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Nicolas Coeurdacier, Hélène Rey
and Pablo Winant

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Nicolas Coeurdacier, SciencesPo and CEPR
Hélène Rey, London Business School and CEPR
Pablo Winant, Paris School of Economics

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Centre for Economic Policy Research
77 Bastwick Street, London EC1V 3PZ, UK
Tel: (44 20) 7183 8801, Fax: (44 20) 7183 8820
Email: cepr@cepr.org, Website: www.cepr.org

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ABSTRACT

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Nicolas Coeurdacier
Department of Economics
London Business School
Regent's Park
London NW1 4SA

Email: ncoeurdacier@london.edu

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Hélène Rey
Department of Economics
London Business School
Regent's Park
London NW1 4SA

Email: hrey@london.edu

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Pablo Winant
Paris Jourdan Sciences Economiques
48 Boulevard Jourdan
75014 Paris
FRANCE

Email: pablo.winant@gmail.com

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The Risky Steady-State

Nicolas Coeurdacier
SciencesPo and C.E.P.R.

Hélène Rey
London Business School, C.E.P.R.

Pablo Winant
Paris School of Economics

May 18, 2011

Abstract

We propose a simple quantitative method to linearize around the risky steady state of a small open economy. Unlike when the deterministic steady state is used, the net foreign asset position is well defined. We allow for both stochastic income and stochastic interest rate.

1 Approximation around a risky steady-state

1.1 The risky steady-state

It is common practice in dynamic macroeconomics to consider the limit behavior of the economy when agents do not anticipate the effect of future shocks. This approximation is referred to as the perfect foresight path of the economy.

The corresponding equilibrium is called the deterministic steady-state. To take optimal decisions rational agents observe the gap with the steady-state values and choose a decision rule which maximizes intertemporal utility of returning to the steady-state.

By contrast, risk-averse agents are aware of the existence of future shocks hitting the economy. As a result, they anticipate the convergence of economic variables to some stochastic steady-state, which is defined as the ergodic distribution of these variables. The properties of this ergodic distribution are mathematically much more challenging than the deterministic steady-state of the perfect foresight case.

In order to avoid these difficulties and to restore some intuition about the convergence behavior of the economy, we propose to define a risky-steady state as follows. The risky steady-state is the point where agents choose to stay at a given date if they expect future risk and if the realization of shocks is 0 at this date.[1] More formally, given a decision rule $Y_t = g^r\left(Y_{t-1}, \epsilon_t\right)$ defining optimal

[1] A similar concept has been introduced in Juillard and Kamenik [2005].

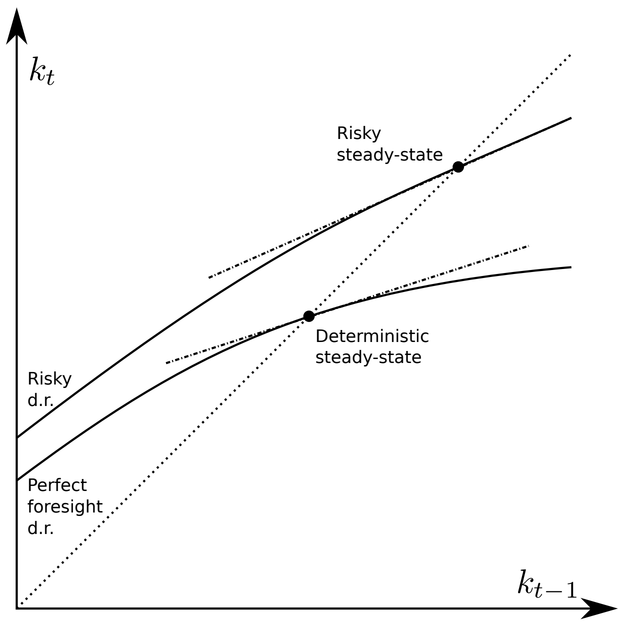

Figure 1: Decision rules for capital accumulation with and without expected shocks.

decisions for states Y_{t-1} and shocks ϵ_t, the risky steady-state satisfies:

$$\bar{Y} = g^r\left(\bar{Y}, 0\right)$$

Throughout the paper, for any variable u, we denote by $\bar{u}$ its value at the risky steady-state. As its name suggests, the risky steady-state incorporates information about future expected risk and corresponding optimal decisions. Consider for instance a standard stochastic neoclassical growth model where anticipated volatility leads to precautionary capital accumulation (see Phelps [1962] and Mirrlees [1965]). The level of the stock of capital is higher in the risky steady-state than in the deterministic steady-state, as shown in figure 1.

1.2 Linear approximation

Most standard dynamic macroeconomics problems can be summarized by a function f and a process of random innovations (ϵ_t) with covariance matrix Σ. The solution is a process (Y_t) such that

$$E_t\left[f\left(Y_{t+1}, Y_t, Y_{t-1}, \epsilon_t\right)\right] = 0 \qquad (1)$$

The local behavior of an economic model around the deterministic steady-state is well known (see Kim et al. [2008] and Schmitt-Grohé and Uribe [2004]).

Under the assumption that shocks are small enough, the perturbation approach consists in finding the deterministic steady-state Y^* such that $f(Y^*, Y^*, Y^*) = 0$, then to compute a Taylor expansion of the perfect-foresight path, and finally, to correct for the presence of expected risk. Two great advantages of this methodology are the fact that it uses only derivatives of f taken at Y^*, to construct the approximations and the the fact that it involves only linear algebra operations.

Nevertheless, if the deterministic steady-state, or the perfect foresight path is not properly defined, this method will fail. As we show below, it is the case in a small open economy model where equilibrium wealth is not defined (see Schmitt-Grohé and Uribe [2003]), or in portfolio choice problems for which portfolios are indeterminate in the deterministic steady-state and along the perfect foresight path (see Devereux and Sutherland [2010] and Tille and van Wincoop [2010]).

For this reason, we are interested in characterizing directly the local behavior around the risky steady-state. As it implies a joint approximation of the steady-state and of the dynamic properties, it can be referred to as an approximation around the risky steady-state. We propose in this sub-section a simple way to build an approximation. In the next section we study some properties of this simple solution.

Let us assume for simplicity that some exogenous variables X_t follow an AR(1) process: $X_t = \rho_X \left(X_{t-1} - \bar{X}\right) + \epsilon_t$

The endogenous variables $Y_t$follow the decision rule

$$Y_t = g(Y_{t-1}, X_t) = g(S_t)$$

where the state-space is $S_t = (Y_{t-1}, X_t)$.

Abbreviating $(X_{t+1}, Y_{t+1}, X_t, Y_t, X_{t-1}, Y_{t-1})$ by V_{t+1} we need to solve the optimality condition

$$E_t\left[f(V_{t+1})\right] = 0$$

which has the same dimension as vector Y_t, provided that we have one Euler equation for each control variable.

In order to take risk into account we replace this original equation by its second order-expansion Φ around expected future variables :

$$\begin{aligned} \Phi &= f(E_t V_{t+1}) + \frac{1}{2} E_t \left[f''.\left[V_{t+1} - E_t V_{t+1}\right]^2 \right] \\ &= 0 \end{aligned} \tag{2}$$

where second order derivatives are taken at point $E_t V_{t+1}$.

Our strategy consists in postulating a linear decision rule for Y_t around the unknown risky steady-state $\bar{Y}$:

$$Y_t = \bar{Y} + R_Y\left(S_t - \bar{S}\right)$$

and to identify the risky steady-state $\bar{Y}$ and the coefficients R_Y jointly.

This is done by solving numerically the two following local conditions:

$$\Phi\left(\bar{S}\right) = 0$$
$$\frac{\partial \Phi}{\partial S_t} = 0$$

The intuition on these conditions will be more easily understood in the next section example. The condition $\Phi\left(\bar{S}\right) = 0$ characterizes the risky-steady state. It is analogous to the condition $f\left(Y^*, Y^*, Y^*\right) = 0$ defining the deterministic steady-state.

2 Intertemporal consumption decisions in a small open economy

In this section we consider a simple model of intertemporal consumption choice in a small open economy. Consider a representative agent maximizing the following life-time utility function: $U = \sum_0^\infty \frac{c_t^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$.
We assume that this agent receives an endowment process y_t and can save an amount w_t at an exogenous world interest rate r_t according to the budget constraint:

$$c_t = y_t + w_{t-1} r_t - w_t$$

The exogenous process (y_t, r_t) is specified as:

$$\log\left(\frac{y_t}{\bar{y}}\right) = \rho_y \log\left(\frac{y_{t-1}}{\bar{y}}\right) + \epsilon_t$$
$$\log\left(\frac{r_t}{\bar{r}}\right) = \rho_r \log\left(\frac{r_{t-1}}{\bar{r}}\right) + \eta_t$$

where (ϵ_t, η_t) is an i.i.d. normal process with covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_y^2 & \zeta\sigma_r\sigma_y \\ \zeta\sigma_r\sigma_y & \sigma_r^2 \end{bmatrix}$$

.

From the maximization program, we can derive the usual Euler equation :

$$\beta E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} r_{t+1}\right] = 1$$

The deterministic steady-state (c^*, w^*) would be defined by :

$$c^* = \bar{y} + w^*\left(\bar{r} - 1\right)$$
$$\beta\bar{r} = 1$$

These two equations do not define equilibrium values c^* and w^* uniquely. Instead, they imply a counterfactual relation between two independent structural parameters β and $\bar{r}$. This does not imply that the original model is not valid, but it indicates a limitation of the usual perturbation approach.

As we will show, it is still possible to get an approximation of the solution if we compute the risky steady-state and the coefficient for the wealth dynamics at the same time. Optimal net foreign assets w_t can be written as a function of the state space (w_{t-1}, y_t, r_t). Let us postulate a linear relation:

$$w_t = \bar{w} + W_w \hat{w}_{t-1} + W_y \hat{y}_t + W_r \hat{r}_t \tag{3}$$

In this equation $\bar{w}$ is the unknown steady-state value for net foreign holdings, $(\hat{w}_{t-1}, \hat{y}_t, \hat{r}_t)$ the deviations from the risky this value and (W_w, W_y, W_r) three coefficients to be determined. The The Euler equation equivalent of equation above can be approximated similarly to equation (2) to yield:

$$\begin{aligned} \frac{1}{\beta}\left(\frac{E_t\left[c_{t+1}\right]}{c_t}\right)^{\gamma} &= E_t\left[r_{t+1}\right]\left(1+\frac{\gamma(\gamma+1)}{2}\frac{Var_t\left(c_{t+1}\right)}{E_t\left[c_{t+1}\right]^2}\right) \\ &\quad -\gamma\frac{Cov_t\left(c_{t+1}, r_{t+1}\right)}{E_t\left[c_{t+1}\right]} \end{aligned} \tag{4}$$

Evaluated at the risky steady state, this equation becomes:

$$\frac{1}{\beta} = \bar{r}\left(1+\gamma(\gamma+1)\frac{\overline{var_t}\left(c_{t+1}\right)}{\bar{c}^2}\right) - \gamma\frac{\overline{cov_t}\left(c_{t+1}, r_{t+1}\right)}{\bar{c}} \tag{5}$$

where $\overline{cov_t}\left(u_{t+1}, v_{t+1}\right) = cov_t\left(u_{t+1}, v_{t+1} | u_t = \bar{u}, v_t = \bar{v}\right)$ and $\overline{var}_t\left(u_{t+1}\right) = \overline{cov}_t\left(u_{t+1}, u_{t+1}\right)$ denote conditional second order moments evaluated at the risky steady-state.

In the absence of risk the return on investment must be equal to the inverse of time preference. But a foreign asset whose returns are positively correlated with consumption is less able to provide consumption smoothing which is reflected in the risk-premium term $\gamma\frac{\overline{cov_t}(c_{t+1}, r_{t+1})}{\bar{c}}$. The second term $\gamma(\gamma+1)\frac{\overline{var_t}(c_{t+1})}{\bar{c}^2}$ comes from precautionary savings. It denotes the desire to save more when the variance of consumption growth is higher.

2.1 Precautionary saving

In order to disentangle the two effects, let assume first that financial returns are riskless. In this case, the covariance term disappears from the dynamic and from the static equations ((4) and (5)). This case has been studied in Clarida [1987] and Carroll [2001] which have proved that a solution exists if and only if $\beta r < 1$. This condition is also a necessary condition implied by equation (5).

Carroll [2001] describes how poor agents accumulate wealth until they reach a satisfying level of precautionary savings and then use the riskless asset to soften fluctuations in their income.

σ_y	$\bar{w}$	W_w	W_y	$\bar{c}$	$\sigma(c)$
0.01	-14.878	0.999	0.546	0.4	0.005
0.025	-0.0	0.999	0.546	1.0	0.011
0.05	24.796	0.999	0.546	2.0	0.023

Table 1: Decision rules for different levels of risk
Solution is computed with $\beta = 0.96$, $\gamma = 2.0$, $\bar{y} = 1.0$, $\rho_y = 0.95$, $\zeta = 0$ and $\bar{r} = \frac{1}{\beta} - 0.0013$ (such that $\bar{w} = 0$ with $\sigma_y = 0.025$).
By construction W_w is the biggest eigenvalue of the decision rule.

Using the budget equation we see that the variance of consumption is exactly given by :

$$\begin{aligned} Var_t\left(c_{t+1}\right) &= Var\left(y_{t+1} - w_{t+1}\right) \\ &= \left(1 - W_y\right)^2 \sigma_y^2 \end{aligned} \tag{6}$$

which is a constant given our first order decision rule (equation (3)). When expected consumption $E_t(c_{t+1})$ and wealth increase following a positive income shock, the desire to hold precautionary savings decreases and the riskless asset is less desirable (equation 4). It acts as a stabilizing force pushing consumption towards a target level. A similar point was also made by Daniel [1997] in a two-countries settings with riskless interest rates: they noted the formal similarity between equation (4) without the covariance term and Euler equations arising with Uzawa-type preferences.

As a result the evolution law of the state space (w_{t-1}, y_t, r_t) has only stable eigenvalues (for $\beta\bar{r} < 1$, see Carroll [2001] for a proof). Note that this result contradicts the common belief in small open economy applications that consumption follows a unit-root and net foreign asset positions are non-stationary. Various tools have been used in this literature to make the problem stationary (see Ghironi [2006] and Schmitt-Grohé and Uribe [2004]). We show here that the non-stationarity is an artefact of the approximation around a deterministic steady-state instead of the risky one.

Precautionary motives induce a well defined risky steady-state for net foreign assets which, in our example, depends on the level of aggregate income risk in the economy. Riskier countries will tend to accumulate more wealth than safer ones. Table 1 shows the decision rules for various levels of income risk. Nevertheless, small changes in the aggregate risk have a strong impact on the risky steady-state for net foreign assets (see column 1 in table 1), a feature that will disappear if foreign assets are risky.

σ_y	$\bar{w}$	W_w	W_y	$\bar{c}$	$\sigma(c)$
0.01	-0.161	0.944	0.516	0.995	0.032
0.025	0.0	0.945	0.52	1.0	0.032
0.05	0.607	0.95	0.537	1.017	0.034

Table 2: Decision rules for different levels of risk
Solution is computed with $\beta = 0.96$, $\gamma = 2.0$, $\bar{y} = 1.0$, $\rho_y = \rho_r = 0.9$, $\sigma_r = 0.025$, $\zeta = 0$ and $\bar{r} = \frac{1}{\beta} - 0.014$ (such that $\bar{w} = 0$ with $\sigma_y = 0.025$)

2.2 Stochastic world interest rate

We now extend the model with precautionary savings à la Carroll [2001] to allow for stochastic world rate of returns.[2] When foreign assets are risky, an additional stabilizing force on the consumption path is at work: following positive income shocks, agents will increase their stock of foreign assets. This will increase the covariance of their consumption with the world stochastic interest rate (term $\gamma \frac{Cov_t(c_{t+1}, r_{t+1})}{E_t[c_{t+1}]}$) in equation (4) and reduce the demand for foreign assets. This stabilizing mechanism reduces the persistence of shocks on net foreign assets compared to the non stochastic case.

As shown in column 3 of table 2, the wealth dynamics moves further away from having a unit-root. Note again that net foreign assets would be non-stationary in standard approximation methods around the deterministic steady-state. Moreover with stochastic returns, small changes in aggregate risk do not translate anymore in a large dispersion of net foreign assets at the risky steady-state (see column 2 in table 2). This has also to do with the covariance term of equation (4) but evaluated at the risky-steady state: having a large amount of foreign assets at the risky steady-state is less desirable as this increases the covariance of consumption with the world rate of return.

3 Conclusion

We develop a new way of approximating standard dynamic stochastic macroeconomics models by solving simultaneously for a linear dynamics of state variables **and** the risky steady-state. The risky steady-state is the equilibrium at which state variables stay constant in presence of expected future shocks but when the innovations for these shocks turn out to be zero.

We study the properties of this approximation around the risky steady-state in a small open economy model of intertemporal consumption decisions with stochastic incomes and stochastic world interest rates. Contrary to standard approximation around the deterministic steady-state, net foreign assets are well defined at the risky steady-state and are stationary.

We believe that such a method can be applied more broadly to models

[2] Chamberlain and Wilson [2000] studies a similar problem and shows under which conditions on structural parameters a bounded solution for the consumption path exists.

involving portfolio decisions where standard perturbation methods have shown some limitations. Moreover, the welfare implications for risk-sharing can be quite different in these types of models since uncertainty directly affects steady-state variables.

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