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ABSTRACT

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We study a sequential all-pay auction with two contestants who are privately informed about a parameter (ability) that affects their cost of effort. In the model, contestant 1 (the first mover) exerts an effort in the first period which translates into an observable output but with some noise, and contestant 2 (the second mover) observes this noisy output. Then, contestant 2 exerts an effort in the second period, and wins the contest if her output is larger than or equal to the observed noisy output of contestant 1; otherwise, contestant 1 wins. We study two variations of this model where contestant 1 either knows or does not know the realization of the noise before she chooses her effort. Contestant 2 does not know the realization of the noise in both variations. For both variations, we characterize the subgame perfect equilibrium and investigate the effect of a random noise on the expected highest effort in this contest.

JEL Classification: D44
Keywords: noisy outputs and sequential contests

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Sequential All-Pay Auctions with Noisy Outputs

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July 20, 2011

Abstract

We study a sequential all-pay auction with two contestants who are privately informed about a parameter (ability) that affects their cost of effort. In the model, contestant 1 (the first mover) exerts an effort in the first period which translates into an observable output but with some noise, and contestant 2 (the second mover) observes this noisy output. Then, contestant 2 exerts an effort in the second period, and wins the contest if her output is larger than or equal to the observed noisy output of contestant 1; otherwise, contestant 1 wins. We study two variations of this model where contestant 1 either knows or does not know the realization of the noise before she chooses her effort. Contestant 2 does not know the realization of the noise in both variations. For both variations, we characterize the subgame perfect equilibrium and investigate the effect of a random noise on the expected highest effort in this contest.

Keywords: Sequential contests, noisy outputs.

JEL classification: D44, O31, O32

1 Introduction

In an all-pay auction with a single prize, the contestant with the highest effort (output) wins the entire prize, but all the contestants bear the cost of their effort. In the economic literature, all-pay

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auctions are usually studied under complete information where each contestant's type (her valuation for winning or her ability) is common knowledge (see, for example, Hillman and Riley 1989; Leininger 1991; Baye et al. 1996; Che and Gale 1998, 2000; and Siegel 2009), or under incomplete information where each contestant's type is private information and only the distribution of the contestants' types is common knowledge (see, for example, Amman and Leininger 1996; Krishna and Morgan 1997; Gavious et al. 2003; Moldovanu and Sela 2001, 2006; and Moldovanu et al. 2010). In the all-pay auction with either complete information or incomplete information efforts translate deterministically into observable outputs such that the contestant who made the highest effort is also the one with the highest output and this contestant wins the contest. However, in real-life contests, the relationship between the contestant's effort and her observable output is usually not deterministic. Rather, it is frequently the case that there is some noise in the process that maps efforts into measured outputs. Contests with outputs which are not deterministically determined by efforts have received some attention in the literature. For example, Lazear and Rosen (1981) considered a contestant's output to be a stochastic function of the unobservable effort and the identity of the most productive agent to be determined by an external shock. This model is known in the literature as a rank-order tournament and was later extended and generalized by several authors, e.g., Green and Stokey (1983), Nalebuff and Stiglitz (1983), Rosen (1986), Krishna and Morgan (1998) and Akerlof and Holden (2008). The all-pay auction under complete information is actually the limiting case of the rank-order tournament when the noise approaches zero. In this paper, similarly to the rank-order tournament, we assume that the output is a stochastic function of the effort, but in contrast to the rank-order tournament model, we analyze sequential all-pay auctions under incomplete information. Thus the novelty of this paper lies in the fact that we combine incomplete information and noisy outputs in the same model.

The paper's outline is as follows. In Section 2 we present our model of a sequential all-pay auction with two contestants who are privately informed about a parameter (ability) that affects their cost of effort. Contestant 1 (the first mover) exerts an effort x_1 in the first period, and contestant 2 (the second mover) observes a noisy output of contestant 1's effort, $x_1 + t$, where t is the noise term. Then, contestant 2 exerts an effort x_2 in the second period, and wins the contest if her effort is larger than or equal to the noisy output of contestant 1, i.e., $x_2 \geq x_1 + t$; otherwise, contestant 1 wins. The random noise t is uniformly distributed on an interval $[-k, k]$

where k describes the magnitude of the random noise and determines its variance.[1] The smaller the value of k is, the higher is the contest's accuracy. This type of sequential contest was previously studied by Segev and Sela (2011) without the presence of a random noise (i.e., when $k = 0$). Here, we characterize the subgame perfect equilibrium of this contest with a random noise and focus on the effect of the noise on the contestants' efforts. The assumption of a noisy output in a sequential contest is relevant in various applications, including sport contests such as athletics and gymnastics, political races in which the candidates compete with each other in a sequence of speeches, and court trials where the lawyers of both sides make their final speeches sequentially.

We present two variations of the model. In the first one (Section 3 - Symmetric Information) we assume that both contestants do not know the realization of the noise when they exert their effort. We show that when the magnitude of the noise, k, increases, then in equilibrium less types of contestant 1 will exert a positive effort in the contest. If the magnitude of the random noise is sufficiently high, contestant 1 will have no incentive to exert any positive effort since anyway she wins with zero effort. Thus, we focus on a more interesting case where the magnitude of the random noise is relatively low (k goes to zero). One could hypothesize that even a small noise can have a large impact on contestant 1's equilibrium effort and dramatically change it with respect to the contest without the noise. In our model, however, we show that the marginal effect of the magnitude of the random noise, k, on the contestants' strategies goes to zero when k goes to zero. Thus, we conclude that the equilibrium behavior in the sequential all-pay auction is robust under the existence of a small noise.

In the second variation of our model (Section 3 - Asymmetric Information) we assume that contestant 1 knows the realization of the noise when exerting her effort, while contestant 2 does not. We thus assume that contestant 1 has more information about the contest than contestant 2. This assumption describes contests in which the first mover has the opportunity to gather information about the contest environment before exerting an effort. This commonly occurs in market situations when one firm identifies the market earlier than the other firm which enables her to evaluate correctly the connection between the effort and the observed output. We show that a positive realization of noise decreases contestant 1's equilibrium effort for any type who

[1] The model can be studied for any symmetric distribution of noise but then a closed-form expression for the subgame perfect equilibrium cannot be derived.

exerts a positive effort while a negative realization increases it with respect to the contest without any noise. Moreover, in equilibrium, the probability that contestant 1 will exert a positive effort in the contest decreases in the absolute value of the noise. Therefore, we conclude that a positive realization of noise decreases the expected output of contestant 1. The effect of a negative noise is ambiguous since, on the one hand, it increases the output of contestant 1 for any type who exerts a positive effort, but on the other, it decreases the incentive to exert a positive effort in the contest. It is worth noting that the effects of a positive and a negative noise with the same absolute value do not balance each other such that the total effect on the contestants' outputs is not necessarily zero. However, similarly to the case when both contestants do not know the realization of the noise, we show that the marginal effect of the magnitude of the random noise, k, on the contestants' strategies goes to zero when k goes to zero. Hence, independent of the information of the contestants on the random noise, the equilibrium behavior in the sequential all-pay auction is robust under the existence of a small noise.

Section 5 concludes. All proofs are in the appendix.

2 The model

We consider a sequential all-pay auction with two contestants where contestant 1 (the first mover) exerts an effort x_1 in the first period, while contestant 2 (the second mover) observes an output of $x_1 + t$ where t represents a random noise that is drawn from a uniform distribution on the interval $[-k, k]$, $k \geq 0$ and this information is common knowledge. The value of k determines the variance of the random noise and the smaller the value of k is the higher is the contest's accuracy. Contestant 2 exerts an effort x_2 in the second period, and wins the contest if the effort x_2 is larger than or equal to $x_1 + t$; otherwise, contestant 1 wins. The valuation of both contestants for the prize is 1. An effort x_i costs $\frac{x_i}{a_i}$ where $a_i \geq 0$ is the ability (or type) of contestant i which is private information to i. Contestant i's ability is drawn independently from the interval $[0, 1]$ according to a cumulative distribution function F_i which is common knowledge. We assume that $F_i, i = 1, 2$ has a positive and continuous density function $F_i' > 0$. Since the ability of the players is distributed on $[0, 1]$ we can assume that the output is limited to this interval and therefore we assume that if $x_1 + t \leq 0$, then contestant 2 observes an output of zero while if $x_1 + t \geq 1$ she observes an output of 1. The goal of the contest designer is to maximize the expected highest output in this contest.

3 Symmetric information

Assume that both contestants do not know the realization of the noise t when exerting their effort. If contestant 1 exerts an effort of $b_1(a_1)$ in the first period, contestant 2 observes a noisy output of $b_1(a_1)+t$. Then contestant 2's equilibrium strategy is given by

$$b_2(a_2) = \begin{cases} 0 & \text{if } \ a_2 < b_1(a_1)+t \\ b_1(a_1)+t & \text{if } \ a_2 \geq b_1(a_1)+t \end{cases} \tag{1}$$

In the following, we assume that $k \leq \frac{1}{2}$ and that F_2 is concave. Then we can show (see the proof of Proposition 1) that contestant 1's equilibrium strategy satisfies $k \leq b_1(a_1) \leq 1-k$. In that case, contestant's 1 maximization problem is given by

$$\max_{b_1}\left\{\int_{-k}^{k} F_2(b_1+t)\frac{1}{2k}dt - \frac{b_1}{a_1}\right\}$$

The F.O.C. is therefore

$$\int_{-k}^{k}\frac{1}{2k}F_2'(b_1+t)\,dt - \frac{1}{a_1} = 0 \tag{2}$$

The S.O.C. is

$$\int_{-k}^{k}\frac{1}{2k}F_2''(b_1+t)\,dt < 0 \tag{3}$$

If F_2 is concave then the S.O.C. holds everywhere. Thus, according to the above analysis, contestant 1's equilibrium strategy is as follows:

Proposition 1 *In the sequential all-pay auction, for every concave distribution function F_2, the equilibrium strategy of contestant 1 is given by $b_1(a_1)=0$ for all $0 \leq a_1 < a_1^*$, and for all $a_1 \geq a_1^*$ it is implicitly defined by*

$$\frac{1}{2k}F_2(b_1(a_1)+k) - \frac{1}{2k}F_2(b_1(a_1)-k) = \frac{1}{a_1} \tag{4}$$

The cutoff type a_1^* is implicitly defined by

$$\frac{1}{2k}\int_{-k}^{k} F_2(b_1(a_1^*)+t)\,dt - \frac{b_1(a_1^*)}{a_1^*} = \frac{1}{2k}\int_{0}^{k} F_2(t)\,dt \tag{5}$$

where $b_1(a_1^*)$ is implicitly defined by (4).

Proof. See Appendix. ■

In the following we use Proposition 1 to illustrate the contestants' behavior in a sequential all-pay auction.

Example 1 *Assume a sequential all-pay auction where contestant* 2*'s type is distributed according to* $F_2(a_2) = \sqrt{a_2}$*. By (4), contestant* 1*'s equilibrium strategy is implicitly given by*

$$\frac{1}{2k}\left(\sqrt{b_1 + k} - \sqrt{b_1 - k}\right) = \frac{1}{a_1}$$

Thus, contestant 1*'s equilibrium strategy is explicitly given by*

$$b_1(a_1) = \begin{cases} 0 & \text{if } 0 \le a_1 < a_1^* \\ \frac{a_1^2}{4} + \frac{k^2}{a_1^2} & \text{if } a_1^* \le a_1 \le 1 \end{cases}$$

The cutoff type a_1^* *is defined by (5)*

$$\frac{1}{2k}\int_{-k}^{k}\left(\frac{a_1^2}{4} + \frac{k^2}{a_1^2} + t\right)^{\frac{1}{2}} dt - \left(\frac{a_1}{4} + \frac{k^2}{a_1^3}\right) = \frac{1}{2k}\int_0^k t^{\frac{1}{2}} dt$$

Therefore

$$a_1^* = c\sqrt{k}$$

where c *is the solution to the equation*

$$4c^3 - 3c^4 + 4 = 0 \Rightarrow c \simeq 1.637\,2$$

Note that contestant 1*'s effort is increasing in* a_1 *for all* $a_1 \ge \sqrt{2k} = 1.414\,2\sqrt{k}$ *and therefore for all* $a_1 \ge a_1^*$*. Moreover, for a given type* a_1 *who exerts a positive effort, the effort is increasing in* k*. Note also that* a_1^* *approaches zero when* $k \to 0$*. In that case, contestant* 1 *will exert an effort of* $b_1(a_1) = \frac{a_1^2}{4}$*. Finally, note that* $a_1^* \le 1$ *iff* $k \le \left(\frac{1}{1.637\,2}\right)^2 = 0.373\,08$*. That is, if* $k > 0.373\,08$ *contestant* 1*, independent of her type, exerts an effort of* $b_1 = 0$*. The following figure presents the equilibrium effort of contestant* 1 *when* $k = 0.1$ *and* $a_1^* \simeq 0.517\,73$*.*

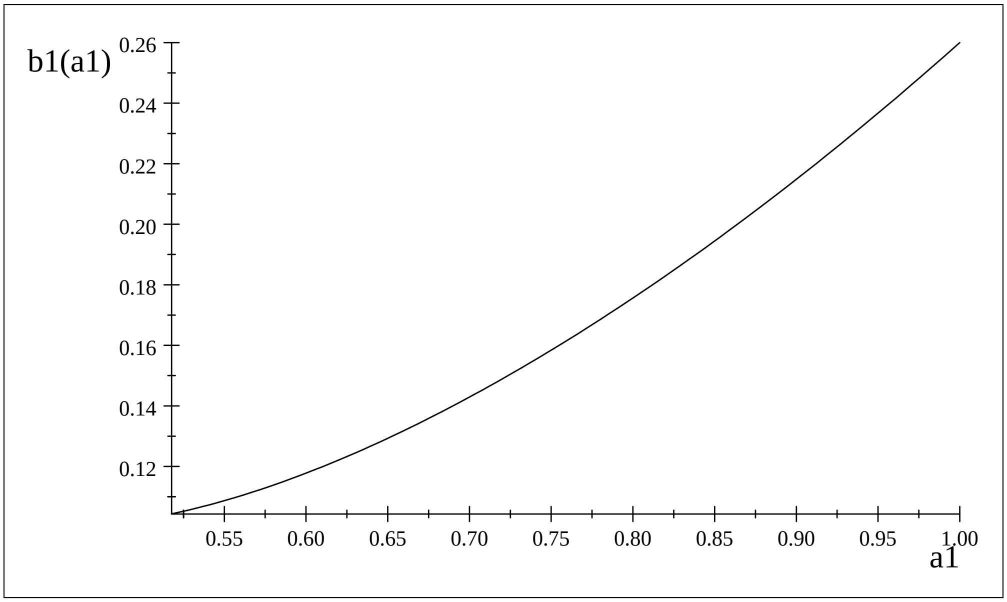

Figure 1: The equilibrium effort function as a function of the type.

Our goal in the following is to examine the effect of the size of the interval of the random noise, k, on the contestants' behavior, and particularly on the expected highest output in the contest. The expected highest output in our model is equal to contestant 1's expected output which is given by

$$TE_1 = \int_{a_1^*}^{1} \left(\frac{1}{2k} \int_{-k}^{k} \left(b_1\left(a_1\right) + t\right) dt \right) f_1(a_1) da_1 = \int_{a_1^*}^{1} b_1\left(a_1\right) f_1(a_1) da_1 \tag{6}$$

where $b_1(a_1)$ is defined by (4). If k is sufficiently large, no type of contestant 1 will exert a positive effort (i.e., $a_1^* = 1$). The following example illustrates contestant 1's expected effort in the contest.

Example 2 *Consider a sequential all-pay auction with two contestants whose types are distributed according to $F_1\left(x\right) = F_2\left(x\right) = \sqrt{x}$. Then, by (6), the expected highest effort is given by*

$$TE_1 = \int_{1.637\sqrt{k}}^{1} \left(\frac{a_1^2}{4} + \frac{k^2}{a_1^2} \right) \frac{1}{2\sqrt{a_1}} da_1 = \frac{1}{20} - \frac{k^2}{3} - 0.01236 k^{\frac{5}{4}}$$

Figure 2 depicts the expected highest effort as a function of k.

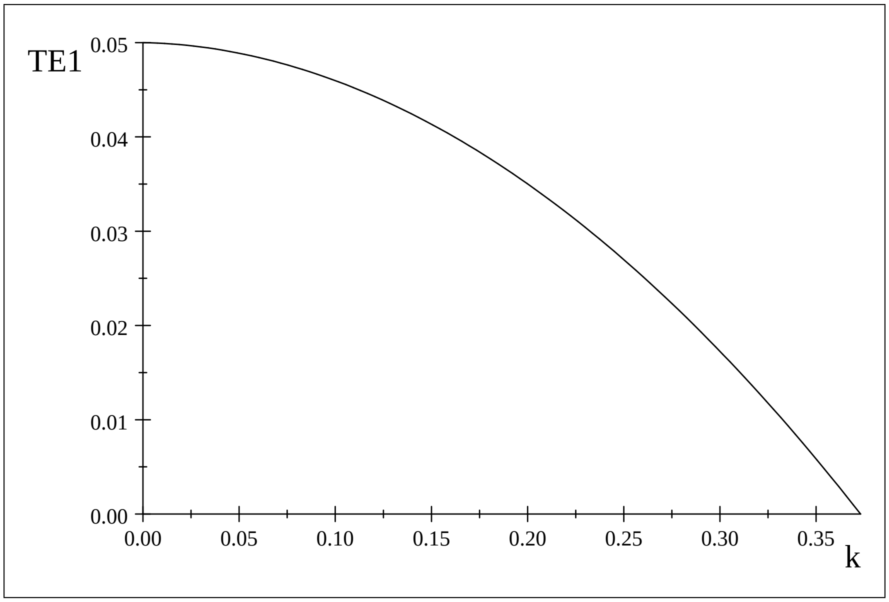

Figure 2: The expected highest effort as a function of k

We next show that the cutoff type a_1^* increases in k for any (concave) distribution of contestant 2's types.

Proposition 2 *The ex-ante probability that contestant* 1 *will exert a positive effort in the sequential all-pay auction decreases in the magnitude of the random noise, i.e.,*

$$\frac{da_1^*}{dk} > 0$$

Proof. See Appendix. ■

The effect of the magnitude of the random noise on the contestants' expected efforts is ambiguous. On the one hand, from the above proposition, it decreases the ex-ante probability that contestant 1 will exert a positive effort, but, on the other, in some cases it increases the effort of contestant 1 for any given type (as in Example 1 where $b_1(a_1)$ is increasing in k). However, the following result shows that if the magnitude of the random noise k is small enough, then it has no effect on the expected effort of contestant 1.

Proposition 3 *In the sequential all-pay auction, if $F_2'(x) \to \infty$ when $x \to 0$, the marginal effect of the magnitude of the random noise, k, on the expected highest output is zero when k approaches zero, i.e.,*

$$\lim_{k \to 0} \frac{dTE_1}{dk} = 0$$

Proof. See Appendix. ■

Note that the condition in the above proposition, $F_2'(x) \to \infty$ when $x \to 0$, holds for all concave distribution functions of the form $F(x) = x^{\gamma}, 0 < \gamma < 1$. Moreover, since contestant 2 only equalizes the output of contestant 1 the effect of the random noise is similar on both contestants. Thus, we conclude that a relatively small noise in the sequential all-pay contest does not result in a dramatic change in the contestants' output. In other words, the sequential all-pay auction is robust under a small noise with respect to the contestants' outputs.

3.1 Asymmetric information

In many market situations the first player to arrive at the market gathers the available information and can successfully evaluate the connection between her effort and the observable output. Assume therefore that contestant 2 does not know the value of the realization of the noise t which is uniformly distributed on the interval $[-k, k]$, while contestant 1 knows the realization of t before she exerts her effort. Note, however, that contestant 2's behavior will not be changed when contestant 1 knows the realization of t. Then, as in the previous section, the equilibrium strategy of contestant 2 is given by

$$b_2(a_2) = \begin{cases} 0 & \text{if } a_2 < b_1(a_1) + t \\ b_1(a_1) + t & \text{if } a_2 \geq b_1(a_1) + t \end{cases}$$

Given a noise of t, contestant 1 with ability a_1 solves the following optimization problem:

$$\max_{b_1} \left\{ F_2(b_1 + t) - \frac{b_1}{a_1} \right\} \tag{7}$$

The F.O.C. is then

$$F_2'(b_1 + t) - \frac{1}{a_1} = 0 \tag{8}$$

The S.O.C. is

$$F_2''(b_1 + t) < 0 \tag{9}$$

We assume again that F_2 is concave. Thus, contestant 1's equilibrium strategy is as follows:

Proposition 4 *In the sequential all-pay auction, for every concave distribution function F_2, the equilibrium strategy of contestant 1 is given by*

$$b_1(a_1) = \begin{cases} 0 & \text{if} \quad a_1 < \overline{a} \\ (F_2')^{-1}(\frac{1}{a_1}) - t & \text{if} \quad a_1 \geq \overline{a} \end{cases} \tag{10}$$

If $0 \leq t \leq (F_2')^{-1}(1)$, the cutoff type $\overline{a} = a^*$ is determined by

$$(F_2')^{-1}(\frac{1}{a^*}) - t = 0 \Leftrightarrow a^* = \frac{1}{F_2'(t)} \tag{11}$$

and if $t < 0$, the cutoff type $\bar{a} = a^{**}$ is determined by

$$F_2((F_2')^{-1}(\frac{1}{a^{**}})) - \frac{1}{a^{**}}\left((F_2')^{-1}(\frac{1}{a^{**}}) - t\right) = 0 \tag{12}$$

Finally, if $t > (F_2')^{-1}(1)$ then $\overline{a} = 1$

Note that $b_1(a_1) + t < 1$ for all $a_1 \leq 1$ and therefore the maximization problem (7) is well defined. Moreover, if $t > (F_2')^{-1}(1)$ then all types of contestant 1 exert a zero effort. If $t < 0$, the cutoff $\bar{a} = a^{**}$ is the type whose expected payoff is equal to zero when a positive effort is exerted. Finally, given the realization of the noise t, contestant 1's equilibrium effort is (weakly) increasing in her type.

By Proposition 4, a positive noise decreases contestant 1's output and a negative noise increases it with respect to the situation without any noise. The noise, either negative or positive, increases the cutoff, that is, it decreases the ex-ante probability that contestant 1 will exert a positive effort. Thus, a positive noise necessarily decreases contestant 1's expected output. However, the effect of a negative noise on contestant 1's expected effort is ambiguous since, on the one hand, it increases the effort, but, on the other, it increases the probability that contestant 1 will exert a zero effort. Note that if contestant 1 exerts a positive effort when t is positive as well as when t is negative with the same absolute value, then by (10) her effort when the noise is negative is higher by $2t$ than when the noise is positive. However, a positive noise and a negative noise, even if they have the same absolute value, by (11) and (12) will affect differently contestant 1's decision whether or not to exert a positive effort. The following result provides a condition on the distribution function of contestant 2's types for which a negative noise encourages a larger set of contestant 1's types to exert a positive effort in the contest than a positive noise with the same absolute value. Thus, this result also provides a condition according to which a negative noise is better than a positive one with the same absolute value from the viewpoint of a designer who wishes to maximize the expected highest output.

Proposition 5 *In a sequential all-pay contest with* $t \leq (F_2')^{-1}(1)$*, assume that the following condition holds*

$$\frac{F_2(t)}{F_2'(t)} \geq 2t \tag{13}$$

Then, if contestant 1 exerts a positive effort with a positive noise t*, she also exerts a positive effort with a negative noise of* $-t$*, i.e., we have*

$$a^{**} \leq a^*$$

In that case, a negative noise of $-t$ *yields a higher expected output of contestant* 1 *than a positive noise of* t*.*

Proof. See Appendix. ■

Condition (13) is satisfied in particular for all concave distribution functions of the forms $F_2(t) = t^\gamma$,for $0 < \gamma \leq \frac{1}{2}$. Thus, for all these distribution functions, a negative noise yields a higher expected output of contestant 1 than a positive noise with the same absolute value.

The expected highest output, given a noise of t, is equal to the expected output of contestants 1 which is given by

$$TE_1(t) = \int_{\bar{a}}^{1} ((F_2')^{-1}(\frac{1}{a_1}) - t)F_1'(a_1)da_1 \tag{14}$$

If t is uniformly distributed on the interval $[-k, k]$ and $k \leq (F_2')^{-1}(1)$, the expected output of contestant 1 is given by

$$\begin{aligned} TE_1 &= \int_0^k \left(\int_{a^*}^1 b_1(a_1) F_1'(a_1)da_1\right)\frac{1}{2k}dt + \int_{-k}^0 \left(\int_{a^{**}}^1 b_1(a_1) F_1'(a_1)da_1\right)\frac{1}{2k}dt \\ &= \int_0^k \frac{1}{2k}\left(\int_{a^*}^1 (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 - t(1-F_1(a^*))\right)dt \\ &\quad + \int_{-k}^0 \frac{1}{2k}\left(\int_{a^{**}}^1 (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 - t(1-F_1(a^{**}))\right)dt \end{aligned} \tag{15}$$

In the following example, we illustrate the equilibrium strategy and the effect of a random noise on the expected highest output in a sequential all-pay auction.

Example 3 *Consider a sequential all-pay auction where both contestants' types are distributed according to* $F(x) = \sqrt{x}$*. Then, by (10), (11) and (12) for* $0 \leq t \leq \frac{1}{4}$*, contestant* 1*'s equilibrium strategy is given by*

$$b_1(a_1) = \begin{cases} \frac{a_1^2}{4} - t & \text{if } a_1 \geq 2\sqrt{t} \\ 0 & \text{if } a_1 < 2\sqrt{t} \end{cases}$$

and for $t < 0$*, it is given by*

$$b_1(a_1) = \begin{cases} \frac{a_1^2}{4} - t & \text{if } a_1 \geq 2\sqrt{-t} \\ 0 & \text{if } a_1 < 2\sqrt{-t} \end{cases}$$

By (15), the expected output of contestants 1 *is*

1) for $t \geq 0$

$$TE_1\,(t \geq 0) = \int_{2\sqrt{t}}^{1} (\frac{a_1^2}{4} - t)\frac{1}{2\sqrt{a_1}} da_1 = \frac{4}{5}\sqrt{2}t^{\frac{5}{4}} - t + \frac{1}{20}$$

and

$$\frac{dTE_1}{dt} = \sqrt{2}t^{\frac{1}{4}} - 1$$

Then for all $0 \leq t \leq \frac{1}{4}$*, we have* $\frac{dTE_1}{dt} \leq 0$*; that is, any positive noise decreases the expected output of contestant 1 compared to the case without any noise.*

2) for $t < 0$

$$TE_1\,(t < 0) = \int_{2\sqrt{-t}}^{1} (\frac{a_1^2}{4} - t)\frac{1}{2\sqrt{a_1}} da_1 = \frac{1}{20} - \frac{6}{5}\sqrt{2}\,(-t)^{\frac{5}{4}} - t$$

and

$$\frac{dTE_1}{dt} = \frac{3}{2}\sqrt{2}\,(-t)^{\frac{1}{4}} - 1$$

Thus, for all $t > -0.04939$*,* $\frac{dTE}{dt} \leq 0$ *and for all* $t < -0.04939$*,* $\frac{dTE}{dt} \geq 0$*. In other words, a small negative noise increases contestant* 1*' expected output, and a large negative noise decreases it. We plot the expected highest effort as a function of the realization of the noise* t *in Figure 3.*

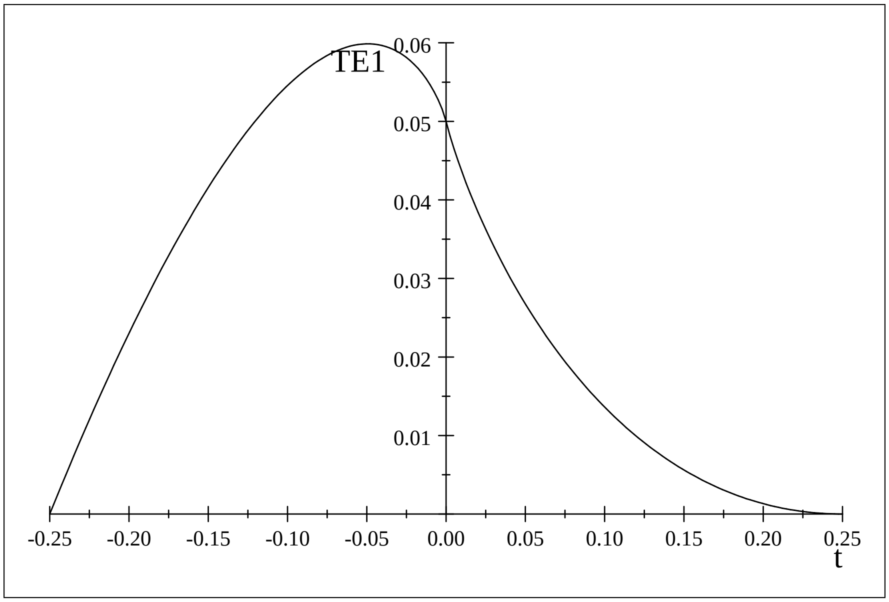

Figure 3: The expected highest effort as a function of t

If t is uniformly distributed on $[-k, k]$ for $k \leq \frac{1}{4}$ then

$$\begin{aligned} TE_1 &= \int_0^k TE_1 (t \geq 0) \frac{1}{2k} dt + \int_{-k}^0 TE_1 (t < 0) \frac{1}{2k} dt \\ &= \frac{1}{2k} \left(\int_0^k \left(\frac{4}{5} \sqrt{2} t^{\frac{5}{4}} - t + \frac{1}{20} \right) dt + \int_{-k}^0 \left(\frac{1}{20} - \frac{6}{5} \sqrt{2} (-t)^{\frac{5}{4}} - t \right) dt \right) = \frac{1}{20} - \frac{4}{45} \sqrt{2} k^{\frac{5}{4}} \end{aligned}$$

In Figure 4 we can see that the expected effort of contestant 1 decreases in the magnitude of the random noise k.

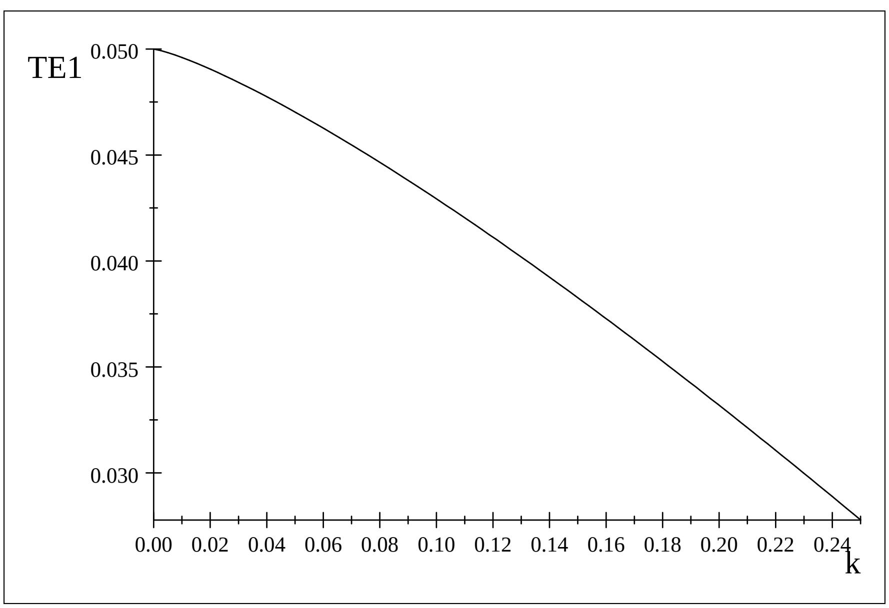

Figure 4: The expected highest effort as a function of k

In the above example, although the random noise is symmetrically distributed around zero, the equilibrium output of contestant 1 is not symmetric, i.e., $b_1(a_1, t) \neq b_1(a_1, -t)$ and, in particular the expected highest effort is not symmetric around zero as can be seen in Figure 3. In the following, we show that when the magnitude of the noise k is small enough the effect of the negative noises will be positive and will balance the negative effect of the positive noises such that the overall effect of random noise on contestant 1's expected effort will be zero.

Proposition 6 *In the sequential all-pay auction, if $F_2'(x) \to \infty$ when $x \to 0$, the marginal effect of the magnitude of the random noise, k, on the expected highest output is zero when k approaches zero, i.e.,*

$$\lim_{k \to 0} \frac{dTE_1}{dk} = 0.$$

Proof. See Appendix. ■

Proposition 6 demonstrates that the sequential all-pay auction is robust under a small noise in contestant 1's output when she knows the realization of the noise before she exerts the effort. Thus, by Propositions 3 and 6 we can conclude that with either symmetric or asymmetric information on the realization of random noise, a relatively small noise has no effect on the expected highest effort.

4 Concluding remarks

We established the existence of a subgame perfect equilibrium in the sequential all-pay auction with noisy outputs. We showed that when the noise is uniformly distributed around zero, this auction is robust in the sense that the marginal effect of small noises on the contestants' expected highest effort is zero. In other words, in a sequential all-pay auction, small noises do not have a dramatic effect on the contestants' output with respect to the contest without any noise. Owing to the complexity of the environment we focused here on a specific distribution of the random noise, namely, the uniform distribution. However, we conjecture that our results will hold for other distributions of random noise as long as they are symmetrically distributed around zero.

5 Appendix

5.1 Proof of Proposition 1

We wish to characterize the equilibrium effort function $b_1(a_1)$ of contestant 1 when the equilibrium effort function of contestant 2 is given by (1). We divide our analysis into the following three cases: 1. $b_1(a_1) < k$ 2. $b_1(a_1) > 1-k$ and 3. $k \leq b_1(a_1) \leq 1-k$.

1) Assume first that $b_1(a_1) < k$. Then contestant $1'$s maximization problem is given by

$$\max_{b_1}\left\{\int_{-k}^{-b_1} F_2(0)\frac{1}{2k}dt+\int_{-b_1}^{k} F_2(b_1+t)\frac{1}{2k}dt-\frac{b_1}{a_1}\right\}=\max_{b_1}\left\{\int_{-b_1}^{k} F_2(b_1+t)\frac{1}{2k}dt-\frac{b_1}{a_1}\right\}$$

The F.O.C. is given by

$$\int_{-b_1}^{k}\frac{1}{2k}F_2'(b_1+t)\,dt-\frac{1}{a_1}=0$$

Thus,

$$\frac{1}{2k}F_2(b_1+k)=\frac{1}{a_1} \tag{16}$$

The S.O.C. is given by

$$\frac{1}{2k}F_2'(0)+\int_{-b_1}^{k}\frac{1}{2k}F_2''(b_1+t)\,dt=\frac{1}{2k}F_2'(b_1+k)$$

Since $F_2'(b_1+k)>0$, the S.O.C. does not hold and therefore the maximum is never achieved at an internal effort $b_1 \in (0,k)$.

2) Assume now that $b_1(a_1) > 1-k$. Then, contestant $1'$s maximization problem is given by

$$\max_{b_1}\left\{\int_{-k}^{1-b_1} F_2(b_1+t)\frac{1}{2k}dt+\int_{1-b_1}^{k}\frac{1}{2k}dt-\frac{b_1}{a_1}\right\}$$

The F.O.C. is

$$\frac{1}{2k}\int_{-k}^{1-b_1} F_2'(b_1+t)\,dt-\frac{1}{a_1}=0$$

Thus,

$$\frac{1}{2k}(1-F_2(b_1-k))=\frac{1}{a_1} \tag{17}$$

Let $a_1=1$. Then if $b_1(1)>1-k$ we obtain

$$\frac{1}{2k}\left(1-F_2\left(b_1(1)-k\right)\right)<\frac{1}{2k}\left(1-F_2\left(1-2k\right)\right)<\frac{1}{2k}\left(1-(1-2k)\right)=1$$

The second inequality is due to our assumption that F_2 is concave. This inequality contradicts equation (17) and therefore $b_1(1) < 1-k$, which implies by the monotonicity of b_1 that $b_1(a_1) < 1-k$ for all $a_1 \leq 1$.

3) When $k \leq b_1(a_1) \leq 1-k$, contestant's 1 maximization problem is given by

$$\max_{b_1} \left\{ \int_{-k}^{k} F_2\left(b_1 + t\right) \frac{1}{2k} dt - \frac{b_1}{a_1} \right\}$$

The F.O.C. is therefore

$$\int_{-k}^{k} \frac{1}{2k} F_2'\left(b_1 + t\right) dt - \frac{1}{a_1} = 0$$

The S.O.C. is

$$\int_{-k}^{k} \frac{1}{2k} F_2''\left(b_1 + t\right)) dt < 0$$

If F_2 is concave then the S.O.C. holds everywhere. Thus, the equilibrium strategy of contestant 1, $b_1(a_1)$, is implicitly determined by

$$\frac{1}{2k} F_2\left(b_1 + k\right) - \frac{1}{2k} F_2\left(b_1 - k\right) = \frac{1}{a_1}$$

where $b_1(a_1)$ is a an increasing function. The cutoff type a_1^* is the type who is indifferent between an effort of zero and an effort given by (4). Therefore it is given by

$$\frac{1}{2k} \int_{-k}^{k} F_2\left(b_1(a_1^*) + t\right) dt - \frac{b_1(a_1^*)}{a_1^*} = \frac{1}{2k} \int_{0}^{k} F_2\left(t\right) dt$$

where $b_1(a_1^*)$ is implicitly defined by (4). By the analysis in case (1) it is possible that an interval of types will find it optimal to exert $b_1 = k$. We show next, however, that no such interval exists. Denote by $\hat{a}_1$ the type who by equation (4) is supposed to exert an effort of $b_1\left(\hat{a}_1\right) = k$. Thus $\hat{a}_1$ is the solution to

$$\frac{1}{2k} F_2\left(2k\right) = \frac{1}{a_1}$$

or

$$\hat{a}_1 = \frac{2k}{F_2\left(2k\right)}$$

Since F_2 is concave, $\hat{a}_1$ is between 0 and 1. If contestant 1 with type a_1 exerts an effort of $b_1 = 0$, then her expected payoff is $\pi_{a_1}\left(0\right) = \frac{1}{2k} \int_0^k F_2\left(t\right) dt$, while if she exerts an effort of k her expected payoff is $\pi_{a_1}\left(k\right) = \frac{1}{2k} \int_{-k}^{k} F_2\left(k + t\right) dt - \frac{k}{a_1}$. Recall that we already showed that she will never exert an effort strictly between 0 and k. The difference between these expected payoffs is given by

$$\begin{aligned}\Delta(a_1) &= \pi_{a_1}(0) - \pi_{a_1}(k) = \frac{1}{2k}\int_0^k F_2(t)\,dt - \frac{1}{2k}\int_{-k}^k F_2(k+t)\,dt + \frac{k}{a_1} \\ &= \frac{k}{a_1} - \frac{1}{2k}\int_k^{2k} F_2(t)\,dt > \frac{k}{a_1} - \frac{1}{2}F_2(2k)\end{aligned}$$

By the definition of $\hat{a}_1$, we obtain that $\Delta(a_1)$ is positive for all $a_1 < \hat{a}_1$. Thus, all types of contestant 1 that are smaller than $\hat{a}_1$ will exert an effort of $b_1 = 0$. Therefore no type $a_1 < \hat{a}_1$ can be indifferent between an effort of zero and an effort given by (4) (since by the monotonicity of the effort function given in (4), $b_1(a_1) < k$). Therefore $a_1^* \geq \hat{a}_1$. Finally, all $a_1 \in [\hat{a}_1, a_1^*]$ prefer an effort of zero over an effort given by (4). This follows from the fact that the L.H.S. of equation (5) is constant while the R.H.S. is increasing in a_1

$$\frac{d}{da_1}\left(\frac{1}{2k}\int_{-k}^k F_2(b_1(a_1)+t)\,dt - \frac{b_1(a_1)}{a_1}\right) = \frac{b_1'(a_1)}{2k}\int_{-k}^k F_2'(b_1(a_1)+t)\,dt - \frac{b_1'(a_1)}{a_1} + \frac{b_1(a_1)}{a_1^2} = \frac{b_1(a_1)}{a_1^2} > 0$$

Therefore all $a_1 \in [\hat{a}_1, a_1^*]$ prefer an effort of zero. Note that all types which are larger than a_1^* will exert an effort according to (4), and particularly, since $b_1(a_1)$ is a monotonically increasing function, all the positive efforts are larger than k. *Q.E.D.*

5.2 Proof of Proposition 2

Recall that a_1^* is implicitly defined as the solution to the equation

$$\frac{1}{2k}\int_{-k}^k F_2(b_1(a_1)+t)\,dt - \frac{b_1(a_1)}{a_1} - \frac{1}{2k}\int_0^k F_2(t)\,dt = 0$$

where $b_1(a_1^*)$ is implicitly defined by (4). Therefore

$$\frac{da_1^*}{dk} = -\frac{\left(-\frac{1}{2k^2}\int_{-k}^k F_2(b_1(a_1)+t)\,dt + \frac{1}{2k}F_2(b_1(a_1)+k) + \frac{1}{2k}F_2(b_1(a_1)-k) + \frac{1}{2k^2}\int_0^k F_2(t)\,dt - \frac{1}{2k}F_2(k)\right)}{b_1'(a_1)\left(\frac{1}{2k}\int_{-k}^k F_2'(b_1(a_1)+t)\,dt - \frac{1}{a_1}\right) + \frac{b_1(a_1)}{a_1^2}}$$

Note that by equation (4) the denominator is positive. Moreover, we have

$$\frac{1}{2k^2}\int_0^k F_2(t)\,dt - \frac{1}{2k}F_2(k) < 0$$

and

$$2k\left(\frac{1}{2}\left(F_2(b_1(a_1)+k) + F_2(b_1(a_1)-k)\right)\right) < \int_{-k}^k F_2(b_1(a_1)+t)\,dt$$

This last inequality is true since F_2 is concave. Therefore, we conclude that $\frac{da_1^*}{dk} > 0$. *Q.E.D.*

5.3 Proof of Proposition 3

If $F_2'(x) \to \infty$ when $x \to 0$, then in the limit when k goes to zero we have the following equilibrium strategy for player 1

$$\frac{1}{2k}F_2(b_1(a_1)+k) - \frac{1}{2k}F_2(b_1(a_1)-k) = \frac{1}{a_1} \Rightarrow \tag{18}$$
$$b_1(a_1) = (F_2')^{-1}\left(\frac{1}{a_1}\right) \quad \text{for all } a_1 \geq 0$$

Note that if $F_2'(0)$ is finite then an interval of types $a_1 \in \left[0, \frac{1}{F_2'(0)}\right]$ will exert zero effort since the solution to the maximization problem

$$\max_s \left\{ F_2(b_1(s)) - \frac{b_1(s)}{a_1} \right\}$$

is $b_1 = 0$. But when $F_2'(x) \to \infty$ when $x \to 0$, all positive types find it optimal to exert a positive effort in the limit when k goes to zero.

By (14) we have

$$\frac{dTE_1}{dk} = \int_{a_1^*}^{1} \frac{db_1}{dk} f_1(a_1)da_1 - \frac{da_1^*}{dk} b_1(a^*) f_1(a^*)$$

Using the implicit condition (4) we obtain that

$$\lim_{k\to 0}\frac{db_1}{dk} = \lim_{k\to 0} - \frac{\left(\frac{1}{2k}(F_2'(b_1+k) + F_2'(b_1-k)) - \frac{1}{2k^2}(F_2(b_1+k) - F_2(b_1-k))\right)}{\frac{1}{2k}(F_2'(b_1+k) - F_2'(b_1-k))}$$

Since $F_2'(b_1) = \lim_{k\to 0}\left(\frac{1}{2k}(F_2(b_1+k) - F_2(b_1-k))\right)$ and $F_2''(b_1) = \lim_{k\to 0}\left(\frac{1}{2k}(F_2'(b_1+k) - F_2'(b_1-k))\right)$, then

$$\lim_{k\to 0}\frac{db_1}{dk} = \frac{\frac{1}{k}(F_2'(b_1) - F_2'(b_1))}{F_2''(b_1)} = 0$$

Moreover, from the above we know that $\lim_{k\to 0} a_1^*(k) = 0$ if $F_2'(x) \to \infty$ when $x \to 0$, and thus we obtain that

$$\lim_{k\to 0}\frac{dTE_1}{dk} = 0$$

Q.E.D.

5.4 Proof of Proposition 5

Given a positive realization and a negative realization of the noise with the same absolute value we let $v = |t| = |-t|$; Then, by (11) when $t > 0$, a^* is determined by

$$(F_2')^{-1}\left(\frac{1}{a^*}\right) = v$$

and by (12) when $t<0$, a^{**} is determined by

$$F_2((F_2')^{-1}(\frac{1}{a^{**}}))-\frac{1}{a^{**}}\left((F_2')^{-1}(\frac{1}{a^{**}})+v\right)=0$$

If we replace a^{**} by a^{*} in the L.H.S. of the last equation we obtain

$$F_2(v)-F_2'(v)2v \tag{19}$$

Thus, if (19) is positive, contestant 1 with type a^* has a positive expected payoff when the realization of the noise is $-v$. Since the expected payoff of contestant 1 increases in her type, and the expected payoff of type a^{**} is zero when the realization of the noise is $-v$, we obtain that $a^{**}\leq a^*$. *Q.E.D.*

5.5 Proof of Proposition 6

The derivative of (15) is

$$\begin{aligned}\frac{dTE_1}{dk} &= \frac{1}{2k}\left(\int_{a^*}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1-k(1-F_1(a^*(k)))\right)\\ &-\int_0^k\frac{1}{2k^2}\left(\int_{a^*}^{1}b_1(a_1)F_1'(a_1)da_1\right)dt\\ &+\frac{1}{2k}\left(\int_{a^{**}}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1+k(1-F_1(a^{**}(-k)))\right)\\ &-\int_{-k}^{0}\frac{1}{2k^2}\left(\int_{a^{**}}^{1}b_1(a_1)F_1'(a_1)da_1\right)dt\end{aligned}$$

Thus,

$$\lim_{k\to0}\frac{dTE_1}{dk}=\lim_{k\to0}\left(\begin{array}{c}\frac{1}{2k}\left(\int_{a^*}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1+\int_{a^{**}}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1\right)-\frac{1}{2}\left(F_1(a^{**}(-k))-F_1(a^*(k))\right)\\ -\frac{1}{2k^2}\left(\int_0^k\left(\int_{a^*}^{1}b_1(a_1)F_1'(a_1)da_1\right)dt+\int_{-k}^{0}\left(\int_{a^{**}}^{1}b_1(a_1)F_1'(a_1)da_1\right)dt\right)\end{array}\right)$$

When $F_2'(x)\to\infty$ when $x\to\infty$ then $\lim_{k\to0}a^*(k)\to0$ and $\lim_{k\to0}a^{**}(k)\to0$. Therefore,

$$\begin{aligned}\lim_{k\to0}\frac{dTE_1}{dk} &= \lim_{k\to0}\left(\frac{1}{k}\int_0^1(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1-\frac{1}{2k^2}\int_{-k}^{k}\left(\int_0^1 b_1(a_1)F_1'(a_1)da_1\right)dt\right)\\ &= \lim_{k\to0}\frac{1}{k}\int_0^1(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1-\lim_{k\to0}\frac{1}{k}\left(\int_0^1 b_1(a_1)F_1'(a_1)da_1\right)dt=0\end{aligned}$$

Q.E.D.

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