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COMPETITIVE QUANTITY DISCOUNTS

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ABSTRACT

Competitive quantity discounts*

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Abstract

We analyze the effects of competition with quantity discounts in a duopoly model with asymmetric firms. Consumers are privately informed about demand, so firms use quantity discounts as a price discrimination device. However, a dominant firm may also use quantity discounts to weaken or eliminate its competitor. We analyze the effects of quantity discounts on firms' profits and consumer surplus. Our main finding is that quantity discounts can decrease social welfare (i.e., the sum of producers' and consumers' surplus) for a small set of parameter values.

1 Introduction

Many firms offer quantity discounts, i.e., lower prices for large purchases. It is generally recognised that these discounts can simply be a way to pass economies of scale on to buyers, or to allow firms to better extract the surplus of their customers. But there may be other rationales as well, and antitrust authorities are sometimes concerned that dominant firms may use non-linear prices to eliminate or soften competition.[1]

*We thank Piercarlo Zanchettin and seminar participants at the EARIE 2010 conference in Istanbul for useful comments.

[1] See Kolay, Shaffer and Ordover (2004) and Beard, Ford and Kaserman (2007). The competition policy debate has also centred around other forms of loyalty discounts, including bundled discounts (where price discounts are conditioned on the customer's total purchases of various products supplied by the firm), market-share discounts (i.e., discounts conditioned on the firm's share of the customer's total purchases), and exclusivity discounts. Still other forms of loyalty rebates are conditioned on the customer's past purchasing history. For an overview of the current debate, see Heimler (2005), Kobayashi (2005), Spector (2005), Faella (2006), Ordover and Shaffer (2007), and Ahlborn and Bailey (2008).

One theory contends that quantity discounts provide a cost-effective way to deprive a rival of economies of scale so as to drive it out of business. This theory implies that in assessing the competitive harm of quantity discounts the courts should apply the demanding standards required in predatory pricing cases. That is, plaintiffs must be able to demonstrate that the incumbent had suffered a loss by pricing below marginal cost or average variable cost, and that it had a reasonable prospect of recouping this loss in the future. This theory is consistent with the current policy in the U.S., where the courts have typically been reluctant to prohibit single-product quantity discounts under the antitrust laws.[2]

The European case law, by contrast, is implicitly or explicitly based on the notion that quantity discounts can have exclusionary effects even if they are not part of a predatory strategy. In *Michelin II*, for instance, the European Court of Justice did not require a proof that price was less than unit cost, and ruled that any quantity discount that does not reflect cost efficiencies is presumed to be abusive if practiced by a dominant firm.[3]

Are the concerns of the European antitrust authorities well grounded? Are the concerns of the European antitrust authorities well grounded? To answer this question, we consider a model where two asymmetric firms supply differentiated products and compete in non-linear prices. We contrast the non-linear pricing equilibrium with the one that would obtain if firms were constrained to use linear prices. The model is timeless – a static, one-shot game of price competition – and there are no economies of scale. As a result, there is no room for predatory pricing. However, consumers are privately informed about their demand, so firms may use quantity discounts as a price discrimination device.

Martimort and Stole (2009) have characterised the equilibrium with non-linear prices in the symmetric case. Although they have not developed the comparison with the case where firms are restricted to use linear prices, it is easy to show that with symmetric firms quantity discounts always increase social welfare (i.e., the sum of producers' and consumers' surplus). Thus, the symmetric model provides little support to the view that quantity discounts can be anti-competitive.

In the asymmetric case, by contrast, we find that anti-competitive effects are not only possible, but also likely, at least in some senses. First, quantity discounts often harm the smaller firm, which can be excluded by the dominant firm more easily than with linear prices. Second, while in the symmetric case quantity discounts harm consumers only when the products are weak substitutes, if the firms are sufficiently asymmetric consumers are harmed for any degree of

[2] In two recent decisions – *Brooke Group* and *Concord Boats* – the U.S. courts have explicitly applied to cases involving quantity discounts the standards required in predatory pricing cases: see Klein and Lerner (2008).

[3] The 2008 Commission's *Guidance Paper* has reiterated that loyalty discounts can have anticompetitive effects "without necessarily entailing a sacrifice for the dominant undertaking" (§ 37). It has also stressed, however, that the possible exclusionary effects of quantity discounts can only materialize when firms are sufficiently asymmetric. However, *Michelin II* and the subsequent cases where quantity discounts were found abusive all involved all-units discounts. Incremental discounts are generally treated more leniently: see Waelbroeck (2005).

product substitutability. Third, in the asymmetric case quantity discounts can decrease social welfare for a range of parameter values.

Thus, the analysis does provide some support to the concerns of the European antitrust authorities. However, we also argue that identifying the circumstances in which a ban of quantity discounts can improve social welfare may be an extremely challenging task. After presenting our main results, we discuss their implications for competition policy more fully in the concluding section.

There is an extensive literature on monopolistic non-linear pricing: see Mussa and Rosen (1978) for a pioneering contribution, and Wilson (1994) for a comprehensive treatment. The analysis of the oligopoly case, however, is much less developed.[4] Yin (2004) analyses a model in which firms compete in two-part tariffs. Kolay, Shaffer and Ordover (2004) model all-units discounts assuming that firms can offer only piecewise linear price schedules. Martimort and Stole (2009) are the first to allow for any non-linear price schedule. Another recent contribution is Hoerning and Valletti (2010), who focus on the special case where total demand is fixed.

Another strand of the literature (Armstrong and Vickers, 2001; Rochet and Stole, 2002) focuses on models of one-stop shopping, where consumers patronize only one firm. In this case, firms effectively compete in utility space, and in equilibrium they offer two-part tariffs with a marginal price equal to marginal cost. Armstrong and Vickers (2010) extend the analysis to the case in which consumers can purchase from both firms at an extra cost. However, they focus on the case of multiproduct firms, so their contribution properly belongs to the literature on bundled discounts (see also Greenlee, Reitman and Sibley, 2008). There is also a small literature on market-share discounts, which includes Majumdar and Shaffer (2009), Ordover and Shaffer (2007), and Calzolari and Denicolò (2009). Finally, turning back to quantity discounts, Beard, Ford and Kaserman (2007) have argued that quantity discounts can be used as an imperfect substitute for exclusive contracts in models where the incumbent can contract with the buyers before an entrant enters the market. Their analysis highlights an additional channel whereby quantity discounts can have anti-competitive effects, one that is not considered in this paper.

The rest of the paper is organized as follows. Section 2 sets up the model. In section 3 we derive the equilibrium with linear prices, and in section 4 that with non-linear prices. Section 5 compares these two modes of competition in terms of profits, consumer surplus, and social welfare. Section 6 analyzes the case of a selective ban imposed only on the dominant firm. Section 7 discusses the policy implications of our results and concludes the paper. All proofs are in the appendix.[5]

[4] Most of the literature on common agency games with incomplete information has focused on the *intrinsic* common agency case, where the agent must choose between participating with all principals or none at all. This framework suits such cases as a firm that is regulated by several regulatory agencies. As a rule, however, a consumer can choose to buy from only a subsets of the firms. In the jargon of the common agency literature, this corresponds to the case of *delegated* common agency.

[5] A Web Appendix (available at http://www2.dse.unibo.it/calzolari/) provides the detailed

2 The model

Two firms, denoted by $i = A, B$, supply differentiated products to a final consumer. The consumer's utility function in monetary terms, $u(q_A, q_B, \theta)$,[6] depends on consumption of the two goods q_A and q_B and a parameter, θ, which is the consumer's private information. Following Martimort and Stole (2009), we posit a quadratic utility function

$$u(q_A, q_B, \theta) = \theta(q_A + q_B) - \frac{1-\gamma}{2}(q_A^2 + q_B^2) - \gamma q_A q_B \tag{1}$$

and assume that θ is uniformly distributed over the interval $[0, 1]$. These assumptions serve to get explicit solutions, without which it would be impossible to perform a thorough comparison between the two modes of competition (i.e., linear *v.* non-linear prices).[7]

The parameter $\gamma \in [0, \frac{1}{2})$ captures the degree of substitutability between the goods. The goods are independent when $\gamma = 0$ and perfect substitutes in the limiting case $\gamma = \frac{1}{2}$. The factor $\frac{1-\gamma}{2}$ that multiplies the middle term in (1) serves to prevent changes in γ to affect the size of the market, as in Shubik and Levitan (1980).

Firms have constant marginal costs c_A and c_B, with $c_A \leq c_B$. There are no fixed costs. With no loss of generality, we normalize c_A to zero and denote $c_B = c \geq 0$. Thus, the parameter c captures the degree of asymmetry among the firms.[8] Since firm B is never active when $c > 1$, we can focus on the case $c \leq 1$, again with no loss of generality.

Firms simultaneously and independently offer a price schedule. With linear prices, price schedules must take the simple form $P_i(q_i) = p_i q_i$, so that firm i's strategy is simply its price $p_i \in \Re_+$. When firms can use quantity discounts, a strategy for firm i is a function $P_i(q_i) : [0, q_{\max}] \to \Re_+$ where q_i is the quantity that firm i is willing to supply, $P_i \geq 0$ is the corresponding total payment requested, and $q_{\max}$ is an upper bound large enough that no consumer may ever want to consumer more than $q_{\max}$.

Each firm maximises its expected profits

$$\pi_i = E_\theta[P_i(q_i(\theta)) - c_i q_i(\theta)], \tag{2}$$

and the consumer maximises his net utility given his type θ

$$U(q_A, q_B, \theta) = u(q_A, q_B, \theta) - P_A(q_A) - P_B(q_B). \tag{3}$$

derivation of the closed form solutions and all the numerical calculations. The software used is Mathematica by Wolfram Research Inc..

[6] In an alternative interpretation of the model, A and B are manufacturers that sell their products through a common retailer, and the function u is the retailer's gross profit.

[7] The linear-quadratic-uniform specification of the model is not special, as it corresponds to the case in which each consumer has a linear demand for the goods. Martimort and Stole (2009) show that several qualitative properties of the solution hold for any suitably well-behaved utility function.

[8] The parameter c may also capture demand asymmetries. For example, one could add a linear term $(\alpha_A q_A + \alpha_B q_B)$ in the utility function, where $\alpha_A - \alpha_B$ can be interpreted as a measure of vertical product differentiation. Setting $c_A - \alpha_A = 0$ and $c = c_B - \alpha_B \geq 0$ one re-obtains the formulation used in the paper.

Given the timing of the game, it is natural to focus on subgame perfect equilibria where the consumer maximises U for any possible pair of price schedules submitted by the firms.

In what follows, we shall say that a firm is *active* if its equilibrium output is strictly positive, and that it is *excluded* if its equilibrium output is zero.

3 Linear prices

We start from the case in which firms compete in linear prices. We seek the Bertrand equilibrium of the industry described in the previous section.

Although individual demand functions are linear, the total demand functions obtained aggregating over the heterogeneous consumers are non linear. This complicates the calculation of the equilibrium prices considerably. We have:

Proposition 1 *When firms compete in linear prices, there is a unique equilibrium in pure strategies. If $c \geq c^{\ell}$, where*

$$c^{\ell} \equiv \frac{3 - 5\gamma}{3(1 - \gamma)} \tag{4}$$

firm B is not active and firm A charges the monopoly linear price $p_M = \frac{1}{3}$. If instead $c < c^{\ell}$, both firms are active. The equilibrium prices can be calculated explicitly and are reported in the Appendix. They satisfy $p_A^* \leq p_B^*$, with a strict inequality when $c > 0$. In the symmetric case $c = 0$, equilibrium prices reduce to

$$p_A^* = p_B^* = \frac{1 - 2\gamma}{3 - 4\gamma}. \tag{5}$$

Proof. See the Appendix.

When firm B is active, in equilibrium there are three groups of consumers: low demand consumers, $\theta \in [0, p_A^*]$, who do not buy any product; intermediate demand consumers, $\theta \in [p_A^*, \hat{\theta}^{\ell}]$, who buy only product A; and high demand consumers, $\theta \in [\hat{\theta}^{\ell}, 1]$, who buy both products. Notice that the inframarginal consumers $\theta < \hat{\theta}^{\ell}$ are "captive," in the sense that a small increase in p_A leads these consumers to reduce q_A, but not to purchase product B.

This latter observation is crucial to understand a surprising property of the equilibrium, namely, the absence of a limit pricing region. Another (and related) surprising property is that firm B is excluded only if the cost c is greater than the monopoly price $p_M^{\ell} = \frac{1}{3}$. One would expect, by contrast, that if the products are sufficiently close substitutes firm B should be excluded even if the cost gap c is small, because firm A engages in limit pricing. This is, indeed, what happens in models of product differentiation with homogeneous consumers (see, for instance, Zanchettin 2006).

The intuitive reason why firm A does not engage in limit pricing is that in our model both firms compete for high-type consumers, but low-type consumers are effectively captive to firm A. By reducing its price, firm A may increase its profit on consumers of type $\theta \in [\hat{\theta}^{\ell}, 1]$, but it will certainly decrease its profit on its captive consumers as long as p_A is below the monopoly price $p_M = \frac{1}{3}$. This effect makes firm A less aggressive and is stronger, the closer is $\hat{\theta}^{\ell}$ to 1.

Consider, in particular, under what circumstances a limit pricing strategy might be optimal for firm A. Such a strategy requires setting p_A such that $\hat{\theta}^{\ell} = 1$. But when $\hat{\theta}^{\ell}$ is close to 1, almost all consumers are captive, so firm A will have an incentive to charge the monopoly price p_M. This means that firm A will drive B out of the market only if it can do so by charging the monopoly price.[9]

Another property of the equilibrium is that an increase in the degree of product differentiation may increase p_A^* when γ is already sufficiently large. When γ increases, the products become better substitutes. As a result, competition is more intense, decreasing both prices. However, in our model a countervailing effect is at work: when $c > 0$, an increase in γ enlarges the set of consumers that purchase only product A. This reinforces firm A's incentive to exploit these captive consumers, reducing the intensity of competition. This latter effect can prevail on the standard effect of greater substitutability when γ is large enough, leading to an increase in p_A^*.

4 Non linear prices

Now we turn to the case where firms can offer non-linear price schedules $P_i(q_i)$. The difficulty in finding the equilibrium with non-linear prices is that the strategy space is very large. We overcome this difficulty by using a guess-and-check strategy. We start by guessing a specific functional form of the equilibrium price schedules, so that they are fully identified by a few parameters. If the initial guess is correct, the equilibrium of the original game will coincide with that of a restricted game where firms can choose only those parameters. The equilibrium of the restricted game then becomes the candidate equilibrium of the original game. Finally, we verify that the candidate equilibrium strategies satisfy the best response property *over the unrestricted strategy space*. The drawback of the guess-and-check strategy is that it fails to locate equilibria that do not conform to the initial guess, if there are any.

To simplify the exposition, we present separately the equilibrium when firm B is active and when it is excluded. We start by determining the conditions under which firm B is, indeed, active. Recall that in any standard screening environment like ours equilibrium quantities must satisfy two properties: monotonicity (firms sell larger quantities to higher types), and no distortion at the top. These properties imply that firm B will stay active as long as the mar-

[9] Only in the limiting case $\gamma = \frac{1}{2}$ does one obtain the familiar limit pricing equilibrium, where firm B prices at c and firm A prices at $c - \epsilon$ and serves the entire market. When γ approaches $\frac{1}{2}$, p_B^* converges to c from above and p_A^* converges to c from below. The equilibrium quantity Q_B^* converges to zero, but is positive for any $\gamma < \frac{1}{2}$.

ginal willingness to pay for product B of consumer $\theta = 1$ when he consumes the efficient stand-alone quantity of product A (which is $\frac{1}{1-\gamma}$) exceeds the marginal cost c.[10] Since

$$u'_{q_B}\left(\frac{1}{1-\gamma}, 0, 1\right) = \frac{1-2\gamma}{1-\gamma}(\equiv c^d), \tag{6}$$

it follows that firm B is active if and only if $c < c^d$. The threshold c^d decreases with the degree of product substitutability γ, is equal to 1 when $\gamma = 0$ and converges to 0 when γ approaches 1/2.

Since the utility function is quadratic, we guess that firm B will offer a quadratic price schedule

$$P_B(q_B) = \alpha_{0,B} + \alpha_{1,B} q_B + \alpha_{2,B} q_B^2 \quad \text{for } 0 \leq q_B \leq q_{\max}. \tag{7}$$

As for firm A, which being more efficient may serve some consumers who purchase only product A, we guess that its price schedule comprises various piecewise quadratic branches. To be more specific, we guess that in addition to an upper branch, intended for high demand consumers who purchase both products, which is

$$P_A^d(q_A) = \alpha_{0,A}^d + \alpha_{1,A} q_A + \alpha_{2,A} q_A^2 \quad \text{for } \tilde{q}_A \leq q_A \leq q_{\max}, \tag{8}$$

firm A may strategically engage in limit pricing (to induce some intermediate demand consumers to purchase only product A) and in monopoly pricing (to exploit those low demand consumers who are effectively captive).

The monopoly part of price schedule can be easily calculated using standard monopolistic screening techniques and is

$$P_A^m(q_A) = \frac{1}{2} q_A - \frac{1-\gamma}{4} q_A^2 \quad \text{for } \bar{q}_A \leq q_A \leq \tilde{q}_A, \tag{9}$$

modulo a fixed fee. The limit pricing schedule must lead intermediate demand consumers to purchase a limit quantity $q_A^{\lim}(\theta)$ that induces consumers to purchase only product A. The limit quantity $q_A^{\lim}(\theta)$ is such that the marginal willingness to pay for product B is just equal to the marginal price of product B at $q_B = 0$, i.e. $u'_{q_B}(q_A^{\lim}(\theta), 0, \theta) = P_B'^*(0)$. It is easy to show that this implies

$$P_A^{\lim}(q_A) = \alpha_{1,B} q_A - \left(\frac{1}{2} - \gamma\right) q_A^2 \quad \text{for } 0 \leq q_A \leq \bar{q}_A, \tag{10}$$

again *modulo* a fixed fee.

Summarizing, our guess is that firm A will submit a price schedule that combines the duopoly, limit pricing and monopoly branches described above. Proposition 2 describes the equilibrium that conforms to this guess.

[10] Recall that there are no fixed costs.

To state the proposition, it is convenient to define the price schedule which obtains in the symmetric equilibrium where $c = 0$ (the case analysed by Martimort and Stole, 2009), which is:[11]

$$P^*(q) = \alpha_1^* q - \frac{\alpha_1^*}{2} q^2, \tag{11}$$

where

$$\alpha_1^* = \frac{1}{4}\left[3(1-\gamma) - \sqrt{1 - 2\gamma + 9\gamma^2}\right]. \tag{12}$$

We are now ready to state:

Proposition 2 *When firm B is active, i.e., if $c < c^d$, the following is an equilibrium in non-linear prices. Firm B offers the price schedule*

$$P_B^*(q_B) = P^*(q_B) + c\left(1 - \alpha_1^* \frac{1-\gamma}{1-2\gamma}\right) q_B \quad \textit{for } 0 \le q_B \le 1 - c\frac{1-\gamma}{1-2\gamma},$$

and firm A offers the price schedule

$$P_A^*(q) = \begin{cases} P_A^m(q_A) & \textit{for } 0 \le q_A \le \tilde{q}_A^* \\ \alpha_{0,A}^{\lim *} + P_A^{\lim}(q_A) & \textit{for } \tilde{q}_A^* \le q_A \le \hat{q}_A^* \\ \alpha_{0,A}^{d*} + P^*(q_A) + c\frac{\gamma\alpha_1^*}{1-2\gamma} q_A & \textit{for } \hat{q}_A^* \le q_A \le 1 + c\dfrac{\gamma}{1-2\gamma}. \end{cases}$$

The constants $\alpha_{0,A}^{\lim *}$, $\alpha_{0,A}^{d*}$, $\tilde{q}_A^*$ and $\hat{q}_A^*$ are such that firm A's price schedule is continuous and differentiable at $\tilde{q}_A^*$ and $\hat{q}_A^*$; the precise expressions are given in the Appendix.

Proof. See the Appendix.

In equilibrium, consumers are divided into three groups, as in the linear pricing case: low demand consumers, who do not buy any product; intermediate demand consumers, who buy only product A; and high demand consumers, who buy both products. Unlike the linear pricing case, however, firm A now engages in limit pricing for some intermediate demand consumers consumers. Thus, firm A's price schedule comprises three parts, in accord with our guess: the monopoly part, which is intended for consumers who purchase only product A and are not contested; the limit pricing part, which applies to consumers who in equilibrium purchase only product A but are contested by firm B; and the duopoly part, which is intended for consumers that purchase also product B. Firm B, by contrast, will serve only consumers who purchase also product A.

[11] Martimort and Stole (2009) use a slightly different parametrization of the utility function, which however is equivalent to ours. See Calzolari and Denicolò (2009) for the derivation of the symmetric equilibrium with out parametrization.

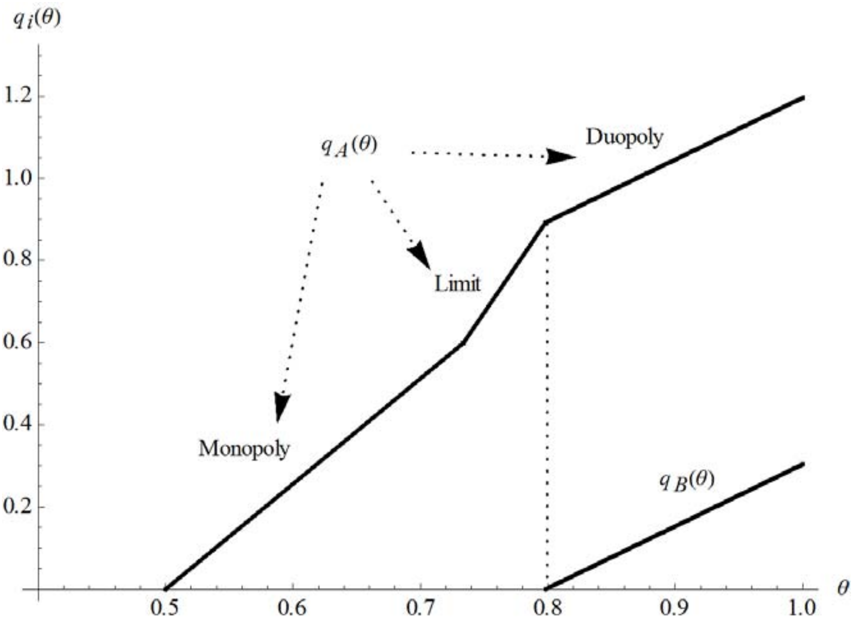

Figure 1: Equilibrium quantities with firm B active (for $\gamma = 0.22$ and $c = 0.5$).

The equilibrium quantities are

$$\begin{aligned} q_A^*(\theta) &= \begin{cases} \max\left\{0, \dfrac{2\theta-1}{1-\gamma}, \dfrac{\theta - \left[\alpha_1^* + c\left(1-\alpha_1^* \frac{1-\gamma}{1-2\gamma}\right)\right]}{\gamma}\right\} & \text{for } \theta \leq \hat{\theta}^* \\ q^*(\theta) + c\dfrac{\gamma}{1-2\gamma} & \text{for } \theta > \hat{\theta}^*. \end{cases} \\ q_B^*(\theta) &= q^*(\theta) - c\frac{1-\gamma}{1-2\gamma}, \end{aligned} \tag{13}$$

where

$$q^*(\theta) = \frac{\theta - \alpha_1^*}{1-\alpha_1^*} \tag{14}$$

is the symmetric equilibrium quantity, and

$$\hat{\theta}^* = \alpha_1^* + c\,(1-\alpha_1^*)\,\frac{1-\gamma}{1-2\gamma}. \tag{15}$$

Equilibrium quantities are depicted in Figure 1.

Various comments are in order. First, the equilibrium price schedules satisfy the no-fixed fee property $P_i^*(0) = 0$. As argued by Wilson (1994), this property must always hold in the absence of fixed costs. Second, the equilibrium price schedules are everywhere concave. This means that firms always offer quantity discounts and never charge quantity premia. Third, the equilibrium satisfies the no-distortion-at-the-top property, which implies that type $\theta = 1$ must purchase the efficient quantities $q_i^{fb}(1)$.[12] For the consumer $\theta = 1$ to be willing to purchase the efficient quantities, the slopes of the equilibrium price schedules at the efficient quantities must equal the marginal costs, i.e. $P_i'(q_i^{fb}) = c_i$. Fourth, marginal prices are never smaller than marginal costs. This confirms that there is no rule for predatory pricing in our model.[13] Fifth, the intensity of quantity discounts (as measured by the degree of concavity of the price schedules, $\frac{P''(q)q}{P'(q)}$) is the same for both firms when consumers purchase both products, and is smaller than under monopoly. Finally, an increase in c shifts up the price schedules of both firms (since $\alpha_1^* < \frac{1-2\gamma}{1-\gamma}$), demonstrating that prices are strategic substitutes even in the presence of quantity discounts.

When firm B is not active, we can posit, without any substantial loss of generality, that it stands ready to supply consumers at marginal cost. That is, firm B offers a price schedule $P_B(q_B) = cq_B$. We have:

Proposition 3 *When firm B is not active, i.e., if $c \geq c^d$, in equilibrium it offers the competitive price schedule $P_B^*(q_B) = cq_B$. Firm A offers the price schedule*

$$P_A^*(q) = \begin{cases} P_A^{\lim}(q_A) & \text{for } 0 \leq q_A \leq \tilde{q}_A^* \\ \alpha_{0,A}^{m*} + P_A^m(q_A) & \text{for } \tilde{q}_A^* \leq q_A \leq \frac{1}{1-\gamma} \end{cases} \tag{16}$$

where the constants $\tilde{q}_A^*$ and $\alpha_{0,A}^{m*}$ are such that firm A's price schedule is continuous and differentiable at $\tilde{q}_A^*$; the precise expressions are given in the Appendix.

Proof. See the Appendix.

Obviously, firm A's price schedule now comprises only two parts, the monopoly part and the limit pricing part. The twist with respect to the case in which firm B is active is that now the limit pricing schedule applies to low demand consumers and the monopoly schedule to high demand ones.

This new pattern is a consequence of the downward distortion in quantities caused by rent extraction under asymmetric information. Consider the marginal willingness to pay for product B of a consumer who purchases the monopoly

[12] The efficient quantities are implicitly defined by the conditions

$$u'_{q_i}(q_i^{fb}, q_j^{fb}, \theta) = c_i$$

and hence are

$$q_A^{fb}(\theta) = \theta + \frac{\gamma c}{1-2\gamma}$$
$$q_B^{fb}(\theta) = \theta - \frac{c(1-\gamma)}{1-2\gamma}.$$

[13] However, the no-distortion-at-the-top property means that marginal prices are equal to marginal costs for type $\theta = 1$. Thus, if a firm approximates the true equilibrium price schedule with a piecewise linear schedule, it might inadvertently price below marginal cost over a range of quantity levels.

quantity of product A:

$$u'_{q_B}(q_A^m(\theta), 0, \theta) = \frac{\gamma + (1-3\gamma)\theta}{1-\gamma}. \tag{17}$$

This expression is increasing in θ when $\gamma < \frac{1}{3}$, whereas it is decreasing in θ when $\gamma > \frac{1}{3}$. It follows that if A charged the monopoly price schedule, B's threat of entry would be stronger for higher types when the products are weak substitutes, and for lower types when the products are close substitutes. But we know that firm B is active in equilibrium precisely when the good are weak substitutes, in which case firm A can engage in monopoly pricing for low types. In contrast, when the products are close substitutes firm B is not active, and monopoly pricing will apply to high types.

The equilibrium quantities now are $q_B^*(\theta) = 0$ and

$$q_A^*(\theta) = \max\left\{0, \frac{\theta - c}{\gamma}, q_A^m(\theta)\right\}. \tag{18}$$

We conclude the presentation of the non-linear pricing equilibrium by noting that certain branches of firm A's price schedule may vanish for a range of parameter values. In particular, when B is active the monopoly part of A's price schedule vanishes (that is, $\tilde{q}_A^* = 0$) when $c > \tilde{c}$, where

$$\tilde{c} \equiv \frac{1}{2}\frac{(1-2\alpha_1^*)(1-2\gamma)}{1-\alpha_1^*(1-\gamma)-2\gamma}.$$

Likewise, when B is not active the limit pricing part of A's price schedule vanishes (that is, $\tilde{q}_A^* = 0$) when $c > \frac{1}{2}$.

Figure 2 illustrates how the parameter space (c, γ) splits into four regions that correspond to the possible equilibrium patterns: the *monopoly-limit-duopoly region,* $R_{MLD} \equiv \left\{(c,\gamma) : c^d > c > \tilde{c}\right\}$, where firm B is active but some consumers are captive to firm A; the *limit-duopoly region,* $R_{LD} \equiv \left\{(c,\gamma) : c < \min\left\{c^d, \tilde{c}\right\}\right\}$, where firm B is active and there are no captive consumers; the *limit-monopoly region,* $R_{LM} \equiv \{(c,\gamma) : c^d \le c < \frac{1}{2}\}$, where firm B is not active but still contends some consumers to firm A; and the *monopoly region,* $R_M \equiv \{(c,\gamma) : c \ge \max\{c^d, \frac{1}{2}\}\}$, where firm A is an unconstrained monopolist.

5 Comparison

This section compares the equilibria with linear and non-linear prices in terms of profits, consumer surplus, and welfare. Although we have obtained explicit solutions for the equilibrium with both linear and non-linear prices, the expressions for equilibrium profits, consumer surplus and social welfare are cumbersome and analytically intractable. Therefore, we resort to numerical calculations. These are reported in the Web Appendix which is available at http://www2.dse.unibo.it/calzolari/. Since there are only two parameters in

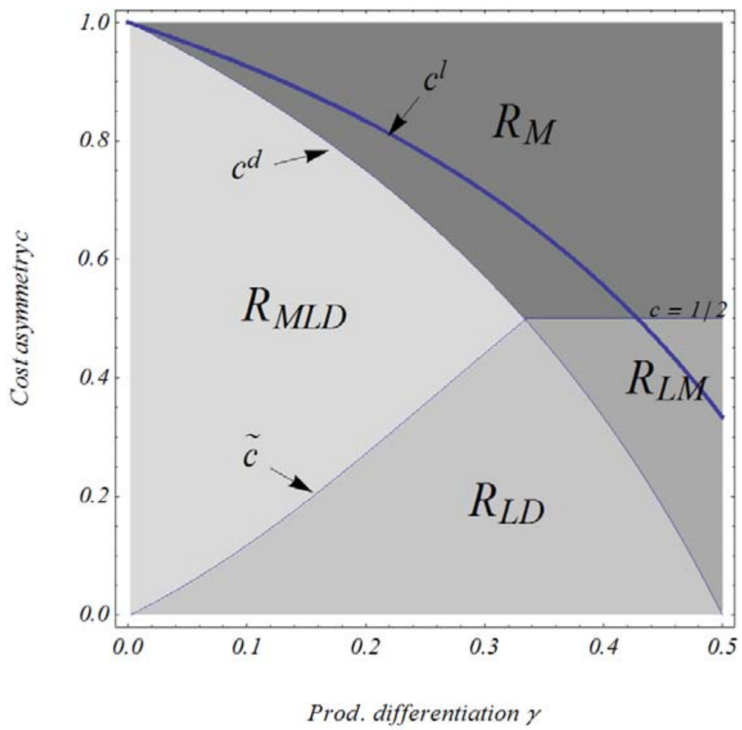

Figure 2: The four regions with non linear pricing.

our model, the degree of product substitutability γ and the degree of asymmetry c, we can conveniently present the outcome of the numerical calculations by a series of figures.[14]

5.1 The symmetric case

It is useful to start from the symmetric case $c = 0$, where we can easily determine the effect of quantity discounts in terms of the unique parameter, the degree of product substitutability γ.

Proposition 4 *In the symmetric case, there exist two thresholds, $\gamma_f \simeq 0.25$ and $\gamma_c \simeq 0.14$, such that firms are better off with quantity discounts if the products are weak substitutes ($0 \leq \gamma \leq \gamma_f$), and consumers are better off if the products are close substitutes ($\gamma_c \leq \gamma < \frac{1}{2}$). Social welfare is always larger with quantity discounts.*

Proof. See the Web Appendix.

[14] The figures have been drawn by Mathematica, using the closed form solutions for equilibrium variables and payoffs: see the Web Appendix.

These conclusions follow from two opposing effects of quantity discounts, which are at work also in the asymmetric case. On one hand, quantity discounts intensify competition, reducing profits and benefiting consumers. On the other hand, they also allow firms to better extract consumer surplus, increasing profits and harming consumers. The former effect prevails when the products are close substitutes, in which case firms are caught in a prisoners' dilemma: both would gain from a ban of quantity discounts, but if quantity discounts are permitted, each firm has a unilateral incentive to use them. The latter effect prevails when the products are relatively independent.[15] However, competition in non-linear prices increases social welfare for any degree of product differentiation. Thus, under symmetry the model provides little support to the view that quantity discounts can be anti-competitive.

5.2 The asymmetric case

The asymmetric case is more interesting from a competition policy perspective, as it opens the possibility that the dominant firm may use quantity discounts to eliminate (or weaken) its competitor.

5.2.1 Exclusion

Quantity discounts increase the likelihood that the less efficient firm is excluded from the market. However, this outcome results from inefficient participation of the less efficient firm under linear prices, not from inefficient exclusion with quantity discounts.[16]

Proposition 5 *When firms compete with non-linear prices exclusion of firm B is efficient. When they compete with linear prices, exclusion is inefficiently low since firm B remains active also for parameters values where exclusion is socially efficient.*

The proof is very simple. Consider the circumstances under which exclusion is socially efficient. Because the products are differentiated, the product supplied by the less efficient firm should be consumed unless c is large enough. To be precise, the condition is that c should be lower than the maximum marginal willingness to pay for product B when the efficient quantity of product A is already being consumed. The maximum is achieved at $\theta = 1$ and is equal to c^d, so exclusion is socially desirable only if $c \geq c^d$. This means that with quantity discounts, firm B is excluded precisely when exclusion is socially efficient.[17]

[15] This result depends on the specific functional forms we have used in this paper. When the products are weak substitutes, quantity discounts tend to have the same effects as under monopoly. As is well known, however, these effects may well be ambiguous for different specifications of demand.

[16] However, recall that we have ruled out fixed costs. With fixed costs, the picture would be less clear. Since there is no definite relation between a firm's marginal contribution to social welfare and its profits, both over- and under-participation seem possible.

[17] This property of the equilibrium follows from the no-distortion-at-the-top property, which implies that type $\theta = 1$ consumes the efficient quantity of product A.

With linear prices, by contrast, firm A prices above marginal cost, so consumer $\theta = 1$ buys less than the efficient quantity of product A. Since the products are substitutes, the demand for product B is inefficiently high. Hence, with linear prices firm B is active when $c^d \leq c < c^\ell$, even if it should actually be excluded.

5.2.2 Profits

As we have seen above, in the symmetric case both firms benefit from quantity discounts if the products are weak substitutes. In the asymmetric case, the picture is quite different. Figures 3 and 4 depict the region of parameter values where the profit of the more and the less efficient firm, respectively, are lower with quantity discounts. It appears that the more efficient firm is much more likely to benefit from quantity discounts than the less efficient one.

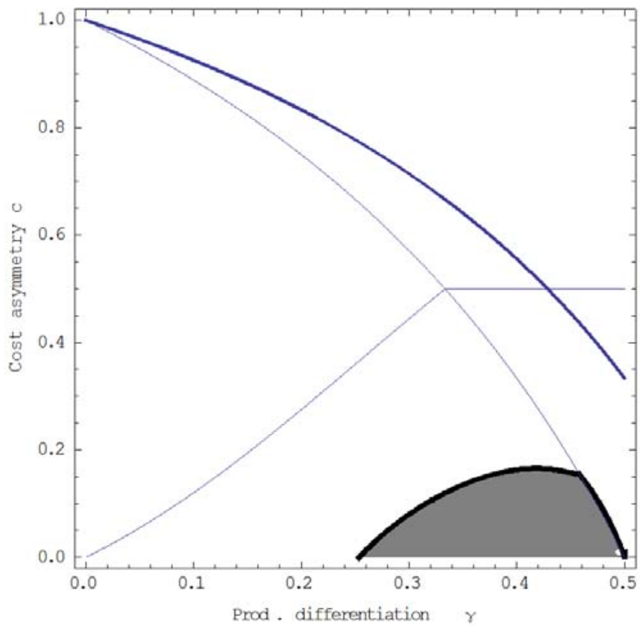

Figure 3: Quantity discounts decrease the profit of the more efficient firm in the gray region.

As in the symmetric case, an increase in the degree of substitutability γ reduces the likelihood that firms may gain from quantity discounts. An increase in the degree of asymmetry c, by contrast, increases the likelihood that quantity discounts benefit the more efficient firm and harm the less efficient one. Unless firms are almost symmetric, or products almost independent, quantity discounts harm the less efficient firm.

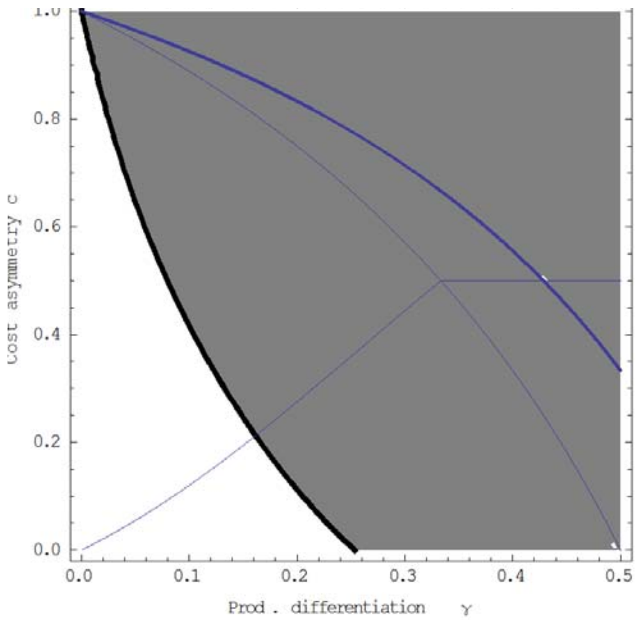

Figure 4: Quantity discounts decrease, or do not affect, the less efficient firm's profit in the gray region.

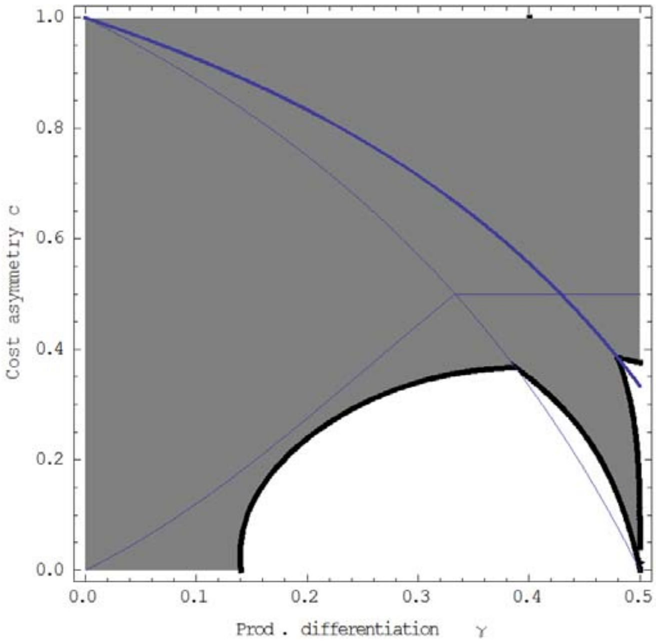

Figure 5: Quantity discounts decrease consumer surplus in the gray region.

5.2.3 Consumer surplus

The effect of quantity discounts on consumer surplus is illustrated in Figure 5. An increase in the degree of asymmetry increases the likelihood of a negative effect. Consumers may benefit from quantity discounts only if the degree of asymmetry is low and the products are close substitutes.

5.2.4 The Carlton and Waldman test

Though the maximisation of consumer surplus is sometimes advocated as the proper goal of competition policy, this criterion may be misleading in price discrimination cases. Carlton and Waldman (2008) argue that

> [...] an antitrust claim involving exclusion requires that there be (i) harm to a rival, (ii) harm to consumers and (iii) a linkage between the harm to the rival and the harm to consumers. For example, a monopolist who switches from simple monopoly pricing to discriminatory pricing may harm consumers but because no rival is affected should not (and is not) regarded as violating the antitrust laws. (Carlton and Waldman, 2008, p. 1 [Roman numbering added])

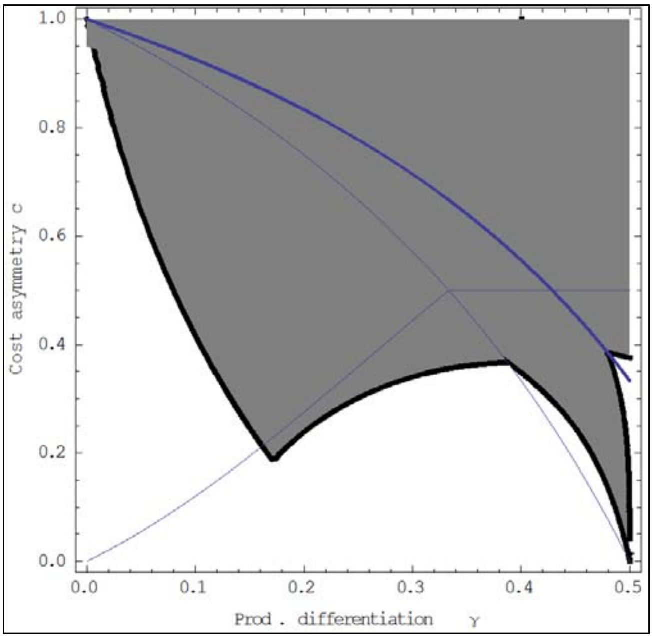

Figure 6: Conditions (i) and (ii) of the Carlton and Waldman test are satisfied in the grey region.

Figure 6 represents the region where conditions (i) and (ii) of the Carlton and Waldman test are both met. If condition (iii) is also satisfied (which, however, should not be taken for granted), the Carlton and Waldman test seems to imply that quantity discounts may be anti-competitive in a sizeable region of the parameter space.

5.2.5 Social welfare

Consider, finally, the classic criterion of social welfare maximisation, where social welfare is defined as the sum of producers' and consumers' surplus. Unlike the symmetric case, there does exist a region where quantity discounts decrease social welfare – the grey area in Figure 7. However, social welfare can decrease only for intermediate values of c and γ, so identifying the circumstances in which a ban is desirable may be a formidable task in practice.

The reason why prohibiting quantity discounts cannot increase social welfare for "extreme" values of the parameters is easy to grasp. When c is low, firms are almost symmetric and hence the results for the symmetric case apply. When c is high, the less efficient firm exerts little competitive pressure and so the market resembles a monopoly, where quantity discounts are welfare increasing. When γ is close to zero, the products are almost independent. There are effectively two separate monopolies, so quantity discounts are again welfare increasing. When γ is large, the products are close substitutes, and the positive "competition-

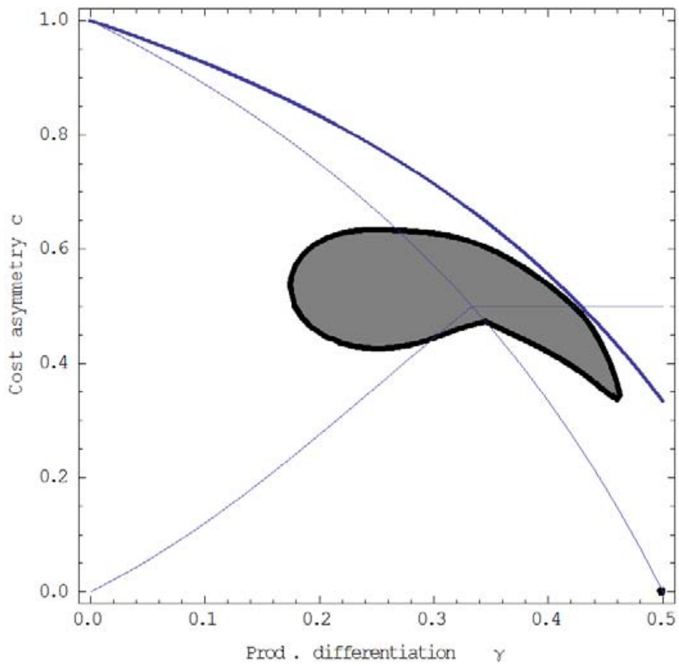

Figure 7: Quantity discounts decrease social welfare in the grey region.

enhancing" effect of quantity discounts prevails.

For intermediate values of c and γ, the picture is less clear. To illustrate how quantity discounts can decrease welfare, consider for instance the point $c = \frac{1}{2}$ and $\gamma = \frac{1}{4}$, where social welfare is higher with linear prices. With non-linear prices, firm B's output is zero and firm A behaves as an unconstrained monopolist. With linear prices, by contrast, firm B is active. As we have seen above, the intuitive reason is that firm A prices above marginal cost and so high type consumers purchase an inefficiently low amount of product A. Although firm B is not very efficient, it exerts some competitive pressure on firm A. As a result, firm A lowers its price, and a much larger set of types than under non-linear prices is served. This increases consumer surplus significantly, leading to a welfare improvement.

6 A selective ban

So far we have compared the case in which both firms can engage in quantity discounts to the case where both are constrained to use linear prices. But competition policy can regulate the unilateral behavior of dominant firms only. In the legal jargon, the less efficient firm, which cannot possibly have a dominant position, has no "special responsibility," and hence is always free to use quantity discounts. Thus, it is interesting to compare the case in which both firms can engage in quantity discounts to the case where only the dominant firm is constrained to use linear prices.

In this case, firm B is excluded when $c \geq c^{\ell}$, as in the case of linear prices. The argument is exactly the same and will not be repeated here. It is also obvious that if firm B is excluded, the equilibrium (where firm A is a *de facto* monopolist) is the same as under linear prices. We therefore focus on the case in which firm B is active.

Proposition 6 *With a selective ban, when firm B is active, i.e., if $c < c^{\ell}$, the following is an equilibrium. Firm A sets a linear price p_A^{**}, which can be calculated explicitly and is reported in the Appendix, and firm B offers the price schedule*

$$P_B^{**}(q_B) = \frac{1}{2}\left[1 + \frac{p_A^{**}\gamma - c(1-\gamma)}{1-\gamma}\right] q_B - \frac{1-2\gamma}{4(1-\gamma)} q_B^2. \quad \textit{for } 0 \leq q_B \leq 1.$$

Proof. See the Appendix.

Although firm B alone can now use quantity discounts, low demand consumers still purchase only product A, irrespective of the level of c. The intuition is that quantity discounts entail not only greater marginal prices, but also bigger quantity distortions for lower types. This means that the set of consumer types who will purchase exclusively from firm A will be even larger than in the fully non-linear pricing equilibrium, all else equal.

With a selective ban firm B still charges no fixed fee, but the no-distortion-at-the-top property no longer holds. The intuitive reason is that firm A now prices above marginal costs, so consumer $\theta = 1$ purchases an inefficiently low amount of product A. Firm B does not further distort the consumption of consumer $\theta = 1$; that is, the marginal price faced by consumer $\theta = 1$ is equal to c. However, since this consumer buys an inefficiently low amount of product A and the products are substitutes, he will purchase an inefficiently large amount of product B.

The qualitative effects of a selective ban on producers' and consumers' surplus and social welfare are as follows. Firm A is almost always harmed by a selective ban: it can benefit only in a small subset of the (already small) region where non-linear pricing would generate a prisoners' dilemma for the firms. Firm B, by contrast, always benefits from a selective ban. There is also an increase in the region of the parameter space in which banning quantity discounts increases social welfare, as shown in Figure 8. However, the region is still relatively small and difficult to identify.

7 Conclusion

Our analysis has revealed that quantity discounts can have anti-competitive effects even if they are not part of a predatory strategy. This means that

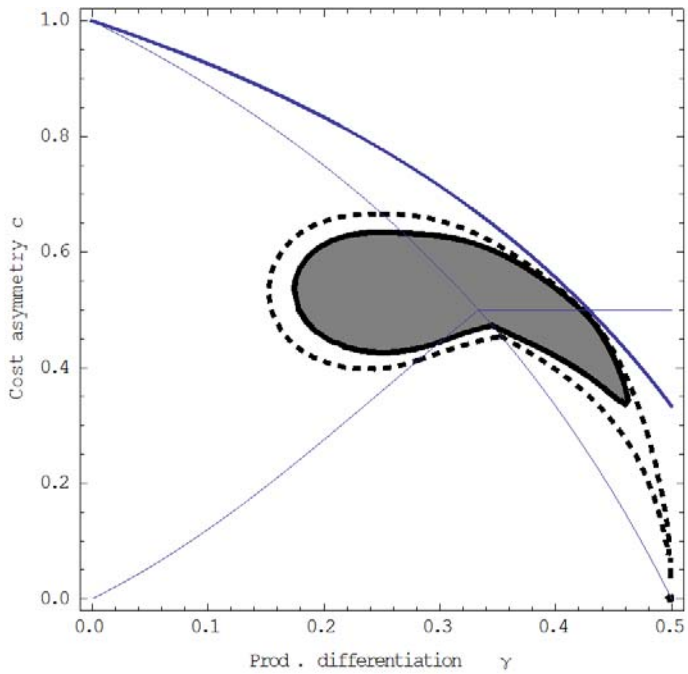

Figure 8: Quantity discounts by the inefficient firm only decrease social welfare in the dashed contour region.

competition policy should not necessarily apply to quantity discount cases the standards required in predatory pricing cases.

Having said this, we must stress that even though we have used a fully specified model, most of our welfare results are ambiguous. In particular:

(a) Quantity discounts facilitate the exclusion of the less efficient firm when the products are close substitutes. However, in the absence of fixed costs the less efficient firm is excluded only when exclusion is, indeed, socially efficient.

(b) Quantity discounts decrease consumer surplus when firms are highly asymmetric *or* the products are weak substitutes.

(c) Using the Carlton and Waldman test (according to which exclusionary abuses require (i) harm to a rival, (ii) harm to consumers and (iii) a linkage between the two), quantity discounts should be prohibited when firms are highly asymmetric *and* the products are fairly close substitutes.

(d) Quantity discounts decrease social welfare (i.e., the sum of producers' and consumers' surplus) for certain intermediate values of the degree of asymmetry and the degree of product substitutability. However, identifying the circumstances in which banning quantity discounts increases social welfare can be an exceedingly difficult task for antitrust authorities.

Ultimately, the question of whether, and in what circumstances, quantity discounts should be considered abusive rests on the welfare criterion adopted by antitrust authorities. Given all this uncertainty, abstaining from intervention may well be the optimal policy after all.

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Appendix

Proof of Proposition 1.

Consumer θ's demand for product i is

$$q_i = \frac{\gamma p_j - (1-\gamma)p_i}{1-2\gamma} + \theta$$

if $q_j > 0$, and

$$q_i = \frac{\theta - p_i}{1-\gamma}$$

if $q_j = 0$.

Next we aggregate the individual demands into the total demand functions. To proceed, we assume that $p_B \geq p_A$ as firm B has a greater unit cost than firm A. Later, we verify that this inequality holds in equilibrium.

When $p_B \geq p_A > 0$, there are three groups of consumers: low-θ types, $\theta \in [0, p_A]$, who do not buy any product; intermediate-θ types, $\theta \in [p_A, \hat{\theta}^{\ell}]$, who buy only product q_A; and high-θ types, $\theta \in [\hat{\theta}^{\ell}, 1]$, who buy both products. The critical threshold $\hat{\theta}^{\ell}$ is implicitly defined by the condition $q_B = 0$, which gives:

$$\hat{\theta}^{\ell} = \frac{(1-\gamma)p_B - \gamma p_A}{1-2\gamma}$$

Summing across consumers one obtains the total demand for the two products:

$$\begin{aligned}
Q_A &= \int_{p_A}^{\hat{\theta}^{\ell}} \frac{\theta - p_A}{1-\gamma} d\theta + \int_{\hat{\theta}^{\ell}}^{1} \left[\theta + \frac{\gamma p_B - (1-\gamma)p_A}{1-2\gamma}\right] d\theta \\
&= \frac{(1-\gamma)(p_B - p_A)^2 + [1 - 2\gamma - (1-\gamma)p_B + gp_A]\,[1 - 2\gamma + (1+\gamma)p_B + \gamma p_A]}{2(1-2\gamma)^2}
\end{aligned}$$

$$\begin{aligned}
Q_B &= \int_{\hat{\theta}^{\ell}}^{1} \left[\theta + \frac{\gamma p_A - (1-\gamma)p_B}{1-2\gamma}\right] d\theta \\
&= \frac{[1 - 2\gamma - (1-\gamma)p_B + \gamma p_A]^2}{2(1-2\gamma)^2}
\end{aligned}$$

Firms' profits are $\pi_A = p_A Q_A$ and $\pi_B = (p_B - c)\, Q_B$, respectively.

The best response function of firm B is

$$p_B = \frac{1 - 2\gamma + 2c(1-\gamma) + \gamma p_A}{3(1-\gamma)}.$$

This has a positive intercept and is increasing, with a slope equal to $\frac{\gamma}{3(1-\gamma)} < 1$. The best response function of firm A is

$$p_A = \frac{2[1-(3-\gamma)\gamma] + 2\gamma^2(1-p_B) - \sqrt{\Xi}}{3[1-(3-\gamma)\gamma]}$$

where

$$\Xi = 1+\gamma\left\{-3+\gamma+4\gamma^3+2(1-2\gamma)\left[-3+\gamma(5+\gamma)\right]p_B + \left[3-12\gamma+\gamma^2(12+\gamma)\right]p_B^2\right\}.$$

A's best response function has a positive intercept too. It is always increasing, with a slope lower than one.

Equilibrium prices are

$$p_A^* = \frac{2(3-\gamma)(1-2\gamma)(3-5\gamma)+10\gamma^2(1-\gamma)c - 3\sqrt{\Omega}}{(3-4\gamma)[9-4(6-\gamma)\gamma]},$$

where

$$\begin{aligned}\Omega \quad \equiv \quad & \{3-2\gamma\left[5+\gamma(2\gamma-5)\right]\}^2 - 8(1-\gamma)\gamma(1-2\gamma)\{3-\gamma\left[7-\gamma(2+\gamma)\right]\}c + \\ & +4(1-\gamma)^2\gamma\{3-\gamma\left[12-\gamma(12+\gamma)\right]\}c^2,\end{aligned}$$

and

$$p_B^* = \frac{1-2\gamma+2c(1-\gamma)+\gamma p_A^*}{3(1-\gamma)}.$$

The calculations leading to these results are reported in the Web Appendix, which also shows that $p_B^* \geq p_A^*$.

The above analysis assumes that firm B is active. It is immediate to check that the optimal linear price for firm A when it is a pure monopolist is $p_M = \frac{1}{3}$. Now consider firm B's best response to $p_A = \frac{1}{3}$. This is

$$p_B = \frac{5+6c-\frac{2}{(1-\gamma)}}{9}.$$

It is easy to verify that $p_B - c \geq 0$ iff $c \leq c^\ell$. This means that when $c < c^\ell$ firm B is active, whereas when $c \geq c^\ell$ firm A can engage in monopoly pricing. In other words, firm A's linear limit price coincides with the linear monopoly price. ■

Proof of Proposition 2.

We start by reporting the equilibrium value of the coefficients $\alpha_{0,A}^d$, $\alpha_{0,A}^{\lim}$, $\hat{q}_A$ and $\tilde{q}_A$. As stated in the proposition, these must guarantee that the smooth pasting conditions hold, that is, that A's price schedule be continuous and continuously differentiable at $\hat{q}_A$ and $\tilde{q}_A$. Continuity of firm A's price schedule at $\hat{q}_A$ and $\tilde{q}_A$ requires

$$\alpha_{0,A}^d + \alpha_{1,A}^*\hat{q}_A - \frac{\alpha_{1,A}^*}{2}\hat{q}_A^2 = \alpha_{0,A}^{\lim} + \alpha_{1,B}^*\hat{q}_A - \left(\frac{1}{2}-\gamma\right)\hat{q}_A^2$$

and

$$\alpha_{0,A}^{\lim} + \alpha_{1,B}^{*}\tilde{q}_A - \left(\frac{1}{2} - \gamma\right)\tilde{q}_A^2 = \frac{1}{2}\tilde{q}_A - \frac{1-\gamma}{4}\tilde{q}_A^2,$$

respectively. Differentiability of A's price schedule is equivalent to continuity of the equilibrium quantity, which requires

$$\hat{q}_A = q_A^{d*}(\hat{\theta}) = q_A^{\lim}(\hat{\theta})$$

and

$$\tilde{q}_A = q_A^{\lim}(\tilde{\theta}) = q_A^{m}(\tilde{\theta}).$$

When $c > \tilde{c}$, where

$$\tilde{c} \equiv \frac{1}{2}\frac{(1-2\alpha_1^*)(1-2\gamma)}{1-\alpha_1^*(1-\gamma)-2\gamma},$$

the solution to this system of equations is

$$\alpha_{0,A}^{\lim\,*} = -\frac{[(1-2\gamma)(1-2c-2\alpha_1^*)+2\alpha_1^* c(1-\gamma)]^2}{4(1-2\gamma)^2(1-3\gamma)} < 0,$$

$$\alpha_{0,A}^{d*} = \alpha_{0,A}^{\lim} - \frac{c^2(1-2\gamma-\alpha_1^*)}{2(1-2\gamma)^2} < 0,$$

$$\hat{q}_A^* = \frac{c}{1-2\gamma} > 0$$

and

$$\tilde{q}_A^* = \frac{2\alpha_1^* c}{1-2\gamma} - \frac{(1-2\alpha_1^*)(1-2c)}{1-3\gamma} > 0$$

When instead $c \leq \tilde{c}$, firm B's cost is so small that the monopoly region vanishes, $\tilde{q}_A^* = 0$, and firm A is always constrained by its competitor. The fixed fee $\alpha_{0,A}^{\lim}$ also vanishes, so that the no-fixed-fee property $P_A^*(0) = 0$ continues to hold, and we have

$$\alpha_{0,A}^{d*} = -\frac{c^2(1-2\gamma-\alpha_1^*)}{2(1-2\gamma)^2}.$$

The cutoff quantity is $\hat{q}_A^*$ is the same as above.

Next we show that the price schedules specified in the proposition satisfy the best response property in the set of all feasible price schedules (that is, not necessarily piecewise linear-quadratic). The verification procedure is as follows. Given P_j^*, firm i faces a monopolistic screening problem where type θ has an indirect utility function

$$v_i^*(q_i, \theta) = \max_{q_j \geq 0}\left[u(q_i, q_j, \theta) - P_j^*(q_j)\right],$$

which accounts for any benefit he can obtain by optimally trading with firm j. Since u is quadratic and P_j^* is piecewise quadratic, v_i^* is also piecewise quadratic. It may have kinks, but we shall show that any such kink preserves concavity, so the indirect utility function is globally concave.

However, the consumer's reservation utility $v_i^*(0,\theta)$ is now type dependent. Thus, in order to apply the standard approach of pointwise maximisation of the virtual surplus function, we will have to check not only that the sorting condition is satisfied, but also that the equilibrium rent increases with θ more quickly than $v_i^*(0,\theta)$ so that the consumer's participation constraint $v_i^*(q_i,\theta) \geq v_i^*(0,\theta)$ binds only at the lowest participating type $\hat{\theta}_i$ (see Jullien, 2000).

This latter property, however, is implied by the sorting condition. To show this, notice that when the indirect utility function is differentiable, the sorting condition is satisfied if the following condition holds

$$\int_0^{q_i} \frac{\partial^2 v_i^*(u,\theta)}{\partial\theta\partial q_i} du \geq 0.$$

This is equivalent to

$$\frac{\partial v_i^*(q_i,\theta)}{\partial\theta} \geq \frac{\partial v_i^*(0,\theta)}{\partial\theta}.$$

But this inequality implies that if $v_i^*(q_i,\theta_i) \geq v_i^*(0,\theta_i)$ then the participation constraint is, indeed, satisfied for any $\theta \geq \hat{\theta}_i$.

Thus, provided that the sorting condition holds, firm i's problem reduces to finding a function $q_i^+(\theta)$ that pointwise maximises the "indirect virtual surplus"

$$s_i(q_i,\theta) = v_i^*(q_i,\theta) - c_i q_i - (1-\theta)\frac{dv_i^*}{d\theta}.$$

We check ex-post, as usual, that the maximiser $q_i(\theta)$ satisfies the standard monotonicity condition. The verification of the best response property is completed by checking that $q_i^+(\theta) = q_i^*(\theta)$.

Consider, then, firm A's best response to the equilibrium price schedule of firm B, $P_B^*(q_B)$. The indirect utility function is piecewise quadratic, with two branches corresponding to the case in which the $\arg\max_{q_B \geq 0}\left[u(q_A,q_B,\theta) - P_B^*(q_B)\right]$ is 0 or is strictly positive, and a kink between the two branches:

$$v_A^*(q_A,\theta) = \begin{cases} \theta q_A - \frac{1-\gamma}{2}q_A^2 & \text{if } q_B = 0 \text{ or, equivalently, } q_A \geq q_A^{\lim}(\theta) \\ A_0 + A_1 q_A + A_2 q_A^2 & \text{if } q_B > 0 \text{ or, equivalently, } q_A < q_A^{\lim}(\theta). \end{cases}$$

The coefficients A_0, A_1 and A_2 can be calculated as

$$\begin{aligned} A_0 &= \frac{\left[(\theta-c)(1-2\gamma) - \alpha_1^*(1-c(1-\gamma)-2\gamma)\right]^2}{2(1-\gamma-\alpha_1^*)(1-2\gamma)^2}, \\ A_1 &= \gamma\frac{c(1-2\gamma)+\alpha_1^*(1-c(1-\gamma)-2\gamma)}{(1-\gamma-\alpha_1^*)(1-2\gamma)} + \theta\frac{1-2\gamma-\alpha_1^*}{1-\gamma-\alpha_1^*} \\ A_2 &= -\frac{1-2\gamma+\alpha_1^*(1-\gamma)}{2(1-\gamma-\alpha_1^*)} < 0. \end{aligned}$$

On both branches of the indirect utility function, the coefficients of the quadratic terms are negative. In addition, it can be checked that

$$\left.\frac{\partial^2 v_A^*(q_A,\theta)}{\partial q_A^2}\right|_{q_A \leq q_A^{\lim}(\theta)} = A_2 \geq \left.\frac{\partial^2 v_A^*(q_A,\theta)}{\partial q_A^2}\right|_{q_A > q_A^{\lim}(\theta)} = -(1-\gamma),$$

so the function v_A^* is globally concave. It can also be checked that the sorting condition is satisfied since

$$\frac{\partial^2 v_A^*}{\partial\theta\partial q_A} = \begin{cases} 1 & \text{if } q_A \geq q_A^{\lim}(\theta) \\ \dfrac{1-2\gamma-\alpha_1^*}{1-\gamma-\alpha_1^*} > 0 & \text{if } q_A < q_A^{\lim}(\theta). \end{cases}$$

We can therefore obtain A's best response by maximising the virtual surplus function $s_A(q_A, \theta)$. Like the indirect utility function, the virtual surplus function is piecewise quadratic with a kink. The maximum can occur in either one of the two quadratic branches, or at the kink. To be precise:

$$q_A^+(\theta) = \begin{cases} \dfrac{2\theta-1}{1-\gamma} & \text{if } \gamma < \frac{1}{3} \text{ and } \frac{1}{2} \leq \theta \leq \theta_1 \\ q_A^{\lim}(\theta) & \text{if } \gamma < \frac{1}{3} \text{ and } \theta_1 \leq \theta \leq \theta_2 \text{ or if } \gamma \geq \frac{1}{3} \text{ and } \alpha_{1,B}^* \leq \theta \leq \theta_2 \\ \dfrac{\theta-\alpha_1^*}{1-\alpha_1^*} + \dfrac{c\gamma}{1-2\gamma} & \text{if } \theta \geq \theta_2, \end{cases}$$

where

$$q_A^{\lim}(\theta) \equiv \frac{\theta-\alpha_1^*}{\gamma} + c\frac{\alpha_1^*\gamma - (1-\alpha_1^*)(1-2\gamma)}{(1-2\gamma)\gamma},$$

$$\theta_1 = \frac{\alpha_1^*(1-\gamma)-\gamma}{1-3\gamma} + c(1-\gamma)\frac{1-\alpha_1^*(1-\gamma)-2\gamma}{1+\gamma(6\gamma-5)} (= \tilde{\theta}^*)$$

and

$$\theta_2 = \frac{c(1-\gamma)+\alpha_1^*(1-2\gamma-c(1-\gamma))}{1-2\gamma} (= \hat{\theta}^*).$$

When $\gamma \geq \frac{1}{3}$ or $\gamma < \frac{1}{3}$ and $\theta_1 < \frac{1}{2}$, which happens for $c < \tilde{c}$, the optimum is never achieved on the upper branch of the indirect utility function. In other words, firm A's best response never involves setting the quantity at the monopoly level.

Finally, one can easily check that $q_A^+(\theta) = q_A^*(\theta)$, which implies that in order to implement the quantities $q_A^+(\theta)$ firm A must, indeed, offer the equilibrium price schedule $P_A^*(q_A)$.

Consider now firm B. The procedure is the same as form firm A, but now we must distinguish between two cases, depending on whether A's price schedule comprises also the lower (monopoly) branch or not. Consider first the case in which $\tilde{q}_A^* = 0$ and hence there is no monopoly branch of A's price schedule.

The indirect utility function of a consumer who trades with firm B then is

$$v_B^*(q_B, \theta) = \begin{cases} \theta q_B - \frac{1-\gamma}{2} q_B^2 & \text{if } q_A = 0 \text{ or, equivalently, if } q_B \geq q_B^{\lim}(\theta) \\ \hat{B}_0 + \hat{B}_1 q_B + \hat{B}_2 q_B^2 & \text{if } 0 < q_A \leq \hat{q}_A \text{ or, equivalently, if } \check{q}_B(\theta) \leq q_B < q_B^{\lim}(\theta) \\ B_0 + B_1 q_B + B_2 q_B^2 & \text{if } q_A > \hat{q}_A \text{ or, equivalently, if } 0 < q_B \leq \check{q}_B(\theta) \end{cases}$$

where

$$q_B^{\lim}(\theta) = \frac{\theta-\alpha_1^*}{\gamma} - \frac{\alpha_1^* c}{1-2\gamma}$$

$$\check{q}_B(\theta) = \frac{\theta-\alpha_1^*-c(1-\alpha_1^*)}{\gamma} + \frac{\alpha_1^* c}{1-2\gamma}.$$

The coefficients of the lower branches of the indirect utility functions are

$$\begin{aligned} \hat{B}_0 &= \frac{(\theta - c)^2}{2\gamma} \\ \hat{B}_1 &= c \\ \hat{B}_2 &= -\frac{1 - 2\gamma}{2} \end{aligned}$$

and

$$\begin{aligned} B_0 &= \frac{2\theta - 1}{2(1-\gamma)} \\ B_1 &= \theta - \frac{\gamma}{1-\gamma} \\ B_2 &= -\frac{1-\gamma}{2}. \end{aligned}$$

All branches are concave, and global concavity can be checked by comparing the left and right derivatives of $v_B^*(q_B, \theta)$ at the kinks. The sorting condition can also be checked as for firm A. We can therefore find B's best response by pointwise maximisation of the virtual surplus function.

One can easily check that there is never an interior maximum on the upper or intermediate branch of the virtual surplus function. This is equivalent to saying that firm B is active only when firm A supplies the duopoly quantity $q_A^{d*}(\theta)$. Pointwise maximisation of the virtual surplus function leads to

$$q_B^+(\theta) = \frac{\theta - \alpha_1^*}{1 - \alpha_1^*} - c\frac{1-\gamma}{1-2\gamma}.$$

This coincides with $q_B^*(\theta)$, thereby confirming that $P_B^*(q_B)$ is the best response.

The case where firm A's price schedule comprises also the lower (monopoly) branch is similar. The indirect utility function $v_B^*(q_B, \theta)$, and hence the virtual surplus $s_B(q_B, \theta)$, now comprise four branches (all quadratic). The equation of the fourth branch, which corresponds to $0 < q_A < \tilde{q}_A^*$, is

$$v_B^*(q_B, \theta) = \tilde{B}_0 + \tilde{B}_1 q_B + \tilde{B}_2 q_B^2$$

where

$$\begin{aligned} \tilde{B}_0 &= \frac{(2\theta - 1)^2}{4(1-\gamma)} \\ \tilde{B}_1 &= \frac{\theta + \gamma(1 - 3\gamma)}{1-\gamma} \\ \tilde{B}_2 &= -\frac{1 - \gamma(2+\gamma)}{2(1-\gamma)}. \end{aligned}$$

However, it turns out that the optimum still lies on the lower branch where $q_A > \hat{q}_A$ and that it entails $q_B^+(\theta) = q_B^*(\theta)$. This observation completes the proof of Proposition 2. ■

Proof of Proposition 3.

Like in the proof of Proposition 2, we start by determining the equilibrium values of the coefficients $\alpha_{0,A}^m$ and $\tilde{q}_A$. They are determined by a smooth pasting condition that imposes the continuity and differentiability of the price schedule at $\tilde{q}_A$. Continuity requires that

$$c\tilde{q}_A - \left(\frac{1}{2} - \gamma\right)\tilde{q}_A^2 = \alpha_{0,A}^m + \frac{1}{2}\tilde{q}_A - \frac{1-\gamma}{4}\tilde{q}_A^2,$$

whereas differentiability, which is equivalent to the continuity of the equilibrium quantity, requires

$$\tilde{q}_A = q_A^{\lim}(\tilde{\theta}) = q_A^m(\tilde{\theta}),$$

where now

$$q_A^{\lim}(\theta) = \frac{\theta - c}{\gamma}.$$

The solution is

$$\tilde{q}_A^* = \frac{2c-1}{1-3\gamma} > 0$$

and

$$\alpha_{0,A}^{m*} = \frac{(2c-1)^2}{4(1-3\gamma)} > 0$$

when $c < \frac{1}{2}$, and $\tilde{q}_A^* = \alpha_{0,A}^{m*} = 0$ when $c \geq \frac{1}{2}$.

The rest of proof is similar to the proof of Proposition 2. Now, however, it suffices to find firm A's best response to $P_B^*(q_B) = cq_B$. The indirect utility function is:

$$v_A^*(q_A,\theta) = \begin{cases} \theta q_A - \frac{1-\gamma}{2}q_A^2 & \text{if } q_B = 0 \text{ or, equivalently, } q_A \geq q_A^{\lim}(\theta) \\ A_0 + A_1 q_A + A_2 q_A^2 & \text{if } q_B > 0 \text{ or, equivalently, } q_A < q_A^{\lim}(\theta), \end{cases}$$

where now

$$\begin{aligned} A_0 &= \frac{(\theta - c)^2}{2(1-\gamma)} \\ A_1 &= \frac{\theta(1-2\gamma) + c\gamma}{1-\gamma} \\ A_2 &= -\frac{1-2\gamma}{2(1-\gamma)}. \end{aligned}$$

On both branches of the indirect utility function, the coefficients of the quadratic terms are negative. In addition, it can be checked that

$$\left.\frac{\partial^2 v_A^*(q_A,\theta)}{\partial q_A^2}\right|_{q_A \leq q_A^{\lim}(\theta)} = A_2 \geq \left.\frac{\partial^2 v_A^*(q_A,\theta)}{\partial q_A^2}\right|_{q_A > q_A^{\lim}(\theta)} = -(1-\gamma),$$

so the function v_A^* is globally concave. It can also be checked that the sorting condition is satisfied since

$$\frac{\partial^2 v_A^*}{\partial\theta\partial q_A} = \begin{cases} 1 & \text{if } q_A \geq q_A^{\lim}(\theta) \\ \frac{1-2\gamma}{1-\gamma} > 0 & \text{if } q_A < q_A^{\lim}(\theta). \end{cases}$$

We can therefore obtain A's best response by maximising the virtual surplus function $s_A(q_A,\theta)$. Like the indirect utility function, the virtual surplus function is piecewise quadratic with a kink. The maximum can occur on the upper branch, or at the kink. It can never occur on the lower branch, where

$$q_A < q_A^{\lim}(\theta) = \frac{\theta - c}{\gamma}.$$

To show this, notice that the optimum on the upper branch is

$$q_A^-(\theta) = 2\left[2\theta - 1 + c\gamma\left(1 - 2\gamma\right)\right].$$

However, it can easily be checked that $q_A^-(\theta) > \frac{\theta-c}{\gamma}$, a contradiction.

When $\gamma < \frac{1}{3}$ or $\gamma \geq \frac{1}{3}$ and $c > \frac{1}{2}$, the maximisation of the virtual surplus leads to

$$q_A^+(\theta) = \frac{2\theta - 1}{1-\gamma}(\equiv q_A^m(\theta)).$$

When instead $\gamma \geq \frac{1}{3}$ and $c \leq \frac{1}{2}$, we have

$$\begin{array}{ll} q_A^+(\theta) = q_A^{\lim}(\theta) & \text{if } c \leq \theta \leq \theta_3 \\ q_A^+(\theta) = q_A^m(\theta) & \text{if } \theta_3 \leq \theta \leq 1 \end{array}$$

where

$$\theta_3 = \frac{\gamma - c(1-\gamma)}{3\gamma - 1}(= \tilde{\theta}^*).$$

Finally, one can easily check that $q_A^+(\theta) = q_A^*(\theta)$, which implies that in order to implement the quantities $q_A^+(\theta)$ firm A must, indeed, offer the equilibrium price schedule $P_A^*(q_A)$.

To complete the proof, it suffices to check that $q_A^+(\theta) \geq q_A^{\lim}(\theta)$, which implies that the demand for product B when $p_B = c$ vanishes. ■

Proof of Proposition 6.

To calculate the equilibrium with a selective ban, consider firm B's best response function first. For any given p_A, define the indirect utility function as

$$v(q_B,\theta) = \max_{q_A \geq 0}\left[u - p_A q_A\right]$$

This indirect utility function accounts for any benefit the consumer can obtain by purchasing product A at the constant price p_A. The maximum is achieved at

$$q_A = \max\left\{\frac{\theta - p_A - \gamma q_B}{1-\gamma}, 0\right\}.$$

As discussed in the main text, in equilibrium firm B will serve only consumers who purchase positive amounts of product A. Thus, there is no loss of generality in focusing on the solution where $q_A > 0$, where the indirect utility function becomes

$$v(q_B,\theta) = \frac{\theta^2 + (2\theta - p_A - q_B)\left[(1-2\gamma)q_B - p_A\right]}{2(1-\gamma)}.$$

It is easy to verify that the indirect utility function satisfies the sorting condition $\frac{\partial^2 v}{\partial\theta\partial q_B} > 0$, and that the participation constraints binds for the lowest participating type. (Although there is a type-dependent reservation utility $\frac{\theta^2 - p_A(2\theta - p_A)}{2(1-\gamma)}$, this does not affect the solution.) Thus, firm B's best response can be determined as the solution to a well-behaved monopolistic screening problem. Define the virtual surplus function as

$$\begin{aligned} s(q_B, \theta) &= v(q_B, \theta) - (1-\theta)\frac{\partial v(q_B, \theta)}{\partial \theta} - cq_B \\ &= \frac{3\theta^2 - 2\theta + (4\theta - p_A - q_B - 2)\left[(1-2\gamma)q_B - p_A\right]}{2(1-\gamma)} - cq_B. \end{aligned}$$

The optimal quantity $q_B^+(\theta)$ is found by pointwise maximisation of the virtual surplus function, yielding

$$q_B(\theta) = 2\theta - 1 + \frac{p_A\gamma - c(1-\gamma)}{1-2\gamma}.$$

One can finally calculate the price schedule that supports the quantity $q_B^+(\theta)$, given that firm A prices at p_A:

$$P_B(q_B) = \frac{1 - 2\gamma + c(1-\gamma) + \gamma p_A}{2(1-\gamma)} q_B - \frac{1-2\gamma}{4(1-\gamma)} q_B^2.$$

This if firm B's best response function. Notice that the coefficient of the quadratic term is independent of p_A. Thus, the pricing game is effectively equivalent to a standard Bertrand game where firms choose linear prices, with the twist that firm B is actually choosing the coefficient of the linear term of a quadratic pricing function

$$P_B(q_B) = p_B q_B - \frac{1-2\gamma}{4(1-\gamma)} q_B^2.$$

In this pricing game, firm B's best response function is

$$p_B = \frac{1 - 2\gamma + c(1-\gamma) + \gamma p_A}{2(1-\gamma)},$$

as we have just seen. When firm B offers a non linear price $p_B q_B - \beta_2 q_B^2$ where $\beta_2 = \frac{1-2\gamma}{4(1-\gamma)}$ and firm A offers a linear price p_A, firm A's quantity is

$$Q_A = \int_{p_A}^{\hat{\theta}^{mix}} \frac{\theta - p_A}{1-\gamma} d\theta + \int_{\hat{\theta}^{mix}}^{1} \frac{\theta(1 - 2\gamma + 2\beta_2) - p_A(1 - \gamma + 2\beta_2) + \gamma p_B}{1 - 2\gamma + 2\beta_2(1-\gamma)} d\theta$$

where the first integral corresponds to types buying only from firm A, the second to types buying from both firms, and

$$\hat{\theta}^{mix} = \frac{(1-\gamma)p_B - \gamma p_A}{1-2\gamma}.$$

maximising $\pi_A = p_A Q_A$ with respect to p_A gives firm A's best response:

$$p_A = \frac{2[1-3\gamma+2\beta_2(1-2\gamma)]+2\gamma^2(2-p_B)-1/2\sqrt{\Psi}}{6\beta_2(1-2\gamma)+3[1-(3-\gamma)\gamma]}$$

where

$$\begin{aligned}\Psi &= 12[(3-\gamma)\gamma-2\beta_2(1-2\gamma)-1][1+2\beta_2-(4-2p_B+p_B^2+4\beta_2)\gamma+(2-p_B)^2\gamma^2]+ \\ &\quad +16[1-2\beta_2(1-2\gamma)-\gamma(3+(2-p_B)\gamma)]^2.\end{aligned}$$

The equilibrium price of firm A is (see the Web Appendix for details):

$$p_A^{**} = \frac{4-\gamma\left\{16-\gamma\left[19+3c(1-\gamma)-6\gamma\right]\right\}-\sqrt{2}\sqrt{\Theta}}{6-\gamma\left[24(1-\gamma)+5\gamma^2\right]}$$

where

$$\begin{aligned}\Theta &= (1-\gamma)\{2-\gamma[2c(1-2\gamma)(3-6\gamma+2\gamma^3)-c^2(1-\gamma)(3+2\gamma(-6+\gamma(6+\gamma)))+ \\ &\quad +(1-\gamma)(-11+2\gamma(7+2\gamma(-5+2\gamma)))]\}.\end{aligned}$$

Substituting this expression into B's best response one gets the coefficient of the linear term in B's price schedule.

Finally, substituting the equilibrium prices into $\hat{\theta}^{mix}$, the Web Appendix verifies that $\hat{\theta}^{mix} < 1$, and hence firm B is active, if and only if $c < c^{\ell}$.■