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Gabrielle Demange

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Gabrielle Demange, Paris School of Economics, EHESS and CEPR

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Centre for Economic Policy Research
77 Bastwick Street, London EC1V 3PZ, UK
Tel: (44 20) 7183 8801, Fax: (44 20) 7183 8820
Email: cepr@cepr.org, Website: www.cepr.org

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ABSTRACT

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JEL Classification: G01, G21 and G28
Keywords: bankruptcy, contagion, contagion in financial networks a threat index, financial linkages and systemic risk

Gabrielle Demange
EHESS
Paris School of Economics
48, Boulevard Jourdan
75014 Paris
FRANCE

Email: demange@pse.ens.fr

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Contagion in financial networks : a threat index

Gabrielle Demange[*]

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Abstract An intricate web of claims and obligations ties together the balance sheets of a wide variety of financial institutions. Under the occurrence of default, these interbank claims generate externalities across institutions and possibly disseminate defaults and bankruptcy. Building on a simple model for the joint determination of the repayments of interbank claims, this paper introduces a measure of the threat that a bank poses to the system. Such a measure, called threat index, may be helpful to determine how to inject cash into banks so as to increase debt reimbursement, or to assess the contributions of individual institutions to the risk in the system. Although the threat index and the default level of a bank both reflect some form of weakness and are affected by the whole liability network, the two indicators differ. As a result, injecting cash into the banks with the largest default level may not be optimal.

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1 Introduction

An intricate web of claims and obligations ties together the balance sheets of a wide variety of financial institutions, banks, hedge funds, and various intermediaries. Some argue that these interbank claims have played a large role in the dissemination of the financial crisis of 2007-2008. As such, interbank claims are an important concern for both bankers and regulators and there is a general call for addressing their role in the risk of the system, the so-called 'systemic' risk. Following the recommendation made by the G20, the new framework proposed by the Basel committee (Basel III) plans to identify some 'systemically important financial institutions' from which higher standards will be required. Indeed regulation has so far typically been defined at the unit level, determined

[2]Paris School of Economics, EHESS, Paris, France. E-Mail: `demange@pse.ens.fr`. I would like to thank Jean-Edouard Colliard and Jean-Charles Rochet for helpful comments.

by the balance sheet of the bank under consideration. A prominent example is the Value at Risk indicator (VaR), which is based on a statistical assessment of a bank's payoffs independently of what is happening to other banks. Various proposals have been made to modify this measure to account for the contribution of a bank to systemic risk. But this contribution to systemic risk can be understood in a variety of ways, each one potentially leading to different assessments and cost evaluations. Tarashev, Borio, and Tsatsaronis (2010) distinguish between the role of a bank in spreading the losses and amplifying the default of other banks from that in participating in the systemic events, defined as those in which a large fraction of simultaneous defaults arise, say due to correlated portfolios. CoVaR for example, proposed by Adrian and Brunnermeier (2008), is a measure similar to VaR but conditional on systemic events and falls in the second category.

Mutual interbank liabilities introduce linkages in defaults when they occur. My purpose is to propose a measure of the impact that a bank's default imposes on its creditors accounting for these linkages. The measure is derived from an explicit criterion, based on the aggregate size of the balance sheets or, equivalently in our model, the overall debt repayments. Such a measure may be helpful to determine how to inject cash into banks so as to optimally increase the reimbursement of debts, or to assess the contributions of individual institutions to the risk in the system.

The analysis builds on a simplified description of a banking system. Banks have claims on each other and the result of the activities of each bank with the non-financial sector is summarized by a single number, called the *net worth*. The model extends Eisenberg and Noe (2001) (hereafter EN) by allowing net worth to be negative. A negative net worth represents a net liability towards the non-banking sector, resulting for example from a low return on investments or losses in derivative assets. As a result, bankruptcy will be a possible outcome.

Due to the interbank liabilities, the capacity of a bank to repay its interbank liabilities depends not only on its realized net worth but also on the capacity of its debtors, calling for a joint determination of the repayments. A clearing mechanism on the proportions of the liabilities repaid by banks -repayment ratios- solves this loopback. Banks' ratios form a kind of equilibrium in which each bank in default reimburses as much as it can given others' repayments and limited liability. Creditors outside the banking system have priority over those inside and trigger bankruptcy if their claims are not fully repaid. A clearing ratio vector exists and conditions for its uniqueness are provided. Two types of situations with default may arise. In the first situation, defaults are limited to the interbank liabilities; they affect the size of the banks' balance sheets but have no impact outside the financial sector. In the second situation, some banks default on their obligations to the outside sector and go bankrupt; the clearing mechanism then not only determines the volume of the repayment flows within the financial system but also the losses incurred by the creditors outside the system. Bankruptcy never arises when each bank is a net creditor to the non-financial sector (each net worth is non-negative) but surely arises when the aggregate banking sector is indebted towards the non-financial sector (aggregate net worth is negative). In other cases, net worth is positive on aggregate but not for each bank, the outcome depends on the liabilities structure. Larger gross liabilities, keeping net levels unchanged, diminish the occurrence of bankruptcy and, therefore, protect creditors outside the banking sector.

The default ratio of a bank towards other banks (the complement to its repayment ratio) reflects its weakness. However, it is not the most appropriate indicator of the threat posed by the bank on the system. The threat index of a bank proposed in this paper measures the decrease in the overall debt payments following a decrease in its net worth. If the bank is not defaulting, its index is null. If it is in default, a further decrease in its net worth can only lower its repayments to its creditors, which, in turn, either lower their repayments for those in default, or increase the loss to outside creditors for those which are bankrupt, and so forth. The initial decrease propagates along chains of defaulting creditors, possibly with cycle, triggering further decrease in payments and the threat index measures this overall impact. Although the threat index and the default ratio of a bank are both measures of its weakness and are affected by the liabilities network, they differ in general, and are, in some precise sense, dual to each other. A bank's default ratio is determined by the ability of its debtors to repay their debts; a bank's threat index on the other hand is determined by the impact its default inflicts on its creditors, and hence by the financial health of its creditors. The discrepancy between the two indices, default ratio and threat index (partially) depends on the asset-liability structure, in particular its asymmetry. Furthermore the determinants of the threat indices are shown to differ substantially between the two situations with or without bankruptcy.

The threat index is useful to determine a 'targeting policy' that injects an amount of cash into banks so as to improve effective payments as much as possible. Cash should be injected into the defaulting banks with the largest threat index. As a result, due to the discrepancy between threat indices and default levels, injecting cash into the banks that appear the weakest, those with the largest default ratio, may be sub-optimal.

The literature on financial contagion is growing. Empirical studies have examined the potential for contagion in real banking systems. Most often the contagion risk of a bank is defined as the expected number of subsequent failures (possibly weighted by their size) following its initial failure.

The simulations are calibrated on real payment systems (see e.g. Furfine 2003 on Fedwire) or on interbank networks (e.g. Upper and Worms 2004, Elsinger et al. 2004, Degryse and Nguyen 2004 for Germany, Austria, Belgium respectively). These studies concluded that systemic risk was extremely limited, in the sense that the probability of a large number of failures triggered by the single initial failure of a bank was almost null. A difficulty, however, is that data on bilateral exposures is limited. This makes difficult to assess the impact of the liability structure and furthermore the techniques to 'fill' the missing data possibly underestimates contagion.[1]

The interbank liability structure has also been examined from an ex ante point of view by evaluating the trade-off between risk sharing and risk spreading induced by the cross-liabilities on contagion. Indeed, interbank liabilities have two opposing effects on contagion: they increase the opportunities for sharing liquidity shocks among counter-parties but also facilitate the channels through which default spreads. Theoretical works have examined various questions, starting with Allen and Gale (2000) who take an optimality point of view, and Freixas, Parigi, and Rochet (2000) who analyze the role of a central bank. Though our analysis is ex post, the fact that an increase in gross liabilities, keeping net levels unchanged, diminishes the occurrence of bankruptcy points to an insurance role for the liabilities.

Gai and Kapadia (2008) analyze how the network structure affects the trade-off between risk sharing and risk spreading by using a random graph in which the expected number of links of a bank is given by a parameter, identical for all banks. They find that financial systems exhibit a robust-yet-fragile tendency: while greater connectivity reduces the likelihood of widespread default, the impact on the financial system, should problems occur, can be on a significantly larger scale than hitherto.[2] The analysis however relies on an a priori symmetric network and measures systematic risk by the expected contagion size given an expected bank picked at random. Given the observed heterogeneity of financial institutions, both in their size and connections, an important question is to assess the externalities initiated by a *given* bank. This is precisely the purpose of the threat index.

Finally, the paper relates to the large literature that studies the interactions and externalities channeled through a network of connections.

Firstly, the threat index provides an assessment of a position in a network, and, as such, is related

[1] Cifuentes, Ferrucci, and Shin (2005) introduce a further mechanism of contagion through asset sales amplified by regulatory solvency constraints and mark-to-market rules.

[2] This result however is driven by the assumption that a bank's total amount of assets and liabilities is kept fixed, irrespective of the network structure. In practice the amount of interbank assets and liabilities is related positively with the number of links. Intuitively, this correlation should hamper the risk-sharing benefit of forming links.

to the power indices introduced in the sociological literature by Katz (1953) or Bonacich (1987). It turns out that the threat index can be seen as an extension of the Bonacich index to a richer setting in which links are directed and assigned values, here given by the proportions of the liabilities of a bank, and the relevant network is endogenous, restricted to the links between defaulting banks. Despite this similarity, the approaches differ as the threat index is based on an explicit objective.

Secondly, targeting policies have been investigated in alternative network models in which individual actions generate externalities channelled through a network. In a criminal network for example, the 'key player' to remove, the one whose arrest triggers the largest decrease in global criminal activity, may not be the one the more active (see Ballester, Calvó-Armengol, Zenou 2005). Similar insights hold in our model since the more threatening banks may not be the weakest.

The paper is organized as follows. Section 2 presents the model, defines the clearing mechanism and proves the existence of a clearing ratio vector as well as conditions ensuring its uniqueness. Section 3 introduces and analyzes the threat index when no bank is indebted towards the non-financial sector (i.e. net worth values are positive). A targeting policy, which would inject cash in specific defaulting banks is analyzed, as well as a solidarity policy, which would force safe banks to increase their repayments beyond their nominal liabilities. Finally, some comparative static exercises on the impact of liabilities are performed. Section 4 extends the analysis to the situation where bankruptcy is unavoidable by allowing net worth to be negative. Section 5 concludes and Section 6 gathers some proofs.

2 A contagion model

2.1 The framework

There are n financial institutions, called banks for simplicity. Denote $N = \{1, \cdots, n\}$. Banks draw some risky revenues from their activities with the non-financial sector and are linked through claims on each other.

We are at an ex-post stage. The result of all the operations of a bank with the non-banking sector is summarized by a single value, z_i for bank i. This value, called the *net worth*, is the accounting value of the assets and liabilities on the non-financial sector once the payoffs from previous investments are revealed. As a result, the net worth level can be negative when a low return on investment is realized. A negative net worth corresponds to a liability towards creditors outside the financial system.

The interbank assets and liabilities are described by a $n \times n$ matrix $\boldsymbol{\ell} = (\ell_{ij})$ where ℓ_{ij} represents the magnitude of i's nominal debt obligation towards j; hence ℓ_{ii} is null for each i. The total nominal liabilities of bank i are

$$\ell_i^* = \sum_j \ell_{ij}. \tag{1}$$

Thus a bank may have two types of creditors, outside the banking system if its net worth z_i is negative, and within it if ℓ_i^* is positive. Partial default is possible within the financial system, and is determined by a clearing mechanism. Instead any default to the outside sector leads to bankruptcy. Note however that the net worth allows for compensation between the assets and liabilities outside the financial sector.

Let us make a couple of remarks on the liability structure, sometimes referred to as a network. When dealing with a large number of banks, the pattern of their relationships is quite stable and specific, with some banks having regular and large relationship while others have none. In such a situation, the interpretation of financial interlinkages as a network, where banks are nodes and bilateral exposures are the links, is very compelling. The liability structure depends on the situation under consideration, in particular on the maturity of the debts. In payment systems, liabilities are often both ways, reflecting common clienteles for example. Not only both ℓ_{ij} and ℓ_{ji} can be simultaneously positive but they are likely to be both positive or both null. In long term arrangements however, some patterns are more directed, such as the ones described in the Austrian banking system, with almost a pyramidal structure.

Notation. $\mathbb{I}$ denotes the $n \times n$ identity matrix, $\mathbf{0}$ and $\mathbb{1}$ denote respectively a n-vector of 0 and 1. Given a n-vector $\boldsymbol{\theta}$ and S a subset of indices, $\boldsymbol{\theta}_S$ denotes the vector obtained from $\boldsymbol{\theta}$ by keeping the components indexed by S. Similarly, $A_{S\times T}$ denotes the matrix obtained from a matrix A by keeping the rows indexed by i in S and the columns indexed by j in T. A^t denotes the transpose of A.

2.2 Clearing repayment ratio vectors

The capacity for the banks to repay their debts depends on the net worth levels and the mutual liabilities. Given the realized net worth levels, $\boldsymbol{z}$, and the mutual liabilities, $\boldsymbol{\ell}$, repayments must be determined. Some banks may be unable to repay fully their debts and default possibly propagates due to the mutual liabilities. Priorities to repayments should be defined. Priority is given to creditors over stockholders under limited liability. Furthermore, when a defaulting bank is indebted towards both the non-financial and financial sectors, priority is given towards the non-financial one. The clearing mechanism is based on these priorities and shown to determine the ratios in a unique way in most situations, in particular if banks are enough connected. An alternative justification for the mechanism is that clearing ratios maximize some objective for the financial system. This will serve as a basis for defining the threat indices, as specified in the next sections.

To simplify the presentation, we consider first the case where each bank is indebted, namely each ℓ_i^* is positive. Start by assuming that all banks fully repay their debts to i. Due to the limited liability of stockholders, bank i will fully repay its debts only if the amount to be reimbursed, ℓ_i^*, is less than $z_i + \sum_j \ell_{ji}$; otherwise i defaults. Default may be partial meaning that a bank in difficulty pays a fraction of its liability. This fraction, between 0 and 1, is called *repayment ratio* or simply ratio, denoted by θ_i. Partial default arises when θ_i is strictly between 0 and 1, and bankruptcy when θ_i is null. A ratio vector $\boldsymbol{\theta} = (\theta_i)$ specifies the repayment ratio of each bank, where θ_i is between 0 and 1. The *default ratio* is then defined as $1 - \theta_i$. Due to the mutual liabilities, the capacity for a bank to repay its debts depends on the repayments made by its debtors, thereby introducing linkages between the repayment ratios of the banks and calling for their joint determination.

The clearing mechanism provides such a determination. It specifies three requirements on the ratio vector: limited liability, creditors' priority, and a bankruptcy condition. These conditions are stated in terms of the net asset and net equity of the banks.

The *net asset value* of a bank is defined as the accounting sum of the net worth and the loans repayments by other banks. Formally, denoting bank i's net asset by $a_i(\boldsymbol{\theta})$ given repayment vector $\boldsymbol{\theta}$, we have

$$a_i(\boldsymbol{\theta}) = z_i + \sum_j \theta_j \ell_{ji}. \tag{2}$$

Note that i's net asset value is independent of its own repayment. When z_i is non-negative, the net asset value, which is surely non-negative, is the amount available to reimburse the liabilities to other banks. When z_i is negative, a positive net asset value is what is left for reimbursing the liabilities to other banks after outside creditors have been repaid; a negative net asset value represents the loss to outside creditors if they seize the repayments $\sum_j \theta_j \ell_{ji}$, hence amounts to their minimal loss.

The *net equity* of a bank is defined as the accounting residual value that results from the realized operations of the bank with all other parties, outside or inside the banking sector. Formally, denoting bank i's net equity by $e_i(\boldsymbol{\theta})$ given repayment vector $\boldsymbol{\theta}$ we have

$$e_i(\boldsymbol{\theta}) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^*, \tag{3}$$

which is the sum of the net worth and the net payments within the banking system to i. If non-negative, net equity is the value accruing to stockholders, hence corresponds to the standard notion of equity. It is convenient to consider the same expression even if it takes a negative value.

Net asset values and net equity are related by

$$e_i(\boldsymbol{\theta}) = a_i(\boldsymbol{\theta}) - \theta_i \ell_i^*. \tag{4}$$

Definition 1 *Given $(\boldsymbol{z}, \boldsymbol{\ell})$, a vector $\boldsymbol{\theta} = (\theta_i)$ in $[0,1]^n$ is said to be a clearing ratio vector if it satisfies for each i*

(limited liability)

$$a_i(\boldsymbol{\theta}) \geq 0 \textit{ implies } a_i(\boldsymbol{\theta}) \geq \theta_i \ell_i^* \textit{ equivalently } e_i(\boldsymbol{\theta}) \geq 0 \tag{5}$$

(priority of creditors over stockholders)

$$a_i(\boldsymbol{\theta}) \geq 0 \textit{ implies either } \theta_i = 1 \textit{ or } a_i(\boldsymbol{\theta}) = \theta_i \ell_i^* \textit{ equivalently } e_i(\boldsymbol{\theta}) = 0 \tag{6}$$

(bankruptcy rule or priority of outside creditors)

$$a_i(\boldsymbol{\theta}) < 0 \textit{ implies } \theta_i = 0. \tag{7}$$

In the 'normal' situation, all banks reimburse and stockholders end up with a non-negative amount, $e_i(\mathbb{1}) = z_i + \sum_j \ell_{ji} - \ell_i^* \geq 0$ for each i. One checks that $\mathbb{1}$ is a clearing ratio vector.

The conditions bear on each bank's ratio as a function of its net asset value. These conditions are well defined since the bank's net asset value is not affected by its own repayment ratio, but only depends on its position with the non-financial sector and the ratios of other banks.

Limited liability states that stockholders are not required to repay more than the net asset value in the bank, assuming it positive. The priority of creditors over stockholders requires a bank to repay its liabilities to other banks as much as it can out of its net asset when positive. The bankruptcy condition requires that a bank repays nothing to other banks if its net asset value is negative.

The two rules of creditors' priority and bankruptcy both deal with default and interact only when a bank has two types of creditors, outside and inside the financial sector. Indeed, when bank i is not indebted towards the non-financial sector, z_i is non-negative, its net asset value is surely non-negative and the bankruptcy condition does not apply. When bank i is indebted towards the non-financial sector, z_i is negative, the rules give the priority to the creditors outside the financial sector. This is because, as seen earlier, a positive net asset value $a_i(\boldsymbol{\theta})$ represents the amount that is left once the outside creditors have been fully repaid and a negative asset value represents the minimum loss to the outside creditors, who seize the repayments $\sum_j \theta_j \ell_{ji}$ and leave nothing to the financial creditors. Thus banks get repaid from i only on the residual value after full repayment of outside creditors.

To sum up, a bank first defaults on its liabilities to the financial system, and second, if full default on its interbank liabilities is not sufficient to fulfill its obligations to the outside sector, the bank is declared bankrupt and defaults on its outside creditors.

Now let us consider the possibility that some banks have no liabilities towards other banks (but they may have lent to them). Although their repayment ratios have no meaning nor consequence, it is convenient to define them in a way consistent with the above definitions. This avoids to distinguish between the indebted and non-indebted banks. The ratio of bank without liability is set equal to 1 if its net asset value is non-negative and set to 0 if the net asset value is negative. With this convention, the ratio is uniquely defined set to its maximal value 1 when the asset is null. This avoids artificial and non relevant cases of non-uniqueness, as will be clear later on.

2.3 Existence, complementarity, uniqueness

In a setting with non-negative net worth values, net asset values are all non negative and there is no need to consider bankruptcy. This is the case considered by EN, who proved the existence of a clearing ratio as well as some other properties under limited liability and absolute priority of creditors. These properties extend under some conditions when bankruptcy is a possible outcome.

Proposition 1

1. *There is a greatest clearing ratio vector. The values of net equities are the same at the clearing ratio vectors (if there are several).*

2. *If aggregate net worth $\sum_i z_i$ is positive, then one bank at least fully repays its debts, $\theta_i = 1$ for some i. If it is negative, then one bank at least is bankrupt, $\theta_i = 0$ for some i.*

According to point *1*, there is surely a clearing ratio vector. The proof is in the appendix. It relies on the fact that repayment ratios are complements: the higher the ratios of other banks, the more a given bank is able to repay. This is due to the monotonicity of the net asset values: increasing i's ratio can only increase other banks' asset values and has no impact on its own.[3] Complementarity implies that there is a greatest ratio vector among those that satisfy the limited liability and bankruptcy conditions. This vector can be shown to be a clearing vector, hence is surely the greatest clearing vector.

[3] The $a_j(\boldsymbol{\theta})$ for $j \neq i$ are non-decreasing in θ_i and $a_i(\boldsymbol{\theta})$ is independent of θ_i.

Point *2* relies on the following *aggregation formula*, which is useful for the sequel:

$$\text{for any } \boldsymbol{\theta} \quad \sum_{i \in N} e_i(\boldsymbol{\theta}) = \sum_{i \in N} z_i. \tag{8}$$

This identity is trivially obtained since net equities are linear in $\boldsymbol{\theta}$ and the payments within N cancel out. Apply (8) at a clearing ratio vector. When aggregate net worth $\sum_i z_i$ is positive, there is surely a bank with positive net equity. Creditors' priority requires that bank to fully repay its debts. When aggregate net worth $\sum_i z_i$ is negative, there is surely a bank with negative net equity. This occurs only when the bank is bankrupt and repays nothing to other banks. ∎

The following corollary treats the situations with positive net worth levels.

Corollary 1 *When the net worth of each bank is positive, no bank is bankrupt and each equity is non-negative at a clearing ratio.*

The corollary is immediate. Since each bank's net asset value is positive, no bank can be bankrupt. The corollary allows us to interpret the clearing mechanism as performing a redistribution within the banking system in case of default. Recall that no default arises when each equity value $e_i(\mathbb{1})$ is non-negative. If instead some of these values are negative, though some banks default, each net equity ends up to be non-negative and corresponds to standard equity. Thanks to the aggregation formula (8), aggregate equity is equal to the aggregate net worth flowing in, hence not affected by default. The clearing ratio determines the distribution of this aggregate among the stockholders of the various banks.

The argument extends to situations in which some banks are indebted to the non-banking sector provided no bank ends up bankrupt. But in case of bankruptcy, the negative net equity of a bankrupt bank corresponds to a loss to the outside creditors: the aggregate net worth is split between the stockholders of safe banks and the losses incurred by the outside creditors of the bankrupt banks. We will see in Section 4 that the clearing mechanism then minimizes these losses.

Banks' status The property that the values of net equities are identical across clearing ratio vectors has a number of implications on the status of the banks and their repayment ratios. First, the banks are naturally partitioned into three classes according to the sign of their net equity: the safe banks, those with a positive net equity value, the defaulting banks with a null one, and the bankrupt banks with a negative one. From the conditions on a clearing ratio vector, the safe banks fully repay their debts and the bankrupt banks repay nothing. Formally, at any clearing ratio, denoting by S, D, and B respectively the set of safe, defaulting and bankrupt banks, we have

$$S = \{i, e_i(\boldsymbol{\theta}) > 0\},\ D = \{i, e_i(\boldsymbol{\theta}) = 0\},\ \text{and } B = \{i, e_i(\boldsymbol{\theta}) < 0\}. \tag{9}$$

Surely

$$\theta_i = 1 \text{ for each } i \text{ in } S \text{ and } \theta_i = 0 \text{ for each } i \text{ in } B. \tag{10}$$

For a non-indebted bank, the ratio is also fixed with our convention ($\theta_i = 1$ if $e_i(\boldsymbol{\theta}) \geq 0$ and $\theta_i = 0$ if $e_i(\boldsymbol{\theta}) < 0$). For indebted defaulting banks, however, their ratios are determined by the fact that their equity is null given the others' ratios. This opens up the possibility of multiple clearing vectors.

Uniqueness of a clearing ratio vector As we have just seen, clearing vectors may differ only for the indebted defaulting banks. Under some conditions stated in the next proposition, the equality of the ratios in S and B across clearing vectors 'propagates' to their debtors and results in a unique clearing ratio vector. One condition is a weaker form of irreducibility for the liability network.

Proposition 2 *The clearing ratio vector is unique in the following cases:*

(a) each net worth level is positive

(b) aggregate net worth is not null and each strict subset of indebted banks has a financial creditor outside the subset.

Case (a) is shown by EN. Case (b) requires that no subset of banks are indebted only between each other. When each bank is indebted, this condition is equivalent to the irreducibility of the liability network, namely, there is a chain of liabilities from any bank to any other one. When some banks are not indebted, the condition is satisfied under the following situations. There is a 'universal creditor', that is a bank, say bank 1, that has lent to all other banks and has no liability: $\ell_{j1} > 0$ and $\ell_{1j} = 0$ for any $j > 1$. Another situation is a pyramidal structure as is described in Section 3.1.

Let us give an intuition for the proof, which is detailed in Section 6. Let $\boldsymbol{\theta}$ be a clearing ratio vector distinct from the greatest one $\boldsymbol{\theta}^+$ and T be the set of banks for which the ratios differ. As argued above, the bankrupt and the safe banks, respectively those with negative and positive equity values, coincide at any clearing ratio vector, and have the same ratios, respectively 0 or 1. Furthermore, the ratios also coincide for non-indebted banks. Thus T contains only banks that are both indebted and defaulting. Hence T is not the whole set N (because there is surely a safe or a bankrupt bank under the assumption of a non-null aggregate net worth, as stated in Proposition 1).

The key point is to show that T must have no financial creditor outside T. By definition, each bank in T receives the same amount from banks outside T under the two clearing ratio vectors. So thanks to the property that net equities are independent of the clearing vector, their repayment to banks outside T must be identical under the two vectors. But, because repayments can only be larger under $\boldsymbol{\theta}^+$ than under $\boldsymbol{\theta}$, this implies that T has no creditor outside T. This is impossible under either assumption stated in the proposition. ■

Assume each bank is indebted. A weaker notion than irreducibility, called connectedness, requires that each strict subset of banks has a creditor or a debtor outside the subset, or, equivalently there is a path of loans-liabilities between any two banks.[4] Our next example shows that connectedness is not sufficient to guarantee uniqueness. The system has four banks. Banks 1 and 2 have a negative net worth and banks 3 and 4 a positive one: $\boldsymbol{z} = (-1, -1, 1.5, 1.5)$. The liabilities matrix is:

$$\boldsymbol{\ell} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

All nominal liabilities are of the same amount 1. Bank 1 has a liability to bank 2 and vice versa. Bank 3 has a liability to banks 1 and 4, and bank 4 to 2 and 3. We show that any ratio vector in which 3 and 4 fully repay their debts and 1 and 2 have equal ratios is a clearing vector. Let $\boldsymbol{\theta}$ with $\theta_1 = \theta_2, \theta_3 = \theta_4 = 1$. Bank 1's equity, $e_1(\boldsymbol{\theta}) = -1 + \theta_2 + \theta_3 - \theta_1$, is null so bank 1 is allowed to default; bank 3's equity, $e_3(\boldsymbol{\theta}) = 1.5 + \theta_4 - 2\theta_3 = 0.5$, is positive so bank 3 is safe (and similarly for banks 2 and 4 by symmetry).

Observe that the liability matrix is reducible since $\{1, 2\}$ have no liabilities towards 3 and 4. But the liabilities-loans graph is connected since $\{1, 2\}$ has 3 and 4 as debtors. At a clearing vector, banks 1 and 2 may appear as almost bankrupt or almost safe as their ratio may take any value between 0 and 1 (but whatever value their equity is null).

3 The threat indices when each net worth is positive

This section assumes positive net worth, $\boldsymbol{z} \gg \mathbf{0}$. No bank is bankrupt in that case (Corollary 1).

3.1 Defining threat indices

To define the threat imposed by a defaulting bank on the system, let us take the perspective of the whole system of banks. Assume that the system's objective is defined by the aggregate size of the

[4] In graph terminology, irreducibility considers directed paths in the liabilities network whereas connectedness considers non-directed paths.

balance sheets. A justification is that the activity of the banks is linked to the size of their balance sheets, or there is an implicit cost to banks' defaults, proportional to the non repaid amount. (Note that this objective differs from aggregate equity, which is constant anyway as long as there is no bankruptcy.) The aggregate size of the balance sheets is given by the sum of the overall asset over the non-financial sector and the debt repayments $\sum_i \ell_i^* \theta_i$. Since the first term is taken as exogenous, the aggregate size of the balance sheets is maximized by maximizing debt repayments.

There is an alternative justification for this objective, taking a payment system point of view. Let the system guarantee the repayments so that its loss is given by the non-repaid amounts, $\sum_i \ell_i^*(1-\theta_i)$. Its objective is to minimize its loss, which amounts to maximize debt repayments.

Thus whatever interpretation, the objective is to maximize the effective payments $\sum_i \ell_i^* \theta_i$. The bank's threat index measures the loss in terms of this objective induced by a decrease in the bank's net worth at the margin or, taking the opposite side, the index measures the benefit to the payment system of rescuing the bank say in the form of additional capital. The analysis is made rather simple by the fact that clearing ratios maximize the total payments within the banking system under some constraints. Specifically, due to the complementarity in repayment ratios, the clearing ratio vector solves the following program, as proved by EN[5]

$$\mathcal{P} \quad : \quad \max_{\boldsymbol{\theta}} \sum_i \ell_i^* \theta_i$$

$$0 \leq \theta_i \leq 1 \text{ for each } i \tag{11}$$

$$\theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} \leq z_i \text{ for each } i. \tag{12}$$

The constraints reflect those on debt contracts, (11), according to which a bank can only reimburse (the condition θ_i non-negative) and only up to its nominal liabilities (the condition θ_i less than 1) and those on equities, the limited liability condition (12). Observe that creditors' priority is not a constraint. It is surely satisfied at the optimal solution because the objective is strictly increasing in each repayment ratio and these ratios are complements. The creditors' priority can thus be seen as forcing the clearing ratio vector to maximize the total payments within the banking system under the limited liability condition.

We consider the value of the program $\mathcal{P}$ as the net worth levels or the liabilities vary. To make this dependence clear, let $V(\boldsymbol{z}, \boldsymbol{\ell})$ denote the value of $\mathcal{P}$ associated to $(\boldsymbol{z}, \boldsymbol{\ell})$. At differentiable points of V, the envelope theorem applies; the impact of a (marginal) increase in the net worth of a bank

[5]Debt payments constitute the more natural and simple objective but they can be replaced by any function that is increasing in the ratios θ_i. By the same argument, the clearing ratio solves the associated program. Threat indices are then defined as the marginal impact on the value of the program following a marginal increase in the net worth.

is assessed by considering the multiplier μ_i associated to the equity constraint (12):

$$\frac{\partial V}{\partial z_i} = \mu_i.$$

Thus, a marginal increase of one unit in i's net worth increases the value V of the payments within the system by μ_i units. Alternatively a marginal decrease of one unit in i's net worth decreases the payments by μ_i units. This is why we call the multiplier μ_i a 'threat' index (but could have called it a credit multiplier as well).

The next proposition states that the function V is indeed differentiable at 'most' points and provides an expression for the multipliers. These properties are especially useful for deriving an optimal targeting policy, as investigated in Section 3.3.

Non differentiability arises when some banks are at the 'boundary' between the status of safe and defaulting banks. Specifically a boundary bank fully repays its debt but ends up with a null equity. Observe that typically there is no boundary bank since small perturbations in the values of $\boldsymbol{z}$ or $\boldsymbol{\ell}$ make each bank either safe or truly defaulting.

Proposition 3 *The value function V of program $\mathcal{P}$ is concave in $(\boldsymbol{z}, \boldsymbol{\ell})$. It is differentiable at each point for which there is no boundary bank with a derivative vector $(\frac{\partial V}{\partial z_i})$ given by the unique $\boldsymbol{\mu}$ that is null on S and solves on D*

$$\ell_i^* \mu_i - \sum_{j \in D} \ell_{ij} \mu_j = \ell_i^* \quad \textit{for each } i \textit{ in } D. \tag{13}$$

The multiplier μ_i is called i's threat index.

Thus the threat indices are uniquely defined and the function V is differentiable almost everywhere. The case where there are boundary banks is investigated in section 3.2. The proposition is proved by applying well known results on linear programming and duality. Standard complementarity relationships between primal and dual variables apply to the repayment ratios (the solutions to the primal) and the threat indices (the solutions to the dual). This explains why the threat index of a safe bank is null. Otherwise, for a bank with a repayment ratio θ_i strictly smaller than 1, the multiplier is non-null and even larger than 1.

With a complete liabilities network in which liability ℓ_{ij} is positive for each distinct i and j, the fact that the linear system (13) has a unique solution follows from well known results on diagonally dominant matrices.[6] When the liability structure is incomplete, the system is shown to be invertible

[6] Whatever strict subset D of N, the sum of row i of $\boldsymbol{\ell}_{D\times D}$ is strictly smaller than ℓ_i^*. Matrix $(dg(\boldsymbol{\ell}^*) - \boldsymbol{\ell})^t_{D\times D}$ is a Leontieff diagonally dominant matrix, hence is invertible, with a positive inverse.

for the subset of defaulting banks, though it may not be for all subsets of N. This will be developed in the next section when interpreting the indices.

Comparing clearing ratios and threat indices A comparison of the determinants of the repayment ratios and the threat indices of the defaulting banks makes clear that they are not necessarily aligned. To see this, let us write down the conditions satisfied by the clearing ratio and threat index vectors, assuming no boundary bank.

The clearing ratio vector is of the form $(\boldsymbol{\theta}_D, \mathbb{1}_S)$ because the safe banks have a ratio equal to 1. Since the equity of a defaulting bank is null, $\boldsymbol{\theta}_D$ satisfy the system of linear inequalities

$$\ell_i^* - \sum_{j\in D}\theta_j\ell_{ji} - \sum_{j\in S}\ell_{ji} \quad \leq z_i \qquad \text{for each } i \text{ in } S \tag{14}$$

$$\theta_i\ell_i^* - \sum_{j\in D}\theta_j\ell_{ji} - \sum_{j\in S}\ell_{ji} \quad = z_i \quad \text{for each } i \in D. \tag{15}$$

Thus, clearing ratios and threat indices satisfy

$$\theta_i = 0,\, \mu_i = 1 \qquad \text{for each } i \text{ in } S \tag{16}$$

$$\theta_i\ell_i^* - \sum_{j\in D}\theta_j\ell_{ji} = z_i + \sum_{j\notin D}\ell_{ji},\; \ell_i^*\mu_i - \sum_{j\in D}\ell_{ij}\mu_j = \ell_i^* \quad \text{for each } i \text{ in } D. \tag{17}$$

Whereas the distress of a bank as measured by its repayment ratio depends on the distress of its debtors (through the ℓ_{ji}), the threat the bank imposes on the payment system depends on the threat of its creditors (through the ℓ_{ij}). Thus the impact of the liability structure differs except under strong symmetry of the liabilities/loans. Also, the repayment ratios are affected by the precise values taken by the net worth whereas the indices depend on these values only through the banks' status (the interpretation of the threat indices will explain why this is the case).

Let us illustrate with a pyramidal structure in which ratios and indices can be computed recursively.

Pyramidal network In a pyramidal network, liabilities go in the same direction without loop, from the top or from the bottom. Consider first the situation where these liabilities are directed from the top, as represented in Figure 1, in which an arrow from i to j represents a positive liability of i towards j. This describes a situation in which chains of intermediaries collect funds for the bank at the top, say bank 1. Each intermediary lends only to its unique superior; it collects funds from the financial institutions directly below it or from non-banking firms for those without 'subordinate'.

Let the level of an intermediary be defined as the number of links between it and top bank 1. The clearing ratio vector is easily computed recursively starting from the top. Bank 1 has no claims

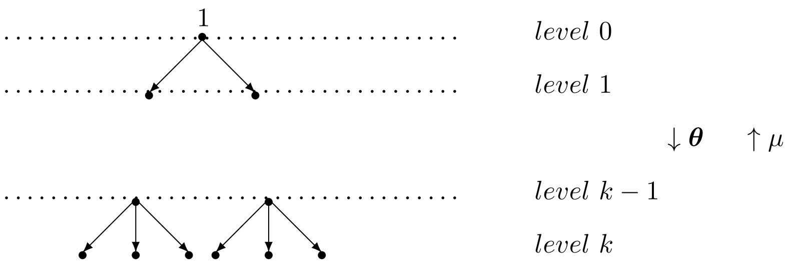

Figure 1: Pyramidal network

on other banks so its repayment ratio is determined by its net worth as the minimum of 1 and z_1/ℓ_1^*. The repayment ratio of the banks at level 1 can now be computed since they receive payments only from bank 1 and these are known. The computation proceeds: at step $k+1$, the repayment ratios of the banks at level k are determined since the repayments of all their debtors are known (i's ratio is the minimum of 1 and $(z_i + \sum_{j \in D} \theta_j \ell_{ji})/\ell_i^*$. The clearing vector is obtained after k steps and the status of the banks are determined.[7]

Knowing the banks' status, the threat indices are computed recursively in a similar way, starting from the bottom instead of the top. Assume first that there are no boundary banks. Set the indices of all safe banks to zero. The threat index of a bank at level k is either equal to 0 (the bank is safe) or to 1 (the bank is in default but has no creditor). At each further step, the threat index of a bank only depends on those of its creditors, which have been determined at the previous step. Hence the indices are computed recursively for the defaulting banks using expression (13): $\ell_i^* \mu_i - \sum_{j \in D} \ell_{ij} \mu_j = \ell_i^*$. The computation is especially simple when the pyramidal structure reduces to a single debt chain, in which each bank i has a liability towards $i+1$. Then ℓ_{ii+1} is equal to ℓ_i^* and all other liabilities are null. The threat index of a defaulting bank is simply equal to 1 plus the number of its consecutive direct or indirect creditors that are defaulting. Clearly, the orders given by the default ratios and the threat indices may differ.

If there are boundary banks, the computation can be performed by either considering these boundary banks as defaulting banks, so as to obtain the maximal threat indices, or by considering them as safe banks, so as to obtain the minimal indices.

Similar recursive computations can be performed in the reverse situation in which all the liabilities point toward the top. This may represent a 'conglomerate' in which at each level a unit has lent

[7]When some net worth levels are negative, the same computation applies simply by setting a ratio equal to zero for a bank with a negative net asset value.

funds to its direct subordinates. In this situation, the top has a null threat index. This situation can be qualified as less prone to contagion than the previous one because a single default cannot touch all banks.

In general there are cycles in the network and a recursive computation is not possible. Indices can however be interpreted by considering these cycles, as the next section shows.

3.2 Interpreting the threat indices

This section first explains the expression for the indices. Then it treats the common situation in which there is a lack of data on bilateral interbank exposures and the liabilities structure is estimated through the log-fitting model. Finally we compare the threat indices with the centrality indices introduced in the sociological literature.

It is convenient to work with the relative liabilities. For i indebted, $\ell_i^* > 0$, the proportions of its liabilities to creditors are described by its relative liabilities: $\pi_{ij} = \frac{\ell_{ij}}{\ell_i^*}$ for each j. Surely a defaulting bank is indebted. Let $\boldsymbol{\pi}_{D\times D}$ denote the $D \times D$ matrix[8] $(\pi_{ij})_{i\in D, j\in D}$. Dividing equation (13) by ℓ_i^*, the threat indices among the defaulting banks satisfy:

$$\mu_i = 1 + \sum_{j\in D} \pi_{ij}\mu_j, \text{ for each } i \text{ in } D \tag{18}$$

or in matrix form (where $\mathbb{I}$ denotes the identity matrix)

$$(\mathbb{I} - \boldsymbol{\pi})_{D\times D}\boldsymbol{\mu}_D = \mathbb{1}_D,$$

Only the relative liabilities *within* the set D determine the indices. The threat index of a defaulting bank i is equal to 1 plus the weighted sum of the threat indices of its creditors weighted by the amount of the obligations of i to them. The following Lemma allows us to provide an interpretation for the indices. (Lemma 1 is also used in the proof of Proposition 3 to show that $\boldsymbol{\mu}_D$ is uniquely defined).

Lemma 1 *Let $\boldsymbol{z} \gg \mathbf{0}$ and D be the default set at a clearing ratio vector $\boldsymbol{\theta}$. The matrix $(\mathbb{I} - \boldsymbol{\pi})_{D\times D}$ is invertible, with a positive inverse given by the converging infinite sum:*

$$(\mathbb{I} - \boldsymbol{\pi})^{-1}_{D\times D} = \mathbb{I}_{D\times D} + \boldsymbol{\pi}_{D\times D} + \boldsymbol{\pi}^{(2)}_{D\times D} + ... + \boldsymbol{\pi}^{(p)}_{D\times D} + ... \tag{19}$$

The proof is in Section 6. For a complete liability structure ($\ell_{ij} > 0$ for each distinct i and j), (19) follows from well known results on positive matrices because the total of each row of $\boldsymbol{\pi}_{D\times D}$ is strictly

[8] There is a slight abuse of notation because the matrix $\boldsymbol{\pi}$ is not defined on $N \times N$ if some banks are not indebted. But these banks are surely not defaulting.

smaller than 1 (since D is a strict subset of N). For an incomplete liability structure, though the result may not hold for any subset, it surely holds for the set D of defaulting banks. The argument relies on the fact that each subset of defaulting banks has a creditor outside it[9] (this is because the z_i are positive and each defaulting bank has null equity, thus, some payments must go out from the subset). This implies that the matrix $\boldsymbol{\pi}_{D\times D}$ is productive. Then (19) follows from standard results.

Lemma 1, especially expression (19), is useful to interpret Proposition 3 and the expression (18) for the threat indices. Assume first that there is no boundary bank. Consider the impact of an increase of δ units in the net worth of a bank on the subsequent payments. We show that for δ small enough, the impact is proportional to δ: the factor of proportionality gives the value μ_i. A safe bank, which already fulfills its obligations, keeps the additional cash: there is no impact on payments ($\mu_i = 0$). A defaulting bank instead uses the additional cash (at least partially) for reimbursing its debts. Its creditors in default in turn use (part of) the additional cash to repay their debts, and so on. The initial additional cash thus triggers a sequence of additional reimbursements along the creditors which are themselves in default. For δ small enough, no bank's status is modified. Hence, each amount that is received by a defaulting bank is entirely used to repay its debts. This leads to the following computation.

The additional cash received by the defaulting bank i, which is entirely used for reimbursement, generates a first increase of δ units in the payments. This explains why μ_i is larger than 1. Each i's creditor j receives the share π_{ij} of δ, entirely used for reimbursement by those in default, thereby generating a first 'indirect' additional payment in the system equal to $(\sum_{j\in D}\pi_{ij})\delta$. The sum term is the $i-$th element of $\boldsymbol{\pi}_{D\times D}\mathbb{1}_D$. By the same argument, each of the $\pi_{ij}\delta$ units received by the defaulting i's creditor j generates $\sum_{k\in D}\pi_{jk}$ extra units of payments. So, summing over all defaulting creditors of i, the 'second' indirect additional increase equals $\sum_{j\in D}(\sum_{k\in D}\pi_{jk})\pi_{ij}\delta$, or exchanging the order of summation, $\sum_{k\in D}(\sum_{j\in D}\pi_{ij}\pi_{jk})\delta$. Since the element in square brackets is the (i,k) element of the matrix $\boldsymbol{\pi}^{(2)}_{D\times D} = \boldsymbol{\pi}_{D\times D}\times\boldsymbol{\pi}_{D\times D}$, the 'second' indirect impact is δ times the $i-$th component of $\boldsymbol{\pi}^{(2)}_{D\times D}\mathbb{1}_D$. Iterating, the additional indirect impact along a path of p banks, each one

[9] Without an outside creditor for each subset, invertibility may fail, as in the following example where subset T is made of indebted banks and

$$\boldsymbol{\pi}_{T\times T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Rows totals are 1 for banks 1 and 2 and 0 for bank 3. Hence bank 3 has a creditor outside T, and $T = \{1,2,3\}$ as well. But banks 1 and 2 have no creditor outside T and furthermore are not indebted to 3: $\{1,2\}$ has no creditor outside it. Since the vector $x = (1,1,0)$ satisfies $x = x\boldsymbol{\pi}_{T\times T}$ the matrix $(\mathbb{I}-\boldsymbol{\pi})_{T\times T}$ is not invertible.

defaulting and debtor to its successor, is δ times the $i-$th component of $\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D$. Summing all indirect impacts gives that the initial increase of δ units of i's cash-flow generates an overall increase in the payments given by δ times the i-th component of the infinite sum $\sum_{p\geq 0}\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D$. Thus, this sum is equal to μ_i. Considering all defaulting banks, we obtain

$$\boldsymbol{\mu}_D = (\mathbb{I}_{D\times D} + \boldsymbol{\pi}_{D\times D} + \boldsymbol{\pi}^{(2)}_{D\times D} + ... + \boldsymbol{\pi}^{(p)}_{D\times D} + ...)\mathbb{1}_D$$

or $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}\boldsymbol{\mu}_D = \mathbb{1}_D$, the equation (18).

The above argument explains why the indices are determined by the *relative* liability structure *within* the set D only. As long as the banks' status do not change, which holds true for small enough δ and no boundary bank, a cash injection triggers automatic increases in the payments entirely determined by the liability shares of the recipient defaulting banks; in particular these increases do not depend upon net worth levels (again given unchanged banks' status).

The argument extends to the situation where there are some boundary banks by distinguishing an increase in the net worth of a boundary bank from a decrease. Increasing the net worth in a boundary bank has no impact on the payments because it already repays its debt. If instead the net worth is decreased, its ratio is necessarily lowered and the same argument as above allows us to compute the impact on the payments (other boundary banks (if any) are treated as defaulting banks because each one can only receive less reimbursements). This explains why the value function V is not differentiable. Furthermore, for boundary bank i

$$\frac{\partial V}{\partial z_i^+} = 0, \frac{\partial V}{\partial z_i^-} \geq 1,$$

in which the expressions denote respectively the right and left derivatives of V. V may not be differentiable with respect to the net worth of a defaulting bank either. Let us consider a defaulting bank with a chain of defaulting creditors leading to a boundary bank. An increase in the net worth of the defaulting bank makes the boundary bank become safe while a decrease makes it default, generating a further decrease in payments. The extremal values for the threat indices are obtained by applying expression (13) either to the whole default set (for the maximal values) or to the set obtained by eliminating the boundary banks from D (for the minimal ones).

Log-fitting model We compute here the threat indices when there is a lack of data on bilateral interbank exposures. Most often the log-fitting method is used to estimate the missing data given the available information on some total exposures. The method is justified if the missing data are independent conditional on the current information. Explicit formula are obtained in the following simple situation.

Let the total amount of liabilities l_i^* and loans $l_{*i} = \sum_{j \in N} l_{ji}$ be known for each bank. Thus the sums in each row and each column of matrix $\boldsymbol{\ell}$ are known. Without any specific information on bilateral exposures, the estimated proportions of i's liabilities are independent of i, thus equal to the overall proportions of the loans. The bilateral exposures are estimated by

$$\pi_{ij} = \frac{\ell_{*j}}{\sum_{j \in N} \ell_{*j}}, \ell_{ij} = \frac{\ell_i^* \ell_{*j}}{\sum_{j \in N} \ell_{*j}}, \ell_{ij} = \ell_i^* \pi_{ij}.$$

Since the values π_{ij} are independent of i, the expression (18) for the threat indices of defaulting banks, $\mu_i = 1 + \sum_{j \in D} \pi_{ij} \mu_j$, implies that index μ_i is independent of i in D. Straightforward computation gives

$$\mu_i = 1 + \frac{\sum_{j \in D} \ell_{*j}}{\sum_{j \in S} \ell_{*j}} \text{ for each } i \in D. \tag{20}$$

According to this expression, the log-fitting model does not discriminate the banks within each status class. The common value for the threat index of defaulting banks is equal to 1 plus the loans distributed by the defaulting banks relative to those distributed by the safe banks. The banks which have distributed a large amount of loans are the ones that are the more severely hurt, independently of which banks are defaulting. This explains why the total amount of loans distributed by the defaulting banks influences the value of μ_d. If we dropped the log-fitting framework, say because of additional information on some links (thus keeping fixed the totals ℓ_i^* and ℓ_{*j}), the threat indices would be more dispersed, presumably with a maximal value larger than μ_d or even a general shift upwards in a sense to be made precise. This question deserves more study.

Links with centrality indices Threat indices have a flavor of the 'centrality' indices introduced in the network literature by Katz/Bonacich index (1953) and (1987) within the network restricted to the defaulting banks. In the following specific case, they coincide. Let G be the incidence matrix of the liabilities network, which has 1 if ℓ_{ij} is positive and 0 otherwise. Let all the positive liabilities ℓ_{ij} have an identical level, and furthermore let each bank have the same number of creditors, say p, hence the same total liabilities. The matrix $\boldsymbol{\pi}$ is then proportional to the incidence matrix of the liabilities network : $\boldsymbol{\pi} = \frac{1}{p} G$. Thus given D

$$\boldsymbol{\mu}_D = \left(\mathbb{I} - \frac{1}{p} G_{D \times D} \right)^{-1} \mathbb{1}_D.$$

For a network with nodes D and adjacency matrix g, the Bonacich index is defined as $\beta = (\mathbb{I} - ag)^{-1} g \mathbb{1}_D$ where a is an 'attenuation' parameter, which captures the importance of indirect links. Taking for g the adjacency matrix $G_{D \times D}$ of the sub-network linking defaulting banks, simple computation[10] shows that $\boldsymbol{\mu}_D$ is a linear transformation of the Bonacich index with attenuation parameter $1/p$: the more numerous creditors each bank has, the more the impact of default is dissipated along a chain of creditors. There are two important differences with the standard framework however: the subset of relevant nodes is endogenous, given by the set of defaulting banks, and the attenuation parameter is here determined, defined by the reciprocal of the number of total creditors of a bank.

3.3 Targeting policy

A targeting policy aims to inject a given amount of cash so as to increase the effective payments V within the banking system as much as possible. We first consider a policy that injects cash in the banking system.

Cash injection The targeting policy is easily derived from Proposition 3.

Proposition 4 *A marginal injection of cash that optimally increases V is targeted towards the defaulting banks with the largest value for the threat index. These targeted banks are kept identical as long as the banks' status remain unchanged.*

PROOF The first assertion straightforwardly follows from Proposition 3. The function V is concave and $\frac{\partial V}{\partial z_i} = \mu_i$ for each i at each $\boldsymbol{z}$ for which no bank is at the boundary. Since the marginal impact of cash injection towards i increases the value V of the payments within the system by μ_i units, cash should be allocated to the banks with the largest μ_i. As for the last statement, recall that the threat index $\boldsymbol{\mu}$ is independent of the precise values of $\boldsymbol{z}$ provided that they lead to the same banks' status. ∎

The policy is especially simple since there is no need to modify the targets while injecting cash as long as no defaulting bank is transformed into a safe one. Recall that the orders given by the default ratios and the threat indices may differ. Thus the targets may not be the banks with the largest default ratios.

Also, the targets might not be the largest banks. As clear from the expression (18), the threat index of a defaulting bank is determined by the *share* of its liabilities towards its creditors that are in default. Accordingly, the benefit of injecting cash at the margin is unrelated to size. This should not be interpreted however that size does not matter. First, it is only at the margin that size does not matter. While injecting a large amount of cash, the targeting policy requires to adjust the targeted

[10] Writing $\mathbb{I} = (\mathbb{I} - ag) + ag$, we have $(\mathbb{I} - ag)^{-1}\, \mathbb{1}_D = \mathbb{1}_D + a\,(\mathbb{I} - ag)^{-1}\, g\mathbb{1}_D = 1 + a\beta$.

banks when some defaulting banks become safe. It is plausible that a small bank becomes safe before a large one, hence cash will go to a large defaulting bank at some stage. Second, the expression for the threat indices is conditional on the realized default set and one may suspect that a large bank, or more precisely a bank with large liabilities, generates more default among its creditors when it defaults. If true, its index is likely to be large, conditional on its default. A full analysis of these issues requires to specify how the net worth and the liabilities are likely to be related, for example by considering an ex ante stage in which banks choose their investments.

Solidarity policy Alternative policies based on different tools than cash injection can be contemplated. One tool is to force banks to pay more than their liabilities, which amounts to increase the upper-bound of 1 on the repayment ratio. Such a tool is effective only on the safe banks, which are the only ones to be able to increase their payment. Hence the targeted banks differ from those under capital injection. The associated policies can be qualified as 'solidarity policies' since they involve transfers from safe to defaulting banks. An optimal solidarity policy depends on the precise constraints one wants to impose on these transfers. Whatever these constraints are, one needs to assess the impact on the payments V of increasing the upper-bound on the repayment ratio, which is given by the multiplier associated to the constraint $\theta_i \leq 1$. To facilitate comparison with the threat index and work with the same unit, we write the constraint $\theta_i \leq 1$ as $\theta_i \ell_i^* \leq \ell_i^*$. We call the associated multiplier λ_i the *solidarity index*. For the same reasons as for the threat indices, solidarity indices are not uniquely defined when some banks are at the boundary. So let us assume no boundary bank. Increasing the upper-bound on the ratio of a defaulting bank has no effect: λ is null on D. For the safe banks, assuming no boundary banks, the values of λ on S are given by

$$\ell_i^* \lambda_i = \sum_{j \in D} \ell_{ij} \mu_j + \ell_i^*, \text{ for each } i \text{ in } S. \tag{21}$$

The solidarity index of a safe bank is easy to interpret. Increasing the upper-bound on the ratio of a safe bank say by one percent has a direct effect of increasing the payment by one percent of ℓ_i^*, and an indirect effect on the banks that receive these payments. This indirect effect is similar to an increase in the net worth of each creditor in the proportion given by the relative liabilities. This explains expression (21) (accounting for the fact that the threat indices μ_j are null outside D).

3.4 Comparative statics in liabilities

There is a concern about the impact of large cross-liabilities on the stability of the system. This section analyzes this impact on the effective payments V.

Consider first an increase in the liability between two banks, ℓ_{ij} of i to j. The impact on the creditor j is akin to an additional unit of net worth, the amount of which depends on how much debtor i can pay. Thus, the status of both banks matter to determine the impact on the payments. Indeed, thanks to the envelope theorem, and assuming no boundary bank, the payment function V is differentiable with respect to ℓ_{ij} with a derivative given by

$$\frac{\partial V}{\partial \ell_{ij}} = \theta_i[1 - \mu_i + \mu_j]. \tag{22}$$

The interpretation is as follows. Let us consider an increase of one unit in liability ℓ_{ij}. To simplify the notation, let the 'unit' of increase be small enough so that its impact is equal to the marginal impact. If both banks are safe, payments increase by 1 unit as expected. To understand other cases, let us distinguish between the status of bank i.

If i is safe but j defaults, the payments increase by $1 + \mu_j$: there is an additional payment of 1 unit by i to j and this unit has the same impact on defaulting j as an additional unit of net worth, hence an additional increase by μ_j.

If i defaults, the impact here is more subtle. As i already exhausts its repayment capacity, an additional liability has no impact on its overall repayment, but there is a change in its distribution. Specifically, the impact can be decomposed into two parts, a direct effect and a composition effect. First, i sends an additional amount θ_i to bank j, and this has a direct effect equal to $\theta_i(1 + \mu_j)$ arguing as above. Second, as bank i is constrained, it has θ_i less units to repay its original debts, as if its net worth was diminished by θ_i: the composition effect is $-\theta_i\mu_i$. The sum of the two effects is $\theta_i[1 + \mu_j - \mu_i]$, which is (22).

Now, to understand better the impact of liabilities on effective payments, we consider identical increases in the joint liabilities of two banks, which leave unchanged their net liabilities. Also we compare the increase in the effective payments with the increase in nominal liabilities, as measured by the net increase:

$$\theta_i[1 - \mu_i + \mu_j] - 1. \tag{23}$$

The net increase is null when both banks are safe, positive for i safe and j in default, negative in the opposite situation, and ambiguous when both banks are defaulting.

Let us first consider an identical increase in the joint liabilities of two banks, namely both ℓ_{ij} and ℓ_{ji} are increased by an identical marginal amount. The marginal overall net increase in the payments is

$$(\theta_i + \theta_j) + [\mu_j - \mu_i](\theta_i - \theta_j) - 2.$$

When both banks are safe, the net increase is null as expected. When one bank is safe, net payments never decrease: For i safe, $\theta_i = 1$ and $\mu_i = 0$, the net increase equals $(1-\theta_j)(\mu_j - 1)$, which is non-negative since each term in the product is. The intuition is that increasing each liability calls for more transfers across banks, which results in safe banks paying more per unit of additional liability than those in default. When both banks are defaulting however, the impact on payments is ambiguous because of the composition effect we have identified previously. With similar repayment ratios or similar threat indices, the impact is indeed negative: the direct effects identified above, which increase repayments between the two banks, cancel out and we are left with the negative composition effect, according to which other banks get less.

Finally, let us consider an equal increase in all liabilities, which would follow for example from a softening of regulation constraints. The net increase in the payments, adding up the impact of all pairs, is[11]

$$\sum_i (1-\theta_i)[n(\mu_i - \overline{\mu}) - (n-1)] \tag{24}$$

where $\overline{\mu} = \frac{1}{n}\sum_i \mu_i$ is the average value of the μ_i. When there is a single defaulting bank, its threat index is 1, and the expression is null. The extra payments received by the defaulting bank are entirely sent back for reimbursement but generate no further payments. When there are several defaulting banks, the sign can be positive or negative. Let us illustrate in the log-fitting model (section 3.2). As seen in from (20)), the threat indices of the defaulting banks are all identical given by the value $\mu_d = \frac{\sum_{j\in N} \ell_{*j}}{\sum_{j\in S} \ell_{*j}}$ where ℓ_{*j} is the amount of loans distributed by j.. Hence the term within the square bracket in (24) is independent of i and, assuming at least a defaulting bank (i.e. a θ_i smaller than 1) a net increase is positive if

$$\mu_d > \frac{n-1}{n-d} \quad \text{or} \quad \frac{\sum_{j\in D} \ell_{*j}}{d} > \frac{d-1}{d} \frac{\sum_{j\in S} \ell_{*j}}{n-d} \tag{25}$$

where d and $n-d$ are respectively the number of defaulting and safe banks. The inequality requires the average loan per defaulting bank to be larger than the average loan per safe bank by the factor $\frac{d-1}{d}$. As seen earlier, increasing liabilities has a positive impact on a pair formed with a safe and a defaulting bank which is increasing in μ_d and has a negative one on a pair with defaulting banks which is independent of their identical index μ_d. Under (25) the threat index is large enough so that the positive impact dominates.

[11]The sum of the net increase (23) over all pairs of distinct elements i j is $-(n-1)\sum_i(1-\theta_i) - \sum_i \theta_i[(n-1)\mu_i - \sum_{j\neq i}\mu_j]$. Since $[(n-1)\mu_i - \sum_{j\neq i}\mu_j] = n[\mu_i - \overline{\mu}]$, the overall net increase is $-(n-1)\sum_i(1-\theta_i) - n\sum_i \theta_i(\mu_i - \overline{\mu})$. Adding the null term $n\sum_i(\mu_i - \overline{\mu})$ and rearranging yields (24).

With an identical amount of loans per bank, the ℓ_{*j} are equal across j, the condition (25) is surely met: increasing liabilities is beneficial. Such a situation corresponds to a priori similar institutions, which are engaged into symmetrical interbank relationships. Due to shocks in their activities, they may end up in an asymmetrical situation, with some of them defaulting. However, independently of the realized net worth levels, and the subsequent status for the firms, more links are better for net reimbursements. Thus there is a benefit 'ex post', which implies an 'insurance' benefit ex ante, taking the expectation over all values of the net worth.

4 Threat indices in case of bankruptcy

Bankruptcy is unavoidable in some circumstances, for example if aggregate net worth is negative (Proposition 1). In that case no ratio vector satisfies the constraints of the program $\mathcal{P}$ and the clearing ratio vector cannot be a solution. This section shows that the clearing vector is a solution of a modified program that takes into account bankruptcy when it occurs. The threat indices are then defined as measuring the incremental benefit in the objective due to an additional unit of cash into banks.

A negative net equity represents a loss, which is borne by the creditors if the bank is declared bankrupt, or by the stockholders or whatever entity called to help such as taxpayers if the bank is bailed out. The objective of the system takes into account of these losses. Specifically, given a cost c per unit of loss, the objective is given by the overall repayments net of the cost of the losses: $\sum_i \theta_i \ell_i^* - c(\sum_i \delta_i)$ where δ_i denotes the loss inflicted by i. The program is

$$\mathcal{Q}_c \quad : \quad \max_{(\boldsymbol{\theta},\boldsymbol{\delta})} \sum_i \theta_i \ell_i^* - c(\sum_i \delta_i)$$

$$0 \leq \theta_i \leq 1,\ 0 \leq \delta_i \ \text{ for each } i \tag{26}$$

$$\theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} \leq z_i + \delta_i \ \text{ for each } i \tag{27}$$

The constraint (27) writes as $0 \leq e_i(\boldsymbol{\theta}) + \delta_i$. At an optimal solution, it is surely binding for a bank i with a positive δ_i. Thus, δ_i is positive when net equity is negative, hence, it indeed corresponds to the loss inflicted by i to the creditors outside the financial system or, alternatively, to the additional amount of equity that have to be put into the bank to rescue it.

Observe that the constraints of the two programs $\mathcal{P}$ and $\mathcal{Q}_c$ differ. In particular, $\mathcal{P}$ is not the program $\mathcal{Q}_0$ obtained for c null. (In $\mathcal{Q}_0$, the equity constraint can be relaxed at no cost; ratios equal to 1 are feasible and maximize the flows). On the contrary, when net worth levels are positive, the solutions to $\mathcal{P}$ coincide with those to $\mathcal{Q}_c$ for large enough c, as followed from the next proposition. The result is intuitive: since bankruptcy is avoidable, it is optimal to avoid it when the cost associated to bankruptcy losses is large enough. More generally, under a mild condition, even when bankruptcy must occur, the clearing repayment ratio vector solves $\mathcal{Q}_c$ for c large enough.

Let us associate to a ratio vector $\boldsymbol{\theta}$ the vector $\boldsymbol{\delta}$ of the loss to outside creditors : $\delta_i = \max[0, -e_i(\boldsymbol{\theta})]$.

Proposition 5 *Let us assume one of the cases of proposition 2. Assume furthermore that there is a safe bank at the (unique) clearing repayment ratio vector. The clearing repayment ratio vector and its associated loss solves $\mathcal{Q}_c$ for c large enough.*

Let no default bank be at the boundary with a ratio equal to 1 or 0. The threat indices, defined as the multipliers associated to the equity constraint (27), are uniquely defined by

$$\mu_i = 0 \textit{ for each } i \textit{ in } S, \; \mu_k = c \textit{ for each } k \textit{ in } B \tag{28}$$

$$\mu_i - \sum_{j \in D} \pi_{ij}\mu_j = 1 + c \sum_{k \in B} \pi_{ik} \textit{ for each } i \textit{ in } D. \tag{29}$$

Thus the programs $\mathcal{Q}_c$ all admit the same solution, the clearing vector, whatever value for c large enough. A consequence is that the clearing vector solves a lexicographic objective: first minimize the loss to the creditors outside the financial sector, second, if there are multiple solutions to the minimization problem, choose one that maximizes the payment flows within the financial system. In particular, when bankruptcy can be avoided, the minimal loss is null, and no bank is bankrupt at a clearing ratio.[12] In that case, any ratio that satisfies the constraints of $\mathcal{P}$ produces the minimal loss, and the criterion on payments selects among these ratios and we fall back exactly on $\mathcal{P}$.

Without a safe bank, an extreme case, the clearing ratio vector minimizes the aggregate loss to outside creditors but is not a solution to $\mathcal{Q}_c$ even for large c because it does not maximize the flows within the financial sector. This can be shown as follows. At any ratio vector, the loss to the outside sector is at least as large as its aggregate debt: Summing (27) over all the banks $\sum_i(\delta_i + z_i) \geq 0$ at a feasible solution. When there is no safe bank at a clearing ratio vector the equity values all satisfy $e_i(\boldsymbol{\theta}) + \delta_i = 0$. Hence the aggregate loss to outside creditors is minimal. However the clearing ratio may not maximize the flows within the financial sector. Let for example each bank have a negative net equity under full repayment: $e_i(\mathbb{1}) = z_i + \sum_j \ell_{ji} - \ell_i^* < 0$ for each i. The optimal solution to $\mathcal{Q}_c$ is to have banks fully repay their debts and to inject in each of them exactly the amount necessary to do so (i.e. $\delta_i = -e_i(\mathbb{1})$). The aggregate outside loss is minimal equal to $-\sum_i z_i$ (since $\sum_i e_i(\mathbb{1}) = \sum_i z_i$) and the payment flows within the financial system is clearly maximal.

[12] This extends the insight of Corollary 1, according to which, when each net worth is positive, the clearing mechanism avoids bankruptcy possibly by performing a redistribution within the banking system through partial default.

Consider now the threat indices. Arguing as in the previous section, a bank's threat index measures the incremental benefit in the objective due to an additional unit of cash in the bank. The objective becomes dominated by the cost associated with capital injection as c becomes large, and this explains why the behavior of the threat indices largely depends on whether there are bankrupt banks. Without bankrupt banks, an empty set B, expression (29) for the threat indices coincides with (13) found in the previous section. Threat indices reflect the impact of the banks' net worth on the payment system and do not depend on c because there is no capital injection. With bankrupt banks instead, the threat indices adjust to the value of the cost c (although the solution to $\mathcal{Q}_c$ stays constant for large enough c). The expression (29) can be interpreted as in the previous section by considering the additional flow of repayments that an increase in the net worth of a defaulting bank induces. With bankruptcy, what matters is not only the payments flowing along the defaulting banks but also how much of this flow reaches the bankrupt banks because this allows to diminish capital injection. As c increases, the latter becomes predominant and the threat index measures the payments reaching the bankrupt banks. Specifically, the limits $\widehat{\mu}_i$ for each i of the threat indices per unit of cost, μ_i/c, as c increases satisfy

$$\widehat{\mu}_i - \sum_{j\in D} \pi_{ij}\widehat{\mu}_j = \sum_{k\in B} \pi_{ik} \text{ for each } i \text{ in } D. \tag{30}$$

Let us write (30) in matrix form. The right hand side is the proportion of i's liabilities towards bankrupt banks. Stacking over defaulting banks gives $\boldsymbol{\pi}_{D\times B}\mathbb{1}_B$, the vector of relative liabilities of banks in D to banks in B. Thus equation (30) writes as $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}\widehat{\mu}_D = \boldsymbol{\pi}_{D\times B}\mathbb{1}_B$. This gives the following expression for $\widehat{\mu}_D$:

$$\widehat{\mu}_D = \boldsymbol{\pi}_{D\times B}\mathbb{1}_B + \boldsymbol{\pi}_{D\times D}\boldsymbol{\pi}_{D\times B}\mathbb{1}_B + \boldsymbol{\pi}^{(2)}_{D\times D}\boldsymbol{\pi}_{D\times B}\mathbb{1}_B ... + \boldsymbol{\pi}^{(p)}_{D\times D}\boldsymbol{\pi}_{D\times B}\mathbb{1}_B ... \tag{31}$$

Each term in the sum corresponds to the amounts received by bankrupt banks following an increase in the net worth values of defaulting banks, either directly (for the first term) or indirectly through a chain of p defaulting banks (for the $p+1$-th term). Let defaulting bank i receive an additional unit of cash. The unit is entirely used for reimbursement. Each bankrupt bank k receives π_{ik}, thereby generating a direct total flow into bankrupt banks equal to $\sum_{k\in B}\pi_{ik}$. This term is the i-th component of the vector $\boldsymbol{\pi}_{D\times B}\mathbb{1}_B$, the first element in the sum on the right hand side of (31). Non bankrupt banks also receive additional payment, π_{ij} for j, and for those which are defaulting, they will pass this to their creditors: defaulting j pays an amount of $\pi_{ij}\sum_{k\in B}\pi_{jk}$ to the bankrupt banks. Hence there is a total of $\sum_{j\in D}\pi_{ij}\sum_{k\in B}\pi_{jk}$ reaching the bankrupt banks through an intermediary defaulting bank. This term is equal to the i-th component of $\boldsymbol{\pi}_{D\times D}\boldsymbol{\pi}_{D\times B}\mathbb{1}_B$, the second element in the sum on the right hand side of (31). Iterating, the amount received by the bankrupt banks after flowing through a chain of p defaulting banks is the i-th component of $\boldsymbol{\pi}^{(p)}_{D\times D}\boldsymbol{\pi}_{D\times B}\mathbb{1}_B$. Finally, the total amount received by bankrupt banks is obtained by summing over all p, which gives the right hand side of (31).

An alternative interpretation of expression (31) is in stochastic term. Interpret $\boldsymbol{\pi}$ as a transition matrix in which element π_{ij} is the probability of reaching j from i (by definition the sum $\sum_j \pi_{ij}$ is equal to 1). In this interpretation, the element i, j of the matrix $\boldsymbol{\pi}^{(p)}_{D\times D}$ is the probability of reaching j from i in p steps while staying all along in D, and the i-th component of the vector $\pi_{D\times B}\mathbb{1}_B = (\sum_{k\in B}\pi_{ik})_{i\in D}$ is the probability of reaching in one step an element of B from i. Thus the i-th element of $\boldsymbol{\pi}^{(p)}_{D\times D}\boldsymbol{\pi}_{D\times B}\mathbb{1}_B$, which is $\sum_j \pi^{(p)}_{D\times D}(i,j)(\sum_{k\in B}\pi_{ik})$, is the probability of reaching a bankrupt bank for the first time in $p+1$ steps starting from i and staying all along in D, that is before reaching a safe bank. Such an interpretation of $\boldsymbol{\mu}$ could be useful because it allows to rely on standard probability techniques.

Finally, we conclude with some remarks on the role of the liabilities and the loss to the outside creditors. The clearing mechanism minimizes this loss *given* the constraints associated to the liabilities. One interesting question is how this loss is affected by the relationships between banks. To address it, consider first a single aggregate bank, whose net worth is equal to the aggregate net worth, $\sum_i z_i$. Since liabilities within banks cancel out, its net worth coincides with its equity. Hence the aggregate bank is bankrupt if and only if the aggregate net worth is negative. In that case, the loss incurred by the non financial sector, $-\sum_i z_i$, is surely less than or equal to the loss in the disaggregated system. This follows from the aggregation formula, written as

$$-\sum_{i\in B} e_i(\theta) = \sum_{i\notin B} e_i(\theta) - \sum_i z_i$$

and the fact that $\sum_{i\notin B} e_i(\theta) \geq 0$. Similarly, if aggregate net worth is non-negative, the loss incurred by the non financial sector is null with a consolidated bank, but may be positive with separate banks if some are bankrupt.

Now the question is whether large enough liabilities allow the disaggregated system to achieve the overall minimal loss by facilitating the transfers across banks. Consider increasing gross liabilities without changing their net values, namely each ℓ_{ij} is increased by an identical amount. We conjecture that indeed the disaggregated system behaves as the aggregate one for a large enough increase. For example, if aggregate net worth is non-negative, the aggregate bank is not bankrupt, no single bank is bankrupt for large enough gross liabilities. Similarly, if aggregate net worth is negative, though surely some banks must be bankrupt, the loss incurred by the non-financial sector would be minimal equal to $-\sum_i z_i$.

5 Concluding remarks

This work represents a contribution to our understanding of the impact of interbank liabilities. It accomplishes two tasks: first, it introduces a clearing mechanism between financial institutions which may be indebted towards the non-financial sector and go bankrupt; second, it defines a threat index which reflects the impact that each bank has on the overall debt repayments. The analysis is at an ex post stage when the payoffs stemming from the activities of the banks with entities outside the financial system are realized. Given the banks' net worth values resulting from these payoffs, the clearing mechanism determines simultaneously the repayments of the banks to other banks and the possible losses to the outside creditors in a way consistent with three requirements. A clearing vector is shown to exist and conditions for its uniqueness are provided. The threat index of a bank computes how a modification of its net worth, say through cash injection, modifies the (weighted) overall repaid amounts, both within the banks and towards the non-banking sector. The threat indices may substantially differ from the default levels. As a result, injecting cash into the banks that appear the weakest, those with the largest default ratio, may be sub-optimal.

The threat index reflects an externality imposed by a defaulting bank on the debt repayments of all other banks. While the default level of a bank depends on its assets and the safety of its debtors, its threat index depends on its liabilities and the safety of its creditors. A bank thus may not assess properly the externality it will impose on the system when it decides on its interbank relationships, since it is concerned with the safety of its debtors and not with that of its creditors. This raises the issue of which regulatory tools could help in improving incentives. Such an issue should be addressed by taking an ex ante perspective, in a model in which the liabilities and the investment decisions, which generate future net worth levels, are chosen.

6 Proofs

Proof of Proposition 1 To prove the existence of a clearing ratio vector, let us consider the following set of ratio vectors, called the feasible set. A vector $\boldsymbol{\theta}$ in $[0,1]^n$ is said to be *feasible* if it satisfies $\theta_i \ell_i^* \leq \max(a_i(\boldsymbol{\theta}), 0)$ for each bank i. The limited liability and the bankruptcy conditions are satisfied at a feasible vector, but not necessarily creditors' priority : for a positive $a_i(\boldsymbol{\theta})$, feasibility only requires $\theta_i \ell_i^* \leq a_i(\boldsymbol{\theta})$ so that the ratio θ_i may be strictly lower than 1 and at the same time equity be strictly positive: the ratio is 'too low'. Creditors' priority however is satisfied for all banks at a maximal element of the feasible set thanks to the monotonicity of asset values, as we now show.

By definition, a maximal element $\boldsymbol{\theta}^+$ is such that increasing a component makes the ratio vector infeasible. To show that creditors' priority is satisfied, we only need to consider banks whose net asset values are non-negative $a_i(\boldsymbol{\theta}^+) \geq 0$. Let θ_i^+ be strictly lower than 1 and prove that i's equity is null. Recall that increasing i's ratio does not affect its asset value and can only increase other banks' asset values. Thus an increase in ratio θ_i^+ makes the vector infeasible only if $\theta_i^+ \ell_i^*$ is equal to $\max(a_i(\boldsymbol{\theta}^+), 0)$, which writes $\theta_i^+ \ell_i^* = a_i(\boldsymbol{\theta}^+)$ since $a_i(\boldsymbol{\theta}^+) \geq 0$. So i's equity is null: creditor's priority is satisfied. This proves that $\boldsymbol{\theta}^+$ is a clearing vector.

We prove that there is a greatest feasible vector. As it is maximal, this implies that it is a clearing ratio vector, which can only be the greatest one. The monotonicity of asset values implies that feasible ratio vectors are complements, in the sense that taking the maximum component by component of two feasible vectors yields a feasible vector. Let $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ be both feasible and $\boldsymbol{\theta} \vee \boldsymbol{\theta}' = (\max(\theta_i, \theta_i'))$ their supremum. By monotonicity, $a_i(\boldsymbol{\theta} \vee \boldsymbol{\theta}')$ is at least as large as each of the values $a_i(\boldsymbol{\theta})$ and $a_i(\boldsymbol{\theta}')$. Feasibility follows: $\max(a_i(\boldsymbol{\theta} \vee \boldsymbol{\theta}'), 0) \geq \max(a_i(\boldsymbol{\theta}), 0) \vee \max(a_i(\boldsymbol{\theta}'), 0) \geq (\theta_i \vee \theta_i')\ell_i^*$.

We now prove that, in case of multiple clearing ratio vectors, the net equity of each bank is the same at each one. It suffices to compare the values of net equity at a clearing ratio vector $\boldsymbol{\theta}$ with those at the greatest clearing ratio vector $\boldsymbol{\theta}^+$ and show $e_i(\boldsymbol{\theta}^+) = e_i(\boldsymbol{\theta})$ for each i.

Thanks to the aggregation formula (8), the sums of the net equity values over all banks are equal to aggregate net worth whatever repayment ratio vector, so $\sum_i e_i(\boldsymbol{\theta}^+) = \sum_i e_i(\boldsymbol{\theta})$. Thus to prove that net equities are equal, it is enough to show

$$e_i(\boldsymbol{\theta}^+) \geq e_i(\boldsymbol{\theta}) \text{ for each } i. \tag{32}$$

Let us consider two cases depending on the value θ_i. Recall that $e_i(\boldsymbol{\theta}^+) = a_i(\boldsymbol{\theta}^+) - \theta_i^+ \ell_i^*$ and similarly at $\boldsymbol{\theta}$. Also note that $\boldsymbol{\theta}^+ \geq \boldsymbol{\theta}$ implies $a_i(\boldsymbol{\theta}^+) \geq a_i(\boldsymbol{\theta})$ for each i.

For $\theta_i = 1$, θ_i^+ is also equal to 1, thus inequality $e_i(\boldsymbol{\theta}^+) \geq e_i(\boldsymbol{\theta})$ follows from the inequality $a_i(\boldsymbol{\theta}^+) \geq a_i(\boldsymbol{\theta})$ and $\theta_i^+ = \theta_i$.

For $0 \leq \theta_i < 1$, i's net equity under $\boldsymbol{\theta}$ is non-positive, $e_i(\boldsymbol{\theta}) \leq 0$. If $\theta_i^+ > 0$, i's net equity under $\boldsymbol{\theta}^+$ can only be non-negative (because of the bankruptcy condition), so surely $e_i(\boldsymbol{\theta}^+) \geq e_i(\boldsymbol{\theta})$. If $\theta_i^+ = 0$, it must be that $\theta_i = 0$ as well; in that case net equities are given by the net asset values, hence again $e_i(\boldsymbol{\theta}^+) \geq e_i(\boldsymbol{\theta})$.

Point *2* has been proved in the text. ■

It is useful for the sequel to prove the following two lemmas.

Lemma A1 *Given a ratio $\boldsymbol{\theta}$ and a nonempty subset T of N, the following equality holds*

$$\sum_{i\in T} e_i(\boldsymbol{\theta}) = \sum_{i\in T} z_i + \sum_{i\in T, j\notin T} \theta_j \ell_{ji} - \sum_{i\in T, j\notin T} \theta_i \ell_{ij}. \tag{33}$$

Proof The proof is trivial by summing net equities values over T since the payments within T cancel out. ■

Formula (33) says that the aggregate net equity of the banks in T is equal to their aggregate net worth plus the net payment from banks outside T, i.e., the difference between the payments received by T from $N-T$ and those made by T to $N-T$.

Lemma A2 *Let T be a nonempty subset of N such that each i in T has null equity and positive net worth: $e_i(\boldsymbol{\theta}) = 0$ and $z_i > 0$. Then T is not the whole set N and has a creditor in $N-T$.*

Proof By contradiction, let T have no outside financial creditor: $\ell_{ij} = 0$ for each i in T and j not in T. (33) applied to T at $\boldsymbol{\theta}$ implies $\sum_{i\in T} e_i(\boldsymbol{\theta}) = \sum_{i\in T} z_i + \sum_{i\in T, j\notin T} \theta_j \ell_{ji} = 0$, in contradiction with each z_i strictly positive. ■

Proof of Proposition 2 Let us prove the uniqueness of a clearing vector under the assumptions stated in the proposition. By contradiction, let $\boldsymbol{\theta}$ be a clearing ratio vector distinct from the greatest one $\boldsymbol{\theta}^+$ and define T as the non-empty set of banks i for which $\theta_i^+ - \theta_i > 0$.

We know that the safe banks (those with a positive net equity) and the bankrupt banks (those with a negative net equity) coincide at both $\boldsymbol{\theta}^+$ and $\boldsymbol{\theta}$, hence their ratios coincide as well, respectively equal to 1 or 0 (see equation (10)); also the ratios coincide for non-indebted banks since by convention they are either 1 for those with a non-negative asset value or 0 otherwise. Hence all banks in T are indebted and have a null net equity.

We show that no bank in T has a creditor outside T. Apply (33) to the clearing vectors $\boldsymbol{\theta}^+$ and $\boldsymbol{\theta}$. The equity values coincide, as well as the received payments by T from $N-T$; hence the repayments made by T must coincide as well at the two ratios. Formally, $e_i(\boldsymbol{\theta}^+) = e_i(\boldsymbol{\theta})$ for each i in T and $\theta_j^+ = \theta_j$ for each j in $N-T$ imply

$$\sum_{i\in T, j\notin T} \theta_i^+ \ell_{ij} = \sum_{i\in T, j\notin T} \theta_i \ell_{ij}.$$

Since by definition of T, $\theta_i^+ > \theta_i$ for each i in T, the above equation implies $\ell_{ij} = 0$ for each i in T and j not in T: no bank in T has a creditor outside T.

Thus, T is a non-empty subset of N composed of banks which have null equity, are indebted, and have no financial creditors outside T. This is impossible in case (a) where $\boldsymbol{z} \gg \mathbf{0}$, by applying Lemma A2. In case (b), observe that T is not the whole set N because aggregate net worth is not null: either it is positive and there are some safe banks, or it is negative and there are bankrupt banks (by Proposition 1). This gives the desired contradiction. ■

Proof of Lemma 1 Let D be the set of defaulting banks at a clearing ratio. To show that $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}$ is invertible with an inverse given by the infinite sum $\mathbb{I}_{D\times D}+\boldsymbol{\pi}_{D\times D}+\boldsymbol{\pi}^{(2)}_{D\times D}+...+\boldsymbol{\pi}^{(p)}_{D\times D}+....$, we prove that an iterate of the matrix $\boldsymbol{\pi}_{D\times D}$ has all its rows totals smaller than 1: $\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D \ll \mathbb{1}_D$. The result then follows from standard results on productive matrices: the spectral radius of $\boldsymbol{\pi}^{(p)}_{D\times D}$ is strictly smaller than 1 hence also that of $\boldsymbol{\pi}_{D\times D}$.

Each bank in D has null equity, so, from Lemma A2, each subset of D has an outside creditor. Interpret $\boldsymbol{\pi}$ as a transition matrix in which element π_{ij} is the probability of reaching j from i. The (i,j) element of the matrix $\boldsymbol{\pi}^{(q)}_{D\times D}$ gives the probability of reaching j from i in q steps along a path included in D. Hence the sum $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)$ is the probability of the paths of length q that start from i and are included in D. Such a sum is non-increasing in q since a path included in D of length $q+1$ has necessarily its first q elements included in D.[13] So, once the inequality $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)<1$ holds for q it holds for all larger values than q. Thus, $\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D \ll \mathbb{1}_D$ holds if for each i in D there is q for which $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)<1$.

By contradiction, assume that for some i in D we have $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)=1$ for each q. All the paths starting from i are included in D. Let C be composed with all the elements that can be reached from i. By construction, C has no outside creditor and is included in D, hence all its elements have null equity. Applying Lemma A2 gives the desired contradiction. ■

Proof of Proposition 3 Recall that $V(\boldsymbol{z})$ is the value of the program $\mathcal{P}$. First assume all banks to be indebted. Writing the constraint $\theta_i\le 1$ as $\theta_i\ell^*_i\le\ell^*_i$, the program $\mathcal{P}$ is equivalent to $\mathcal{P}'$:

$$\mathcal{P} \quad : \quad \max_{\boldsymbol{\theta}}\sum_i \ell^*_i\theta_i$$

$$0\le\theta_i\le 1 \text{ for each } i \tag{34}$$

$$\theta_i\ell^*_i-\sum_j\theta_j\ell_{ji}\le z_i \text{ for each } i \tag{35}$$

The derivative of V with respect to z_i is given by the multiplier associated to the i-th constraint (35) when the multiplier is unique.

The program $\mathcal{P}$ has a finite solution: the feasible set is non-empty (it contains $\boldsymbol{\theta}=\mathbf{0}$ since $\boldsymbol{z}$ is positive) and is compact. From well known results on linear programming, the multipliers associated

[13] Formally $\sum_{j\in D}\pi^{(q+1)}_{D\times D}(i,j) = \sum_{j\in D}\pi^{(q)}_{D\times D}(i,k)\sum_{k\in D}\pi(k,j)$ which is equal, exchanging sums, to $\sum_{k\in D}\pi^{(q)}_{D\times D}(i,k)(\sum_{j\in D}\pi(k,j))$ Since the term in bracket $\sum_{j\in D}\pi(k,j)$ is not larger than 1 for each k, we finally obtain $\sum_{j\in D}\pi^{(q+1)}_{D\times D}(i,j)\le\sum_{k\in D}\pi^{(q)}_{D\times D}(i,k)$, the desired inequality.

to the constraints are the solutions to the dual program of $\mathcal{P}$, and furthermore, the values of the primal and dual coincide. We first show that the dual program of $\mathcal{P}$ is

$$\mathcal{D} \quad : \min_{(\lambda,\mu)\geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^*$$

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j \geq 0 \text{ for each } i. \tag{36}$$

and furthermore that the constraints of the dual (36) are binding. To show this, recall that the dual of $\max \ell^* \cdot \boldsymbol{\theta}$ under $A\boldsymbol{\theta} \leq b$, $\boldsymbol{\theta} \geq 0$ is $\min b \cdot \boldsymbol{\gamma}$ under $A^t\boldsymbol{\gamma} \geq \ell^*$, and $\boldsymbol{\gamma} \geq 0$. Apply this to $\mathcal{P}$ with A is the $2n \times n$ matrix and b is the $2n$-vector

$$A = \begin{pmatrix} dg(\ell^*) - \ell^t \\ dg(\ell^*) \end{pmatrix}, \; b = \begin{pmatrix} \boldsymbol{z} \\ \ell* \end{pmatrix}$$

Writing the $2n$-vector $\boldsymbol{\gamma}$ as $\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix}$, the objective of the dual is to minimize $\sum_i z_i\mu_i + \sum_i \lambda_i\ell_i^*$ under the constraints

$$\begin{pmatrix} dg(\ell^*) - \ell & dg(\ell^*) \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} \geq \ell^*.$$

Spelling out the i-th constraint of the dual yields $\ell_i^*\mu_i - \sum_j \ell_{ij}\mu_j + \ell_i^*\lambda_i \geq \ell_i^*$ which is (36).

We now show that these constraints are binding:

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j = 0 \text{ for each } i. \tag{37}$$

By contradiction suppose $(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j > 0$ for some i. $\lambda_i + \mu_i$ must be strictly positive. If $\lambda_i > 0$, λ_i can be decreased without affecting the other constraints and the objective is decreased. If $\lambda_i = 0$, then μ_i must be strictly positive. A small decrease in μ_i is feasible because it relaxes the constraints (36) for the banks distinct from i and, by assumption, constraint (36) is not binding for i. A decrease in μ_i results in a decrease in the objective, a contradiction again.

Now, let S be the set of safe banks, for which (35) is strict. By the slackness conditions, $\mu_i = 0$ for i in S. Equation (37) immediately gives that their solidarity indices satisfy (21). Let us assume that there are no boundary banks. All banks that are not in S have a repayment ratio strictly smaller than 1. By the slackness conditions, their solidarity indices λ_i are null. Using $\mu_i = 0$ for i in S and $\lambda_i = 0$ for i in D, equations (37) write as $\mu_i\ell_i^* - \sum_{j\in D} \ell_{ij}\mu_j = \ell_i^*$: this proves (13). The fact that the system (13) has a unique solution, which is furthermore positive, follows from Lemma 1.

Now, assume that some banks are not indebted. They are surely safe and the program $\mathcal{P}$ is equivalent to $\mathcal{P}'$ defined by considering the indebted banks only, which gives the values of their threat indices. It then suffices to set the threat indices of non-indebted banks equal to 0. ∎

Proof of Proposition 5 First assume all banks to be indebted. The program $\mathcal{Q}_c$ is equivalent to $\mathcal{Q}'_c$:

$$\mathcal{Q}_c \quad : \quad \max_{\boldsymbol{\theta},\boldsymbol{\delta}\geq 0}[\sum_i \ell_i^*\theta_i] - c[\sum_i \delta_i]$$

$$\theta_i\ell_i^* \leq \ell_i^* \text{ for each } i \quad (34)$$

$$\theta_i\ell_i^* - \sum_j \theta_j\ell_{ji} \leq z_i + \delta_i \text{ for each } i \quad (27)$$

Applying the duality theorem as in the proof of Proposition 3, the dual is

$$\mathcal{D}_c \quad : \min_{(\lambda,\mu)\geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^*$$

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j \geq 0 \text{ for each } i \tag{38}$$

$$\mu_i \leq c \text{ for each } i. \tag{39}$$

From well known results, if $(\boldsymbol{\theta}, \boldsymbol{\delta})$ and $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are feasible vectors respectively for the primal and the dual and they satisfy the complementary slackness conditions, each one is a solution respectively for the primal and the dual. The slackness conditions are

$$\begin{cases} (a) & \theta_i = 0 & \text{or} & (\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j = 0 \\ (b) & \theta_i = 1 & \text{or} & \lambda_i = 0 \\ (c) & \delta_i = 0 & \text{or} & \mu_i = c \\ (d) & \delta_i + e_i(\boldsymbol{\theta}) = 0 & \text{or} & \mu_i = 0 \end{cases} \tag{40}$$

Let $\boldsymbol{\theta}$ be a clearing ratio vector, and S, D, and B be respectively the sets of safe, defaulting, and bankrupt banks. By assumption S is non-empty. Let $\boldsymbol{\delta}$ represent the loss vector to outside creditors: δ_i is null for i not in B and $\delta_i = -e_i(\boldsymbol{\theta})$ for i in B. To prove that, for c large enough, $(\boldsymbol{\theta}, \boldsymbol{\delta})$ is a solution to $\mathcal{Q}_c$, we display feasible multipliers so that the complementary slackness are satisfied. For elements in B and S, the slackness conditions are trivially satisfied as follows:

for each i in B, set $\mu_i = c$, $\lambda_i = 0$;

for each i in S, set $\mu_i = 0$ and λ_i so that $\ell_i^*\lambda_i = \ell_i^* + \sum_j \ell_{ij}\mu_j$ once each μ_j outside S is specified.

For i in D, both δ_i and $e_i(\boldsymbol{\theta})$ are null so (c) and (d) are satisfied. Take $\lambda_i = 0$ so that (b) holds as well. Now to have (a), dividing by the total liabilities, it suffices to prove that there are μ_i not greater than c that satisfy the linear system for $\boldsymbol{\mu}_D$

$$\mu_i - \sum_{j\in D} \pi_{ij}\mu_j = 1 + c\sum_{k\in B} \pi_{ik}, \text{ for each } i \in D$$

or in matrix form

$$[\mathbb{I} - \boldsymbol{\pi}]_{D\times D}\boldsymbol{\mu}_D = \mathbb{1}_D + c\boldsymbol{\pi}_{D\times B}\mathbb{1}_B. \tag{41}$$

The matrix $[\mathbb{I} - \boldsymbol{\pi}]_{D\times D}$ is invertible thanks to the assumptions: D is not the whole set N since there is a safe bank, and the irreducibility assumption implies that $[\mathbb{I} - \boldsymbol{\pi}]_{D\times D}$ is invertible for any strict subset of N. Thus (41) defines $\boldsymbol{\mu}_D$ in a unique way. Dividing by c we have

$$\frac{1}{c}\boldsymbol{\mu}_D = [\mathbb{I} - \boldsymbol{\pi}]^{-1}_{D\times D}[\frac{1}{c}\mathbb{1}_D + \boldsymbol{\pi}_{D\times B}\mathbb{1}_B].$$

This implies that $\frac{1}{c}\boldsymbol{\mu}_D$ tends to $\widehat{\boldsymbol{\mu}}_D = [\mathbb{I} - \boldsymbol{\pi}]^{-1}_{D\times D}\boldsymbol{\pi}_{D\times B}\mathbb{1}_B$ as c increases. Hence each component of $\boldsymbol{\mu}_D$ is less than c for c large enough if each component of $\widehat{\boldsymbol{\mu}}_D$ is strictly smaller than 1. Interpret $\boldsymbol{\pi}$ as a transition matrix in which element π_{ij} is the probability of reaching j from i. As seen in the text, the i-component of $\widehat{\boldsymbol{\mu}}_D$ gives the probability of reaching a bank in B from i *before* reaching a bank in S. Under the irreducibility assumption this probability is strictly smaller than 1, this ends the proof.

Now, assume that some banks are not indebted. The above argument then extends as follows. The program $\mathcal{Q}_c$ is independent of their ratios, so that $\mathcal{Q}_c$ is equivalent to $\mathcal{Q}'_c$ where $\boldsymbol{\theta}$ specifies the ratios for the indebted banks and the constraints (26) are required only for them. Thus, for a non-indebted bank i, there is no multiplier λ_i and the slackness conditions (a) and (b) are not required. Recall that at a clearing ratio, its ratio is set to 1 if its net asset value is non-negative and to 0 if its is negative. One easily checks that treating the bank as a safe bank with $\mu_i = 0$ in the first case and as a bankrupt bank with $\mu_i = c$ in the second case, the slackness conditions (c) and (d) hold. ■

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