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ABSTRACT

Growth and the Optimal Carbon Tax: When to Switch from Exhaustible Resources to Renewables?*

Optimal climate policy is studied in a Ramsey growth model. A developing economy weighs global warming less, hence is more likely to exhaust fossil fuel and exacerbate global warming. The optimal carbon tax is higher for a developed economy. We analyze the optimal time of transition from fossil fuel to renewables, amount of fossil fuel to leave in situ, and carbon tax. Subsidizing a backstop without an optimal carbon tax induces more fossil fuel to be left in situ and a quicker phasing in of renewables, but fossil fuel is depleted more quickly. Global warming need thus not be alleviated.

JEL Classification: D90 and E13
Keywords: carbon tax, exhaustible resources, global warming, green paradox, growth, intergenerational inequality aversion, renewables and second best

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1. Introduction

Global warming is a pressing issue, but to date not much progress has been made on implementing the substantial carbon taxes that are needed to reduce demand for fossil fuels and stabilize the atmospheric concentration of CO_2. Attention has therefore been directed at subsidizing clean alternatives for fossil fuels ('backstops') such as solar or wind energy to reduce demand for fossil fuels and curb global warming. Sinn (2008ab) has forcefully argued that such second-best policies are counterproductive. They encourage owners of oil and gas fields to pump more quickly, so that fossil fuel reserves get exhausted more quickly and global warming is accelerated. This so-called Green Paradox has spawned various papers that attempt to make these arguments more formal (Hoel, 2008; Gerlagh, 2009; Grafton, Kompas and Long, 2010). However, if following the Stern Review (2007) the social discount rate that is used is low and global warming damages are acute, it is optimal to not fully exhaust fossil fuel reserves in which case a lower price of the backstop induces a bigger fraction of fossil fuel reserves to remain unexploited (van der Ploeg and Withagen, 2010). The Green Paradox does not occur then.

So far, analysis of the Green Paradox has been thoroughly partial equilibrium. The main objective of this paper is therefore to provide a general equilibrium analysis of this phenomenon within the context of the Ramsey growth model and allow for capital accumulation. To ensure that the initial cost advantage of using fossil fuel rather than alternatives such as solar or wind energy gradually diminishes as reserves are depleted, we suppose that the cost of extracting fossil fuel rises as fewer reserves remain. We also suppose that global warming damages are convex. As more fossil fuel is depleted and thus more CO_2 is emitted into the atmosphere, marginal damages of burning of fossil fuel increase. Since the extraction cost advantage of fossil fuel fades away while marginal damages of global warming increase as reserves are depleted, there might be a moment before fossil fuel reserves are fully exhausted when it is optimal to switch to the backstop. In that case, fossil fuel will be left in situ which curbs global warming. However, for underdeveloped economies with relatively scarce capital consumption will be low and thus the marginal utility of consumption will be high and the marginal global warming damages will as a consequence be low. It may then not be optimal to leave fossil fuel in situ; instead, all fossil fuel will be fully depleted and the Green Paradox raises its ugly head again.

This dilemma for underdeveloped economies reminds one of Bertolt Brecht's dictum: 'Erst kommt das Fressen: Dann kommt die Moral' (from *Die Dreigroschenoper)*.

The switch to the renewable backstop is more likely to occur in a socially optimal than in a market outcome which does not internalize global warming damages. Furthermore, the market economy will always fully exhaust fossil fuel reserves if the cost of supplying the backstop exceeds the marginal cost of extracting the last drop of fossil fuel. The amount of fossil fuel reserves that is left in situ in the optimal outcome is determined by the condition which says that the cost advantage of the last drop of fossil fuel over the backstop exactly equals the present value of marginal global warming damages. We then show that more reserves are left in situ if the social rate of discount is low, the climate challenge is acute, and the initial stock of fossil fuel reserves is high. The stock that is left in situ is higher if the economy is more developed and consumption is relatively close to its steady state. Conversely, if the economy is still underdeveloped and consumption low, less fossil fuel reserves will be left in situ. We also show that the optimal carbon tax rate rises as the economy proceeds along its development path.

We thus extend the classic Ramsey model of economic growth to allow for natural exhaustible resources, renewable backstops and global warming damages. Our analysis is related to the famous DSHH growth model with investment in manmade capital and natural exhaustible resources as factors of production (Solow, 1974; Stiglitz, 1974; Dasgupta and Heal, 1974), but differs in that we allow for renewable backstops and global warming damages and thus do not have to concern ourselves with sustainability issues. Our analysis is also related to earlier studies on pollution in the Ramsey model (e.g., van der Ploeg and Withagen, 1991). We build on studies that study capital accumulation, fossil fuel depletion and backstops (e.g., Tahvonen, 1997; Tsur and Zemel, 2003, 2005) and pollution and climate change in model with depletion of exhaustible resources but without investment and growth (e.g., Krautkraemer, 1985, 1998; Withagen, 1994; Hoel and Kverndokk, 1996). Our analysis extends our earlier results on the second-best effects of subsidizing clean backstops on fossil fuel exhaustion, speed and duration of phasing in of the backstop, and the effects on green and total welfare (van der Ploeg and Withagen, 2010) to allow for saving, investment and capital accumulation. We are thus able to offer a more rigorous, general-equilibrium analysis of the Green Paradox (Sinn, 2008ab). Our main result is that the Green Paradox is more likely to occur in a less developed than in a mature economy.

A recent paper by Golosov et al. (2010) is in some respects close to ours. It looks at backstops in a Ramsey or Dasgupta-Heal-Solow-Stiglitz type model with stock-dependent extraction costs, albeit that they model global warming externalities as directly impacting aggregate production whereas we allow for them as losses to social welfare. Their main findings are that a constant carbon tax rate does not affect fossil fuel use at all; the time path of the optimal carbon tax has an inverse U-shape, where the eventual decline of the carbon tax results from their assumption of natural decay of the concentration of CO2 in the atmosphere and that fossil fuel is fully exhausted. Golosov et al. (2010) also take the moment of transition to the backstop as exogenous, which greatly facilitates the analysis at the cost of realism. For some of their analytical results they also suppose that there is 100 per cent depreciation of manmade capital at the end of each discrete period (which gives a closed-loop expression for the value function) and that extraction costs do not depend on the remaining stock of fossil fuel reserves.

Our concerns are, in contrast, threefold: first, what are the determinants of the date that the renewable backstop kicks in and the date that fossil fuel is phased out altogether; second, is the backstop going to be used on its own or alongside fossil fuel and what is the optimal sequencing of fossil fuel and renewables; and third, how much fossil fuel if indeed any is it socially optimal to leave in situ. We thus devote considerable attention to the intricate connections between the fossil-fuel economy and the carbon-free economy and to endogenously determining the time when the backstop kicks in. We also analyze the difference between the market outcome and the socially optimal outcome, and find that the optimal carbon tax rises during the phase where fossil fuel is used and ends up being zero when the backstop has replaced fossil fuel completely. Furthermore, the size of the optimal carbon tax depends on the stage of economic development of the economy. An economy which has a low level of economic development starts off with a low carbon tax rate which rises as the economy develops which echoes the prescriptions of those who argue in favor of a rising ramp for the carbon tax (e.g., Nordhaus, 2007). However, a mature economy sets off with a high carbon tax rate.

To keep our analysis as simple as possible, we suppose perfectly competitive markets, market clearing, and exogenous labor supply. We also assume that renewables are a perfect substitute for fossil fuel in production. Furthermore, we suppose that renewables are a perfect substitute for consumption and investment goods. We could allow resource costs to be strictly convex, but we leave analysis of the resulting continuum of backstops coming on stream as fossil fuel prices rise with time as this complicates the analysis due to the various regimes that will result (cf., van der Ploeg and Withagen, 2010).

The outline of the paper is as follows. Section 2 presents the model and the optimality conditions for the social optimum. Section 3 characterizes the carbon-free economy. Section 4 analyzes the case of a developing economy where the current capital stock is substantially below the stock that will prevail in the carbon-free economy and where the capital stock will be higher at the time of the switch to renewables despite fossil fuel having run out. In that case, there will be an initial phase of only fossil fuel extraction followed by a phase with only using the backstop. Section 5 studies the case where the current capital stock will have to fall once fossil fuel reserves run out and the economy has to switch to the backstop which occurs if the backstop is relatively expensive. In that case, there may be a phase where fossil fuels and the backstop are used simultaneously. Section 6 first shows that a rising carbon tax rate ensures that the decentralized market outcome yields the socially optimal outcome and that the optimal carbon tax rate is higher for a mature than for a developing economy. It then discusses second-best climate policy for the decentralized market economy when an optimal carbon tax is infeasible and why this is more likely to lead to a Green Paradox for a developing than for a mature economy. Section 7 presents policy simulations to illustrate the time path of the optimal carbon tax and how it depends on the stage of economic development, the aversion to intergenerational inequality and the rate of time preference, and the Green Paradox. Section 8 concludes and highlights policy implications.

2. The model

Fossil fuel depletion is given by:

(1) $$\dot{S} = -R, \quad S(0) = S_0 \quad \text{and} \quad \int_0^\infty R(t)dt \le S_0,$$

where R denotes fossil fuel use, S the stock of remaining fossil fuel reserves, and S_0 the given initial stock of fossil fuel reserves. Supposing that renewables and consumption goods are perfect substitutes, the material balance equation of the economy and the investment dynamics are given by:

(2) $\dot{K} = F(K, X+R) - bX - G(S)R - C - \delta K, \quad K(0) = K_0,$

where *C*, *X* and *K* denote consumption, backstop use and the man-made capital stock, respectively, δ denotes the depreciation rate of manmade capital stock, *b* is the resource cost of one unit of the backstop, *G*(*S*) is the resource cost needed to extract one unit of fossil fuel, and K_0 is the given initial capital stock. Due to the presence of fixed factors such as land or labor, the production function *F*(.) has decreasing returns to scale with respect to capital and energy inputs, *X*+*R*. The production function satisfies the familiar Inada conditions and supposes that renewables and fossil fuels are perfect substitutes in production. We assume that the initial cost of extracting one unit of fossil fuel, $G(S_0)$, is less than the resource cost of the backstop, *b*, and that unit extraction costs rises as fewer fossil fuel reserves are left.

Assumption 1: Extraction cost per unit of fossil fuel satisfies $0 < G(S_0) < b$ and $G'(S) < 0, \forall 0 \leq S \leq S_0$.

Hence, as fossil fuel reserves are depleted, the backstop may with time become cheaper than fossil fuel.

Social welfare is given by the following intertemporal welfare function:

(3) $\int_0^\infty U(C(t)) - D(E_0 + S_0 - S(t)) \ \exp(-\rho t) dt, \quad U' > 0, U'' < 0, D' > 0, D'' > 0.$

This formulation supposes that there is no natural degradation of CO2 in the atmosphere, so that the stock of CO2 in the atmosphere is simply equal to the initial stock plus the accumulated sum of past CO2 emissions, $E = E_0 + S_0 - S$. The function capturing global warming damages, *D*(.), is convex; the utility function, *U*(.), is concave. The social planner maximizes (3) subject to (1), (2) and the non-negativity constraints for *X* and *R*. Positivity of *C* is ensured via the usual assumption that $\lim_{C \to 0} U'(C) = \infty$.

Letting λ be the marginal social value of manmade capital and μ the marginal social value of fossil fuel reserves, the Hamiltonian function for this problem reads:

(4) $$H \equiv U(C) - D(E_0 + S_0 - S) + \lambda \left[F(K, X+R) - C - bX - G(S)R - \delta K \right] - \mu R.$$

We thus obtain the following necessary optimality conditions:

(5)
(a) $U'(C) = \lambda,$

(b) $F_X(K, X+R) \le b$ and $X \ge 0,$ c.s.,

(c) $F_R(K, X+R) - G(S) - \mu / \lambda \le 0$ and $R \ge 0,$ c.s.,

(d) $\rho\lambda - \dot{\lambda} = \left[F_K(K, X+R) - \delta \right] \lambda,$

(e) $\rho\mu - \dot{\mu} = D'(E) - G'(S)R\lambda,$

(f) $\lim_{t\to\infty} \left[\lambda(t)K(t) + \mu(t)S(t) \right] \exp(-\rho t) = 0.$

Conditions (5a) and (5d) give the Keynes-Ramsey rule for the growth in consumption:

(6) $$\dot{C} / C = \sigma \left[F_K(K, X+R) - \delta - \rho \right],$$

where $\sigma \equiv -U'(C) / CU''(C) > 0$ denotes the elasticity of intertemporal substitution. The coefficients of relative intergenerational consumption inequality aversion and relative risk aversion equal $1/\sigma$. Growth in consumption is thus high if the return on capital is high and consumers are relatively patient, especially so if there is not much intergenerational inequality aversion. We define the marginal social benefit of an extra unit of fossil fuel reserves in terms of resource units as $\eta \equiv \mu / \lambda.$ Its dynamics follow from (5a), (5d) and (5e), and are described by the differential equation:

(7) $$\frac{\dot{\eta}}{\eta} = F_K(K, X+R) - \delta - \frac{D'(E)}{U'(C)\eta} + \frac{G'(S)R}{\eta}.$$

If we define $p = F_R$ and $r = F_K - \delta,$ we can interpret ρ as the social price of fossil fuel and r as the net rate of return on capital. Then, along an interval of time where fossil fuel is used we have from (5c) that $p = G(S) + \eta$ and thus from (5a) and (7) the following intertemporal efficiency condition:

(7′) $\dot{p} = r\ \ p - G(S)\ \ - \dfrac{D'(E)}{U'(C)}$ or $\dfrac{d\ \ p - G(S)\ }{dt} = r\ \ p - G(S)\ \ - \dfrac{D'(E)}{U'(C)} + G'(S)R.$

The first equation of (7′) says that the return on leaving a marginal unit of fossil fuel in the earth (i.e., the capital gains on fossil fuel reserves) should equal the return from depleting a marginal unit of fossil fuel, selling it and obtaining a market rate of return on it (i.e., the net return on capital on price minus cost of extraction) *minus* the marginal global warming damages resulting from burning this unit of fossil fuel. In case there are no global warming externalities and no extraction costs, (7′) becomes the familiar Hotelling rule which says that the capital gains on fossil fuel should equal the market rate of interest. Note that the Hotelling rent on fossil fuel η, i.e., the difference between the social price of fossil fuel p and the cost of fuel extraction $G(S)$, vanishes when the “laissez-faire” market economy relying on fossil fuel only approaches the moment in time where the renewable backstop is introduced (cf., Heal, 1976).

3. The backstop economy

Let us assume $X(t) > 0$ for some interval of time. Along this interval $F_X(K, X + R) = b$ and thus energy use increases in the use of manmade capital and decreases in the resource cost of the backstop:

(8) $X + R = V(K, b), \quad \text{with } V_K = -F_{KV} / F_{VV} > 0 \text{ and } V_b = 1/F_{VV} < 0.$

Define output net of fuel costs as $\tilde{F}(K,b) \equiv F(K, V(K,b)) - bV(K,b),$ where $\tilde{F}_K = F_K > 0$ and $\tilde{F}_b = -V(K,b) < 0.$ Hence, we obtain the state equations for manmade capital and consumption:

(2′) $\dot{K} = \tilde{F}(K,b) - \ \ G(S) - b\ \ R - \delta K - C, \quad K(0) = K_0,$

and

(6′) $\dot{C}/C = \sigma\left[\tilde{F}_K(K,b) - \delta - \rho\right].$

If production is given by the Cobb-Douglas function $F(K, X+R) = K^{\alpha}(X+R)^{\beta}$ with $\alpha, \beta > 0,$

$\alpha + \beta < 1,$ we have for $X > 0$ that $\beta K^{\alpha}(X+R)^{\beta-1} = b,\ \tilde{F}(K,b) = (1-\beta)K^{\frac{\alpha}{1-\beta}}\left(\frac{\beta}{b}\right)^{\frac{\beta}{1-\beta}}$ and

the Keynes-Ramsey rule for the growth in consumption becomes:

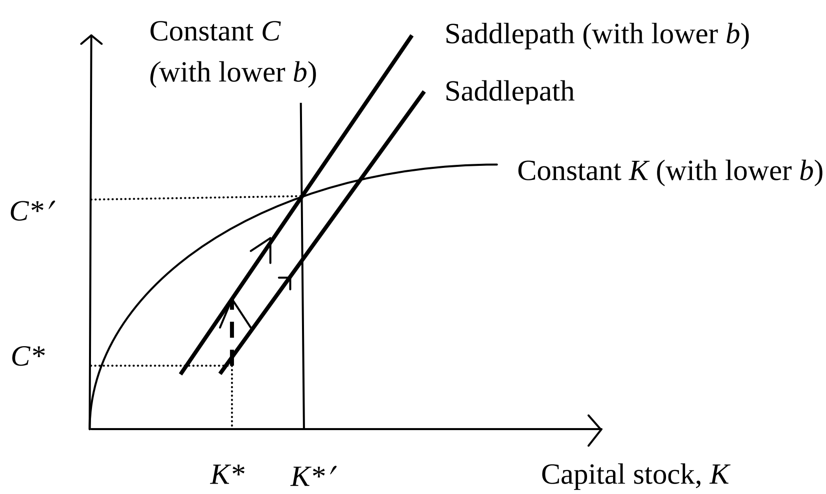
$$(6'') \qquad \dot{C}/C = \sigma\left[\alpha K^{\frac{\alpha+\beta-1}{1-\beta}}\left(\frac{\beta}{b}\right)^{\frac{\beta}{1-\beta}} - \delta - \rho\right].$$

Suppose now that we are in a world without the non-renewable resource (i.e., $R = 0$). The resulting carbon-free economy simplifies to (6′) and

$$(2'') \qquad \dot{K} = \tilde{F}(K,b) - \delta K - C\ .$$

It amounts to a version of the familiar Ramsey model of economic growth. The relevant phase diagram is given in fig. 1 where we have used the Inada conditions.

Figure 1: Impact of a reduction in the cost of the backstop on growth in the carbon-free economy



The system converges towards the steady state from any initial stock of capital. Note that the steady-state levels of the capital stock and consumption are decreasing functions of the cost of the backstop. For the Cobb-Douglas production function, the steady state for the carbon-free economy is defined by:

(9)
$$K^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{\beta}{b}\right)^{\frac{\beta}{1-\alpha-\beta}} \equiv K^*(b), \quad K_b^* < 0 \text{ and}$$
$$C^* = (1-\beta) K^{*\frac{\alpha}{1-\beta}} \left(\frac{\beta}{b}\right)^{\frac{\beta}{1-\beta}} - \delta K^* \equiv C^*(b), \quad C_b^* < 0.$$

An unanticipated, permanent reduction in the resource cost of the backstop (lower *b*) shifts the constant-*K* locus upwards and the constant-*C* locus to the right. Starting from the initial steady state (*K**, *C**), consumption jumps up on impact and then increases steadily towards the new steady state *C**′ along the new saddlepath denoted by:

(10) $$C = \Theta(K,b), \quad \Theta_K > 0, \ \Theta_b < 0.$$

Capital also grows steadily towards its new steady-state value *K**′. On impact use of the backstop increases and with time it increases further as the capital stock expands. The new steady state has bigger use of the backstop, a higher capital stock and higher consumption.

We conclude this section by showing that there must be a switch from using only fossil fuels to a phase where the backstop is used (possibly together with fossil fuel) provided the following assumption holds.

Assumption 2: $dF(K,V(K,b))/dK = F_K(K,V) - b\frac{F_{KV}(K,V)}{F_{VV}(K,V)} \to \infty$ as $K \to 0$.

In case that only fossil fuel is extracted, we have $F_R(K,R) = G(S) + \eta \le b$ and thus $R \ge V(K,b)$. For a given capital stock, fossil fuel in this phase thus exceeds the amount of fuel that is used when only the backstop is used or the backstop and fossil fuel are used simultaneously. Assumption 2 holds for a Cobb-Douglas production function.[1]

Lemma 1: It is suboptimal to have only use of fossil fuel throughout. If extraction of fossil fuel ends at time T with some fossil fuel left in situ, then the rent of extracting a further drop of fossil fuel must equal the present-value of global warming damages:

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[1] If $R > 0$ then $R = \left(\frac{G(S)+\eta}{\beta}\right)^{1/(\beta-1)} K^{\alpha/(1-\beta)}$ and $F_K = \alpha\left(\frac{G(S)+\eta}{\beta}\right)^{\beta/(\beta-1)} K^{(1-\alpha-\beta)/(\beta-1)}$. Since $G(S)+\eta \le b$, β < 1 and $\alpha+\beta$ < 1, we have that $dF/dK \to \infty$ as $K \to 0$ and thus assumption 2 holds.

(11) $$\eta(t) = \frac{D'(E_0 + S_0 - S(T))}{\rho U'(C(t))}, \quad \forall t \geq T \quad \text{if} \quad S(T) > 0.$$

Proof: If there would be only fossil fuel use throughout, then R must go to zero which implies that K must go to zero. But then it follows from assumption 2 and (6) that consumption is non-decreasing, which is infeasible. Once extraction stops, we have from (5e) that $\rho\mu(t) - \dot{\mu}(t) = D'(E_0 + S_0 - S(T))$ which yields upon integration $\mu(t) = \frac{D'(E_0 + S_0 - S(T))}{\rho} + \left[\mu(T) - \frac{D'(E_0 + S_0 - S(T))}{\rho}\right] \exp \rho(t-T)$ for $t \geq T$. If $\frac{D'(E_0 + S_0 - S(T))}{\rho} \neq \mu(T)$, the transversality condition (5f) is violated unless $S(T) = 0$. Hence, given that $\eta = \mu / U'(C)$, (11) must hold. Q.E.D.

4. Social optimum: development and switch towards a carbon-free economy

The most relevant case might be where the economy is on a development path. Although there are various mature economies, many parts of the global economy (China, India, Brazil, etc.) are still developing and far from their steady state. It is thus of interest to consider what happens if the global capital stock is low compared with the steady-state capital stock and consequently fuel demand is low as well. In this section we therefore make the following assumption.

Assumption 3: The initial stock of manmade capital is below the steady-state stock of manmade capital that prevails in the carbon-free economy, $K_0 < K^*$.

The following proposition shows that optimal energy use implies a first phase of using only fossil fuel and a final phase of using only the backstop.

Proposition 1: As long as $K(t) < K^*$ holds, there will be no simultaneous use of fossil fuel and the backstop. The optimal path has an initial phase where only fossil fuel is used. The economy converges asymptotically to the steady state (C^*, K^*).

Proof: Along intervals of simultaneous use, $0 = \frac{d\ G(S) + \mu/\lambda}{dt} = \frac{\rho\mu - D'(E)}{\lambda} + \frac{(F_K - \delta - \rho)\mu}{\lambda}$. If $K(t) < K^*$, then the second term on the right-hand side is positive which implies that the first term must be negative. But this implies that $\dot{\mu} < 0$ (from (5e), noting that $G'(S) < 0$) and μ becomes arbitrarily negative. This is not allowed. Hence, there can never be simultaneous use of fossil fuel and the backstop. Making use of lemma 1, we confirm that the optimal sequence has an initial phase of using only fossil fuel. Eventually, we arrive in the steady state because there cannot always be only use of fossil fuel. Q.E.D.

The proposition does not claim that the phase where only fossil fuel is used is directly followed by a phase where only the backstop is used. In following sections we discuss the possibility that capital overshoots its steady-state value in which case there will be an intermediate phase where fossil fuel and the backstop are used alongside each other but at capital levels higher than the carbon-free steady state.

4.1. Should society start with the backstop from the outset?

The first interval can be degenerate in which case the economy relies on the backstop from the beginning. Let us consider this in more detail for the case that the initial stock of capital is small, $K_0 < K^*$. We take the initial resource stock and the initial CO2 stock as given. For every $0 \le S \le S_0$, we determine $C = \Omega(S)$ from the condition that society is indifferent between using the backstop and fossil fuel:

(12) $$b = G(S) + \frac{D'(E_0 + S_0 - S)}{\rho U'(C)} \quad \Rightarrow \quad C = \Omega(S) \text{ with } \Omega' = \frac{\rho U'^2}{D'U''\left(G' - D'/\rho U'\right)} > 0.$$

We thus see that a higher stock of fossil fuel permits society to consume more. This is also the case if the resource cost of renewables is high, society's rate of time preferences is high and the initial stock of carbon in the ground and in the atmosphere is low; we suppress these arguments in $\Omega(.)$. Next we determine $K = \Gamma(S)$ by identifying it with the stock of capital corresponding with $C = \Omega(S)$ being on the stable manifold of the solution to the carbon-free economy (2″) and (6′). Since the stable saddlepath of the carbon-free economy, given by (10), slopes upwards (see fig. 1), we can solve $\Theta(K) = \Omega(S)$ to give $K = \Gamma(S)$ with $\Gamma_S(S) = \Omega_S / \Theta_K > 0$ so that a higher stock of fossil fuel also permits a higher stock of manmade capital. The shape and location of $\Gamma(S)$ are determined by several factors. First, if $b > G(0)$ then, as can be seen from (12),

$\Omega(0) = C_{\min} > 0$, implying $\Gamma(0) > 0$. If, more realistically, $b < G(0)$, then $\Omega(0) = \Gamma(0) = 0.$ Second, both $C^* > \Omega(S_0)$ and $C^* \leq \Omega(S_0)$ are possible, depending on the parameters of the model. Hence, we have either $K^* > \Gamma(S_0)$ or $K^* < \Gamma(S_0)$. If at some instant of time, the stock of fossil fuels is S and the stock of CO2 is $E_0 + S_0 - S$ and if the stock of manmade capital happens to be $K = \Gamma(S),$ it is optimal to have only backstop use forever as it is easily verified that all necessary conditions are satisfied.[2]

If the initial capital stock is large relative to the initial stock of fossil fuel, i.e., $K_0 > \Gamma(S_0),$ but still $K_0 < K^*$, then it is also optimal to have only backstop use indefinitely. The intuition stems from the fact that a relatively high initial stock of capital implies that the marginal utility of consumption is relatively low and the social cost of global warming relatively high.

However, if the economy is in its early stages of development and the stock of fossil fuel is still large, i.e., $K_0 < \Gamma(S_0),$ it is suboptimal to start with the backstop. If it would be optimal to start with the backstop, then condition (11) of lemma 1 gives $\eta(t) = D'(E_0) / \rho U'(C(t)), \forall t \geq 0$ but $\Omega(0) < \Omega(S_0)$ then implies that $b > G(S_0) + \eta(0)$ contradicting optimality conditions (5a) and (5b). Hence, initially there must be only fossil fuel use. Further, the next proposition demonstrates that capital use rises along the optimal path, at least as long as it is smaller than K^*.

Proposition 2: Capital use rises along an optimal path as long as it is smaller than K^*.

Proof: Let us assume that along the interval with only fossil fuel, capital is decreasing for some period of time. Since the economy will eventually approach the steady state and we start in an underdeveloped economy, the decrease will not be permanent. Hence, there exist instants of time $t_1 < t_2$ with $K(t_1) = K(t_2),\ S(t_1) > S(t_2)$ such that

$\dot{K}(t_1) = F(K(t_1), R(t_1)) - G(S(t_1)R(t_1) - \delta K(t_1) - C(t_1) < 0$ and either (i)

$\dot{K}(t_2) = F(K(t_1), R(t_2)) - G(S(t_2))R(t_2) - \delta K(t_1) - C(t_2) > 0$ or (ii)

$\dot{K}(t_2) = F(K(t_1), X(t_2)) - bX(t_2) - \delta K(t_1) - C(t_2) > 0.$ Recall that consumption is increasing with time, so that $C(t_1) < C(t_2)$. In case (i) we therefore have $F(K(t_1), R(t_1)) - G(S(t_1)R(t_1) <$

[2] We have $b < G(S(t)) + \mu(t) / \lambda(t),$ since lemma 1 gives $\mu(t) = D'(E_0) / \rho,\ G(S(t)) = G(S_0)$ and λ is decreasing.

$F(K(t_1), R(t_2)) - G(S(t_2)R(t_2).$ This gives rise to a contradiction in the following way. We have $F_R(t_i) = G(t_i) + \mu(t_i)/\lambda(t_i)$ from the necessary conditions and we also have $K(t_2) = K(t_1)$ but $S(t_2) < S(t_1).$ Hence, the shadow price of the resource, relative to the shadow price of capital is high and we obtain $p(t_2) = G(S(t_2)) + \mu(t_2)/\lambda(t_2) > p(t_1) = G(S(t_1)) + \mu(t_1)/\lambda(t_1).$ This implies $R(t_2) < R(t_1)$. But then welfare can be improved by marginally decreasing $R(t_1)$ (more net production, less pollution). Hence, we were not on an optimal path. In case (ii) we have $F(K(t_1), X(t_2)) - bX(t_2) >$ $F(K(t_1), R(t_1)) - G(S(t_1)R(t_1).$ This also implies that $R(t_2) < R(t_1)$ and the same contradiction is reached. Indeed, we now have $F_R(t_2) = b > G(S(t_1))$ and $F_R(t_1) < b$ from the necessary conditions. Q.E.D.

In this proposition we make the proviso of manmade capital being below its steady-state value. Starting with capital stock below its steady-state value, it could be that it monotonically approaches the steady-state value, but it is also possible that capital grows, overshoots and then falls back to its steady state. Indeed, if the initial stock of CO2 is low, initial capital is close to K^* and initial extraction costs are low (high S_0), it is optimal to extract fossil fuel to have high consumption and build up capital to a level larger than K^* . In section 5 we discuss the optimal paths that result when starting out from large capital stocks.

4.2. Should society deplete all fossil fuel reserves?

The next question is whether all fossil fuel reserves are depleted before the backstop phase starts. As time proceeds the cost advantage of using fossil fuel diminishes as less accessible reserves have to be depleted and the present value of global warming damages increases $(G' < 0, D'' > 0).$ Fossil fuel continues to be used until the instant of time T where the economy switches over to a carbon-free economy when *either* the marginal cost advantage of using fossil fuel over the backstop exactly equals the present value of future marginal global warming damages (i.e., (11) is satisfied) *or* the stock of fossil fuel has been fully exhausted.

(i) Leave fossil fuel in situ if $\Gamma(0) < K_0 < \Gamma(S_0)$

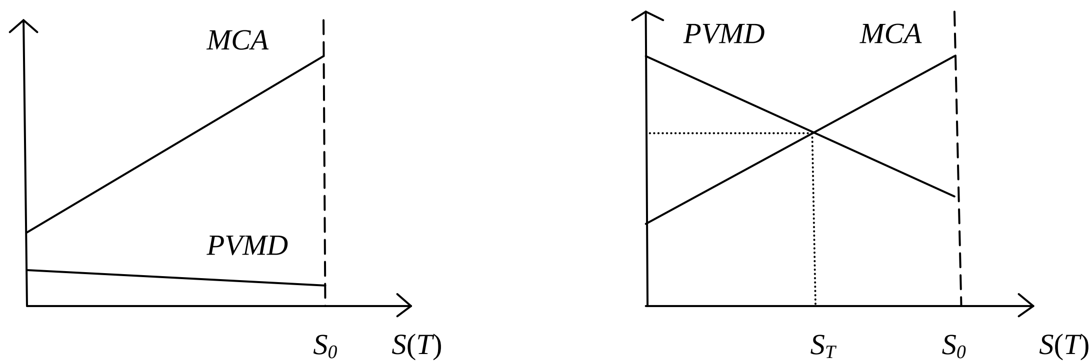
Let us denote by S_0^0 the initial stock of fossil fuel, together with an initial stock of CO2 equal to $E_0 + S_0 - S_0^0,$ that would warrant immediate use of the backstop, starting with the initial capital stock K_0. The stock S_0^0 is determined as follows. With K_0 corresponds C_0 on the stable branch of the carbon-free economy converging to the steady state. Then S_0^0 follows from (12) upon insertion of C_0. We claim that at least S_0^0 will be left in situ. To see this, suppose $S(T) < S_0^0$ remains in situ and that extraction ceases at T. From $S(T)$ we determine $C(T) = \Omega(S(T))$ by means of (12). Then $K(T) = \Gamma(S(T)) < K_0$. Hence, if less fossil fuel is left in the ground, the stock of manmade capital must have decreased which contradicts proposition 2. Hence, with relatively high initial stocks of capital, fossil fuel must be left in the ground.

(ii) Full exhaustion of fossil fuel if $K_0 < \Gamma(0)$

Full exhaustion will occur if the initial capital stock is relatively small, since then the marginal damages of global warming are small relative to the marginal utility of consumption. We thus conclude that developing economies are more likely to fully exhaust fossil fuel reserves than mature economies. Closer inspection of the condition $b = G(S) + \frac{D'(E_0 + S_0 - S)}{\rho U' \ \Omega(S)}$ gives more insight into these two cases. Fossil fuel is expensive from a social perspective if the marginal cost of global warming and extraction costs are high (i.e., not much fossil fuel is left in the ground) and if the marginal utility of consumption is low (i.e., consumption and thus the stock of fossil fuel is high). It is thus difficult to have a clear-cut conclusion. However, if there is no intergenerational inequality aversion (i.e., $U(.)$ is linear and σ infinity with U' equal to the constant φ, say), one can say unambiguously that it is then better to leave some fossil fuel in situ if the solution to $b = G(S) + \frac{D'(E_0 + S_0 - S)}{\rho\varphi}$ yields a positive value of S. If $U(.)$ is concave enough and intergenerational inequality aversion is large (σ small), it is only optimal to leave some fossil fuel in situ provided the cost of the backstop does not exceed the social cost of extracting the last drop of fossil fuel, i.e., $b > G(0) + D'(E_0 + S_0) / \rho U'(C)$ does not hold, for some low value of consumption, $C < C^*$. However, if there is substantial intergenerational inequality aversion (σ small enough), $U'(C(0))$ will be very large in the early stages of economic development and it may thus well be that case that $b > G(0) + D'(E_0 + S_0) / \rho U'(C)$ holds for some small value of C. In that case, it may be optimal to fully exhaust fossil fuel even though the backstop is cheaper than the social cost of extracting the last drop of fossil fuel if future generations are going to enjoy much higher consumption. For a given value of $U'(C(T))$, the cases of full and partial exhaustion are portrayed in panels (a) and (b) of fig. 2.

Figure 2: Fossil fuel reserves left in situ at time of switch to carbon-free economy

(a) Full exhaustion **(b) Partial exhaustion**



If in line with the Stern Review (2007) a more prudent (i.e., lower) rate of time preference is used, the lower ρ gives rise to an upward shift of the locus describing present value of the marginal global warming damages ($PVMD = D'(E_0 + S_0 - S) / \rho U'(C)$ and thus more fossil fuel left in situ provided that case (b) prevails (i.e., the initial CO2 concentration and initial fossil fuel reserves are high and the economy is well developed with a high level of consumption and low level of marginal utility at the time of the switch). A lower cost of the backstop shifts down the locus describing the marginal cost advantage of fossil fuel ($MCA = b - G(S)$), which increases the incentive to switch to the backstop and thus also increases the amount of fossil fuel left in situ. The amount of fossil fuel left in situ at the time of the switch to the carbon free society is thus:

$$(12') \qquad S(T) = \text{Max}\left[0, S_T(\overset{+}{C(T)}, \overset{-}{b}, \overset{+}{E_0 + S_0}, \overset{-}{\rho})\right], \text{ where } S_T \text{ follows from } b - G(S_T) = \frac{D'(E_0 + S_0 - S_T)}{\rho U'\ C(T)}.$$

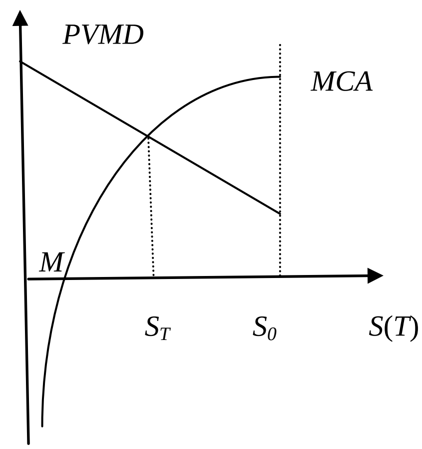
A richer and more patient world with an acute climate challenge and a relatively cheap carbon-free alternative thus decides to leave a bigger fraction of available fossil fuel reserves untouched.

If we add to assumption 1 that the extraction cost of extracting one unit of fossil fuel goes to infinity as fossil fuel reserves approach full exhaustion, we never get full exhaustion.

Assumption 1′: Extraction cost per unit of fossil fuel satisfies $0 < G(S_0) < b,\ G'(S) < 0, G''(S) > 0,$ $\forall 0 \le S \le S_0$ and $b < G(0)$.

With assumption 1′ the condition $b = G(S) + D'(E_0 + S_0 - S) / \rho U'\ \Omega(S)$ always yields a non-negative solution for S. As fig. 2′ indicates, it is sufficient to have $\lim_{S \to 0} G(S) = \infty$ to ensure partial exhaustion with $S(T) > 0$ regardless of the value of b, which is what we do in the policy simulations of section 7.

Figure 2′: Fossil fuel reserves left in situ with ever-increasing extraction costs



4.3. Fossil-fuel economy and optimal timing of transition to the carbon-free economy

Here we couple our Ramsey-DHSS model for the fossil-fuel phase (see sections 2 and 4.1) with our version of the Ramsey model with an infinitely elastic supply of renewables (see section 3) using our way of determining the optimal amount of fossil fuel to be left in situ (see section 4.2), and show how the optimal time of transition T to the carbon-free economy is calculated. The endogenous determination of T is what sets our analysis of climate change and growth apart from earlier analysis (Golosov et al., 2010). The carbon-free phase is described by equations (2″) and (6″) starting with the, as yet, unknown initial condition $K(T)$. Here we suppose $K(T) < K^*$. The level of consumption at the time of the switch to the carbon-free economy must be on the saddlepath (10) of the carbon-free economy. The fossil-fuel economy (1), (2′), (6′) and (7′),

$$\dot{S} = -R, \quad \dot{K} = \tilde{F}(K,p) - \; G(S) - b \; R - \delta K - C, \quad \dot{C} = \sigma C\left[\tilde{F}_K(K,p) - \delta - \rho\right] \text{ and}$$

(13) $$\dot{p} = r\left[p - G(S)\right] - \frac{D'(E_0 + S_0 - S)}{U'(C)} \text{ with}$$

$$\tilde{F}(K,p) = (1-\beta)K^{\frac{\alpha}{1-\beta}}\left(\frac{\beta}{p}\right)^{\frac{\beta}{1-\beta}}, R = \left(\frac{\beta K^{\alpha}}{p}\right)^{\frac{1}{1-\beta}} \text{ and } r = \alpha K^{\frac{\alpha+\beta-1}{1-\beta}}\left(\frac{\beta}{p}\right)^{\frac{\beta}{1-\beta}},$$

must be solved for the interval [0,*T*] given initial conditions $S(0) = S_0$ and $K(0) = K_0$, and terminal conditions $p(T) = b$ and $C(T) = \Theta(K(T), b)$ (from (10)). The last condition couples the four-dimensional state-space system for the fossil-fuel economy with the two-dimensional one for the carbon-free economy for a given transition time *T*. The *S(T)* and *C(T)* thus obtained are used to solve the in-situ condition (12′) or (12″) for the optimal transition time *T*. We nest an integration algorithm for the dynamic system (13) within a Gauss-Newton algorithm for solving the optimal value of *T* (see appendix). The optimal stock of fossil fuel left in situ thus depends on how much is consumed at the time of the switch to the backstop.

5. Social optimum: downsizing the economy and the switch towards a carbon-free economy

Now replace assumption 3 and consider a mature economy with a relatively high initial capital stock compared to the one that prevails in the steady-state of the carbon-free economy.

Assumption 3′: The initial stock of manmade capital is above the steady-state stock of manmade capital that prevails in the carbon-free economy, $K_0 > K^*$.

This assumption may seem less relevant at first blush, but may apply if running out of fossil fuel reserves in the absence of a cheap enough alternative renewable source of energy implies that the carbon-free backstop economy ends up with much lower levels of the capital stock K^* and production. We have shown that it is not possible to use only fossil fuel forever or to use fossil fuel at some positive rate forever, but we will see that it is feasible to use fossil fuel at an ever-decreasing rate. Hence, the optimal program necessarily ends up in (C^*, K^*). To establish what happens in that final phase, we distinguish three mutually exclusive possibilities.

Proposition 3 (i): If $K_0 > K^*$ and the backstop is cheaper than the initial social cost of fossil fuel (in terms of the marginal value of steady-state consumption), i.e.,

(14i) $$b < G(S_0) + \frac{D'(E_0)}{\rho U'(C^*)},$$

the backstop is used forever and fossil fuel is never used. The economy is given by (2″) and (6′).

Proof: If condition (14i) holds, $b < G(S) + \frac{D'(E_0 + S_0 - S)}{\rho U'(C)}$ for all $0 < S \leq S_0$ and all $C > C^*$. So, there exists a path satisfying all the necessary conditions with only use of the backstop. The necessary conditions are also sufficient. In spite of the fact that fossil fuel is not extracted, $e^{-\rho t}\mu(t)S_0 \to 0$ as $t \to \infty$ as $\mu(t) = D'(E_0)/\rho,$ so the transversality condition is satisfied. Q.E.D.

Proposition 3 (ii): If $K_0 > K^*$ and the backstop is more expensive than the social cost of fuel (in terms of the marginal value of steady-state consumption) for all stocks of fossil fuel reserves lower than the initial stock, i.e.,

(14ii) $$b > G(S) + \frac{D'(E_0 + S_0 - S)}{\rho U'(C^*)}, \quad \forall 0 < S < S_0,$$

there is a final phase with use of only the backstop. Furthermore, at the start of this final phase all fossil fuel reserves are fully exhausted.

Proof: There cannot be a final phase with only fossil fuel. There cannot be a final phase with simultaneous use, because that requires $b = G(S) + \frac{D'(E_0 + S_0 - S)}{\rho U'(C^*)}$ for some $S \geq 0.$ This can be seen as follows. With simultaneous supply in an infinite final interval of time, one has $b = G(S(t)) + \mu(t)/\lambda(t) \to G(S(\infty)) + \mu(\infty)/\lambda^*$ with $\lambda(t) = U'(C(t))$ and $\lambda^* = U'(C^*).$ Moreover, $\dot{\mu}(t) = \rho\mu(t) - D'(E(t)) + \lambda(t)R(t)G'(S(t)) \to \rho\mu(\infty) - D'(E_0 + S_0 - S(\infty)).$ Therefore, μ converges to a constant, and this constant should be $\frac{D'(E_0 + S_0 - S(\infty))}{\rho}$ with $S(\infty) \geq 0.$ Hence, there is a final phase with only the backstop. If, at the start of the final interval, some fossil fuel would be left, the same amount of fossil fuel is left in the steady state. It then pays to extract fossil fuel, which contradicts the fact that we use the backstop only.   Q.E.D.

Proposition 3 (iii): If $K_0 > K^*$, a necessary condition for having a final phase with simultaneous use of fossil fuel and the backstop is that there exists a positive level of fossil fuel reserves smaller than the initial level where the social cost of fossil (in terms of the marginal value of steady-state consumption) exactly equals the resource cost of the backstop:

$$\text{(14iii)} \quad \exists\, 0 < S^* < S_0 \text{ such that } G(S^*) + \frac{D'(E_0 + S_0 - S^*)}{\rho U'(C^*)} = b.$$

The economy converges towards the steady state of the carbon-free economy, (K^*, S^*, C^*). However, simultaneous use throughout only occurs by fluke.

Proof: The first part follows directly from propositions 3(i) and (ii). As the use of fossil fuel asymptotically tends to zero, the economy must converge to (K^*, S^*, C^*). The final part follows from the following argument. If there is simultaneous use throughout, the social price of energy must satisfy $p = F_R = b = G(S) + \eta$. It thus follows from equation (7′) with $\dot{p} = 0$ that

$$\text{(15)} \qquad b - G(S) = \frac{D'(E_0 + S_0 - S)}{r(K,b)U'(C)}, \quad \text{with } r(K,b) \equiv F_K(K, V(K,b)) - \delta,$$

where $r_K = F_{KK} < 0, r_b = -V_K < 0.$ Equation (12) can be solved to give the optimal remaining stock of fossil fuel reserves and, using the depletion equation (1), fossil fuel use:

$$\text{(16)} \qquad S = S(\overset{+}{K}, \overset{+}{C}, \overset{-/+}{b}, E_0 \overset{+}{+} S_0) \quad \Rightarrow \quad R = -\dot{S} = -S_K \dot{K} - S_C \dot{C}.$$

Upon substituting equation (16) into equation (2′) and using equation (6′), we obtain:

$$\text{(17)} \qquad \dot{K} = \frac{\tilde{F}(K,b) + \ G(S) - b \ S_C \sigma C \left[\tilde{F}(K,b) - \delta - \rho \right] - \delta K - C}{1 - (G(S) - b) S_K}, \quad K(0) = K_0.$$

The economy with simultaneous use of fossil fuel and the backstop is thus described by the differential equations (16) and (17) with initial conditions for the predetermined states, $S(0) = S_0$ and $K(0) = K_0$, and the differential equation (6′) with $C(0)$ adjusting to be on the stable manifold. And it is also needed that $S(t) \geq 0, \forall t$. At time zero, equation (15) becomes $b - G(S_0) = D'(E_0) / r(K_0, b)U'(C(0))$ and gives the initial value of *C*(0) in terms of the initial capital stock and the initial fossil fuel stock. A solution with simultaneous use throughout then exists only if this value of *C*(0) happens to coincide with the value of *C*(0) that is consistent with being on the stable manifold defining the saddlepath of the two-dimensional state-space system (17) and (6′). This only occurs by fluke. Q.E.D.

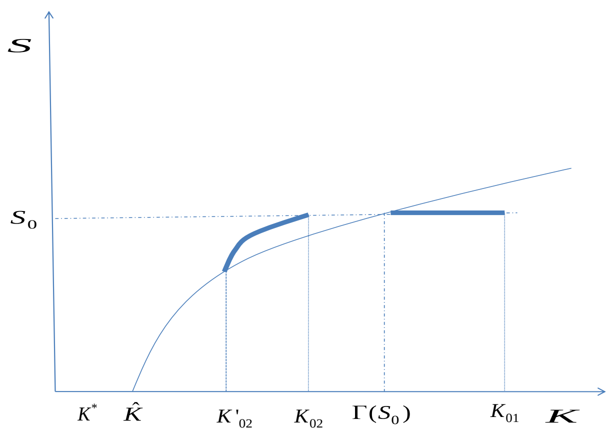
We now give a full characterization of the optimal paths.

5.1. Final phase with use of only the backstop after full exhaustion of fossil fuel reserves

We start with the case where condition (14ii) holds, so that there is final phase with only use of the backstop and at the start of this phase all fossil fuel is exhausted. According to (14ii) we have that at the steady-state level of consumption the backstop is more expensive than fossil fuel, even after full exhaustion: $b > G(0) + \frac{D'(E_0 + S_0)}{\rho U'(C^*)}$. This implies that the sum of the initial fossil fuel stock and the CO2 stock is bounded from above. We define $\hat{C}$ by $b = G(0) + \frac{D'(E_0 + S_0)}{\rho U'(\hat{C})}$ and $\hat{K}$ as the capital stock on the stable branch of the carbon-free economy corresponding with $\hat{C}$, that is $\hat{K} = \Theta^{-1}(\hat{C})$ (see (10)). Exhaustion of the stock of fossil fuel should take place at the moment where the capital stock reaches $\hat{K}$. Let the initial stocks of CO2 and fossil fuel be given. Using equations (17) and (6), we can then determine the initial capital stock such that, starting from these initial values, it is optimal to have simultaneous use of the backstop and fossil fuel with exhaustion taking place at the moment the capital stock reaches $\hat{K}$. Along this path all necessary conditions are satisfied and thus it is optimal in view of our the assumptions that we have made about concavity of *U*(.) and convexity of *D*(.); see fig. 3 below.

If the actual initial stock of capital equals $\Gamma(S_0),$ then indeed it is optimal to start with simultaneous use of fossil fuel and the backstop until $\hat{K}$ is reached. By construction, at time zero the cost of the backstop equals the social marginal cost of fossil fuel: $b = G(S_0) + \mu(0) / \lambda(0).$ If the initial capital stock K_0 happens to be larger than $\Gamma(S_0),$ then capital is less scarce implying that its initial shadow price $\lambda(0)$ is smaller and thus $b < G(S_0) + \mu(0)/\lambda(0)$. As is intuitively clear, it is then optimal to start with the backstop only. The marginal cost of doing so is small, relative to the total marginal social cost of fossil fuel. It is then optimal to use only the backstop until $K = \Gamma(S_0)$ is reached.

Figure 3: Characterization of optimal paths in case *K_0* > *K**



The situation is a bit more complicated for a small initial capital stock. In the previous case only one of the three state variables (capital, fossil fuel stock and CO2 stock) was changing. But with a small capital stock, i.e., $K_0 < \Gamma(S_0)$, we should start with only fossil fuel, but the locus of points where it is optimal to start with simultaneous use is changing too. Therefore, one cannot simply argue that the optimal trajectory will lead us from K_{02} to K'_{02} following the $\Gamma(S)$ curve in fig. 3. At $K = K'_{02}$ the fossil fuel stock corresponds with the start of simultaneous use, but the stock of CO2 is higher than the initial one E_0. Therefore, the marginal damage costs are too high and it would be optimal to start with the backstop. Nevertheless, there will be a phase with simultaneous use before the backstop kicks in. Heuristically, suppose $E_0 + S_0$ is left unchanged at say ϑ. Along the optimal path, S then gradually tends towards zero and E tends towards ϑ. Once the economy is close enough to $\hat{K}$, the marginal social cost of fossil fuel has become very high and it is better to use fossil fuel simultaneously with the backstop.

Summing up, if the backstop is expensive, we end with using only the backstop after a phase of simultaneous use. Initially, we have only fossil fuel if it is abundant or the stock of CO2 is small relative to the initial stock of capital, or only the backstop if capital is relatively abundant.

In the previous section we mentioned that with a large initial resource stock, low initial CO2 stock and initial capital close to the steady state value K^*, it would be optimal for capital to overshoot its steady state by using only fossil fuel. If this is indeed occurs, our arguments imply that capital does not monotonically approach its steady state but first increases and then decreases. One cannot exclude the possibility that, starting from the steady state with an abundant resource stock, it is optimal to have a direct transition from fossil fuel to the backstop.

Proposition 3 (ii) states that all fossil fuel must be *fully* exhausted before the final phase with only the backstop. The end of the simultaneous phase, T_2, thus follows from $S(K(T_2), C(T_2), b, E_0 + S_0) = 0.$ What determines the time T_1 at which the backstop is phased in alongside fossil fuel? If it is optimal to have a simultaneous phase, it cannot be optimal for this time to be zero as $S_0 = S(K_0, C(0), b, E_0 + S_0)$ only holds by fluke. Hence, if it is optimal to have a transitional simultaneous phase, then T_1 must be strictly positive and there must thus be an initial phase of only using fossil fuel. It is easy to construct an example where we should use fossil fuel initially. Suppose $E_0 = 0$ and $D'(0) = 0.$ If there would be an initial interval of time with use of the backstop only, then along the interval $\frac{d(G(S) + \mu / \lambda)}{dt} = \frac{\rho\mu - D'(E)}{\lambda} + \frac{(F_K - \delta - \rho)\mu}{\lambda} = \frac{(F_K - \delta)\mu}{\lambda} > 0.$ If we would start with only the backstop, then we need $b < G(S_0) + \mu(0) / \lambda(0).$ But along this regime $G(S) + \mu / \lambda$ is increasing and no transition to another regime can be made. Since initial simultaneous use only holds by fluke, we should now start with only fossil fuel.

5.2. Final phase with simultaneous phase of fossil fuel and the backstop

Now consider the optimal path with a final phase of simultaneous use of fossil fuel and the backstop, which holds under condition (14iii) of proposition 3(iii). This is the most realistic case. It will become increasingly difficult and extremely costly to extract the last drop of oil. We determine $S_0 > S^* > 0$ from $b = G(S^*) + \frac{D'(E_0 + S_0 - S^*)}{\rho U'(C^*)}$ and depict the locus of points for which it is optimal to have simultaneous use throughout, but now subject to the boundary condition that we end in (K^*, S^*, C^*). This yields the same picture as in fig. 3 with K^* and $\hat{K}$ coinciding and the curve ending in $(S^*, K^*) > (0,0)$. The analysis is straightforward; for a high initial capital stock, we have initial use of the backstop only; otherwise, only fossil fuel is used.

Expression (15) says that along a path of simultaneous use, the cost of the backstop must equal the extraction cost plus the present value of the global warming damages resulting from depleting a marginal unit of fossil fuel. With simultaneous use there cannot be a steady state with constant *C* and *K*, since that would require from (13) a constant *r*, a constant *p* and hence a constant *S*. Since this contradicts positive extraction of fossil fuel, there can only be simultaneous use in a transitional phase if the conditions of proposition 2 (ii) are satisfied. Of course, if the conditions of proposition 2 (iii) are satisfied, one might get a final phase where both fossil fuel and the backstop are used simultaneously. This only occurs from the outset by fluke. In that case, as one approaches the steady state (K^*, C^*, S^*), less fossil fuel is extracted from the earth. Asymptotically fossil fuel extraction goes to zero, but reserves need not vanish in the long run.

6. Climate policy, the market and the Green Paradox

6.1. Realizing the socially optimal outcome in the market economy

The market economy does not internalize the global warming externalities induced by CO_2 emissions (and thus corresponds to the social optimum with $D(E) = 0$). However, we suppose that the market economy is in an intertemporal equilibrium, which means that all markets clear at all instants of time. Hence, expectations of mining firms with respect to future prices are always realized, and not just temporarily, which can lead to either over- or under-exploitation of non-renewable resources (Dasgupta and Heal, 1978). Let us now consider extraction in the decentralized economy. We maintain assumption 1 saying that the initial extraction cost of fossil fuel is less than the cost of the backstop, $b > G(S_0)$. As we have seen, if the cost of extracting the last drop of fossil fuel is also less than the cost of the backstop, $b > G(0)$, all fossil fuel will eventually be fully exhausted.

Proposition 5: The market economy without taxes or subsidies never has simultaneous use of fossil fuel and the backstop. The sequence of use is: first only fossil fuel for the interval [0, *T*], after which the backstop takes over indefinitely. At the transition, we have $S(T)=0$ if $b>G(0)$. Otherwise, $b=G(S(T))$. The economy converges asymptotically to the steady state (C^*, K^*).

Proof: Along intervals of simultaneous use, $p(T) = b$ and (7′) must hold. As $D'(E) = 0$, we must have $b-G(S)=0$ (cf., equation (15)). But this implies that S is constant, which implies that $R = 0$. Hence, intervals of simultaneous use are not optimal in the "laissez-faire" economy. Since $G(S_0) < b$, it is optimal to have an initial interval with only fossil fuel use followed by an interval where the backstop is used forever, so that the carbon-free economy converges to the steady state (C^*, K^*). If $b > G(0)$, there will be full exhaustion in finite time. However, with assumption 1′ there will be partial exhaustion given by (12″). Q.E.D.

In case assumption 1′ holds, the market outcome, indicated by *M*, corresponds to the intersection of the *MCA* curve with the horizontal axis in fig. 2′; the socially optimal outcome S_T corresponds to the intersection of the MCA curve with the *PVMD* curve. Clearly, the market outcome depletes fossil fuel reserves to a lower level than the socially optimal outcome because it does not internalize global warming externalities. Although the PVMD curve in fig. 2′ is drawn for $K(0) < K^*$, proposition 5 also applies to the case $K(0) > K^*$. Crucial for the result of a higher $S(T)$ in the social optimum is that the market fails to internalize marginal global warming damages.

Proposition 6: The government can reproduce the social optimum outcome by levying a carbon tax τ during the fossil-fuel phase [0, *T*] equal to the social cost of carbon:

$$(18)\qquad \tau(t) = \int_t^{\infty}\left[\frac{D'\left(E_0+S_0-S(s)\right)}{U'\left(C(s)\right)}\right]e^{-\int_t^s r(s')ds'}ds, \quad \forall t\in[0,T].$$

The optimal carbon tax (18) rises over the time interval [0, *T*] if $K_0 < K^*$.

Proof: The consumer maximizes $\int_0^{\infty} e^{-\rho t}U(C)dt$ subject to its budget constraint $\dot{A}=rA+Y-C$, where *A* denotes household assets, *r* the market interest rate, and *Y* wages plus profits. This yields the Euler equation $\dot{C} = \sigma C(r - \rho).$ Consider first the case where firms internalize the effect of resource depletion of extraction costs. Firms thus maximize the present value of their expected stream of profits, $\int_t^\infty \Big[F\big(K(s), R(s) + X(s)\big) - \big(G(S(s) + \tau(s)\big) R(s) + Z(s) - bX(s) - I(s) \Big] e^{-\int_t^s r(s')ds'} ds,$ subject to the investment accumulation equation $\dot{K} = I - \delta K$, the resource depletion equation $\dot{S} = -R$, and $R, X \geq 0,$ where Z stands for the carbon tax revenues that are refunded in a lump-sum manner to firms and I stands for aggregate investment. Defining the Hamiltonian function $\hat{H} \equiv F\big(K, R + X\big) - \big(G(S) + \tau\big)R + Z - bX - I + \hat{\lambda}\big(I - \delta K\big) - \hat{\mu} R,$ we obtain the first-order conditions $-1 + \hat{\lambda} = 0,\ F_R - G(S) - \tau - \hat{\mu} \leq 0$

and $R \geq 0,$ c.s., $F_X \leq b$ and $X \geq 0,$ c.s., $r\hat{\lambda} - \dot{\hat{\lambda}} = F_K - \delta\hat{\lambda}$ and $r\hat{\mu} - \dot{\hat{\mu}} = -G'(S)R.$ This gives rise to the efficiency conditions $F_K(K,R) = r + \delta$ and $F_R \leq G(S) + \tau + \hat{\mu}$ and $R \geq 0,$ c.s., where it follows from integration that $\hat{\mu}(t) = \int_t^\infty \big[-G'\big(S(s)\big) R(s) \big] e^{-\int_t^s r(s')ds'} ds,\ \forall t \in [0,T].$ In equilibrium the government budget must be balanced, $Z = \tau R,$ and wage plus profit income equals $Y = F(K,R) - \big(G(S) + \tau\big)R - bX.$ Now turning our attention to the socially optimal outcome described by equations (1), (2), (6), (7) and (7′), we find

$$\eta(t) = \int_t^\infty \left[\frac{D'\big(E_0 + S_0 - S(s)\big)}{U'\big(C(s)\big)} - G'\big(S(s)\big) R(s) \right] e^{-\int_t^s r(s')ds'} ds, \quad \forall t \in [0,T]$$ from integration of (7′).

Comparing the first-order conditions of the decentralized market and the socially optimal outcome, we match them by setting $\tau = \eta - \hat{\mu}$ which gives (18). Since with time K and C rise, r and $U'(C)$ fall with time. Given that S falls with time, τ given by (18) rises with time. Q.E.D.

The optimal tax on carbon given in (18) thus equals the social cost of carbon, i.e., the present discounted value of all future marginal global warming damages using the market rate of interest and not the rate of time preference. The contribution of depleting an extra unit of fossil fuel at any point of time to the social cost of carbon rises with time on account of the increasing stock of the CO2 in the atmosphere (provided global warming damages are convex) and due to the development of the economy, rising path of consumption and the consequent falling marginal utility of consumption. Furthermore, the interest rate used to discount marginal global warming damages and extraction cost externalities falls with time as the economy develops and the capital stock rises (see proposition 2). This also tends to increase the social cost of carbon and the optimal carbon tax with time. Hence, the optimal carbon tax rises with time.

If the economy is in a low stage of development, consumption and the capital stock are low, the interest rate is high and the marginal value of consumption is high so that the social cost of carbon and the optimal carbon tax are low. As the economy develops, the social cost of carbon and optimal carbon tax rise. Once renewables kick in, the optimal carbon tax drops to zero.[3] Hence, the magnitude of the optimal carbon tax depends on the state of economic development.

6.2. Second-best outcome if carbon tax is infeasible

Here we investigate what happens if it is for political or other reasons infeasible to levy a carbon tax. In that case, the Green Paradox states that subsidizing the backstop fuel with the aim of curbing demand for fossil fuels and emissions of CO2 may be counterproductive. This paradox has been studied before in partial equilibrium model without capital accumulation (Sinn, 2008ab; Hoel, 2008; Gerlach, 2009; Grafton et al., 2010; van der Ploeg and Withagen, 2010). We extend this earlier work to a growth context. The Green Paradox highlights the second-best effects of introducing a backstop subsidy υ financed by a lump-sum tax, to phase out fossil fuel more quickly and mitigate global warming if a carbon tax is ruled out, $\tau = 0$. We are interested in the effects of a backstop subsidy on consumption, accumulation of manmade capital, growth and economic development. There is no case for a subsidy (or tax) on the backstop once fossil fuel is no longer in use. However, we suppose that the government is able to commit and keeps the backstop policy once fossil fuel is no longer used. This is necessary, because as soon as the government removes a backstop subsidy the private sector might start using fossil fuel again. In the following we introduce a "small" subsidy or tax into the "laissez-faire" market economy and then see whether or not this brings us closer to the social optimum. By "small" we mean that the sign of $b-\upsilon-G(0)$ is not reversed.

The carbon –free phase

[3] However, if the economy starts off at a relatively high degree of economic development, it is optimal for the carbon tax to start off high and then diminish with time. In fact, if there is natural decay of CO2 in the atmosphere, it may be optimal for the carbon tax rate to first rise, then to stabilize and finally to fall (cf., Golosov, et al., 2010).

Equation (2″) is unaffected and the Euler equation for the carbon-free phase becomes:

$$(6''')\quad \dot{C}/C=\sigma\left[\tilde{F}_K(K,b-\upsilon)-\delta-\rho\right]=\sigma\left[\alpha K^{\frac{\alpha+\beta-1}{1-\beta}}\left(\frac{\beta}{b-\upsilon}\right)^{\frac{\beta}{1-\beta}}-\delta-\rho\right].$$

We thus see that with a backstop subsidy ($\upsilon > 0$) the economy converges to a steady state with higher capital.[4] Whether steady state consumption is higher or lower depends on the parameters.[5] It can be shown, however, that with infinitesimally small subsidies steady-state consumption is higher. In any case, the introduction of a subsidy in an economy that is already carbon free leads to a downward jump in the rate of consumption.

Fossil-fuel phase: small initial capital stock

Consider first the case of a small initial capital stock, $K_0 < K^*$ (section 4). In the market economy the sequence of energy use is that first only fossil fuel is used followed by use of the backstop only. For a small initial stock of manmade capital, overshooting does not occur and the same sequence will occur in the social optimum. If $b > G(0)$, the “laissez-faire” market economy fully exhausts fossil fuel reserves, even though it may be socially optimal to leave some fossil fuel in the soil as has been demonstrated before. Subsidizing the backstop, while keeping $b-\upsilon > G(0)$, has adverse effects.[6] Fossil fuel is pumped up more quickly, the resource is fully depleted, and the transition to the carbon-free economy occurs more quickly, so that green welfare falls (Green Paradox). Moreover, overall social welfare also falls, because of the distorting effect of the subsidy. However, in the situation under consideration, with a low initial stock of manmade capital and a low initial level of consumption, the marginal utility of consumption is high relative to marginal global warming damages. The “laissez-faire” economy thus does not perform too badly compared to the optimal outcome, especially if it is in a low stage of development.

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[4] This is clear for the Cobb-Douglas case from the steady-state condition $F_K = \alpha K^{*(\alpha+\beta-1)/(1-\beta)}\left(\frac{b-\upsilon}{\beta}\right)^{\beta/(\beta-1)} = \delta+\rho.$

[5] For the Cobb-Douglas case we have $C^* = F(K^*, X^*) - bX^* - \delta K^* = K^{*\alpha/(1-\beta)}\left(\frac{b-\upsilon}{\beta}\right)^{\beta/(\beta-1)}\left(1-\frac{b\beta}{b-\upsilon}\right)-\delta K^*.$

[6] If the government cannot commit, it will abolish the subsidy after exhaustion of fossil fuel. This implies that the subsidy declines over time and becomes zero at the transition, because otherwise a price jump occurs in the energy price which would lead fossil fuel owners not to fully exhaust their stock. Counter-intuitively, in this case a tax on the backstop could be welfare improving.

With assumption 1′ (or if $b < G(0)$) we always have $b = G(S_T)$ for some $S_T > 0$. The economy without the backstop subsidy will then leave some fossil fuel unexploited, albeit less than in the social optimum. Now, a backstop subsidy has the effect of leaving more fossil fuel unexploited and in this sense brings the economy closer to the first best. The subsidy is beneficial for green welfare (no Green Paradox). Still, aggregate welfare is likely to be lower than in the social optimum, because the distorting effect of the subsidy will dominate the effect on green welfare, since the marginal damage in the case at hand is low. But fine tuning of the subsidy might be cumbersome. We conclude that for an economy with a low level of development the Green Paradox need not occur as with low levels of consumption and high marginal utility of consumption the valuation of global warming damages is low. So for developing economies a second-best renewables subsidy makes more sense than for mature economies. (As we have seen already, a first-best carbon tax is lower for a developing than for a mature economy).

Fossil-fuel phase: big initial capital stock

Now consider the situation of a high initial capital stock, $K_0 > K^*$, as characterized in section 5. Although in the social optimum several outcomes can occur, in the market economy we have fossil fuel first and then only the backstop. We focus here on the case of extraction costs becoming very large as the stock gets depleted, which corresponds to condition (14iii). In this case, there are several regimes. Suppose, for reasons of exposition, that it is optimal to have simultaneous use throughout, where exhaustion takes place only asymptotically. Then the unregulated market will over-extract fossil fuel. Depletion will take place too fast. A subsidy on the backstop then helps to slow down extraction. But the subsidy should be designed in such a way that there is some extraction of fossil fuel. This poses a complicated dilemma. For simultaneous use it is necessary that $b - \nu(t) = G(S(t))$, so that the economy should balance on a knife edge. The design of such a policy will be as difficult as the design of the optimal carbon tax, and be less effective. With a subsidy also redistributional issues come into play. A permanent constant subsidy might lead to under-exploitation of the stock of fossil fuel. So, it is difficult to make definitive statements in this case. The situation becomes more complicated if the first-best is to have use of the backstop initially or to use only fossil fuel initially. In the latter case a subsidy on the backstop will lead to too fast extraction of fossil fuel.

If the backstop is very cheap and condition (14i) holds, fossil fuel is never used in an optimum even though in the "laissez-faire" outcome all fossil fuel is exhausted if $b > G(0)$. Marginally subsidizing the backstop is detrimental because this speeds up extraction; a tax on the backstop is in order as in a well developed economy the marginal damage of emissions is large relative to the marginal utility of consumption and CO2 emissions must be avoided. The tax on the backstop postpones exhaustion of fossil fuel. More drastic is to subsidize the backstop to such an extent that $b - \upsilon < G(0)$ holds. In that case, some fossil fuel will be left in the ground.

If the backstop is expensive and condition (14ii) holds, fossil fuel will be fully exhausted in the first-best economy. Without a backstop subsidy or tax, we have $b > G(0)$. Several possibilities occur in the social optimum, which require different policies. If the economy starts with a high level of CO2 concentration, the optimum sequence is to have the backstop first, then simultaneous use of fossil fuel and the backstop, and finally only the backstop again. The subsidy policy should then be initially directed towards low fossil fuel extraction levels. Since in the socially optimal outcome no fossil fuel is left in situ, care is required. A subsidy on the backstop speeds up extraction initially, whereas extraction should be slowed down. So, a tax is in order for that purpose. But, a constant tax is not appropriate during the phase where simultaneous use is desirable. It may thus be beneficial to subsidize the backstop to such an extent that fossil fuel is not used, especially if climate damages are severe.

We conclude from the above that backstop subsidies may be counterproductive, which in some cases means a substantial welfare loss and in others a minor loss. Moreover, constancy of the subsidy or tax is most likely not credible and inefficient, and non-constant policies are hard to design. This makes a compelling case for a rising carbon tax which is relatively easy to design, and the backstop policy may in any case be a very inefficient way of redistributing income.

7. Policy simulations

For the policy simulations we restrict attention to the situation of underdevelopment and a not too expensive backstop and simulate both the social optimum and the decentralized market outcome described in sections 4 and section 6 under assumption 3, $K_0 < K^*$. The algorithm that has been used is described in the appendix. Normalizing so that S_0 = 20, we set E_0 = 24.[7] Following the Stern Review (2007) we use a low discount rate, $\rho = 0.014.$ We use the CES utility function $U(C) = C^{1-1/\sigma} / (1-1/\sigma)$ with a ballpark estimate of the elasticity of intertemporal substitution equal to $\sigma = 0.5$ [8] and explore the sensitivity with respect to σ to gain insight into the effect of intergenerational inequality aversion on global warming and economic growth. The shares of labor and of oil/gas in GDP have been set at $\alpha = 0.2$ and $\beta = 0.1.$ The average lifetime of manmade capital has been set at twenty years, so $\delta = 0.05.$ The initial stock of capital is set at half the value that prevails in the steady state of the carbon-free economy, i.e., $K_0 = K^*/2$. We set b = 0.4[9] and the initial cost of extracting one unit of fossil fuel at $G(S_0)$ = 0.2. In line with assumption 1′, we suppose that unit extraction costs become infinitely large as more and more fossil fuel is extracted and capture this with the specification $G(S) = \gamma S_0/S$ with γ = 0.2. This implies that in the market outcome where global warming externalities are not internalized, half of the initial stock of fossil fuel is left in situ at the time of the switch to the renewable backstop, $S(T) = S_0/2 = 10$. When global warming externalities are internalized, a bigger stock will be left in situ. The cost of extracting the last drop of fossil fuel thus exactly equals the cost of

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[7] In 2000 there were oil and gas reserves in the crust of the earth corresponding to 469 and 1,121 Giga tons of carbon, respectively, whereas there had been emitted 224 plus 111 Giga tons of carbon into the atmosphere resulting from burning, respectively, oil and gas (Edenhofer and Kalkuhl, 2009). Normalizing so that S_0 = 20, we set E_0 = 24 (rather than 335x20/1,590 = 4.2) to allow for the substantial CO2 concentration that was already in the atmosphere for non-anthropogenic reasons.

[8] Some argue that the elasticity of intertemporal substitution σ is very low with an implied coefficient of relative risk aversion of about 10 (e.g., Mehra and Prescott, 1985; Campbell and Mankiw, 1989; Obstfeld, 1994); others argue that σ is one or greater than one with a much smaller and more realistic implied coefficient of relative risk aversion (e.g., Hansen and Singleton, 1982). $\sigma = 0.5$ implies a coefficient of relative risk aversion 2 which seems a little high. Rather than breaking the link between risk aversion and intertemporal substitution to allow for an elasticity of intertemporal substitution in the range 0.05 to 1 and a coefficient of relative risk aversion in the range 0.4 to 1.4 (Epstein and Zin, 1991), we will explore the sensitivity of our results with respect to different values of σ.

[9] Solar and wind energy are more expensive than fossil fuel, especially if one looks beyond marginal production costs once capacity is installed and considers the costs needed to increase capacity, deal with intermittence and repair offshore wind mills. Wind energy can be at least three times as expensive as ‘grey’ electricity (Wikipedia). As far as the electricity industry is concerned, costs of renewables have fallen substantially: solar energy is currently 50% more expensive than conventional electricity; wind energy has the same cost and is (apart from the problem of intermittence) competitive; and biomass, CCS coal/gas and advanced natural gas combined cycle have mark-ups of 10%, 60% and 20%, respectively (Paltsev et al., 2009). These mark-ups are measured from a very low base and may not be so impressive when they account for a much larger market share. Hence, we use a 100% mark-up.

renewables in the market outcome, but will be less than in the socially optimal outcome. For global warming damages we use the specification $D(E) = \kappa E^2 / 2$ with $\kappa = 0.00012$.[10]

Although our simplified theoretical model ignores labor-augmenting technical progress, population growth, imperfect substitution between fossil fuel and renewables, and many other features, our specification illustrates some key trade-offs between climate change and growth.

7.1. The optimal carbon tax needed to attain the first-best outcome in the market economy

Comparing the socially optimal and the "laissez-faire" decentralized market outcome, we find that optimally internalizing global warming externalities implies that renewables get phased in more quickly, namely at time 22.0 rather than 42.3, and that 14.9 rather than 10.0 units of fossil fuel are left in situ at the time of the switch to the carbon-free economy. Switching more quickly to the carbon-free economy and leaving more fossil fuel in situ is an effective way to curb CO2 emissions and global warming. However, the economy ends up with less capital (3.44 rather than 3.57) and a little less consumption (0.818 rather than 0.829) in the long run. Fig. 4 plots the time paths of the key variables for these two outcomes (optimal solid lines, market dashed lines) and also gives the time path of the optimal carbon tax that ensures that the market properly internalizes the global warming externality. Both manmade capital and consumption rise for most of the fossil-fuel phase and then continue to rise to their new steady-state levels during the carbon-free phase; note that in the "laissez-faire" outcome manmade capital and consumption decline during the carbon-free phase. In this benchmark simulation there is no overshooting of capital in the optimum, but the market does overshoot. Consumption and manmade capital are smaller for the optimal than for the "laissez-faire" outcome, because the use of fossil fuel is cut back more quickly to limit global warming. This is encouraged by the steeper time path for the price of fossil fuel in the socially optimal outcome and is achieved by a rising time profile for the optimal carbon tax rising from 0.24 to 0.31. This contrasts with the inverse U-shape for the time profile of the optimal carbon tax found in Golosov et al. (2010), where the eventual decline of the carbon tax results from their assumption of natural decay of CO2 in the atmosphere.

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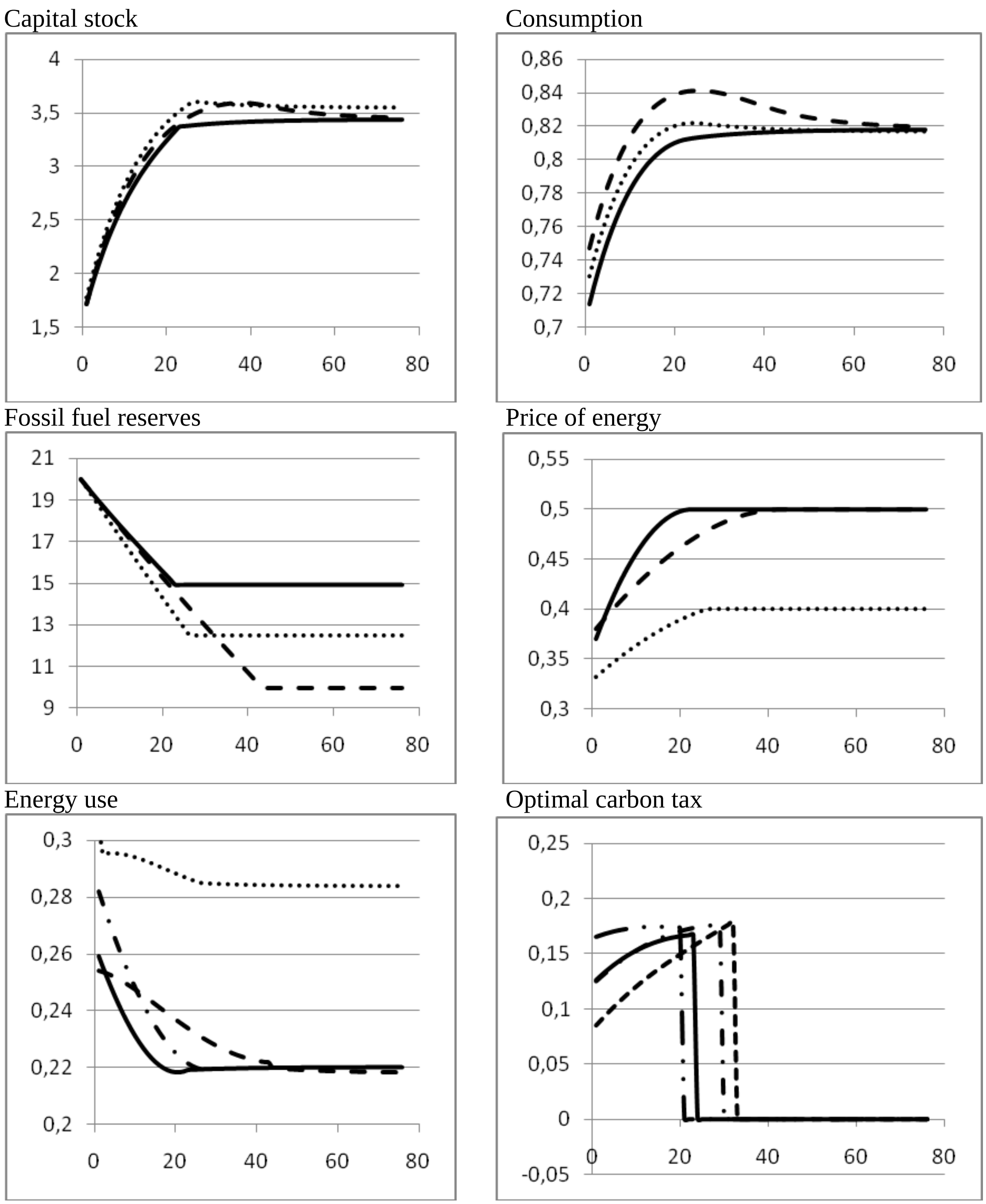
[10] Peer-reviewed estimates of the social cost of carbon for 2005 have an average of $43 per ton of carbon and a standard deviation of $83 dollar per ton of carbon, and these estimates are likely to increase by 2 to 4 percent per year (Yohe et al., 2007). A ballpark estimate of the social cost of carbon is $30 dollar per ton (Nordhaus, 2007).

Figure 4: Simulation trajectories



Key:	solid lines – optimum (benchmark):	T = 22.0, $S(T)$ = 14.9;
	dashed lines – "laissez-faire" market outcome:	T = 42.3, $S(T)$ = 10;
	dotted lines – no carbon tax and backstop subsidy:	T = 25.3, $S(T)$ = 12.5;
	double dots and dashes – optimum for developed economy:	T = 19.7, $S(T)$ = 15.3;
	dots and dashes – optimum with higher inequality aversion:	T = 28.9, $S(T)$ = 13.1;
	short dashes – optimum with higher rate of time preference:	T = 31.8, $S(T)$ = 12.

In contrast to the Green Solow model put forward by Brock and Taylor (2010), our model is framed in terms of the Ramsey model and has CO2 pollution from burning exhaustible fossil fuels rather than from being a by-product of production with the possibility of CO2 abatement. Within our framework there are initially substantial CO2 emissions per unit of output, because initially the marginal utility of consumption is large compared to the marginal damages of accumulated CO2 emissions. Over time the economy develops by rapid accumulation of manmade capital which compensates the falling use of fossil fuel resulting from the rising price of fossil fuel and growth tapering off as the economy develops. Consequently, CO2 emissions are initially high and then fall rapidly over time. Once the economy has switched to a clean backstop, CO2 emissions are reduced to zero and the accumulated pollution in the atmosphere is stabilized. In contrast, in the Green Solow model there is an Environmental Kuznets Curve resulting from the economy allocating a growing fraction of resources to abatement.

To gain better insight into the role of the state of economic development, we also simulated our model for a more developed economy starting off with $K_0 = 0.9\ K^*$. It takes now less time, 19.7, before renewables are phased in and more fossil fuel is left in situ, 15.3 rather than 14.9. The optimal carbon tax (double dots and dashes) is now higher than for the less developed economy (starting with $K_0 = K^*/2$, solid lines), since a richer economy values private consumption less, has a lower interest rate, and leaves more fossil fuel in situ and thus the social cost of carbon and the optimal carbon tax are higher. The optimal carbon tax thus depends on the state of development of the economy. Manmade capital now overshoots: it rises from 3.09 to 3.65 at the time of the switch and then falls back to its long-run value, 3.44. Consumption also overshoots. Strictly speaking, this simulation is not an optimum as overshooting implies that there must be a final phase where fossil fuel and the backstop are used simultaneously. But the simulation does make clear that overshooting is optimal.

7.2. Effects of intergenerational inequality aversion and time preference

When the elasticity of intergenerational inequality aversion ($1/\sigma$) is increased from 2 in the benchmark to 4, we find that the time to phase out fossil fuel and switch to renewables in the socially optimal outcome is postponed from instant 22.0 to 28.9, and the stock of fossil fuel that is left in situ at the end of the fossil-fuel phase is decreased from 14.9 to 13.1. Both these factors tend to increase CO2 emissions and global warming, as may be expected when intergenerational inequality aversion is higher and thus more priority is given to current, relatively poor generations who have to shoulder most of the burden of combating climate change rather than to distant, relatively rich generations. This way the economy develops faster initially at the expense of global warming, albeit that the steady-state levels of the capital stock and consumption are not affected by a higher degree of intergenerational inequality aversion. Fig. 4 indicates that the optimal carbon tax rate for this case (dots and dashes) is higher and lasts longer. Still, fossil fuel use is higher and lasts longer than when intergenerational equality aversion is not so high.

One might argue that the private sector employs a higher rate of time preference than the government, say a rate of time preference of 0.03 rather than 0.014 for the "laissez-faire" decentralized market economy. The time of the switch towards renewables is then reduced by a tiny amount from 42.34 to 42.30 whilst the stock of fossil fuel that remains in situ remains 10.0. However, as the economy is impatient and consumes more upfront and thus invests less, it ends up in the long run with much less manmade capital (2.58 rather than 3.57) and somewhat lower consumption (0.80 rather than 0.83).[11] If the government were to employ the higher rate of discount of 0.03 as well, it will pursue a more aggressive climate change policy. Renewables are phased in more quickly than in the "laissez-faire" outcome at time instant 31.8, but a lot more slowly than when the government would have employed a precautionary discount rate of 0.014 as in the benchmark (at time instant 22.0). Furthermore, fossil fuel left in situ, 12.0, is less than with a prudent discount rate of 0.014, but more than in the "laissez-faire" market outcome. Fig. 4 indicates that the optimal carbon tax for the case of a low discount rate (solid line) is higher than that for a high discount rate (short dashes) but lasts for a shorter period.

7.3. Second-best outcome: subsidizing the backstop does not lead to the Green Paradox

If we subsidize the costs of the backstop on global warming and development by an amount equal to $\upsilon = 0.1$ and finance it with lump-sum taxes, the date of switching from fossil fuel to renewables becomes 25.3, later than in the "laissez-faire" decentralized market outcome and earlier than in the socially optimal outcome. Furthermore, the amount of fossil fuel left in situ

[11] The reason that C* changes only a little compared with K* is that the share of capital is much smaller than the combined share of all the non-energy factors (capital and labor) in value added.

increases from 10 in the "laissez-faire" economy to 12.5, which is less than in the socially optimal outcome. Fig. 4 also plots the time paths of the key macroeconomic and resource variables under the backstop subsidy (dotted lines). Like in the "laissez-faire" market outcome, capital and consumption rise during the fossil-fuel era and then decline during the carbon-free phase. The path for consumption is higher than in the social optimum, but lower than in the "laissez-faire" decentralized market economy. As a result of the backstop subsidy, the price of energy is much lower both during the fossil-fuel and the carbon-free phase. As a result, private agents are encouraged to use much more fossil fuel in production than even in the "laissez-faire" outcome. This is what underpins the inexorable logic of the Green Paradox: despite renewable being phased in more quickly and more fossil fuel being left in situ, private agents pump up fossil fuel much more vigorously. The net effect on global warming damages is, however, ambiguous: for our numerical example global warming damages are reduced from 2.19 in the "laissez-faire" market outcome to 1.98 (more than in the social optimum, 1.73). Hence, despite the Green Paradox of pumping up more fossil fuel, global warming damages need not increase under the backstop subsidy as renewables are phased in more quickly and more fossil fuel is left in situ. As the backstop subsidy has a distorting effect on private decisions, private welfare falls from -67.8 in the "laissez-faire" to -69.0 in the market outcome with the backstop subsidy. The backstop subsidy thus boosts green welfare, but curbs social welfare from -70.0 to -70.9. Clearly, the backstop subsidy also performs worse than the social optimum with the optimal carbon tax.

8. Conclusion

We have coupled a Ramsey-DHSS growth model of a fossil fuel economy with a Ramsey growth model with an infinitely elastic supply of renewables to address optimal climate change policy. A key feature of our model is that we endogenously determine the optimal time of transition from the fossil fuel to the carbon-free economy and the optimal amount of fossil fuel to be left in the crust of the earth. We have shown that the social cost of carbon and thus the optimal carbon tax rises with the state of economic development. The intuition for this result is that in the initial phases of development when capital and consumption are low, the interest rate is high, and the marginal value of consumption is high and the stock of carbon pollution is still modest, it is optimal for fossil fuel use and CO2 emissions to be relatively high. As the carbon tax rises, fossil fuel use diminishes. We have also shown that the optimal carbon tax increases the amount of fossil fuel that is left in the crust of the earth and also brings forward the date that fossil fuel is phased out and carbon-free renewables are phased in compared with the "laissez-faire" decentralized market outcome. Although capital and consumption grow during the fossil-fuel phase as the economy develops, they fall towards their long-run values during the carbon-free phase. A lower rate of time preference or a lower degree of intergenerational inequality aversion imply a higher priority to future generations than the relatively poor, present generations, hence induces a higher optimal carbon tax and thus leads to more fossil fuel to be left in situ and quicker phasing in of renewables. If a carbon tax is infeasible and the government subsidizes renewables instead, renewables are phased in more quickly and more fossil fuel is left in situ. Despite fossil fuel being pumped up more vigorously (a manifestation of the Green Paradox), total global warming damages need not increase. Starting off with a relatively low level of economic development compared to the steady state of the carbon-free economy means that it is optimal to have a phase where only fossil fuel is used followed by a phase where only renewables are used. However, an intermediate phase where renewables are used alongside fossil occurs if starting off with a relatively degree of economic development.

We have made a number of simplifying assumptions to highlight the importance of endogenous determination of the time that the economy switches from fossil fuel to renewables and the optimal amount of fossil fuel to be left in situ. In practice, there may be an upward-sloping supply schedule of renewables (e.g., Sinn, 2008ab; van der Ploeg and Withagen, 2010). There may also be technical progress in renewables (e.g., Bovenberg and Smulders, 1996; Popp, 2002; Bosetti, et al., 2009; Acemoglu et al., 2010) leading to a gradual decline in the price of renewables, thus bringing forward the date of the switch from fossil fuel to renewables and kick-starting a green innovation machine. If there is also technical progress and population growth in the economy, then the economy will be non-stationary and convergence to a steady state is not warranted. We leave these issues for further research.

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Appendix: Spectral decomposition algorithm

The carbon-free phase starts at time T and is given by (2″) and (6″) starting with the initial condition $K(T)$. $C(T)$ must be on the saddlepath (10) of the carbon-free economy, which we linearize as follows:

$$C(T) = \Theta \; K(T) \; \cong C^*(b) + \theta \; K(T) - K^*(b) \;, \; \Theta' = \theta \equiv \frac{1}{2}\rho + \frac{1}{2}\sqrt{(\rho+\delta)^2 + 4\sigma\left(\frac{1-\alpha-\beta}{1-\beta}\right)(\rho+\delta)\frac{C^*}{K^*}} > 0.$$

We linearize (13) around (S^*, K^*, C^*, b) with S^* from $b = G(S^*) + \dfrac{D'(E_0 + S_0 - S^*)}{\rho U'(C^*)}$. Defining $X^* = V(K^*, b),$ we get the state-space system $\dot{\underset{\sim}{x}} = \mathrm{A}\underset{\sim}{x} + \underset{\sim}{a},$ where $\underset{\sim}{x} \equiv (S - S^*, K - K^*, C - C^*, p - b)'$ and $\underset{\sim}{a} \equiv (-X^*, K^{*\alpha} X^{*\beta} - G(S^*) - \delta K^* - C^*, 0, 0)'$. Spectral decomposition gives $\mathrm{A} = \mathrm{M}\Lambda\mathrm{M}^{-1} = \mathrm{N}^{-1}\Lambda\mathrm{N}$ where the diagonal matrix Λ contains the eigenvalues in descending order and the matrix M contains the eigenvectors. Defining the canonical variables $\underset{\sim}{y} = \mathrm{N}\underset{\sim}{x}$ yields $\dot{\underset{\sim}{y}} = \Lambda\underset{\sim}{y} + \underset{\sim}{n}, \quad \underset{\sim}{n} \equiv \mathrm{N}\underset{\sim}{a}.$ The system has two eigenvalues with positive real part, collected in the vector $\underset{\sim}{\lambda}_u$, and two with negative real part, collected in $\underset{\sim}{\lambda}_s$, and thus satisfies the saddlepoint property. We thus have $\Lambda = \mathrm{diag}(\lambda_{u1}\lambda_{u2}, \lambda_{s1}, \lambda_{s2}),$ so we get:

$$\text{(A2)}\quad \begin{aligned} y_{ui}(t) &= e^{\lambda_{ui}(t-T)}\, y_{ui}(T) + \bar{n}_{ui} - \bar{n}_{ui}, \quad \bar{n}_{ui} \equiv n_{ui}/\lambda_{ui}, \quad i=1,2, \\ y_{si}(t) &= e^{\lambda_{si}t}\, y_{si}(0) + \bar{n}_{si} - \bar{n}_{si}, \quad \bar{n}_{si} \equiv n_{si}/\lambda_{si}, \quad i=1,2, \quad \forall t \in [0,T]. \end{aligned}$$

Decomposing so that $\mathrm{M} = \begin{pmatrix} \mathrm{M}_{su} & \mathrm{M}_{ss} \\ \mathrm{M}_{uu} & \mathrm{M}_{us} \end{pmatrix}$ and $\underset{\sim}{x} = (\underset{\sim}{x}_s{}', \underset{\sim}{x}_u{}')'$, we write the initial conditions as follows:

$$\text{(A3)}\quad \underset{\sim}{x}_s(0) = \mathrm{M}_{ss}\underset{\sim}{y}_s(0) + \mathrm{M}_{su}\begin{pmatrix} e^{-\lambda_{u1}T} & 0 \\ 0 & e^{-\lambda_{u2}T} \end{pmatrix}\left[\underset{\sim}{y}_u(T) + \underset{\sim}{\bar{n}}_u\right] - \underset{\sim}{\bar{n}}_u = \underset{\sim}{x}_{so} \equiv (S_0 - S^*, K_0 - K^*)'.$$

The terminal conditions $C(T) = \Theta\;\; K(T)$ given above and $p(T) = b$ are written as follows:

$$\text{(A4)}\quad \begin{aligned} &\mathrm{E}\left(\mathrm{M}_{ss}\left\{\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T} \end{pmatrix}\left[\underset{\sim}{y}_s(0) + \underset{\sim}{\bar{n}}_s\right] - \underset{\sim}{\bar{n}}_s\right\} + \mathrm{M}_{su}\underset{\sim}{y}_u(T)\right) + \mathrm{M}_{us}\left\{\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T} \end{pmatrix}\left[\underset{\sim}{y}_s(0) + \underset{\sim}{\bar{n}}_s\right] - \underset{\sim}{\bar{n}}_s\right\} \\ &+\mathrm{M}_{uu}\underset{\sim}{y}_u(T) = \underset{\sim}{0} \text{ from } \mathrm{E}\underset{\sim}{x}_s(T) + \underset{\sim}{x}_u(T) = \underset{\sim}{0}, \quad \mathrm{E} = \begin{pmatrix} 0 & -\theta \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The initial and terminal conditions (A3) and (A4) can be solved as follows:

$$\begin{pmatrix} \underset{\sim}{y}_s(0) \\ \underset{\sim}{y}_u(T) \end{pmatrix} = \begin{pmatrix} \mathrm{M}_{ss} & \mathrm{M}_{su}\begin{pmatrix} e^{-\lambda_{u1}T} & 0 \\ 0 & e^{-\lambda_{u2}T} \end{pmatrix} \\ (\mathrm{M}_{us} + \mathrm{EM}_{ss})\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T} \end{pmatrix} & \mathrm{M}_{uu} + \mathrm{EM}_{su}\begin{pmatrix} e^{-\lambda_{u1}T} & 0 \\ 0 & e^{-\lambda_{u2}T} \end{pmatrix} \end{pmatrix}^{-1} \begin{pmatrix} \underset{\sim}{x}_{s0} + \mathrm{M}_{su}\begin{pmatrix} 1-e^{-\lambda_{u1}T} & 0 \\ 0 & 1-e^{-\lambda_{u2}T} \end{pmatrix}\underset{\sim}{n}_u \\ \mathrm{M}_{us} + \mathrm{EM}_{ss}\begin{pmatrix} 1-e^{\lambda_{s1}T} & 0 \\ 0 & 1-e^{\lambda_{s2}T} \end{pmatrix}\underset{\sim}{n}_s \end{pmatrix}.$$

Given this we can calculate $\underset{\sim}{y}$ from (A2) and thus finally obtain $\underset{\sim}{x} = \mathrm{M}\underset{\sim}{y}$ for all instants of time $t \in [0,T]$. The resulting solution trajectories satisfy the necessary initial and terminal conditions, (A3) and (A4). To obtain the instant of time at which the economy switches from fossil fuel to the backstop, we use the condition (12″) and thus solve for time T from $b = G(x_{s1}(T) + S^*) + \dfrac{D'(E_0 + S_0 - x_{s1}(T) - S^*)}{\rho U'(x_{u1}(T) + C^*)}$. This procedure implies that $S(T)$ depends on $C(T)$. Given $K(T)$ obtained from the fossil-fuel economy, the carbon-free economy can be found from multiple shooting or directly from linearization:

$$\begin{aligned} &\dot{K}(t) = K^* + (\rho - \theta)\left[K(t) - K^*\right] \quad \Rightarrow \quad K(t) = K^* + e^{(\rho-\theta)(t-T)}\left[K(T) - K^*\right] \quad \text{and} \\ &C(t) = C^* + \theta e^{(\rho-\theta)(t-T)}\left[K(T) - K^*\right], \quad \forall t \geq T, \text{ where } \theta > \rho. \end{aligned}$$

We have also tried a fourth-order Runge-Kutta algorithm to solve (1), (2′), (6′) and (7′) given $K(0) = K_0$, $S(0) = S_0$ and guesses for $C(0)$ and $p(0)$; and Gauss-Newton iterations to adjust $C(0)$, $p(0)$ and T to satisfy $p(T) = b$, $b - G(S_T) = \dfrac{D'(E_0 + S_0 - S_T)}{\rho U'\;\; C(N)}$, and $C(T) = \Theta\;\; K(T)$. This was numerically sensitive, hence we report the results from our linearized model.