DISCUSSION PAPER SERIES

No. 8183

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Ella Segev and Aner Sela

*INDUSTRIAL ORGANIZATION*

Centre for Economic Policy Research

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Available online at: www.cepr.org/pubs/dps/DP8183.asp

ISSN 0265-8003

# SEQUENTIAL ALL-PAY AUCTIONS WITH HEAD STARTS

**Ella Segev, Ben Gurion University of the Negev**
**Aner Sela, Ben Gurion University of the Negev and CEPR**

Discussion Paper No. 8183
January 2011

Centre for Economic Policy Research
77 Bastwick Street, London EC1V 3PZ, UK
Tel: (44 20) 7183 8801, Fax: (44 20) 7183 8820
Email: cepr@cepr.org, Website: www.cepr.org

CEPR Discussion Paper No. 8183

January 2011

## ABSTRACT

# Sequential All-Pay Auctions with Head Starts

We study a sequential all-pay auction where heterogeneous contestants are privately informed about a parameter (ability) that affects their cost of effort. In the case of two contestants, contestant 1 (the first mover) makes an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then makes an effort in the second period. Contestant 2 wins the contest if his effort is larger than or equal to the effort of contestant 1; otherwise, contestant 1 wins. This model is then generalized to any number of contestants where in each period of the contest, $1 \leq j \leq n$, a new contestant joins and chooses an effort. Contestant $j$ observes the efforts of all contestants in the previous periods and then makes an effort in period $j$. He wins if his effort is larger than or equal to the efforts of all the contestants in the previous periods and strictly larger than the efforts of all the contestants in the following periods. This generalized model is studied also with a "stopping rule" according to which the contest ends as soon as a contestant exerts an effort strictly smaller than the effort of the previous contestant. We characterize the unique sub-game perfect equilibrium of these sequential all-pay auctions and analyze the use of head starts to improve the contestants' performances.

JEL Classification: D44, O31 and O32
Keywords: all-pay auctions, head starts and sequential contests

Ella Segev
Department of Industrial Engineering and Management
Ben-Gurion University of the Negev
Beer-Sheva 84105
ISRAEL

Email: ellasgv@bgu.ac.il

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Aner Sela
Department of Economics
Ben-Gurion University of the Negev
Beer-Sheva 84105
ISRAEL

Email: anersela@bgu.ac.il

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=156699

Submitted 21 December 2010

# 1 Introduction

Most of the literature in contest theory has focused on contests where players simultaneously submit their efforts, although in many contest settings, effort choices are made sequentially rather than simultaneously. These settings include, for example, sports in which contestants participate one after the other and observe the achievements of the contestants before them, court trials where evidence is presented sequentially, employee searches where applicants arrive sequentially, and many aspects of the political arena - such as the parties presidential nominating conventions. The differences between simultaneous and sequential contests have been addressed in the literature by several researchers.[1] Leininger (1993), Morgan (2003) and Baik and Shogren (1992) investigated the question of which form of contest, sequential or simultaneous, naturally arises most often in competitive situations. They studied two-player models where contestants compete in the (generalized) Tullock contest and each contestant is able to choose between two dates to make their efforts. If the contestants choose different dates, a sequential contest occurs, but if they choose the same date the contest will be a simultaneous one. They all showed that sequential contests may arise endogenously in equilibrium.[2] Despite these interesting findings, while numerous studies have dealt with simultaneous all-pay auctions (all-pay contests) only a few have studied sequential all-pay auctions. In this type of contest, both simultaneous and sequential, each player submits a bid (effort) and the player who submits the highest bid wins the contest, but, independently of success, all players bear the cost of their bids. Various applications of all-pay auctions have been made to rent-seeking and lobbying in organizations, R&D races, political contests, promotions in labor markets, sports competitions, trade wars, and military and biological wars of attrition. All-pay auctions have been studied either under complete information where each player's type (valuation for winning the contest or ability) is common knowledge[3] or under incomplete information

[1] Dixit (1987) studied a sequential Tullock contest and examined whether the ability to commit to an effort choice before other contestants choose their effort while assuming that they can then observe this choice is advantageous or not. Linster (1993) analyzed two-player sequential Tullock contests and showed that if the stronger player is the first (second) mover in the sequential contest the players' total effort is larger (smaller) than in the simultaneous contest.

[2] Hamilton and Slutsky (1990) Deneckere and Kovenock (1992) and Mailath (1993) studied sequential oligopoly games and showed that sequential choices of quantities in a Cournot competition can be the equilibrium outcome of non-cooperative play.

[3] All-pay auctions under complete information have been studied, among others, by Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993, 1996), Che and Gale (1998) and Siegel (2009)).

where each player's type is private information and only the distribution from which the players' types is drawn is common knowledge.[4] Most studies dealing with sequential all-pay auctions assume a two-stage contest under complete information. Leininger (1991) modeled a patent race between an incumbent and an entrant as a sequential asymmetric all-pay auction under complete information, and Konrad and Leininger (2007) characterized the equilibrium of the all-pay auction under complete information in which a group of players choose their effort 'early' and the other group of players choose their effort 'late'. The assumption of incomplete information complicates the analysis of the sequential all-pay auction but also makes it more relevant and interesting. In this work, we study a sequential all-pay auction under incomplete information where the ability of each contestant is his private information. We consider first a sequential all-pay auction with two contestants where contestant 1 (the first mover) makes an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then makes an effort in the second period. Contestant 2 wins the contest if his effort is larger than or equal to the effort of contestant 1; otherwise, contestant 1 wins. This particular type of sequential all-pay auction has various applications. For example, in some industries (e.g. the audit industry) it is customary for clients to give the incumbent firm the right to make a final offer for an engagement after learning about the offers of its rival.[5]

In our model, contestant 2 has an obvious advantage over contestant 1, for which reason contestant 1 exerts a relatively low effort and sometimes, depending on the distribution of his opponent's abilities, he might even prefer not to participate in the contest at all (it is worth noting that this feature of our model can explain why players sometimes choose to stay out of a contest). Given the low effort of contestant 1 in the first period as well as the rules of the contest according to which contestant 2 needs only to equalize the effort of contestant 1 in order to win, we have a relatively low expected total effort as well as a low expected highest effort. However, a designer who wishes to maximize the expected total effort or the expected highest effort

[4] All-pay auctions under incomplete information have been studied, among others, by Hillman and Riley (1989), Amman and Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006) and Moldovanu et al. (2010)).

[5] The concept of Stackelberg games in which players choose their strategies sequentially was introduced and analyzed also by computer scientists such as Garg and Narahari (2008), Luh et al. (1984) and others. All these authors impose a hierarchical decision making structure on a simultaneous game to describe sequential choices of strategies. The solution concept they use is a Stackelberg equilibrium where the leaders use "secure strategies" which secure them a minimal payoff while the followers use an optimal response strategy.

can change the rules of the sequential all-pay auction to make it more profitable by explicitly or implicitly favoring contestant 1 over contestant 2. In other words, he can give contestant 1 a head start.

There are numerous examples of real-life contests in which players are given head starts. A common situation occurs in the labor market when a favored applicant is given a head start and then other applicants are required to do much better than the favored one in order to win the job. Therefore contests with head starts may raise the contestants' expected total effort or alternatively their expected highest effort. Kirkegaard (2009) studied asymmetric all-pay auctions with head starts under incomplete information where players simultaneously choose their efforts. He showed that the total effort increases if the weak contestant is favored with a head start, but if the contestants are sufficiently heterogenous, then in some cases the weak contestant should be given both a head start and a handicap.[6] In our sequential all-pay auction, contestant 2 has an advantage over contestant 1 because of the timing of their play. In order that the first mover will exert a higher effort we therefore assume that contestant 1 is given a multiplicative head start. That is, contestant 2 will win the contest if his effort $x_2$ is larger or equal to $tx_1$, where $x_1$ is the effort of contestant 1 and $t$ is a constant larger than 1. We provide sufficient conditions under which by imposing a head start for contestant 1 the designer of the contest can significantly increase the expected efforts of both contestants and particularly the expected total effort as well as the expected highest effort. The optimal head start can be high enough such that several types of contestant 1 will win for sure, since no type of contestant 2 will have a chance to win against them. As such, head starts may also play the role of a winning bid in a sequential all-pay auction when contestant 1 has an incentive to participate independently of the distribution of his opponent's type.

We then turn to study a sequential all-pay auction with $n > 2$ players. In this model in each period of the contest, $1 \leq j \leq n$, a new contestant joins and chooses an effort. Contestant $j, j = 1, ..., n$ observes the efforts of all contestants in the previous $j - 1$ periods and then makes an effort in period $j$. Contestant $j$ wins if his effort is larger than or equal to the efforts of all the contestants in the $j - 1$ previous periods and strictly larger than the efforts of all the contestants in the following $n - j$ periods. In real-life contests with more than two contestants, however, the contest designer may not want to wait until all the contestants join.

[6] Siegel (2010) provided an algorithm that constructs the unique equilibrium in simultaneous all-pay auctions with head starts in which players do not choose weakly-dominated strategies.

He might decide that as soon as he observes an effort that is lower than the previous contestant's effort the contest ends and the player before the last one wins the contest. For example, in an employee search the employer may decide to interview applicants as long as their performances (weakly) increase but as soon as an applicant's performance is lower than than the previous applicant's performance the search ends and the highest contestant (the one before the last) gets the job. This might be optimal especially in cases where the contest designer bears a cost for waiting (such as the cost of an unmanned job) or a cost for adding contestants (such as a search cost). Such circumstances are our motivation for studying another version of the sequential all-pay auction with $n > 2$ players that includes a "stopping rule." In this contest a contestant wins if his effort is larger or equal to the effort of the contestant in the previous period and, moreover, strictly larger than the effort of the contestant in the following period. Therefore the contest ends as soon as a contestant makes an effort that is strictly smaller than the effort of the contestant in the previous period. The winner is the contestant who participated one period before the last one and his effort is necessarily higher than or equal to all previous contestants' efforts.

We also study this $n$-player model (with and without a "stopping rule" ) with head starts.[7] The analysis of the sequential all-pay auction with $n$ players and head starts is quite complicated since a head start which is given to the contestant in period $k$ affects the equilibrium strategies of all the contestants in the following periods $j \geq k$. Furthermore, in contrast to the model with two players, the use of head starts in the sequential all-pay auction with $n > 2$ players may decrease the number of active periods since players may choose to withdraw, and therefore may lower the contestants' expected highest and total effort. However, we provide sufficient conditions under which there always exists some non-trivial head starts that increase the expected total effort. Furthermore, we show that using head starts for any subset of contestants who play in the first $n-1$ periods increases the expected highest effort independently of whether the contest ends after $n$ periods or even earlier. Hence, our analysis establishes a key role for head starts in sequential all-pay auctions and

[7] When the head starts are relatively large so that they play the role of a winning bid, our sequential all-pay auctions are related to sequential second price auctions with a buy price (see, e.g., Milgrom 2003) in which buyers arrive one after the other without knowing their place in the queue. When a bidder arrives, he can either buy the object at the publicly announced "buy price" and end the auction, or place a bid lower than the buy price. If no bidder takes the buy price a second price auction determines the outcome.

particularly in sequential contests under incomplete information.

The rest of the paper is organized as follows: Section 2 presents the two-player sequential all-pay auction. Section 3 presents the $n$-player sequential all-pay auction with a "stopping rule" and Section 4 presents the general form of the $n$-player sequential all-pay auction. Furthermore, Sections 2, 3 and 4 all characterize the unique sub-game perfect equilibrium with and without head starts and provide conditions under which the use of head starts improves the contestants' performance. Section 5 concludes. All proofs are in the Appendix.

## 2 The two-player model

We consider first a sequential all-pay auction with two contestants where contestant 1 (the first mover) makes an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then makes an effort in the second period. Contestant 2 wins the contest if his effort ($x_2$) is larger than or equal to the effort of contestant 1 ($x_1$); otherwise, contestant 1 wins. Both contestants' valuation for the prize is 1. An effort $x_i$ causes a cost $\frac{x_i}{a_i}$ where $a_i \geq 0$ is the ability (or type) of contestant $i$ which is private information to $i$. Contestants' abilities are drawn independently. Contestant $i$'s ability is drawn from the interval $[0, 1]$ according to a distribution function $F_i$ which is common knowledge. We assume that $F_i, i = 1, 2$ has a positive and continuous density function $F_i' > 0$.

We begin the analysis by considering the equilibrium effort function of contestant 2 in the second period. We assume that if both contestants make the same effort then contestant 2 is the winner. Therefore contestant 2 makes the same effort as contestant 1 as long as his type $a_2$ is larger than or equal to the effort of contestant 1; otherwise he stays out of the contest. Formally, the equilibrium effort of contestant 2 is given by:

$$b_2(a_2) = \begin{cases} 0 & \text{if} \quad 0 \leq a_2 < b_1(a_1) \\ b_1(a_1) & \text{if} \quad b_1(a_1) \leq a_2 \leq 1 \end{cases}$$

where we assume that contestant 1 uses a strictly monotonic equilibrium effort function $b_1(a_1)$. Applying the revelation principle, contestant 1 with ability $a_1$ chooses to behave as an agent with ability $s$ that solves the following optimization problem:

$$\max_s \left\{ F_2(b_1(s)) - \frac{b_1(s)}{a_1} \right\} \tag{1}$$

The F.O.C. is then

$$a_1 F_2'(b_1(s))b_1'(s) - b_1'(s) = 0 \tag{2}$$

and the S.O.C. is

$$a_1 F_2''(b_1(s))(b_1'(s))^2 + a_1 F_2'(b_1(s))b_1''(s) - b_1''(s) = a_1 F_2''(b_1(s))(b_1'(s))^2 < 0$$

Note that if $F_2$ is convex, the S.O.C does not hold and then $b_1(a_1) = 0$ for all $a_1$ is the solution of the maximization problem (1). Thus, in the following we assume that $F_2$ is concave ($F_1$ is not necessarily concave). Then the S.O.C. holds and in equilibrium, the maximization problem (1) must be solved by $s = a_1$. Thus we obtain that the equilibrium effort of contestant 1 with type $a_1$ is

$$b_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 \leq \tilde{a} \\ (F_2')^{-1}\left(\frac{1}{a_1}\right) & \text{if} \quad \tilde{a} \leq a_1 \leq 1 \end{cases} \tag{3}$$

where the cutoff $\tilde{a}$ is defined by $\max\left\{\frac{1}{F_2'(0)}, 0\right\}$. This cutoff depends on the distribution of the second player's ability. If $F_2'(0)$ is a finite number then types $0 \leq a_1 \leq \tilde{a}$ do not find it optimal to exert a positive effort. As was mentioned above, for the class of convex distribution functions we have $\tilde{a} = 1$ such that contestant 1 chooses to stay out of the contest (in the following we will solve this problem by providing an incentive (a head start) for contestant 1 to participate in the contest). However, if contestant 2's distribution function $F_2$ is concave, we have a real competition in the sequential all-pay auction even without head starts.

The expected efforts of contestants 1 and 2 are

$$\begin{aligned} TE_1 &= \int_{\tilde{a}}^{1} (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \\ TE_2 &= \int_{\tilde{a}}^{1} \left[1 - F_2((F_2')^{-1}(\frac{1}{a_1}))\right](F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \end{aligned}$$

Note that contestant 2 makes the same effort as contestant 1 or else makes an effort of zero. Therefore the expected highest effort is equal to the expected effort of contestant 1 and is given by

$$HE = \int_{\tilde{a}}^{1} (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \tag{4}$$

The expected total effort is given by[8]

$$TE = TE_1 + TE_2 = \int_{\tilde{a}}^{1} \left[2 - F_2((F_2')^{-1}(\frac{1}{a_1}))\right](F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \tag{5}$$

[8] We assume that the contest designer does not discount the effort in the second period. We discuss this generalization and its implication in the results in Section 5.

**Example 1** *Consider a sequential all-pay auction with two contestants whose abilities are distributed according to the distribution functions $F_1(x) = F_2(x) = x^{0.5}$. By (3), the equilibrium effort function of contestant* 1 *in the sequential all-pay auction is*

$$b_1(a_1) = \frac{a_1^2}{4} \quad \textit{for all } a_1 \geq 0$$

*Therefore by (4) the expected highest effort is given by*

$$HE = \int_0^1 \frac{a_1^2}{4} \frac{1}{2\sqrt{a_1}} da_1 = 0.05$$

*and by (5) the expected total effort is*

$$TE = \int_0^1 \left(2 - \sqrt{\frac{a_1^2}{4}}\right) \frac{a_1^2}{4} \frac{1}{2\sqrt{a_1}} da_1 = \frac{23}{280} \approx 0.0821$$

In Example 1, the contestants' expected highest effort as well as their expected total effort are significantly lower than in the standard all-pay auction where both contestants simultaneously choose their efforts. In the next subsection we change the rules of the sequential all-pay auction by adding head starts to improve the contestants' performance in the contest.

## 2.1 Head starts

In our sequential all-pay auction, contestant 2 has an advantage over contestant 1 because of the timing of their play. Thus, contestant 1's effort is relatively low and sometimes, depending on the distribution of contestant 2's abilities, will choose to stay out of the contest. In that case there is no real competition. Thus we examine whether the players' performance can be enhanced by using a head start for contestant 1. We assume that contestant 2 will win the contest if his effort $x_2$ is larger than or equal to $tx_1$ where $x_1$ is the effort of contestant 1 and $t$ is a constant larger than 1. The equilibrium effort of contestant 2 is then given by

$$\beta_2(a_2) = \begin{cases} 0 & \text{if} \quad 0 \leq a_2 < t\beta_1(a_1) \\ t\beta_1(a_1) & \text{if} \quad t\beta_1(a_1) \leq a_2 \leq 1 \end{cases}$$

where we assume that contestant 1 uses a strictly monotonic equilibrium effort function $\beta_1(a_1)$. Applying the revelation principle, contestant 1 with ability $a_1$ chooses to behave as an agent with ability $s$ that solves the following optimization problem:

$$\max_s \left\{ F_2(t\beta_1(s)) - \frac{\beta_1(s)}{a_1} \right\} \tag{6}$$

The F.O.C. is

$$a_1 F_2'(t\beta_1(s))t\beta_1'(s) - \beta_1'(s) = 0$$

and the S.O.C. is

$$aF_2''(t\beta_1(s))(t\beta_1'(s))^2 + a_1 F_2'(t\beta_1(s))t\beta_1''(s) - \beta_1''(s) = a_1 F_2''(t\beta_1(s))(t\beta_1'(s))^2 < 0$$

Thus, if $F_2$ is concave, in equilibrium, the above maximization problem must be solved by $s = a_1$. Then we obtain the following condition

$$a_1 F_2'(t\beta_1(a_1))t - 1 = 0$$

and the equilibrium effort of contestant 1 with type $a_1$ is

$$\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 \le \widehat{a} \\ \frac{1}{t}\left(F_2'\right)^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \widehat{a} \le a_1 \le a^* \\ \frac{1}{t} & \text{if} \quad a^* \le a_1 \le 1 \end{cases} \tag{7}$$

where $\widehat{a}$ is defined as $\max\left\{\frac{1}{tF_2'(0)}, 0\right\}$ and $a^*$ is either equal to 1 or determined by the solution to the following equation

$$\begin{aligned} t\beta_1(a^*) &= (F_2')^{-1}(\frac{1}{a^* t}) = 1 \\ &\Rightarrow a^* = \min\left\{1, \frac{1}{tF_2'(1)}\right\} \end{aligned}$$

Note that $a^* \ge \hat{a}$ since $a^* = 1$ or $a^* = \frac{1}{tF_2'(1)}$, while $\hat{a}$ is either zero or $\hat{a} = \frac{1}{tF_2'(0)}$ and $F_2'$ is a decreasing function. Furthermore, if $1 \le t \le \frac{1}{F_2'(1)}$, then $a^* = 1$ and only if $t > \frac{1}{F_2'(1)}$ does there exist a cutoff type $0 < a^* < 1$ and an interval of types $a^* \le a_1 \le 1$ who exert the effort $b_1(a^*) = \frac{1}{t}$ and win for sure (this serves as a winning bid).

The expected efforts of contestants 1 and 2 are given by

$$\begin{aligned} TE_1(t) &= \int_{\hat{a}}^{a^*} \frac{1}{t}(F_2')^{-1}(\frac{1}{a_1 t})F_1'(a_1)da_1 + \int_{a^*}^{1} \frac{1}{t}F_1'(a_1)da_1 \\ TE_2(t) &= \int_{\hat{a}}^{a^*} \left[1 - F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right](F_2')^{-1}(\frac{1}{a_1 t})F_1'(a_1)da_1 \end{aligned}$$

The expected total effort is therefore

$$
\begin{aligned}
TE(t) &= TE_1(t) + TE_2(t) && (8)\\
&= \int_{\hat{a}}^{a^*} \left[\frac{1}{t} + 1 - F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right] (F_2')^{-1}(\frac{1}{a_1 t}) F_1'(a_1) da_1 + \int_{a^*}^{1} \frac{1}{t} F_1'(a_1) da_1
\end{aligned}
$$

Note that the expected effort of contestant 1 is not always higher than the expected effort of contestant 2 as was the case without a head start and therefore the expected highest effort is not equal to the expected effort of contestant 1. The expected highest effort is given by

$$
\begin{aligned}
HE(t) &= \int_0^1 \int_0^1 \max\{\beta_1(a_1), \beta_2(a_2)\} F_2'(a_2)\, da_2 F_1'(a_1)\, da_1 && (9)\\
&= \int_{\hat{a}}^{a^*} \left[F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right] \frac{1}{t} (F_2')^{-1}(\frac{1}{a_1 t}) F_1'(a_1) da_1 && (10)\\
&\quad + \int_{\hat{a}}^{a^*} \left[1 - F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right] (F_2')^{-1}(\frac{1}{a_1 t}) F_1'(a_1) da_1 + \int_{a^*}^{1} \frac{1}{t} F_1'(a_1) da_1
\end{aligned}
$$

The first term describes those types of contestant 2 who choose to stay out of the contest ($0 \le a_2 < t\beta_1(a_1)$) in which case the highest effort is equal to that of contestant 1, $\beta_1(a_1) = \frac{1}{t}(F_2')^{-1}(\frac{1}{a_1 t})$. The second term describes those types of contestant 2 who equalize the effort of contestant 1 multiplied by $t$ in which case the highest effort is equal to $t\beta_1(a_1) = (F_2')^{-1}(\frac{1}{a_1 t})$. The last term describes those types of contestant 1 who win for sure by choosing the winning bid.

Below we discuss the equilibrium behavior of the contestants when the distribution function of contestant 2's types is convex rather than concave (again, there is no restriction on the distribution of contestant 1's types). When $F_2$ is convex and a head start $t > 1$ is given to contestant 1 then the equilibrium effort of contestant 2 is once again

$$
\beta_2(a_2) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < t\beta_1(a_1) \\ t\beta_1(a_1) & \text{if} \quad t\beta_1(a_1) \le a_2 \le 1 \end{cases}
$$

while the equilibrium effort of contestant 1 is given by

$$
\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 < \frac{1}{t} \\ \frac{1}{t} & \text{if} \quad \frac{1}{t} \le a_1 \le 1 \end{cases}
$$

Note that when $F_2$ is convex and a head start is given to contestant 1 some of contestant 1's types participate in the contest. In this case the expected total effort and the expected highest effort are the same and are both equal to contestant 1's expected effort.

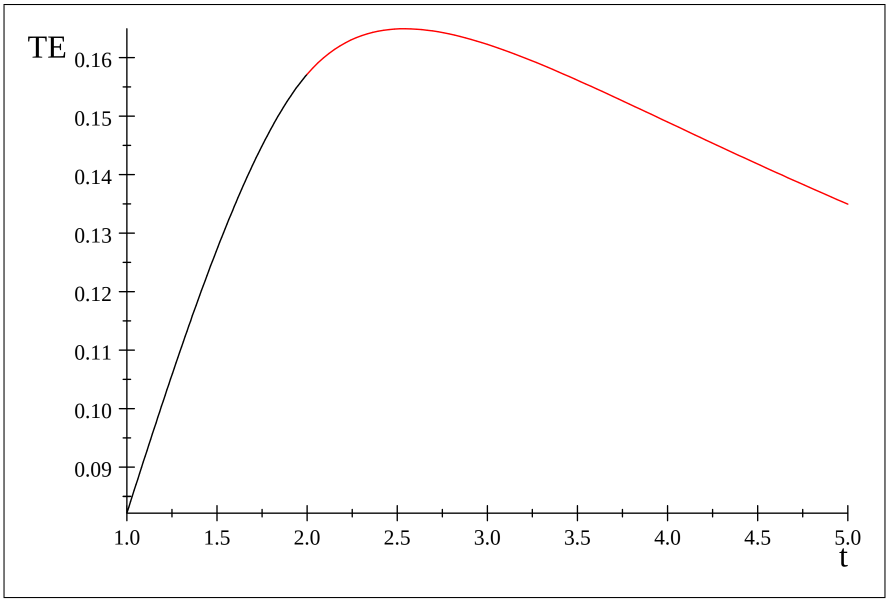
**Example 2** *Consider a sequential all pay auction with two contestants where $F_1(x) = F_2(x) = x^{0.5}$. By (7), the equilibrium effort function of contestant 1 is given by*

$$\beta_1(a_1) = \begin{cases} \frac{1}{t}(F_2^{'})^{-1}(\frac{1}{a_1 t}) = \frac{t a_1^2}{4} & \text{if} \quad 0 \le a_1 \le \min\left\{\frac{2}{t}, 1\right\} \\ \frac{1}{t} & \text{if} \quad \min\left\{\frac{2}{t}, 1\right\} < a_1 \le 1 \end{cases}$$

*The expected total effort is given by*

$$\begin{aligned} TE &= \int_0^{\min\left\{\frac{2}{t},1\right\}} \left(\frac{a_1^2 t}{4}\right) \frac{1}{2\sqrt{a_1}} da_1 + \int_{\min\left\{\frac{2}{t},1\right\}}^1 \left(\frac{1}{t}\right) \frac{1}{2\sqrt{a_1}} da_1 \\ &+ \int_0^{\min\left\{\frac{2}{t},1\right\}} \left( \int_{\frac{a_1^2 t^2}{4}}^1 \left(\frac{a_1^2 t^2}{4}\right) \frac{1}{2\sqrt{a_2}} da_2 \right) \frac{1}{2\sqrt{a_1}} da_1 \end{aligned}$$

*The following figure presents the expected total effort as a function of $t$.*




*The optimal head start that yields the highest expected total effort in the sequential all-pay auction is therefore*

$$t_{total} = \frac{7}{4}\left(199 - 5\sqrt{1561}\right) = 2.541\,9$$

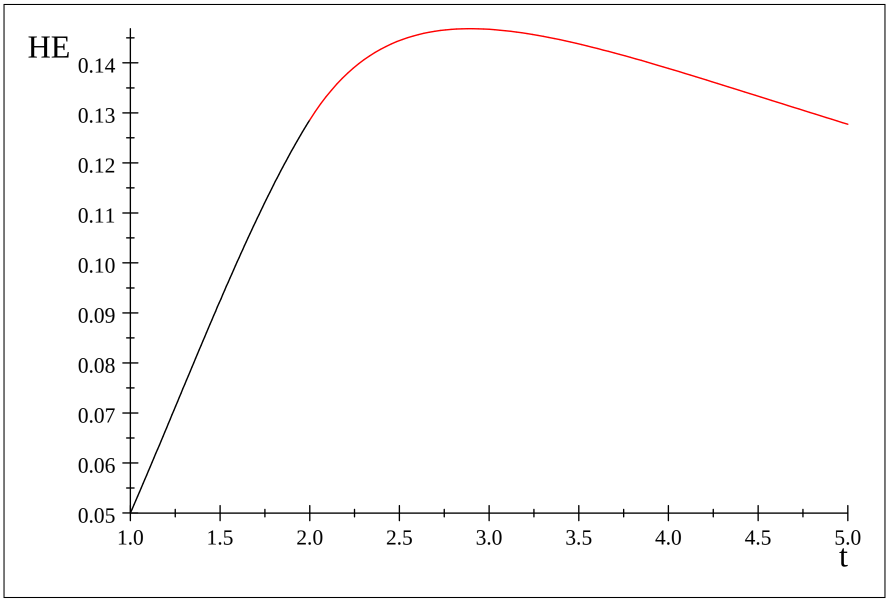
*and the expected total effort is then*

$$TE(t_{total}) = 0.16492$$

*The expected highest effort is*

$$\begin{aligned} HE &= \int_0^{\min\left\{\frac{2}{t},1\right\}} \left( \int_0^{\frac{a_1^2 t^2}{4}} \left(\frac{a_1^2 t}{4}\right) \frac{1}{2\sqrt{a_2}} da_2 + \int_{\frac{a_1^2 t^2}{4}}^1 \left(\frac{a_1^2 t^2}{4}\right) \frac{1}{2\sqrt{a_2}} da_2 \right) \frac{1}{2\sqrt{a_1}} da_1 \\ &+ \int_{\min\left\{\frac{2}{t},1\right\}}^1 \frac{1}{t} \frac{1}{2\sqrt{a_1}} da_1 \end{aligned}$$

*The following figure presents the expected highest effort as a function of t.*




*The optimal head start that yields the highest expected highest effort in the sequential all-pay auction is therefore*

$$t_{high} = \frac{1}{\left(\frac{1}{180}\sqrt{10}\sqrt{317} + \frac{7}{36}\sqrt{2}\right)^2} = 2.894\,5$$

*and the expected highest effort is then*

$$HE(t_{high}) = 0.1468$$

From Examples 1 and 2 we can see that the optimal head start significantly increases the contestants' expected highest effort as well as their expected total effort.

We now turn to examine the conditions under which a head start is efficient in the sequential all-pay auction, namely, those conditions on the distribution of the contestants' abilities that ensure that a head start increases the expected highest effort or the expected total effort. The following condition is required for establishing the effects of a head start on contestant 1's equilibrium effort.

**Condition 1** *The equilibrium effort function of contestant 1 in the sequential all-pay auction without a head start given by (3) is strictly convex for all $\tilde{a} \leq a_1 \leq 1$*

If Condition 1 is satisfied, any head start $t$ close to 1 increases the expected effort of contestant 1 since then, for $t > 1$ and $\tilde{a} \leq a_1 \leq 1$ we have $b_1(a_1) < \frac{1}{t}b_1(ta_1) = \beta_1(a_1)$. Given that without any head start, the expected highest effort is equal to the expected effort of contestant 1, we obtain the following result about the positive effect of head starts on the expected highest effort in the contest.

**Proposition 1** *If Condition 1 holds, then the expected highest effort in the two-player sequential all-pay auction with a head start* $1 < t \leq \frac{1}{F_2'(1)}$ *is higher than the expected highest effort in the sequential all-pay auction without any head start.*

**Proof.** See Appendix. ■

Now we examine the effect of head starts on the expected effort of contestant 2. On the one hand, the effort of every type of contestant 1 increases when a head start is given and therefore contestant 2 should also increase his effort if he wants to win the contest. But, on the other hand, by giving a head start to contestant 1, low types of contestant 2 will prefer to stay out of the contest since the minimal effort which is required from them in order to win is relatively high.

The following conditions are required for establishing the effect of a head start on the effort of contestant 2.

**Condition 2** *The function* $G(x) = (1 - F_2(x))\, x$ *is convex.*

**Condition 3** *The highest equilibrium effort of contestant 1 (the effort of type* $a_1 = 1$*) in the contest without a head start is lower than* $x^* = \arg\max_{x \in [0,1]} G(x)$*. Formally,*

$$b_1(1) = (F_2')^{-1}(1) < x^*$$

Using conditions 1, 2 and 3 we obtain a positive effect of relatively small head starts on the expected effort of contestant 2 as well.

**Proposition 2** *If Conditions 1,2 and 3 hold, then for* $t > 1$ *sufficiently close to* $1$*, the expected effort of contestant 2 increases in* $t$*.*

**Proof.** See Appendix. ■

Note that all the three conditions 1,2 and 3 hold for a large class of distribution functions including, for example, every concave distribution function of the form $F(x) = x^{\gamma}, 0 < \gamma < 1$. The combination of

Proposition 1 and Proposition 2 yields the result that the use of a head start in the sequential all pay auction is efficient for a designer who wishes to maximize the expected total effort.

**Proposition 3** *If Conditions 1,2 and 3 hold, then the expected total effort in the two-player sequential all-pay auction with a head start $t > 1$ which is sufficiently close to 1 is higher than the expected total effort in the two-player sequential all-pay auction without any head start.*

By Theorem 3, a head start $t > 1$ that is sufficiently close to 1 increases the expected highest effort as well as the expected total effort. However, we cannot conclude that the optimal head start for a designer who wishes to maximize the expected highest or total effort is close to 1. Note that for $1 < t \leq \frac{1}{F_2'(1)}$ the effort of every type of contestant 1 is higher than in the contest without a head start. However, for $t > \frac{1}{F_2'(1)}$ the effort of low types of contestant 1 is higher than in the contest without a head start but the effort of the high types in the contest with a head start is not necessarily higher than their efforts in the contest without a head start. In this case, the head start serves as a winning bid and therefore some high types will choose the winning bid but not any bid above it as they might have done without the head start. Nevertheless, as we can see from Example 2, the optimal head starts (those that imply the highest expected total effort and the highest expected highest effort) might be obtained for a head start satisfying $t > \frac{1}{F_2'(1)}$ although such a head start does not necessarily increase the effort of all possible contestants' types.

# 3 The $n$-player model with a "stopping rule"

We consider now a sequential all-pay auction with $n > 2$ contestants and a head start $t \geq 1$ (the case of $t = 1$ will be referred to as a contest without a head start). Contestant $j, 1 \leq j \leq n-1$ makes an effort $x_j$ in period $j$ and contestant $j+1$ observes this effort and then makes an effort $x_{j+1}$ in period $j+1$. Contestant $j$ wins a prize equal to 1 iff $x_j \geq tx_{j-1}$ and $tx_j > x_{j+1}$, $j = 2, ..., n-1$ (contestant 1 wins iff $tx_1 > x_2$, and contestant $n$ wins iff $x_n \geq tx_{n-1}$). Therefore the contest ends in period $k < n$ if the contestant in that period exerts an effort strictly lower than $tx_{k-1}$. This serves as a "stopping rule" for the contest. The use of a stopping rule makes sense in contests in which adding a new contestant is costly for the contest designer (either because time is costly or because it involves a cost to bring in a new contestant).

An effort $x_i$ causes a cost $\frac{x_i}{a_i}$ for contestant $i$, where $a_i \geq 0$ is the ability (or type) of contestant $i$ which is private information to $i$. As previously, contestant $i$'s ability is drawn (independently of the other contestants' abilities) from the interval $[0,1]$ according to a distribution function $F_i$ which is common knowledge. We assume that $F_i$ has a positive and continuous density $F_i' > 0, i = 1, 2, ..., n$.

Note that contestant $n$ faces the same problem as that of contestant 2 in the two-player model. Thus, the equilibrium effort of contestant $n$, if called to play, is given by

$$\beta_n(a_n) = \begin{cases} 0 & \text{if} \quad 0 \leq a_n < t\beta_{n-1}(a_{n-1}) \\ t\beta_{n-1}(a_{n-1}) & \text{if} \quad t\beta_{n-1}(a_{n-1}) \leq a_n \leq 1 \end{cases}$$

We assume that contestant $i$, $i = 2, ..., n-1$ uses a strictly monotonic equilibrium effort function $\beta_i(a_i)$. If contestant $i$ observes an effort $\beta_{i-1}(a_{i-1})$ of the previous contestant, and $t\beta_{i-1}(a_{i-1})$ is higher than his type, he will stay out of the contest. Otherwise, applying the revelation principle, player $i$, $i = 2, ..., n-1$ with ability $a_i$ chooses to behave as an agent with ability $s$ to solve the following optimization problem:

$$\max_s \left\{ F_{i+1}(t\beta_i(s)) - \frac{\beta_i(s)}{a_i} \right\} \tag{11}$$

$$s.t \quad \beta_i(s) \geq t\beta_{i-1}(a_{i-1})$$

Then, all types that find it optimal to participate (namely, $a_i \geq t\beta_{i-1}(a_{i-1})$), but for whom the constraint in the above maximization problem is binding, will exert the effort of $t\beta_{i-1}(a_{i-1})$ such that contestant $i'$s equilibrium effort, $i = 2, ..., n-1$ is given by:

$$\beta_i(a_i) = \begin{cases} 0 & \text{if} \quad 0 \leq a_i < t\beta_{i-1}(a_{i-1}) \\ t\beta_{i-1}(a_{i-1}) & \text{if} \quad t\beta_{i-1}(a_{i-1}) \leq a_i < \overleftarrow{a}_i \\ \frac{1}{t}\left(F_{i+1}'\right)^{-1}\left(\frac{1}{ta_i}\right) & \text{if} \quad \overleftarrow{a}_i \leq a_i < \overrightarrow{a}_i \\ \frac{1}{t} & \text{if} \quad \overrightarrow{a}_i \leq a_i \leq 1 \end{cases} \tag{12}$$

where $\overleftarrow{a}_i$ and $\overrightarrow{a}_i$ are defined by

$$\frac{1}{t}\left(F_{i+1}'\right)^{-1}\left(\frac{1}{t\overleftarrow{a}_i}\right) = t\beta_{i-1}(a_{i-1})$$

and

$$(F_{i+1}')^{-1}\left(\frac{1}{t\overrightarrow{a}_i}\right) = 1$$

respectively. Note that $t\beta_{i-1}(a_{i-1}) < \overleftarrow{a}_i$ since $t\beta_{i-1}(a_{i-1}) = \frac{1}{t}\left(F'_{i+1}\right)^{-1}\left(\frac{1}{t\overleftarrow{a}_i}\right) < \overleftarrow{a}_i$ (the effort function is always smaller than the type). However it is not necessarily true here that $\overleftarrow{a}_i < \overrightarrow{a}_i$, and then the third range of (12) does not exist. Moreover it is also not necessarily true that $\overrightarrow{a}_i \leq 1$ and then the fourth range of (12) does not exist.

Contestant 1 solves the same maximization problem as in the two-player model and therefore

$$\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 \leq \overline{a}_{1,t} \\ \frac{1}{t}\left(F'_2\right)^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \overline{a}_{1,t} \leq a_1 \leq \overrightarrow{a}_1 \\ \frac{1}{t} & \text{if} \quad \overrightarrow{a}_1 \leq a_1 \leq 1 \end{cases} \tag{13}$$

where $\overline{a}_{1,t}$ is defined as previously as $\max\left\{0, \frac{1}{tF'_2(0)}\right\}$ and $\overrightarrow{a}_1$ is defined as $\overrightarrow{a}_1 = \min\left\{\frac{1}{tF'_2(1)}, 1\right\}$.

The expected effort of contestant 1 is then given by

$$TE_1 = \int_{\overline{a}_{1,t}}^{1} (F'_2)^{-1}(\frac{1}{a_1})F'_1(a_1)da_1$$

and the expected effort of contestant $i$, $i = 2, ..., n$ is given by

$$TE_i = \int_0^1 \int_{t\beta_1(a_1)}^1 ... \int_{t\beta_{i-2}(a_{i-2})}^1 \int_{t\beta_{i-1}(a_{i-1})}^1 \beta_i(a_i)F'_i(a_i)da_iF'_{i-1}(a_{i-1})da_{i-1}...F'_2(a_2)da_2F'_1(a_1)da_1$$

The expected total effort is therefore

$$TE = \sum_{i=1}^{n} TE_i \tag{14}$$

For a given realization of the players' abilities $a_1, .., a_n$ we define

$$HE(a_1, .., a_n) = \max_{1\leq i\leq n} \beta_i(a_i)$$

Then, the expected highest effort is given by

$$HE = \int_0^1 \int_0^1 ... \int_0^1 \int_0^1 HE(a_1, .., a_n)\, F'_n(a_n)da_nF'_{n-1}(a_{n-1})da_{n-1}...F'_2(a_2)da_2F'_1(a_1)da_1 \tag{15}$$

An immediate result is the following.

**Proposition 4** *The expected effort of contestant $i$, $i = 1, 2, ..., n$ in the sequential all-pay auction with $n+1$ contestants is always higher than or equal to his expected effort in the sequential all-pay auction with $n$*

*contestants. Thus, in particular, the expected total effort as well as the expected highest effort increase if new contestants are added to the contest.*

***Proof.*** *See Appendix.* ■

The following example illustrates the effects of head starts in a three-player sequential all-pay auction.

**Example 3** *Consider a sequential all-pay auction with three contestants and* $F_i(x) = x^{0.5}, i = 1,2,3,$ *and assume that contestants 1 and 2 are given a head start* $t \leq 4$. *By (13), the equilibrium effort of contestant 1 is given by*

$$\beta_1(a_1) = \begin{cases} \frac{ta_1^2}{4} & \text{if} \quad 0 \leq a_1 < \min\left\{\frac{2}{t}, 1\right\} \\ \frac{1}{t} & \text{if} \quad \min\left\{\frac{2}{t}, 1\right\} \leq a_1 \leq 1 \end{cases}$$

*By (12), the equilibrium effort of contestant 2 is as follows: if* $2 \leq t \leq 4$ *and* $0 \leq a_1 < \frac{2}{t\sqrt{t}}$ *then*

$$\beta_2(a_2) = \begin{cases} 0 & \text{if} \quad 0 \leq a_2 < \frac{t^2a_1^2}{4} \\ \frac{t^2a_1^2}{4} & \text{if} \quad \frac{t^2a_1^2}{4} \leq a_2 < \sqrt{t}a_1 \\ \frac{ta_2^2}{4} & \text{if} \quad \sqrt{t}a_1 \leq a_2 \leq \frac{2}{t} \\ \frac{1}{t} & \text{if} \quad \frac{2}{t} \leq a_2 \leq 1 \end{cases}$$

*if* $\frac{2}{t\sqrt{t}} \leq a_1 < \frac{1}{\sqrt{t}}$ *then*

$$\beta_2(a_2) = \begin{cases} 0 & \text{for} \quad 0 \leq a_2 < \frac{t^2a_1^2}{4} \\ \frac{t^2a_1^2}{4} & \text{for} \quad \frac{t^2a_1^2}{4} \leq a_2 < \sqrt{t}a_1 \\ \frac{ta_2^2}{4} & \text{for} \quad \sqrt{t}a_1 \leq a_2 \leq 1 \end{cases}$$

*and if* $\frac{1}{\sqrt{t}} \leq a_1 < \frac{2}{t}$ *then*

$$\beta_2(a_2) = \begin{cases} 0 & \text{for} \quad 0 \leq a_2 < \frac{t^2a_1^2}{4} \\ \frac{t^2a_1^2}{4} & \text{for} \quad \frac{t^2a_1^2}{4} \leq a_2 \leq 1 \end{cases}$$

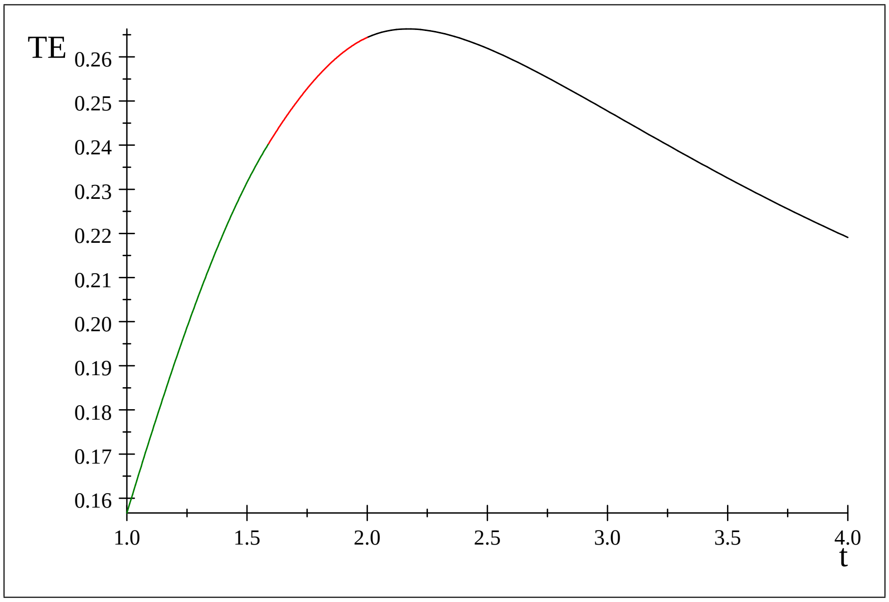
*The equilibrium effort of contestant 3 is as follows: if* $0 \leq a_1 < \frac{2}{t\sqrt{t}}$ *and* $\frac{t^2a_1^2}{4} \leq a_2 < \sqrt{t}a_1$ *then*

$$\beta_3(a_3) = \begin{cases} 0 & \text{if} \quad 0 \leq a_3 < \frac{t^3a_1^2}{4} \\ \frac{t^3a_1^2}{4} & \text{if} \quad \frac{t^3a_1^2}{4} \leq a_3 < 1 \end{cases}$$

*and if* $0 \leq a_1 < \frac{2}{t\sqrt{t}}$ *and* $\sqrt{t}a_1 \leq a_2 \leq \frac{2}{t}$ *then*

$$\beta_3(a_3) = \begin{cases} 0 & \text{if} \quad 0 \leq a_3 < \frac{t^2a_2^2}{4} \\ \frac{t^2a_2^2}{4} & \text{if} \quad \frac{t^2a_2^2}{4} \leq a_3 < 1 \end{cases}$$

*In all other cases,* $\beta_3(a_3) = 0$*. The following figure presents the total effort as a function of the head start* $t$*.*



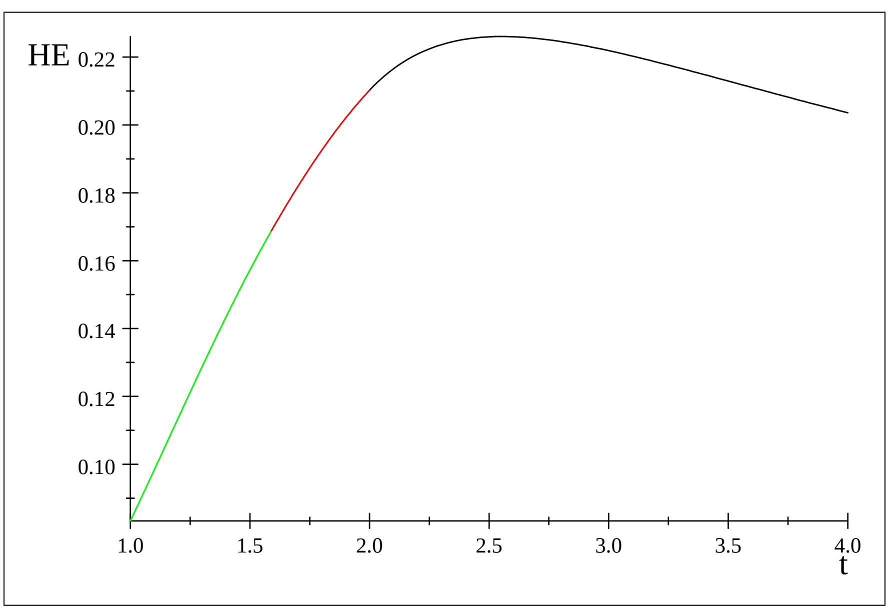

*Thus, if we give a head start to contestants* 1 *and* 2*, the optimal head start that maximizes the expected total effort is*

$$t_{total} = 2.1706$$

*and the highest expected total effort is then*

$$TE(t_{total}) = 0.2663$$

*The following figure presents the expected highest effort as a function of the head start* $t$*.*



*Thus, if we give a head start to contestants* 1 *and* 2*, the optimal head start that maximizes the expected highest effort is*

$$t_{highest} = 2.4305$$

*and the highest expected highest effort is then*

$$HE(t_{highest}) = 0.2255$$

The analysis of the expected total effort as well as the expected highest effort in this model with a head start $t > 1$ is quite complicated since as we can see from the equilibrium analysis a head start which is given to the contestant in period $k$ affects the equilibrium strategies of all consecutive contestants $j \geq k$. Furthermore, in contrast to the model with two contestants, the use of a head start in the sequential all-pay auction with $n > 2$ contestants may decrease the number of active contestants and therefore may decrease the contestants' expected total effort and the expected highest effort. However, as we show in the following there are sufficient conditions on the distribution functions of the contestants' types according to which the use of head starts is profitable for a designer who wishes to maximize the expected highest effort as well as the expected total effort in the sequential all-pay auction with any number of contestants.

We first need a generalization of Condition 1 for all the contestants who participate in the first $n - 1$ periods.

**Condition 4** *The equilibrium effort function of contestant* $i$, $i = 1, ..., n-1$ *in the sequential all-pay auction without a head start* ($t = 1$) *given by (13) and (12) is strictly convex for all* $\overleftarrow{a}_i \leq a_i < 1$.

Using Condition 4 we can show our main result.

**Theorem 1** *If Condition 4 holds, then the expected highest effort of the contestants in the sequential all-pay auction with* $n$ *players and a head start* $t > 1$ *sufficiently close to 1 is higher than the expected highest effort in the sequential all-pay auction without a head start.*

**Proof.** See Appendix. ■

According to Theorem 1, if every contestant is given a head start with respect to his next opponent then the expected highest effort in the sequential contest with head starts is higher than in the sequential contest without head starts. Moreover, by the proof of Theorem 1 this result holds even if a head start is given only for a subset of the contestants. In the following, we will assume that the head start is not necessarily given to all the contestants. In particular, we only give a head start to player $n-1$. Then similarly to the two-player model we assume the following conditions.

**Condition 5** *The function $G_n(x) = (1 - F_n(x))\, x$ is convex.*

**Condition 6** *The equilibrium highest effort of contestant $n-1$ (the effort of type $a_{n-1} = 1$) in the contest without a head start is lower than $x_n^* = \arg\max_{x\in[0,1]} G_n(x)$. Formally,*

$$\beta_{n-1}(1) = (F_n')^{-1}(1) < x_n^*$$

By the same arguments as in the proof of Proposition 3 we obtain

**Proposition 5** *If Conditions 4,5 and 6 hold, then the expected total effort in the sequential all-pay auction with a head start to contestant $n-1$, $t > 1$ which is sufficiently close to 1 is higher than the expected total effort in the sequential all-pay auction without a head start.*

By Proposition 5, if a head start is given to contestant $n-1$ only, the expected total effort increases, but obviously this is not the optimal allocation of head starts that maximizes the expected total effort. Furthermore, the optimal allocation of head starts may include different head starts for contestants according to their timing of play.

# 4 The generalized $n$-player model

We consider now a generalized sequential all-pay auction with $n > 2$ contestants with a head start $t \geq 1$ but without a "stopping rule." In this generalized model, contestant $j, 1 \leq j \leq n$, observes the efforts of contestants $1, 2, ..., j-1$ in the previous periods and then makes an effort $x_j$ at period $j$. Contestant $j$ wins a prize equal to 1 iff $x_j \geq tx_i$ for all $i < j$ and $tx_j > x_i$ for all $i > j$ . In the case without head starts $(t = 1)$ contestant $j$ wins if his effort is larger than or equal to the efforts of all the contestants in the previous periods and his effort is larger than the efforts of all the contestants in the following periods.

By the same arguments as in the previous section, the equilibrium effort of contestant $n$ is given by

$$\beta_n(a_n) = \begin{cases} 0 & \text{if} \quad 0 \le a_n < \max_{j<n} t\beta_j(a_j) \\ \max_{j<n} t\beta_j(a_j) & \text{if} \quad \max_{j<n} t\beta_j(a_j) \le a_n \le 1 \end{cases}$$

Contestant $i$, $i$, $i = 2, ..., n-1$ uses a strictly monotonic equilibrium effort function $\beta_i(a_i)$ which is the solution to the following optimization problem:

$$\max_s \left\{ H_i(t\beta_i(s)) - \frac{\beta_i(s)}{a_i} \right\} \tag{16}$$

$$s.t \quad \beta_i(s) \ge t\beta_j(a_j) \text{ for all } j < i$$

where $H_i(x) = \Pi_{j=i+1}^n F_j(x)$. If $H_i(x)$ is concave then contestant $i'$s equilibrium effort, $i = 2, ..., n-1$ is given by

$$\beta_i(a_i) = \begin{cases} 0 & \text{if} \quad 0 \le a_i < \max_{j<i} t\beta_j(a_j) \\ \max_{j<i} t\beta_j(a_j) & \text{if} \quad \max_{j<i} t\beta_j(a_j) \le a_i < \overline{a}_i \\ \frac{1}{t}(H_i')^{-1}\left(\frac{1}{ta_i}\right) & \text{if} \quad \overline{a}_i \le a_i < \overleftrightarrow{a}_i \\ \frac{1}{t_i} & \text{if} \quad \overleftrightarrow{a}_i \le a_i \le 1 \end{cases} \tag{17}$$

where $\overline{a}_i$ and $\overleftrightarrow{a}_i$ are defined by

$$\max_{j<i} t\beta_j(a_j) = \frac{1}{t}(H_i')^{-1}\left(\frac{1}{t\overline{a}_i}\right) \text{ and } (H_i')^{-1}(\frac{1}{\overleftrightarrow{a}_i t}) = 1$$

Note that it is not necessarily true that $\max_{j<i} t\beta_j(a_j) \le \overline{a}_i$, in which case the second range of (17) does not exist. Moreover, it is not necessarily true that $\overleftrightarrow{a}_i \le 1$ and then the third range of (17) does not exist. Contestant 1 solves the same maximization problem as in the two-player model and therefore

$$\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 \le \overline{a}_1 \\ \frac{1}{t}(H_1')^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \overline{a}_1 \le a_1 \le \overleftrightarrow{a}_1 \\ \frac{1}{t_1} & \text{if} \quad \overleftrightarrow{a}_1 \le a_1 \le 1 \end{cases}$$

where $\overline{a}_1$ is defined as previously as $\max\left\{0, \frac{1}{tH_1'(0)}\right\}$ and $\overleftrightarrow{a}_1$ is defined as $\overleftrightarrow{a}_1 = \frac{1}{tH_1'(1)}$.

If $H_i(x) = \Pi_{j=i+1}^n F_j(x)$ is convex, then contestant $i'$s equilibrium effort, $i = 2, ..., n-1$ is given by

$$\beta_i(a_i) = \begin{cases} 0 & \text{if} \quad 0 \le a_i < \max_{j<i} t\beta_j(a_j) \\ \max_{j<i} t\beta_j(a_j) & \text{if} \quad \max_{j<i} t\beta_j(a_j) \le a_i < \overline{a}_i \\ \frac{1}{t_i} & \text{if} \quad \overline{a}_i \le a_i \le 1 \end{cases}$$

and contestant 1's equilibrium effort is given by

$$\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 < \frac{1}{t_1} \\ \frac{1}{t_i} & \text{if} \quad \frac{1}{t_1} \leq a_i \leq 1 \end{cases}$$

The expected total effort and the expected highest effort are defined as in the previous section. Then, exactly the same arguments we used to establish Theorem 1 hold here and we obtain that our main result also holds for the generalized all-pay auction.

**Theorem 2** *If Condition 4 holds, then the expected highest effort of the contestants in the generalized all-pay auction with* $n$ *players and a head starts* $t > 1$ *sufficiently close to 1 is higher than the expected highest effort in the generalized all-pay auction without a head start.*

## 5 Concluding remarks

We presented various models of sequential all-pay auctions in which contestants arrive one by one and where each contestant observes the effort of the previous contestants before making his effort. We characterized the equilibrium behavior of the contestants and derived expressions for the expected total and highest efforts. Then we analyzed the implications of using a head start mechanism in which early contestants are favored over later ones. These head starts, on the one hand, encourage early contestants to exert higher efforts but, on the other, may cause later contestants to withdraw from the contest. We demonstrated that in our model the allocation of head starts increases the expected highest effort as well as the expected total effort.

If we will assume in our sequential all-pay auctions that the contest designer incurs some cost for each contestant - either because of the discount of time or because of the cost of adding a new contestant - then the results in this paper will still hold. In the two-player model, for example, the contest designer values more the effort made in the first period and therefore he will want to increase the first mover's expected effort by increasing the head start. Thus, we can show that every head start that increases the expected total effort or the expected highest effort without any discount of time will also increase these terms when the second mover's effort is discounted by some fixed factor between zero and one. This argument remains true also for the $n$-player models with and without a "stopping rule".

In this paper we assumed throughout that the contestants have asymmetric distribution functions for their types $F_i, i = 1, ..., n$ but are given the same head start $t$. It can be easily verified that all the results in this paper hold for asymmetric head starts $t_i, i = 1, ..., n$ as long as they are sufficiently close to 1.

# A Appendix

## A.1 Proof of Proposition 1

The expected highest effort in the two-player model without a head start is equal to contestant 1's expected effort, while the expected highest effort in the two-player model with a head start is larger than or equal to contestant 1's expected effort. Thus, in order to prove that a head start increases the expected highest effort it is sufficient to show that a head start increases contestant 1's expected effort. However, what we actually show is even stronger. In that for every type of contestant 1 who made a positive effort when there was no head start, this effort increases when a head start is given. Therefore we show that

$$\beta_1(a_1) \geq b_1(a_1) \text{ for all } 0 \leq a_1 \leq 1 \text{ and } 1 \leq t \leq \frac{1}{F_2'(1)}$$

Note that if Condition 1 holds then since $b_1(a_1)$ is increasing in $a_1$ and $\tilde{a} \geq 0$ then for all $t > 1$,

$$\beta_1(a_1) = \frac{1}{t}(F_2')^{-1}\left(\frac{1}{ta_1}\right) > (F_2')^{-1}\left(\frac{1}{a_1}\right) = b_1(a_1)$$

Likewise, the lowest type of contestant 1 who is active in the two-player model with a head start is lower than the lowest active type of contestant 1 in the two-player model without any head start. Formally, $\widehat{a} = \max\left\{\frac{1}{tF_2'(0)}, 0\right\} \leq \widetilde{a} = \max\left\{\frac{1}{F_2'(0)}, 0\right\}$ for any $t \geq 1$. Thus, we have

$$\begin{aligned}
\beta_1(a_1) &= \frac{1}{t}(F_2')^{-1}\left(\frac{1}{ta_1}\right) > (F_2')^{-1}\left(\frac{1}{a_1}\right) = b_1(a_1) \quad \text{for all } \widetilde{a} \leq a_1 \leq 1 \\
\beta_1(a_1) &= \frac{1}{t}(F')^{-1}\left(\frac{1}{ta_1}\right) > b_1(a_1) = 0 \quad \text{for all } \widehat{a} \leq a_1 \leq \widetilde{a} \\
\beta_1(a_1) &= b_1(a_1) = 0 \quad \text{for all } 0 \leq a_1 \leq \widehat{a}
\end{aligned}$$

and the expected effort of contestant 1 with a head start $t$ is higher than his expected effort without any head start. *Q.E.D.*

## A.2 Proof of Proposition 2

The expected effort of contestant 2 given an effort $\beta_1(a_1,t) > 0$ of contestant 1 is

$$E_2(t,a_1) = (1 - F_2(t\beta_1(a_1,t)))\, t\beta_1(a_1,t)$$

The expected effort of contestant 2 is then

$$TE_2(t) = \int_0^1 E_2(t,a_1)F_1'(a_1)da_1 = \int_{\widehat{a}}^1 E_2(t,a_1)F_1'(a_1)da_1 > 0$$

The function $t\beta_1(a_1,t) = (F_2')^{-1}\left(\frac{1}{a_1 t}\right)$ is increasing in $a_1$ as well as in $t$. By Condition 3 we know that $(F_2')^{-1}(1) < x^*$. Therefore we obtain that, for $t > 1$ close enough to 1 and for all $a_1 \leq 1$,

$$t\beta_1(a_1,t) \leq t\beta_1(a_1 = 1, t) = (F_2')^{-1}\left(\frac{1}{t}\right) < x^*$$

Thus by Condition 2 we have

$$\frac{dE_2(t,a_1)}{dt} > 0$$

So far we showed that for all types $\widehat{a} \leq a_1 \leq 1$ for which contestant 1 exerts a positive effort the expected effort of contestant 2 increases in $t$ as long as $t$ is sufficiently close to 1. By Condition 1, the interval of types of contestant 1 who exert a positive effort increases in $t$, i.e., $\frac{d\widehat{a}}{dt} = \frac{d}{dt}\max\left\{\frac{1}{tF_2'(0)}, 0\right\} \leq 0$ and therefore, if $t$ is sufficiently close to 1 we established that

$$\frac{d}{dt}TE_2(t) = \frac{d}{dt}\int_0^1 E_2(t,a_1)F_1'(a_1)da_1 = \frac{d}{dt}\int_{\widehat{a}}^1 E_2(t,a_1)F_1'(a_1)da_1 > 0$$

*Q.E.D.*

## A.3 Proof of Proposition 4

By (13) and (12), the expected efforts of contestants $1,..,n-1$ are the same in the sequential all-pay auction with either $n$ or $n+1$ contestants. The expected effort of contestant $n$ in a contest with $n$ contestants is given by

$$\beta_n(a_n) = \begin{cases} 0 & \text{if} \quad 0 \leq a_n < t\beta_{n-1}(a_{n-1}) \\ t\beta_{n-1}(a_{n-1}) & \text{if} \quad t\beta_{n-1}(a_{n-1}) \leq a_n \leq 1 \end{cases}$$

while his expected effort in a contest with $n+1$ players is given by

$$\beta_n(an) = \begin{cases} 0 & \text{if} \quad 0 \le a_i < t\beta_{n-1}(a_{n-1}) \\ t\beta_{i-1}(a_{n-1}) & \text{if} \quad t\beta_{n-1}(a_{n-1}) \le a_n < \overleftarrow{a}_i \\ \frac{1}{t}\left(F'_{n+1}\right)^{-1}\left(\frac{1}{ta_n}\right) & \text{if} \quad \overleftarrow{a}_n \le a_n < \overrightarrow{a}_n \\ \frac{1}{t} & \text{if} \quad \overrightarrow{a}_n \le a_n \le 1 \end{cases}$$

Since for all $\overleftarrow{a}_n < a_n \le \overrightarrow{a}_n$ we have $\frac{1}{t}\left(F'_{n+1}\right)^{-1}\left(\frac{1}{ta_n}\right) \ge t\beta_{n-1}(a_{n-1})$ and for all $\overrightarrow{a}_n \le a_n \le 1$ we have $\frac{1}{t} > t\beta_{n-1}(a_{n-1})$, the result is obtained. *Q.E.D.*

## A.4 Proof of Theorem 1

By Condition 4, the function $\left(F'_{i+1}\right)^{-1}\left(\frac{1}{a_i}\right)$ is strictly convex and therefore, for $t > 1$ and $i = 1, ..., n-1$ we have

$$\frac{1}{t}\left(F'_{i+1}\right)^{-1}\left(\frac{1}{ta_i}\right) > \left(F'_{i+1}\right)^{-1}\left(\frac{1}{a_i}\right) \tag{18}$$

Denote the equilibrium effort of contestant $i$ with a type $a_i$ in the contest without any head start by $\beta_i(a_i, t=1) = b_i(a_i)$, then we have

**Lemma 1** *For $t > 1$ close enough to 1 if the equilibrium effort of contestant $i$ with a type $a_i$ is positive, then this equilibrium effort is higher than or equal to his equilibrium effort in the contest without a head start ($t = 1$). Formally, for $i = 1$ if $a_1 \ge \overline{a}_{1,t}$ then*

$$\beta_1(a_1, t) \ge b_1(a_1)$$

*and for $i = 2, ..., n$ if $a_i \ge t\beta_{i-1}(a_{i-1}; t)$ then*

$$\beta_i(a_i; t) \ge b_i(a_i)$$

**Proof:** By (13) and (12) if $t = 1$ contestant $i$'s equilibrium efforts $i = 2, ..., n-1$ are given by

$$b_i(a_i) = \begin{cases} 0 & \text{if} \quad 0 \le a_i < b_{i-1}(a_{i-1}) \\ b_{i-1}(a_{i-1}) & \text{if} \quad b_{i-1}(a_{i-1}) \le a_i < \overline{a}_i \\ \left(F'_{i+1}\right)^{-1}\left(\frac{1}{a_i}\right) & \text{if} \quad \overline{a}_i \le a_i \le 1 \end{cases} \tag{19}$$

and

$$b_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 \le \overline{a}_1 \\ (F_2')^{-1}\left(\frac{1}{a_1}\right) & \text{if} \quad \overline{a}_1 \le a_1 \le 1 \end{cases}$$

while for $t$ close enough to 1

$$\beta_i(a_i;t) = \begin{cases} 0 & \text{if} \quad 0 \le a_i < t\beta_{i-1}(a_{i-1};t) \\ t\beta_{i-1}(a_{i-1};t) & \text{if} \quad t\beta_{i-1}(a_{i-1};t) \le a_i < \overleftarrow{a}_i \\ \frac{1}{t}\left(F_{i+1}'\right)^{-1}\left(\frac{1}{ta_i}\right) & \text{if} \quad \overleftarrow{a}_i \le a_i < 1 \end{cases}$$

and

$$\beta_1(a_1;t) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 \le \overline{a}_{1,t} \\ \frac{1}{t}(F_2')^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \overline{a}_{1,t} \le a_1 \le 1 \end{cases}$$

For $i = 1$ since $\overline{a}_{1,t} \le \overline{a}_1$ the result follows from (18). We prove the rest of the lemma by induction on $i$.

For $i = 2$,

$$b_2(a_2) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < (F_2')^{-1}\left(\frac{1}{a_1}\right) \\ (F_2')^{-1}\left(\frac{1}{a_1}\right) & \text{if} \quad (F_2')^{-1}\left(\frac{1}{a_1}\right) \le a_2 < \overline{a}_2 \\ (F_3')^{-1}\left(\frac{1}{a_2}\right) & \text{if} \quad \overline{a}_2 \le a_2 \le 1 \end{cases}$$

where $\overline{a}_2 = \frac{1}{F_3'\left((F_2')^{-1}\left(\frac{1}{a_1}\right)\right)}$ and

$$\beta_2(a_2;t) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < (F_2')^{-1}\left(\frac{1}{ta_1}\right) \\ (F_2')^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad (F_2')^{-1}\left(\frac{1}{ta_1}\right) \le a_2 < \overleftarrow{a}_2 \\ \frac{1}{t}(F_3')^{-1}\left(\frac{1}{ta_2}\right) & \text{if} \quad \overleftarrow{a}_2 \le a_2 < 1 \end{cases}$$

where $\overleftarrow{a}_2 = \frac{1}{tF_3'\left(t(F_2')^{-1}\left(\frac{1}{ta_1}\right)\right)}$.

We thus need to show that for $a_2 \ge (F_2')^{-1}\left(\frac{1}{ta_1}\right)$ we have $\beta_2(a_2;t) > b_2(a_2)$. To do this we consider two cases. Case 1) If $a_2 < \overline{a}_2$ since the effort function is increasing in the type we obtain that $\beta_2(a_2;t) \ge (F_2')^{-1}\left(\frac{1}{ta_1}\right) > (F_2')^{-1}\left(\frac{1}{a_1}\right) = b_2(a_2)$. Case 2) If $\overline{a}_2 \le a_2 \le 1$ then $b_2(a_2) = (F_3')^{-1}\left(\frac{1}{a_2}\right)$ and we have two sub-cases. Case (2a): if $a_2 < \overleftarrow{a}_2$ then by construction $\beta_2(a_2;t) = (F_2')^{-1}\left(\frac{1}{ta_1}\right) > \frac{1}{t}(F_3')^{-1}\left(\frac{1}{ta_2}\right)$ and therefore by (18) we obtain, $\beta_2(a_2;t) = (F_2')^{-1}\left(\frac{1}{ta_1}\right) > \frac{1}{t}(F_3')^{-1}\left(\frac{1}{ta_2}\right) > (F_3')^{-1}\left(\frac{1}{a_2}\right) = b_2(a_2)$. Case (2b): if $\overleftarrow{a}_2 \le a_2 < 1$ then $\beta_2(a_2;t) = \frac{1}{t}(F_3')^{-1}\left(\frac{1}{ta_2}\right) > (F_3')^{-1}\left(\frac{1}{a_2}\right) = b_2(a_2)$.

Assume by induction that the lemma is true for all $i = 2, ..., l-1$. We need to show that for all $a_l \ge t\beta_{l-1}(a_{l-1};t)$ we have $b_l(a_l) < \beta_l(a_l;t)$. By the induction assumption, we know that $b_{l-1}(a_{l-1}) <$

$t\beta_{l-1}(a_{l-1};t)$. Thus, similarly to the case of $i=2$ we have two cases. Case 1) If $a_l < \overline{a}_l$ then $\beta_l(a_l;t) \geq t\beta_{l-1}(a_{l-1};t) > b_{l-1}(a_{l-1}) = b_l(a_l)$. Case 2) If $\overline{a}_l \leq a_l \leq 1$ then $b_l(a_l) = \left(F'_{l+1}\right)^{-1}\left(\frac{1}{a_i}\right)$ and we have two sub-cases. Case (2a): if $a_l < \overleftarrow{a}_l$ then by ( 18) we obtain $\beta_l(a_l;t) = t\beta_{l-1}(a_{l-1};t) > \frac{1}{t}\left(F'_{l+1}\right)^{-1}\left(\frac{1}{ta_l}\right) > \left(F'_{l+1}\right)^{-1}\left(\frac{1}{a_i}\right) = b_l(a_l)$. Case (2b): if $\overleftarrow{a}_l \leq a_l < 1$ then $\beta_l(a_l;t) = \frac{1}{t}\left(F'_{l+1}\right)^{-1}\left(\frac{1}{ta_l}\right) > \left(F'_{l+1}\right)^{-1}\left(\frac{1}{a_i}\right) = b_l(a_l)$.■

We use Lemma 1 to prove the theorem. For a given realization of the contestants' abilities: $a_1,...,a_n$ we denote by $HE(a_1,...,a_n)$ the highest effort. Notice that this effort can be made by more than one contestant. Therefore we denote by $j_0 = j(a_{1,...,}a_n)$ the first (i.e. the lowest indexed) contestant that makes this highest effort. Formally, if $l \in \arg\max_{1\leq i\leq n} b_i(a_i)$ then $l \geq j_0$.

It is sufficient to prove that when a head start $t$ sufficiently close to 1 is given to the contestants, then for any given realization the highest bid increases i.e.,

$$HE_t(a_1,...,a_n) = \max_{1\leq i\leq n}\beta_i(a_i;t) \geq b_{j_0}(a_{j_0}) = HE(a_1,...,a_n)$$

Since $HE_t(a_1,...,a_n) \geq \beta_{j_0}(a_{j_0};t)$ it is enough to show that

$$\beta_{j_0}(a_{j_0};t) \geq b_{j_0}(a_{j_0})$$

This last inequality was proved in Lemma 1 but only if $\beta_{j_0}(a_{j_0};t) > 0$. Thus, it remains to show that $\beta_{j_0}(a_{j_0};t) > 0$ or equivalently that $a_{j_0} \geq t\beta_{j_0-1}(a_{j_0-1};t)$. First note that we must have

$$b_{j_0}(a_{j_0}) = \left(F'_{j_0+1}\right)^{-1}\left(\frac{1}{a_{j_0}}\right) \text{ and } a_{j_0} \geq \overline{a}_{j_0}$$

Otherwise, either $b_{j_0}(a_{j_0}) = 0$ (but then obviously this cannot be the highest bid), or $b_{j_0}(a_{j_0}) = b_{j_0-1}(a_{j_0-1})$ which contradicts the definition of $j_0$ as the lowest indexed player who submits the highest effort. By (19)

$$b_{j_0-1}(a_{j_0-1}) < \overline{a}_{j_0} \tag{20}$$

and from Lemma 1

$$b_{j_0-1}(a_{j_0-1}) < \beta_{j_0-1}(a_{j_0-1};t) \tag{21}$$

Moreover, since

$$\lim_{t\to 1} t\beta_{j_0-1}(a_{j_0-1};t) = b_{j_0-1}(a_{j_0-1})$$

then by (20) and (21) we can find $t > 1$ close enough to 1 such that

$$b_{j_0-1}\left(a_{j_0-1}\right) < t\beta_{j_0-1}\left(a_{j_0-1};t\right) < \overline{a}_{j_0} \leq a_{j_0}$$

*Q.E.D.*

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