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ON THE STRATEGIC DISCLOSURE OF FEASIBLE OPTIONS IN BARGAINING

Geoffroy de Clippel and Kfir Eliaz

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Geoffroy de Clippel, Brown University
Kfir Eliaz, Brown University and CEPR

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Centre for Economic Policy Research
77 Bastwick Street, London EC1V 3PZ, UK
Tel: (44 20) 7183 8801, Fax: (44 20) 7183 8820
Email: cepr@cepr.org, Website: www.cepr.org

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ABSTRACT

On the Strategic Disclosure of Feasible Options in Bargaining*

Most of the economic literature on bargaining has focused on situations where the set of possible outcomes is taken as given. This paper is concerned with situations where decision-makers first need to identify the set of feasible outcomes before they bargain over which of them is selected. Our objective is to understand how different bargaining institutions affect the incentives to disclose possible solutions to the bargaining problem, where inefficiency may arise when both parties withold Pareto superior options. We take a first step in this direction by proposing a simple, stylized model that captures the idea that bargainers may strategically withhold information regarding the existence of feasible alternatives that are Pareto superior. We characterize a partial ordering of "regular" bargaining solutions (i.e., those belonging to some class of "natural" solutions) according to the likelihood of disclosure that they induce. This ordering identifies the best solution in this class, which favors the "weaker" bargainer subject to the regularity constraints. We also illustrate our result in a simple environment where the best solution coincides with Nash, and where the Kalai-Smorodinsky solution is ranked above Raiffa's simple coin-toss solution. The analysis is extended to a dynamic setting in which the bargainers can choose the timing of disclosure.

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Geoffroy de Clippel
Department of Economics
Brown University
64 Waterman St.
Providence, RI 02912
USA

Email: declippel@brown.edu

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Kfir Eliaz
Department of Economics
Brown University
64 Waterman St
Providence, RI 02912
USA

Email: Kfir_Eliaz@brown.edu

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On the Strategic Disclosure of Feasible Options in Bargaining

Geoffroy de Clippel* Kfir Eliaz†

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Abstract

Most of the economic literature on bargaining has focused on situations where the set of possible outcomes is taken as given. This paper is concerned with situations where decision-makers first need to identify the set of feasible outcomes before they bargain over which of them is selected. Our objective is to understand how different bargaining institutions affect the incentives to disclose possible solutions to the bargaining problem, where inefficiency may arise when both parties withold Pareto superior options. We take a first step in this direction by proposing a simple, stylized model that captures the idea that bargainers may strategically withhold information regarding the existence of feasible alternatives that are Pareto superior. We characterize a partial ordering of "regular" bargaining solutions (i.e., those belonging to some class of "natural" solutions) according to the likelihood of disclosure that they induce. This ordering identifies the best solution in this class, which favors the "weaker" bargainer subject to the regularity constraints. We also illustrate our result in a simple environment where the best solution coincides with Nash, and where the Kalai-Smorodinsky solution is ranked above Raiffa's simple coin-toss solution. The analysis is extended to a dynamic setting in which the bargainers can choose the timing of disclosure.

1. INTRODUCTION

Bargaining theory aims to understand how parties resolve conflicting interests on which outcome to implement. The economic literature has focused so far on the case where the set of feasible alternatives is obvious, e.g. sharing some monetary value. There are, however, many situations where this set is not commonly known. In these situations, the bargainers themselves must propose feasible solutions to their conflict, and creativity to identify new options is likely to be mutually beneficial.

One common situation with these features is the selection of a candidate, or a group of candidates, for a task by several parties with conflicting interests: deciding which candidate to hire for a vacant post, choosing a candidate to run for office, deciding on the composition of some external committee, choosing an arbitrator, etc. Quite often it is

*Department of Economics, Brown University. Email: declippel@brown.edu. Financial support from the Deutsche Bank through the Institute for Advanced Study is gratefully acknowledged.

†Department of Economics, Brown University. Email: kfir_eliaz@brown.edu

not commonly known which potential candidates are suitable for the task, and which are actually willing to be nominated. Hence, the parties responsible for making the decision must propose names of candidates from which a selection can be made. Another example is that of international conflicts, where different parties may have conflicting interests on say, how to address the development of a nuclear program by hostile country, or how to fight against terrorism or how to resolve an ethnic conflict. Possible solutions may involve different forms of sanctions, a variety of military operations or the creation of new reforms or laws. The parties who wish to resolve the conflict would need to suggest concrete plans of actions.

Notice how monetary transfers are often not an option in the examples we have just described. But even in cases where monetary transfers are feasible, such as in labor-management negotiations, it is often important for the parties themselves to identify and propose the specific details and dimensions over which compromises could be made (e.g., pension plans, overtime wages, paid holidays, tenure clocks, etc.). All these examples share the feature that in order for the parties to resolve their decision problem or their dispute, they need to come up with a concrete set of feasible options to choose from.

Such conflicts of interests have received much attention in the more applied or popular literature (see e.g. Fisher et al. (1991), or the webpage of the Federal Mediation and Conciliation Service). For example, one of the key steps in what is known as *interest-based* (or *integrative*, or *win-win*, or *mutual gains*, or *principled*) bargaining technique is for both parties to suggest feasible options, before implementing an agreed-upon objective criterion (e.g. "traditional practices," "what a court would decide," "comparing the options' market value," "fairness," etc.) to evaluate them. However, private incentives may go against the systematic disclosure of win-win options: rational parties would anticipate what is the potential impact on the final outcome of disclosing an option. Thus, even if a party is aware of an alternative that is Pareto improving, it may decide to withhold that information in the hope that another party will reveal a more profitable option that it is not aware of. The popular literature on negotiations suggests that such strategic concerns are real and may impede negotiations. For instance, the disputing parties are instructed to suggest, one at a time and *as rapidly as possible* (which may be interpreted as a way to limit strategic considerations), a number of solutions that might meet the needs of the parties. It is often emphasized that evaluating the proposed options is irrelevant at this stage, as selection will occur only after a satisfactory number of options have been proposed or the parties have exhausted their ideas.[1]

While classical bargaining theory has taken the set of feasible agreements as an *exogenous* variable, this paper explores how options emerge *endogenously*. In particular, our objective is to understand how different bargaining institutions affect the incentives to disclose possible solutions to the bargaining problem, where inefficiency may arise when both parties withold Pareto superior options. We take a first step in this direc-

[1]One vivid example of this appears in Haynes (1986), who discusses the role of mediators when implementing an interest-based approach to divorce and family issues: "*If the mediator determines that the parties are withholding options with a covert strategy in mind, the mediator can cite a similar situation with another couple and describe different options they considered. This can helps break the logjam by forcing the couple to examine the options and including them on their list, thereby creating a greater level of safety for other options that are developed after one goes up on the board.*" In our model, though, feasible options can only be disclosed by the bargainers themselves - there will be no mediator with extra information to break the logjam.

tion by proposing a simple, stylized model that captures the idea that bargainers may strategically withhold information regarding the existence of feasible alternatives that are Pareto superior. Section 2 presents our basic model, which investigates the case where each bargainer knows only about one feasible solution to the bargaining, and his decision problem is whether or not to disclose his information. More specifically, there are two bargainers, who each has learned (in the sense of obtaining verifiable evidence) about the feasibility of some option. Neither bargainer knows what option the other has learned about, but they both have a common prior on the payoffs associated with the potentially feasible options. The bargainers first decide (simultaneously) whether or not to disclose their options, and then in the second stage, they apply a bargaining solution, which is modeled as a function that assigns to every set of disclosed options a lottery on the union of this set and the disagreement point. Attention is restricted to a class of bargaining solutions (referred to as "regular") with some reasonable properties, which in our basic set-up contains all the classical solutions such as Raiffa, Kalai-Smorodinsky and Nash. We interpret the "regularity" properties of a bargaining solution as "descriptive" properties in the sense that parties to a dispute would want to use bargaining procedures that possess these properties (they may be viewed as "normative" properties when one takes the set of agreements as given). We emphasize that our model abstracts from many details that accompany real-life negotiations (such as those described above) and may fit some situations better than others. Its purpose is not to give a one-to-one mapping of reality but rather, to provide a tractable framework that enables us to isolate the effect of the bargaining procedure on the incentives to disclose feasible options.

Focusing on the symmetric Bayesian Nash equilibria of the disclosure game, we show in Section 3 that each bargaining solution induces a unique, strictly positive, probability of no-disclosure (and hence, disagreement). This probability uniquely determines the ex-ante welfare of the bargainers in our model. In Section 4, we define a partial ordering on regular bargaining solutions, and show that being superior according to that ordering implies a higher degree of efficiency in the symmetric equilibrium of the disclosure game. In the simple environment we begin with, this partial ordering implies that the level of inefficiency is systematically lower when the Nash solution is applied than when the Kalai-Smorodinsky solution is applied, and in turn lower than when the Raiffa solution is applied. Moreover, in this environment the Nash solution induces the *minimal* level of inefficiency among all regular solutions. As a dual result, we also derive an upper-bound on the level of inefficiency that is possible when picking the final option according to a regular bargaining solution. In addition, we show that our partial ordering induces a lattice structure on the set of regular bargaining solutions: given any pair of regular solutions, we can construct a new pair of regular solutions, one which is more efficient than each of the original solutions, and another, which is less efficient.

When evaluating the efficiency of bargaining solutions, our approach is to take as given the set of solutions that are used (with the interpretation that most disputes are resolved via some regular bargaining solution), and ask which procedures perform better in terms of disclosure. An analagous approach is taken in the literature that examines the incentives to engage in costly information acquisition under different committee designs or under different auction formats (see Persico (2000, 2004)). An alternative, implementation-theoretic approach, which is *not* taken in this paper, is to try and internalize the incentive to disclose information by designing an optimal mechanism that assigns to every pair of bargainer types a probability of disclosure and a probability distribution over the feasible outcomes.[2]

In Section 5, we address the question of disclosure over time, where pure inefficiency now becomes a delay. In equilibrium, a bargainer immediately discloses an option if it is relatively favorable to him, and will delay disclosure for less favorable options, where the rate of delay is independent of the bargaining solution. However, the likelihood of disclosing immediately varies with the solution, and it turns out that the normative comparison derived for the one-shot game carries over to the dynamic game: delay is uniformly lower if the solution that is applied is larger according to the incomplete ordering defined in Section 4, and the Nash solution is thus optimal if the objective is to minimize the level of inefficiency. While in general, we cannot compare the bargainers' welfare in the static and dynamic game, we show that for a uniform distribution of bargainer types, the ex-ante expected welfare under the Raiffa, Nash and Kalai-Smorodinsly solutions are strictly lower in the dynamic game than in the static game. We conclude Section 5 by examining a variant of the dynamic disclosure game where bargainers have the possibility to react immediately after the other had disclosed his option, i.e. before the bargaining solution is applied. In that case, the equilibrium probability of immediate disclosure is independent of the bargaining solution. Furthermore, for every regular bargaining solution and for every bargainer type, the timing of disclosure is delayed relative to the original dynamic disclosure game.

Section 6 extends the analysis of the static game to a more general environment, where we characterize the most efficient regular bargaining solution (which reduces to the Nash solution in the simple emvironment of Section 2). This solution has the property that whenever two options have been disclosed, it seeks to maximize the expected payoff of the "weaker" bargainer (in the sense that his minimal payoff from the two disclosed options is lower than the minimal payoff of the other bargainer) subject to the constraint that the stronger bargainer obtains as close as possible to half of the maximal attainable surplus.

The final section of the paper, Section 7, provides some concluding remarks. Some proofs are relegated to the Appendix.

2. MODEL

Consider two bargainers who each learn about the feasibility of an option, represented in the space of utilities as a pair of non-negative real numbers (x_1, x_2). The set X, of all payoff pairs associated with the potentially feasible options, has the following properties. First, no element in X Pareto dominates another. Second, X is symmetric in the sense that if $(x_1, x_2) \in X$ then $(x_2, x_1) \in X$. We normalize the lowest and highest payoffs that any bargainer can achieve to zero and one, respectively. We will first focus on the case in which X is the line joining $(1, 0)$ to $(0, 1)$. In Section 6, we will discuss how our analysis can be extended to more general sets X with the above properties.

Bargainers do not know what option his opponent has learned is feasible. His beliefs regarding the payoffs from his opponent's option is described by a common symmetric density f on X with full support. Symmetry here means that $f(x_1, x_2) = f(x_2, x_1)$, for

[2] To illustrate the difference betwen these two approaches, compare Persico (2004) that studies the incentives to acquire information under prevalent committee designs (specifically, threshold voting rules), with Gerardi and Yariv (2008) that characterize the ex-ante optimal collective decision-making procedure.

each $(x_1, x_2) \in X$. For notational simplicity, individual i's type will be summarized by his own payoff in the option he is aware of. This is without loss of generality since the other component is the complementary number that guarantees a sum of 1.

The two bargainers play the following game. First, in the disclosure stage, they decide independently whether or not to disclose the feasibility of the option they are aware of. We assume that when bargainer i discloses an option, then the payoffs associated with that option, $(x, 1-x)$, become common knowledge (henceforth, we identify an option with the payoffs it induces).[3] Second, in the bargaining stage, an outcome is selected according to a lottery (referred to as the "bargaining solution") over the set of disclosed options and the disagreement outcome, which is assumed to give a zero payoff to both players. The bargaining solution may be a reduced-form to describe the equilibrium outcome of some specific bargaining procedure, or to describe the outcome following arguments in an unstructured bargaining situation, as those investigated, for instance, in the axiomatic literature.

We denote by $b(x, y)$ the pair of expected payoffs for the bargainers when applying the bargaining solution b if bargainer i disclosed the option $(x, 1-x)$ and bargainer j disclosed the option $(1-y, y)$. If only one bargainer, say i, disclosed an option $(x, 1-x)$, then the pair of expected payoffs is denoted $b(x, \emptyset)$. The bargaining solution is *regular* if the following properties are satisfied for $i = 1, 2$.

1. (Ex-post Efficiency) $b(x, \emptyset) = (x, 1-x)$ and for all $x, y \in [0, 1]$, there exists $\alpha \in [0, 1]$ such that $b(x, y) = \alpha(x, 1-x) + (1-\alpha)(1-y, y)$.
2. (Symmetry) $b_i(x, y) = b_i(1-y, 1-x)$ and $b_i(x, y) = b_j(y, x)$.
3. (Monotonicity) $x' \geq x, y' \leq y \Rightarrow b_i(x', y') \geq b_i(x, y)$, for all x, x', y, and y' in $[0, 1]$.

Our analysis will be limited to regular bargaining solutions, except where stated otherwise. It will prove useful to note that the three regularity conditions have the following implication.

Lemma 1 *If b is a regular bargaining solution, then for all x, x' such that $x' > x$, there exists a subset Y of $[0, 1]$ with strictly positive measure such that $b_i(x', y) > b_i(x, y)$, for all y in Y.*

Proof: See the appendix. ■

Discussion

Before we analyze the equilibria of this game, we comment on several key features of the model. Our model addresses situations in which there is a very large set of potentially feasible options, but parties do not know a priori which ones are actually feasible and/or what are their associated payoffs. The parties may have learned about the feasibility of an option and its associated payoff by chance, or they might have actively searched through the set of potential options until they discovered one which is actually feasible and

[3] Types are verifiable once disclosed, and hence an agent cannot report anything else than what he knows.

identified its associated payoffs. In that latter case, our work should be understood as a building block of a more elaborate model. Investigating the incentives to search in the first place remains an interesting open question. The density f can be interpreted as encoding the bargainers' subjective beliefs regarding what their opponent might know, and/or as the objective distribution of payoffs associated to feasible options in a situation where they do not know how these payoffs maps to physical options. As an illustration of the latter interpretation, consider two parties with conflicting objectives who need to agree on a person to hire. While the parties may know the distribution of the potential candidates' characteristics in the population, they may not necessarily know the characteristics of a specific candidate, and whether a candidate is interested in being considered for the position.

In addition, we consider those situations where the parties can only select from a set of concrete options for which there is verifiable evidence attesting to their feasibility (and from which the payoffs can be inferred). For example, when a group of individuals need to make a hiring decision for a vacant post, they can only choose among a list of candidates that was presented to them. Even though they might know the distribution of talents in the population, they will not consider the possibility of hiring a randomly drawn candidate. Similarly, when heads of countries meet to decide on a response to terrorism, they will only consider those concrete plans ofaaa actions that were presented to them.

Our analysis focuses on situations where the bargainers are completely symmetric ex-ante. Any asymmetry between the two bargainers is either at the interim stage because of their realized type and the actions they decide to take, or it is at the ex-post stage as a result of the bargaining solution. We, therefore, assume that the players make their disclosure decisions simultaneously (i.e., we do not impose any exogenous sequence of moves). This may be interpreted as a situation in which the two bargainers have scheduled a meeting to discuss the alternative solutions to their bargaining problem, and prior to the meeting, each bargainer needs to decide whether or not to bring all the documents that provide a detailed description of the option he knows.. Alternatively, the bargaining solution may represent the decision of an arbitrator, who requests the two parties to send him the evidence they have. In other words, we take the view, that the disclosure stage is unstructured, and that any pre-assigned sequence of disclosure cannot be enforced.[4]

The regularity conditions are meant to capture common features of prevalent bargaining procedures. These are interpreted as properties that most bargainers would find appealing, so much so that they would see their violation as a reason for not using the procedure to resolve their conflict. In this sense, we interpret the regularity conditions as descriptive properties of bargaining solutions, which were designed without taking into account their implication on disclosure. In Section 4 we discuss the case in which these conditions are relaxed.

Finally, as will become clear in the next section, our analysis will not change qualitatively (but will become messier) if we allowed for the possibility that a bargainer may fail to learn about any option.

[4]This view is motivated by case studies of real-life negotiations, where one rarely reads about a pre-specified order by which the parties are asked to disclose their evidence. Section 5 analyzes the case where one bargainer may disclose before another, but the timing of disclosure will be endogenous.

3. POSITIVE ANALYSIS OF THE DISCLOSURE GAME

A mixed-strategy for player i in the disclosure stage is a measurable function σ_i : $[0,1] \to [0,1]$, where $\sigma_i(x)$ is the probability that i announces his option while of type x. A pair of mixed strategies, one for each bargainer, forms a *Bayesian Nash equilibrium* (BNE) of the disclosure game if the action it prescribes to each type of each player is optimal against the strategy to the opponent. The BNE is *symmetric* if both bargainers follow the same strategy.

The key variables to consider to identify the BNEs of the game are the players' expected net gain of revealing over withholding when of a specific type and given the opponent's strategy:

$$ENG_1(x, \sigma_2) = x \int_{y=0}^{1} (1 - \sigma_2(y)) f(y) dy + \int_{y=0}^{1} \sigma_2(y)[b_1(x, y) - (1 - y)] f(y) dy,$$

for each type $x \in [0,1]$ and each strategy σ_2. The expected net gain of player 2 is similarly defined. We start by establishing two key properties of this function: it is strictly increasing in one's own type (independently of the opponent's strategy), and strictly decreasing in the likelihood of disclosure by the opponent.

Lemma 2 *1. $ENG_i(x, \sigma_{-i})$ is strictly increasing in x.*

2. If $\hat{\sigma}_{-i}(y) \geq \sigma_{-i}(y)$, for each $y \in [0,1]$, then $ENG_i(x, \hat{\sigma}_{-i}) \leq ENG_i(x, \sigma_{-i})$.

Proof: We assume $i = 1$. A similar argument applies to player 2. The fact that it is non-decreasing in x follows immediately from the monotonicity condition on b. If $\{y \in X | \sigma_2(y) < 1\}$ has a strictly positive measure, then it is strictly increasing in x via its first term. Otherwise, the function is strictly increasing in x, as a consequence of Lemma 1.

The second property follows from the fact that $b_1(x, y) - (1 - y) \leq x$, for each $(x, y) \in [0,1]^2$, which itself follows from the fact that $b_1(x, y) \leq \max\{x, 1 - y\}$, since b selects a convex combination between $(x, 1 - x)$ and $(1 - y, y)$. ■

Using this lemma, we can characterize the symmetric BNE of the disclosure game.[5]

Proposition 1 *The disclosure game has a unique symmetric BNE in which every player discloses his option if and only if his type is greater or equal to a threshold*

$$\theta = \sup\{x \in [0, \frac{1}{2}] \mid xF(x) - \int_{y=x}^{1} (b_1(x, y) - (1 - y)) f(y) dy < 0\}. \tag{1}$$

Hence, a positive measure of types withhold their information in the symmetric equilibrium. In addition, if the symmetric BNE is the unique BNE, then it is also the unique profile of strategies that survive the iterated elimination of strictly dominated strategies.

[5]Notice that the existence of a BNE is guaranteed even without any requirement of continuity on b.

<u>Proof</u>: The first property from Lemma 2 implies that there exists a best response to any strategy, and that any such best response coincides almost everywhere with a threshold strategy. More precisely, if σ_i^* is a best response against σ_{-i}, then there exists a unique $\theta_i \in [0,1]$ such that σ_i^* coincides almost everywhere with the threshold strategy $\sigma_i^{\theta_i}$, where $\sigma_i^*(x) = 0$, for each x such that $x < \theta_i$, and $\sigma_i^*(x) = 1$, for each $x \in [0,1]$ such that $x > \theta_i$. The existence of a symmetric BNE is thus equivalent to the existence of a fixed point to the correspondence that associates i's optimal threshold strategy to each of the opponent's threshold strategies, or $\theta_i = BR_i(\theta_{-i})$ for short. This will follow from Brouwer's fixed-point theorem after showing that BR_i is continuous. Let thus $(\theta(k))_{k\in\mathbb{N}}$ be a sequence of real numbers in $[0,1]$ that converges to some θ. Suppose on the other hand that $BR_i(\theta(k))$ converges to some $\theta' \neq BR_i(\theta)$. To fix ideas, we'll assume that $\theta' > BR_i(\theta)$ (a similar reasoning applies if the inequality is reversed). Hence there exists K such that $\frac{BR_i(\theta)+\theta'}{2} < BR_i(\theta(k))$, for all $k \geq K$, and

$$ENG_i(\frac{BR_i(\theta)+\theta'}{2}, \sigma^{\theta(k)}) < 0.$$

Taking the limit on k, we get

$$ENG_i(\frac{BR_i(\theta)+\theta'}{2}, \sigma^{\theta}) \leq 0,$$

by continuity of the integral with respect to its bounds, but which thus leads to a contradiction, since $\frac{BR_i(\theta)+\theta'}{2} > BR_i(\theta)$. Hence BR_i is indeed continuous, and admits a fixed-point.

The first property of Lemma 2, and the definition of the expected net gain, imply that $BR_1(\theta_2) = \sup\{x \in [0,\frac{1}{2}] \mid xF(x) - \int_{y=x}^{1}(b_1(x,y)-(1-y))f(y)dy < 0\}$. We will thus be done showing that all symmetric BNE's must satisfy (1) after proving that the threshold must fall below 1/2. We start by proving that the expected net gain from disclosing for type $\frac{1}{2}$ is zero when his opponent always discloses his type. To see this, note that the expected net gain from disclosing of some player, say 1, when his type is $\frac{1}{2}$ and $\sigma_2(y) = 1$ for all $y \in [0,1]$ is given by

$$\int_{y=0}^{1}[b_1(\frac{1}{2},y)-(1-y)]f(y)dy$$

This expression can be decomposed into two parts: one where the opponent's type is below $\frac{1}{2}$, and another where his type is above $\frac{1}{2}$. The second component may be rewritten as follows. First, using the symmetry of b we replace $b_1(\frac{1}{2},y) = b_2(y,\frac{1}{2})$. Since b selects a point on the line joining $(x,1-x)$ and $(y,1-y)$, we have that $b_2(y,\frac{1}{2}) = 1 - b_1(y,\frac{1}{2})$. By symmetry of b, we have that $b_1(y,\frac{1}{2}) = b_1(\frac{1}{2},1-y)$. Define the random variable $y' \equiv 1-y$ with density function f'. Note that by the symmetry of f, we have that $f'(y') = f(y)$ for any $y' \in [0,\frac{1}{2}]$ and $y = 1-y'$. It follows that the net expected gain of type $\frac{1}{2}$ equals

$$\int_{y=0}^{\frac{1}{2}}[b_1(\frac{1}{2},y)-(1-y)]f(y)dy - \int_{y'=0}^{\frac{1}{2}}[1-b_1(\frac{1}{2},y')-y')]f(y')dy,$$

which is equal to zero. Part 1 of Lemma 2 implies that $BR_i(0) = 1/2$. Part 2 of Lemma 2 implies that $BR_i(\theta) \leq 1/2$, for all $\theta \in [0,1]$, as desired.

We now show that the symmetric BNE must be unique. Suppose, on the contrary, that there were two symmetric BNE's. Let θ and θ' be the two corresponding common thresholds that the two players are using. Assume without loss of generality that $\theta' > \theta$, and let $\hat{\theta}$ be a number that falls between θ and θ'. Lemma 2 and the definition of the thresholds imply:

$$0 < ENG_1(\hat{\theta}, \sigma_2^{\theta}) \leq ENG_1(\hat{\theta}, \sigma_2^{\theta'}) < 0,$$

which is impossible. This establishes the uniqueness of the symmetric BNE.

Finally, let Σ be the set of strategies, for either player,[6] that survive the iterated elimination of strictly dominated strategies. Let then

$$\theta = \sup\{x \in [0,1] | (\forall \sigma \in \Sigma) : \sigma = 0 \text{ almost surely on } [0,x]\}$$

$$\theta' = \inf\{x \in [0,1] | (\forall \sigma \in \Sigma) : \sigma = 1 \text{ almost surely on } [x,1]\}.$$

Obviously, $\theta \leq \theta'$. Observe also that $\theta \leq BR_i(BR_i(\theta))$ if the disclosure game admits a unique BNE. Otherwise, the function that associates $x - BR_i(BR_i(x))$ to each x between 0 and θ is strictly positive at θ and non-positive at 0, and hence admits a zero by the intermediate values theorem. Let thus θ^* be an element of $[0, \theta)$ such that $\theta^* = BR_i(BR_i(\theta^*))$. Notice that the pair of strategies $(\sigma^{\theta^*}, \sigma^{BR_2(\theta^*)})$ then forms a BNE, which implies that $\sigma^{\theta^*} \in \Sigma$ and contradicts the definition of θ.

Any strategy in Σ for i's opponent has him withhold his information for almost every type between 0 and θ. The more his opponent reveals, the lower i's expected net gain, according to lemma 1. Hence if player i wants to disclose his type when his opponent uses σ^{θ}, then a fortiori he wants to disclose it when his opponent plays some strategy in Σ (because there is more disclosure with σ^{θ} than with any strategy from Σ). This means that against any strategy in Σ, player i's best response satisfies that he discloses his type whenever it is above $BR_i(\theta)$. Hence $\theta' \leq BR_i(\theta)$.

The second property in Lemma 2 implies that BR_i is non-increasing, and hence $BR_i(\theta') \geq BR_i(BR_i(\theta))$. In the same way we proved that $\theta' \leq BR_i(\theta)$, Lemma 2 and the definition of θ implies that $\theta \geq BR_i(\theta')$, and hence $\theta \geq BR_i(BR_i(\theta))$, by transitivity. Combining this with our earlier observation, we conclude that $\theta = BR_i(BR_i(\theta))$ and hence the pair of strategies $(\sigma^{\theta}, \sigma^{BR_2(\theta)})$ forms a BNE. Uniqueness of the BNE implies that this is in fact the symmetric BNE. Hence we must also have that $\theta = BR_i(\theta)$, which implies that $\theta' = \theta$, and we are done proving that the unique symmetric BNE is also the unique profile of strategies that survive the iterated elimination of strictly dominated strategies when the disclosure game admits a unique BNE. ■

Because the two bargainers are completely symmetric (ex-ante) in our set-up (both have equal bargaining abilities - symmetric b - and both are equally likely to discover the feasibility of any given alternative - symmetric f), our analysis focuses on the symmetric BNE. The disclosure game may also have asymmetric BNEs in addition to the unique symmetric one. Our next result establishes that a large class of bargaining solutions will induce an inefficient outcome at *any* BNE of the disclosure game.

[6] Indeed, the set of strategies that survive the iterated elimination of strictly dominated strategies is the same for both players because the game is symmetric.

A bargaining solution b is *strictly compromising* if it never selects the best option of a player when they disagree on their most preferred alternative: $b_1(x,y)$ is different from both x and $1-y$, for all $x,y \in [0,1]$ such that $x \neq 1-y$.

Proposition 2 *If b is strictly compromising, then inefficiency occurs with positive probability at any BNE of the disclosure game.*

Proof: Assume there exists a BNE in which player 1 always discloses his type. Then, as shown in the proof of Proposition 1, player 2's expected net gain when of type 1/2 is equal to zero. Lemma 2 implies that it is a strictly dominant action for player 2 to reveal his type whenever it is larger than 1/2. Hence, player 1's expected net gain from disclosure when his type is x and player 2's type is lower or equal to $\frac{1}{2}$ is at most $xF(1/2)$, which equals $\frac{1}{2}x$ by the symmetry of f. Define

$$\lambda = \int_{y=1/2}^{3/4} [(1-y) - b_1(\frac{1}{4}, y)] f(y) dy.$$

This is player 1's expected net loss from disclosure when his type is $\frac{1}{4}$ and player 2's types between $y = 1/2$ up to 3/4. By our assumption that b is strictly compromising, $\lambda > 0$. In addition, by the monotonicity of b, player 1's expected net loss from disclosure when his type is $x < \frac{1}{4}$ and player 2's types between $y = 1/2$ up to 3/4 is at least λ. Let

$$\delta(x) \equiv \int_{1-x}^{1} [b_1(x,y) - (1-y)] f(y) dy$$

This is player 1's expected net gain from disclosing when his type is x and player 2's type is higher than $1-x$. Since b is strictly compromising,

$$\delta(x) < \int_{1-x}^{1} [x - (1-y)] f(y) dy.$$

Hence, for any $x > 0$, we have that

$$\delta(x) < \int_{1-x}^{1} x f(y) dy = x[1 - F(1-x)] < \frac{1}{2}x$$

where the last inequality follows from symmetry of f. It follows that player 1's expected net gain from disclosure when his type is $x < \frac{1}{4}$ is smaller or equal to

$$\frac{1}{2}x - \lambda + \frac{1}{2}x = x - \lambda$$

(note we have not even taken into account the expected loss that occurs when player 2's type is between $\frac{1}{4}$ and $1-x$). Hence, any type $x < \min\{\frac{1}{4}, \lambda\}$ would strictly prefer not to disclose his type, a contradiction. ■

We now illustrate the mechanics of the disclosure game with some classical bargaining solutions and a uniform distribution f.

Raiffa

Perhaps the most natural bargaining solution when only two options are available is to simply toss a coin. This is precisely the definition of Raiffa's discrete bargaining solution (see Luce and Raiffa (1957, Section 6.7)):

$$b_R(x,y) = (\frac{x+(1-y)}{2}, \frac{(1-x)+y}{2})$$

for all $x, y \in [0,1]$. Recall from the proof of Proposition 1 that the best response to any strategy is a threshold strategy, and hence one may restrict attention to best responses in terms of the thresholds. Because the Raiffa solution is continuous, player 1's best response threshold θ_1 as a function of player 2 threshold θ_2 is obtained by looking for the root of player 1's expected net gain function:

$$ENG_1^{b_R}(\theta_1, \sigma_2^{\theta_2}) = \theta_1\theta_2 + \int_{y=\theta_2}^{1} \frac{\theta_1 - (1-y)}{2} dy = 0$$

or

$$\frac{\theta_1 + \theta_2 + \theta_1\theta_2}{2} - \frac{1+\theta_2^2}{4} = 0$$

which gives for $i = 1, 2$ and $j \neq i$:

$$\theta_i = BR_i(\theta_j) = \frac{(1-\theta_j)^2}{2(1+\theta_j)}$$

One can thus conclude that the disclosure game admits a unique BNE, which is the symmetric equilibrium with common threshold $-2 + \sqrt{5} \sim 0.236$.

Kalai-Smorodinsky

Consider now Kalai and Smorodinsky's (1975) bargaining solution. When applied to two points on the line X, it will pick the lottery so as to equalize the two players' utility gains relative to the best feasible option for them (usually called the "utopia point"). Formally:

$$b_{KS}(x,y) = (\frac{\max(x, 1-y)}{\max(x,1-y) + \max(1-x,y)}, \frac{\max(1-x,y)}{\max(x,1-y) + \max(1-x,y)}).$$

Using the fact that the Kalai-Smorodinsky solution is continuous, equation (1) characterizing the unique symmetric BNE becomes:

$$\theta_{KS}^2 + \int_{y=\theta_{KS}}^{1-\theta_{KS}} [\frac{1-y}{1-\theta_{KS}+1-y} - (1-y)]dy + \int_{y=1-\theta_{KS}}^{1} [\frac{\theta_{KS}}{\theta_{KS}+y} - (1-y)]dy = 0.$$

Re-arranging, developing, and making the change of variables $z = 1-y$ in the first part of the second term yields:

$$\theta_{KS}^2 - \int_{y=\theta_{KS}}^{1} (1-y)dy + \int_{z=\theta_{KS}}^{1-\theta_{KS}} \frac{z}{1-\theta_{KS}+z} dz + \int_{y=1-\theta_{KS}}^{1} \frac{\theta_{KS}}{\theta_{KS}+y} dy = 0.$$

Using integration by parts, this equation reduces to[7]

$$\frac{3\theta_{KS}^2 - 2\theta_{KS} + 1}{2} - (1-\theta_{KS})ln(2-2\theta_{KS}) - \theta_{KS}^2 ln(1+\theta_{KS}) = 0.$$

Solving this equation numerically yields that θ_{KS} is approximately 0.22.

Nash

Consider now Nash's (1950) bargaining solution. When applied to two points on the line X this solution picks the lottery that brings the players' utilities as close as possible to $(1/2, 1/2)$. Formally:

$$b_N(x,y) = \begin{cases} (x, 1-x) & \text{if } \min\{\frac{1}{2}, 1-y\} \leq x \leq \max\{\frac{1}{2}, 1-y\} \\ (1-y, y) & \text{if } \min\{\frac{1}{2}, 1-x\} \leq y \leq \max\min\{\frac{1}{2}, 1-x\} \\ (\frac{1}{2}, \frac{1}{2}) & \text{otherwise} \end{cases}$$

As in the previous examples, one may restrict attention to best responses in terms of thresholds, and the Nash solution being continuous, player 1's best response threshold θ_1 as a function of player 2's threshold θ_2 is obtained by looking for the root of player 1's expected net gain function. Following an earlier reasoning, we know that it is a dominant strategy for both players to reveal their types when above 1/2, and hence one can restrict atttention to to cases where θ_1 and θ_2 are no greater than 1/2. The root is thus characterized by the following equation:

$$ENG_1^{b_N}(\theta_1, \sigma_2^{\theta_2}) = \theta_1\theta_2 + \int_{y=\theta_2}^{1/2} [\frac{1}{2} - (1-y)]dy + \int_{y=1/2}^{1-\theta_1} 0dy + \int_{y=1-\theta_1}^{1} [\theta_1 - (1-y)]dy = 0$$

or

$$\frac{\theta_1^2}{2} + \theta_1\theta_2 + \frac{\theta_2}{2} - \frac{\theta_2^2}{2} - \frac{1}{8} = 0$$

which gives for $i = 1, 2$ and $j \neq i$:

$$\theta_i = BR_i(\theta_j) = -\theta_j + \sqrt{2\theta_j^2 - \theta_j + \frac{1}{4}}$$

One can thus conclude that the disclosure game admits three BNEs, two in which one player reveals all his types while the other reveals only when his type is above 1/2,[8] and the unique symmetric equilibrium where the common threshold equals $(-1+\sqrt{3})/4 \sim 0.183$.

[7] Integrating by parts, one gets $\int \frac{w}{\alpha + w} = w - \alpha ln(\alpha + w)$, for each α such that $\alpha + w > 0$. Hence the sum of the third and fourth terms is equal to $[z - (1-\theta)ln(1-\theta+z)]_{z=\theta}^{1-\theta} + \theta[y - \theta ln(\theta+y)]_{y=1-\theta}^{1}$, or $1 - 2\theta - (1-\theta)ln(2-2\theta) + \theta[\theta - \theta ln(1+\theta)]$.

[8] Notice that the Nash bargaining solution is not strictly compromising when both options falls on the same side of X compared to $(1/2, 1/2)$, and this explains why one gets efficient asymmetric equilibria without contradicting the content of Remark 1. The fact that it is not strictly compromising probably makes the Nash solution less convincing as a positive description of reasonable bargaining outcomes, which is related to Luce and Raiffa's (1957) and Kalai and Smorodinsky's (1975) criticisms of the Nash solution, but it does provides good incentives for the participants to disclose their information regarding feasible options (more on this in the next Section).

4. NORMATIVE ANALYSIS: HOW TO FAVOR DISCLOSURE?

We now introduce a partial ordering on bargaining solutions that will allow us to compare their performance in terms of efficiency when taking the disclosure game into account. For any two bargaining solutions b and b', we will write $b' \succeq b$ whenever the following condition holds: $(\forall x \leq 1/2)(\forall y \geq x) : b'_1(x, y) \geq b_1(x, y)$ (the symmetry of b and b' also imply that $(\forall y \leq 1/2)(\forall x \geq y) : b'_2(x, y) \geq b_2(x, y)$).

Proposition 3 *If $b' \succeq b$, then the probability of inefficiency in the symmetric equilibrium of the disclosure game associated with b' is smaller or equal to the probability of inefficiency in the symmetric equilibrium of the disclosure game associated with b.*

Proof: Recall from the proof of Proposition 1 that the unique symmetric BNE of the disclosure game associated with any regular bargaining solution involves threshold strategies, whose common threshold falls in the interior of $[0, \frac{1}{2}]$. Let θ be the threshold associated to b, and θ' be the threshold associated to b'. Notice that player 1's expected net gain of revealing over withholding under b when of type θ' while the opponent plays the threshold strategy associated to θ' is non-positive:

$$ENG_1^b(\theta', \sigma_2^{\theta'}) \leq 0. \tag{2}$$

Indeed, this inequality actually holds pointwise, since $b' \succeq b$ and player 2 withholds his information when $y < \theta'$, and is thus preserved through summation.

We are now ready to conclude the proof by showing that $\theta \geq \theta'$ (indeed, the probability of ending up with an inefficient outcome, i.e. the disagreement point, is equal to the square of the BNE threshold). Suppose, on the contrary, that $\theta < \theta'$, and let $\tilde{\theta}$ be a number that falls between θ and θ'. Remember our first observation in the proof of Proposition 1 that a player's expected net gain is increasing in his own type. Inequality (2) thus implies that

$$ENG_1^b(\tilde{\theta}, \sigma_2^{\theta'}) < 0.$$

Remember also the second observation from the proof of Proposition 1, namely that a player's expected net gain does not increase when the opponent reveals more, and hence

$$ENG_1^b(\tilde{\theta}, \sigma_2^{\theta}) < 0,$$

but this contradicts the fact that the threshold strategies associated to θ forms a BNE of the disclosure game associated to b (as it should be optimal for player 1 to reveal his option at $\tilde{\theta}$ since it is larger than θ). ■

Corollary 1 *The probability of inefficiency in the symmetric equilibrium of the disclosure game associated with any regular bargaining solution is larger or equal to the probability of inefficiency in the symmetric equilibrium of the disclosure game associated with the Nash bargaining solution.*

Proof: This follows from the previous Proposition, after proving that $b_N \succeq b$, for any regular bargaining solution b. Let x be a number smaller or equal to $1/2$, and let us prove that $b_N(x, y)$ is more advantageous to player 1 than $b(x, y)$, for all $y \geq x$. This is obvious when $y \geq 1/2$ since the Nash bargaining solution picks the right-most option in that region. Since b is symmetric, it must be that $b(x,x) = (1/2, 1/2)$. Monotonicity implies that $b_1(x,y) \geq 1/2$ for each $y \in [x, 1/2]$, hence, the desired inequality when compared to the Nash bargaining solution which always picks 1/2 in that region. ■

A key property of the Nash solution, which helps explain why it maximizes disclosure (within the class of regular solutions) is that this solution favors the "weak" party in the bargaining.

Definition. *Given any pair of options,* $(x, 1-x)$ *and* $(1-y, y)$*, bargainer* 1 *is said to be in a weaker bargaining position than bargainer* 2 *if* $\min\{x, 1-y\} < \min\{1-x, y\}$*, and vice versa if the former is smaller than the latter.*

In other words, a bargainer is in a better position if the worst payoff he can get, given the disclosed options, is higher than the worst payoff of the other bargainer. Note that all regular bargaining solutions indeed give a higher final payoff to the bargainer who is stronger in the above sense.[9]

Let the *utilitarian sum* be the maximal sum of expected payoffs over all payoffs on the line connecting the disclosed options. Note that when the utility frontier is linear, the sum of expected payoffs is constant. Note also that in our setting the term 'utilitarian' does not imply interpersonal comparisons since the symmetry we impose in the space of utilities amounts to a normalization of the Bernoulli functions.[10] The Nash solution then maximizes the expected payoff of the weaker bargainer, subject to the constraint that the stronger bargainer receives at least half of the utilitarian sum. An alternative way to describe the Nash solution is to say that it selects the Pareto optimal point (i.e., on the line connecting the payoffs associated with the disclosed options) that gives the strongest bargainer an expected payoff that is as "close as possible" to half the utilitarian surplus. As we show in Section 6, this defines the most efficient bargaining solution when the utility frontier is not necessarily linear.

The minimal amount of disclosure

A dual to Corollary 1 gives us an upper bound on the probability of inefficiency associated with any regular bargaining solution. Consider the bargaining solution that maximizes the maximum of the two players' payoffs,

$$b_{MM}(x,y) = \begin{cases} (x, 1-x) & \text{if } x < y \\ (1-y, y) & \text{if } y < x \\ (1/2, 1/2) & \text{if } x = y \end{cases}$$

(in other words, this solution picks the point that is the furthest from $(1/2, 1/2)$, i.e., it minimizes the product of the bargainers' payoffs). It is easy to check that b_{MM} is regular. It is obvious that $b \succeq b_{MM}$, for any regular bargaining solution b, since b_{MM} picks the left-most point in X whenever player 1 reports an option $x \leq 1/2$ and player 2 reports

[9]Indeed, suppose, for instance, that 1 is weaker than 2 and that $x \leq 1-y$ to fix ideas (a similar argument applies in the other cases). In that case, $x \leq y$. Symmetry implies that $b_1(x,x) = 1/2$. Monotonicity implies that $b_1(x,y) \leq 1/2$, and hence $b_1(x,y) \leq b_2(x,y)$.

[10]Similarly, the Kalai-Smorodinsky solution is a scale-covariant solution, but can be described as the egalitarian principle applied to the problem where the utopia point has been normalized to (1,1).

an option $y > x$ (both solution equal $(1/2, 1/2)$ when they both report x, by symmetry). Proposition 3 allows us to conclude that the probability of inefficiency at the symmetric equilibrium in the disclosure game associated with any regular bargaining solution is smaller or equal to the probability of inefficiency at the symmetric equilibrium in the disclosure game associated with the above bargaining solution. Simple computations in the case of a uniform f yields that the common threshold in the unique symmetric BNE is equal to $1 - \sqrt{1/2} \sim 0.293$.

Kalai-Smorodinsky vs. Raiffa

We also have $b_{KS} \succeq b_R$, and hence, the equilibrium outcome associated with the Kalai-Smorodinsky solution is never less efficient than the one associated with the Raiffa solution. To see this, let $x \leq 1/2$ and $y \geq x$. We need to prove that player 1's payoff under the Kalai-Smorodinsky solution is larger than his payoff under the Raiffa solution when he reports x and his opponent reports y. Consider first the case where $y \leq 1 - x$, for which the relevant inequality to check is

$$\frac{1-y}{(1-y)+(1-x)} \geq \frac{x+(1-y)}{2}.$$

Simple algebra shows that this inequality is equivalent to $0 \geq x(1-x) - y(1-y)$, which indeed holds true since the function $f(z) = z(1-z)$ is symmetric around $1/2$, increasing before $1/2$ and decreasing after $1/2$. Similarly, the relevant inequality to check when $y \geq 1 - x$ is

$$\frac{x}{x+y} \geq \frac{x+(1-y)}{2}.$$

Simple algebra shows that this inequality is equivalent to $0 \geq -x(1-x) + y(1-y)$, which again holds true because of the properties of the function.

Combining bargaining solutions

Proposition 3 implies an algorithm that transforms any pair of regular solutions into a regular solution, which is at least as efficient as each of the two original solutions. A similar procedure yields a regular solution, which is less efficient than each of the original solutions. In other words, the partial ordering of solutions according to their efficiency induces a lattice structure over regular bargaining solutions.

For any pair of regular bargaining solutions, b and b', let $b \vee b'$ be the bargaining solution defined as follows. First,

$$(b \vee b')_1(x,y) = \begin{cases} \max\{b_1(x,y), b'_1(x,y)\} & \text{if } \min\{x, 1-y\} \leq \min\{1-x, y\} \\ \min\{b_1(x,y), b'_1(x,y)\} & \text{if } \min\{x, 1-y\} \geq \min\{1-x, y\} \end{cases}$$

where $(b \vee b')_2(x,y) = 1 - (b \vee b')_1(x,y)$. Second, $(b \vee b')(x, \emptyset) = (x, 1-x)$ and similarly, $(b \vee b')(\emptyset, y) = (1-y, y)$. In an analogous way we define $b \wedge b'$, where the only difference is the following: if bargainer 1 is weak, then $(b \wedge b')_1(x,y)$ equals $\min\{b_1(x,y), b'_1(x,y)\}$, and if bargainer 2 is weak , then $(b \wedge b')_1(x,y)$ equals $\max\{b_1(x,y), b'_1(x,y)\}$.

Proposition 4 *(i) $b \vee b'$ and $b \wedge b'$ are regular solutions, (ii) $b \vee b' \succeq b$ and $b \vee b' \succeq b'$, and (iii) $b \succeq b \wedge b'$ and $b' \succeq b \wedge b'$.*

Proof: To establish (i), first notice that $b \vee b'$ and $b \wedge b'$ are well-defined, as $\min\{x, 1-y\} = \min\{1-x, y\}$ if and only if $x = y$, in which case $b_1(x,y) = b'_1(x,y) = \frac{1}{2}$. Next, it is easy to check that $b \vee b'$ and $b \wedge b'$ satisfy efficiency and symmetry. Next, consider a pair of real-valued functions, $\phi(\cdot)$ and $\varphi(\cdot)$, defined over some subset of $\mathbb{R}$. If both $\phi(\cdot)$ and $\varphi(\cdot)$ are non-decreasing then so are $\max\{\phi(\cdot), \varphi(\cdot)\}$ and $\min\{\phi(\cdot), \varphi(\cdot)\}$. Also, if α is a real number such that $\phi(\alpha) = \varphi(\alpha)$, then $h(\cdot)$, where $h(z) = \phi(z)$ if $z \leq \alpha$ and $h(z) = \varphi(z)$ if $z \geq \alpha$, is also non-decreasing. Hence $b \vee b'$ and $b \wedge b'$ is monotone (apply these simple facts to x and y in turn). We conclude by establishing (ii) and (iii). For any $x \leq 1/2$ and $y \geq x$, we have that $\min\{x, 1-y\} \leq \min\{1-x, y\}$, in which case $(b \vee b')_1(x,y)$ is equal to $\max\{b_1(x,y), b'_1(x,y)\}$. By the definition of the partial order $\succeq$, it follows that $b \vee b' \succeq b$ and $b \vee b' \succeq b'$. A similar argument implies that $b \succeq b \wedge b'$ *and* $b' \succeq b \wedge b'$. ■

Regularity and disclosure

As mentioned above, the regularity conditions may be interpreted as reasonable properties of a bargaining solution, which is meant to reach a compromise between parties with conflicting preferences. However, in the simple environment of the previous subsections, these properties restrict the extent to which bargainers would be willing to disclose options in equilibrium. This is easily seen by noting that a dictatorial bargaining solution guarantees efficiency in our model as it becomes a weakly dominant strategy to always disclose. Disclosure is also weakly dominant under an (ex-post) inefficient bargaining solution that implements disagreement unless *both* bargainers disclose their options.

However, these solutions would *not* guarantee efficiency in the following two extensions of our model: (*i*) introducing an exogenous probability p that a bargainer has no option to disclose, and (*ii*) expanding the set of potentially feasible options such that the option known to one bargainer may be Pareto inferior to the option known to the other bargainer. While the first extension can be easily accommodated, the second extension is more challenging. For example, consider the case where each bargainer independently draws a type from a distribution on $[0,1]^2$. The difficulty here is that a bargainer's net expected gain from disclosing is not necessarily increasing in his type, and hence, proving existence of a symmetric pure-strategy equilibrium is not straightforward. Furthermore, it is not clear how such equilibria (if they exist) would look like (i.e., what would be the analogue of the cutoff strategies of the "one-shot" game).

While monotone bargaining solutions may be appealing to parties in conflict, they restrict the extent to which bargainers are willing to disclose feasible options. To see why a non-monotonic solution may out-perform any regular solution, consider the bargainer solution that selects the disclosed option, which maximizes the payoff for the weak bargainer. Note this is a "deterministic" variant of the Nash solution, which always picks the option that maximizes the product of the players' payoffs without ever trying to compromise through the use of lotteries. This solution violates the monotonicity condition that is part of the definition of a regular solution. Indeed, it picks $(1/2, 1/2)$ if the set of available options is $\{(1/3, 2/3), (1/2, 1/2)\}$, and $(1/3, 2/3)$ if the set of available options is $\{(1/3, 2/3), (3/4, 1/4)\}$. Player 1's payoff thereby decreases, while the available options become more favorable to him. When f is uniform, the one-shot disclosure game induced by this solution has a symmetric BNE where the probability of disclosure in equilibrium is given by

$$\sigma(x) = \begin{cases} -\frac{1}{3} + \frac{4}{3}(\frac{1}{2-4x})^{\frac{3}{2}} & if \quad x \leq \frac{1}{4} \\ 1 & if \quad x > \frac{1}{4} \end{cases}$$

The aggregate probability with which a bargainer withholds his information is equal to 0.138, and hence, the overall probability of inefficiency is lower than under *any* regular bargaining solution.

Note that the equilibrium threshold induced by the above bargaining solution is actually higher than the threshold induced by the monotonic Nash solution. The reason the non-monotonic solution is more efficient stems from the fact that every type discloses with some positive probability. This highlights the difficulty in characterizing the most efficient bargaining solution among those that are symmetric and ex-post efficient, but not necessarily monotone. Providing such a characterization remains an open problem.

5. DISCLOSURE OVER TIME

One may argue that players would not remain silent if the outcome of the static game is inefficient because none of them spoke up. It is thus important to discuss the dynamic extension of our game. The bargainers now decide when to speak, and the solution is implemented as soon as at least one option has been disclosed. For simplicity, we will restrict attention right away to symmetric pure-strategy Bayesian Nash equilibria. A strategy is a measurable function $\tau : [0,1] \to \mathbb{R}_+ \cup \{\infty\}$, which determines for each type x the time $\tau(x)$ at which to reveal x.[11] Measurability means that the inverse image of any Lebesgue measurable set (in particular any interval) is Lebesgue measurable: $\tau^{-1}(T) = \{x \in [0,1] | \tau(x) \in T\}$ is Lebesgue measurable if T is Lebesgue measurable. It guarantees that a player's expected utility when his opponent is known to reveal over some given interval of time, is well-defined. Utilities are discounted exponentially over time following a discount factor $\delta < 1$. The outcome when player 1 is of type x, while player 2 is of type y, and they both implement the strategy τ, is x at time $\tau(x)$ if $\tau(x) < \tau(y)$, y at time $\tau(y)$ if $\tau(x) > \tau(y)$, and $b(x,y)$ at time $\tau(x)$ if $\tau(x) = \tau(y)$. The strategy τ is part of a *symmetric Bayesian Nash equilibrium* if, for every type $x \in [0,1]$, the expected net gain of revealing at any time $t \geq 0$ different from $\tau(x)$ is non-positive, where a player's expected net gain - let's say player 1 to fix notations - is given by the following formula when $t > \tau(x)$ (a similar formula applies in the other case):

$$ENG_1(t \text{ vs. } \tau(x), x) = x(e^{-\delta t} - e^{-\delta \tau(x)}) \int_{y \in \tau^{-1}(]t,\infty])} f(y) dy$$

$$+ \int_{y \in \tau^{-1}(t)} (e^{-\delta t} b_1(x,y) - e^{-\delta \tau(x)} x) f(y) dy$$

$$+ \int_{y \in \tau^{-1}(]\tau(x),t[)} (e^{-\delta \tau(y)}(1-y) - e^{-\delta \tau(x)} x) f(y) dy$$

[11] $\tau(x) = \infty$ means that the player never discloses his option when of type x.

$$+\int_{y\in\tau^{-1}(\tau(x))} e^{-\delta\tau(x)}((1-y)-b_1(x,y))f(y)dy.$$

This dynamic disclosure game is similar in spirit to a war of attrition since both parties incur the cost of delay when neither gives in. There are two important distinctions between the two games. First, in contrast to the war of attrition, when a player discloses no sooner than his rival, that player's payoff depends on the rival's type. Second, the players' preferences are not quasi-linear in the cost of delay. Both distinctions imply that the standard techniques used to solve the war of attrition do not apply here.

We will need the following additional assumption on b to establish the uniqueness of the symmetric BNE:[12]

$$b_1(x,\frac{1}{2}) < x, \forall x > 1/2, \text{ and } b_1(x,\frac{1}{2}) > x, \forall x < 1/2. \tag{3}$$

The weak inequality is implied by the first regularity condition. Requiring a strict inequality is a mild additional requirement which is satisfied by all the classical solutions (Kalai-Smorodinsky, Nash and Raiffa). Notice, on the other hand, that the max-max solution (b_{MM}) does not satisfy this additional condition.

Proposition 5 *Let τ^* be the strategy defined as follows:*

$$\tau^*(x) = \begin{cases} 0 & \text{if } x \geq \theta \\ \int_x^\theta \frac{(1-2y)f(y)}{\delta y F(y)}dy & \text{if } x \leq \theta, \end{cases}$$

where

$$\theta = \sup\{x \in [0,1/2] | \int_{y=x}^1 (b_1(x,y)-(1-y))f(y)dy < 0\}. \tag{4}$$

Then (τ^*,τ^*) forms a symmetric Bayesian Nash equilibrium of the dynamic disclosure game. If b satisfies condition (3),[13] then it is essentially[14] the unique symmetric BNE of the game.

Proof: We prove that the strategy τ^* is indeed part of a symmetric BNE. The proof of uniqueness is relegated to the Appendix.

We start by showing that reporting at $\tau^*(x)$ is optimal, for any $x \in [0,\theta[$. Consider first the possibility of revealing at positive times. The function τ^* being invertible on $[0,\theta[$, we can identify any positive time with the type speaking at that time. The expected utility from revealing at $\tau^*(z)$ when of type x is equal to

$$U(z|x) := xF(z)e^{-\delta\tau^*(z)} + \int_{y=z}^1 (1-y)e^{-\delta\tau^*(y)}f(y)dy,$$

[12] A similar pair of conditions necessarily hold for player 2 as well, as a consequence of the second regularity condition.

[13] We conjecture that the uniqueness result remains valid even without this extra condition, but this remains an open problem.

[14] Formally, (τ,τ) is a symmetric BNE if and only if $\tau = \tau^*$ on $]0,1]$ and $\tau(0) \geq \tau^*(0)$. If $\int_x^\theta \frac{(1-2y)f(y)}{\delta y F(y)}dy$ does not diverge when x tends to zero, then there are multiple equilibria but they differ only in the zero type action.

for each $z \in [0, \theta[$. This expression is differentiable, and the derivatives is equal to

$$xf(z)e^{-\delta\tau^*(z)} - \delta x(\tau^*)'(z)F(z)e^{-\delta\tau^*(z)} - (1-z)f(z)e^{-\delta\tau^*(z)},$$

or

$$\frac{(1-z)}{z}f(z)(x-z)e^{-\delta\tau^*(z)}$$

after rearranging the terms and using the definition of τ^* to compute $(\tau^*)'$. We see that the first order condition is satisfied at $z = x$, and that the derivative is positive when $z < x$ and negative when $x < z$. Hence there is no profitable deviation to a positive time different from $\tau^*(x)$, when of type x. Deviating to report at zero is not profitable either, as the expected payoff in that case is

$$xF(\theta) + \int_{y=\theta}^{1} b_1(x,y)f(y)dy$$

which is equal to

$$U(\theta|x) + \int_{y=\theta}^{1} (b_1(x,y) - (1-y))f(y)dy.$$

For any $\epsilon > 0$ small enough, using the third regularity condition, this expression is lower or equal to

$$U(\theta|x) + \int_{y=\theta-\epsilon}^{1} (b_1(\theta-\epsilon,y) - (1-y))f(y)dy + \int_{y=\theta-\epsilon}^{\theta} ((1-y) - b_1(\theta-\epsilon,y))f(y)dy.$$

The second term is negative, for all $\epsilon > 0$, by definition of θ. Hence, taking the limit when ϵ decreases to zero, we get that the expected utility of reporting at zero is no greater than $U(\theta|x)$, which in turn, by our previous reasoning, is smaller than the expected utility of reporting at $\tau(x)$. This establishes the optimality of τ^*, for any type strictly in between 0 and θ.

Consider now a type $x \in [\theta, 1]$. The expected utility of revealing at a time t is equal to $U(z|x)$, where z is the unique real number in $[0, \theta[$ such that $\tau^*(z) = t$. Our earlier reasoning regarding U's derivative implies that this expected utility is no larger than $U(\theta|x)$ (since $z < \theta \leq x$), which is equal to $xF(\theta) + \int_{y=\theta}^{1}(1-y)f(y)dy$. This, in turn, is no larger than

$$xF(\theta) + \int_{y=\theta}^{1} (1-y)f(y)dy + \int_{y=\theta+\epsilon}^{1} (b_1(\theta,y) - (1-y))f(y)dy,$$

for all $\epsilon > 0$, by definition of θ. Taking the limit when ϵ tends to zero, the last expression is then equal to the expected utility of type $x$when revealing at zero. We have thus proved the optimality of τ^* for any type no smaller than θ. ■

The equilibrium behavior in the dynamic game is a natural variant of the equilibrium behavior in the static game studied previously. Indeed, there is a threshold above which players reveal their options, while lower types now reveal with delay instead of withholding their information forever due to the rules of the game. It turns out that the partial ordering identified in Proposition 3 continues to predict the efficiency of disclosure in the dynamic game as well.

Proposition 6 *Let b and b' be two regular bargaining solutions that satisfy (3), and let τ and τ' be the strategies in the symmetric BNE of the dynamic disclosure game associated to b and b' respectively. If $b' \succeq b$, then $\tau'(x) \leq \tau(x)$, for each $x \in [0, 1]$.*

Proof: Given the characterization of the symmetric BNE in Proposition 5, we see that proving $\tau'(x) \leq \tau(x)$, for each $x \in [0, 1]$, is equivalent to proving $\theta' \leq \theta$, where θ and θ' are the thresholds defined in (4) for b and b' respectively. Suppose, to the contrary of what we want to prove, that $\theta' > \theta$. Then for any $\epsilon > 0$ small enough so that $\theta' - \epsilon > \theta$, we have:

$$\int_{y=\theta'-\epsilon}^{1} (b_1(\theta' - \epsilon, y) - (1 - y))f(y)dy \geq 0,$$

by definition of θ. Since $b' \succeq b$, we must also have

$$\int_{y=\theta'-\epsilon}^{1} (b'_1(\theta' - \epsilon, y) - (1 - y))f(y)dy \geq 0,$$

but this contradicts the definition of θ'. Hence $\theta' \leq \theta$, as desired. ■

Hence, disclosure is faster with Nash, than with Kalai-Smorodinsky, than with Raiffa, and any pair of regular solutions can be combined as in the previous Section to derive a solution where disclosure is faster, and another where disclosure is slower.

Dynamic vs. static

In comparing between the dynamic and the static versions of the disclosure game, we begin by showing that for any regular bargaining solution satisfying condition (3), the lowest type to disclose in the static game is lower than the lowest type who discloses immediately in the dynamic game.

Proposition 7 *Let b be a regular bargaining solution that satisfies (3). Let θ_S and θ_D be the thresholds given by (1) and (4), respectively (i.e., the former is the cutoff of the one-shot simulatenous game, while the latter is the cutoff of the dynamic game). Then $\theta_S \leq \theta_D$.*

Proof: Assume $\theta_S > \theta_D$ and let $\hat{\theta}$ be a player type between θ_S and θ_D. Consider the static disclosure game first. Assume player j uses the symmetric equilibrium strategy associated with the threshold θ_S. Then for type x of player i, the expected net gain from disclosing, given by

$$xF(\theta_S) + \int_{\theta_S}^{1} [b_i(x, y) - (1 - y)]f(y)dy$$

is positive for all $x > \theta_S$ and negative for all $x < \theta_S$. In particular, it is negative for $x = \hat{\theta} < \theta_S$. Since $\hat{\theta}F(\theta_S)$ is strictly positive, it follows that

$$\int_{\theta_S}^{1} [b_i(\hat{\theta}, y) - (1 - y)]f(y)dy < 0$$

Because $\theta_S \leq \frac{1}{2}$, we have that $1 - y > \hat{\theta}$ for all $\hat{\theta} \leq y \leq \theta_S$. Hence, $b_i(\hat{\theta}, y) \leq (1 - y)$ for all $\hat{\theta} \leq y \leq \theta_S$. Therefore,

$$\int_{\hat{\theta}}^{1} [b_i(\hat{\theta}, y) - (1 - y)] f(y) dy < 0$$

This contradicts the definition of $\theta_D < \hat{\theta}$ in Proposition 7. ■

Example 1. *To illustrate Proposition 7, we compute the equilibrium threshold of the dynamic game associated with the Raiffa (θ_D^R), Kalai-Smorodinsky (θ_D^{KS}) and Nash solutions (θ_D^N) for a uniform distribution. By the continuity of these bargaining solutions, the three thresholds are given by the solutions between* 0 *and* $\frac{1}{2}$ *to the following equations:*

$$\int_{y=\theta_D^R}^{1} \frac{\theta_D^R - (1 - y)}{2} dy = 0$$

which yields $\theta_D^R = \frac{1}{3}$ *(compared with* $\theta_S^R = 0.24$ *in the static game),*

$$\int_{y=\theta_D^{KS}}^{1-\theta_D^{KS}} [\frac{1 - y}{1 - \theta_D^{KS} + 1 - y} - (1 - y)] dy + \int_{y=1-\theta_D^{KS}}^{1} [\frac{\theta_D^{KS}}{\theta_D^{KS} + y} - (1 - y)] dy = 0$$

which yields $\theta_D^{KS} \approx 0.34$ *(compared with* $\theta_S^{KS} \approx 0.22$ *in the static game), and*

$$\int_{y=\theta_D^N}^{1/2} [\frac{1}{2} - (1 - y)] dy + \int_{y=1-\theta_D^N}^{1} [\theta_D^N - (1 - y)] dy = 0$$

which yields $\theta_D^N = \frac{1}{4}$ *(compared with* $\theta_S^N \approx 0.18$ *in the static game).*

Proposition 7 raises the following question: given a regular bargaining solution satisfying (3), are bargainers better off in the symmetric Nash equilibrium of the static game or the dynamic game? We address this question in the special case where f is uniform. Let b be a (regular) bargaining solution, and let θ_S be the common threshold for disclosing in the static game. Observe that the ex-ante expected sum of payoffs is equal to

$$1 - \theta_S^2 \tag{5}$$

since the sum of the bargainers' payoffs equals 1 when at least one of them dicloses his option, and 0 otherwise. Since the two bargainers are ex-ante symmetric, the ex-ante expected payoff of each is equal to $(1 - \theta_S^2)/2$.

A similar reasoning implies that the sum of bargainers' ex-ante expected payoffs in the symmetric BNE of the dynamic game is equal to

$$1 - \theta_D^2 + \int_{x=0}^{\theta_D} \int_{y=0}^{\theta_D} e^{-\delta \tau(\max\{x,y\})} dx dy \tag{6}$$

where

$$\tau(x) = \int_{x}^{\theta_D} \frac{1 - 2y}{\delta y^2} dy = -\frac{1}{\delta \theta_D} + \frac{1}{\delta x} - \frac{2}{\delta} \ln \theta_D + \frac{2}{\delta} \ln x \tag{7}$$

for $x \leq \theta_D$ and uniform f. In the Appendix we show that (6) is then equal to

$$1 - \theta_D^2 - 2e^{1/\theta_D} \cdot \theta_D^2 \cdot E_i(-\frac{1}{\theta_D}) \tag{8}$$

where $E_i(x)$ denotes the *exponential integral*.[15] Again, symmetry implies that the each bargainer's ex-ante expected payoff is 1/2 of this expression. Note it does not depend on the discount factor.

Substituting into (5) and (8) the equilibrium thresholds computed earlier yields that a bargainer's ex-ante expected payoff (which equals half of the sum of expected payoffs) in the static game is higher than his expected payoff in the dynamic game for each of the three bargaining solutions. Specifically, the ex-ante expected payoff for the Raiffa solution is 0.472 in the dynamic game and 0.474 in the static game; for Kalai-Smorodinsky, the ex-ante expected payoff is 0.473 in the dynamic game and 0.476 in the static game; and finally, for the Nash solution, the ex-ante expected payoffs in the dynamic and static games are 0.481 and 0.483, respectively. Though the magnitudes are not large, the quantitative result is interesting. Without thinking much about the problem, one could think that the dynamic procedure should perform better than the static one because bargainers have an opportunity to speak if nobody has spoken right away. Of course this need not be so because changing the procedure changes the bargainers' incentives to disclose their option, and will in fact make them less likely to disclose right away, as shown in Proposition 7. These computations for a uniform distribution illustrates that this negative effect may overcome the positive effect of letting the bargainers more time to speak. It remains an open question whether this is true for all regular bargaining solutions and for all symmetric distributions.

Dynamic disclosure with an opportunity to react

As a natural variant of our dynamic game, we study a situation where bargainers have one last chance to disclose their option right after the other has proken, i.e. right before b is implemented. Note that the strategies in this game are richer than those of the original dynamic game. As in the original game, they specify the latest period in which a bargainer would disclose if the other party has not done so. But in addition, for every history which ended with disclosure by the other party, a bargainer's strategy also specifies whether or not he would disclose as a function of the other party's disclosed type and the period of disclosure. To eliminate notational complications and unlikely off-equilibrium behavior, we focus on a slightly refined notion of BNE. Indeed, we will assume that type x discloses right after the other party has disclosed a type y if and only if $y > 1 - x$. In other words, we focus on equilibria in which a bargainer discloses immediately after the other party has disclosed whenever it is optimal for him to do so (whenever the the payoff from the other party's option is lower than the payoff from his own option). Given this restriction, strategies in a refined BNE are measurable functions $\tau : [0, 1] \to \mathbb{R}_+ \cup \{\infty\}$, that describe when a player discloses his option as a function of his type.

[15] For real, nonzero values of x, the exponential integral $E_i(x)$ is defined as $-\int_{-x}^{\infty} e^{-t}/t dt$.

Proposition 8 *The modified dynamic disclosure game admits an essentially unique refined symmetric Bayesian Nash equilibrium. The equilibrium disclosure strategy* t^* *for both players is the following:*

$$\mathsf{t}^*(x) = \begin{cases} 0 & \text{if } x \geq 1/2 \\ \int_x^{1/2} \frac{(1-2y)f(y)}{\delta y F(y)} dy & \text{if } x \leq 1/2. \end{cases}$$

Proof: We prove that the strategy t^* is indeed part of a symmetric BNE. The proof of uniqueness is relegated to the Appendix.

We start by showing that reporting at $\mathsf{t}^*(x)$ is optimal, for any $x \in [0, 1/2[$. Consider first the possibility of revealing at positive times. The function t^* being invertible on $[0, 1/2[$, we can identify any positive time with the type speaking at that time. The expected utility from revealing at $\mathsf{t}^*(z)$ when of type x is equal to

$$U(z|x) := xF(z)e^{-\delta \mathsf{t}^*(z)} + \int_{y=z}^{1-x} (1-y)e^{-\delta \mathsf{t}^*(y)} f(y)dy + \int_{y=1-x}^{1} b_1(x,y)e^{-\delta \mathsf{t}^*(y)} f(y)dy,$$

for each $z \in [0, 1/2[$. It is easy to check that this expression is differentiable, and has the same derivative as the similar expression in the proof of Proposition 5 (because the third term does not depend on z), i.e.

$$\frac{(1-z)}{z} f(z)(x-z)e^{-\delta \mathsf{t}^*(z)}.$$

We see that the first order condition is satisfied at $z = x$, and that the derivative is positive when $z < x$ and negative when $x < z$. Hence there is no profitable deviation to a positive time different from $\mathsf{t}^*(x)$, when of type x. Deviating to report at zero is not profitable either, as the expected payoff in that case is

$$\frac{x}{2} + \int_{y=1/2}^{1} b_1(x,y) f(y)dy$$

which is equal to

$$U(1/2|x) + \int_{y=1/2}^{1-x} (b_1(x,y) - (1-y)) f(y)dy,$$

and the second term of this expression is non-positive, as $y \leq 1-x$ implies that $x \leq 1-y$ and hence $b_1(x,y) \leq 1-y$.

Consider now a type $x \in [1/2, 1]$. The expected utility of revealing at a positive time t, corresponding to a $z < 1/2$, is equal to

$$xF(z)e^{-\delta \mathsf{t}^*(z)} + \int_{y=z}^{1-x} (1-y)e^{-\delta \mathsf{t}^*(y)} f(y)dy + \int_{y=1-x}^{1} b_1(x,y)e^{-\delta \mathsf{t}^*(y)} f(y)dy,$$

if $z \leq 1-x$, and to

$$xF(z)e^{-\delta \mathsf{t}^*(z)} + \int_{y=z}^{1} b_1(x,y)e^{-\delta \mathsf{t}^*(y)} f(y)dy,$$

if $z \geq 1-x$. The expression when $z \leq 1-x$ is non-decreasing in z, as was $U(z|x)$ when x was smaller than $1/2$. The expression when $z \geq 1-x$ is also non-decreasing because its derivative is equal to

$$(\frac{1-z}{z}x - b_1(x,z))f(z)e^{-\delta t^*(z)}.$$

Notice that $z \geq 1-x$ implies $x \geq 1-z$ and hence $b_1(x,z) \leq x$. On the other hand, $z \leq 1/2$ implies that $(1-z)/z \geq 1$, and hence $b_1(x,z) \leq (x(1-z))/z$, which implies that the last derivative in non-negative, as desired. The expected utility of revealing at a positive t is thus no larger than when taking the limit of that expected utility when z tends to $1/2$, i.e. $\frac{x}{2} + \int_{y=1/2}^{1} b_1(x,y)f(y)dy$. But this is exactly the expected utility the player gets by revealing at zero, which shows that there are no profitable deviations when $x \in [1/2, 1]$ either. ■

By Proposition 8, the timing of disclosure in the unique refined symmetric BNE is *independent* of the bargaining solution. Furthermore, independently of the bargaining solution, every type above $\frac{1}{2}$ delays the latest time at which he would disclose, relative to his timing of disclosure in the original dynamic game (where a bargainer cannot disclose immediately after his rival). Hence, for every bargaining solution, the ex-ante expected payoff of a bargainer is lower in this dynamic game than in the original game discussed above. Again, one sees that more opportunities to speak can in fact be damaging in terms of welfare.

6. MORE GENERAL UTILITY FRONTIERS

In this section we investigate how our analysis of the static disclosure game would change, if the utility frontier X was not linear. We first note that all of our general results (i.e., those that did not involve the specific bargaining solutions of Raiffa, Kalai-Smorodinsky and Nash) in Sections 3 and 4 continue to hold for any bounded utility frontier $u_2 = g(u_1)$, which is symmetric (if X contains a point where bargainer 1 gets x and bargainer 2 gets y, then X also contains the point where 1 gets y and 2 gets x) and has no Pareto comparisons (to adapt the results to the more general environment, one needs to replace $1-y$ with $g^{-1}(y)$).

Extending the utility frontier beyond a line with slope -1 has several implications. First, while the Raiffa solution is monotone independently of the shape of X, the Nash and Kalai-Smorodinsky solutions need not be.[16] Second, the Nash, Kalai-Smorodinsky and Raiffa solutions may no longer be comparable according to the partial ordering characterized in Proposition 3.[17] Finally, as we show below, the Nash bargaining solution

[16]To see this in the case of the Nash solution, for instance, consider some concave frontier g and let $u^* = g(u^*)$. Take a pair of symmetric options $(x, g(x))$ and $(g(x), x)$, where $x < u^* < g(x)$. The Nash solution gives each bargainer an expected payoff of $\frac{1}{2}[x + g(x)]$, which is strictly lower than u^*. Next consider the pair of options (u^*, u^*) and $(g(x), x)$. The Nash solution associated with these options is (u^*, u^*), and hence, bargainer 2's expected payoff went up, even though (u^*, u^*) is worse for him than $(x, g(x))$.

[17]For example, consider the following convex utility frontier: $g(u_1) = 1 - 2u_1$, if $u_1 \leq \frac{1}{3}$, and $g(u_1) = \frac{1}{2} - \frac{1}{2}u_1$, otherwise. For the pair $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{5}{12}, \frac{7}{24})$ it easy to show that the Kalai-Smorodinsky solution gives a higher expected payoff to bargainer 1 than the Nash solution, which in turn gives a higher expected payoff than the Raiffa solution. However, for the pair $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{7}{24}, \frac{5}{12})$, the Raiffa solution gives bargainer 1 a higher expected payoff than the Kalai-Smorodinsky solution, which in turn, gives a higher expected payoff than the Nash solution.

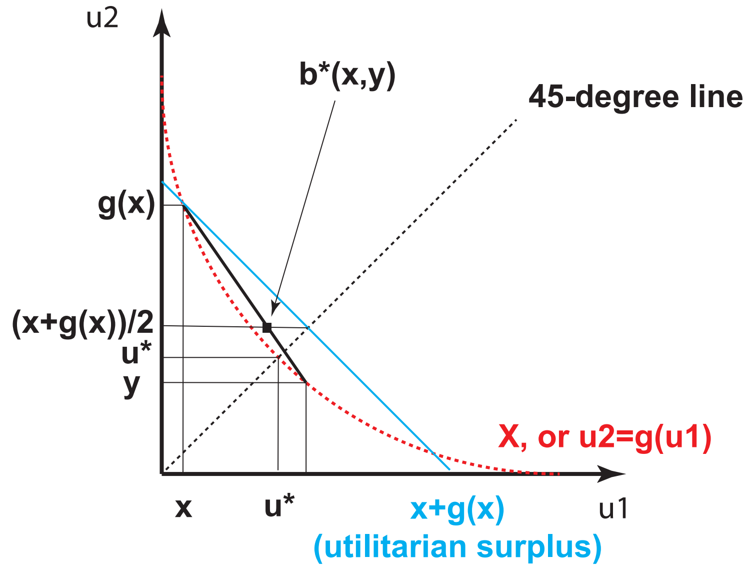

Figure 1

is no longer the most efficient.

Characterizing the most efficient regular bargaining solution for any symmetric and decreasing g remains an open question. However, Proposition 4 implies an algorithm that transforms any pair of regular solutions into a regular solution, which is at least as efficient as each of the two original solutions. By imposing additional structure on g, we are able to say more than this. In particular, if we assume that g is differentiable and is either convex or concave, then we are able to characterize the most efficient regular bargaining solution.

Consider first the case in which g is differentiable and *convex*. Our first observation is the set of regular bargaining solution still contains the well-known bargaining solutions that we discussed.

Proposition 9 *The Raiffa, Nash and Kalai-Smorodinsky solutions are all regular on X.*

Proof: See the Appendix. ■

Let b^* be the bargaining solution defined as follows (see the discussion following Corollary 1). If two options were disclosed, it selects the lottery over these two options, which maximizes the expected payoff to the weak bargainer, subject to the constraint that the expected payoff to the strong bargainer is at least half the utilitarian surplus (see Figure 1). If only one option was disclosed, it selects that option with certainty. Observe that b^* describes the Nash solution when g is linear.

Lemma 3 *b^* is a regular bargaining solution.*

Proof: See the Appendix. ■

We now prove that b^* is the most efficient (in the sense of minimizing the probability that no option is disclosed) regular bargaining solution when g is differentiable and convex. To understand the intuition for this result, note that in order to motivate bargainers to disclose, the solution needs to favor the weakest bargainer (this follows from Proposition 3). However, if the solution is too biased in favor of the weakest bargainer, it may violate monotonicity. For example, suppose $g(x) > x$, $g^{-1}(y) > y$, and $y > x$ (as in Figure 1 again). Then bargainer 1 is weaker than 2. Since $g^{-1}(y) > x$, bargainer 1 would like the solution to pick a point as close as possible to $(g^{-1}(y), y)$. Suppose there is a regular bargaining solution b that picks this point. Then by symmetry it would give each bargainer an expected payoff of $\frac{1}{2}[x + g(x)]$ when the two options are $(x, g(x))$ and $(g(x), x)$. But this violates monotonicity since bargainer 2's payoff in $(g^{-1}(y), y)$ is higher than in $(g(x), x)$, but his expected payoff from the bargaining solution is actually lower since $y < \frac{1}{2}[g(x) + x]$. Thus, in order to satisfy the monotonicity constraint, we cannot give the second bargainer less than $\frac{1}{2}[g(x) + x]$, which is half the utilitarian surplus in this case.

Proposition 10 *b^* is the most efficient regular bargaining solution.*

Proof: As pointed out in the beginning of this section, Proposition 3 is one of the results that carry over to any symmetric X with no Pareto comparisons. Hence the result will follow after showing that $b^* \succeq b$, for all regular bargaining solution b, which amounts to show $b_1^*(x, y) \geq b_1(x, y)$, for all $x \leq u^*$ and all $y \geq x$, where u^* is the real number such that $u^* = g(u^*)$. We may also assume without loss of generality that $y \leq g(x)$, as otherwise our argument applies by renaming $(x, g(x))$ $(g^{-1}(y), y)$, and vice-versa. We will be done after showing that monotonicity on b implies that $b_2(x, y)$ is no smaller than half the utilitarian surplus (since $b_1^*(x, y)$ is player 1's largest feasible payoff under that constraint). The utilitarian surplus is achieved at $(x, g(x))$, since g is convex. Changing $(g^{-1}(y), y)$ into $(g^{-1}(x), x)$ does not increase player 2's payoff (since $x \leq y$), and leads to a payoff for player 2 that is equal to half the utilitarian surplus of the original problem (the new problem being solved by symmetry). ■

To analyze the case where g is differentiable and concave, we use the following "duality" argument. For any payoff pair $(u, g(u))$ we define a dual pair $(v, h(v))$ where $v \equiv 1 - u$ and $h(v) \equiv 1 - g(1 - v)$. It follows that $h(v)$ is differentiable, decreasing and convex. Let b be a regular bargaining solution defined on the set of disclosable payoffs, $\{(v, h(v)) : v \in [0, 1]\}$. Define the "dual solution" to b as follows: for any pair of disclosed payoff pairs, $(u, g(u))$ and $(g^{-1}(u'), u')$,

$$d_i(u, u') = 1 - b_i(1 - u, 1 - u')$$

This mapping from the bargaining solution b to its dual solution d preserves the regularity of the solutions as well as their ranking in terms of efficiency.

Proposition 11 *(i) If b is regular, then so is d, and (ii) for any pair of regular bargaining solutions, (b, b') and their dual solutions (d, d'), we have that $b \succeq b'$ implies $d \succeq d'$.*

Proof: (i) By construction, the dual solution d is symmetric and ex-post efficient. To establish monotonicity, suppose we move from the payoff pair $(u, g(u))$ and $(g^{-1}(u'), u')$ to $(u^*, g(u^*))$ and $(g^{-1}(u'), u')$. If $u^* > u$, then $1 - u^* < 1 - u$. Because b is monotone,

$$b_i(1 - u^*, 1 - u') \leq b_i(1 - u, 1 - u')$$

Hence,

$$d_i(u^*, u') \geq d_i(u, u')$$

Essentially the same argument applies if we were to change $(g^{-1}(u'), u')$ holding fixed $(u, g(u))$.

(ii) Define ϕ as the value in $[0, 1]$ that satisfies $\phi = g(\phi)$. We have to show that

$$d'_1(u, u') \geq d_1(u, u'),$$

for all $u \leq \phi$ and all $u' \in [u, g(u)]$. By definition of d, this is equivalent to showing that

$$b'_1(v, v') \leq b_1(v, v'),$$

where $v := 1 - u$ and $v' := 1 - u'$. Since $b(v, v')$ and $b'(v, v')$ belong to the same segment with negative slope, this is equivalent to

$$b'_2(v, v') \geq b_2(v, v').$$

Symmetry of b implies that this is equivalent to

$$b'_2(h(v'), h(v)) \leq b_2(h(v'), h(v)).$$

This inequality is indeed verified, since $b' \succeq b$ (notice indeed that $h(v) \leq h(1 - \phi)$ and $h(v') \in [h(v), v]$, since $u \leq \phi$ and $u' \in [u, g(u)]$). ■

Let d^* be the dual of b^*, the regular bargaining solution defined above, which is most efficient when g is convex. By Proposition 11, d^* is the most efficient regular bargaining solution when g is concave. Note that for each of the well-known bargaining solutions, Raiffa, Nash and Kalai-Smorodinsky, defined over a convex g, there is a dual regular bargaining solution when g is concave. However, apart for Raiffa, these dual solutions do not correspond to the definition of the original bargaining solution (e.g., the dual of Nash does not select the payoff pair, which maximizes the product of the bargainers' payoffs).

7. CONCLUDING REMARKS

Most of the economic literature on bargaining and collective decision-making has focused on situations where the set of possible outcomes is taken as given. It may include a pre-determined list of candidates to be voted an offer, or it may consist of the possible allocations of surplus among the negotiating parties. The non-cooperative literature studies what outcomes would emerge as a function of the bargaining procedure, the bargainers' attitudes towards risk and delay and the information they have about these attitudes. The axiomatic literature may be viewed as proposing bargaining procedures that satisfy certain desirable properties when the set of possible outcomes is taken as given. This paper is concerned with situations where decision-makers first need to identify the set of feasible outcomes before they bargain over which of them is selected. How do different bargaining procedures - which may be normatively appealing when the set of possible outcomes is given - affect the incentives of the parties to propose feasible solutions to their conflict? Which type of procedures provide the most incentives to disclose relevant information on options that are feasible?

This paper makes a first step towards addressing these questions. We characterize a partial ordering of regular bargaining solutions (i.e., those belonging to some class of "natural" solutions) according to the likelihood of disclosure that they induce. This ordering identifies the best solution in this class, which favors the weaker bargainer subject to the regularity constraints. We also illustrate our result in a simple environment where the best solution coincides with Nash, and where the Kalai-Smorodinsky solution is ranked above Raiffa's simple coin-toss solution. The analysis is then extended to a dynamic setting in which the bargainers can choose the timing of disclosure.

There are several directions in which the next steps can be taken. One direction would be to weaken the monotonicity requirement and search for the best (in terms of disclosure) bargaining solution among those that are efficient and symmetric. We conjecture that the best solution maximizes the expected payoff of the weaker bargainer (as in the example given in Section 4). A second challenging direction is to consider situations in which the bargainers are aware of a *set* of feasible options and need to decide which of these to disclose. This direction can be explored using the framework of the "one-shot" disclosure game, where each player independently draws a subset of options from some feasible set (either the line or the square $[0,1]^2$). The difficulty here lies in constructing a simple type space, which accommodates a tractable analysis. An alternative direction, which may be more tractable, is to analyze a dynamic disclosure game in discrete time, where in every period, each bargainer randomly draws an option from some feasible set and must to decide whether or not to disclose one of his options.

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Appendix

Proof of Lemma 1. Consider a pair of types x, x' such that $x' > x$ as in the statement of the lemma, and let $\bar{x}$ be any real number that falls strictly in between x and x'. Notice that

$$x = b_1(x, 1-x) \leq b_1(\bar{x}, 1-x) = b_1(x, 1-\bar{x}) \leq b_1(\bar{x}, 1-\bar{x}) = \bar{x},$$

where the two inequalities follow from the monotonicity condition, and the equality follows from the symmetry condition. Notice also that $x < \bar{x}$, and that $b_1(\bar{x}, 1-\bar{x}) \leq b_1(x', 1-\bar{x})$ by monotonicity. So, if there is a positive measure of $\bar{x}$'s strictly between x and x' such that $b_1(\bar{x}, 1-x) = x$, then we are done proving the property since $b_1(x, 1-\bar{x}) < b_1(x', 1-\bar{x})$, for all such $\bar{x}$'s. Let us conclude the proof by an argument ad absurdum. If the property we want to prove is wrong, then it must thus be that $b_1(\bar{x}, 1-x) > x$, for almost all $\bar{x}$ strictly in between x and x'. Monotonicity implies that $x < b_1(x', 1-x)$, or $x < b_1(x, 1-x')$ by symmetry, in that case. Monotonicity again implies that $x^* < b_1(x^*, 1-x')$, for all $x^* \in [x, b_1(x', 1-x)[$. At the same time, it must be that $b_1(x^*, 1-x) \leq x^*$ since a bargaining solution picks a lottery defined over disclosed options. Hence $b_1(x^*, 1-x) < b_1(x^*, 1-x')$, or $b_1(x, 1-x^*) < b_1(x', 1-x^*)$ by symmetry, for all such x^*'s, and the property that we want to prove in fact holds, giving us the contradiction that we wanted. ■

Uniqueness of the Symmetric BNE in Proposition 5

Let b be a regular bargaining solution that satisfies condition (3), and let τ be a strategy that is part of a symmetric BNE in the original dynamic game. We have to show that $\tau = \tau^*$. We proceed in various steps.

Step 1 $\int_{y \in \tau^{-1}(\infty)} f(y)dy = 0.$

Proof: Suppose, to the contrary of what we want to prove, that $\int_{y \in \tau^{-1}(\infty)} f(y)dy > 0$. Let $x > 0$ be such that $\tau(x) = \infty$. Player 1's expected net gain from revealing at a time t instead of ∞ is:

$$e^{-\delta t} x \int_{y \in \tau^{-1}(\infty)} f(y)dy + \int_{y \in \tau^{-1}(]t,\infty[)} (e^{-\delta t} x - e^{-\delta \tau(y)}(1-y)) f(y)dy$$
$$+ \int_{y \in \tau^{-1}(\{t\})} e^{-\delta t}(b_1(x,y) - (1-y)) f(y)dy,$$

which is equal to $e^{-\delta t}$ times

$$x \int_{y \in \tau^{-1}(\infty)} f(y)dy + \int_{y \in \tau^{-1}(]t,\infty[)} (x - e^{-\delta(\tau(y)-t)}(1-y)) f(y)dy$$
$$+ \int_{y \in \tau^{-1}(\{t\})} (b_1(x,y) - (1-y)) f(y)dy,$$

which is greater or equal to

$$x \int_{y \in \tau^{-1}(\infty)} f(y)dy - \int_{y \in \tau^{-1}(]t,\infty[)} f(y)dy,$$

since both x and $b_1(x,y)$ are non-negative, and both $1-y$ and $e^{-\delta(\tau(y)-t)}(1-y)$ are no larger than 1. The first term of this last expression is strictly positive, and independent of t, while the second can be made as small as needed by taking t large enough, as

$$\lim_{t\to\infty}\int_{y\in\tau^{-1}(]t,\infty[)} f(y)dy = 0,$$

by the measurability of τ. $\square$

Step 2 *If $t \in]0,\infty[$, then $\int_{y\in\tau^{-1}(t)} f(y)dy = 0$.*

Proof: Let $\bar{x}$ be the supremum of $\tau^{-1}(t)$, and $\underline{x}$ be the infimum of $\tau^{-1}(t)$. For expositional convenience, we start by assuming that both the infimum and the supremum are reached in $\tau^{-1}(\infty)$, but we will show at the end of the proof how our argument extends to the more general case.

We start by assuming that $\underline{x} \leq 1-\bar{x}$. Hence $1-y \geq b_1(\underline{x},y)$, for all $y \in \tau^{-1}(t)$. In addition, $1-y > b_1(\underline{x},y)$ for each $y \in \tau^{-1}(t)$ such that $y < 1/2$, as a consequence of the third regularity condition (Monotonicity), and the fact that $b_1(\underline{x},\underline{x}) = 1/2$. We now prove that $\int_{y\in\tau^{-1}(t)\cap[0,1/2[} f(y)dy = 0$. Otherwise, the previous reasoning implies that $\int_{y\in\tau^{-1}(t)}((1-y)-b_1(\underline{x},y))f(y)dy > 0$. Given that τ is a measurable function, we know that

$$\lim_{k\to\infty}\int_{y\in[0,1] \text{ s.t. } t<\tau(y)\leq t+\frac{1}{k}} f(y)dy = \int_{y\in[0,1] \text{ s.t. } t<\tau(y)\leq \lim_{k\to\infty} t+\frac{1}{k}} f(y)dy = 0,$$

and hence one can always find a k as large as necessary such that there is a very small probability for the other player to speak in between t and $t+\frac{1}{k}$. Player 1's expected net gain of revealing at $t+\frac{1}{k}$ instead of t when of type $\underline{x}$ is

$$\underline{x}(e^{-\delta(t+\frac{1}{k})}-e^{-\delta t})\int_{y\in\tau^{-1}(]t+\frac{1}{k},\infty])} f(y)dy + \int_{y\in\tau^{-1}(t+\frac{1}{k})}(e^{-\delta(t+\frac{1}{k})}b_1(\underline{x},y)-e^{-\delta t}\underline{x})f(y)dy$$

$$+\int_{y\in\tau^{-1}(]t,t+\frac{1}{k}[)}(e^{-\delta\tau(y)}(1-y)-e^{-\delta t}\underline{x})f(y)dy + \int_{y\in\tau^{-1}(t)} e^{-\delta t}((1-y)-b_1(\underline{x},y))f(y)dy,$$

which is larger or equal to $e^{-\delta t}$ times

$$\underline{x}(e^{-\delta/k-1})\int_{y\in\tau^{-1}(]t+\frac{1}{k},\infty])} f(y)dy - \underline{x}\int_{y\in\tau^{-1}(]t,t+\frac{1}{k}])} f(y)dy$$

$$+\int_{y\in\tau^{-1}(t)}((1-y)-b_1(\underline{x},y))f(y)dy,$$

as it is indeed easy to check that the integrand of the second and third terms from the previous expression are both larger or equal to $-\underline{x}e^{-\delta t}$. The first two terms of the last expression can be made as small as needed by choosing a k large enough, while the third one is strictly positive independently of k, and hence the possibility of a profitable deviation, which contradicts the fact that τ is part of a symmetric BNE. Hence we have proved, by contradiction, that $\int_{y\in\tau^{-1}(t)\cap[0,1/2[} f(y)dy = 0$, and hence that

$\int_{y\in\tau^{-1}(t)} f(y)dy = \int_{y\in\tau^{-1}(t)\cap[1/2,1]} f(y)dy$. If $\bar{x} \leq 1/2$, then we are done proving that $\int_{y\in\tau^{-1}(t)} f(y)dy = 0$. Let's thus assume that $\bar{x} > 1/2$.

Notice that $\bar{x} \geq b_1(\bar{x}, y)$, for each $y \in \tau^{-1}(t)$ such that $y \geq 1/2$. In fact, $\bar{x} > b_1(\bar{x}, y)$ for each $y \in \tau^{-1}(t)$ such that $y > 1/2$, as a consequence of condition (3), the second regularity condition, and the fact that $b_1(\bar{x}, \bar{x}) = 1/2$. Hence $\int_{y\in\tau^{-1}(t)}(\bar{x} - b_1(\bar{x}, y))f(y)dy > 0$ if $\int_{y\in\tau^{-1}(t)\cap]1/2,1]} f(y)dy > 0$. In that case, one can construct a profitable deviation to a $t' < t$ for type $\bar{x}$ (similar argument to the one developed in the previous paragraph). To avoid this contradiction, on must accept that $\int_{y\in\tau^{-1}(t)\cap]1/2,1]} f(y)dy = 0$. Combined with the result of the previous paragraph, one concludes that $\int_{y\in\tau^{-1}(t)} f(y)dy = 0$, as desired.

A similar argument applies in the case where $\underline{x} \geq 1 - \bar{x}$, except that one must start to work with $\bar{x}$ to show that $\int_{y\in\tau^{-1}(t)\cap]1/2,1]} f(y)dy = 0$, and then work with $\underline{x}$ to conclude.

We now consider the case where $\underline{x}$ and $\bar{x}$ do not necessarily belong to $\tau^{-1}(t)$. Again, we provide the argument only for the case were $\underline{x} \leq 1 - \bar{x}$, a similar argument applying if the inequality is reversed. Let $(\underline{x}_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\tau^{-1}(t)$ that converges to $\underline{x}$, and let $(\bar{x}_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\tau^{-1}(t)$ that converges to $\bar{x}$ such that $\underline{x}_n \leq 1 - \bar{x}_n$, for each n. For notational simplicity, let α_n be the following real number:

$$\alpha_n := \int_{y\in\tau^{-1}(t)\cap[\underline{x}_n,\bar{x}_n]} ((1-y) - b_1(\underline{x}_n, y))f(y)dy,$$

for each $n \in \mathbb{N}$. Notice first that these numbers are non-decreasing in n. Indeed, consider $m < n$. We have:

$$\begin{aligned}\alpha_n = &\int_{y\in\tau^{-1}(t)\cap[\underline{x}_n,\underline{x}_m]} ((1-y) - b_1(\underline{x}_n, y))f(y)dy \\ &+ \int_{y\in\tau^{-1}(t)\cap[\underline{x}_m,\bar{x}_m]} ((1-y) - b_1(\underline{x}_n, y))f(y)dy \\ &+ \int_{y\in\tau^{-1}(t)\cap[\bar{x}_m,\bar{x}_n]} ((1-y) - b_1(\underline{x}_n, y))f(y)dy.\end{aligned}$$

Since $\underline{x}_n \leq 1 - \bar{x}_n$, we must have $b_1(\underline{x}_n, y) \leq 1 - y$, for each $y \in [\underline{x}_n, \bar{x}_n]$, and hence the first and the third terms must be non-negative. The third regularity condition also implies that the second term is larger or equal to α_m, since $\underline{x}_m \geq \underline{x}_n$, and hence $\alpha_n \geq \alpha_m$, as desired.

We now show that $\int_{y\in\tau^{-1}(t)\cap[0,1/2]} f(y)dy = 0$. Otherwise, there exists N such that $\int_{y\in\tau^{-1}(t)\cap[0,1/2]\cap[\underline{x}_n,\bar{x}_n]} f(y)dy > 0$, for each $n \geq N$. The reasoning that we did at the beginning of the proof when the infimum and the supremum are reached implies that $\alpha_n > 0$, for each $n \geq N$, and in particular $\alpha_N > 0$. Notice that

$$\int_{y\in\tau^{-1}(t)} ((1-y) - b_1(\underline{x}_n, y))f(y)dy = \alpha_n + \int_{y\in\tau^{-1}(t)\setminus[\underline{x}_n,\bar{x}_n]} ((1-y) - b_1(\underline{x}_n, y))f(y)dy,$$

for each $n \geq N$. The first term is larger or equal to α_N, which is strictly larger than 0 and independent of n, while the second term converges towards zero as n increases, since the integrand is bounded and $\int_{y\in\tau^{-1}(t)\setminus[\underline{x}_n,\bar{x}_n]} f(y)dy$ converges towards zero, and we are done proving that the expression on the left-hand side must be strictly positive for n large enough. As before, this implies that player 1 of type $\underline{x}_n$ prefers to disclose his type slightly later than at t, thereby contradicting the definition of a BNE. It must thus be the case that $\int_{y\in\tau^{-1}(t)\cap[0,1/2]} f(y)dy = 0$, as desired.

Adapting the argument to show that $\int_{y\in\tau^{-1}(t)\cap[1/2,1]} f(y)dy = 0$ when the infimum and the supremum are not reached, and thereby conclude the proof, is similar and left to the reader. □

Step 3 *τ is strictly decreasing with respect to time in the following sense: if $x' > x$ and $\tau(x) > 0$, then $\tau(x') < \tau(x)$; if $x' > x$ and $\tau(x) = 0$, then $\tau(x') = 0$.*

Proof: Let $x, x' \in [0,1]$ be such that $x' > x$, and $\int_{y\in[0,1] \text{ s.t. } \tau(y)>\tau(x)} f(y)dy > 0$.

Suppose that $\tau(x') > \tau(x)$. In that case, $ENG_1(\tau(x') \text{ vs. } \tau(x), x') \geq 0$, since (τ, τ) is a BNE, and hence[18] $ENG_1(\tau(x') \text{ vs. } \tau(x), x) > 0$, thereby contradicting the optimality of reporting at $\tau(x)$ when of type x. Hence, one must conclude that $\tau(x') \leq \tau(x)$.

Suppose now that $\tau(x) > 0$. We know from the previous paragraph that $\tau(x'') \leq \tau(x)$, for all $x'' \in]x, x'[$. Steps 1 and 2 imply that there exists $x'' \in]x, x'[$ such that $\tau(x'') < \tau(x)$. The reasoning from the previous paragraph implies that $\tau(x') \leq \tau(x'')$, and hence $\tau(x') < \tau(x)$.

We have thus established the two desired properties, but under the assumption that $\int_{y\in[0,1] \text{ s.t. } \tau(y)>\tau(x)} f(y)dy > 0$. We now show that this inequality must in fact hold for any $x > 0$. Suppose first that x is such that $\tau(x) = 0$. If the inequality does not hold, then it means that the opponent will reveal his type with probability 1 at time 0. Then it is easy to check that $\int_{y\in[0,1]} b_1(x,y)f(y)dy < \int_{y\in[0,1]}(1-y)f(y)dy$, for any $x \in [0,1]$ that is small enough. A reasoning similar to the one developed in the second paragraph of the proof of Step 2 would imply a contradiction, namely that a slight delay is a profitable deviation for any such x. Consider now an x such that $\tau(x) > 0$, let $t^* = \inf_{y\in[0,x]} \tau(y)$, and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $[0,x]$ such that $(\tau(x_k))_{k\in\mathbb{N}}$ decreases towards t^* as k tends to infinity. Since τ is measurable, we have:

$$\lim_{k\to\infty} \int_{y\in\tau^{-1}(]\tau(x_k),\infty])} f(y)dy = \int_{y\in\tau^{-1}(]\lim_{k\to\infty}\tau(x_k),\infty])} f(y)dy = \int_{y\in\tau^{-1}(]t^*,\infty])} f(y)dy.$$

Notice that the right-most expression must be strictly positive. We just proved this if $t^* = 0$, while, if $t^* > 0$, then the opponent does not speak before t^* if his type is no greater than x, and the probability of him speaking at t^* is zero, by Steps 1 and 2. Hence there exists $K \in \mathbb{N}$ such that $\int_{y\in[0,1] \text{ s.t. } \tau(y)>\tau(x_k)} f(y)dy > 0$, for all $k \geq K$. The result from the previous paragraph implies that $\tau(x) \leq \tau(x_k)$, for all such k's, and hence $\tau(x) = t^*$, and $\int_{y\in[0,1] \text{ s.t. } \tau(y)>\tau(x)} f(y)dy > 0$, as desired.

Finally, $\tau(x') < \tau(0)$, for all $x' > 0$. Otherwise we can find $x' \in]0,1]$ such that $\tau(0) \leq \tau(x')$. The previous argument in the proof implies that all the types strictly between 0 and x' report after $\tau(x')$ and hence also after $\tau(0)$. The first argument in the proof then implies that $\tau(x') < \tau(0)$, the desired contradiction. □

[18]The second term in the definition of the expected net gain, as stated before the statement of this proposition, is zero, by Step 2.

Step 4 *Let* $\alpha = \inf\{x \in [0,1] | \tau(x) = 0\}$. *Then* τ *is continuous on* $]0, \alpha[$, *and* $lim_{x \to \alpha_-} \tau(x) = 0$.

Proof: Let $x \in]0, \alpha]$, and let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $[0, x]$ that converges to x. Step 3 implies that $\tau(x_k) \geq \tau(x)$, for all $k \in \mathbb{N}$. Suppose, to the contrary of what we want to prove, that there exists $\eta > 0$ and $K \in \mathbb{N}$ such that $\tau(x_k) > \tau(x) + \eta$, for all $k \geq K$. This implies that no type reveals after $\tau(x)$ and before $\tau(x) + \eta$. Indeed, suppose on the contrary that there exists y such that $\tau(y) \in]\tau(x), \tau(x) + \eta[$. Step 3 implies that y is strictly smaller than x, and hence there exists $k \geq K$ such that $y < x_k < x$. Step 3 implies that $\tau(x_k) < \tau(y) < \tau(x) + \eta$, which contradicts the definition of K. Consider now a type y for which $\tau(y)$ is very close to the $\inf\{\tau(z) | \tau(z) \geq \tau(x) + \eta\}$ (i.e. y is smaller than x, but very close to it). Then revealing a bit earlier, let's say at $\tau(x) + \frac{\eta}{2}$ instead of $\tau(y)$, is a profitable deviation since the loss, coming from the opponent's types between y and x, can be made as small as needed, while the gain is larger than the gain from getting y earlier by at least $\eta/2$ units of time for all the opponent's type who reveal after $\tau(y)$ (y is strictly positive if close enough to x, and so there is a positive probability that the opponent reveals after $\tau(y)$). This contradicts the optimality of revealing y at $\tau(y)$, and hence we have established the left-continuity on $]0, \alpha[$, and that $lim_{x \to \alpha_-} \tau(x) = 0$. A similar reasoning applies to show the right-continuity on $]0, \alpha[$. $\square$

Step 5 $\tau(x) = 0$ *if and only if* $x \in [\theta, 1]$, *where*

$$\theta = \sup\{x \in [0, 1/2] | \int_{y=x}^{1} (b_1(x,y) - (1-y)) f(y) dy < 0\}.$$

Proof: Observe first that the function $g : [0, 1/2] \to \mathbb{R}$ that associates $\int_x^1 (b_1(x,y) - (1-y)) f(y) dy$, to any $x \in [0, 1/2]$, is strictly increasing. Suppose that $x' > x$. We have:

$$g(x') = \int_{y=x'}^{1} (b_1(x',y) - (1-y)) f(y) dy \geq \int_{y=x'}^{1} (b_1(x,y) - (1-y)) f(y) dy$$

$$> \int_{y=x}^{1} (b_1(x,y) - (1-y)) f(y) dy = g(x).$$

The weak inequality follows from the third regularity condition, while the strict inequality follows from the fact that $b_1(x,y) - (1-y) < 0$, for each $y \in]x, x'[$, as $1 - y > 1/2$ and $b_1(x,y) \leq 1/2$ (as a consequence of the second and third regularity conditions), for all such y's. Notice also that $g(0) < 0$. Indeed, $b_1(0,y) \leq 1 - y$, for all $y \in [1/2, 1]$, by the first regularity condition, and $b_1(0,y) \leq 1/2 < 1 - y$, for all $y \in [0, 1/2[$, by the second and third regularity conditions. Notice finally that $g(1/2) \geq 0$, as $b_1(1/2, y) \geq 1 - y$, for each $y \in [1/2, 1]$, by the first regularity condition. Hence θ is well-defined, $g(x) < 0$, for each $x \in [0, 1/2]$ such that $x < \theta$, and $g(x) > 0$, for each $x \in [0, 1/2]$ such that $x > \theta$.

We now prove that $\tau(x) > 0$, for each $x < \theta$. Otherwise, there exists $x < \theta$ such that $\tau(x) = 0$. Then $g(x) < 0$, and hence

$$\int_{y=\alpha}^{1} b_1(x,y) < \int_{y=\alpha}^{1} (1-y) f(y) dy,$$

where $\alpha = \inf\{y \in [0,1] | \tau(y) = 0\}$, because $b_1(x,y) \leq 1-y$, for each $y \in [\alpha, x]$, by the first regularity condition. A reasoning similar to the one we did in the second paragraph proof of Step 2 implies that a bargainer of type x can improve his payoff by reporting at some small positive time rather than at zero, thereby contradicting the optimality of τ. Hence $\tau(x) > 0$, for each $x < \theta$, as desired.

We now prove that $\tau(x) = 0$, for each $x > \theta$. First notice that $\tau(x) = 0$, for each $x > 1/2$. Suppose, on the contrary, that $\tau(x) > 0$, for some $x > 1/2$. The expected net gain of reporting at 0 instead is strictly positive, as $b_1(x,y) - (1-y) \geq 0$, for all the opponent's types y that report at 0, and $x > 1-y$, for all the opponent's types $y > x$ that report at a positive time lower than $\tau(x)$. So $\tau(x) = 0$, for each $x > 1/2$, and we have proved the statement for $\theta = 1/2$. Suppose now that $\theta < 1/2$. As before, let $\alpha = \inf\{y \in [0,1] | \tau(y) = 0\}$. We know that $\alpha \leq 1/2$. Suppose, to the contrary of what we want to prove, that $\alpha > \theta$. Let then x be smaller than α, but very close to it. Hence $\tau(x) > 0$. The expected net gain of revealing at zero instead is equal to:

$$\int_{y=\alpha}^{1} (b_1(x,y)-(1-y))f(y)dy + \int_{y=x} \alpha(x - e^{-\delta\tau(y)}(1-y))f(y)dy + x(1-e^{-\delta\tau(x)}) \int_{y=0}^{x} f(y)dy,$$

which is greater or equal to

$$\int_{y=\alpha}^{1} (b_1(x,y) - (1-y))f(y)dy + \int_{y=x} \alpha(x - e^{-\delta\tau(y)}(1-y))f(y)dy,$$

which is equal to

$$\int_{y=x}^{1} (b_1(x,y) - (1-y))f(y)dy + \int_{y=x} \alpha(x - e^{-\delta\tau(y)}(1-y) - b_1(x,y) + (1-y))f(y)dy.$$

Notice that the first term is $g(x)$, which is strictly positive if $x > \theta$, and increasing with x. The second term, on the other hand, can be made as small as desired, by choosing x large enough, so as to be as closed as needed to α. Hence the expected net gain for such a type to reveal at zero is strictly positive, which contradicts the optimality of τ. This concludes the proof that $\tau(x) = 0$, for each $x > \theta$.

Finally, we prove that $\tau(\theta) = 0$. We have proved that $\theta = \alpha$. If $\tau(\theta) > 0$, then $\tau(x) \geq \tau(\theta)$, for all $x < \alpha$, by Step 3, and $\lim_{x \to \alpha_-} \tau(x) > 0$, which would contradict Step 4. Hence $\tau(\theta) = 0$, and we are done proving Step 5. $\Box$

Step 6 *τ is differentiable on $]0, \theta[$, and*

$$\tau'(x) = \frac{(1-2x)f(x)}{\delta x F(x)},$$

for all $x \in]0, \theta[$.

Proof: Let $x \in]0, \theta[$. The expected net gain of revealing at $\tau(x+\epsilon)$ instead of $\tau(x)$ is equal to:

$$\int_{y=x}^{x+\epsilon} (xe^{-\delta\tau(x+\epsilon)} - (1-y)e^{-\delta\tau(y)})f(y)dy + x(e^{-\delta\tau(x+\epsilon)} - e^{-\delta\tau(x)}) \int_{y=0}^{x} f(y)dy,$$

which is also equal to

$$-\int_{y=x}^{x+\epsilon}(1-y)e^{-\delta\tau(y)}f(y)dy+x(e^{-\delta\tau(x+\epsilon)}F(x+\epsilon)-e^{-\delta\tau(x)}F(x)).$$

In order for τ to be optimal, it must be that this expression is non-positive. Dividing by ϵ, and taking the limit when ϵ decreases to 0, we get:

$$-e^{-\delta\tau(x)}(1-2x)f(x)-x\delta\lim_{\epsilon\to 0_+}[\frac{\tau(x+\epsilon)-\tau(x)}{\epsilon}]e^{-\delta\tau(x)}F(x)\leq 0.$$

A similar reasoning applied to the case that type $x+\epsilon$ is not better off by reporting at $\tau(x)$ gives

$$e^{-\delta\tau(x)}(1-2x)f(x)+x\delta\lim_{\epsilon\to 0_+}[\frac{\tau(x+\epsilon)-\tau(x)}{\epsilon}]e^{-\delta\tau(x)}F(x)\leq 0.$$

Combining the two previous inequalities, we conclude that

$$\lim_{\epsilon\to 0_+}[\frac{\tau(x+\epsilon)-\tau(x)}{\epsilon}]=-\frac{(1-2x)f(x)}{\delta xF(x)}.$$

A similar reasoning with $\epsilon<0$ implies that

$$\lim_{\epsilon\to 0_-}[\frac{\tau(x+\epsilon)-\tau(x)}{\epsilon}]=-\frac{(1-2x)f(x)}{\delta xF(x)},$$

which concludes the proof of this step. □

Step 7 *$\tau=\tau^*$.*

Proof: Step 5 establishes that $\tau=\tau^*$ on $[\theta,1]$. Step 6 implies that $\tau=C+\tau^*$ on $[0,\theta[$, for some real number C. The fact that $\lim_{x\to\theta_-}\theta(x)=0$, implies that $C=0$, and establishes that $\tau=\tau^*$ on $[0,1]$. ■

Uniqueness of the Refined Symmetric BNE in Proposition 8. Let b be a regular bargaining solution, and let $\mathfrak{t}$ be a strategy that is part of a refined symmetric BNE in the dynamic game with an opportunity to react. We have to show that $\mathfrak{t}=\mathfrak{t}^*$. We proceed in various steps.

Step 1 *$\int_{y\in\mathfrak{t}^{-1}(t)\cap[0,1/2]}f(y)dy=0$, for all $t\in\mathbb{R}_+$.*

Proof: Player 1's expected net gain of revealing at $t'>t$ instead of t, when of type x, is equal to

$$\int_{y\in\mathfrak{t}^{-1}(]t',\infty])}\min\{x,b_1(x,y)\}(e^{-\delta t'}-e^{-\delta t})f(y)dy$$
$$+\int_{y\in\mathfrak{t}^{-1}(t')}(b_1(x,y)e^{-\delta t'}-\min\{x,b_1(x,y)\}e^{-\delta t})f(y)dy$$
$$+\int_{y\in\mathfrak{t}^{-1}(]t,t'[)}(\max\{1-y,b_1(x,y)\}e^{-\delta\mathfrak{t}(y)}-\min\{x,b_1(x,y)\}e^{-\delta t})f(y)dy$$

$$+\int_{y\in\mathfrak{t}^{-1}(t)}(\max\{1-y,b_1(x,y)\}-b_1(x,y))e^{-\delta t}f(y)dy,$$

which is larger or equal to

$$\int_{y\in\mathfrak{t}^{-1}(]t',\infty])}\min\{x,b_1(x,y)\}(e^{-\delta t'}-e^{-\delta t})f(y)dy-\int_{y\in\mathfrak{t}^{-1}(]t,t'])}\min\{x,b_1(x,y)\}e^{-\delta t}f(y)dy$$

$$+\int_{y\in\mathfrak{t}^{-1}(t)}(\max\{1-y,b_1(x,y)\}-b_1(x,y))e^{-\delta t}f(y)dy,$$

since the integrand of the second and third terms are both larger or equal to $-\min\{x,b_1(x,y)\}$.

Suppose, to the contrary of what we want to prove, that $\int_{y\in\mathfrak{t}^{-1}(t)\cap[0,1/2]}f(y)dy>0$, for some $t\geq 0$. Let's focus on one of the types x that reveal at t, and that is small enough so that $\int_{y\in\mathfrak{t}^{-1}(t)\cap[x,1/2]}f(y)dy>0$. Notice that $\max\{1-y,b_1(x,y)\}\geq b_1(x,y)$, for any $y\in[0,1]$, and that $\max\{1-y,b_1(x,y)\}>b_1(x,y)$, for any $y\in[x,1/2[$. Indeed, the second regularity condition implies that $b_1(x,x)=1/2$, and the third regularity condition implies that $b_1(x,y)\leq 1/2<1-y$, for all such y's. Hence the third term in the lower bound on Player 1's expected net gain of revealing at t' instead of t is strictly positive, and independent of t'. The first two terms, on the other hand, can be made as small as needed by choosing t' close enough to t (see Step 2 in the previous proof in this Appendix for a similar argument), thereby leading to a contradiction of the optimality of $\mathfrak{t}$. $\square$

Step 2 *Let $x,x'\in[0,1]$ be such that $x'<1/2<x$. If $\int_{y\in\mathfrak{t}^{-1}([\mathfrak{t}(x),\infty])}f(y)dy>0$, then $\mathfrak{t}(x')\geq\mathfrak{t}(x)$.*

Proof: Let $t=\mathfrak{t}(x)$ and $t'=\mathfrak{t}(x')$. Suppose, to the contrary of what we want to prove, that $t>t'$. Player 1's expected net gain of revealing at t instead of t', when of type x, is equal to

$$\int_{y\in\mathfrak{t}^{-1}(]t,\infty])}\min\{x,b_1(x,y)\}(e^{-\delta t}-e^{-\delta t'})f(y)dy$$

$$+\int_{y\in\mathfrak{t}^{-1}(t)}(b_1(x,y)e^{-\delta t}-\min\{x,b_1(x,y)\}e^{-\delta t'})f(y)dy$$

$$+\int_{y\in\mathfrak{t}^{-1}(]t',t[)}(\max\{1-y,b_1(x,y)\}e^{-\delta\mathfrak{t}(y)}-\min\{x,b_1(x,y)\}e^{-\delta t'})f(y)dy$$

$$+\int_{y\in\mathfrak{t}^{-1}(t')}(\max\{1-y,b_1(x,y)\}-b_1(x,y))e^{-\delta t'}f(y)dy.$$

We now prove that this expected net gain does not decrease when replacing x by x'. The third regularity condition implies that $\min\{x,b_1(x,y)\}$ is non-decreasing in x, for all $y\in[0,1]$. Hence $\min\{x,b_1(x,y)\}(e^{-\delta t}-e^{-\delta t'})$ is non-increasing in x, as $t>t'$. If $t=\infty$, then $b_1(x,y)e^{-\delta t}-\min\{x,b_1(x,y)\}e^{-\delta t'}=-\min\{x,b_1(x,y)\}e^{-\delta t'}$, which again is non-increasing in x, independently of y. If t is finite, then the integral in the second term is equal to the integral when $y\geq 1/2$, by Step 1. The integrand in that case is equal to $b_1(x,y)(e^{-\delta t}-e^{-\delta t'})$. The integrand has the same functional form when x is replaced by

x', for all y's such that $1-y \leq x'$, which is thus no smaller than what it was with x, by the third regularity condition. Consider now some y such that $1-y \in]x', 1/2[$. We have:

$$b_1(x,y)(e^{-\delta t} - e^{-\delta t'}) \leq b_1(1-y,y)(e^{-\delta t} - e^{-\delta t'}) = \min\{b_1(1-y,y), 1-y\}(e^{-\delta t} - e^{-\delta t'})$$
$$\leq \min\{b_1(x',y), x'\}(e^{-\delta t} - e^{-\delta t'}) \leq b_1(x',y)e^{-\delta t} - \min\{b_1(x',y), x'\}e^{-\delta t'},$$

where the two first inequalities follow from the third regularity condition, since $x' < 1-y < x$, and the equality follows from the fact that $b_1(1-y,y) = 1-y$. Let's consider now the integrand of the third term. First, if $1-y > x$, then it is equal to $(1-y)e^{-\delta \mathfrak{t}(y)} - xe^{-\delta t'}$. Then $1-y > x'$ a fortiori, and therefore the integrand is equal to $(1-y)e^{-\delta \mathfrak{t}(y)} - x'e^{-\delta t'}$ when x is replaced by x', which is strictly greater than the previous expression. If $1-y < x'$, then the integrand for x' is equal to $b_1(x',y)(e^{-\delta \mathfrak{t}(y)} - e^{-\delta t'})$, which is no smaller than the integrand for x, which is equal to $b_1(x,y)(e^{-\delta \mathfrak{t}(y)} - e^{-\delta t'})$. A similar comparison holds when $x' < 1-y < x$:

$$\max\{1-y, b_1(x,y)\}e^{-\delta \mathfrak{t}(y)} - \min\{x, b_1(x,y)\}e^{-\delta t'} = b_1(x,y)(e^{-\delta \mathfrak{t}(y)} - e^{-\delta t'})$$
$$\leq (1-y)(e^{-\delta \mathfrak{t}(y)} - e^{-\delta t'}) \leq \max\{1-y, b_1(x',y)\}e^{-\delta t} - \min\{x', b_1(x',y)\}e^{-\delta t'}.$$

Finally, Step 1 implies that we can restrict attention to $y \geq 1/2$ in the fourth term. In that case, the integrand is equal to zero when of type x, while the integrand for x' is non-negative.

Given that $\int_{y \in \mathfrak{t}^{-1}([t,\infty])} f(y)dy > 0$, there must be a positive probability that player 2 discloses an option y for which $1-y > x$ strictly after t' and strictly before t. Notice indeed that all the terms associated to other y's in player 1's expected net gain of revealing at t instead of t', when of type x, are non-positive, and in fact must sum up to a strictly negative number when player 2 discloses an option with positive probability after t. Remember our reasoning from the previous paragraph that the integrand involving y's such that $1-y > x$, and that are disclosed strictly after t' and strictly before t, are strictly increasing when replacing x by x'. If $\mathfrak{t}$ is part of a symmetric BNE, then it must be that player 1's expected net gain of disclosing his option at t instead of t' is non-negative when of type x, but our reasoning also shows that the same expected net gain is strictly larger for x' if $t > t'$, thereby contradicting the optimality of $\mathfrak{t}$. We have thus shown that $t \leq t'$, as desired. $\square$

Step 3 $\mathfrak{t}(x) = 0$, *for almost all* $x \in]1/2, 1]$, *i.e.* $\int_{x \in]1/2,1] \cap \mathfrak{t}^{-1}(]0,\infty])} f(x)dx = 0$.

Proof: Let X be the set of x's in $]1/2, 1]$ such that $\int_{y \in \mathfrak{t}^{-1}([\mathfrak{t}(x),\infty])} f(y)dy > 0$, and $\bar{X}$ be its complement in $]1/2, 1]$. Let also t be the infimum of $\mathfrak{t}(x)$ when x varies in $\bar{X}$, and let $(t_k)_{k \in \mathbb{N}}$ be a decreasing sequence of non-negative real number such that $(t_k)_{k \in \mathbb{N}}$ converges to t, and $t_k = \mathfrak{t}(x_k)$ for some $x_k \in \bar{X}$, for each $k \in \mathbb{N}$. We have:

$$\int_{x \in \bar{X}} f(x)dx \leq \int_{y \in \mathfrak{t}^{-1}([t,\infty])} f(y)dy = \lim_{k \to \infty} \int_{y \in \mathfrak{t}^{-1}([t_k,\infty])} f(y)dy = 0.$$

We will now show that $\mathfrak{t}(x) = 0$, for all $x \in X$. This will allow us to conclude the proof, since the probability of a player not revealing his option at $t = 0$ when of a type

$x \in]1/2, 1]$ will then be known to be no larger than the probability of $\bar{X}$, which we have just shown is null.

Let thus $x \in]1/2, 1]$ be such that $\int_{y \in \mathfrak{t}^{-1}([t(x),\infty])} f(y)dy > 0$. Suppose, to the contrary of what we want to prove that $t = \mathfrak{t}(x) > 0$. Player 1's expected net gain of revealing at 0 instead of t is equal to

$$\int_{y \in \mathfrak{t}^{-1}(]t,\infty])} \min\{x, b_1(x,y)\}(1 - e^{-\delta t}) f(y)dy$$
$$+ \int_{y \in \mathfrak{t}^{-1}(t)} (\min\{x, b_1(x,y)\} - b_1(x,y)e^{-\delta t}) f(y)dy$$
$$+ \int_{y \in \mathfrak{t}^{-1}(]0,t[)} (\min\{x, b_1(x,y)\} - \max\{1-y, b_1(x,y)\}e^{-\delta \mathfrak{t}(y)}) f(y)dy$$
$$+ \int_{y \in \mathfrak{t}^{-1}(0)} (b_1(x,y) - \max\{1-y, b_1(x,y)\}) f(y)dy.$$

The integrand in the first term is clearly strictly positive. The integral in the second term can be restricted to those y's that are no smaller than 1/2, by Step 1, and the integrand is equal to $b_1(x,y)(1 - e^{-\delta t})$. Again, this is strictly positive. We know from Step 2 that y must be at least 1/2 be be revealed strictly before t. Hence $1 - y < x$ for all such y's, and the third integrand is equal to $b_1(x,y)(1 - e^{-\delta \mathfrak{t}(y)})$, which is strictly positive when $\mathfrak{t}(y) > 0$, while the integrand in the fourth term is null. Given that there is a positive probability that the other player discloses his option at or after t, one concludes that player 1's expected net gain of revealing at 0 instead of t is strictly positive, which contradicts the optimality of $\mathfrak{t}$. Hence $\mathfrak{t}(x) = 0$, and we are done with the proof. □

Step 4 $\mathfrak{t}$ *is strictly decreasing on* $[0, 1/2[$.

Proof: Consider $x' < x < 1/2$, and let $t = \mathfrak{t}(x)$ and $t' = \mathfrak{t}(x')$. Let's start by assuming that $\int_{y \in \mathfrak{t}^{-1}([\mathfrak{t}(x),\infty])} f(y)dy > 0$. It is straightforward to check that the proof of Step 2 goes through in this case as well, after noticing that the second term in the expected net gain of revealing at t instead of t' is null when $t > t'$, as the probability of a player revealing at a strictly positive time is null thanks to Steps 1 and 3. Hence $\mathfrak{t}(x') \geq \mathfrak{t}(x)$.

We may assume that $\mathfrak{t}(x) > 0$, as otherwise almost all types between x and 1/2 disclose at 0, contradicting Step 1. We know from the previous paragraph that $\mathfrak{t}(x'') \leq \mathfrak{t}(x)$, for all $x'' \in]x', x[$. Step 1 implies that there exists $x'' \in]x', x[$ such that $\mathfrak{t}(x'') > \mathfrak{t}(x)$. The reasoning from the previous paragraph implies that $\mathfrak{t}(x') \geq \mathfrak{t}(x'')$, and hence $\mathfrak{t}(x') > \mathfrak{t}(x)$.

We have thus established the desired property, but under the additional assumption that $\int_{y \in \mathfrak{t}^{-1}([t(x),\infty])} f(y)dy > 0$. We now show that this inequality must in fact hold for any $x \in]0, 1/2[$, thereby proving the result by applying our previous arguments to x's that are as close to 1/2 as needed. Let x^* be the supremum of the x's in $[0, 1/2[$ for which there is a strictly positive probability of disclosure on or after $\mathfrak{t}(x)$. We thus have to show that $x^* = 1/2$. Suppose on the contrary that $x^* < 1/2$. Let then $t^* = \inf_{y \in]x^*,1/2[} \mathfrak{t}(y)$, and let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $]x^*, 1/2[$ such that $(\mathfrak{t}(x_k))_{k \in \mathbb{N}}$ decreases towards t^*, as k tends to infinity. Since $\mathfrak{t}$ is measurable, we have:

$$\lim_{k \to \infty} \int_{y \in \mathfrak{t}^{-1}([\mathfrak{t}(x_k),\infty])} f(y)dy = \int_{y \in \mathfrak{t}^{-1}([\lim_{k \to \infty} \mathfrak{t}(x_k),\infty])} f(y)dy = \int_{y \in \mathfrak{t}^{-1}([t^*,\infty])} f(y)dy.$$

Notice that the right-most expression must be strictly positive, since $]x^*, 1/2[\subseteq \mathfrak{t}^{-1}([t^*, \infty])$, by definition of t^*. Hence there exists $K \in \mathbb{N}$ such that $\int_{y\in[0,1] \text{ s.t. } \mathfrak{t}(y)\geq \mathfrak{t}(x_k)} f(y)dy > 0$, for all $k \geq K$, leading to the desired contradiction, given the definition of x^*. $\square$

Step 5 $\mathfrak{t} = \mathfrak{t}^*$.

Proof: We start by strengthening the result from Step 3, by showing that $\mathfrak{t}(x) = 0$, for all $x > 1/2$. Suppose, to the contrary of what we want to prove, that $\mathfrak{t}(x) > 0$, for some $x > 1/2$. Let us compute type x's expected net gain of revealing at $\mathfrak{t}(x)$ instead of 0. This expression is the same as the one written in the proof of Step 2, if one takes $t = \mathfrak{t}(x)$ and $t' = 0$. Notice also that the second term in the formula is null, since almost all types above 1/2 reveal at zero (cf. Step 3), and the revelation strategy followed by types smaller than 1/2 is strictly decreasing (cf. Step 4). The fourth term is zero as well, because $y \geq 1/2$ if revealed at zero (cf. Step 4), and $\max\{1-y, b_1(x,y)\} = b_1(x,y)$, for all such y's. Hence the expected net gain can be rewritten as follows:

$$\int_{y\in\mathfrak{t}^{-1}([t,\infty])} \min\{x, b_1(x,y)\}(e^{-\delta t}-1)f(y)dy + \int_{y\in\mathfrak{t}^{-1}([0,t]),\ y\geq 1-x} b_1(x,y)(e^{-\delta\mathfrak{t}(y)}-1)f(y)dy$$
$$+\int_{y\in\mathfrak{t}^{-1}([0,t]),\ y\leq 1-x} ((1-y)e^{-\delta\mathfrak{t}(y)} - x)f(y)dy.$$

Notice that this expression is strictly negative if $\int_{y\in\mathfrak{t}^{-1}([0,t]),\ y\leq 1-x} f(y)dy = 0$, which would contradict the optimality of revealing at $t = \mathfrak{t}(x) > 0$ when of type x. Consider now the expected net gain for a type $x' \in]1/2, x[$ to reveal at t instead of 0. A simple rearrangement of terms in the integrals implies that it is equal to

$$\int_{y\in\mathfrak{t}^{-1}([t,\infty])} \min\{x', b_1(x',y)\}(e^{-\delta t}-1)f(y)dy + \int_{y\in\mathfrak{t}^{-1}([0,t]),\ y\geq 1-x} b_1(x',y)(e^{-\delta\mathfrak{t}(y)}-1)f(y)dy$$
$$+\int_{y\in\mathfrak{t}^{-1}([0,t]),\ y\leq 1-x} ((1-y)e^{-\delta\mathfrak{t}(y)} - x')f(y)dy$$
$$+\int_{y\in\mathfrak{t}^{-1}([0,t]),\ 1-x\leq y\leq 1-x'} [((1-y) - b_1(x',y))e^{-\delta\mathfrak{t}(y)} + (b_1(x',y) - x')]f(y)dy.$$

The first two terms are no smaller than their counterpart with x instead of x'. The third term, on the other hand, is strictly larger than its counterpart, since $\int_{y\in\mathfrak{t}^{-1}([0,t]),\ y\leq 1-x} f(y)dy > 0$. The fourth term, finally, is non-negative since $y \leq 1-x'$ implies $x' \leq b_1(x',y) \leq 1-y$. Type x's expected net gain of revealing at t instead of 0 being non-negative, it must now be strictly positive for type x'. Hence all the types in $]1/2, x']$ would reveal after 0, thereby contradicting Step 3. This establishes that $\mathfrak{t}(x) = \mathfrak{t}^*(x)$, for all $x > 1/2$.

Next, one can follows the arguments in the proofs of Steps 4 and 6 in the previous proof in this Appendix to show that $\mathfrak{t}$ is continuous of $]0, 1/2[$, that $\lim_{x\to 1/2_-} \mathfrak{t}(x) = 0$, and that $\mathfrak{t}$ is differentiable on $]0, 1/2[$ with

$$\mathfrak{t}'(x) = \frac{(1-2x)f(x)}{\delta x F(x)},$$

for each $x \in]0, 1/2[$. One can then follow the argument from the proof of Step 7 in the previous proof in this Appendix to show that $\mathfrak{t} = \mathfrak{t}^*$. ■

Derivation of the sum of bargainers' ex-ante expected payoffs in the dynamic disclosure game. Recall that the sum of bargainers' ex-ante expected payoffs in the symmetric BNE of the dynamic game is equal to

$$1 - \theta_D^2 + \int_{x=0}^{\theta_D} \int_{y=0}^{\theta_D} e^{-\delta\tau(\max\{x,y\})} dxdy \tag{9}$$

where

$$\tau(x) = \int_x^{\theta_D} \frac{1-2y}{\delta y^2} dy = -\frac{1}{\delta\theta_D} + \frac{1}{\delta x} - \frac{2}{\delta} \ln\theta_D + \frac{2}{\delta} \ln x \tag{10}$$

for $x \leq \theta_D$ and uniform f. Note that the last term in (9) is equal to

$$\int_{x=0}^{\theta_D} \int_{y=0}^{x} e^{-\delta\tau(x)} dxdy + \int_{x=0}^{\theta_D} \int_{y=x}^{\theta_D} e^{-\delta\tau(y)} dxdy \tag{11}$$

Note that the second term in the sum above may be rewritten as follows:

$$\int_{x=0}^{\theta_D} \int_{y=x}^{\theta_D} e^{-\delta\tau(y)} dxdy = \int_{y=0}^{\theta_D} \int_{x=0}^{y} e^{-\delta\tau(y)} dxdy$$

Hence, (11) may be rewritten as

$$2\int_{x=0}^{\theta_D} \int_{y=0}^{x} e^{-\delta\tau(x)} dxdy = 2\int_{x=0}^{\theta_D} xe^{-\delta\tau(x)} dx$$

From (10) it follows that

$$e^{-\delta\tau(x)} = e^{1/\theta_D} \cdot e^{-1/x} \cdot \theta_D^2 \cdot x^{-2}$$

Therefore,

$$2\int_{x=0}^{\theta_D} xe^{-\delta\tau(x)} dx = 2\int_{x=0}^{\theta_D} [e^{1/\theta_D} \cdot \theta_D^2 \cdot \frac{e^{-1/x}}{x}] dx = -2e^{1/\theta_D} \cdot \theta_D^2 \cdot E_i(-\frac{1}{\theta_D})$$

Substituting this expression into (9) yields:

$$1 - \theta_D^2 - 2e^{1/\theta_D} \cdot \theta_D^2 \cdot E_i(-\frac{1}{\theta_D})$$

■

Proof of Proposition 9. By definition, all three bargaining solutions are symmetric and ex-post efficient. It remains to verify that they are also monotone. By definition, the Raiffa solution is monotone regardless of whether g is convex or not. To show that the Nash solution is monotone, let $(x, g(x))$ and $(y, g(y))$ be two payoff pairs on the utility frontier $u_2 = g(u_1)$ such that $y > x$. The line connecting these two points is given by

$$u_2 = g(y) + \alpha(x, y) \cdot (y - u_1)$$

where

$$\alpha(x,y) \equiv \frac{g(x)-g(y)}{y-x}$$

Let $b_i^N(x,y)$ be player i's payoff at the Nash solution associated with $(x,g(x))$ and $(y,g(y))$. The first bargainer's payoff under the Nash solution is as close as possible to half the intercept of the line going through $((x,g(x))$ and $(y,g(y))$, and hence

$$b_1^N(x,y) = \begin{cases} \phi(x,y) & if \quad x<\phi(x,y)<y \\ x & if \quad \phi(x,y)\leq x \\ y & if \quad \phi(x,y)\geq y, \end{cases}$$

where

$$\phi(x,y) \equiv \frac{g(y)}{2\alpha(x,y)} + \frac{y}{2}.$$

Consider first a change from $x=z$ to $x=z'$ such that $y>z'>z$. We need to show that $b_1^N(z',y)\geq b_1^N(z,y)$ and $b_2^N(z',y)\leq b_2^N(z,y)$. Note that because $\alpha(z',y)<\alpha(z,y)$ we have that $\phi(z',y)>\phi(z,y)$. A priori there are nine cases to consider, with z and z' falling in the three different areas that define b_1^N. It is straightforward to show that monotonicity does occur, or that the combination of conditions are impossible, in all except perhaps the following two cases. If z falls in the first region ($x<\phi(z,y)<y$), while z' falls in the second region ($\phi(z',y)\leq z'$), then $b_1^N(z,y)=\phi(z,y)\leq\phi(z',y)\leq z'=b_1^N(z',y)$, and we are done proving monotonicity in that case. Also, it is impossible for z to fall in the third area, and for z' to fall in in the first or second area, since this would lead to the contradiction $y\leq\phi(z,y)<\phi(z',y)<y$. It follows that $b_1^N(z',y)\geq b_1^N(z,y)$. An analogous argument shows that $b_2^N(z',y)\leq b_2^N(z,y)$, and that monotonicity is satisfied when y changes from $y=z$ to $y=z'$ such that $z'>z>x$.

As for the Kalai-Smorodinsky solution, let $(x,g(x))$ and $(y,g(y))$ be two points on the frontier satisfying $y>x$ (and hence, $g(x)>g(y)$). The KS solution to $(x,g(x))$ and $(y,g(y))$ is given by the intersection of the line connecting the two points with the ray going from the origin to the "utopia" point $(y,g(x))$.

Suppose we increase y to y'. By the definition of KS, it is clear that the expected payoff of player 1 assigned by KS will increase. It is not clear what happens to the expected payoff of player 2. Let u_2 be the expected payoff of player 2 in the KS solution to $(x,g(x))$ and $(y,g(y))$. Let u_2' be player 2's expected payoff at the solution assigned to $(x,g(x))$ and $(y',g(y'))$. We want to show that $u_2>u_2'$.[19]

The KS solution to $(x,g(x))$ and $(y,g(y))$ is given by the equation

$$\frac{y}{g(x)}u_2 = x + [\frac{y-x}{g(x)-g(y)}][g(x)-u_2]$$

Let

$$\delta \equiv \frac{y}{g(x)}$$

(the inverse of the slope of the ray) and

$$\mu \equiv \frac{y-x}{g(x)-g(y)}$$

[19] Renaming variables implies that the subsequent reasoning also applies when decreasing y to y', as long as y' remains above x. In that case, $u_2'>u_2$, as needed for monotonicity.

(the inverse of the absolute value of the slope of the line connecting the two points on the frontier). In a similar way, define

$$\delta' \equiv \frac{y'}{g(x)}$$

and

$$\mu' \equiv \frac{y' - x}{g(x) - g(y')}$$

We can therefore solve for u_2 and u_2' :

$$u_2 = \frac{x + \mu g(x)}{\delta + \mu}$$

and

$$u_2' = \frac{x + \mu' g(x)}{\delta' + \mu'}$$

Assuming $y' > y$, we want to show that $u_2 > u_2'$, or

$$\frac{x + \mu g(x)}{\delta + \mu} > \frac{x + \mu' g(x)}{\delta' + \mu'}$$

which is equivalent to (since the denominators are positive)

$$x(\delta' + \mu' - \delta - \mu) + g(x)(\mu\delta' - \mu'\delta) > 0$$

Since $g(x) = y'/\delta' = y/\delta$, this is equivalent to

$$x(\delta' + \mu' - \delta - \mu) + y'\mu - y\mu' > 0$$

which may be rewritten as

$$\mu(y' - x) - \mu'(y - x) + x(\delta' - \delta) > 0$$

Plugging in the expressions for $(\mu, \mu', \delta, \delta')$ gives

$$\frac{(y - x)(y' - x)}{g(x) - g(y)} - \frac{(y' - x)(y - x)}{g(x) - g(y')} + \frac{x(y' - y)}{g(x)} > 0$$

Placing the first two terms under the same denominator, it thus amounts to show

$$\frac{(y - x)(y' - x)[g(y) - g(y')]}{[g(x) - g(y)][g(x) - g(y')]} + \frac{x(y' - y)}{g(x)} > 0$$

The inequality indeed holds, as $y' > y > x$ and $g(x) > g(y) > g(y')$.

The fact that player 1's payoff increases (decreases) and player 2's payoff decreases (increases) when increasing x to x' whenever both x and x' fall above y, follows from the previous argument, after observing that the Kalai-Smorodinsky solution is anonymous. ■

Proof of Lemma 3. Efficiency and symmetry follow by construction. Monotonicity follows by construction in the following cases:
(i) Start from two points on the same side of the 45 degree line $u_2 = u_1$ and change only one of the points such that both still remain on the same side of $u_2 = u_1$.
(ii) Start from $(x, g(x))$ and $(z, g(z))$ such that $g(x) > x$, $g(z) < z$ and $g(x) \geq z$. Fix $(x, f(x))$ and change $(z, f(z))$ into $(z', f(z'))$ such that it is still the case that $g(x) \geq \max\{z', g(z')\}$.

Monotonicity is more difficult to show in the last remaining case (all other cases follow by symmetry): starts from $(x, g(x))$ and $(z, g(z))$ such that $g(x) > x$, $g(z) < z$, $g(x) > z$ and $g(z) > x$, then change $(x, g(x))$ into $(x', g(x'))$ such that $g(x') > z$.

We will prove monotonicity by checking the sign of the derivative of b_1^* with respect to its first component in that last region. It is helpful to do the following change of variable. For each $(x, g(x))$ falling in that last region, let α be the absolute value of the slope of the line joining $(z, g(z))$ to $(x, g(x))$. Vice versa, each $\alpha > 1$ determines a unique $(x, g(x))$ that falls in that region (at the intersection of X and the line of slope $-\alpha$ that goes through $(z, g(z))$). Let $\delta = x + g(x)$ (note that this is the utilitarian surplus). Then, for each $\alpha > 1$, we have:

$$\delta(\alpha)/2 = g(z) + \alpha(z - b_1^*(x(\alpha), g(z))),$$

or

$$b_1^*(x(\alpha), g(z)) = z - \frac{\delta(\alpha) - 2g(z)}{2\alpha}.$$

Let now ϵ be any small strictly positive number. We have:

$$\frac{b_1^*(x(\alpha+\epsilon), g(z)) - b_1^*(x(\alpha), g(z))}{\epsilon} = \frac{\delta(\alpha)\alpha + \delta(\alpha)\epsilon - 2g(z)\epsilon - \alpha\delta(\alpha+\epsilon)}{2\alpha(\alpha+\epsilon)\epsilon}.$$

Taking the limit as epsilon tends to zero, this expression is equal to

$$-\frac{\delta'(\alpha)}{2\alpha} + \frac{\delta(\alpha) - 2g(z)}{2\alpha^2}$$

(δ is differentiable because g is). Notice that $\delta(\alpha + \epsilon)$ is larger than the sum of the components of the vector at the intersection of this new line (going through $(z, g(z))$ and with angle $-\alpha - \epsilon$) and the vertical line going through $(x, g(x))$. This is so because the intersection of the new line with the utility frontier falls on the left of x, and the slope $\alpha + \epsilon$ is larger than 1 (i.e. any decrease in the first component is more than matched by an increase in the second component). The sum of the components of the vector associated to the new line is $x + g(x) + (z - x)\epsilon$. Therefore,

$$\delta'(\alpha) = \lim_{\epsilon \to 0} \frac{\delta(\alpha+\epsilon) - \delta(\alpha)}{\epsilon} \geq \lim_{\epsilon \to 0} \frac{x + g(x) + (z-x)\epsilon - x - g(x)}{\epsilon} = z - x.$$

Hence

$$\frac{db_1^*(x(\alpha), g(z))}{d\alpha} \leq \frac{-\alpha(z-x) + \delta(\alpha) - 2g(z)}{2\alpha^2} = \frac{x - g(z)}{2\alpha^2} \leq 0,$$

where the equality follows from the fact that $\alpha(z, x) = g(x) - g(z)$ and $\delta(\alpha) = x + g(x)$, and the last inequality follows from the fact that $x \leq g(z)$ (because $g(x) \geq z$, $g(z) < z$ and $g(x) > x$). Finally, $dx/d\alpha$ being strictly negative, it must be that b_1^* varies monotonically with x, as desired. ■