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ABSTRACT

On the optimal supply of liquidity with borrowing constraints*

We characterize policies for the supply of liquidity in an economy where agents have a precautionary savings motive due to random production opportunities and the presence of borrowing constraints. We show that a socially efficient provision of liquidity involves a trade-off between insurance and production incentives. Two scenarios are studied: if no aggregate information is available to the policy maker, constant flat expansions are socially beneficial if unproductive spells are sufficiently long. If some aggregate information is available, a socially beneficial state-dependent policy prescribes expanding the supply of liquidity in recessions and contracting it in expansions.

JEL Classification: E5
Keywords: Friedman rule, heterogenous agents, incomplete markets, liquidity, precautionary savings and state dependent policy.

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1 Introduction

It is known that borrowing constraints may amplify the effect of shocks compared to a complete markets benchmark. Several contributions have explored this mechanism, under a variety of assumptions concerning the market structure, the equilibrium definition, and the precise nature of the underlying borrowing friction, such as e.g. Scheinkman and Weiss (1986); Carlstrom and Fuerst (1997); Kocherlakota (2000); Kiyotaki and Moore (2008); Guerrieri and Lorenzoni (2009). These analyses typically treat the policy, i.e. the supply of liquidity, as fixed. We use an analytically tractable model of production and savings, in the presence of a borrowing constraint, to study how to regulate the supply of liquidity. A novel feature is that in our model the degree of propagation of shocks, which generates a business cycle, is linked to the distribution of wealth. An interesting question is whether policy should depend on this feature of the business cycle.

The agents in our model face random oscillations in their ability to produce, between productive and unproductive periods. The economy is populated by agents of two types, defined by their productive state. Types are assumed to be perfectly negatively correlated, so that in each instant only one type is productive. Since the state is not observable agents cannot issue private debt, i.e. they face a borrowing constraint. We follow the seminal work of Scheinkman and Weiss (1986) and focus on a competitive equilibrium in which unproductive agents can trade a liquid asset (e.g. "money") for some consumption.

We extend Scheinkman and Weiss's analysis, which assumes a constant supply of liquid assets, by letting the government control the liquidity through lump-sum transfers.[1] We provide an analytical characterization of the price and expected return of the liquid asset and aggregate production, as functions of the liquid asset growth rate and the wealth distribution. Moreover, we characterize the dynamics of the wealth distribution as a function of liquid as-

[1] As in Levine (1991) we assume that the government does not know which agent is productive, so that the transfers are equal across agents. See Kehoe, Levine and Woodford (1990) for a thorough discussion of this assumption and in particular Levine (1991) for a careful derivation of the equal-treatment restriction from first principles.

set growth and the history of shocks. These objects give a complete analytical description of the dynamics of the economy. Since the liquid asset is the only savings instrument, an unproductive agent consumes exchanging assets for goods in the centralized market. As the state can be reversed, the value of the asset is positive for productive agents too, who are hence willing to work and produce goods to be traded in the centralized market. A feature of the competitive equilibrium that we study is that rich-productive agents are relatively less interested in trading goods for assets than poor-productive agents. It follows that trade volumes, aggregate production, the asset price and the risk premium depend on the distribution of wealth, which evolves through time following the history of shocks. The time series generated by this economy will thus display a "business cycle".

We study policies for the provision of liquidity under two different informational assumptions. In the first setting, nor the individual state nor aggregate variables are observable to the policy maker, therefore the policy is state independent. In the second one, individual states are not observable but the policy maker observes some aggregates features of the economy, such as aggregate production (or the asset price), so that the policy can be state dependent, i.e. depend on the "business cycle". Since the liquid asset growth rate affects the distribution of wealth, policy has real effects which involve two important margins: the first one is that an increase in the supply of liquidity provides insurance to agents who incur in a long spell of unproductive periods (who end-up having low liquidity and low consumption). The second margin is the classic cost of inflation: an expansionary policy lowers the return on liquid assets, lowering productive agents' incentives to save and produce.

We adopt an ex-ante welfare criterion and characterize the welfare implications of different policies. It is shown that the optimal policy trades off insurance vs. production incentives. When the informational assumptions are such that only a state independent policy can be implemented, we show that the insurance motive outweighs the production incentives provided that the average length of unproductive spells is high. That is, flat expansions are optimal when the productive state is sufficiently persistent. This result is reminiscent of the optimal (steady) expansions characterized by Levine (1991) and Molico (2006). When some aggregate information is available to the policy maker, we show that a state dependent policy dominates a flat policy, i.e. one that does not depend on the cycle. We find that when unproductive agents are sufficiently poor, a situation that corresponds to a recession in the model, the optimal policy prescribes expanding the supply of liquid assets. When the liquidity of unproductive agents is large enough, a situation that corresponds to a boom in the economy, an optimal policy prescribes to contract the supply of liquidity. This policy rule trades off the provision of insurance versus the cost of discouraging production through a low return on savings. Unlike a flat policy, the state dependent policy is able to decouple these two opposing forces, at least partially. To the best of our knowledge our characterization of the ex-ante optimal state-dependent rule is novel in the literature, though it is related to the properties of ex-post monetary expansions that are discussed in e.g. Section 3 of Scheinkman and Weiss (1986) and Section 6 of Brunnermeier and Sannikov (2010), to which we return below.

Related literature

A few previous contributions discuss environments where a flat expansion in the money supply is efficient in an economy with incomplete markets and where money serves an essential role. Levine's (1991) seminal paper considers an endowment economy where the agents' (bounded) utility functions change randomly according to whether they are "buyers" or "sellers", a state that follows an exogenous Markov process. In Levine's model sellers sell their entire endowment, which amounts to a restriction on the agents' marginal utilities over the set of feasible trades. Because of this assumption policy can provide insurance at no cost since altering the relative price has no effect on welfare in the corner solution. Kehoe, Levine and Woodford (1990) extend Levine's setting to allow for internal solutions where sellers do not necessarily sell all their endowment. For reasons of tractability, they restrict attention to equilibria in which what happens in each period is independent of history (two-state markov equilibria). Compared to these papers, our contribution is to analyze the question of the optimal policy (state independent and dependent) in the context of a production economy, and to focus on equilibria in which the decisions in each period depend on the whole history of shocks, as summarized by the distribution of asset holdings.

A central feature of our model is that business cycles, and the magnitude of fluctuations, depend on the tightness of the borrowing constraint. This echoes the results of Guerrieri and Lorenzoni (2009) and Guerrieri and Lorenzoni (2011), who explore the effects of borrowing constraints on business cycles in a model with liquid assets. The first paper shows that in a model where there is a complementarity between consumption and production decisions a tight borrowing constraint magnifies aggregate shocks.[2] The second paper evaluates the effects of a credit crunch, an exogenous tightening of the borrowing constraint, to show that the interest rate decreases and output drops. In our model, as the borrowing constraint becomes tighter, economic fluctuations become more severe. The relative simplicity of our setup allows us to investigate the optimal (state dependent and independent) provision of liquidity.

Our paper is also related to Molico (2006), who shows that mild monetary expansions can be beneficial in a search model of money. In his model, agents meet randomly bilaterally. Once agents meet, they exchange goods for money. The price paid by the buyer results from bargaining and depends on the amount of money held by each agent upon entering the pairwise meeting. Therefore, the distribution of money is generically non-degenerate and so monetary injections, via lump-sum, can improve the terms of trade for poor buyers. Our model departs from Molico (2006) in a few important ways. First, he evaluates the benefits of expansions by comparing across stationary distributions while we do not (that is, we do not abstract from the transitional paths). Second, we characterize competitive equilibrium rather than a search equilibrium. Third, we evaluate the optimal policy and we allow the

[2] A related mechanism is explored by Chamley (2010), who shows that a liquidity shock to a fraction of the population can propagate and generate a liquidity trap.

planner to use the distribution of money as an input.[3]

Finally, our analysis shares several features with Brunnermeier and Sannikov (2010, 2011) quantitative general equilibrium models, such as the assumption of incomplete markets / financial frictions, the random productive abilities of agents, and the equilibrium interaction between the wealth distribution, the value of liquidity and total production. Compared to our model, these authors add several interesting elements to be able to directly analyze the role of financial intermediation: specialized financial intermediaries, money and physical capital. We see our approaches as complementary. The simple structure of our economy allows us to derive an analytical characterization of the equilibrium and of the (ex-ante) socially efficient policy. We think our simple framework is useful to interpret the workings of the more realistic, but more involved, quantitative results of larger general equilibrium models. For instance, our characterization of the nature of the ex-ante optimal policy provides a simple interpretation of the result, discussed in Section 6 of Brunnermeier and Sannikov (2010), that monetary expansions may (ex-post) alleviate the effects of recession by redistributing wealth towards poor but productive agents.

Overview of the analysis

Section 2 defines the economic environment: the agent's utility functions, production possibilities, and the asset transfer scheme. With fluctuating productive opportunities, the agents face an insurance problem. It is shown that without uncertainty the Townsend (1980), Bewley (1980) result on the optimality of the "Friedman rule" holds: agents are fully insured and the efficient complete markets allocation can be sustained.

Section 3 defines an equilibrium and the agent's optimality conditions, in the stochastic environment. In a similar fashion to Bewley (1983), it is shown in Section 3.1 that a steady

[3]Our paper also relates to Chiu and Molico (2010), who evaluate the welfare costs of inflation in a search model of money as in Lagos and Wright (2005) but with endogenous participation in the centralized market (fixed cost of participation). Participation depends on the current amount of money holdings and realization of participation cost, therefore making the distribution of money an endogenous object. Precautionary motives are not present in this model and therefore inflation is not optimal. Chiu and Molico (2010) deals with a stationary environment and do not evaluate the optimality of state dependent policies.

contraction on the supply of assets provides that the unique ergodic set of money holdings is such that markets shut down and therefore there is no equilibrium with assets. The reasoning behind the result is simple: due to the individual uncertainty, the agents need to hold large amounts of liquid assets to satisfy their tax needs. If an agent is relatively poor, she will fail to comply her tax obligations with positive probability, as her relative wealth will decrease as long as she is unproductive. We show that there is no equilibrium with assets where tax obligation are always met. Under a contractionary policy, the wealth distribution is degenerate at a single value and consumption allocations are those of autarky. In Section 3.3 we discuss the second best nature of the policy. That is, we show that there is no policy able to complete the markets so that agents enjoy the consumption level of the complete markets allocation. We also show that generalizing the policy rule to depend on the measure of wealth, which encompasses the history of shocks and transactions, does not affect this result.

Section 4 characterizes analytically the equilibrium quantities and prices as a function of the growth rate of assets and the distribution of wealth. In this section we discuss the beneficial effects of an expansionary policy. In particular, we show that it provides a floor to consumption, therefore bounding below the instantaneous utility of unproductive agents. In Section 5 we discuss the optimality of policy. In Section 5.1 we evaluate the optimality of an expansionary policy in a setup where individual states nor aggregate variables are observable and therefore only state independent policy is possible. Here, we show that flat expansions in the asset base are optimal when the average length of unproductive spells is high. In Section 5.2 we depart from the assumption of constant growth rate of the asset base to allow the government to follow a state dependent policy. To do this we assume that the government observes the wealth in the hands of unproductive agents but does not observe transactions nor identities. We find that the optimal policy expands the asset base when unproductive agents are poor and contracts the base otherwise. Interestingly, there is a bound to the size of the expansions as given by a strong connection of policy in the corners. Section 6 concludes.

2 The model

This section describes the model economy: agents' preferences, production possibilities, and markets. Two useful benchmarks are presented: the (efficient) allocation with complete markets and the optimal monetary policy with no uncertainty.

We consider two types of infinitely lived agents (with a large mass of agents of each type), indexed by $i = 1, 2$, and assume that at each point in time only one type of agent can produce. We further restrict attention to the case where agents of the same type play the same action at every point in time so that we can discuss the model in terms of two representative agents, one of each type. The productive agent transforms labor into consumption one for one, the unproductive agent cannot produce. The productivity of labor is state dependent: the duration of productivity spells is random, exponentially distributed, with mean duration $1/\lambda > 0$. Money, an intrinsically useless piece of paper, is distributed at each time t between the two agents so that $m_t^1 + m_t^2 = m_t$. The growth rate of the money supply at time t is μ_t, so that the money supply follows $m_t = m_0 e^{\mu_t t}$ with m_0 given. As in Scheinkman and Weiss (1986) and Levine (1991), we let the individual state of an agent to be private information, precluding agents from issuing private debt.[4] Then, agents face a borrowing constraint restricting their unique savings instrument, money, to be non-negative.

Let $\rho > 0$ denote the time discount rate, ω denote a history of shocks and money supply levels, and $s(t, \omega) = \{1, 2\}$ be an indicator function denoting which agent is productive for a given history ω and current time t. Agent i chooses consumption c^i, labor supply l^i, and depletion of money balances $\dot{m}^i$, in order to maximize her (time-separable) expected discounted utility,

$$\max_{\{c^i(t,\omega), l^i(t,\omega), \dot{m}^i(t,\omega)\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} \left[u\left(c^i(t,\omega)\right) - l^i(t,\omega)\right] \, dt \right\} \tag{1}$$

[4] Having a large mass of agents of each type is important for the argument as it implies that a single agent cannot infer the productive state of a different agent given his own state. Note that if there were only 2 agents, both agents know the state of the other agent, therefore individual states are not any more private information implying that debt contracts are easy to write and fulfill.

subject to the constraints

$$\dot{m}^i(t,\omega) \leq \left[l^i(t,\omega) + \tau(t,\omega) - c^i(t,\omega)\right] / \tilde{q}(t,\omega) \qquad \text{if } s(t,\omega) = i \tag{2}$$

$$\dot{m}^i(t,\omega) \leq \left[\tau(t,\omega) - c^i(t,\omega)\right] / \tilde{q}(t,\omega) \text{ and } \; l^i(t,\omega) = 0 \qquad \text{if } s(t,\omega) \neq i \tag{3}$$

$$m^i(t,\omega) \geq 0 \;\; , \;\; l^i(t,\omega) \geq 0 \;\; , \;\; c^i(t,\omega) \geq 0 \tag{4}$$

where $\tilde{q}(t,\omega)$ denotes the price of money, i.e. the inverse of the consumption price level, $\tau(t,\omega)$ denotes a government lump-sum transfer to each agent, and expectations are taken with respect to the processes s and m conditional on time $t = 0$.

A monetary policy with $\mu_t > 0$ is called expansionary, a policy with $\mu_t < 0$ is called contractionary. It is immediate that when the money supply is constant for all t (i.e. $\mu_t = 0 \; \forall \; t$) the economy is the one analyzed by Scheinkman and Weiss (1986). For any history ω the monetary policy μ_t determines the transfers to the agents τ_t through the government budget constraint,

$$\tilde{q}_t \; \mu_t \; m_t = 2\tau_t \tag{5}$$

which states that transfers are financed by printing money. The government transfer scheme implies that in the case of a contractionary policy agents must use their money holdings to pay taxes (i.e. $\tau < 0$).

Notice that the government cannot differentiate transfers across agent-types. This follows from the assumption that the identity of the productive type is not known to the government. Levine (1991) shows in a similar setup that, because of the private information assumption, the best mechanism is linear and can be understood as monetary policy. In Appendix B we explore the type of allocations that can be achieved using tax policy by changing government powers (commitment vs. no commitment), types of available taxes (lump-sum vs. distortionary), and government knowledge about the state (agent's type observable vs. not observable).

Next we state two important remarks. The first one characterizes a symmetric efficient allocation with complete markets (the proof is standard so we omit it):

Remark 1 *Assuming complete markets and an ex-ante equal probability of each state ($s = 1, 2$), the symmetric efficient allocation prescribes the same constant level of consumption, $\bar{c}$, for both agents, where $\bar{c}$ solves $u'(\bar{c}) = 1$.*

Thus without borrowing constraints the efficient allocation solves a static problem, and it encodes full insurance: agents consume a constant amount (equal since we assume ex-ante identity) and the aggregate output is constant.

The next remark characterizes the optimal monetary policy in the case of no uncertainty. This helps highlighting the essential role of uncertainty in our problem. In particular, consider the case where each agent is productive for T periods, and then becomes unproductive for the next T periods. Without loss of generality, for the characterization of the stationary equilibria, let us assume that the economy starts in period $t = 1$ with agent 1 being productive and agent 2 owning all the money, so that $m_1^1 = 0$. We have

Remark 2 *Let $u(c^i) = {c^i}^{1-\theta}/(1-\theta)$. The Euler equation for $\dot{m}_t^i$ gives $\rho = \frac{\dot{q}_t}{q_t} - \theta \frac{\dot{c}_t^i}{c_t^i} - \mu$ which is solved by the "Friedman rule": $\mu_t = -\rho$, and $c_t^i = \bar{c}$ for $i = 1, 2$ and all t.*

See Appendix A for the proof. This remark, together with the efficient allocation described in Remark 1, shows that without uncertainty this economy replicates Townsend (1980), Bewley (1980) result on the optimality of the "Friedman rule".

3 Monetary policy in the stochastic model

This section defines the monetary equilibrium in the original model with stochastic production opportunities. We look for a markovian equilibrium where $\tilde{q}(t, \omega) = \tilde{q}(m(t, \omega), m^i(t, \omega), s(t, \omega))$. With a slight abuse of notation this implies $c^i(t, \omega) = c^i(m(t, \omega),\ m^i(t, \omega),\ s(t, \omega))$, and $l^i(t, \omega) = l^i(m(t, \omega), m^i(t, \omega), s(t, \omega))$, and $\dot{m}^i(t, \omega) = \dot{m}^i(m(t, \omega), m^i(t, \omega), s(t, \omega))$. In other words, we are looking for an equilibrium that depends solely on three states: the level of the money supply, the distribution of money holdings, and the current state.

We assume that the government commits to a monetary policy at time 0. We will explore two different setups. In the first one the government chooses the optimal constant anticipated monetary policy (that is, $\mu_t = \mu$), which follows as private information prevents the government to obtain any useful information from agents. In the second setup we let the government observe some easily obtainable economy aggregates, such as output or the price level. Therefore, the government will be able to tie its policy to one of these aggregates.

By a simple quantity theory argument it is easily seen that the nominal variables of this economy are homogenous of degree one in the *level* of money, which acts as a numeraire. Hence we simplify the state space using that the nominal price of money is homogenous of degree minus one in the level of the money supply, i.e. $\tilde{q}(m, m^i, s) = \frac{1}{m} q(x^i, s)$, where $x^i \equiv \frac{m^i}{m}$ with $i = 1, 2$ and $s = 1, 2$. The variable $x^i \in [0, 1]$ is the share of total money balances in the hands of agent i, i.e. a measure of the wealth distribution. Likewise the consumption rule is homogeneous of degree zero in the level of the money supply $c^i(m, m^i, s) = c^i(x^i, s)$.

Note that the state of the economy can then be summarized by the wealth share in the hands of the unproductive agent, which we will denote using the variable x (with no superscript).[5] We focus on equilibria where the price of money depends on the share of money in the hands of the unproductive agent (but not on the *identity* of this agent). We will simply denote this price by the function $q(x)$, i.e. a function of the wealth share of the unproductive agent. Next we define a monetary equilibrium.

Definition 1 *Let x be the wealth share of the unproductive agent. A monetary equilibrium is a price function $\tilde{q}(t, \omega) = \frac{1}{m_t}\, q(x)$, with $q : [0, 1] \to \mathbb{R}^+$ and a stochastic process $x(t, \omega)$ with values in $[0, 1]$, such that a consumer i maximizes expected discounted utility (equation (1)) subject to the budget constraints (2) and (3), non-negativity (4), the government budget con-*

[5] This is so because shocks are perfectly positively correlated for agents of a given type and perfectly negatively correlated across agent's types. Therefore, keeping track of the share of wealth of a given agent entails keeping track of the entire wealth distribution.

straint (5), and market clearing constraint $c^1(1-x,1)+c^1(x,2)=l^1(1-x,1)$ *are satisfied.*

Until Section 5.2 we restrict our attention to the model where the government chooses the constant anticipated monetary policy μ. The characterization of the optimal policy will prove useful in understanding what the government can do once she is allowed to observe some aggregate measures of overall activity in the economy.

3.1 No monetary equilibrium with contractionary policy

This section shows that with uncertainty there is no monetary equilibrium for any constant anticipated contractionary monetary policy. This result relates to Bewley (1983) who showed that in a neoclassical growth model with incomplete markets there is no monetary equilibrium if the interest rate is lower than the discount rate.[6]

A contractionary policy $\mu<0$ requires agents to pay lump sum taxes ($\tau<0$). Consider the case where $\mu<0$, agent 1 has fraction of money balances x_t^1, and the current state of the economy is $s(t,\omega)=2$, which means that agent 1 is unproductive. If x_t^1 is low enough, given that $\lambda>0$ and finite, the agent will fail to comply with the monetary authorities with non-zero probability. On the other hand, consider the case where $x_t^1=1$. In this case the agent is able to comply with her tax obligations with certainty, as she can make her consumption profile to be arbitrarily low. This implies that there exists a threshold $\zeta\in(0,1)$ such that for $x^i\geq\zeta$ the agent is able to cover her lifetime tax needs with probability one. Note that the threshold must be independent of the current state s_t as with positive probability the states are reversed. In the next lemma we characterize this threshold.

Lemma 1 *If* $\mu<0$*, for any state of the world* $s(t,\omega)$*, there is a unique threshold:* $\zeta=1/2$*, and a unique ergodic set where* $x_t^1=x_t^2=\frac{1}{2}\ ,\ \forall\ t$*, that ensures tax solvency.*

[6] In his model, as in ours, agents face idiosyncratic shocks and there is a lump sum tax obligation that has to be covered by the agents.

See Appendix C for the proof. Intuitively, given the uncertain duration of the productivity spell, the only value of money holdings that ensures compliance with tax obligations for both types of agents is $x^1 = x^2 = 1/2$. At this point, for any history of shocks, the identical lump-sum (negative) transfers reduce the money holdings of both agents proportionally, leaving the wealth distribution unaffected. This leads us to

Proposition 1 *Let $\mu < 0$: In the ergodic set there is no stationary monetary equilibrium and consumption allocations are autarkic.*

The proof of Proposition 1 follows from noting that Lemma 1 implies no trade in the ergodic set. Productive agents have an unsatisfied demand for money and unproductive ones have an unsatisfied demand for consumption goods.

3.2 The marginal value of money

Solving the model requires characterizing the marginal value of money, given by the lagrange multiplier for $\dot{m}^i$ in the problem defined in (1): $\tilde{\gamma}(m, x^i, s)$. Without loss of generality, let us look at this problem from the perspective of agent 1. Let $\tilde{\gamma}(m, x^1, 1)$ and $\tilde{\gamma}(m, x^1, 2)$ denote the (un-discounted) multipliers associated to the constraints in equation (2) and (3), respectively, so that e.g. $\tilde{\gamma}(m, x^1, 2)$ measures the marginal value of money when the money supply is m, agent 1 holds a share x^1 of it and she is unproductive. Likewise, the multiplier $\tilde{\gamma}(m, x^1, 1)$ measures the marginal value of money when the money supply is m, agent 1 holds a share x^1 of it and she is productive. The first order conditions with respect to $l(t, \omega)$ and $c(t, \omega)$ give

$$\tilde{\gamma}(m, x^1, 1) \quad = \quad \frac{1}{m} q(x^1, 1) \quad \text{and} \quad \tilde{\gamma}(m, x^1, 2) = \frac{1}{m} q(x^1, 2)\ u'\left(c^1(x^1, 2)\right) \tag{6}$$

where we used the homogeneity with respect to m of the price $\tilde{q}$ discussed above. As usual, these conditions equate marginal costs and benefits of an additional unit of money. For a productive agent the marginal benefit, $\tilde{\gamma}(m, x^1, 1)$, equals the cost of obtaining that unit, i.e. the disutility of work to produce and sell a consumption amount $q(x^1, 1)/m$, in nominal terms. For an unproductive agent the marginal cost is the value of the forgone unit of money $\tilde{\gamma}(m, x^1, 2)$, while the benefit is the additional units of consumption that can be bought with it, given by the product of the price q (consumption per unit of money) times the marginal utility of consumption. The homogeneity of $q(m, m^1, 1)$ with respect to m implies that the lagrange multiplier $\tilde{\gamma}$ is also homogenous, i.e. $\tilde{\gamma}(m, x^1, 1) = \gamma(x^1, 1)/m$. We can then rewrite the first order conditions in real terms as

$$\gamma(x^1, 1) \;=\; q(x^1, 1) \qquad \text{and} \qquad \gamma(x^1, 2) \;=\; q(x^1, 2)\; u'\left(c^1(x^1, 2)\right) \tag{7}$$

Notice that if $s(t, \omega) = 1$, then $c^1(t, \omega) = \bar{c}$, where $\bar{c}$ solves $u'(\bar{c}) = 1$. It is shown in Appendix D that for $x \in (0, 1)$ the lagrange multipliers $\gamma(x^i, s)$ solve the following system of differential equations, which we present following the Hamilton-Jacobi-Bellman representation. Without loss of generality we present the system from the perspective of agent 1:

$$(\rho + \mu)\gamma(x^1, 1) \;=\; \lambda(\gamma(x^1, 2) - \gamma(x^1, 1)) + \gamma_x(x^1, 1)\dot{x}^1(x^1, 1) \tag{8}$$

$$(\rho + \mu)\gamma(x^1, 2) \;=\; \lambda(\gamma(x^1, 1) - \gamma(x^1, 2)) + \gamma_x(x^1, 2)\dot{x}^1(x^1, 2) \tag{9}$$

To provide some intuition consider the first equation: when the agent is productive and holds a share of money x^1 the value flow (discounted by the nominal rate $(\rho + \mu)$) is equal to the change in the marginal value due to the evolution of her money holdings, $\gamma_x(x^1, 1)\dot{x}^1(x^1, 1)$, and to the expectations of the change in value in case the state switches and the agent becomes unproductive: $\lambda(\gamma(x^1, 2) - \gamma(x^1, 1))$.

The next lemma characterizes the functions $\gamma(x^1, 1)$ and $\gamma(x^1, 2)$ that solve this system:

Lemma 2 *Consider $x^1 \in (0, 1)$ and $\mu \geq 0$, then $0 < \gamma(x^1, 1) < \gamma(x^1, 2)$, $\gamma_x(x^1, 2) < 0$, $\gamma_x(x^1, 1) < 0$, $\dot{x}^1(x^1, 2) < 0$, and $\lim_{x^1 \downarrow 0} \dot{x}^1(x^1, 2) = 0$. Moreover $\frac{\partial c^1(x^1, 2)}{\partial x^1} > 0$.*

See Appendix E for the proof. The lemma characterizes the functions $\gamma(x^1, 1), \gamma(x^1, 2)$

in a monetary equilibrium, which happens for $\mu \geq 0$. It is shown that for any wealth share x^1 money is more valuable to unproductive agents than to productive agents, i.e. $\gamma(x^1, 1) < \gamma(x^1, 2)$. The fact that $\gamma(x^1, 1)$ and $\gamma(x^1, 2)$ are decreasing implies that the value of money for an agent is decreasing in her wealth. Moreover, it is shown that an unproductive agent will be depleting her wealth share, at any level of wealth x^1, i.e. $\dot{x}^1(x^1, 2) < 0$. The limit $\lim_{x \downarrow 0} \dot{x}(x, 2) = 0$ shows that as the wealth of the unproductive agent approaches zero it is optimal for her to spend all the money transfer. Notice also that the consumption of an unproductive agent is increasing in her wealth; as the agent gets poorer she hedges against running out of resources by reducing her consumption.

The fact that the function $\gamma(x^1, 1)$ is decreasing implies that the price of money is increasing in the wealth of the unproductive agent. To see this, define x to be the wealth share of the unproductive agent (as in the equilibrium definition given above), assumed to be type 2, so that the wealth share of type 1 is $1 - x$. Then, using equation (7), the price of money is $q(x) \equiv q(1 - x, 1) = q(x, 2)$, which will be used throughout the paper. Notice then that $\partial q(x)/\partial x = -\gamma_x(1 - x, 1) > 0$, which shows that the price of money $q(x)$ is increasing in x. Intuitively, as the unproductive agent get uses the money to consume her wealth decreases and the price of money goes down, i.e. less consumption is bought by 1 unit of money. Such dynamics of the terms of trade reflects the fact that as the productive agent accumulates wealth his interest in the exchange becomes small. This result shows that the terms of trade are history dependent, which in turn generate history dependent aggregate fluctuations in production.

3.3 The second best nature of policy

In this section we show that there is no monetary policy μ where both types of agents consume the allocation of the complete markets economy. We state this in the next proposition.

Proposition 2 *If $\rho > 0$ and λ is finite there is no policy rule, i.e. no value of μ, such that $c^1(x^1, 1) = c^1(x^1, 2) = \bar{c}$.*

See Appendix F for a proof. In order to prove the proposition we start by noting that the complete markets allocation sustains constant price and marginal value of money. We then use this result to show that no monetary equilibrium in the economy with uncertainty can be supported. When no monetary equilibrium exists, agents do not trade and therefore are not able to insure at all. This implies that they cannot attain constant consumption as prescribed in the complete markets allocation which completes the proof.

To prove that no monetary equilibrium can be supported with a constant marginal value of money we note that this requires either a zero marginal value of money, and a zero price of money, or a contractionary policy at the discount rate $-\rho$. The first trivially cannot constitute a monetary equilibrium as one of the conditions for a monetary equilibrium is positive price level and marginal value of money (see Definition 1 and Lemma 2). When $\mu = -\rho$ we know from Section 3.1 that this cannot constitute a monetary equilibrium as we restricted the discount rate ρ to be positive. This last result contrasts with the optimal policy under known fluctuations (see Remark 2) where the complete markets allocation can be attained by deflating at the Friedman rate. The difference, as discussed in Section 3.1, is that the length of unproductive spells is random here and therefore agents cannot trade in order to comply with their tax obligations. Both cases imply a collapse of the monetary economy resulting in no trade, money is valueless, and as a result there is no monetary equilibrium.

The state reversal parameter λ being finite is also important for the result to hold. Note that if $\lambda \uparrow \infty$, equation (8) provides that $\gamma(x^1, 1) = \gamma(x^1, 2)$, or that the marginal value of money is independent on whether the agent is productive or not. Furthermore, we can use this condition together with the first order conditions (see equation (7)) to obtain $c^1(x^1, 2) = 1$, and therefore the complete markets allocation is attained. Intuitively, this follows because insurance motives decrease to zero as agents are certain that their states are going to be reversed.

Interestingly, the result that the complete markets allocation cannot be attained does not hinge on μ being constant, as we state in the next corollary.

Corollary 1 *Let x denote the share of money in the hands of unproductive agents and let $\mu = \mu[x]$. If $\rho > 0$ and λ is finite there is no function $\mu[x]$ such that $c^1(x^1, 1) = c^1(x^1, 2) = \bar{c}$.*

The proof of the corollary is discussed in Appendix F.

4 Expansionary policy

In this section we study the case where $\mu > 0$. As in Scheinkman and Weiss (1986) we specialize to the case with logarithmic utility function, $u(c) = \ln(c)$, that will let us characterize analytically many results.[7] We begin by completing the model solution by finding expressions for the evolution of money holdings $\dot{x}^1(x^1, s)$ and imposing the boundary conditions to the system of differential equations for the marginal value of money $\gamma(x^1, s)$, and describing a sufficient condition for a monetary equilibrium to exist. We also characterize some properties of the optimal consumption rule and we describe its local behavior when money growth and current money holdings x^1 are sufficiently small: these results are useful for an important claim of the paper developed in Section 5. We also provide a characterization of the expected return on money balances, a useful object in describing the model dynamics.

We want to find expressions for the evolution of the share of outstanding money balances x^i over time. Consider the law of motion for the share of money held by type 1 when unproductive:

$$\dot{x}^1(x^1, 2) = \frac{\dot{m}^1}{m} - \frac{m^1}{(m)^2}\dot{m} = \mu\left(\frac{1}{2} - x^1\right) - \frac{c^1(x^1, 2)}{q(x^1, 2)} = \mu\left(\frac{1}{2} - x^1\right) - \frac{1}{\gamma(x^1, 2)} \tag{10}$$

[7] Scheinkman and Weiss prove equilibrium existence and uniqueness for the log utility case. Hayek (1996) presents equilibrium existence and uniqueness results for the more general case of CRRA utility preferences with risk aversion greater than the log case.

where we used the budget constraint of the unproductive agent, equation (3), the government budget constraint, equation (5), and the first order condition in equation (7). Because there are two types of agents and shocks are perfectly negatively correlated it is obvious that if unproductive agents' money holdings are x productive agents are holding $1-x$. Therefore, $\dot{x}^1(1-x,1)+\dot{x}^1(x,2)=0$ so that the accumulation rate of money holdings is perfectly negatively correlated across agent's types. This condition, together with equation (10) can be used to produce the evolution of money holdings for productive agents.

The equation describing the evolution of money holdings for unproductive agents, i.e. equation (10), shows that the optimal change of real money holdings, on top of the government transfer, depends on the value of money for the unproductive agent relative to the value of money for the productive agent: $\gamma(x^1,2)/q(x^1,2)=\gamma(x^1,2)/q(1-x^1,1)$. Notice from the first order conditions that the smaller is the consumption level of the unproductive agent, the higher is the value of $\gamma(x^1,2)/q(x^1,2)$, hence the smaller will be the (absolute) real value of money transfers.

For expositional purposes we rewrite the system of ODEs as a function of the aggregate state and we substitute in for $\dot{x}^1(x^1,s)$ the corresponding expressions from equation (8) and (9),

$$\gamma_x(1-x^1,1)\left[\mu\left(x^1-\frac{1}{2}\right)+\frac{1}{\gamma(x^1,2)}\right]=(\rho+\lambda+\mu)\gamma(1-x^1,1)-\lambda\gamma(1-x^1,2) \tag{11}$$

$$\gamma_x(x^1,2)\left[\mu\left(\frac{1}{2}-x^1\right)-\frac{1}{\gamma(x^1,2)}\right]=(\rho+\lambda+\mu)\gamma(x^1,2)-\lambda\gamma(x^1,1) \tag{12}$$

where for each unproductive agent with money holdings x^1 there is a productive agent with money holdings $1-x^1$. Note that this is a system of delay differential equations.[8] See equation (11). It can be seen that the derivative of the marginal value of money for productive agents $\gamma_x(1-x^1,1)$ depends on the marginal value of money for this agent in past times,

[8]Notice in particular that the delay is non-constant, which prevents an analytical solution in closed form. Notice however that the functions $\gamma(1-x^1,1),\gamma(x^1,2)$ are analytical, and that the above system allows to completely characterize these functions given initial values for $\gamma(\frac{1}{2},1),\gamma(\frac{1}{2},2)$.

$\gamma(x^1, 2)$.

Now we derive the boundary conditions for this problem. The boundaries concern the state in which the unproductive agent has no money. In this case an unproductive agent spends the whole money transfer to finance her consumption. Appendix E gives a formal proof of this statement. The budget constraint gives $c_t^1 = \tau_t = q(0,2)\mu/2$. Using equation (7) for the case of log utility gives

$$\gamma(0,2) = \frac{2}{\mu} \qquad \text{with} \qquad \lim_{\mu \downarrow 0} \gamma(0,2) = \infty \tag{13}$$

where the limit obtains because of Inada conditions. This is an important result in our analysis. An expansionary policy provides an upper bound to the marginal utility of money holdings because the price of money is always finite in an equilibrium and therefore the agent enjoys positive consumption even when she is very poor. When there is no money growth (i.e. as $\mu \downarrow 0$), the agent is not able to consume in poverty and therefore Inada conditions imply that the marginal utility of money approaches infinity.

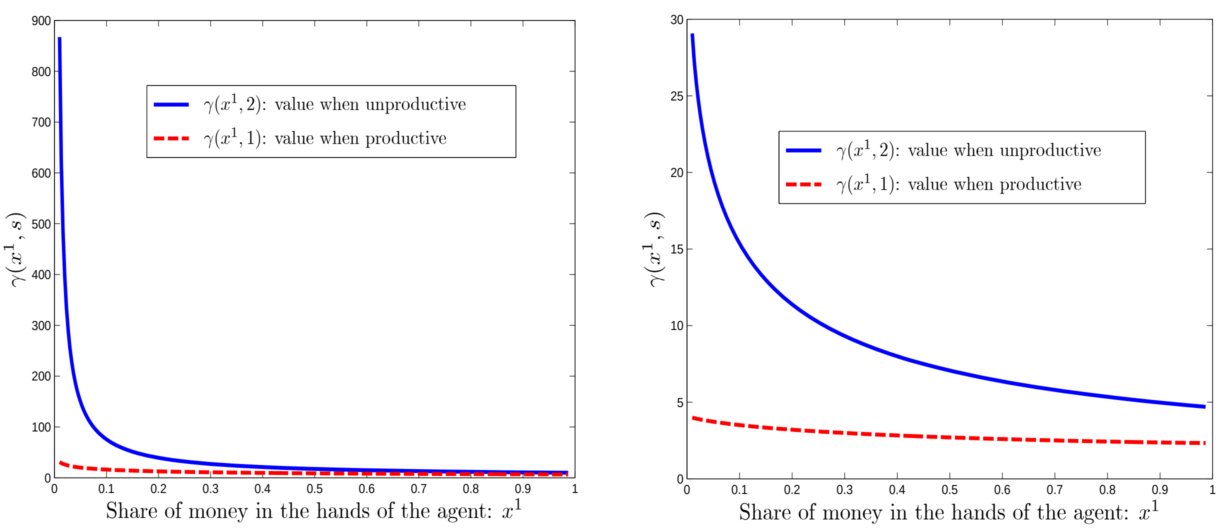
The second boundary condition is also associated to the state in which the unproductive agent has no money and the productive agent has it all. Evaluating equation (11) at $x^1 = 0$ gives

$$\lambda\, \gamma(1,2) = (\rho + \lambda + \mu)\, \gamma(1,1) \tag{14}$$

as $\dot{x}^1(1,1) = 0$ (see Lemma 2). The reason that the share of money remains constant in this state is the following. The unproductive agent, who has a zero share of money, will spend all the money she receives from the government transfer in consumption goods. Intuitively, as the unproductive agent is impatient and there is a positive probability of becoming productive, saving a part of the transfer when $x^1 = 0$ would be optimal if she expected to remain unproductive and that the value of money will go up in the future. In a rational expectations equilibrium, however, conditional on remaining unproductive this agent expects the value of money to go down. Recall that x measures the fraction of money in hands of unproductive agents. The amount of consumption goods bought by a unit if money, measured by $q(x)$, is increasing in x. Hence as $x \downarrow 0$, i.e. as the unproductive agent runs out of money, the real value of money falls.

To illustrate the solution we present a few plots of the important objects in this economy. Throughout the paper we assume that the baseline case uses $\lambda = 0.1$ and $\rho = 0.05$. The parameter choices, and the graphical analysis, is solely for expositional purposes. In Figure 1 we present the solution to the system of delay ODEs that provide the marginal value of money. The left panel plots these values assuming $\mu = 0$ and the right panel assumes $\mu = 0.05$. The properties listed in Lemma 2 are clearly seen and satisfied. Also, the value of money when unproductive when wealth approaches zero (i.e. $x^1 \downarrow 0$) is much higher for the case where $\mu = 0$, as prescribed by equation (13).

Figure 1: The value of money: $\gamma(x^1, 1), \gamma(x^1, 2)$



Parameters: $\lambda = 0.10$, $\rho = 0.05$. Left panel: $\mu = 0$. Right panel: $\mu = 0.05$.

We now discuss the optimal consumption when unproductive. This is an important object because unproductive agents need to spend money to consume and therefore monetary policy will have a strong impact on their consumption behavior. Notice that

$$c^1(x^1, 2) = \frac{\gamma(1 - x^1, 1)}{\gamma(x^1, 2)} \quad \text{with} \quad \frac{\partial c^1(x^1, 2)}{\partial x^1} > 0 \tag{15}$$

which follows from Lemma 2. The fact that consumption when unproductive is increasing in money holdings, implies that for any given money growth rate $\mu > 0$ consumption is smallest when the agent money holdings are zero, in particular at $x^1 = 0$ we have

$$c^1(0, 2) = \frac{\mu}{2}\gamma(1, 1) \quad , \quad \text{with} \quad \lim_{\mu \downarrow 0} \frac{\mu}{2}\gamma(1, 1) = 0 \tag{16}$$

where the limit obtains from the agent's budget constraint. This is important for the welfare analysis that will follow because it shows that monetary transfers provide the unproductive agent with a lower bound to the consumption level. Without transfers, an agent with no money cannot consume.

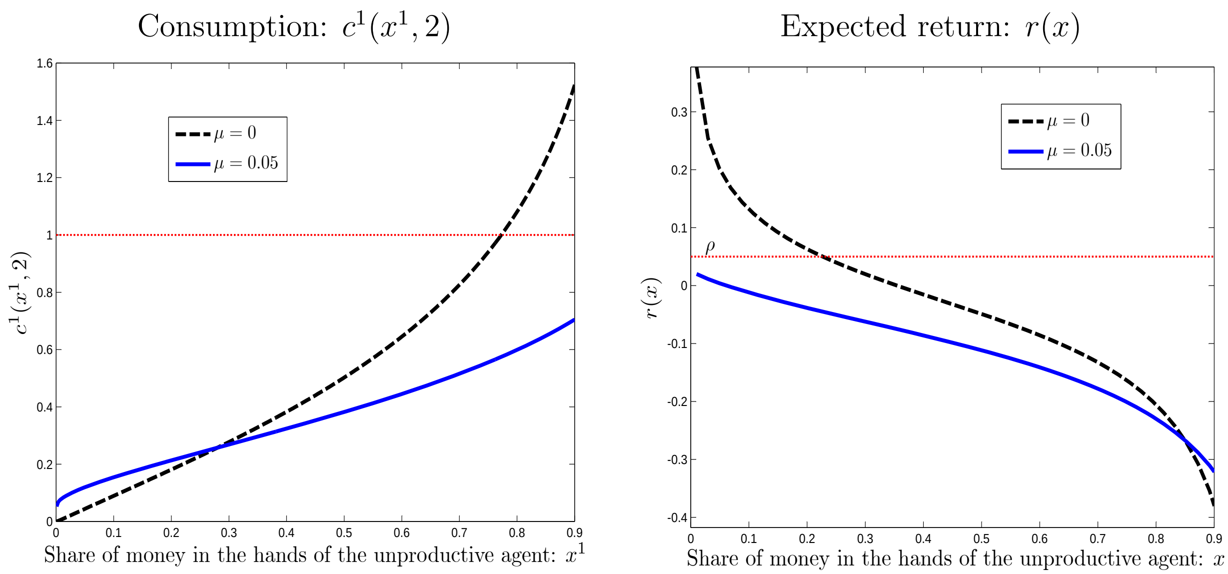
A plot of the consumption function is given in the left panel of Figure 2. For $\mu = 0$, it can be seen that consumption when unproductive goes to zero as $x^1 \downarrow 0$. Note also that the consumption when unproductive may be above the consumption when productive (i.e. $\bar{c} = 1$) for high enough wealth. To see why note that as $x \uparrow 1$ the productive agent owns almost zero money, and therefore is eager to accumulate money to insure against the possibility of a switch of the state. So the price of money is high (i.e. the price of consumption is low), and the productive agent is willing to produce a lot to refill his low money holdings.A comparison of the consumption function at zero and positive money growth (the two curves in the picture) illustrates the tradeoff of inflation: increasing the money growth rate (from 0 to 5 percent in the figure) provides higher consumption to relatively poor unproductive agents but decreases the consumption of unproductive agents with "high" wealth. An optimal choice of the money growth rate trades off these effects. The next lemma characterizes the consumption behavior as the unproductive agent approaches poverty.

Lemma 3 *For any $\mu > 0$ we have: $\lim_{x \downarrow 0} \frac{c^1_x(x^1,2)x^1}{c^1(x^1,2)} = Q_1$, where $Q_1 = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu}\left(1 - \frac{\gamma(0,1)}{\gamma(0,2)}\right) >$*

1 *and* Q_1 *is finite.*

See Appendix G for the proof. Lemma 3 states that when money holdings, x, are "low", the unproductive agent's elasticity of consumption with respect to money holdings approaches a value that is larger than one. This implies that, around low values for money holdings, i.e. $x \approx 0$, the consumption of a poor unproductive agent decreases at a rate that is higher than the rate at which x falls, so that the boundary $x = 0$ is never hit.

Figure 2: Consumption when unproductive and the return on money



Parameters: $\lambda = 0.10$, $\rho = 0.05$.

Another interesting variable is the expected return on money or, equivalently, the expected interest rate. This object is useful in understanding the workings of the model and illustrates how the market incompleteness affects the economy by generating a risk premium. Let $r(x)$ denote the expected (net) return on money,

$$r(x) \equiv \mathbb{E}\left[\frac{\dot{\tilde{q}}(x)}{\tilde{q}(x)} - 1 \middle| x_t = x\right]$$

which is just the expected growth rate of the price of money. Using that $\tilde{q}(x) = \gamma(1-x, 1)/m$

(see equation (6)), the expected return on money can be written as (see Appendix H)

$$r(x) = \rho + \lambda \frac{\gamma(x,1)}{\gamma(1-x,1)} \left(1 - \frac{1}{c^1(1-x,2)}\right) . \tag{17}$$

In a complete markets setting, such as the one described in Remark 2, consumption is constant (at $\bar{c} = 1$) and the expected return on money equals the time discount, ρ. With incomplete markets the return on money is history dependent. There are two channels through which history affects the interest rate. First, through changes in x when the identity of the productive agent doe not change. These effects are "small", in the sense they are continuos in time and proportional to $\dot{x}$. The second channel operates through jumps in the value of x, which occur when the identity of the productive agent changes. That is why the probability of a change in identity, λ, appears in equation (17). The latter effect can be quantitatively large and dominate all other variations. When the current unproductive agents is rich (a high value of x), a state switch implies that the new unproductive agents is poor. This implies that the price of consumption might increase if the state switches (i.e. the price of the asset falls), so that the expected return on money is low.[9] Alternatively, if the current unproductive agent is poor (small value of x), a state switch implies the new unproductive agents is rich. This yields a high expected return on money. This intuitive interpretation of the dynamics of the expected return on money abstracts from the change in consumption profiles that occur if there is no state switch, but much of the action in the return happens in the corners as state reversals imply large swings in the distribution of wealth and therefore in consumption. The expected return on money is plotted in the right panel of Figure 2. It is shown that the expected return is high when unproductive agents are poor. Moreover, a more expansionary policy implies a flatter profile for the expected return, as the changes in consumption after a state switch are smaller.

[9] Recall that the asset price is $q(x) = \gamma(1-x,1)$, which is increasing in x.

4.1 The stationary density of the wealth distribution

This section completes the characterization of the equilibrium by computing the stationary density of wealth (money holdings). This density is useful because it shows that, independently of the monetary policy followed, there is a "non-negligible" amount of histories for which unproductive agents approach extreme poverty. This explain why an expansionary policy, providing a floor to consumption (and utility) of unproductive agents who incur in these histories, might be desirable ex-ante.

Let $F(x,s)$ denote the CDF for the share of money holdings in state s with density $f(x,s) = \frac{\partial F(x,s)}{\partial x}$. The density function of the invariant distribution is derived from the usual Kolmogorov Forward Equation (KFE) after imposing for stationarity. Appendix I derives the KFE for our model with poisson jumps in the state, which gives

$$0 = f_x(x^1,s_i)\dot{x}^1(x^1,s_i) + f(x^1,s_i)\frac{\partial \dot{x}^1(x^1,s_i)}{\partial x} + \lambda\left[f(x^1,s_i) - f(x^1,s_{-i})\right] \tag{18}$$

where $s_i = 1,2$ denotes the current state and s_{-i} the other state. It is immediate that the densities satisfy $f(x^1,2) = f(1-x^1,1)$ since, given the assumed symmetry of the shocks, for each agent with money $x = x^1$ and $s = 2$ there is another agent with money $1-x^1$ and $s = 1$. This allows us to concentrate the analysis on only one density: $f(x^1,2)$. Using the expression for $\dot{x}^1(x^1,2)$ derived in equation (10) and $\dot{x}^1(x^1,2) + \dot{x}^1(1-x^1,1) = 0$ gives

$$\frac{f_x(x^1,2)}{f(x^1,2)} = \frac{\gamma(x^1,2)\left(\lambda-\mu\right) - \frac{(\rho+\lambda+\mu)\gamma(x^1,2)-\lambda\gamma(x^1,1)}{1+\mu\left(x^1-\frac{1}{2}\right)\gamma(x^1,2)}}{1+\mu\left(x^1-\frac{1}{2}\right)\gamma(x^1,2)} - \frac{\lambda}{\mu\left(\frac{1}{2}-x^1\right)+\frac{1}{\gamma(1-x^1,2)}} \equiv \Omega(x^1) \tag{19}$$

It follows that

$$f(x^1,2) = Ce^{\int_{1/2}^{x^1}\Omega(z)\ dz} \qquad \text{where} \qquad C = \left[\int_0^1 e^{\int_{1/2}^{x^1}\Omega(z)\ dz}\ dx^1\right]^{-1} \tag{20}$$

where the constant C ensures that $\int_0^1 f(x^1,2)\ dx^1 = 1$.

The next lemma establishes useful properties of the invariant density function for money holdings:

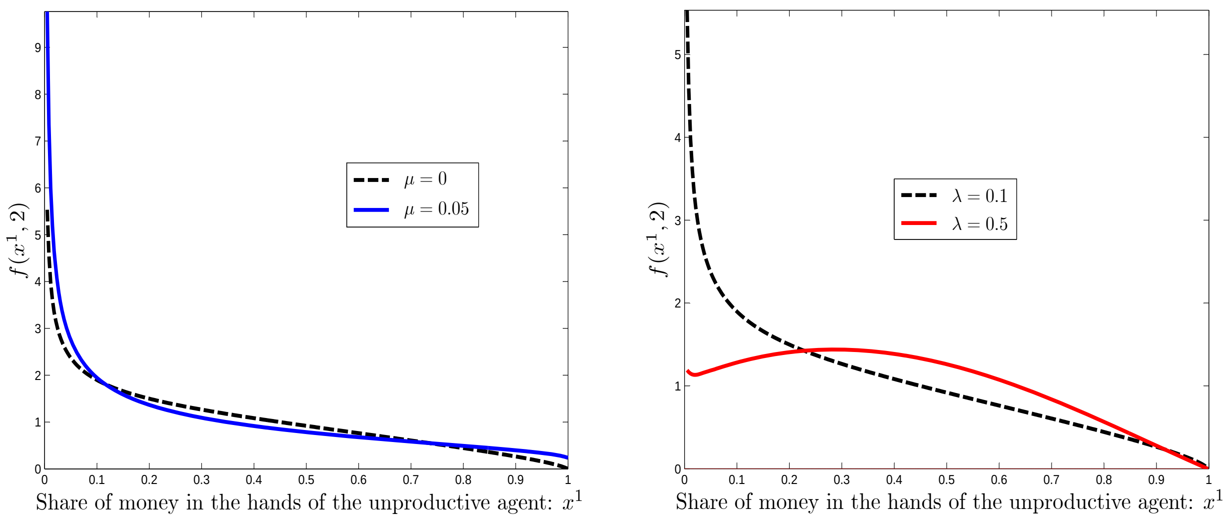
Lemma 4 *For any given $\mu \geq 0$ and $x^1 \in (0,1)$ the invariant density of money holdings $f(x^1, 2)$ has the following properties: (i) continuous and differentiable in x^1 and μ, (ii) $\lim_{x \downarrow 0} f(x^1, 2) = +\infty$, and (iii) $\lim_{x \downarrow 0} f_x(x^1, 2) = -\infty$.*

The details of the proof can be found in Appendix I.1. The lemma exhibits an important property of the model. That is, for any $\mu \geq 0$ the density of unproductive agents diverges to infinity as x converges to zero from above. This asymptote plays an important role in providing the grounds for the optimality of expansionary monetary policies. Note that the asymptote at 0 shows that unproductive agents, unless being hit by a shock that reverse their productive state, deplete their money holdings to the point of almost exhaustion. As positive money growth rates provide a floor to consumption (see equation (16)), the asymptote at 0 provides the basis for the optimal money growth rate being positive.

The left panel of Figure 3 plots the density function $f(x^1, 2)$ for two parametrization, one with zero money growth rate and one with positive money growth rate. In both cases it can be seen that, as shown in Lemma 4, the density function has an asymptote as the share of wealth in hands of an unproductive agent approaches zero. Moreover, the plot shows what happens with the mass "near" zero as we change the money growth rate. As expected, increasing the money growth rate μ increases the amount of histories where unproductive agents have little money. This follows because a higher μ implies higher insurance, and therefore a lower cost of running out of money. In the right panel of Figure 3 we explore the effect of λ on shaping the density function. We do so by analyzing the case where there is no money growth rate (i.e. $\mu = 0$).[10] The figure shows that increasing λ makes the mass close to $x = 0$ lower and the mass at average values of x larger. Intuitively, because the higher is λ the shorter the unproductive state, the probability that money holdings of unproductive agents gets depleted for high λ is lower. This will be important later on in understanding the effect of λ on the optimal monetary policy.

[10] The same qualitative results hold for positive money growth rates.

Figure 3: Invariant distribution of wealth $f(x^1, 2)$



Parameters: $\rho = 0.05$. Left panel: $\lambda = 0.1$. Right panel: $\mu = 0$.

The results of Lemma 4 do not hinge on μ being constant, as we state in the next corollary.

Corollary 2 *Let x denote the share of money in the hands of unproductive agents and let $\mu = \mu[x]$, where $\mu[x]$ is consistent with equilibrium. For any given $\mu[x]$ continuous and $x^1 \in (0, 1)$ the invariant density of money holdings $f(x^1, 2)$ has the following properties: (i) continuous and differentiable in x^1 and μ, (ii) $\lim_{x \downarrow 0} f(x^1, 2) = +\infty$, and (iii) $\lim_{x \downarrow 0} f_x(x^1, 2) = -\infty$.*

The proof of the corollary is a straightforward extension of the proof of Lemma 4 and therefore is omitted.

5 On the optimal supply of liquidity

In this section we define a welfare criterion and explore the properties of different rules for the supply of liquidity (i.e. money). Two scenarios are considered: in the first one it is assumed that no aggregate information is available to guide the policy, so that the rule is "state independent". In the second one we assume the policy can be made contingent on some aggregate variables that summarize the state of the economy, such as total output or the asset price. We refer to this case as the "state dependent" policy.

Let $V(x;\mu)$ denote the discounted present value of the sum of utilities of both types of agents, where agents are given the same Pareto weight. This is a function of the money share of unproductive agents, x. Monetary policy μ affects the value only through changes in the evolution of the state x. Still, we make explicit the dependence of the value function on the policy parameter μ as it will prove to be useful for comparative statics analysis. The continuous time Bellman equation is

$$\rho V(x;\mu) = \ln c^1(x,2;\mu) - 1 - c^1(x,2;\mu) + V_x(x;\mu)\dot{x}(x,2;\mu) + \lambda\left[V(1-x;\mu) - V(x;\mu)\right] \quad (21)$$

where $\dot{x}(x,2;\mu)$ denotes the evolution of money holdings of an unproductive agent, and where we made explicit the dependence of the value function, consumption, and evolution of money holdings on the choice of the money growth rate μ.[11]

We consider the problem from an ex-ante perspective, i.e. assuming that at the beginning of time nature assigns the initial productive states and the planner can choose the initial wealth distribution and a policy rule for money growth. Note that because individual types are not observable, and given the symmetry of the environment (and identical Pareto weights), the planner will give the same amount of liquidity to every agent and therefore at the beginning of time $x = \frac{1}{2}$.[12]

To compare different policies it is useful to define the welfare of a given policy using a certainty equivalent compensating variation. Let α denote the consumption equivalent cost

[11]This ODE follows from writing the problem in discrete time for a time period of length Δ and then taking limits as Δ approaches 0. With a slight abuse of notation let $V(x;\mu)$ denote the value function in discrete time,

$$V(x;\mu) = \Delta\left(\ln c^1(x,2;\mu) - 1 - c^1(x,2;\mu)\right) + (1-\Delta\rho)\left((1-\Delta\lambda)V(\tilde{x};\mu) + \Delta\lambda V(1-\tilde{x};\mu)\right)$$

where Δ is the length of the time period, and $\tilde{x}$ is the share of money in the hands of unproductive agents the next date. That is, $\tilde{x} = x + \dot{x}^1(x,2)\Delta$.

[12]A key assumption here is that the government commits not to do again a "once and for all" complete reshuffling of money balances. It is shown in Lemma 5 below that a policy that keeps the wealth distribution (after the government transfer) constant at $x = 1/2$ is obtained when $\mu \uparrow \infty$. The lemma shows that in this case a monetary equilibrium does not exist and allocations revert to autarky.

of market incompleteness associated with a given policy. That is, α solves the following equation

$$2\ \ln(1-\alpha) - 2 = \rho\, V\left(\frac{1}{2};\mu\right) \tag{22}$$

so that α measures the fraction of the consumption under complete markets that agents would be willing to forego to eliminate the volatility of consumption due to market incompleteness for a given policy rule μ.[13]

5.1 A constant anticipated policy

When the government does not observe the individual states nor any economic outcomes (output, prices), the policy consists of a constant growth rate: $\mu_t = \mu$. Once the initial distribution of money $x = 1/2$ is fixed, the planner chooses the systematic (constant) money growth rate μ to maximize $V(\frac{1}{2};\mu)$. We first show analytically that the planner would never choose to follow a contractionary policy or a policy where inflation is infinite. To do this, we show that $V(\frac{1}{2};\mu)$ diverges to $-\infty$ as μ approaches ∞ or when μ is negative. We then show that, for a fixed discount rate $\rho > 0$, the optimality of expansionary policies ($0 < \mu < \infty$) with respect to a constant money supply ($\mu = 0$) depends on the expected length of unproductive spells $1/\lambda$.

The next lemma shows that ex-ante expected utility diverges as the money growth rate approaches infinity or when a contractionary monetary policy is followed:

Lemma 5 *Under systematic extreme monetary expansions ($\mu \uparrow \infty$) or monetary contractions ($\mu < 0$), there is a break up in trade resulting in autarkic allocations. Then, $V(\frac{1}{2};\mu)$ diverges to $-\infty$ either when $\mu \uparrow \infty$ or $\mu < 0$.*

The economics of lemma, formally elaborated in the proof (see Appendix J), are simple: as the money growth rate becomes very large the marginal value of a unit of money becomes small because inflation erases the value of accumulated money holdings. In the limit, as an

[13] Recall that under complete markets $c^1(x^1,1) = c^1(x^1,2) = 1\ \forall\, x^1$ and $l^1(x^1,1) = 2\ \forall\, x^1$.

"hyperinflation" is implemented (i.e. $\mu \uparrow \infty$) the value of money becomes nil, so that a productive agent has no incentives to accept money in exchange for goods, and the consumption of an unproductive agent is zero. Likewise, for any contractionary policy, i.e. $\mu < 0$, there is no monetary equilibrium and hence no trade, as was shown in Lemma 1. With either type of policy the consumption allocations coincide with those under autarky, so that unproductive agents do not consume. Since agents spend, on average, half of their lives in the unproductive state, their expected utility under this policy diverges to $-\infty$.

For finite non-negative values of money growth rate μ the welfare $V(\frac{1}{2};\mu)$ is finite.[14] Notice that there are only 2 parameters affecting the value of the policy: ρ and λ. We study the optimal choice of μ as we vary ρ and λ in the left panel of Figure 4. The figure shows that the optimal μ increases with ρ. The reason is that agents with a higher discount rate will become poor faster, since their savings motive is weaker. This translates into a higher need for insurance, i.e. a higher growth rate of money. Conversely, when ρ approaches zero the insurance motive vanishes and the optimal choice of μ converges to 0, independently of the value of λ. This happens because absent the time discount the economy can actually sustain first best allocations with $\mu = 0$, which for this limiting case implements the Friedman rule.

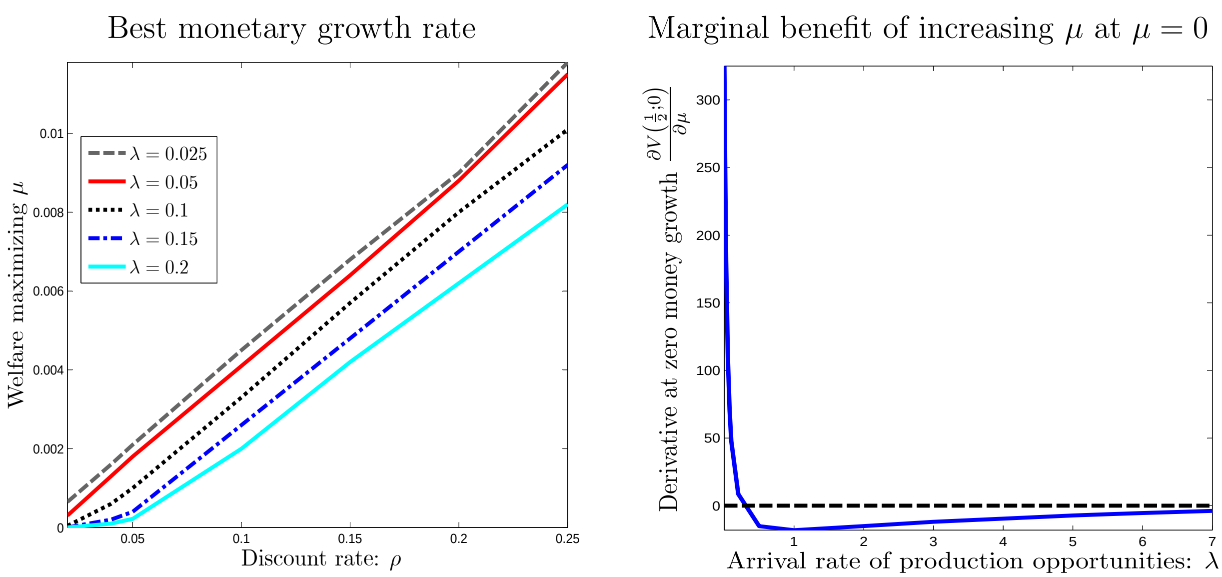
It is also shown that, for a given discount rate, a smaller value of λ increases the optimal rate of money growth. As the expected length of an unproductive spell, given by $1/\lambda$, becomes longer, then the insurance motives become more important and the optimal policy prescribes a higher μ. In the limit, as the duration of the unproductive spells becomes infinitesimal ($\lambda \uparrow \infty$), the insurance problem become irrelevant: it is easily seen from equation (8) and equation (9) that the economy approaches the complete markets allocation (constant value of money) and the optimal money growth is zero.

The optimal policy trades off production incentives and insurance motives. We explore this in the right panel of Figure 4 where we plot the derivative of the value function evaluated at $\mu = 0$ as we vary λ, for the benchmark case where $\rho = 0.05$. Because the value function is

[14]That $V(\frac{1}{2};0)$ is finite follows immediately from Scheinkman and Weiss (1986). For $\mu > 0$ note that $V(\frac{1}{2};\mu) > \frac{\ln\left[\frac{\mu}{2}q(0)-2\right]}{\rho} > -\infty \; \forall \mu > 0$, where $\frac{\mu}{2}\gamma(1,1) = c^1(0,2)$.

continuous and differentiable in μ, and $\lim_{\mu\uparrow\infty} V(\frac{1}{2};\mu) = -\infty$, the optimal policy is expansionary if $\frac{\partial V(\frac{1}{2};0)}{\partial\mu} > 0$ and is constant if $\frac{\partial V(\frac{1}{2};0)}{\partial\mu} < 0$. The plot shows that the optimal policy is expansionary for low values of λ or, in other words, when unproductive spells are long enough. The economics behind the optimal money growth rate being positive for low values of λ is simple: in an economy with borrowing constraints agents will occasionally incur into histories, i.e. long spells of low-productivity, in which they "almost" run out of money. The amount of histories depends on λ. The positive money transfers the government provide a floor to how bad consumption looks in these states.[15] But this insurance provision comes at a cost, as productive agents are less willing to accept money balances in exchange for goods as monetary expansions decrease the return of money. The optimal finite money growth rate strikes a balance between these opposing forces.

Figure 4: On the optimal monetary policy μ



Left panel: Optimal choice of μ as we vary ρ and λ. Right panel: $\frac{\partial V(\frac{1}{2};0)}{\partial\mu}$ as we vary λ (parameter: $\rho = 0.05$).

[15] Notice that self-insurance is hard to achieve in this model because the terms of trade (i.e. the value of money q) worsen as unproductive agents deplete their stock of money.

5.2 A state dependent policy

The analysis of the optimal policy in Section 5.1 assumed that money growth μ was constant. In this section we allow the government to observe some economy aggregates that can be used to condition the policy, under the maintained assumption that the individual state of each agent is not observable. We let x denote the wealth share of the unproductive agent and assume the government can tie the monetary policy to the distribution of wealth, which in this model maps to aggregate production:[16]

$$\mu_t = \mu[x_t]$$

this policy is such that the government observes the share of wealth held by unproductive agents, but cannot identify individual agents nor observes transactions. This implies that the complete markets allocation, where both types of agents consume $\bar{c}$ for every value of wealth and state of nature, cannot be attained (see Corollary 1 in Section 3.3).

Allowing the monetary policy to depend on the distribution of wealth changes the system of ODEs governing the marginal values of money $\gamma(x^i, s)\ \forall\, x^i, s$, and the boundary conditions. The system of ODEs in equations (11) and (12), become

$$\gamma_x[x^1,2]\dot{x}^1[x^1,2] = (\rho + \lambda + \mu[x^1])\gamma[x^1,2] - \lambda\gamma[x^1,1] \tag{23}$$

$$\gamma_x[1-x^1,1](-\dot{x}^1[x^1,2]) = (\rho + \lambda + \mu[x^1])\gamma[1-x^1,1] - \lambda\gamma[1-x^1,2] \tag{24}$$

where

$$\dot{x}^1[x^1,2] \quad = \quad \mu\left[x^1\right]\left(\frac{1}{2} - x^1\right) - \frac{1}{\gamma[x^1,2]} \tag{25}$$

[16] Aggregate production is $1 + c^1(x,2)$ where we already showed that $c^1(x,2)$ is monotone in x.

with boundary conditions

$$\gamma[0,2] \quad = \quad \frac{2}{\mu[0]} \quad , \quad \lambda\gamma[1,2] = (\rho + \lambda + \mu[0])\gamma[1,1] \ . \tag{26}$$

Note that a corollary of Proposition 1 is that the optimal $\mu[x]$ cannot be such that $\mu[x] \leq 0 \ \forall \ x$ as all trade would break down. However, note that a major difference with respect to the case of a constant μ is that it is now feasible to implement $\mu[x] < 0$, at least for some values of x.

We develop our analysis of the state dependent policy by restricting the policy rule and $\mu[x]$ to be the following monotone function of x

$$\mu[x] = \mu_1 + (\mu_0 - \mu_1)\left(e^{-\kappa x} - x e^{-\kappa}\right) \tag{27}$$

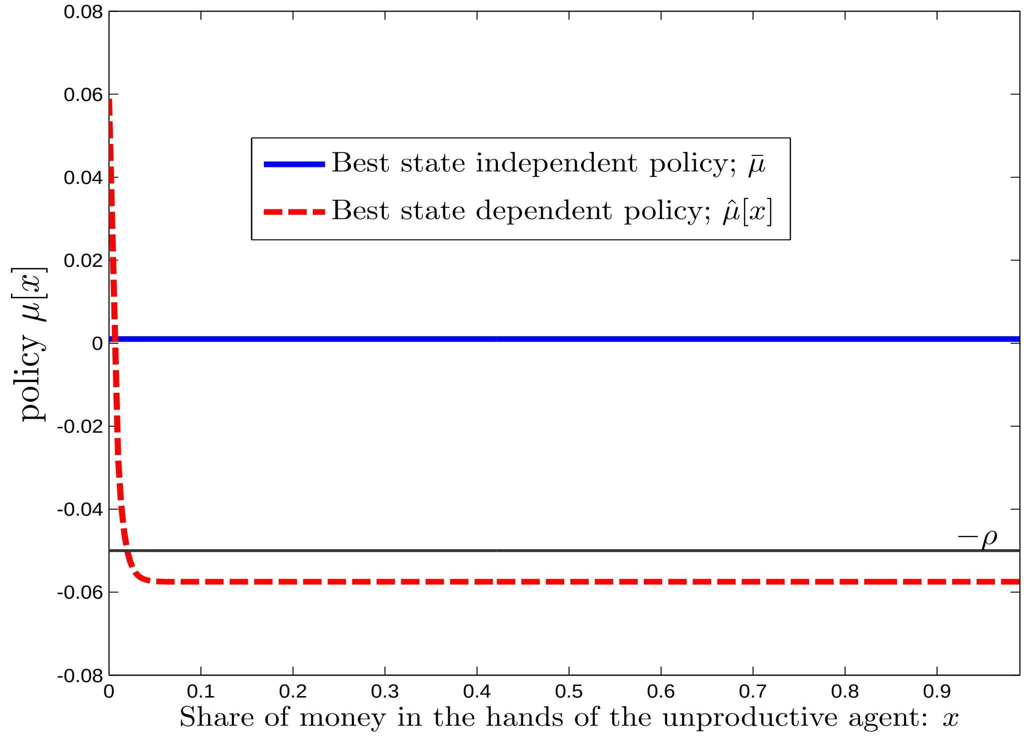
with $\mu[0] = \mu_0$, $\mu[1] = \mu_1$ and $\kappa \geq 0$. Note that if $\kappa = 0$ then $\mu[x]$ is linear, and if $\kappa = 0$ and $\mu_0 = \mu_1$ then $\mu[x]$ is constant over the whole range. This functional assumption for the policy rule allows us to explore the role of state dependent policy by a parsimonious parametrization of the policy choices.[17] The function is monotone decreasing and convex if $\mu_0 > \mu_1$ and $\kappa > 0$. Given the functional form assumed for $\mu[x]$, now the planner chooses a triple of parameters $\{\mu_0, \mu_1, \kappa\}$.

We compute the best policy to maximize the ex-ante expected welfare as defined in equation (21), by searching on a tri-dimensional grid over μ_0, μ_1, and κ. We compare the best constant −and therefore state independent− policy, $\bar{\mu}$, with the best state dependent policy: $\hat{\mu}[x]$. In Figure 5 we present the best state independent and state dependent policies under the baseline parametrization of the model ($\lambda = 0.1$ and $\rho = 0.05$).

The best state independent policy $\bar{\mu}$ consists of a constant expansion of the monetary base of 0.1% per unit of time. As was discussed in the previous subsection this result is due to the fact that the expected duration of the cycle is sufficiently long so that the corner solution of

[17] We discuss in the concluding remarks the challenges involved in solving the unconstrained Ramsey plan *without* imposing a functional form on the policy rule.

Figure 5: Optimal state dependent and state independent policy rules

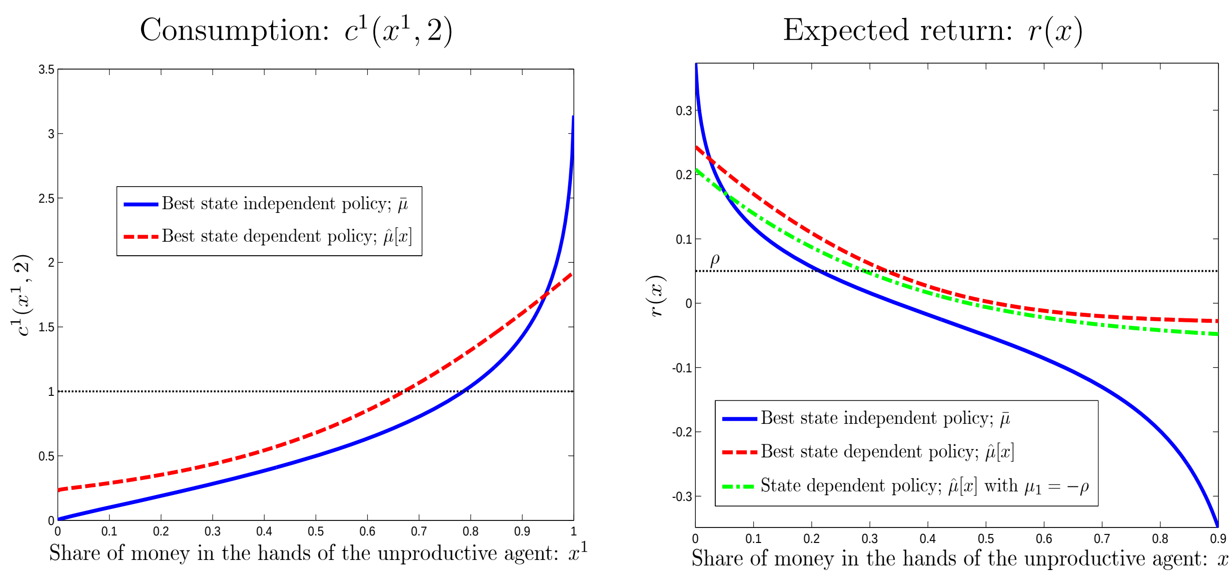


Parameters: $\lambda = 0.10$, $\rho = 0.05$. Optimal state independent policy: $\mu = 0.001$. Optimal state dependent policy: $\mu_0 = 0.059, \mu_1 = -0.0575, \kappa = 137.7$.

$\mu = 0$ is dominated by a strictly positive money growth rate. The best state dependent policy $\hat{\mu}[x]$ is monotone decreasing: it prescribes liquidity injections when unproductive agents are very poor, i.e. $x \approx 0$ (at a nearly 6% rate). As unproductive agents become sufficiently wealthy the policy rule converges quickly to prescribing liquidity contractions: for $x \cong 0.02$ the optimal policy converges to a rate of liquidity contraction slightly below $-\rho$. The state dependent pattern of the policy rule can be equivalently interpreted in terms of the business cycle. Since the aggregate production is $c^1(x,2)+c^1(1-x,1) = c^1(x,2)+1$, then the policy $\mu[x]$ is such that monetary expansions happen when aggregate production is low and monetary contractions when it is high. In other words, the policy is expansionary during recessions and contractionary during expansions. A similar argument for the (ex-post) beneficial effect of liquidity expansions during recessions is developed by Brunnermeier and Sannikov (2010) in the context of a larger quantitative model of a monetary economy.

The comparison of the state independent rule $\bar{\mu}$ with the state dependent policy $\hat{\mu}[x]$ shows that the latter allows the planner to provide more insurance when needed. This can be seen by noting that when x is small, the optimal state dependent policy $\hat{\mu}[x]$ expands the money supply at a faster rate than the optimal state independent policy $\bar{\mu}$. Moreover, the state dependent policy $\hat{\mu}[x]$ provides high production incentives, as shown by the higher return on savings in Figure 6 (red dashed line). This characterization shows that the constant policy is some kind of average of the values prescribed by the state dependent one. This follows since the state dependent rule allows the planner to partially decouple insurance motives and production incentives.

Figure 6: Consumption and the return on money under different policy rules



Parameters: $\lambda = 0.10$, $\rho = 0.05$.

Figure 6 illustrates the profiles of the consumption function and the expected return on money under the two policy rules: $\bar{\mu}$ and $\hat{\mu}[x]$. The figure shows that under the best state dependent policy $\hat{\mu}[x]$ the consumption rule for unproductive agents provides a smoother consumption profile as a function of x than is produced under the constant policy $\bar{\mu}$. The smoother consumption profiles of the state dependent rule also yields a flatter profile for the expected return on money $r(x)$, as shown in the right panel of the figure. This flatter profile reflects the fact that the consumption of an unproductive agent is less extreme under $\hat{\mu}[x]$: the smaller expected changes in consumption (hence marginal utilities) associated to a state switch dampen the risk premia and lead to a smoother expected return of the asset.

Notice that μ_1, the best rate at which the liquidity supply is contracted once the economy is above very small values of x, is smaller than $-\rho$, the rate prescribed by the Friedman rule. We interpret this feature of the policy rule by the forward looking nature of agents' decisions: the agents' expectations of the asset return are a weighted average of future expected periods of monetary expansions and other periods of monetary contractions. The contractions in excess of the rate of time preference partially compensate the expected asset return of the effect due to the anticipated monetary expansions. This raises the mean expected asset return (see Figure 6), and increases the incentives to produce. This is shown in the right panel of the figure which also plots the expected return implied by a state dependent rule which replaces the value of μ_1 prescribed by the best rule with the (negative of the) time discount rate $-\rho$ (green line). It appears that returns are uniformly lower under such a policy.

A summary of some key features of each policy rule is given in Table 1, which reports the welfare costs of market incompleteness and some time-series statistics implied by the two policy rules under our benchmark parametrization with $\lambda = 0.10$ and $\rho = 0.05$. The first row of Table 1 reports the welfare cost of market incompleteness α under both policy rules, using the compensating variation defined in equation (22).[18] It can be seen that the state dependent policy $\hat{\mu}[x]$ is able to provide a much higher welfare to agents: the cost α associated to the constant policy is almost twice the cost of the best state dependent policy.

The other lines of the table report some unconditional moments of the model statistics, computed through computer simulations. These statistics are related to the distribution of money holdings x, discussed in Section 4.1. Those moments thus combine the state contingent behavior of e.g. the consumption and the expected return on money, which depend on the value of x, with the "local time" that the model spends around each value of x. The mean and

[18]Recall that α measures how much of the consumption under the complete markets allocation agents are willing to forego to smooth all the consumption fluctuations that occur under incomplete markets.

Table 1: Summary statistics implied by the policy rules

	State independent $\bar{\mu}$ $\bar{\mu} = 0.001$	state dependent $\hat{\mu}[x]$ $\mu_0 = 0.059, \mu_1 = -0.0575, \kappa = 137.7$
Welfare cost[a] (α)	31.1%	15.3%
$\mathbb{E}\left[\ln\left(c^1(x,2)\right)\right]$	-2.1	-0.7
$\sigma\left[\ln\left(c^1(x,2)\right)\right]$	1.7	0.73
$\mathbb{E}[x]$	0.27	0.35
% of time $x < 0.0001$	7%	34%
mean time to (first) hit $x = 0.0001$	78	17
$\mathbb{E}[r(x)]$	7%	10%
$\mathbb{E}[\mu]$	0.1%	-1.65%

[a] computation does not require simulation
Parameters: $\lambda = 0.1$, $\rho = 0.05$. Details on simulation: We discretize the economy and define a period to be 12 hours. We simulate the economy under both monetary policies 10^6 times.

standard deviation of the unproductive consumption rate indicate why the state dependent policy $\hat{\mu}[x]$ provides higher utility than the constant policy: the mean consumption utility is higher, and less volatile.

Another interesting aspect of $\hat{\mu}[x]$ is that the greater insurance that it provides makes the unproductive agents less fearful of being poor. The fact that more insurance (is provided by the state contingent policy) makes the agent less afraid of getting close to the borrowing constraint can also be seen by analyzing the invariant density of money holdings discussed in Section 4.1, where we showed that there are more histories for which money balances nearly depletes as the provision of insurance increases. Starting form an egalitarian distribution of money holdings (i.e. $x = 1/2$) an unproductive agent depletes most of her money holdings in 17 years under $\hat{\mu}[x]$, while it takes 78 years under $\bar{\mu}$. Likewise, defining a poverty threshold by a small value of wealth, namely $\underline{x} = 0.0001$ in this case, unproductive agents spend 34% of their time being poor (i.e. with $x < \underline{x}$) under $\hat{\mu}[x]$, while only 7% of the time under $\bar{\mu}$.

The last row of Table 1 shows that the average money growth rate under $\hat{\mu}[x]$ is negative, but above the rate prescribed in Friedman's rule (i.e. $-\rho$). Thus, in spite of the fact that $\mu_1 < -\rho$, the *average* money growth rate of the economy is larger than $-\rho$, as the economy spend a significant amount of time at low values of x, where the policy rules prescribes a high money growth rate.

Finally, although the quantitative features of the optimal policy, and the implied behavior of consumption, naturally depend on the benchmark values of $\lambda = 0.1$ and $\rho = 0.05$, the qualitative features of the state dependent optimal policy, such as its decreasing profile, and the implied smoothness of consumption and the expected return on money are general features, which are obtained for any choice of the parameters λ and ρ.

6 Concluding remarks

We studied how to control the supply of liquidity in a model where oscillations in production opportunities are uninsurable and money ("a liquid asset") is the unique available savings instrument. Since these oscillations are correlated across agents and markets are incomplete, the productivity shocks affect the aggregate output level, i.e. the economy displays a business cycle. Two scenarios were considered: the first one assumed that policy was chosen under minimal information, namely a constant unconditional money growth rate. In this case, it was shown that for a sufficiently high duration of the unproductive spells, the optimal policy is expansionary. This characterization is useful to understand the underlying forces underlying the costs and benefits of monetary injections: they provide insurance to poor unproductive agents, at the cost of reducing the return on money, therefore reducing the production incentives.

The second scenario assumed that policy can be made contingent on some aggregate variables, such as total output. In the space of policies that were explored the socially optimal policy prescribes a liquidity expansion when output is low, and liquidity contractions when output is high. This rule follows from the connection between aggregate production (or consumption) and the distribution of wealth. Low aggregate production maps to unproductive agents being poor, so that liquidity expansions are beneficial as they provide insurance. High aggregate production maps to unproductive agents being rich, so that liquidity contractions increase the asset returns, increasing the incentives to produce.

Some interesting extensions are left for future work. First, it would be interesting to provide a more general characterization of the socially optimal policy. Our analysis in Section 5.1 assumed a parametric functional form for the policy rule (monotone and differentiable). More generally, one might want to solve for the state-dependent policy using a Ramsey plan as in Chang (1998), where the planner chooses the policy by picking the best competitive equilibria implied by it. One key difference is that in solving this problem we would not be imposing a functional form on the policy rule. This application is involved because in our problem the set of continuation values will depend on the aggregate state, due to the presence of a physical state variable (wealth). We conjecture that this might be dealt with by applying the ideas developed by Fernandes and Phelan (2000); Phelan and Stacchetti (2001).

Another extension is to improve the quantitative properties of the model. The assumption that the shocks are perfectly correlated makes the model tractable, but perhaps unrealistic. In Appendix A-11 we explore a "Bewley" model with a continuum of agents whose shocks are uncorrelated. In this model aggregate production is constant so that there is no room for studying how to control liquidity over the business cycle. It is shown that in this model a socially efficient policy also involves ex-ante expansions if shocks are sufficiently persistent, for the same reasons discussed in Section 5.1. A more realistic model might combine these two extreme cases, by exploring intermediate cases for the shock's correlations across agents types. This interesting assumption would come at the cost of losing the model's tractability, but might explored by means of numerical simulations, as in the recent related work by

Brunnermeier and Sannikov (2011) and Guerrieri and Lorenzoni (2011).

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Appendix

A Proof of Remark 2

Let $i = 1$ be the index for the unproductive agent, and consider her decision problem. The money supply growth is $\mu = \frac{\dot{m}_t}{m_t}$ and let $\tilde{\xi}^1_t$ denote the lagrange multiplier of the money flow constraint in equation (3). The first order condition with respect to c^1_t gives: $\xi^1_t = \; q_t \; u'(c^1_t)$, where we used the homogeneity of degree -1 in the aggregate money supply m_t for both $\tilde{\xi}^1_t$ and $\tilde{q}_t$, namely: $\tilde{\xi}^1_t = \frac{1}{m_t}\xi^1_t$ and $\tilde{q}_t = \frac{1}{m_t}q_t$. The Euler equation for $\dot{m}^1_t$ gives $\frac{\dot{\tilde{\xi}}^1_t}{\tilde{\xi}^1_t} = \rho$ or $\rho = \frac{\dot{\xi}^1_t}{\xi^1_t} - \frac{\dot{m}_t}{m_t} = \frac{\dot{q}_t}{q_t} - \theta\frac{\dot{c}^1_t}{c^1_t} - \mu$ where the last equality uses $\xi^1_t = q_t u'(c^1_t)$ and that $u''c/u' = -\theta$. Notice that this is solved by $c^1_t = \bar{c} \quad , \quad \dot{c}^1_t = \dot{q}_t = 0, \quad \mu = -\rho.$

The constant level of q is pinned down by imposing that total nominal assets at the beginning of a cycle, which in the stationary equilibrium are held by the unproductive agent, are equal the total nominal consumption and tax expenditures over the cycle of length T. Without loss of generality let's consider the first cycle, starting at time 0, where the unproductive agent holds all the money supply m_0. Using that $m_t = m_0 e^{\mu t}$, $\mu = -\rho$, and $\tilde{q}_t = \frac{1}{m_t}q$, we have

$$\int_0^T \frac{\bar{c}}{\tilde{q}_t}\, dt + \int_0^T \rho m_t \, dt = m_0 \quad , \quad \text{which gives} \quad q = \bar{c}\ \frac{1 - e^{-\rho T}}{\rho e^{-\rho T}} \cong \bar{c}\ \frac{T}{1 - \rho T}.$$

where the approximation is accurate for small T. It is immediate to verify that this allocation also solves the Euler equation of the productive agent and that her money holdings are never negative.

B Fiscal policy under alternative government powers

A central assumption in our analysis is that the government does not know $s(t, \omega)$, i.e. the identity of the productive type. It is useful to explore the consequences of relaxing this assumption to better understand the nature of the monetary policy problem. Without loss of generality, given the symmetry of the states, let us assume that type 1 is productive and type 2 is not productive. Also, for simplicity, let us set the money supply equal to zero in what follows.

We begin by assuming that the government observes the identity of the productive type and is able to tax productive agents and transfer resources to unproductive agents. We consider two taxing technologies. The first one is lump sum taxes: in this case the productive agent pays a flat tax $\bar{\phi} = \bar{c}$, and the government uses the proceedings to finance the consumption of the unproductive agent. It is immediate that under these assumptions the complete markets allocation can be replicated. Alternatively, consider a setup where the only available taxes are distortionary, say proportional to production: then the transfer to the unproductive agent is $\tau = \phi l$. For a generic tax rate $\phi \in (0, 1)$ the consumption of the two agents solves $u'(c^1) = \frac{1}{1-\phi}, u'(c^2) = \frac{1}{\phi}$. An ex-ante optimal policy, maximizing the expected utility of the two types with equal weights, gives $\phi = 1/2$. Under this setting the government fiscal policy provides insurance, consumption is constant through time, though the level of consumption is smaller than under complete markets.

Table 2: Welfare (marginal utilities) under alternative fiscal policy powers

type known		type not known	
lump-sum-tax	distortionary tax	Gvt. commitment	No commitment
$u'(\bar{c}^1) = u'(\bar{c}^2) = 1$	$u'(\tilde{c}^1) = u'(\tilde{c}^2) = 2$	$u'(\bar{c}^1) = u'(\bar{c}^2) = 1$	$u'(\bar{c}^1) = 1 \quad , \quad u'(c^2) = \infty$

Let us next consider a government who does not know the type's identities. In this case the efficient stationary allocation with $c = \bar{c}$ at all times for all types can be sustained if the government has the ability to commit to a trigger policy. Suppose the government credibly announces: "productive types must pay a tax $\bar{c}$ to the government, who will then transfer it to the other types. If at any point in time the tax is not enough to pay for the transfer, the scheme will be shut down and the economy will be left in autarky forever". Assuming the threat is credible (and discounting is finite) then it is in the interest of every individual agent to comply, because deviating implies that the agent consumption is zero when unproductive which, due to Inada conditions, delivers an expected utility of $-\infty$ (since, on average, agents are unproductive for half of the times). The various outcomes sustainable under alternative fiscal policy assumptions are summarized in Table 2.

In what follows, we consider a less powerful government than the one depicted above. We assume the government does not know the identity of productive types, and that it cannot commit to trigger policies. In such a situation fiscal policy, i.e. direct taxation, is powerless. Absent a liquid asset, the resource allocation is autarkic, and individuals experience inefficient fluctuations in utility. We next study the powers of monetary policy, under the maintained assumptions of type-ignorance and no-commitment.

C Proof of Lemma 1

We will first prove by contradiction that $\zeta \notin [0, 1/2)$. Then we will show that $\zeta = 1/2$ is enough to cover the lifetime tax obligations. Suppose that $\zeta < 1/2$. Without loss of generality assume that $x_t^1 \in (\zeta, 1/2)$ and agent 1 is unproductive. Conditional on no reversal of the state, it follows that $x_{t+dt}^1 < x_t^1$. Then for a given $\Delta \in \mathbb{R}^+$, $\Pr\left[x_{t+\Delta}^1 < \zeta\right] > 0$ and therefore the agent will fail to comply with her tax obligations with positive probability. Then, $\zeta \notin [0, 1/2)$. Consider now the case where $x_t^1 = \zeta = 1/2$. As the agent can decide not to trade she can always keep her share of outstanding money balances x^1 above 1/2 and therefore for any $\mu \in (0, 1)$ she will be able to cover her tax needs. That $x^1 = 1/2 \; \forall \; s$ is the ergodic set is trivial. If $x_0^1 < 1/2$ there is a positive probability that an agent fails to pay for her lifetime taxes. An unproductive agent with money holdings $x^1 > 1/2$ is willing to buy goods (and the productive one with $x^1 < 1/2$ willing to take the money) until x^1 reaches 1/2.

D The Euler equation for the marginal utility of money

The Hamilton-Jacobi-Bellman equation implies that the lagrange multiplier $\tilde{\gamma}$ follows

$$
\begin{aligned}
&\mathbb{E}\left\{ e^{-\rho(t+dt)}\tilde{\gamma}(m(t+dt,\omega), x^1(t+dt,\omega), s(t+dt,\omega)) \,\middle|\, m(t,\omega)=m, x^1(t,\omega)=x^1, s(t,\omega)=1 \right\} \\
&\approx e^{-\rho t}\left[-\rho\tilde{\gamma}(m_t,x_t^1,1)dt + \tilde{\gamma}_x(m_t,x_t^1,1)\dot{x}_t^1(x_t^1,1)dt + \tilde{\gamma}_m(m_t,x_t^1,1)\dot{m}_t dt + \tilde{\gamma}(m_t,x_t^1,1)(1-\lambda dt)\right] \\
&+ e^{-\rho t}\tilde{\gamma}(m_t,x_t^1,2)\lambda dt \\
&= \frac{e^{-\rho t}}{m_t}\left[-\rho\gamma(x_t^1,1)dt + \gamma_x(x_t^1,1)\dot{x}_t^1(x_t^1,1)dt - \gamma(x_t^1,1)\frac{\dot{m}_t}{m_t}dt + \gamma(x_t^1,1)(1-\lambda dt) + \gamma(x_t^1,2)\lambda dt\right] \\
&= \frac{e^{-\rho t}}{m_t}\left[\gamma(x_t^1,1) + \gamma_x(x_t^1,1)\dot{x}_t^1(x_t^1,1)dt - \gamma(x_t^1,1)(\rho+\lambda+\mu)dt + \gamma(x_t^1,2)\lambda dt\right]
\end{aligned}
$$

Subtracting $e^{-\rho t}\gamma(x_t^1,1)/m_t$ from both sides, dividing by dt, taking the limit for $dt \downarrow 0$, gives equation (8). An identical logic gives equation (9).

E Proof of Lemma 2

That $\gamma(x^1,s) > 0$ for $s = 1,2$ is implied for all internal solutions from the Khun-Tucker theorem and increasing utility. Next we show that $\gamma_x(x^1,s) < 0$ for $s = 1,2$, $\gamma(x^1,1) < \gamma(x^1,2)$, and $\dot{x}^1(x^1,2) < 0$. The proof follows by conjecturing that $\dot{x}^1(x^1,2) < 0$ in the equilibrium. We then show that $\dot{x}^1(x^1,2) < 0$ is consistent with the equilibrium, which completes the proof.

Conjecture that $\dot{x}^1(x^1,2) < 0$. We first show that there cannot be an equilibrium where $\gamma_x(x,2) \geq 0$. A stability argument in the system of differential equations given in equation (8) and (9) implies that $\gamma(x^1,1)$, and the price level $q(x)$, will diverge, which constitutes a violation of the equilibrium definition (see page 29, Figure 1, in Scheinkman and Weiss (1986)).

Now consider $\dot{x}^1(x^1,2) < 0$ and $\gamma_x(x^1,2) < 0$. Repeating the stability analysis on the system of differential equations given in equation (8) and (9) gives that $\gamma_x(x^1,1) < 0$: if $\gamma_x(x^1,1) \geq 0$, eventually $\gamma_x(x^1,2) > 0$, which constitutes a contradiction. The requirement for $\gamma_x(x^1,1) < 0$ and $\gamma_x(x^1,2) < 0$ is that in equilibrium $\frac{\gamma(x^1,1)}{\gamma(x^1,2)} < \frac{\lambda}{\rho+\lambda+\mu}$ (the locus $\gamma_x(x^1,1) = 0$ provides $\gamma(x^1,1) = \frac{\lambda}{\rho+\lambda+\mu}\gamma(x^1,2)$). Because $\frac{\lambda}{\rho+\lambda+\mu} < 1$, it follows immediately that

$$\gamma(x^1,1) < \gamma(x^1,2) \;\; \forall\; x^1 \;\in\; (0,1) \tag{28}$$

The interpretation is immediate: as the productive agent has more resources, i.e. she can work, the marginal value of wealth is lower for her. It remains to be showed that the conjecture $\dot{x}^1(x^1,2) < 0$ is satisfied in equilibrium and that $\gamma_x(x^1,2) < 0$ is consistent with it. We do both things next.

We show that if $\gamma(x^1,1) < \gamma(x^1,2)$ then $\dot{x}^1(x^1,2) < 0$ and $\gamma_x(x^1,2) < 0$ for $x^1 \in (0,1)$, which will complete the proof. Note that the right hand side of equation (9) and the inequality in equation (28) imply that $\gamma_x(x^1,2)$ and $\dot{x}^1(x^1,2)$ have the same sign. We now argue that the sign must be negative, otherwise the optimality of the consumption plan is violated.

Differentiating the unproductive agent first order condition (equation (7)) gives

$$\frac{\dot{\gamma}(x^1,2)}{\gamma(x^1,2)} = \frac{\dot{q}(x^1,2)}{q(x^1,2)} + \frac{u''\left(c^1(x^1,2)\right)}{u'\left(c^1(x^1,2)\right)}\,\dot{c}^1(x^1,2)$$

or, using that $q(x^1,2) = \gamma(1-x^1,1)$ and $\dot{c}^1 = c^1_x\,\dot{x}^1$,

$$c^1_x(x^1,2)\;\dot{x}^1(x^1,2) = \left[\frac{\dot{\gamma}(x^1,2)}{\gamma(x^1,2)} - \frac{\dot{\gamma}(1-x^1,1)}{\gamma(1-x^1,1)}\right]\;\frac{u'\left(c^1(x^1,2)\right)}{u''\left(c^1(x^1,2)\right)}$$

Using equation (8) and equation (9) to replace the terms in the square parenthesis gives

$$c^1_x(x^1,2)\;\dot{x}^1(x^1,2) = \lambda\left[\frac{\gamma(1-x^1,2)}{\gamma(1-x^1,1)} - \frac{\gamma(x^1,1)}{\gamma(x^1,2)}\right]\;\frac{u'\left(c^1(x^1,2)\right)}{u''\left(c^1(x^1,2)\right)} < 0 \tag{29}$$

where the inequality follows since the term in the square parenthesis is positive, as implied by equation (28). The right hand side of equation (29) implies that consumption is decreasing, i.e. $\dot{c}^1(x^1,2) < 0$. This could happen in one of two ways. First, with $\dot{x}^1(x^1,2) > 0$ and $c^1_x(x^1,2) < 0$. But this violates optimality: consumption is *decreasing* in the agent's wealth share. The agent could deviate from this plan and increase her welfare. The other possibility, consistent with optimality, is that $\dot{x}^1(x^1,2) < 0$ and $c^1_x(x^1,2) > 0$. Then, in the equilibrium, $\gamma_x(x^1,2) < 0$ and $\dot{x}^1(x^1,2) < 0$, which confirms the conjecture.

Finally we show that $\lim_{x^1\downarrow 0}\dot{x}^1(x^1,2) = 0$. The right hand side of equation (29) is strictly negative at all x^1, including the boundary $x^1 = 0$. Since x^1 cannot be negative, this implies that $c^1_x(x^1,2) \uparrow +\infty$ and $\dot{x}^1(x^1,2) \uparrow 0$ as $x^1 \downarrow 0$. Equation (31) in Appendix G can be used to verify that $\lim_{x^1\downarrow 0} c^1_x(x^1,2) = +\infty$.

F Proof of Proposition 2

Conjecture that $c^1(x^1,1) = c^1(x^1,2) = \bar{c}$. From the first order conditions in equation (7),

$$\gamma(x^1,2) = q(x^1,2) = q(1-x^1,1) = \gamma(1-x^1,1)\;\;\forall\, x^1$$

as $u'(\bar{c}) = 1$. When $x^1 = 0$ we have that

$$\gamma(0,2) = \gamma(1,1) \equiv \gamma_a$$

and when $x^1 = 1$ we get

$$\gamma(1,2) = \gamma(0,1) \equiv \gamma_b$$

There are three possibilities: (i) $\gamma_a < \gamma_b$, (ii) $\gamma_a > \gamma_b$, and (iii) $\gamma_a = \gamma_b$. In case (i) we have that $\gamma(0,1) > \gamma(0,2)$ which contradicts Lemma 2. In case (ii) we have that $\gamma(1,1) > \gamma(1,2)$ which again contradicts Lemma 2. The only remaining possibility is (iii) where the marginal value of money is constant for every pair $\{x^1, s\}$. Let $\bar{\gamma}$ denote this value. From equation (8), after imposing the constant value $\bar{\gamma}$ for the marginal value of money, we obtain

$$(\rho + \mu)\bar{\gamma} = 0 \tag{30}$$

which is satisfied if $\mu = -\rho$ or when $\bar{\gamma} = 0$. When $\bar{\gamma} = 0$ note that the price level has to be constant and equal to the marginal value of money $\bar{\gamma}$ which follows from direct inspection of the first order conditions presented in equation (7). Let $\bar{q}$ denote this constant level. Because $\bar{\gamma} = 0$, $\bar{q} = 0$ which cannot constitute an equilibrium as one of the conditions for a monetary equilibrium is positive price $\bar{q}$ (see Definition 1). When $\mu = -\rho$ we know from Section 3.1 that it cannot constitute a monetary equilibrium. This implies that agents do not trade, money is valueless, and as a result there is no monetary equilibrium. This completes the proof of Proposition 2.

This result does not hinge on μ being constant. Let $\mu = \mu[x]$. Note that equation (30) has to hold for every value of x, where x is the share of money in the hands of unproductive agents. This again implies that solution requires either $\bar{\gamma} = 0$ or $\mu[x] = -\rho \ \ \forall\ x$, which we know does not constitute an equilibrium.

G Proof of Lemma 3

That $Q_1 > 1$ follows since $\gamma(0,2) > \gamma(0,1)$. Using equation (15) we write

$$\frac{c^1_x(x^1,2)}{c^1(x^1,2)} = -\frac{\gamma_x(1-x^1,1)}{\gamma(1-x^1,1)} - \frac{\gamma_x(x^1,2)}{\gamma(x^1,2)}$$

From equation (11) (evaluated at $1-x^1$) and equation (12) (evaluated at x^1) we get

$$\frac{\gamma_x(1-x^1,1)}{\gamma(1-x^1,1)} = \frac{\rho+\lambda+\mu-\lambda\frac{\gamma(1-x^1,2)}{\gamma(1-x^1,1)}}{\mu\left(x^1-\frac{1}{2}\right)+\frac{1}{\gamma(x^1,2)}} \quad , \quad \frac{\gamma_x(x^1,2)}{\gamma(x^1,2)} = \frac{\rho+\lambda+\mu-\lambda\frac{\gamma(x^1,1)}{\gamma(x^1,2)}}{\mu\left(\frac{1}{2}-x^1\right)-\frac{1}{\gamma(x^1,2)}} \ .$$

Then, noting that $\gamma(0,2) = \frac{2}{\mu}$, some algebra gives that

$$\lim_{x \downarrow 0} \frac{c^1_x(x^1,2)x^1}{c^1(x^1,2)} = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu}\left(1 - \frac{\gamma(0,1)}{\gamma(0,2)}\right) \equiv Q_1 \tag{31}$$

which proves the lemma.

H The return on money

We define the stochastic interest rate (or return on money) for a small time interval Δ as

$$r(x)\Delta = \mathbb{E}\left[\frac{\tilde{q}_{t+\Delta}}{\tilde{q}_t} - 1 \middle| x_t = x\right] = \mathbb{E}\left[\frac{q_{t+\Delta}}{q_t} - \Delta\mu_t - 1 \middle| x_t = x\right]$$

where $\tilde{q}_{t+\Delta} = \tilde{q}(x_{t+\Delta})$ and $\tilde{q}_t = q(x_t)/m_t$. Without loss of generality consider the case where at time t agent 1 is productive with money holdings given by x^1_t. Then,

$$r(x)\Delta = (1-\lambda\Delta)\frac{\gamma(x^1_{t+\Delta},1)}{\gamma(x^1_t,1)} + \lambda\Delta\frac{\gamma(x^2_{t+\Delta},1)}{\gamma(x^1_t,1)} - 1 - \Delta\mu_t \ ,$$

where we used equation (7). We use that $x^i_{t+\Delta} = x^i_t + \dot{x}^1(x^i_t, i)\Delta$ to do a Taylor expansion of first order of $\gamma(x^i_{t+\Delta}, 1)$ to obtain

$$r(x)\Delta = \frac{\gamma_x(x^1_t,1)\dot{x}^1(x^1_t,1)\Delta}{\gamma(x^1_t,1)} + \lambda\Delta\left(\frac{\gamma(x^2_t,1)+\gamma_x(x^2_t,1)\dot{x}^1(x^2_t,2)\Delta}{\gamma(x^1_t,1)} - \frac{\gamma(x^1_t,1)+\gamma_x(x^1_t,1)\dot{x}^1(x^1_t,1)\Delta}{\gamma(x^1_t,1)}\right) - \Delta\mu_t$$

taking the limit as Δ approaches 0,

$$r(x) = \frac{\gamma_x(x^1_t,1)\dot{x}^1(x^1_t,1)}{\gamma(x^1_t,1)} + \lambda\left(\frac{\gamma(x^2_t,1)}{\gamma(x^1_t,1)} - 1\right) - \mu_t$$

We use equation (23) to get

$$r(x) = \rho + \lambda\left(\frac{\gamma(x^2_t,1)}{\gamma(x^1_t,1)} - \frac{\gamma(x^1_t,2)}{\gamma(x^1_t,1)}\right)$$

or $r(x) = \rho + \lambda\left(\frac{\gamma(x,1)}{\gamma(1-x,1)} - \frac{\gamma(1-x,2)}{\gamma(1-x,1)}\right)$ which immediately yields the expression in equation (17).

I Derivation of the invariant wealth distribution

The CDF for the money holdings, $F(x^i, s, t)$, with density $f(x^i, s, t)$ in states $s = 1, 2$ follows

$$F(x^i,1,t+dt) = (1-\lambda\, dt)F(x^i - \dot{x}^i(x^i,1)\, dt, 1, t) + \lambda\, dt\; F(x^i - \dot{x}^i(x^i,2)\, dt, 2, t)$$
$$F(x^i,2,t+dt) = (1-\lambda\, dt)F(x^i - \dot{x}^i(x^i,2)\, dt, 2, t) + \lambda\, dt\; F(x^i - \dot{x}^i(x^i,1)\, dt, 1, t)$$

Expanding $F(x^i, s, t)$ around x^i gives (we only report the one for $s = 2$)

$$F(x^i,2,t+dt) = (1-\lambda\, dt)\left[F\left(x^i,2,t\right) - f\left(x^i,2,t\right)\dot{x}^i(x^i,2)dt\right] + \lambda dt\;\left[F\left(x^i,1,t\right) - f\left(x^i,1,t\right)\dot{x}^i(x^i,1)dt\right]$$

Subtracting $F(x^i, 2, t)$ from both sides and dividing by dt and taking the limit for $dt \downarrow 0$

$$\lim_{dt\downarrow 0}\frac{F(x^i,2,t+dt) - F(x^i,2,t)}{dt} = \frac{\partial F(x^i,2,t)}{\partial t} = -f\left(x^i,2,t\right)\dot{x}^i(x^i,2) - \lambda\left(F\left(x^i,2,t\right) - F\left(x^i,1,t\right)\right)$$

Using this equation together with the corresponding one for state $s = 1$ and imposing invariance give

$$0 = f\left(x^i,2\right)\dot{x}^i(x^i,2) + f\left(x^i,1\right)\dot{x}^i(x^i,1) \tag{32}$$

Taking the derivative w.r.t. x delivers the Kolmogorov forward equation

$$\frac{\partial\;\partial F(x^i,2,t)}{\partial x\;\partial t} = \frac{\partial f(x^i,2,t)}{\partial t} = \frac{\partial\left[-f\left(x^i,2,t\right)\dot{x}^i(x^i,2) - \lambda\left(F\left(x^i,2,t\right) - F\left(x^i,1,t\right)\right)\right]}{\partial x}$$

which, equated to zero (imposing invariance) gives equation (18).

Using the expression in equation (10), together with $\dot{x}^1(x^1,2) + \dot{x}^1(1-x^1,1) = 0$ to replace

$\dot{x}^1(x^1,1)$ and $\dot{x}^1(x^1,2)$ into equation (18) gives

$$\frac{f_x(x^1,2)}{f(x^1,2)} = \frac{(\lambda-\mu)\gamma(x^1,2)+\frac{\gamma_x(x^1,2)}{\gamma(x^1,2)}-\lambda\gamma(x^1,2)\frac{f(x^1,1)}{f(x^1,2)}}{1+\mu\left(x^1-\frac{1}{2}\right)\gamma(x^1,2)}$$

Using equation (32) to replace the ratio $f(x^1,2)/f(x^1,1)$ in the above expression gives equation (19).

I.1 Proof of Lemma 4

Note that $f(x^1,2)=Ce^{\int_{1/2}^{x}\Omega(z)dz}$, which follows from integrating the ODE in equation (19). Because $\Omega(x^1)$ is continuous and differentiable for every $x^1 \in (0,1)$, it follows immediately that $f(x^1,2)$ is continuous and differentiable for every $x^1 \in (0,1)$.[19]

We next show that $\lim_{x^1\downarrow 0} f(x^1,2)=+\infty$ and $\lim_{x^1\downarrow 0} f_x(x^1,2)=-\infty$. We will do this by looking at the behavior of a different function that is proportional to $f(x^1,2)$ close to 0. Because $\Omega(x^1)$ is continuous as a function of $\gamma(x^1,1)$ and $\gamma(x^1,2)$ there exists a positive constant k_x such that for any x^1

$$d\left(\Omega(x^1),-\frac{(\rho+\lambda+\mu)\frac{2}{\mu}-\lambda\gamma(0,1)}{4(x^1)^2}-\frac{\lambda}{\mu(\frac{1}{2}-x^1)+\frac{1}{\gamma(1,2)}}\right)<k_x\ ,\ \mu>0$$

where we used that $\gamma(0,2)=\frac{2}{\mu}$. Let $k\equiv\max k_x$, and $\hat{\Omega}(x^1)\equiv-\frac{(\rho+\lambda+\mu)\frac{2}{\mu}-\lambda\gamma(0,1)}{4(x^1)^2}-\frac{\lambda}{\mu(\frac{1}{2}-x^1)+\frac{1}{\gamma(1,2)}}=-\frac{Q_1}{2(x^1)^2}-\frac{\lambda}{\mu(1/2-x^1)+\frac{1}{\gamma(1,2)}}$, with $Q_1>1$ as described in Lemma 3. This implies that

$$d\left(\Omega(x^1),\hat{\Omega}(x^1)\right)<k\quad\forall\ x^1\in(0,1),\ \mu>0$$

in other words, the distance of these two functions is uniformly bounded.

Let $\hat{f}(x^1,2)\equiv\hat{C}e^{\int_{1/2}^{x^1}\hat{\Omega}(z)dz}$. Note that

$$0<\lim_{x^1\downarrow 0}\frac{f(x^1,2)}{\hat{f}(x^1,2)}=\frac{C}{\hat{C}}e^{\lim_{x^1\downarrow 0}\int_{1/2}^{x^1}\left[\Omega(z)-\hat{\Omega}(z)\right]dz}<K$$

where K is a positive constant. This result follows because the distance of $\Omega(x^1)-\hat{\Omega}(x^1)$ is uniformly bounded. Then, the limiting behavior of $\hat{f}(x^1,2)$ is the same as the limiting behavior of $f(x^1,2)$.

We now explore the function $\hat{f}(x^1,2)$. Simple integration yields

$$\hat{f}(x^1,2)=\hat{C}\left(\frac{\mu\left(\frac{1}{2}-x^1\right)+\frac{1}{\gamma(1,2)}}{\frac{1}{\gamma(1,2)}}\right)^{\frac{\lambda}{\mu}}e^{\frac{Q_1}{2}\left(\frac{1}{x^1}-\frac{1}{2}\right)}$$

[19] Since $\gamma(x^1,1)$ and $\gamma(x^1,2)$ are continuous and differentiable functions of x^1 in $(0,1)$, inspection of equation (19) shows that $\Omega(x^1)$ is continuous and differentiable in x^1 for any $x^1 \in (0,1)$.

and

$$\hat{f}_x(x^1,2) = -\hat{f}(x^1,2)\left(\frac{\mu}{\mu\left(\frac{1}{2}-x^1\right)+\frac{1}{\gamma(1,2)}}+\frac{Q_1}{2\left(x^1\right)^2}\right)$$

From these expressions it is clear that $\lim_{x^1\downarrow 0}\hat{f}(x^1,2) = +\infty$ and $\lim_{x^1\downarrow 0}\hat{f}_x(x^1,2) = -\infty$. Because we have already established that $0 < \lim_{x^1\downarrow 0}\frac{f(x^1,2)}{\hat{f}(x^1,2)} < K$, it follows immediately that $\lim_{x^1\downarrow} f(x^1,2) = +\infty$ and $\lim_{x^1\downarrow 0} f_x(x^1,2) = -\infty$. Note that these results do not hinge on μ being constant. If $\mu = \mu[x]$ the results still hold as long as $\mu[x]$ is continuous.

J Proof of Lemma 5

That $V(1/2;\mu)$ diverges to $-\infty$ when $\mu < 0$ is a direct implication of Lemma 1. As $x = 1/2$ the Lemma states that there is no monetary equilibrium and therefore there is no trade. As any given agent spends half of her life being unproductive, the agent spends half of her life with zero consumption (because of no trade). It follows immediately that $V(1/2;\mu)$ diverges to $-\infty$ when $\mu < 0$.

We next prove that $V(1/2;\mu)$ diverges to $-\infty$ when $\mu \uparrow \infty$. This proof has two pieces. First that when $\mu \uparrow \infty$ and there is no state reversal, $x = 0$. Second, that the consumption of an agent with no wealth approaches zero as $\mu \uparrow \infty$ (i.e. the real value of the monetary transfer becomes zero).

To prove the first part consider the discrete time version of equation (10), for a small period Δ. Let x^1_t be the (fraction of) money holdings at the beginning of period t, before the lump sum transfer $\mu\Delta/2$ is received and before any purchase is made. Without loss of generality, assume that agent of type 1 is unproductive with current money holdings x^1_t so that,

$$x^1_{t+\Delta}(1+\Delta\mu) = x^1_t + \frac{1}{2}\mu\Delta - \frac{\Delta}{\gamma(x_t,2)}$$

or,

$$x^1_{t+\Delta} = \frac{x^1_t}{1+\Delta\mu} + \frac{\frac{1}{2}\mu\Delta}{1+\Delta\mu} - \frac{\Delta}{\gamma(x_t,2)(1+\Delta\mu)}$$

Using that $\gamma(x,2)$ is decreasing in x (see Lemma 2), and that $\gamma(0,2) = 2/\mu$ (see equation (13)) we can bound above $x^1_{t+\Delta}$,

$$x^1_{t+\Delta} < \frac{x^1_t}{1+\Delta\mu}$$

which immediately implies that

$$\lim_{\mu\uparrow\infty} x^1_{t+\Delta} = 0$$

that is, an agent who remains unproductive begins the period with no money holdings as she spends the whole transfer she received from the government.

We next show that the consumption of an unproductive agent with no wealth approaches zero as $\mu \uparrow \infty$. Formally: $lim_{\mu\uparrow\infty} c^1(0,2) = 0$. To see this notice that From equation (15) we have that $c^1(0,2) = \frac{\gamma(1,1)}{\gamma(0,2)}$. Because $\gamma(x,2)$ decreasing in x (see Lemma 2),

$$c^1(0,2) \leq \frac{\gamma(1,1)}{\gamma(1,2)} = \frac{\lambda}{\rho+\lambda+\mu}$$

where the last step uses equation (14). It is now immediate that $\lim_{\mu\uparrow\infty} c^1(0,2) = 0$.

Notice that the above implies that $\lim_{\mu\uparrow\infty} V(x;\mu) = -\infty$ for any value of x. This is easily seen from the discrete time version of equation (21),

$$V(x;\mu) = \Delta\left(\ln c^1(x,2;\mu) - 1 - c^1(x,2;\mu)\right) + (1-\rho\Delta)((1-\lambda\Delta)V(\tilde{x};\mu) + \lambda\Delta V(1-\tilde{x};\mu))$$

where Δ is the length of the time period and $\tilde{x}$ is the share of money of unproductive agents at the next date. We showed that as $\mu \uparrow \infty$ the unproductive agent will spend all her money, so that $\lim_{\mu\uparrow\infty} \tilde{x} = 0$. Then the agent consumption is expected to be zero if she remains unproductive. Thus the expected utility diverges.

Online Appendices

The Optimum Quantity of Money with Borrowing Constraints

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A-11 Uncorrelated shocks (Bewley economy)

In this section we study a stationary economy where the productivity shocks are uncorrelated across agents. Assume a unit mass of agents, indexed by i, over the $[0,1]$ interval. As before, the productivity state of each agent, s_i follows a Markov process, where λ denotes the rate at which the state switches.

Let m_t^i be the money holdings of agent i at time t, so that the total money supply is $m_t = \int_0^1 m_t^i \, di$. Let τ_t denote the per capita transfer from the government. The government budget constraint is

$$\tau_t = \tilde{q}_t \int_0^1 \dot{m}_t^i \, di = \mu \; m_t \; \tilde{q}_t = \mu \; q_t \tag{A-1}$$

where the last equality uses the homogeneity of $\tilde{q}$ with respect to m. In what follows we let $\mu \geq 0$. The same argument developed in Section 3.1 shows that no monetary equilibrium exists for $\mu < 0$.

Obviously $\int_0^1 x_t^i \, di = 1$ where $x_t^i = m_t^i / m_t$. Notice that in this model with a continuum of agents we have that $x^i \in [0, +\infty)$, where $x^i = 1$ denotes the situation in which a single agent money balances equal the economy's average money, and $x^i \uparrow \infty$ denotes the situation in which one agent holds all of the money. Simple algebra gives that $\dot{x}^i(x^i, s) = \dot{m}_t^i / m_t - x^i \mu$, where we omit the time index for notation simplicity.

The agent's first order conditions of this model are unchanged compared to the previous model (equation (7)). The unproductive agent budget constraint and Euler equation (i.e. when $s = 2$) give

$$\dot{x}^1(x, 2) = \mu \left(1 - x^1\right) - \frac{1}{\gamma(x^1, 2)} \tag{A-2}$$

This equation shows that money growth has no effect on the money share in the case where the agent's money holding equal the average money per capita in the economy, i.e. the ratio is $x^i = 1$.[20]

We assume the economy has a centralized competitive market where one unit of money buys $\tilde{q} = \frac{1}{m} q$ units of consumption. This implies that all productive agents are willing to produce in exchange for money as long as $\gamma(x^1, 1) > q$, and that there is a level of money holdings $\bar{x}$ where productive agents are satiated with money balances: $\gamma(\bar{x}, 1) = q$.

Every agent works to save money balances $\bar{x}$ as soon as she gets productive. So the wealth share jumps from $x^1 < \bar{x}$ to $\bar{x}$ as soon as the agent become productive (this implies that $\dot{x}^1$ is infinite at the time of a jump).[21] A productive agent aims to maintain the wealth share constant at $\bar{x}$ so that $\dot{x}^1(\bar{x}^1, 1) = 0$, which implies that for a productive agent $x^1 = \bar{x}$ then $\dot{m}_t^1 / m_t = \mu \bar{x}$.

Before moving on with solving the model we define an equilibrium for this economy.

Definition 2 *A monetary equilibrium is a price function $\tilde{q}(m) = \frac{1}{m} \, q$, with $q \in \mathbb{R}^+$, and a stochastic process $x(t, \omega)$ with values in $[0, 1]$, such that a consumer i maximizes expected*

[20] Note that the same property holds in the model with two agents, in which the total mass is 2, the index $x^i \in [0,1]$ and an equal distribution of money holdings implies that the ratio of the agent money to the average (per capita) money holdings is $1/2$, so that μ does not affect $\dot{x}^i$ when $x^i = 1/2$.

[21] The assumption of linear disutility of labor is important for this result, as it implies that the productive agent immediately refills his money balances up to $\bar{x}$.

discounted utility (1) subject to the budget constraints (2) and (3) with $q(t,\omega) = q(s)$, non-negativity (4)), the government budget constraint constraint (A-1), and market clearing constraint (A-6).

In an internal solution the lagrange multipliers $\gamma(x^1,1), \gamma(x^1,2)$ solve the system of differential equations that we determined before (equation (8) and (9)), which under the assumptions of this section (using equation (A-2)) gives

$$\begin{aligned} \gamma_x(x^1,1)\dot{x}^1(x^1,1) &= (\rho+\mu)\gamma(x^1,1) - \lambda\left(\gamma(\bar{x},2) - \gamma(\bar{x},1)\right) \\ \gamma_x(x^1,2)\left[\mu\left(1-x^1\right) - \frac{1}{\gamma(x^1,2)}\right] &= (\rho+\mu)\gamma(x^1,2) - \lambda\left(\gamma(x^1,1) - \gamma(x^1,2)\right) \end{aligned}$$

The system decouples in two ODEs which we discuss next.

Internal solution always applies for $x^1 \in (0,1)$ for the unproductive agent. For the productive agent the solution is internal only at $x^1 = \bar{x}$, otherwise the state x^1 records a jump from x^1 to $\bar{x}$. At the replenishment level $\bar{x}$ the equation for the productive agent gives

$$q = \gamma(\bar{x},1) = \frac{\lambda}{\lambda+\mu+\rho}\gamma(\bar{x},2) \tag{A-3}$$

Since the disutility of labor is linear, the marginal utility of money for a productive agent is constant at the level $\gamma(\bar{x},1) = \gamma(x^1,1)$ for $x^1 \in (0,\bar{x})$.[22] The ODE for $\gamma(x^1,2)$ can be rewritten as

$$\gamma_x(x^1,2) = \frac{(\rho+\lambda+\mu)\gamma(x^1,2)^2 - \lambda q\gamma(x^1,2)}{\gamma(x,2)\mu\left(1-x^1\right) - 1} \tag{A-4}$$

with the following boundary conditions at $x^1 = 0$:

$$\gamma(0,2) = \frac{1}{\mu} \tag{A-5}$$

where the boundary stems from the unproductive agent's Euler equation and budget constraint at $x^1 = 0$.

Next we impose that at every t demand equals supply in the asset market where money is exchanged for the consumption good. The asset demand originates from the productive agents who aim to hold an amount of money $\bar{x}$. The supply comes from unproductive agents who exchange money for consumption. Let $f(x^1,2)$ be the density function for the mass of unproductive agents with $\int_0^{\bar{x}} f(x^1,2)dx^1 = 1$, so that $f(x^1,2)$ is the measure of unproductive agent, who account for $1/2$ of the total population under the invariant density. The market clearing equation is (see below)

$$(\mu+2\lambda)(\bar{x}-1) = \int_0^{\bar{x}} \frac{f(x^1,2)}{\gamma(x^1,2)}\, dx^1 \tag{A-6}$$

Finally note that the density function for unproductive agents is obtained from the usual Kolmogorov forward equation. Using $f(x^1,1) = 0$ and equation (A-2) to replace $\dot{x}^1(x^1,2)$

[22] Hence the value (i.e. the utility *level*) for an productive agent with $x^1 \in (0,\bar{x})$ is linear in x^1.

into equation (18) gives

$$\frac{f_x(x^1,2)}{f(x^1,2)} = \frac{(\lambda-\mu)\gamma(x^1,2) - \frac{(\rho+\lambda+\mu)\gamma(x^1,2)-\lambda q}{1+\mu(x^1-1)\gamma(x^1,2)}}{1+\mu\,(x^1-1)\,\gamma(x^1,2)} \tag{A-7}$$

Notice that this problem has four unknowns: $q, \bar{x}$, and two constant of integration, one for the solution of equation (A-4) and one for the density function in equation (A-7). The four equations to solve for the four unknowns are: the boundary conditions in equation (A-3) and equation (A-5), the market clearing condition in equation (A-6), and the density function integrating to a unit mass.

Derivation of market clearing equation

We derive equation (A-6) as the limit of a discrete time model. Recall that each productive agent is aiming to keep his money balances at $m^i/m = \bar{x}$. Consider a time interval of length Δ. In each period there is a fraction $\Delta\lambda$ of the mass of unproductive agents who become productive. The asset demand for one of these agents with wealth x^1 is given by a change in the stock, $\bar{x} - x^1$, and by a flow component that offsets the effect of money growth given by $\mu\Delta(\bar{x}-1)$.The complementary fraction of productive agents, $1-\lambda\Delta$, is formed by agents who were already productive in the previous period and held assets $\bar{x}$. The asset demand for these agents is the flow component that offsets the effect of money growth, given by $\mu\Delta(\bar{x}-1)$. This gives

$$\begin{aligned}(1-\Delta\lambda)\mu\Delta(\bar{x}-1) + \lambda\Delta\int_0^{\bar{x}} \left[\bar{x} - x^1 + \mu\Delta(\bar{x}-1)\right]\, f(x^1,2)\, dx^1 = \\ (1-\Delta\lambda)\int_0^{\bar{x}} \frac{c^1(x^1,2)\Delta}{q}\, f(x^1,2)\, dx^1 + \lambda\Delta\frac{c^1(\bar{x},2)\Delta}{q}\end{aligned}$$

Dividing by Δ and taking the limit as $\Delta \downarrow 0$ gives

$$\mu(\bar{x}-1) + \lambda\int_0^{\bar{x}} \left(\bar{x} - x^1\right)\, f(x^1,2)\, dx^1 = \int_0^{\bar{x}} \frac{c^1(x^1,2)}{q}\, f(x^1,2)\, dx^1$$

Using $\int_0^{\bar{x}} f(x^1,2)dx^1 = 1$ and $c^1(x^1,2) = q/\gamma(x^1,2)$ gives

$$\mu(\bar{x}-1) + \lambda\bar{x} - \lambda\int_0^{\bar{x}} x^1\, f(x^1,2)\, dx^1 = \int_0^{\bar{x}} \frac{1}{\gamma(x^1,2)}\, f(x^1,2)\, dx^1$$

Using that $\int_i x^i di = 1$ or, equivalently, $\frac{1}{2}\int_0^{\bar{x}} x^1\, f(x^1,2)dx^1 + \frac{\bar{x}}{2} = 1$ gives

$$\mu(\bar{x}-1) + \lambda\bar{x} - \lambda\,(2-\bar{x}) = \int_0^{\bar{x}} \frac{1}{\gamma(x^1,2)}\, f(x^1,2)\, dx^1$$

or

$$\mu(\bar{x}-1) + 2\lambda\,(\bar{x}-1) = \int_0^{\bar{x}} \frac{1}{\gamma(x^1,2)}\, f(x^1,2)\, dx^1$$

which gives equation (A-6).

Derivation of invariant wealth distribution

For $x^1 \in [0, \bar{x})$ we have $f(x^1, 1) = 0$. Using equation (A-2) to replace $\dot{x}^1(x^1, 2)$ in equation (18)

$$\frac{f_x(x^1, 2)}{f(x^1, 2)} \quad = \quad \frac{(\lambda - \mu)\gamma(x^1, 2) + \frac{\gamma_x(x^1,2)}{\gamma(x^1,2)}}{1 + \mu\,(x^1 - 1)\,\gamma(x^1, 2)}$$

Using equation (A-4) to replace for $\frac{\gamma_x(x^1,2)}{\gamma(x^1,2)}$ in the above expression gives equation (A-7).

A-11.1 The case with constant money ($\mu = 0$)

It is interesting to analyze the case with $\mu = 0$ for the model with uncorrelated shocks because of its simplicity. We provide a characterization of the equilibrium and we prove existence and uniqueness.

After setting $\mu = 0$ equation (A-4) reduces to the following ODE: $\gamma_x(x^1, 2) = \lambda q \gamma(x^1, 2) - (\rho + \lambda)\gamma(x^1, 2)^2$, with boundary conditions that reduce to $lim_{x^1 \downarrow 0}\gamma(x^1, 2) = \infty$ and $\gamma(\bar{x}, 2) = q\frac{\rho+\lambda}{\lambda}$. Using the ODE and the first boundary provides an expression for the marginal value of money for an unproductive agent as a function of the price level q,

$$\gamma(x^1, 2) = \frac{q\lambda}{\rho + \lambda}\frac{e^{q\lambda x^1}}{e^{q\lambda x^1} - 1}$$

which is strictly positive for every $q > 0$. This implies that the expected utility for an unproductive agent with wealth x^1, denoted by $v(x^1, 2)$, is given by the integral of $\gamma(x^1, 2)$, i.e.

$$v(x^1, 2) = \frac{1}{\rho + \lambda}\log\left(e^{q\lambda x^1} - 1\right) + \bar{v}$$

where $\bar{v}$ is a finite constant.

Using the expression for $\gamma(x^1, 2)$ and equation (A-3) gives an equation that relates $\bar{x}$ and q,

$$\bar{x} = \frac{1}{\lambda q}\log\left(\frac{(\lambda + \rho)^2}{(\lambda + \rho)^2 - \lambda^2}\right) \tag{A-8}$$

which implies a unit elasticity of $\bar{x}$ with respect to the price level q.

Now we turn to evaluate the density of money holdings $f(x^1, 2)$. Setting $\mu = 0$ in equation (A-7) reduces to

$$\frac{f_x(x^1, 2)}{f(x^1, 2)} = \lambda q\left(1 - \frac{\rho}{\lambda + \rho}\frac{e^{q\lambda x^1}}{e^{q\lambda x^1} - 1}\right)$$

where we used the expression we found for $\gamma(x^1, 2)$. This expression is an ODE with solution $f(x^1, 2) = Q_7 e^{q\lambda x^1}\left(e^{q\lambda x^1} - 1\right)^{-\frac{\rho}{\rho+\lambda}}$, where Q_7 is a constant to be determined next. We have that

$$\int_0^{\bar{x}} f(x^1,2)dx^1 = 1$$

We can use this equation to obtain Q_7. Therefore, the density $f(x^1,2)$ is

$$f(x^1,2) = \frac{q\lambda^2}{(\rho+\lambda)\left(e^{q\lambda\bar{x}}-1\right)^{\frac{\lambda}{\rho+\lambda}}} \frac{e^{q\lambda x^1}}{\left(e^{q\lambda x^1}-1\right)^{\frac{\rho}{\rho+\lambda}}}$$

which is readily evaluated using equation (A-8).

An equilibrium exists if there exists a finite price level q such that the market clearing condition (see equation (A-6)) is satisfied. Using that $\mu = 0$ and by substituting the expressions we found for the marginal value of money $\gamma(x^1,2)$, density function $f(x^1,2)$, and that $\bar{x} = \bar{x}(q)$, the market clearing condition reduces to

$$\Upsilon_1(q) = \Upsilon_2(q)$$

where $\Upsilon_1(q) \equiv (\bar{x}(q)-1)\,2\left(\frac{\lambda^2}{(\rho+\lambda)^2-\lambda^2}\right)^{\frac{\lambda}{\rho+\lambda}}$ and $\Upsilon_2(q) \equiv \int_0^{\bar{x}(q)}\left(e^{q\lambda x^1}-1\right)^{\frac{\lambda}{\rho+\lambda}}dx^1$, where both $\Upsilon_1(q)$ and $\Upsilon_2(q)$ are continuous and differentiable functions on q. In order to check for existence and uniqueness of solution we do some analysis on these functions. Properties of $\Upsilon_1(q)$: (i) $\lim_{q\downarrow 0}\Upsilon_1(q) = +\infty$, (ii) $\lim_{q\uparrow\infty}\Upsilon_1(q) < 0$ and finite, (iii) strictly decreasing, (iv) strictly convex. With respect to $\Upsilon_2(q)$ note that it is a function involving the hypergeometric function,

$$\begin{aligned} \Upsilon_2(q) &= \tfrac{\lambda+\rho}{\lambda q}\left(\tfrac{e^{q\lambda\bar{x}(q)}-1}{1-e^{-\lambda q\bar{x}(q)}}\right)^{\frac{\lambda}{\rho+\lambda}} {}_2F_1\left(-\tfrac{\lambda}{\rho+\lambda}, -\tfrac{\lambda}{\rho+\lambda}; \tfrac{\rho}{\rho+\lambda}; e^{-q\lambda\bar{x}(q)}\right) \\ &- \lim_{y\downarrow 0}\tfrac{\lambda+\rho}{\lambda q}\left(\tfrac{e^{q\lambda y}-1}{1-e^{-\lambda qy}}\right)^{\frac{\lambda}{\rho+\lambda}} {}_2F_1\left(-\tfrac{\lambda}{\rho+\lambda}, -\tfrac{\lambda}{\rho+\lambda}; \tfrac{\rho}{\rho+\lambda}; e^{-q\lambda y}\right) \end{aligned}$$

which, using equation (A-8) and that $\lim_{y\downarrow 0}\left(\frac{e^{q\lambda y}-1}{1-e^{-\lambda qy}}\right)^{\frac{\lambda}{\rho+\lambda}} = 1$, reduces to

$$\begin{aligned} \Upsilon_2(q) &= \tfrac{\rho+\lambda}{\lambda q}\left(\left(\tfrac{(\rho+\lambda)^2}{(\rho+\lambda)^2-\lambda^2}\right)^{\frac{\lambda}{\rho+\lambda}} {}_2F_1\left(-\tfrac{\lambda}{\rho+\lambda}, -\tfrac{\lambda}{\rho+\lambda}; \tfrac{\rho}{\rho+\lambda}; \tfrac{(\rho+\lambda)^2-\lambda^2}{(\rho+\lambda)^2}\right)\right. \\ &- \left. {}_2F_1\left(-\tfrac{\lambda}{\rho+\lambda}, -\tfrac{\lambda}{\rho+\lambda}; \tfrac{\rho}{\rho+\lambda}; 1\right)\right) \end{aligned}$$

with the following properties: (i) $\Upsilon_2(q) < 0$, (ii) $\lim_{q\downarrow 0}\Upsilon_2(q) = -\infty$, (iii) $\lim_{q\uparrow\infty}\Upsilon_2(q) = 0$, (iv) $\Upsilon_2(q)$ strictly increasing, and (v) $\Upsilon_2(q)$ strictly concave.

Note now that given the listed properties of $\Upsilon_1(q)$ and $\Upsilon_2(q)$, there exists a unique value q such that $\Upsilon_1(q) = \Upsilon_2(q)$, so that the market clearing condition is satisfied. This implies that when there is constant money in the economy, i.e. $\mu = 0$, there exists a unique monetary equilibrium.

A-11.2 Optimality of expansionary policy

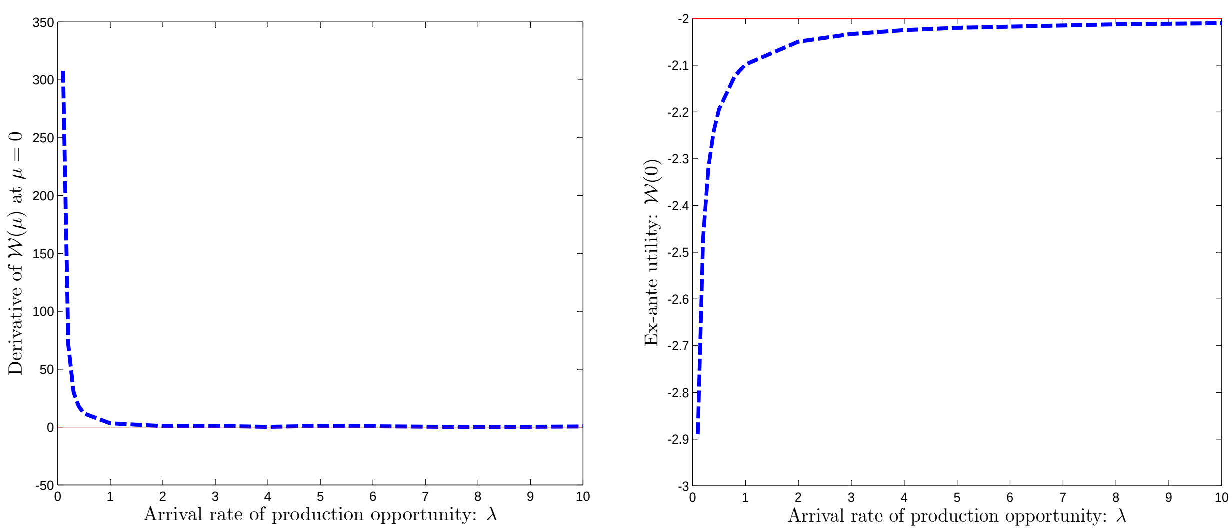
In this section we discuss the optimality of a stationary expansionary policy, for an economy that has reached the invariant distribution of wealth. This exercise exploits the fact that we know the invariant distribution of x: $f(x)$. The government wishes to maximize ex-ante expected welfare under the stationary density,

$$\begin{aligned}\mathcal{W}(\mu) &= \mathbb{E}_x \left\{ u\left(c^1(x,2;\mu)\right) + u\left(c^2(1-x,2;\mu)\right) - l^2(1-x,2;\mu) \middle| \, \mu \right\} \\ &= \int_0^1 f(x,2;\mu) \left[\ln\left(c^1(x,2;\mu)\right) + \ln(1) - \left(1 + c^1(x,2;\mu)\right)\right] \, dx\end{aligned}$$

where the notation emphasizes that the consumption paths and the probability density of money holdings depend on the money growth rate μ, $c^2(1-x,2;\mu) = 1$ because of linear utility of labor, and $l^2(1-x,2;\mu) = 1 + c^1(x,2;\mu)$. The expression for $\mathcal{W}(\mu)$ measures the stationary ex-ante (expected) utility, i.e. the welfare of any given agent before her initial state is realized. Types are given equal weights because the symmetry of the Markov process for the shocks implies that agents are productive 1/2 of the time. It is assumed that initial money holdings x are drawn from the invariant distribution $f(x;2)$.

It is straightforward that monetary contractions (i.e. $\mu < 0$) and extreme expansions (i.e. $\mu \uparrow \infty$) have ex-ante expected utility that diverges.[23] That is, $\mathcal{W}(\mu) \downarrow -\infty$ in both cases.

Figure A-7: Derivative and level of ex-ante expected welfare $\mathcal{W}(\mu)$ at $\mu = 0$



Parameters: $\rho = 0.05$.

[23] The argument is analogous to the one developed in Section 5. In both cases there is a brake in trade.