have $HE_{n=3}(c) = 0.074388 > HE_{n=4}(c) = 0.064331$. The complete proof for when the contestants are ex-ante asymmetric or symmetric including the formal calculations is given in the Appendix. ■

We now present the following definition that we need for the rest of the paper.

Definition 1 *We say that contestant i is stronger than contestant j if F_i first-order stochastically dominates F_j i.e., $F_j(x) \geq F_i(x)$ for all $x \in [0,1]$ and for some x the inequality is strict.*

The intuition behind the result of Proposition 2, at least when the contestants are ex-ante asymmetric, is as follows: When we allocate, for example, a strong contestant in the last stage, the probability of the other contestants in the previous stages of winning the contest becomes quite low and therefore they exert lower efforts. The strong contestant that is allocated in the last stage only equalizes the highest effort of the other contestants, and therefore he makes no real contribution to the expected highest effort. The result is that the allocation of the strong contestant in the last stage may decrease the expected highest effort. However, even if the contestants are ex-ante symmetric we show that increasing the number of contestants can strictly decrease the expected highest effort. We recall that Moldovanu and Sela (2006) proved that in a simultaneous all-pay auction under incomplete information with n contestants, the optimal highest effort is obtained either for n contestants or for any number smaller than n. We similarly show that in the multi-stage sequential all-pay auction when $F_i(x) = x^c$ for all i and $n = 4$, the optimal number of contestants could be either 2 or 3 or 4.

We now derive another interesting result, namely that, when two sets of n ex-ante symmetric contestants are given such that all the contestants in one set are stronger than those in the other set, the set with the weaker contestants might yield a higher expected highest effort in equilibrium. Thus, if a designer wants to maximize the expected highest effort and can choose between a set of strong contestants and a set of weak contestants, it is not necessarily true that he should choose the set of strong contestants.

Proposition 3 *Given two sets of n ex-ante symmetric contestants I_1 and I_2, assume that the ability of each contestant in I_1 is drawn from the distribution function F_1 and that of each contestant in I_2 is drawn from the distribution function F_2 such that F_1 first-order stochastically dominates F_2. Then, the expected highest effort in the multi-stage sequential all-pay auction with the set of contestants I_1 might be either larger or smaller than in the multi-stage sequential all-pay auction with the set of contestants I_2.*

Proof. *This result is easily obtained from the proof of Proposition 2. Note that when the distribution functions are* $F_i(x) = x^{c_i}, 0 < c_i < 1$*, contestant* i *is stronger than* j *if* $c_i > c_j$*. Now, observe that* $HE_{n=2}(c), HE_{n=3}(c)$ *and* $HE_{n=4}(c)$ *are all non-monotonic in* c *and therefore, for example, with three contestants when* $c = \frac{1}{2}$*, we have an expected highest effort that is equal to* $HE_{n=3}\left(c = \frac{1}{2}\right) = 0.07438$*. With a set of three much stronger contestants, however, when, for example,* $c = \frac{4}{5}$*, we have a lower expected highest effort that is equal to* $HE_{n=3}\left(c = \frac{4}{5}\right) = 0.0536$*.* ■

The intuition for the result in Proposition 3 is that when the contestants are relatively strong the contestants that are allocated in the first stages exert low efforts since there is a high probability that their rivals in the last stages will have a high ability and will win the contest. On the other hand, when the contestants are relatively weak, the (high-ability) contestants in the first stages exert relatively high efforts since they believe they have a high probability to win against the contestants in the last stages.

4 The optimal order of contestants

Given that the contestants are ex-ante asymmetric it is interesting to investigate what the optimal allocation of contestants is over the stages of the multi-stage sequential all-pay auction. Should the stronger (weaker) contestants be allocated to the first stages or the last stages? We focus here on the special case where $F_i(x) = x^{c_i}, 0 < c_i < 1$ but we hypothesize that some of the results below will hold for a larger family of distribution functions. We begin with an analysis of a two stage contest.

Proposition 4 *In the two-stage sequential all-pay auction when* $F_i(x) = x^{c_i}$, $0 < c < 1$*, the effort of a sufficiently high ability contestant in the first stage increases in the strength (strong/weak) of the contestant in the second stage, while the effort of a sufficiently low ability contestant decreases in the strength of the contestant in the second stage. Formally, for a sufficiently high value of ability* a *we have*

$$\frac{d}{dc_2} b_1(a, c_2) > 0$$

and for a sufficiently small value of ability a we have

$$\frac{d}{dc_2} b_1(a, c_2) < 0$$

Proof. In the Appendix. ■

The following example illustrates the result of Proposition 4.

Example 1 *Assume a two-stage sequential all-pay auction. If the distribution of contestant 2's ability in the second stage is* $F_2(x) = x^{\frac{1}{2}}$*, then by Proposition 1 the equilibrium effort of contestant 1 in the first stage is given by* $b_1(a) = \frac{a^2}{4}$*. On the other hand, if the distribution of contestant 2's ability in the second stage is* $G_2(x) = x^{\frac{4}{5}}$ *which first order stochastically dominates* F_2*, then the equilibrium effort of contestant 1 in the first stage is given by* $\widetilde{b}_1(a) = 0.327a^5$*. These equilibrium efforts in the first stage (as a function of a) are given in the following figure*

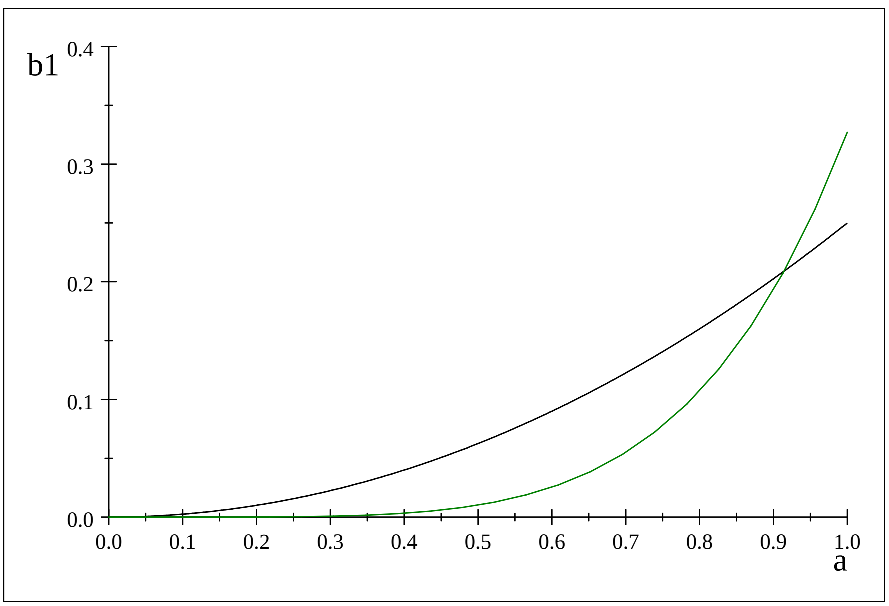

Figure 2: equilibrium bid as a function of the type. b_1 *- in black,* $\widetilde{b}_1$ *- in green*

This figure shows that $\widetilde{b}_1(a) > b_1(a)$ *for all* $a > 0.914$ *and* $\widetilde{b}_1(a) \leq b_1(a)$ *for all* $a \leq 0.914$.

Usually in simultaneous contests and particularly in simultaneous all-pay auctions under incomplete information, most of the total effort comes from the high ability types. According to Proposition 4 the high types exert relatively high efforts in the first stage when there is a strong contestant in the second stage while they exert relatively low efforts when there is a weak contestant in the second stage. However, for our sequential two-stage contest the following result shows that it is optimal to allocate the stronger contestant in the first stage.

Proposition 5 *In a two-stage sequential all-pay auction when* $F_i(x) = x^{c_i}$ *,* $0 < c_i < 1, i = 1, 2$ *the expected highest effort is higher when the stronger contestant is allocated in the first stage, i.e., if* $c_1 > c_2$ *then,*

$$HE_{n=2}(c_1, c_2) \geq HE_{n=2}(c_2, c_1)$$

Proof. *In the Appendix.* ■

From the optimal order in the two-stage sequential all-pay auction we can obtain the following result for the multi-stage sequential all-pay auction.

Proposition 6 *In the multi-stage sequential all-pay auction when* $F_i(x) = x^{c_i}, 0 < c_i < 1,$ $i = 1, ..., n$ *if contestant* j *is sufficiently stronger than all the other contestants (*$c_j >> c_i$ *for all* $i \neq j$*), a designer who maximizes the expected highest effort will not allocate contestant* j *in the last stage.*

Proof. In the Appendix. ■

Following Proposition 6, one might think that if there is a very strong contestant it is better that he be allocated in the first stage to avoid a negative effect on the contestants' efforts in the previous stages. However, when we move the strong contestant to the first stage his effort will be smaller than if he would be allocated in a later stage, and since this contestant makes a meaningful contribution to the expected highest efforts, it is not necessarily optimal that he be allocated in the first stage.

Our next result shows that when we have a multi-stage sequential all-pay auction with n contestants it is always beneficial in terms of the expected highest effort to replace the contestant in the first stage with a different, stronger contestant.

Proposition 7 *Assume a multi-stage sequential all-pay auction when* $F_i(x) = x^{c_i}, 0 < c_i < 1$*. Then, if* $\gamma > c_1$ *we have*

$$HE(\gamma, c_2, c_3, ..., c_n) > HE(c_1, c_2, c_3 ..., c_n)$$

In other words, the expected highest effort $HE(c_1, c_2, c_3..., c_n)$ is increasing in c_1.

Proof. *In the Appendix.* ■

Assume now that we have a set of n contestants $N = \{1, 2, ..., n\}$ with distribution functions $F_i(x) = x^{c_i}, 0 < c_i < 1$. Let $\sigma_K \in R^{|K|}$ be an order of the contestants in a given subset $K \subset N$ with $n-1$ or less contestants, where σ_K^i indicates the identity (c_i) of the contestant in stage i according to the order σ_K. We denote by D_K the set of all such orders for a given subset K. Then $|D_K| = \frac{n!}{(n-|K|)!}$. We say that σ_A is (weakly) preferred over σ_B if the expected highest effort under the order σ_A is higher than or equal to the expected highest effort under the order σ_B, that is, $HE(\sigma_A) \geq HE(\sigma_B)$. We denote the optimal order for a subset $K \subset N$ by $\widetilde{\sigma}_K$, i.e.,

$$\widetilde{\sigma}_K = \arg \max_{\sigma_K \in D_K} HE(\sigma_K)$$

Let T be an optimal subset such that

$$T = \arg\max{}_{K \subset N} HE(\widetilde{\sigma}_K)$$

Then, by Proposition 7 we show that

Proposition 8 *In the multi-stage sequential all-pay auction when $F_i(x) = x^{c_i}, 0 < c_i < 1$, let $T = \arg\max_{K \subset N} HE(\widetilde{\sigma}_K)$. If T is uniquely determined (no other subset K maximizes $HE(\widetilde{\sigma}_K)$) and if s is the strongest contestant, i.e. $c_s = \max\{c_1, ..., c_n\}$, then $s \in T$.*

Moreover, if T is not uniquely determined then the strongest player s must be included in one of the optimal subsets. The immediate conclusion of Proposition 8 is

Conclusion 1 *In the multi-stage sequential all-pay auction when $F_i(x) = x^{c_i}, 0 < c_i < 1$, it is not profitable for a designer who wishes to maximize the expected highest effort to exclude the strongest contestant.*

To sum up, we showed that the expected highest effort may be non-monotonic in the number of contestants so it might be profitable to exclude some contestants. However, we also showed that the strongest contestant should always be allocated in one of the stages.

5 Concluding remarks

We studied multi-stage sequential all-pay auctions under incomplete information and showed that the contestants' expected highest effort is not necessarily increasing or decreasing in the number of contestants regardless of whether they are ex-ante symmetric or asymmetric. However, while some contestants might be excluded in order to increase the expected highest effort, the strongest contestant will not be among them. Since it is not possible to explicitly calculate the contestants' equilibrium efforts for all distribution functions of the contestants' ability our results are restrictive to a specific family of distribution functions, but we hypothesize that these results hold for a larger family of distribution functions. Moreover, and importantly, by assuming a specific family of distribution functions we obtain general 'negative' results (i.e., non-monotonicity in the number of contestants) about the contestants' behavior. Our analysis sheds light on the multi-stage sequential all-pay auction but it also indicates the complexity of this sequential contest form. Thus, there remains much work to be done in order to understand this interesting model.

6 Appendix

6.1 Proof of Proposition 1

The equilibrium effort function of contestant n, which is given by (1), is straightforward. Assume then that contestants $1, ..., i-1, i+1, ..., n$ behave according to the strategies given by (2) and (3). Recall that $d_j = \Pi_{l=j+1}^{n}(1-c_l)$ for $j = 1, ..., n-1$ and $d_n = 1$. Then contestant i solves the following maximization problem

$$\max_{b}\left\{\Pi_{j=i+1}^{n}F_j\left(b^{d_j}\right) - \frac{b}{a_i}\right\}$$

$$s.t \quad b \geq \max_{j=1,\ldots,i-1} b_j$$

By deriving this maximization problem w.r.t. b we obtain that, since $H_i(x) = \Pi_{j=i+1}^{n}F_j\left(x^{d_j}\right)$ is concave for $i = 2, ..., n$ (note that by reordering we have $H_i(x) = x^{1-d_i}$) then indeed $b_i(a_i) = ((1-d_i)a_i)^{\frac{1}{d_i}}$ maximizes the expected payoff of contestant i conditional on

$$b_i(a_i) = ((1-d_i)a_i)^{\frac{1}{d_i}} \geq \gamma_i \Leftrightarrow a_i \geq \frac{1}{(1-d_i)}\gamma_i^{d_i}$$

Moreover, when $0 \leq a_j < \gamma_i^{d_i}$ an effort of $b_i = \gamma_i$ yields a negative payoff. Thus, we get for $i = 2, ..., n-1$

$$b_i(a_i; b_1, ..., b_{i-1}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_i < \gamma_i^{d_i} \\ \gamma_i & \text{if} \quad \gamma_i^{d_i} \leq a_i < \min\left\{\frac{1}{(1-d_i)}\gamma_i^{d_i}, 1\right\} \\ \left((1-d_i)\, a_i\right)^{\frac{1}{d_i}} & \text{if} \quad \min\left\{\frac{1}{(1-d_i)}\gamma_i^{d_i}, 1\right\} \leq a_i \leq 1 \end{cases}$$

Finally, for contestant 1, $\gamma_1 = 0$ and we obtain

$$b_1(a_1) = \left((1-d_1)\, a_1\right)^{\frac{1}{d_1}} \text{ for } 0 \leq a_1 \leq 1$$

■

6.2 Proof of Proposition 2

We prove this proposition using the special case in subsection 2.1 for which we have a closed-form solution of the contestants' behavior in equilibrium. Thus, $F_j(x) = x^{c_j}, 0 < c_j < 1$, $d_j = \Pi_{l=j+1}^{n}(1-c_l)$ for $j = 1, ..., n-1$ and $d_n = 1$. Then, the expected highest effort in a two-stage contest is

$$HE_{n=2}(c_1, c_2) = \int_0^1 \left((c_2 a_1)^{\frac{1}{1-c_2}}\right) c_1 a_1^{c_1-1} da_1 = \frac{c_2^{\frac{1}{1-c_2}} c_1 (1-c_2)}{c_1(1-c_2)+1} = \frac{c_1 d_1 (1-d_1)^{\frac{1}{d_1}}}{c_1 d_1 + 1} \tag{4}$$

and the expected highest effort in the three-stage contest is (note that d_1 here is different from d_1 in the above expression for two contestants)

$$\begin{aligned} HE_{n=3}(c_1, c_2, c_3) &= \int_0^{\frac{c_3^{1-c_2}}{(1-d_1)}} \left(\int_0^{\frac{((1-d_1)a_1)^{\frac{1}{(1-c_2)}}}{c_3}} \left(((1-d_1)\, a_1)^{\frac{1}{d_1}}\right) c_2 a_2^{c_2-1} da_2 \right) c_1 a_1^{c_1-1} da_1 \\ &+ \int_0^{\frac{c_3^{1-c_2}}{(1-d_1)}} \left(\int_{\frac{((1-d_1)a_1)^{\frac{1}{(1-c_2)}}}{c_3}}^1 \left((c_3 a_2)^{\frac{1}{(1-c_3)}}\right) c_2 a_2^{c_2-1} da_2 \right) c_1 a_1^{c_1-1} da_1 \\ &+ \int_{\frac{c_3^{1-c_2}}{(1-d_1)}}^1 \left(((1-d_1)\, a_1)^{\frac{1}{(1-c_2)(1-c_3)}}\right) c_1 a_1^{c_1-1} da_1 \\ &= \frac{c_3^{\frac{1+c_1 d_1}{d_2}} c_2 d_2}{(1+c_1 d_1)(1+c_2 d_2 + c_1 d_1)(1-d_1)^{c_1}} + \frac{c_1 d_1 (1-d_1)^{\frac{1}{d_1}}}{(1+c_1 d_1)} \end{aligned} \tag{5}$$

For example when $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ and $c_3 = \frac{4}{5}$ we have $HE_{n=2}(c_1, c_2) = 0.05 > HE_{n=3}(c_1, c_2, c_3) = 0.04365$. Note that the third contestant we add is a very strong contestant (his distribution of ability first-order stochastically dominates the other contestants' distribution functions) and the expected highest effort decreases when he is added to the contest.

When contestants are ex-ante symmetric i.e., $c_1 = c_2 = ... = c_n = c$ we have $d_j = (1-c)^{n-j}$ and then the expected highest efforts in the contest with two, three and four contestants are given by

$$HE_{n=2}(c) = \int_0^1 \left((ca_1)^{\frac{1}{1-c}}\right) ca_1^{c-1} da_1 = \frac{c^{\frac{2-c}{1-c}}(1-c)}{c(1-c)+1}$$

$$\begin{aligned} HE_{n=3}(c) &= \frac{c^{\frac{c(1-c)^2+2-c}{1-c}}(1-c)}{\left(1+c(1-c)^2\right)\left(1+c(1-c)+c(1-c)^2\right)\left(1-(1-c)^2\right)^c} \\ &\quad + \frac{c(1-c)^2\left(1-(1-c)^2\right)^{\frac{1}{(1-c)^2}}}{\left(1+c(1-c)^2\right)} \end{aligned}$$

$$
HE_{n=4}(c) = \left(\begin{array}{c}
\frac{(1-c)^3 c^{\frac{(2-c)\left(1+c(1-c)^2\right)}{1-c}}}{\left(1+c(1-c)+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} + \frac{c^{\frac{(2-c)\left(1+c(1-c)^2\right)}{1-c}}(1-c)^2}{(1+c(1-c))(2-c)\left(1-(1-c)^3\right)^c} \\
-\frac{c^{\frac{(1+c(1-c))\left(c^2+3(1-c)\right)}{1-c}}(1-c)^4}{(1+c(1-c))\left(1+c(1-c)+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
+\frac{(1-c)^4 c^{\frac{2+2c-6c^2+4c^3-c^4}{1-c}}}{(2-c)\left(1+c(1-c)^3\right)\left(1-(1-c)^2\right)^c\left(1-(1-c)^3\right)^c} \\
-\frac{(1-c)^4\left(1-(1-c)^2\right)^{\frac{c}{(1-c)}} c^{\frac{\left(2+2c-6c^2+4c^3-c^4\right)}{(1-c)}}}{(2-c)\left(1+c(1-c)+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
+\frac{(1-c^c)(1-c)^2 c^{\frac{(2-c)\left(1+c-2c^2+c^3\right)}{1-c}}}{\left(1+c(1-c)+c(1-c)^2\right)\left(1-(1-c)^2\right)^c\left(1-(1-c)^3\right)^c} \\
-\frac{(1-c^c)\left(1-(1-c)^2\right)^{\frac{1+c(1-c)}{(1-c)^2}}(1-c)^5 c^{\frac{\left(1+c-c^2\right)\left(3-3c+c^2\right)}{1-c}}}{\left(1+c(1-c)+c(1-c)^2\right)\left(1+c(1-c)+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
+\frac{(1-c)c^{\frac{(2-c)\left(1+c-2c^2+c^3\right)}{1-c}}}{(1+c(1-c))\left(1-(1-c)^2\right)^c\left(1-(1-c)^3\right)^c} - \frac{(1-c)^2 c^{\frac{(2-c)\left(1+c-2c^2+c^3\right)}{1-c}}}{(2-c)(1+c(1-c))\left(1-(1-c)^3\right)^c} \\
-\frac{c^{\frac{\left(1+c-c^2\right)\left(3-3c+c^2\right)}{1-c}}(1-c)^3}{(1+c(1-c))\left(1+c(1-c)+c(1-c)^2\right)\left(1-(1-c)^2\right)^c\left(1-(1-c)^3\right)^c} \\
+\frac{(1-c)^6\left(1-(1-c)^2\right)^{\frac{1+c(1-c)}{(1-c)^2}} c^{\frac{4-c-5c^2+4c^3-c^4}{1-c}}}{(1+c(1-c))\left(1+c(1-c)+c(1-c)^2\right)\left(1+c(1-c)+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
+\frac{c(1-c)^2\left(1-(1-c)^2\right)^{\frac{1+c-3c^2+3c^3-c^4}{(1-c)^2}}}{\left(1+c(1-c)^2\right)\left(1-(1-c)^3\right)^c} - \frac{c^{\left(1+c(1-c)^2\right)}(1-c)^2\left(1-(1-c)^2\right)^{\frac{1}{(1-c)^2}}}{\left(1+c(1-c)^2\right)\left(1-(1-c)^3\right)^c} \\
-\frac{c^2(1-c)^5\left(1-(1-c)^2\right)^{\frac{1+c(1-c)^3}{(1-c)^2}}}{\left(1+c(1-c)^2\right)\left(1+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
+\frac{c^{\frac{\left(1+c-c^2\right)\left(3-3c+c^2\right)}{1-c}}(1-c)^5}{\left(1+c(1-c)^2\right)\left(1-(1-c)^2\right)^c\left(1-(1-c)^3\right)^c\left(1+c(1-c)^2+c(1-c)^3\right)} \\
+\frac{c(1-c)^3\left(1-(1-c)^2\right)^{\frac{1+c(1-c)^3}{(1-c)^2}}}{\left(1+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
\\
-\frac{c^{\frac{1+(1-c)+c(1-c)^2+c(1-c)^3}{(1-c)}}(1-c)^3}{\left(1-(1-c)^2\right)^c\left(1+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
+\frac{c(1-c)^2\left(1-(1-c)^2\right)^{\frac{1+c(1-c)^3}{(1-c)^2}}}{\left(1+c(1-c)^2\right)\left(1-(1-c)^3\right)^c} - \frac{c^{\left(1+c(1-c)^2\right)}(1-c)^2\left(1-(1-c)^2\right)^{\frac{1}{(1-c)^2}}}{\left(1+c(1-c)^2\right)\left(1-(1-c)^3\right)^c} \\
\\
-\frac{c^2(1-c)^5\left(1-(1-c)^2\right)^{\frac{\left(1+c(1-c)^3\right)}{(1-c)^2}}}{\left(1+c(1-c)^2\right)\left(1+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^3\right)^c} \\
+\frac{c^{\frac{\left(1+c-c^2\right)\left(3-3c+c^2\right)}{(1-c)}}(1-c)^5}{\left(1+c(1-c)^2\right)\left(1+c(1-c)^2+c(1-c)^3\right)\left(1-(1-c)^2\right)^c\left(1-(1-c)^3\right)^c} \\
+\frac{c(1-c)^3\left(1-(1-c)^3\right)^{\frac{1}{(1-c)^3}}}{\left(1+c(1-c)^3\right)} - \frac{c(1-c)^3\left(1-(1-c)^2\right)^{\frac{1+c(1-c)^3}{(1-c)^2}}}{\left(1+c(1-c)^3\right)\left(1-(1-c)^3\right)^c}
\end{array} \right)
$$

Given the above expressions of the highest efforts when the contestants are ex-ante symmetric, if $c = \frac{1}{2}$ we have

$$HE_{n=3}(c) = \frac{4}{297} 2^{\frac{3}{4}} \sqrt{3} + \frac{9}{256} = 0.07438\,8$$

while

$$\begin{aligned} HE_{n=4}(c) &= \frac{11\,021}{4533\,760} 2^{\frac{1}{8}} \sqrt{7} - \frac{2573}{7254\,016} 2^{\frac{5}{8}} \sqrt{7} - \frac{9}{448} 2^{\frac{3}{8}} \sqrt{7} - \frac{20}{11\,781} 2^{\frac{5}{8}} \sqrt{3}\sqrt{7} \\ &\quad + \frac{2673}{144\,704} 3^{\frac{1}{4}} \sqrt{7} + \frac{436}{39\,501} 2^{\frac{1}{8}} \sqrt{3}\sqrt{7} + \frac{5764\,801}{285\,212\,672} \\ &= 0.06433\,1 \end{aligned}$$

Thus, although the contestants are ex-ante symmetric, the expected highest effort in the four-stage contest is smaller than in the three-stage contest. ■

6.3 Proof of Proposition 4

In the two-stage sequential all-pay auction if the distribution of the contestant's ability in the second stage is $F_2(c) = x^c$, then by (3) the equilibrium effort in the first stage is $b(a) = (ac)^{\frac{1}{1-c}}$. The derivative of $b(a)$ w.r.t. c is

$$\frac{d}{dc} b(a,c) = \frac{c^{\frac{c}{1-c}} a^{\frac{1}{1-c}}}{(1-c)^2} \left(c \ln ac + (1-c) \right)$$

The function $c \ln ac + (1-c)$ is increasing in a and is negative around $a = 0$ but is positive around $a = 1$. Thus, for sufficiently low values of a ($a \geq 0$) we have

$$\frac{d}{dc}(b(a,c) < 0$$

and for sufficiently high values of a ($a \leq 1$) we have

$$\frac{d}{dc}(b(a,c) > 0$$

■

6.4 Proof of Proposition 5

Assume a two-stage sequential all-pay auction when $F_i(x) = x^{c_i}, 0 < c_i < 1$, $c_1 \geq c_2$. Then by (4) we have

$$HE_{n=2}(c_1, c_2) \geq HE_{n=2}(c_2, c_1) \Leftrightarrow \frac{c_2^{\frac{1}{1-c_2}} c_1 (1-c_2)}{c_1(1-c_2)+1} \geq \frac{c_1^{\frac{1}{1-c_1}} c_2 (1-c_1)}{c_2(1-c_1)+1}$$

The inequality above holds iff

$$c_2 c_1 \left(c_2^{\frac{c_2}{1-c_2}} (1-c_2) - c_1^{\frac{c_1}{1-c_1}} (1-c_1) - (1-c_1)(1-c_2) \left(c_1^{\frac{1}{1-c_1}} - c_2^{\frac{1}{1-c_2}} \right) \right) \geq 0$$

or iff

$$\frac{c_2^{\frac{c_2}{1-c_2}}}{c_1^{\frac{c_1}{1-c_1}}} \geq \frac{1-c_1+(1-c_1)(1-c_2)c_1}{1-c_2+(1-c_1)(1-c_2)c_2} = \frac{(1-c_1)(1+c_1-c_1c_2)}{(1-c_2)(1+c_2-c_1c_2)} \tag{6}$$

Note that if $c_2 = c_1$ then we have equality in equation (6). Moreover, the function $\frac{c_2^{\frac{c_2}{1-c_2}}}{c_1^{\frac{c_1}{1-c_1}}}$ is decreasing in c_2, while the function $\frac{(1-c_1)(1+c_1-c_1c_2)}{(1-c_2)(1+c_2-c_1c_2)}$ is increasing in c_2 (for all $c_2 \in [0,1]$ given $c_1 \in [0,1]$). Thus, we get that for $c_2 \leq c_1$ the inequality (6) holds. ■

6.5 Proof of Proposition 6

We consider a multi-stage sequential all-pay auction where $F_i(x) = x^{c_i}, 0 < c_i < 1$, $i = 1, ..., n$. Assume that $c_n \to 1$. Then since $\lim_{c_n \to 1} d_j = \Pi_{l=j+1}^{n}(1-c_l) = 0$ we have for all $0 \leq a_j < 1, 1 \leq j \leq n-2$

$$\lim_{c_n \to 1} \left((1-d_j)\, a_j\right)^{\frac{1}{d_j}} = 0$$

Thus, by (2) we obtain that $\lim_{c_n \to 1} b_j(a_j; b_1, ..., b_{j-1}) = 0$ for all $0 \leq a_j < 1, 1 \leq j \leq n-2$. Let $HE(j,k), 1 \leq j \leq n$, now be the expected highest effort of contestants $j, j+1, ..., k$. The above argument yields that

$$\lim_{c_n \to 1} HE(1,n) = \lim_{c_n \to 1} HE(n-1,n) \tag{7}$$

Given that the order of contestants $n-1$ and n does not affect the equilibrium efforts of the first $n-2$ contestants, by (7) and Proposition 5 according to which if $c_1 > c_2$ then $HE_{n=2}(c_1, c_2) \geq HE_{n=2}(c_2, c_1)$, we obtain that if the stronger player ($c_n \to 1$) will be allocated in stage $n-1$ instead of stage n, independent of the order of the first $n-2$ contestants, the expected highest effort will be higher. ■

6.6 Proof of Proposition 7

We wish to prove that the expected highest effort HE is increasing in c_1. Note that by (1), (2) and (3) when the first contestant is replaced with a stronger contestant (c_1 increases) the equilibrium effort functions of all contestants remain the same (including that of the first contestant). The equilibrium effort function of each contestant depends only on the identity of the contestants in the later stages and therefore the equilibrium strategies are not changed if we replace the contestant in the first stage. In other words, the random variables

$$b_2(a_2; b_1), ..., b_n(a_n; b_1, ..., b_{n-1})$$

are the same when c_1 changes to $\gamma > c_1$. Moreover, it is still true that

$$b_1(a_1) = ((1-d_1)a_1)^{\frac{1}{d_1}} \text{ for } 0 \leq a_1 \leq 1$$

Let $HE(a_1)$ be the expected highest effort given a_1, i.e.,

$$HE(a_1) = \int_0^1 .... \int_0^1 \max\{b_1(a_1), b_2(a_2; b_1), ..., b_n(a_n; b_1, ..., b_{n-1})\} f_n(a_n)\, da_n .....f_2(a_2)\, da_2$$

Then $HE(a_1)$ is not a function of c_1 and does not change when moving from the distribution functions $F_i(x) = x^{c_i}, i = 1, ..., n$ to the distribution functions $F_1(x) = x^{\gamma}, F_i(x) = x^{c_i}, i = 2, ..., n$. We first need the following lemma.

Lemma 1 *$HE(a_1)$ is an increasing function of a_1.*

Proof. *We show that for each realization of $a_2, ..., a_n$ the highest effort is increasing in a_1. Given a realization of $a_1, ..., a_n$ we prove by induction on k, for $k = 2, ..., n$, that the highest effort among contestants $1, ..., k-1$, $\gamma_k(a_1) = \max_{i<k} b_i(a_i; b_1, ..., b_{i-1})$ is increasing in a_1 (keeping the realization of $a_2, ..., a_n$ constant).*

Note that the contestant n's effort is always smaller or equal to γ_n and therefore $HE(a_1) = \gamma_n(a_1)$. For $k = 2$ we have $\gamma_2(a_1) = b_1(a_1)$. Obviously $b_1(a_1)$ is increasing in a_1. Assume by the induction hypothesis that $\gamma_2(a_1), ...\gamma_{k-1}(a_1)$ are increasing in a_1. Then, given a_1, there are five possible cases for the realization of a_{k-1}:

1. *Case A: $0 \leq a_{k-1} < (\gamma_{k-1}(a_1))^{d_{k-1}}$ and$b_{k-1}(a_{k-1}; a_1, ..., a_{k-2}) = 0$.*

Assume that $a_1 < \tilde{a}_1$, then by the induction hypothesis $\left(\gamma_{k-1}(a_1)\right)^{d_{k-1}} \leq \left(\gamma_{k-1}(\tilde{a}_1)\right)^{d_{k-1}}$ and therefore $0 \leq a_{k-1} < \left(\gamma_{k-1}(a_1)\right)^{d_{k-1}} \leq \left(\gamma_{k-1}(\tilde{a}_1)\right)^{d_{k-1}}$ and $b_{k-1}(a_{k-1}; \tilde{a}_1, ..., a_{k-2}) = 0$. In this case $\gamma_k(\tilde{a}_1) = \gamma_{k-1}(\tilde{a}_1) \geq \gamma_{k-1}(a_1) = \gamma_k(a_1)$. Therefore $\gamma_k(\tilde{a}_1) \geq \gamma_k(a_1)$.

2. Case B: $\left(\gamma_{k-1}(a_1)\right)^{d_{k-1}} \leq a_{k-1} \leq \left(\gamma_{k-1}(\tilde{a}_1)\right)^{d_{k-1}}$.

This is a type who exerts a positive effort when the first contestant's type was a_1 but will exerts an effort of zero when the first contestant's type is $\tilde{a}_1$. Note that it is still true in this case that $\gamma_k(a_1) = \gamma_{k-1}(a_1)$ since this type exerts an effort that is equal to γ_{k-1}. Moreover, when a_1 increases to $\tilde{a}_1$ we will still have that $\gamma_k(\tilde{a}_1) = \gamma_{k-1}(\tilde{a}_1)$ (this type exerts an effort of zero and the highest effort is not changed). Therefore we again have that $\gamma_k(\tilde{a}_1) = \gamma_{k-1}(\tilde{a}_1) \geq \gamma_{k-1}(a_1) = \gamma_k(a_1)$.

3. Case C: $\left(\gamma_{k-1}(\tilde{a}_1)\right)^{d_{k-1}} \leq a_{k-1} < \min\left\{\frac{1}{(1-d_{k-1})}\left(\gamma_{k-1}(a_1)\right)^{d_{k-1}}, 1\right\}$.

For this type since $\min\left\{\frac{1}{(1-d_{k-1})}\left(\gamma_{k-1}(a_1)\right)^{d_{k-1}}, 1\right\} < \min\left\{\frac{1}{(1-d_{k-1})}\left(\gamma_{k-1}(a_1)\right)^{d_{k-1}}, 1\right\}$, $b_{k-1}(a_{k-1}; a_1, ..., a_{k-2}) = \gamma_{k-1}(a_1)$, $b_{k-1}(a_{k-1}; \tilde{a}_1, ..., a_{k-2}) = \gamma_{k-1}(\tilde{a}_1)$ and again $\gamma_k(\tilde{a}_1) = \gamma_{k-1}(\tilde{a}_1) \geq \gamma_{k-1}(a_1) = \gamma_k(a_1)$.

4. Case D: $\min\left\{\frac{1}{(1-d_{k-1})}\left(\gamma_{k-1}(a_1)\right)^{d_{k-1}}, 1\right\} \leq a_{k-1} \leq \min\left\{\frac{1}{(1-d_{k-1})}\left(\gamma_{k-1}(\tilde{a}_1)\right)^{d_{k-1}}, 1\right\}$.

For this type $b_{k-1}(a_{k-1}; a_1, ..., a_{k-2}) = \left((1-d_{k-1})a_{k-1}\right)^{\frac{1}{d_{k-1}}}$ and $b_{k-1}(a_{k-1}; \tilde{a}_1, ..., a_{k-2}) = \gamma_{k-1}(\tilde{a}_1)$. But since $a_{k-1} \leq \frac{1}{(1-d_{k-1})}\left(\gamma_{k-1}(\tilde{a}_1)\right)^{d_{k-1}}$ we have that $\left((1-d_{k-1})a_{k-1}\right)^{\frac{1}{d_{k-1}}} \leq \left(\gamma_{k-1}(\tilde{a}_1)\right)$. Thus $\gamma_k(a_1) = \left((1-d_{k-1})a_{k-1}\right)^{\frac{1}{d_{k-1}}}$ while $\gamma_k(\tilde{a}_1) = \gamma_{k-1}(\tilde{a}_1)$ and $\gamma_k(\tilde{a}_1) \geq \gamma_k(a_1)$.

5. Case E: $\min\left\{\frac{1}{(1-d_{k-1})}\left(\gamma_{k-1}(\tilde{a}_1)\right)^{d_{k-1}}, 1\right\} \leq a_{k-1} \leq 1$.

For this type $b_{k-1}(a_{k-1}; a_1, ..., a_{k-2}) = \left((1-d_{k-1})a_{k-1}\right)^{\frac{1}{d_{k-1}}}$, $b_{k-1}(a_{k-1}; \tilde{a}_1, ..., a_{k-2}) = \left((1-d_{k-1})a_{k-1}\right)^{\frac{1}{d_{k-1}}}$ and $\gamma_k(\tilde{a}_1) = \left((1-d_{k-1})a_{k-1}\right)^{\frac{1}{d_{k-1}}} = \gamma_k(a_1)$. This concludes the proof that $\gamma_k(a_1)$ is (weakly) increasing in a_1 and therefore $HE(a_1)$ is increasing in a_1. ■

The expected highest effort can be written as

$$HE = \int_0^1 HE(a_1)\, c_1 a_1^{c_1-1} da_1$$

We wish to prove that this function is increasing in c_1. Since $\frac{d}{dc_1}HE(a_1) = 0$ we have

$$\frac{d}{dc_1}HE = \int_0^1 HE(a_1)(1 + c_1 \ln a_1)\, a_1^{c_1-1} da_1$$

Now

$$1 + c_1 \ln a_1 \geq 0 \Leftrightarrow e^{-\frac{1}{c_1}} \leq a_1 \leq 1$$

Thus, we need to show that

$$\begin{aligned} & \int_0^1 HE(a_1)(1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1 \\ = & \int_0^{e^{-\frac{1}{c_1}}} HE(a_1)(1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1 + \int_{e^{-\frac{1}{c_1}}}^1 HE(a_1)(1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1 > 0 \end{aligned}$$

or equivalently that

$$\int_0^{e^{-\frac{1}{c_1}}} HE(a_1)(-1 - c_1 \ln a_1) a_1^{c_1 - 1} da_1 < \int_{e^{-\frac{1}{c_1}}}^1 HE(a_1)(1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1$$

Since $HE(a_1)$ is increasing in a_1 we have

$$\int_0^{e^{-\frac{1}{c_1}}} HE(a_1)(-1 - c_1 \ln a_1) a_1^{c_1 - 1} da_1 \leq HE\left(e^{-\frac{1}{c_1}}\right) \int_0^{e^{-\frac{1}{c_1}}} (-1 - c_1 \ln a_1) a_1^{c_1 - 1} da_1$$

while

$$\int_{e^{-\frac{1}{c_1}}}^1 HE(a_1)(1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1 \geq HE\left(e^{-\frac{1}{c_1}}\right) \int_{e^{-\frac{1}{c_1}}}^1 (1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1$$

Thus, it is sufficient to prove that

$$\int_0^{e^{-\frac{1}{c_1}}} (-1 - c_1 \ln a_1) a_1^{c_1 - 1} da_1 \leq \int_{e^{-\frac{1}{c_1}}}^1 (1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1$$

and in fact we have equality here since

$$\begin{aligned} & \int_0^{e^{-\frac{1}{c_1}}} (-1 - c_1 \ln a_1) a_1^{c_1 - 1} da_1 \\ = & -\int_0^{e^{-\frac{1}{c_1}}} a_1^{c_1 - 1} da_1 - c_1 \int_0^{e^{-\frac{1}{c_1}}} (\ln a_1) a_1^{c_1 - 1} da_1 = -\frac{1}{c_1} e^{-1} - c_1 \left(-\frac{2}{ec_1^2}\right) \\ = & \frac{1}{ec_1} \end{aligned}$$

and

$$\begin{aligned} & \int_{e^{-\frac{1}{c_1}}}^1 (1 + c_1 \ln a_1) a_1^{c_1 - 1} da_1 \\ = & \int_{e^{-\frac{1}{c_1}}}^1 a_1^{c_1 - 1} da_1 + c_1 \int_{e^{-\frac{1}{c_1}}}^1 \ln a_1 a_1^{c_1 - 1} da_1 = \frac{1}{c_1}\left(1 - e^{-1}\right) + c_1 \left(\frac{-e + 2}{ec_1^2}\right) \\ = & \frac{1}{ec_1} \end{aligned}$$

■

6.7 Proof of Proposition 8

We wish to show that if s is the strongest contestant, i.e. $c_s = \max\{c_1, ..., c_n\}$, and $T = \arg\max_{K \subset N} HE(\widetilde{\sigma}_K)$ and is uniquely determined, then $s \in T$. Assume to the contrary that $s \notin T$. Then $\widetilde{\sigma}_T$ indicates the optimal order of the contestants in T. Also assume that contestant j is the first contestant in that order. Now if we replace contestant j with contestant s, it follows from Proposition 7 that either the expected highest effort is now strictly higher which contradicts the definition of T or the expected highest effort is the same which contradicts the uniqueness of T.■

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