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ABSTRACT

Competition Between Multiple Asymmetric Networks: Theory and Applications*

We present a tractable model of competition between an arbitrary number of interconnected communications networks in the presence of tariff-mediated network externalities, call externalities, and cost and market share asymmetries. On the theory side, we provide a criterion for stability in expectations and determine equilibrium outcomes in linear and two-part tariffs. As applications, we reconsider mobile termination for calls from the fixed network (FTM), and between mobile networks (MTM). We show that there is a partial FTM waterbed effect under linear tariffs, and that with more than two networks some known duopoly results are reversed: Under multi-part tariffs, consumer surplus may decrease (rather than increase), and under linear tariffs both on- and off-net prices may increase with higher MTM termination charges.

JEL Classification: D43, L13 and L51
Keywords: call externality, mobile termination rates, multiple networks, on/off-net pricing, telecommunications network competition, waterbed effect

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1 Introduction

Contribution. Two great obstacles of applying models of telecommunications competition to real-world markets are that most either assume symmetric networks and / or duopoly. To our knowledge, there are no realistic cases that can reasonably be portrayed as a symmetric duopoly, since most telecommunications markets are characterized either by at least three networks which have entered at different points in time, as in mobile telephony, or by one large incumbent and several smaller rivals using different technologies, as is usually the case in fixed telephony. One reason for the assumptions of symmetry and duopoly that is usually advanced is that models with several asymmetric networks are not tractable. Here we attempt to show otherwise, and then follow up on the implications.

While a series of recent papers has presented models of network competition with more than two networks, as listed below, all either have assumed symmetry or have not been able to give closed-form solutions for the equilibrium. In this paper we set out to develop and solve a rather general model of competition between interconnected telecommunications networks. As in Hoernig (2007) for the duopoly case, there are tariff-mediated network externalities, i.e. networks price discriminate between on- and off-net calls, and call externalities, i.e. receiving calls conveys utility, and networks can be asymmetric in size. Still, we go beyond the scope of that paper by allowing for an arbitrary number of networks and asymmetries in network and per-customer fixed costs. While being at the centre of the ongoing debate about the regulation of mobile termination rates (MTRs) in the European Union, cost differences seem to have been largely ignored in the economic literature on network competition, again due to the alleged difficulty of solving for the equilibrium outcomes.

Our model is set up such that it can easily be calibrated to real-world communications markets. This exercise is becoming ever more useful for academics and regulators, as the quality of the assessment of the impact of different regulatory options depends heavily on which features of the relevant market can be captured. In this case it is essential to represent the presence of varying numbers of networks of asymmetric size. A first practical application of our approach is Harbord and Hoernig (2010), which contains a study of different options for regulating mobile termination rates proposed by the UK communications regulator Ofcom, plus estimates of how the merger between two of the UK's five mobile networks (Orange and T-Mobile) might affect consumer surplus.

The paper has two principal parts. First we set out the theory, and then we consider applications to interconnection regulation. In the theory part, we show how to set up and solve network competition models with many asymmetric networks, both for competition in linear and multi-part tariffs. The model and most of the calculations are rendered in matrix notation, exploiting maximally the underlying linear structure which is inherited from the traditional Hotelling model. This vastly reduces the complexity of the derivations and leads to equilibrium conditions in the form of "one-liners".

As a first step, we propose a generalization of the condition of stability in expectations introduced by Laffont, Rey and Tirole (1998b, LRTb) to multiple networks. It imposes an upper limit on the intensity of preferences, as a function of tariff-mediated network externalities, the number of networks and characteristics of consumers' preference space. This stability condition is independent of whether networks compete in linear or multi-part tariffs.

We then derive the Nash equilibrium outcomes in the price competition games with linear and multi-part tariffs, respectively. As concerns off-net prices, we allow networks to set a "uniform off-net price", i.e. the same off-net price to all other networks, or to set different prices to groups of other networks. With linear tariffs, we show that the condition in Hoernig (2007) which links the level of the off-net price to the level of the on-net price in the case of two networks, continues to hold "on average" in the case of many networks. If there is a uniform off-net price to all competing networks then this price is set based on average perceived off-net cost. With multi-part tariffs, we show that identical off-net prices to a group of competing networks are set based on average perceived off-net cost, and as if all other competitors had the same average market share as the members of this group. Naturally, the on-net prices continue to be set at the efficient level independently of cost asymmetries and the number of networks.

Furthermore, we complement our main analysis by deriving market outcomes under uniform pricing, i.e. in the absence of tariff-mediated network externalities, and by considering how market outcomes evolve in the symmetric case as the number of networks becomes large. As concerns the latter, it becomes clear that these outcomes are strongly driven by assumptions about the size of the preference space.

In terms of applications, we explore the implications of our results for the effects of mobile termination rate levels. We first reconsider the "waterbed effect" in fixed-to-mobile (FTM) interconnection, i.e. the phenomenon where profits from fixed-to-mobile termination are handed on to consumers. For competition in multi-part tariffs, we show that the standard result from a symmetric Hotelling duopoly, which is that all termination profits are passed on to mobile subscribers (a "full waterbed effect"), continues to hold even in the presence of multiple asymmetric networks. On the other hand, we demonstrate that with linear tariffs this is no longer true: The waterbed is only partial, i.e. networks retain a part of the termination profits. The extent of this partial waterbed is shown to depend on how competitive the market is, as measured by the number of networks and the intensity of horizontal preferences.

Concerning mobile-to-mobile (MTM) termination rates, we reconsider the question of how the latter affect market equilibrium, and mobile subscribers in particular, for the case of many networks. Gans and King (2001) found for duopoly competition in multi-part tariffs that networks maximize joint profits by setting off-net prices below the efficient level and therefore MTRs below the true cost of termination. We show that as the number of networks increases, joint profit-maximizing off-net prices converge towards the efficient price. The corresponding MTRs only converge to termination cost in the absence of call externalities; otherwise they remain further below cost. More importantly, it also becomes clear from LRTb and Gans and King (2001) that in the duopoly case subscribers gain from higher MTM termination rates: The latter lead to larger off-net prices and therefore stronger tariff-mediated network effects, which in turn make networks compete harder for customers through lower fixed fees. We show that with more than two networks the effect on competitive intensity persists, but that on the other hand the overall effect on subscribers will be reversed: With more networks, a higher portion of calls is made off-net and subscribers will not obtain sufficient compensation for higher off-net prices through lower fixed fees if termination rates go up.

With linear tariffs, we show that with more than two networks a new effect can arise: While the direct cost effect of higher termination rates leads to increased off-net prices independently of the number of networks, and in the duopoly case the on-net price will decreases accordingly, in the presence of at least three networks the on-net price may go up as well. This is a direct result from the reduction in competitive intensity that higher off-net prices cause under linear tariffs.

Related literature. There is now a vast amount of work that has sprung from the seminal contributions of Armstrong (1998) and Laffont, Rey and Tirole (1998a,b). In the following we will mostly concentrate on the papers that consider price discrimination between on- and off-net prices, in the tradition of LRTb. See Laffont and Tirole (2000), Armstrong (2002) and Vogelsang (2003) for surveys about the literature on network competition.

Duopoly network competition in linear tariffs has been considered by Doganoglu and Tauman (2002), Berger (2004), de Bijl and Peitz (2004), Hoernig (2007), and Geoffron and Wang (2008). Duopoly equilibrium results under multi-part tariffs have been derived, among others, by Gans and King (2001), Peitz (2005), Berger (2005), and Hoernig (2007). Jullien et al. (2010) extended this literature to consumers with different subscription elasticities.

Call externalities and their effects have previously been considered in Kim and Lim (2001), Jeon *et al.* (2004), Hermalin and Katz (2004), Berger (2004, 2005), Hoernig (2007), and Armstrong and Wright (2007, 2009), and Cambini and Valletti (2008). Our modelling of asymmetries is related to that introduced by Carter and Wright (1999, 2003), and which has been taken up in de Bijl and Peitz (2004), Peitz (2005) and Hoernig (2007).[1]

Several papers on mobile-to-mobile interconnection have considered more than two competing networks, in different settings where all networks directly compete with each other. Symmetric networks are assumed by Calzada and Valletti (2008)[2] and Armstrong and Wright (2007). Dewenter and Haucap (2005) consider more than two asymmetric networks, but they take market shares as given and thus do not close the model. Closest to our paper is Thompson, Renard and Wright (2007), in using a similar demand specification and considering an arbitrary number of networks which can differ in subscription surplus. Yet, networks in their model do not price-discriminate between on- and off-net calls, and no closed-form solution for the equilibrium is derived.[3]

Gans and King (2000) analyze how mobile networks set fixed-to-mobile termination rates under customer ignorance about which mobile network is being called. Under the

[1] Cambini and Valletti (2004) and Valletti and Cambini (2005) present duopoly models where asymmetries arise through previous investment choices.

[2] Calzada and Valletti (2008) consider asymmetric calling patterns with three networks in an extension section.

[3] Other models of competition between multiple symmetric networks under non-discriminatory pricing are Liu and Sun (2008) with three networks, Jeon and Hurkens (2008), Stennek and Tangerås (2008) and Tangerås (2009). On the other hand, Hurkens and Jeon (2008) only consider two networks under termination-based price discrimination.

assumption that mobile networks' market shares are fixed they consider an arbitrary number of asymmetric networks. On the other hand, they assume symmetric duopoly when modelling competition in multi-part tariffs between networks. Wright (2002) considers the setting of fixed-to-mobile termination rates by an arbitrary number of competing symmetric mobile networks. While he abstracts from mobile-to-mobile calls and uses a more general formulation of subscription demand, his pricing structure is equivalent to multi-part tariffs with call prices set at cost. Thus his results can be compared to the ones derived in our framework. He shows that all profits from fixed-to-mobile termination are passed on to mobile customers, i.e. there is a full waterbed effect, if a common shift in the cost of signing up subscribers does not change equilibrium profits. This is the case for example in Hotelling models under full market coverage. The waterbed effect is less than full for example if the market is less than fully covered.[4] Armstrong (2002, section 3), elaborating on Armstrong (1997), models the setting of fixed-to-mobile termination rates by an indeterminate number of symmetric mobile networks under perfect competition.

This paper has the following structure: Section 2 presents the model, discusses stability in consumer expectations and derives socially optimal prices and market shares. Section 3 presents the Nash equilibrium solutions in linear and nonlinear tariffs, while Section 4 derives more results for the symmetric case. Sections 5 and 6 consider applications to fixed-to-mobile and mobile-to-mobile termination, while Section 7 concludes.

2 Theory: Model Setup

2.1 Demand, Market Shares and Consumer Surplus

The following model is a generalization of the network competition models of Armstrong (1998) and Laffont, Rey, Tirole (1998a,b) to many asymmetric networks. It leads to a demand formulation that is related to those of the "pyramid model" of von Ungern-Sternberg (1991), Armstrong and Wright (2007) and the "spokes model" by Chen and Riordan (2007),[5] but allows explicitly for exogenous asymmetries between networks. All networks directly compete against each other, which for more than three networks is different from the most widely used generalization of the Hotelling model to multiple firms, the circular city model of Salop (1979). The equilibrium concept we employ is Nash equilibrium of the pricing game between networks in either linear or multi-part tariffs with price discrimination between on- and off-net prices.

There are $n \geq 2$ networks, each at one of the n nodes of a complete graph which describes consumers' space of preferences. The total size of the preference space is $S(n)$, with the distance between two nodes being $l(n) = 2S(n)/n(n-1)$. When appropriate, we consider the size of this preference space as a function of the number of networks in the market. Indeed, as we will see below, the exact form of this dependence determines

[4] Genakos and Valletti (2009) perform a similar analysis assuming a logit demand structure.

[5] The pyramid and spokes models are built from specific graphical foundations, while Armstrong and Wright's demand formulation is presented *ad hoc*. Renard et al. (2007) use the same preference space but the rest of their model is very different, with heterogeneous consumers but restricted to uniform call prices.

stability in expectations and market outcomes for a large number of networks. On the other hand, we will drop the dependence on n when it is not needed. For now we only note that the corresponding implied size of the preference space in Armstrong and Wright (2007) is $S(n) = n - 1$ and $S(n) = n(n-1)/2$ in von Ungern-Sternberg (1991) and Chen and Riordan (2007). In Section 4 we will discuss the economic significance of this observation.

The total mass of consumers is normalized to 1, distributed uniformly over the whole preference space with density $1/S(n)$. Market shares are $\alpha_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$. All networks are interconnected, and all consumers subscribe to some network.

Similar to Carter and Wright (1999) and Hoernig (2007), a subscriber of network i receives a gross utility of $w_i + A_i$, where A_i is a fixed surplus from being connected to network i (which may derive from brand value, trust etc.), and w_i is the surplus arising from making calls, as defined below. Without loss of generality, we assume $A_1 \geq A_2 \geq ... \geq A_n$. While A_n must be large enough so that all consumers subscribe, we note that otherwise only the differences $A_i - A_j$ will matter. The disutility of not buying a perfect match is modelled in Hotelling fashion through a linear transport cost td, where $t > 0$ and d is the distance between the subscriber and his network. As $t \to \infty$ each network becomes a local monopoly, while for $t \to 0$ transport cost disappear and the market approaches perfect competition.[6]

Subscribers' utility from a call of length q is $u(q)$, with indirect utility $v(p) = \max_q u(q) - pq$ and demand function $q(p) = -v'(p)$. The price elasticity of demand is $\eta(p) = -pq'(p)/q(p)$. Network i charges a *multi-part* tariff consisting of a fixed charge F_i, and prices per minute of p_{ii} for on-net calls and p_{ij} for off-net calls to network $j \neq i$.[7] When considering *linear* tariffs we set $F_i = 0$. *Uniform* tariffs are obtained by restricting all p_{ij} to be equal to p_{ii}. Let v_{ij}, q_{ij}, u_{ij} be defined as $v(p_{ij})$, $q(p_{ij})$, $u(q_{ij})$. The utility of receiving a call of length q is $\gamma u(q)$, where $\gamma \in [0, 1)$. Assuming an *ex-ante* balanced calling pattern, i.e. each subscriber calls every other subscriber with the same probability,[8] the call surplus on network i is

$$w_i = \sum_{j=1}^{n} \alpha_j \left(v_{ij} + \gamma u_{ji}\right) - F_i = \sum_{j=1}^{n} \alpha_j h_{ij} - F_i.$$

Defining the $(n \times n)-$matrix $h = (h_{ij})_{ij}$ and the $(n \times 1)-$vectors $F = (F_i)_i$ and $\alpha = (\alpha_i)_i$, we can restate the above in matrix form as

$$w = h\alpha - F. \tag{1}$$

The matrix h is a function of prices, and therefore as we will see below at equilibrium prices indirectly a function of marginal costs, mobile termination rates and market shares (the latter only in the presence of call externalities, though).

[6] An equivalent alternative formulation of the model would involve a constant size S of the preference space and transport cost $t(n)$ varying with the number of networks.

[7] While we do not consider calls to the fixed network here, these can be easily implemented in this framework, see for example Harbord and Hoernig (2010).

[8] See Hoernig, Inderst and Valletti (2010) for a general model of unbalanced calling patterns and their implications for the pricing equilibrium and the joint setting of mobile termination rates.

We assume throughout that no segment is cornered by one of the networks, thus the indifferent consumer on segment ij is located in its interior, at a distance x_{ij} from network i defined by

$$w_i + A_i - tx_{ij} = w_j + A_j - t(l(n) - x_{ij}).$$

Solving for x_{ij} yields network i's part of segment ij as

$$x_{ij} = \frac{l(n)}{2} + \frac{1}{2t}\left(w_i + A_i - w_j - A_j\right).$$

Defining $\sigma(n) \equiv 1/[2tS(n)]$ and summing subscribers over segments yields network i's total market share:

$$\begin{aligned}\alpha_i &= \frac{1}{S(n)}\sum_{j\neq i} x_{ij} = \frac{1}{n} + \sigma(n)\sum_{j\neq i}(w_i + A_i - w_j - A_j) \\ &= \alpha_{0i} + \sigma(n)\left((n-1)\,w_i - \sum_{j\neq i} w_j\right), \qquad (2)\end{aligned}$$

where $\alpha_{0i} = 1/n + \sum_{j\neq i}(A_i - A_j)$ is network i's *ex-ante* market share that would prevail under identical call surpluses on all networks.[9] Market shares are thus composed of an idiosyncratic part which captures consumers' relative preference for network i through the α_{0i}, and a second part that indicates how market shares are affected by the tariffs on offer. In the symmetric case $\alpha_{0i} = 1/n$ this expression for market shares is equivalent to the one in Armstrong and Wright (2007, p. 31) for $S(n) = n - 1$.

Let E be the $(n \times 1)$ vector of ones, I the $(n \times n)$ identity matrix, $B = nI - EE'$ an $(n \times n)$ matrix with the values $(n-1)$ on the diagonal and -1 elsewhere, and A the $(n \times 1)$ vector of connection surpluses A_i. Note that $E'B = nE' - (E'E)E = 0$, equally $BE = 0$, and

$$BB = n^2 I - 2nEE' + E\,(E'E)\,E = nB.$$

Let $\alpha_0 = E/n + \sigma BA$ be the $(n \times 1)$ vector of *ex ante* market shares α_{0i}. In matrix notation, market shares become

$$\alpha = \frac{1}{n}E + \sigma(n)B(w + A) = \alpha_0 + \sigma Bw. \qquad (3)$$

In a fully covered market, the latter must add up to 1: Since $E'B = 0$ we have

$$\sum_{i=1}^{n}\alpha_i = E'\alpha = \frac{1}{n}E'E + \sigma E'B\,(w + A) = 1.$$

Plugging utility (1) into (3) leads to the condition

$$\alpha = \alpha_0 - \sigma BF + \sigma Bh\alpha. \qquad (4)$$

[9] These market shares would also result if all networks were to offer the same uniform tariff.

Solving the latter for α yields

$$\alpha = (I - \sigma Bh)^{-1}[\alpha_0 - \sigma BF] = G\alpha_0 - \sigma HF, \tag{5}$$

where we have defined $G = (I - \sigma Bh)^{-1}$ and $H = (I - \sigma Bh)^{-1} B$ with elements H_{ij}. Thus we have found a simple unique solution for market shares *when prices are given*.[10] The following technical Lemma states some properties of G and H which will be useful later on.

Lemma 1 *Assume $(I - \sigma Bh)^{-1}$ exists. We have: $E'G = E'$, $E'H = 0$ and $HE = 0$. In particular, $\sum_{i=1}^{n} H_{ij} = 0$ for all j, and $\sum_{j=1}^{n} H_{ij} = 0$ for all i.*

Proof. First note that $E'(I - \sigma Bh) = E' - \sigma 0h = E'$. Therefore

$$E'G = E'(I - \sigma Bh)(I - \sigma Bh)^{-1} = E'.$$

Note that $GE \neq E$ in general. Furthermore, $E'H = (E'G)B = E'B = 0$ and $HE = G(BE) = 0$ since $BE = 0$. ■

Finally, total consumer surplus consists of the difference between the surplus from pertaining to networks and making calls, and "transport cost" which measures the welfare cost of a less than perfect fit with preferences:

$$\begin{aligned} CS &= \sum_{i=1}^{n}\left[\alpha_i (w_i + A_i) - \sum_{j\neq i}\int_0^{x_{ij}} tz\frac{1}{S(n)}dz\right] \\ &= \sum_{i=1}^{n}\left[\alpha_i (w_i + A_i) - \frac{t}{2S(n)}\sum_{j\neq i} x_{ij}^2\right] \\ &= \alpha'(h\alpha + A - F) - \frac{t}{2S(n)}\sum_{i,j\neq i} x_{ij}^2. \end{aligned}$$

2.2 Stability

One important technical aspect, discussed at length in LRTb for the duopoly case, is the stability of equilibrium in consumer expectations. Since under price discrimination between on- and off-net calls consumers' expected utility from joining a specific network depends on the size of this and the other networks, network effects arise which may lead to unstable equilibrium points. We will want to concentrate on stable equilibrium points because the unstable ones are unlikely to arise in actual play of the game.

The following Lemma shows how the stability condition from LRTb can be generalized to an arbitrary number of asymmetric networks.

[10] This solution does not yield equilibrium market shares explicitly in the presence of call externalities since both G and H will depend on market shares indirectly through prices. We study price choice in the next section.

Lemma 2 *The Nash equilibrium in the price competition game, no matter whether in linear or in multi-part tariffs, is stable in consumer expectations if $\alpha_i \geq 0$ for all $i = 1, ..., n$ and $\sigma < 1/\rho$, where ρ is the spectral radius of Bh (i.e. its eigenvalue with the largest absolute value).*

Proof. The condition that all α_i are non-negative is a pre-condition for a well-defined equilibrium candidate. Now consider, similar to LRTb, for given tariffs a virtual tâtonnement process where consumers observe market shares α_{t-1} and then join networks based on the resulting surplus. This leads to market shares

$$\alpha_t = \alpha_0 + \sigma B \left(h\alpha_{t-1} - F\right) = \left[\alpha_0 - \sigma BF\right] + \sigma Bh\alpha_{t-1}.$$

The effect of market shares at $t-1$ on market shares at time t is given by $d\alpha_t/d\alpha_{t-1} = \sigma Bh$. For this tâtonnement process to converge, it is sufficient that the spectral radius of σBh be less than 1 (while it is necessary that it is less than or equal to 1, see Moulin (1986), p. 135), which is equivalent to the condition stated in the Lemma.[11] ■

One eigenvalue of Bh is always zero, since $E'Bh = 0 * E'$. With *symmetric* prices, we have $h_{on} = h_{ii}$, $h_{of} = h_{ij}$, and the other $(n-1)$ eigenvalues of Bh can be determined explicitly. This gives rise to the following result:

Proposition 1 *With networks competing in linear or multi-part tariffs, the symmetric Nash equilibrium is stable if*

$$\sigma(n) < \frac{1}{n\left|h_{on} - h_{of}\right|}$$

Thus it is less likely to be stable:

1. *for a smaller preference space as measured by $S(n)/n$;*
2. *for a very high or very low mobile termination rate a;*
3. *for a lower transport cost t.*

Proof. With symmetric networks we can write $h = (h_{on} - h_{of})\, I + h_{of}EE'$ and obtain

$$\begin{aligned} Bh &= \left[nI - EE'\right]\left[\left(h_{on} - h_{of}\right) I + h_{of}EE'\right] \\ &= \left(h_{on} - h_{of}\right) B + h_{of}\left[nEE' - E\left(E'E\right)E'\right] \\ &= \left(h_{on} - h_{of}\right) B, \end{aligned}$$

since $E'E = n$. Denote by e_i the ith unit vector in $\mathbb{R}^n$, i.e. the $(n \times 1)$ vector whole *ith* element is equal to one and the others equal to zero. For all $j = 2, ..., n$ we obtain $E'\left(e_1 - e_j\right) = 0$ and

$$\begin{aligned} Bh\left(e_1 - e_j\right) &= \left(h_{on} - h_{of}\right)\left(nI - EE'\right)\left(e_1 - e_j\right) \\ &= n\left(h_{on} - h_{of}\right)\left(e_1 - e_j\right). \end{aligned}$$

[11] The stability condition stated in LRTb only considers the largest positive eigenvalue, not the largest one in absolute value, because it is implicitly assumed that "pessimistic market shares" do not exceed 1. The latter will not be true only in the (admittedly unrealistic) case where off-net prices fall far below on-net prices.

Since the eigenvectors $(e_1 - e_j)$, $j > 1$, are independent of each other, this implies that at least $(n-1)$ eigenvalues of Bh are equal to $n(h_{on} - h_{of})$. Since we know that one eigenvalue is equal to zero, this implies that exactly $(n-1)$ eigenvalues are equal to $n(h_{on} - h_{of})$.

As for the statements on stability, remember that $\sigma(n) = 1/[2tS(n)]$. 1. The equilibrium is stable if $S(n)/n > |h_{on} - h_{of}|/2t$. 2. $|h_{on} - h_{of}|$ is large if a is either very low or very high, as discussed below. 3. Stability implies that t be large enough. ■

While statements 2 and 3 carry over from LRTb, statement 1 is new. It establishes a link between the density of subscribers in preference space and stability in expectations. It is intuitive, though: The more concentrated preferences are, the larger are the network effects created if expectations about network choice change.

It is of interest to note that our stability condition adapted to the parameterizations of Armstrong and Wright (2007) and Chen and Riordan (2007) is $|h_{on} - h_{of}| < 2t(1 - 1/n)$ and $|h_{on} - h_{of}| < n - 1$, respectively. Since in both cases by assumption the size of the preference space $S(n)$ expands at rate n or faster with the number of networks, we obtain the counter-intuitive result that market stability increases (rather than decreases) as the number of networks increases. Put differently, in these models instability is more likely to occur in markets with *few* networks.

Stability of equilibrium implies a further useful result. We can see from (5) that $\partial\alpha_j/\partial F_i = -\sigma H_{ji}$ for all $i, j = 1, ..., n$. That is, the effect of changes in fixed fees on market shares is determined by the elements of the matrix H. One might expect that an increase in F_i decreases α_i and increases all α_j, $j \neq i$, i.e. $H_{ii} > 0$ and $H_{ij} < 0$ for $j \neq i$. The following example[12] shows that this is not true in general: Let $\sigma = 1$ and

$$h = \begin{pmatrix} \frac{403}{804} & \frac{605}{804} & \frac{101}{402} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{25\,301}{40\,602} & \frac{61\,205}{81\,204} & \frac{70\,501}{81\,204} & \frac{1}{2} \\ \frac{605}{804} & \frac{202}{201} & \frac{605}{804} & 1 \end{pmatrix}, \text{ then } H = \begin{pmatrix} -94 & 98 & -202 & 198 \\ -2 & 202 & 0 & -200 \\ 98 & -100 & 202 & -200 \\ -2 & -200 & 0 & 202 \end{pmatrix}.$$

While the eigenvalues of Bh, which are 0, $\frac{199}{201}$ and $\frac{349 \pm \sqrt{1409}}{404}$ are all between zero and one, and therefore stability in expectations holds, we have $H_{11} < 0$ and $H_{12}, H_{14} > 0$ (among other "oddities"). What is happening is that an increase in F_1, whose direct effect is a reduction in α_1 and an increase in α_2 and α_4, induces an opposite feedback through network effects: Since in this example $h_{12}, h_{14} > h_{11}$, network 1 becomes more attractive. When things settle down α_1 has increased, and α_2, α_4 have decreased, as compared to their previous level.

In an important case, though, we can show that $H_{ii} > 0$ for all i. This is the case of uniform off-net pricing, i.e. $p_{ij} = p_{ik}$ for all i and $j, k \neq i$ and we can define $v_{if} \equiv v_{ij}$ and $\gamma u_{if} = \gamma u_{ij}$ as the surplus from making and receiving off-net calls. Furthermore, let

$$\beta_i = \frac{1}{1 - n\sigma(v_{ii} + \gamma u_{ii} - v_{if} - \gamma u_{if})}.$$

Then we obtain the following:

[12] My thanks to Iliyan Georgiev for help in constructing this example.

Lemma 3 *Under uniform off-net pricing and stability in expectations, we have*

1. $H_{ii} = n\left(\beta_i - \beta_i^2 / \sum_{k=1}^n \beta_k\right)$ *and* $H_{ij} = -n\beta_i\beta_j / \sum_{k=1}^n \beta_k$.
2. $H_{ii} > 0$ *for all* $i = 1, ..., n$.

Proof. Let $\Delta_i = v_{ii} + \gamma u_{ii} - v_{if} - \gamma u_{if} = (1 - 1/\beta_i)/n\sigma$. First note that under uniform off-net pricing we can write

$$h = \sum_{i=1}^n \left(\Delta_i e_i e_i' + v_{if} e_i E' + \gamma u_{if} E e_i'\right),$$

and obtain, using $E'E = n$ and $E'e_i = 1$,

$$Bh = (nI - EE')\, h = \sum_{i=1}^n \left[\Delta_i (ne_i - E)\, e_i' + v_{if} (ne_i - E)\, E'\right].$$

1. H as stated in the Lemma can be written equivalently as

$$H = n \sum_{i=1}^n \beta_i e_i \left(e_i' - \frac{1}{\sum_{k=1}^n \beta_k} \sum_{j=1}^n \beta_j e_j'\right).$$

Thus, using $e_i'e_i = 1$ and $e_i'e_j = 0$ for $j \neq i$, we have

$$\begin{aligned} \sigma BhH &= \sum_{i=1}^n (\beta_i - 1)(ne_i - E)\left(e_i' - \frac{1}{\sum_{k=1}^n \beta_k}\sum_{j=1}^n \beta_j e_j'\right) \\ &= H - B, \end{aligned}$$

which is equivalent to $H = (I - \sigma Bh)^{-1} B$ since the stability condition implies that $(I - \sigma Bh)$ is non-singular.

2. We will now proceed in two steps to show that the diagonal elements of H are positive. In a first step we will show that Bh has $(n-1)$ eigenvectors which have real eigenvalues and are orthogonal mutually and to E (The remaining eigenvector has eigenvalue zero but will not be used below.) In a second step we diagonalize H.

a) Let x be a generic eigenvector of Bh, with eigenvalue δ. Then $\delta E'x = E'Bhx = 0$, i.e. $\delta = 0$, or x is orthogonal to E, or both hold. Furthermore, $(I - \sigma Bh)\, x = (1 - \sigma\delta)\, x$ and thus $(I - \sigma Bh)^{-1} x = \lambda x$, i.e. x is an eigenvector of $(I - \sigma Bh)^{-1}$ with eigenvalue $\lambda = (1 - \sigma\delta)^{-1}$. If $E'x = 0$ then also $Bx = nx$.

a1) Assume for now that h has full rank, i.e. at most one Δ_i, $i = 1, ..., n$ can be equal to zero. In this case Bh has rank $n-1$ and therefore $n-1$ non-zero eigenvalues (including multiple ones). Let x be a corresponding eigenvector, then $E'x = 0$, and we multiply x with the following real symmetric matrix:

$$\left(\sum_{i=1}^n \Delta_i \left[ne_ie_i' - Ee_i' - e_iE'\right]\right) x = Bhx = \delta x.$$

Thus x is an eigenvector of a real symmetric matrix, and therefore δ is real and x is orthogonal to all other eigenvectors of Bh with non-zero eigenvalues (If there are multiple eigenvalues then the eigenvectors can be chosen to be orthogonal). Let X be the $(n \times (n-1))$ matrix whose columns x_i are these $(n-1)$ orthogonal eigenvectors. A decisive observation is that no row of X is identical to zero: If row i were equal to zero then these $n-1$ eigenvectors would simultaneously be orthogonal not only to E and to each other, but also to e_i, which is impossible in $\mathbb{R}^n$.

a2) Now let h have less than full rank, i.e. without loss of generality we have $\Delta_i = 0$ for $i = 1, ..., k$ where $1 < k \leq n$. Let $\tilde{\Delta}_i(\varepsilon) = \varepsilon > 0$ for $i = 1, ..., k$, and define $\tilde{h}(\varepsilon)$, which has full rank, accordingly. Thus the results in a1) apply to $B\tilde{h}(\varepsilon)$. In particular, eigenvectors $\tilde{x}(\varepsilon)$ and $\tilde{y}(\varepsilon)$ to non-zero eigenvalues obey $\tilde{x}(\varepsilon)' \tilde{y}(\varepsilon) = 0$ and $E'\tilde{x}(\varepsilon) = 0$. Continuity maintains these properties in the limit $\varepsilon \to 0$, thus Bh has $n-1$ eigenvectors which are orthogonal to each other and to E and have real eigenvalues even if h does not have full rank.

b) Since all eigenvalues δ of Bh are real and we have assumed stability, $\lambda = (1 - \sigma\delta)^{-1}$ is real and positive. Let Λ be the diagonal matrix with $\Lambda_{ii} = \lambda_i = (1 - \sigma\delta_i)^{-1}$ and zeros otherwise.

The columns of X, i.e. the eigenvectors x_i, span the same $(n-1)-$dimensional subspace as the eigenvectors of B (which are vectors $e_i - e_j$ for $j \neq i$), thus $\chi B = B$ where $\chi = X\left(X^T X\right)^{-1} X^T$ is the projection onto the subspace spanned by the x_i. Thus:

$$
\begin{aligned}
H &= (I - \sigma Bh)^{-1} B = (I - \sigma Bh)^{-1} X (X'X)^{-1} X'B \\
&= (I - \sigma Bh)^{-1} XX'B = (I - \sigma Bh)^{-1} XnX' \\
&= nX\Lambda X'.
\end{aligned}
$$

This already implies that H is a symmetric matrix with non-negative eigenvalues, and thus $H_{ii} \geq 0$. Still, note that, for all $i = 1, ..., n$,

$$
H_{ii} = e_i' (nX\Lambda X') e_i = nr_i'\Lambda r_i = n \sum_{j=1}^{n-1} \lambda_j r_{ij}^2 > 0.
$$

where $r_i = X'e_i = (r_{i1}, ..., r_{i,n-1}) \neq 0$ is the ith row of X. ■

Interestingly, we cannot conclude that $H_{ij} < 0$ for all $j \neq i$ even under uniform off-net pricing, but only that $\sum_{j \neq i} H_{ij} < 0$ (from $HE = 0$). This is borne out by the following example: Let $\sigma = 2/11$,

$$
h = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \; H = \begin{pmatrix} 22 & -11 & -11 \\ -11 & \frac{44}{5} & \frac{11}{5} \\ -11 & \frac{11}{5} & \frac{44}{5} \end{pmatrix}.
$$

Here $H_{23} > 0$, i.e. an increase in F_2 decreases α_3. The reason is that an increase in F_2 drives too many consumers towards network 1, which has the self-reinforcing effect of further reducing the attractiveness of both networks 2 and 3.

2.3 Profits, Welfare and the First-Best Allocation

Networks incur a fixed cost per customer of f_i, and have on-net cost $c_{ii} = c_{oi} + c_{ti}$, where the indices o and t stand for origination and termination, respectively. The mobile termination rate on network i is a_i, so that costs of off-net calls from network i to network $j \neq i$ are $c_{ij} = c_{oi} + a_j$. The mobile termination margin is $m_i = a_i - c_{ti}$. Networks' profits are

$$\pi_i = \alpha_i \left(\sum\nolimits_{j=1}^{n} \alpha_j R_{ij} + F_i + Q_i - f_i \right), \tag{6}$$

where $R_{ij} = (p_{ij} - c_{oi} - a_j)q_{ij} + (a_i - c_{ti})q_{ji}$ are the profits from an incoming and an outgoing call between networks i and j. Note that this simplifies to $R_{ii} = (p_{ii} - c_{ii})q_{ii}$, and $R_{ij} = (p_{ij} - c_{ij})q_{ij} + m_i q_{ji}$ for $j \neq i$. Furthermore, $Q_i = m_i q_{fi}$ are fixed-to-mobile termination profits, where q_{fi} denotes the number of incoming call minutes from the fixed network.

Let $J^{ij} = e_i e_j'$ be the $(n \times n)$ matrix with entry 1 at position (i, j) and zeros elsewhere, R be the $(n \times n)$ matrix with entries R_{ij}, and F, Q, f be the $(n \times 1)$-vectors with entries F_i, Q_i, and f_i, respectively. We can express network i's profits in matrix notation as

$$\pi_i = \alpha' J^{ii} \left(R\alpha + F + Q - f \right),$$

and, since $\sum_{i=1}^{n} J^{ii} = I$, joint profits of all networks as

$$\sum_{i=1}^{n} \pi_i = \alpha' \left(R\alpha + F + Q - f \right).$$

Total welfare in the market for mobile telephony is given by

$$\begin{aligned} W &= CS + \sum_{i=1}^{n} \pi_i \\ &= \alpha' \left[(R + h)\alpha + Q + A - f \right] - \frac{t}{2S(n)} \sum_{i,j \neq i} x_{ij}^2 \end{aligned}$$

We can now describe first-best prices and market shares:

Proposition 2 *1. First-best per-minute prices are* $p_{ij}^* = \frac{c_{oi} + c_{tj}}{1+\gamma}$ *for all* $i, j = 1, ..., n$.

2. Let $M \equiv R + h$ *at first-best prices. Then socially optimal market shares in the mobile telephony market are*

$$\alpha^* = \left(I - \sigma B \left(M' + M \right) \right)^{-1} \left[\alpha_0 + \sigma B (Q - f) \right], \tag{7}$$

if asymmetries are small enough. With symmetric network cost, optimal market shares become

$$\alpha^* = \alpha_0 + \sigma B (Q - f).$$

Proof. In the expression for aggregate profits the terms corresponding to mobile-to-mobile termination payments and revenues cancel, so that after some re-ordering of terms with indices ij and ji,

$$\alpha' (R+h)\,\alpha = \sum_{i,j} \alpha_i \alpha_j \left[(p_{ij} - c_{oi} - c_{tj})\, q_{ij} + v_{ij} + \gamma u_{ij} \right].$$

Thus for each pair ij the same surplus maximization problem is posed, with first-order condition

$$q_{ij} + \left(p^*_{ij} - c_{oi} - c_{tj}\right) q'_{ij} - q_{ij} + \gamma u'_{ij} q'_{ij} = 0.$$

Since $u'_{ij} = p^*_{ij}$ at the consumer's optimal choice of call minutes the above result is obtained.

Let $M \equiv R + h$ at prices $(p^*_{ij})_{ij}$. Then we need to maximize social surplus

$$W = \alpha' M \alpha + \alpha' (Q + A - f) - \frac{t}{2S(n)} \sum_{i, j \neq i} x^2_{ij}$$

subject to the conditions $x_{ji} = l(n) - x_{ij}$ and $x_{ij} \geq 0$ for all $j \neq i$, $i = 1, ..., n$. Omitting for the moment the non-negativity constraints, and substituting out x_{ji} in $\alpha_j = \left(\sum_{k \neq j} x_{jk}\right)/S(n)$, we have $\frac{d\alpha}{dx_{ij}} = (e_i - e_j)/S(n)$. Thus, maintaining the substitution of x_{ji}, we have the first-order conditions, for all i and $j \neq i$,

$$S(n)\frac{dW}{dx_{ij}} = (e_i - e_j)' (M' + M)\,\alpha^* + (e_i - e_j)' (Q + A - f) - t x^*_{ij} + t\left(l(n) - x^*_{ij}\right) = 0.$$

Summing the above conditions over $j \neq i$, we obtain

$$B_i (M' + M)\,\alpha^* + B_i (Q + A - f) - 2tS(n)\,\alpha^*_i + \frac{2tS(n)}{n} = 0,$$

where B_i is row i of the matrix B. Stacking these equations leads to

$$B (M' + M)\,\alpha^* + B (Q - f) - \frac{\alpha^*}{\sigma} + \frac{\alpha_0}{\sigma} = 0$$

and the condition

$$(I - \sigma B (M' + M))\,\alpha^* = \alpha_0 + \sigma B (Q - f).$$

Note that $B(M' + M) = 0$ with symmetric network cost because then $M = kEE'$ for some constant k. These results hold as long as all $x_{ij} \geq 0$, which is true if and only if the asymmetries in network cost and $\alpha_0 + \sigma B (Q - f)$ are not too large. ■

First-best call prices follow a simple principle: They are equal to the marginal cost of origination on the calling network plus the marginal cost of termination on the receiving network, correct by the call externality. The existence of the latter implies that welfare is maximized if calls are longer than would be optimal by just considering callers' utility.

The above computation of first-best market shares serves purely as a benchmark with which the competitive outcome can be compared. Note, though, that we have considered the transfer Q from the fixed telephony market as given. In particular, we did not take into account the welfare loss caused by this transfer. These optimal market shares also take as given the *ex ante* market shares α_0 which distinguish networks in the eyes of consumers. Nevertheless, at potential "optimal" values of Q and α_0 the first-best market shares are still given by (7).

With symmetric network cost, since $Bh^* = 0$, by (5) the socially optimal market shares α^* can be induced by introducing fixed fees equal to $F = f - Q + kE$, for some constant k. That is, not the absolute value of fixed fees is relevant, but only the differences between networks count. What needs to be signalled to consumers is the difference in net fixed cost $(f - Q)$ per consumer. If on the other hand network costs are not symmetric, then there is no longer a simple correspondence between conditions (5) and (7), and fixed fees must be chosen such that

$$\sigma BF = \alpha_0 - (I - \sigma Bh)\left(I - \sigma B\left(M' + M\right)\right)^{-1}\left[\alpha_0 + \sigma B\left(Q - f\right)\right].$$

3 Theory: Pricing Equilibrium

In this section we describe equilibrium prices and market shares under both linear and multi-part tariffs. As economic predictions differ significantly between these two types of tariffs (see e.g. Laffont, Rey and Tirole 1998a,b), it seems useful to consider the case of many networks for both types of tariffs.

3.1 Linear Tariffs

Since we consider linear tariffs let $F = 0$. Each network chooses the prices p_{ii} and p_{ij} in order to maximize its profits

$$\pi_i = \alpha_i \left(\sum\nolimits_{j=1}^{n} \alpha_j R_{ij} + Q_i - f_i\right) = \alpha' J^{ii}\left(R\alpha + Q - f\right).$$

We first state a central result about how per-minute prices affect market shares.

Lemma 4 *For any price p_{lk}, $l, k = 1, ..., n$, we have $\frac{d\alpha}{dp_{lk}} = \sigma H \frac{dh}{dp_{lk}}\alpha$, with $\sum_{i=1}^{n} \frac{d\alpha_i}{dp_{lk}} = 0$.*

Proof. From condition (4) we have $(I - \sigma Bh)\,\alpha = \alpha_0$. Taking derivatives on both sides leads to

$$-\sigma B \frac{dh}{dp_{lk}}\alpha + (I - \sigma Bh)\frac{d\alpha}{dp} = 0,$$

from which the result follows (dh/dp_{lk} is $(n \times n)$-matrix of derivatives dh_{ij}/dp_{lk}). Furthermore, $\sum_{i=1}^{n} \frac{d\alpha_i}{dp_{lk}} = E' \frac{d\alpha}{dp_{lk}} = \sigma\left(E'H\right)\frac{dh}{dp_{lk}}\alpha = 0$ since $E'H = 0$. ■

As is common in models of network competition in linear prices, we cannot give explicit expressions for the equilibrium prices. Still, we can show how the equilibrium off-net prices relate to on-net prices. For the sake of generality, we consider the case where network i divides its competitors into separate groups K and charges a uniform off-price p_{iK} to each group. Extreme cases are where each group contains a single member (in which case there is price discrimination between all networks), or where all other networks are in the same group (the case of a uniform off-net price). We obtain the following results on equilibrium prices:

Proposition 3 *1. Network i's equilibrium on-net price satisfies the following condition:*

$$L_{ii} = \frac{p_{ii} - c_{ii}}{p_{ii}} = \frac{1}{\eta} - \frac{\sigma(1+\gamma\eta)H_{ii}}{\eta}\left(\frac{\pi_i}{\alpha_i^2} + \sum_{j=1}^{n} R_{ij}\frac{H_{ji}}{H_{ii}}\right). \tag{8}$$

2. If network i sets uniform prices p_{iK} to different groups K of competing networks, its average off-net Lerner index

$$\bar{L}_{ij} = \frac{\sum_K \sum_{j\in K} \alpha_j (p_{iK} - c_{ij})/p_{iK}}{1-\alpha_i}$$

satisfies the condition

$$\bar{L}_{ij} = \frac{1}{\eta} - \frac{(1+\gamma\eta)^{-1} - \alpha_i}{1-\alpha_i}\left(\frac{1}{\eta} - L_{ii}\right). \tag{9}$$

3. Network i's profits are given by

$$\pi_i = \alpha_i^2\left(\frac{1}{\sigma H_{ii}}\frac{1-\eta L_{ii}}{1+\gamma\eta} - \sum_{j=1}^{n} R_{ij}\frac{H_{ji}}{H_{ii}}\right). \tag{10}$$

Proof. For on-net prices we obtain

$$\frac{dh}{dp_{ii}} = (\gamma p_{ii} q'_{ii} - q_{ii}) J^{ii} = -(1+\gamma\eta) q_{ii} J^{ii}.$$

Thus, using $J^{ij} = e_i e'_j$,

$$\frac{d\alpha}{dp_{ii}} = -\sigma(1+\gamma\eta) q_{ii} H J^{ii}\alpha = -\sigma(1+\gamma\eta) q_{ii}\alpha_i H_{\cdot i},$$

where $H_{\cdot i}$ is the ith column of H. Furthermore, $\frac{dR}{dp_{ii}} = (1-\eta L_{ii}) q_{ii} J^{ii}$, where $L_{ii} = (p_{ii} - c_{ii})/p_{ii}$ is the Lerner index for on-net calls. The first-order condition for profit-maximization with respect to the on-net price is

$$\frac{d\alpha'}{dp_{ii}} J^{ii}(R\alpha + Q - f) + \alpha' J^{ii}\frac{dR}{dp_{ii}}\alpha + \alpha' J^{ii} R\frac{d\alpha}{dp_{ii}} = 0,$$

which simplifies to

$$-H_{ii}\left(R_{i\cdot}\alpha+Q_i-f_i\right)-\alpha_i R_{i\cdot}H_{\cdot i}+\frac{\alpha_i\left(1-\eta L_{ii}\right)}{\sigma\left(1+\gamma\eta\right)}=0 \tag{11}$$

or

$$\pi_i=\alpha_i^2\left(\frac{1}{\sigma H_{ii}}\frac{1-\eta L_{ii}}{1+\gamma\eta}-\sum_{j=1}^{n}R_{ij}\frac{H_{ji}}{H_{ii}}\right). \tag{12}$$

Solving for L_{ii} leads to the condition on the on-net price.

2. Assume that network i sets a uniform off-net price p_{iK} to a set K of other networks. We have

$$\frac{dh}{dp_{iK}}=-q_{iK}J^{iK}+\gamma p_{iK}q'_{iK}J^{Ki}=-q_{iK}\left(J^{iK}+\gamma\eta J^{Ki}\right),$$

where J^{iK} and J^{Ki} are matrices with ones at locations ij and ji where $j\in K$, respectively, and zeros elsewhere. Thus

$$\frac{d\alpha}{dp_{iK}}=-\sigma q_{iK}H\left(J^{iK}+\gamma\eta J^{Ki}\right)\alpha=-\sigma q_{iK}\left(\sum_{j\in K}\alpha_j H_{\cdot i}+\gamma\eta\alpha_i\sum_{j\in K}H_{\cdot j}\right)$$

The first-order condition for a profit maximum becomes

$$\frac{d\alpha'}{dp_{iK}}J^{ii}\left(R\alpha+Q-f\right)+\alpha'J^{ii}\frac{dR}{dp_{iK}}\alpha+\alpha'J^{ii}R\frac{d\alpha}{dp_{iK}}=0,$$

where $\frac{dR}{dp_{iK}}$ has elements $q_{iK}\left(1-\eta L_{ij}\right)$, where $L_{ij}=\left(p_{iK}-c_{ij}\right)/p_{iK}$ at locations ij, $j\in K$, and $m_j q'_{iK}$ at locations ji, $j\in K$. Note that off-net costs c_{ij} may differ between receiving networks j. This first-order condition can be rewritten as

$$\begin{aligned}0 &= -\left(\sum_{j\in K}\alpha_j H_{ii}+\gamma\eta\alpha_i\sum_{j\in K}H_{ij}\right)\left(R_{i\cdot}\alpha+Q_i-f_i\right)\\ &\quad-\alpha_i R_{i\cdot}\left(\sum_{j\in K}\alpha_j H_{\cdot i}+\gamma\eta\alpha_i\sum_{j\in K}H_{\cdot j}\right)+\frac{\alpha_i}{\sigma}\sum_{j\in K}\alpha_j\left(1-\eta L_{ij}\right).\end{aligned}$$

Summing over all sets K, and making use of $\sum_{j\neq i}H_{\cdot j}=-H_{\cdot i}$ from Lemma 1, leads to

$$-H_{ii}\left(R_{i\cdot}\alpha+Q_i-f_i\right)-\alpha_i R_{i\cdot}H_{\cdot i}+\frac{\alpha_i\left(1-\alpha_i\right)\left(1-\eta\bar{L}_{ij}\right)}{\sigma\left(1-\alpha_i-\gamma\eta\alpha_i\right)}=0, \tag{13}$$

where $\bar{L}_{ij}=\sum_{j\neq i}\alpha_j L_{ij}/\left(1-\alpha_i\right)$ is the weighed average Lerner index of off-net prices, or

$$\pi_i=\alpha_i^2\left(\frac{\left(1-\alpha_i\right)\left(1-\eta\bar{L}_{ij}\right)}{\sigma\left(1-\alpha_i-\gamma\eta\alpha_i\right)H_{ii}}-\sum_{j=1}^{n}R_{ij}\frac{H_{ji}}{H_{ii}}\right). \tag{14}$$

Taking the difference between the latter and (12) we obtain

$$\frac{(1-\alpha_i)\left(1-\eta\bar{L}_{ij}\right)}{1-\alpha_i-\gamma\eta\alpha_i}=\frac{1-\eta L_{ii}}{1+\gamma\eta},$$

from which the above result follows. ■

The condition describing on-net prices is the generalization to n asymmetric networks of condition (12) in Laffont, Rey and Tirole (1998b) and (3) in Berger (2004). The result on off-net prices is the generalization to n networks with asymmetric costs, and up to n different off-net prices to groups of networks, of equations (6) in Laffont, Rey and Tirole (1998b), (4) in Berger (2004) and (11) in Hoernig (2007) for two networks. It is remarkable that the relationship between the average level of off-net prices, as measured by $\bar{L}_{ij}$, and on-net prices remains the same even with many asymmetric networks.

The first term in the parenthesis of equilibrium profits (10) translates the type and intensity of competition. Essentially, it describes by how much the marginal customer must be compensated to make him change networks. As a result, higher transport cost t and a larger preference space $S(n)$, i.e. lower σ, both increase equilibrium profits. As we will explain below, expression (10) also holds for both linear and multi-part tariffs, with L_{ii} taking on corresponding values. The second term then describes how marginal changes in market share affect call profits.

If network i charges a uniform off-net price p_{iu} to all other networks, then we can reformulate $\bar{L}_{ij}$ as follows:

$$\bar{L}_{ij}=\frac{\sum_{j\neq i}\alpha_j\left(p_{iu}-c_{ij}\right)/p_{iu}}{1-\alpha_i}=\frac{p_{iu}-\bar{c}_{iof}}{p_{iu}},$$

where $\bar{c}_{iof}=\left(\sum_{j\neq i}\alpha_j c_{ij}\right)/\left(1-\alpha_i\right)$ is the weighted average off-net cost faced by network i. Thus for a uniform off-net price, $\bar{L}_{ij}$ simply becomes the Lerner index based on weighted average off-net cost. Expression (9) allows us now to consider how the differential between on- and uniform off-net prices evolves as a function of asymmetries and / or the number of networks in the market.

Corollary 1 *When networks compete in linear tariffs with uniform off-net prices, the following holds for the differential between equilibrium on- and off-net prices:*

1. *For given α_i, p_{ii} and $\bar{c}_{iof}$, $p_{iu}-p_{ii}$ does not depend on the number of other networks, nor on how asymmetric they are;*

2. *The differential $p_{iu}-p_{ii}$ increases with average off-net cost $\bar{c}_{iof}$ and, if $\gamma>0$, with market share α_i.*

Proof. Note first that (9) implies that $\bar{L}_{ij}\geq L_{ii}$, so that for $\bar{c}_{iof}\geq c_{ii}$ we have $p_{iu}\geq p_{ii}$. When we solve equation (9) for the uniform off-net price p_{iu}, we find

$$p_{iu}-p_{ii}=\frac{\bar{c}_{iof}}{\frac{\gamma(\eta-1)}{(1+\gamma\eta)(1-\alpha_i)}+\frac{(1+\gamma\eta)^{-1}-\alpha_i}{1-\alpha_i}\frac{c_{ii}}{p_{ii}}}-p_{ii}.$$

Thus the differential does not depend directly on n nor on the other networks' characteristics, but it increases in $\bar{c}_{iof}$, and also in α_i if $\gamma > 0$. ■

Thus, similar to Hoernig (2007) for the duopoly case, our prediction is that under linear tariffs the on-/off-net differential is larger on larger networks. Furthermore, with a uniform off-net price the effect of each competing networks' MTR a_j is mediated through the average off-net cost $\bar{c}_{iof}$. In particular, a small network's MTR has a small effect on $\bar{c}_{iof}$ and thus on p_{iu}, which increases the small network's incentives to unilaterally raise its MTR.

Uniform prices. If networks set uniform prices, i.e. if they do not discriminate at all between on-net and off-net calls as in Armstrong (1998) and Laffont, Rey and Tirole (1998a), our approach from above still applies. Let the uniform prices be given by p_i, with corresponding demand $q_i = q(p_i)$, utility $u_i = u(q_i)$ and surplus $v_i = v(p_i)$. We find the following:

Proposition 4 *If networks set uniform prices then the Nash equilibrium in linear tariffs has the following properties:*

1. *Equilibrium uniform prices are defined by the condition*

$$L_i \equiv \frac{p_i - \bar{c}_i}{p_i} = \frac{1}{\eta}\left(1 - \sigma(n-1)\left(\frac{\pi_i}{\alpha_i^2} - (a_i - \bar{a}_{-i})\, q_i - (a_i - c_{ti})\,(\bar{q}_{-i} - q_i)\right)\right), \tag{15}$$

where $\bar{c}_i = \sum_{j=1}^{n} \alpha_i c_{ij}$ is average marginal cost.

2. *Equilibrium profits are*

$$\pi_i = \alpha_i^2\left(\frac{1 - \eta L_i}{\sigma(n-1)} + (a_i - \bar{a}_{-i})\, q_i + (a_i - c_{ti})\,(\bar{q}_{-i} - q_i)\right). \tag{16}$$

Proof. Let $V = (v_i)_i$ and $U = (u_i)_i$. With uniform prices we can write $h = VE' + \gamma EU'$ and obtain

$$(I - \sigma Bh)\, B = B - \sigma BVE'B + \gamma BEU'B = B,$$

since $BE = E'B = 0$. Thus $H = (I - \sigma Bh)^{-1} B = B$, which implies that $H_{ii} = n - 1$ and $H_{ij} = -1$. Then adding (11) and (13) leads to

$$\left(-\frac{H_{ii}}{\alpha_i}\pi_i - \alpha_i \sum_{j=i}^{n} R_{ij} H_{ji}\right) + \frac{\alpha_i}{\sigma}\left(1 - \eta\frac{p_i - \bar{c}_i}{p_i}\right) = 0,$$

or

$$1 - \eta\frac{p_i - \bar{c}_i}{p_i} = \sigma\left(\frac{n-1}{\alpha_i^2}\pi_i + (n-1)\, R_{ii} - \sum_{j=i}^{n} R_{ij}\right)$$

from which the above results follow. Furthermore,

$$\begin{aligned}
\frac{1}{n-1}\sum_{j\neq i} R_{ij} - R_{ii} &= \left(p_i - c_{io} - \bar{a}_{-i}\right) q_i + \left(a_i - c_{ti}\right) \bar{q}_{-i} - \left(p_i - c_{ii}\right) q_i \\
&= \left(a_i - c_{ti}\right) \bar{q}_{-i} - \left(\bar{a}_{-i} - c_{it}\right) q_i \\
&= \left(a_i - \bar{a}_{-i}\right) q_i + \left(a_i - c_{ti}\right)\left(\bar{q}_{-i} - q_i\right)
\end{aligned}$$

where $\bar{a}_{-i} = \frac{1}{n-1}\sum_{j\neq i} a_j$, $\bar{q}_{-i} = \frac{1}{n-1}\sum_{j\neq i} q_j$. ■

Thus under uniform pricing call externalities do not influence equilibrium outcomes (they still determine the first best, though), since call prices are set based on average perceived marginal cost. Note that (15) is the generalization to multiple asymmetric networks of (8) in LRT98a.[13]

Expression (16), on the other hand, is a generalization of equilibrium profits (6) of Andersson and Hansen (2009), who considered a model of uniform pricing under inelastic demand and thus had profits depending the MTR differential $\left(a_i - \bar{a}_{-i}\right) q_i$. We will come back to the economic significance of these equilibrium profits in Section 6.

3.2 Multi-Part Tariffs

In this section, we determine the equilibrium prices, fixed fees and market shares for the case of competition in multi-part tariffs. We find the following:

Proposition 5 *If networks compete in multi-part tariffs,*

1. *On-net prices are set efficiently at $p_{ii} = c_{ii}/(1+\gamma)$.*
2. *The uniform off-net price to a group K of competing networks is*

$$p_{iK} = \frac{\sum_{j\in K} \alpha_j c_{ij}}{\sum_{j\in K} \alpha_j - \frac{|K|}{n-1}\gamma\alpha_i}. \tag{17}$$

3. *Equilibrium fixed fees are given by*

$$F = f - Q + \left(\hat{R} - R\right)\alpha, \tag{18}$$

where $\hat{R}$ is an $(n\times n)$ matrix with elements $\hat{R}_{ii} = \frac{1}{\sigma H_{ii}} - \sum_{j=1}^{n}\frac{H_{ji}}{H_{ii}} R_{ij}$ and $\hat{R}_{ij} = 0$ for $j \neq i$.

Proof. 1. In order to determine equilibrium call prices, we follow the standard procedure of first keeping market shares α constant and solving (2) for F_i,

$$F_i = \sum_{j=1}^{n} \alpha_j v_{ij} + \alpha_i \gamma u_{ii} - \frac{\alpha_i \gamma}{n-1}\sum_{j\neq i} u_{ij} + const,$$

[13] Note that in LRT98a, "π" denotes profit per subscriber, so that for $n=2$ and symmetric networks the two expressions are indeed identical.

where "*const*" denotes terms that do not depend on network i's prices. Substituting this into profits leads to

$$\pi_i = \alpha_i \left(\sum_{j=1}^{n} \alpha_j \left(R_{ij} + v_{ij} \right) + \alpha_i \gamma u_{ii} - \frac{\alpha_i \gamma}{n-1} \sum_{j \neq i} u_{ij} \right) + const. \tag{19}$$

This expression can now be maximized over call prices. As concerns the on-net price, network i solves

$$\max_{p_{ii}} \left\{ R_{ii} + h_{ii} \right\} = \left\{ \left(p_{ii} - c_{ii} \right) q_{ii} + v_{ii} + \gamma u_{ii} \right\},$$

which has first-order condition

$$q_{ii} + \left(p_{ii} - c_{ii} \right) q_{ii}' - q_{ii} + \gamma u_{ii}' q_{ii}' = 0.$$

Since $u_{ij}' = p_{ij}$ at the consumer's optimal choice for all $i, j = 1, ..., n$, we obtain $p_{ii} = c_{ii}/(1+\gamma)$.

2. Assume now that network i wants to set a uniform off-net price p_{iK} towards a group K of other networks, solving

$$\max_{p_{iK}} \left\{ \sum_{j \in K} \left(\alpha_j \left[\left(p_{iK} - c_{ij} \right) q_{iK} + v_{iK} \right] - \frac{\alpha_i \gamma}{n-1} u_{iK} \right) \right\}.$$

Here $q_{iK} = q(p_{iK})$, $v_{iK} = v(p_{iK})$ and $u_{iK} = u(q_{iK})$. Performing similar calculations as above leads to

$$p_{iK} = \frac{\sum_{j \in K} \alpha_j c_{ij}}{\sum_{j \in K} \alpha_j - \frac{|K|}{n-1} \gamma \alpha_i}.$$

3. Now we determine the equilibrium fixed fees. Take the call prices and fixed fees of networks $j \neq i$ as given, and consider the first-order condition of network i's profit maximum in (6) with respect to its fixed fee:

$$\frac{\partial \pi_i}{\partial F_i} = \frac{\partial \alpha_i}{\partial F_i} \left(\sum_{j=1}^{n} \alpha_j R_{ij} + F_i + Q_i - f_i \right) + \alpha_i \left(\sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial F_i} R_{ij} + 1 \right) = 0.$$

From (5), for all $i, j = 1, ..., n$ we have $\frac{\partial \alpha_j}{\partial F_i} = -\sigma H_{ji}$, where H_{ji} is the ji-element of matrix H. The first-order condition can then be solved for F_i as

$$F_i = f_i - Q_i + \alpha_i \left(\frac{1}{\sigma H_{ii}} - \sum_{j=1}^{n} \frac{H_{ji}}{H_{ii}} R_{ij} \right) - \sum_{j=1}^{n} \alpha_j R_{ij}.$$

Letting $\hat{R}$ be an $(n \times n)$ matrix with $\hat{R}_{ii} = \frac{1}{\sigma H_{ii}} - \sum_{j=1}^{n} \frac{H_{ji}}{H_{ii}} R_{ij}$ and $\hat{R}_{ij} = 0$ if $j \neq i$, we can write

$$F = f - Q + \left(\hat{R} - R \right) \alpha. \blacksquare$$

Thus we confirm the standard result of the efficiency of on-net prices under multi-part tariffs for the case of many asymmetric networks. If there are no call externalities ($\gamma = 0$) then $p_{ii} = c_{ii}$, while in the presence of the latter the efficient on-net price is below cost.

As concerns the off-net prices, in the absence of call externalities they are equal to weighted average off-net cost:

$$p_{iK} = \frac{\sum_{j\in K} \alpha_j c_{ij}}{\sum_{j\in K} \alpha_j}.$$

This is a natural generalization of the result for two networks. Furthermore, as in Jeon *et al.* (), Berger (2005) and Hoernig (2007), the off-net prices increase in γ and are above (weighted average) off-net cost if $\gamma > 0$. Expression (17) shows that network i sets its off-net price to a set K of networks as if it was setting a uniform off-net price to all networks, assuming they all have the same average market share as those in the set K.

Two special cases of off-net prices are a uniform off-net price

$$p_{iu} = \frac{\sum_{j\neq i} \alpha_j c_{ij}}{1 - \alpha_i - \gamma\alpha_i},$$

and price discrimination between all networks, with

$$p_{ij} = \frac{\alpha_j c_{ij}}{\alpha_j - \frac{1}{n-1}\gamma\alpha_i}.$$

We now consider equilibrium profits and market shares.

Proposition 6 *Equilibrium profits and market shares are, respectively,*

$$\pi_i^* = \alpha_i^2 \left(\frac{1}{\sigma H_{ii}} - \sum_{j=1}^{n} \frac{H_{ji}}{H_{ii}} R_{ij} \right), \tag{20}$$

$$\alpha^* = \left(I - \sigma B \left[h + R - \hat{R} \right] \right)^{-1} (\alpha_0 + \sigma B (Q - f)). \tag{21}$$

Proof. The expression for profits results from substituting equilibrium fixed fees into (6). Finally, substituting fixed fees into (4) yields the condition for the equilibrium market share. ■

One should take note that the expression for equilibrium profits (20) is very similar to the one in (10). Indeed, this similarity is no coincidence:

Corollary 2 *At the respective on-net prices, the expressions for equilibrium profits (10) under linear tariffs and (20) under multi-part tariffs are formally identical.*

Proof. With $p_{ii} = \frac{c_{ii}}{1+\gamma}$, we have $\frac{1-\eta L_{ii}}{1+\gamma\eta} = 1$. Thus (20) can we written as (10). ■

The same argument holds for expression (14), since at the off-net prices (17) the average Lerner index has value $\bar{L}_{ij} = \gamma\frac{\alpha_i}{1-\alpha_i}$, which again makes the additional term disappear in (14). These observations imply that the fundamental difference between competition in linear and multi-part tariffs lies in how usage prices are set, rather than in the existence or not of a fixed fee. Maybe surprisingly, the expression for equilibrium profits under linear tariffs turns out to be more general than the one under multi-part-tariffs, rather than less, as it applies to both cases (with different levels of retail prices, sure enough).

Note that an alternative expression for equilibrium profits under multi-part tariffs is $\pi_i^* = \alpha' J^{ii} \hat{R} \alpha$, which leads to the handy expression for joint equilibrium profits of

$$\sum_{i=1}^{n} \pi_i = \alpha' \hat{R} \alpha.$$

The right-hand side of (21) depends indirectly on α through $h + R - \hat{R}$ and off-net prices. Contrary to the two-network case, this is true even if there are no call externalities, since in this case the off-net prices are equal to off-net costs weighted by market shares. Only if off-net costs (including mobile termination rates) are symmetric will the dependence on α disappear. In the latter case (21) gives an explicit solution for market shares, but otherwise numerical methods need to be employed.

Uniform prices. Here we again derive equilibrium outcomes under uniform pricing, this time under multi-part tariffs.

Proposition 7 *If networks set uniform prices then the Nash equilibrium in multi-part tariffs has the following properties:*

1. *Equilibrium uniform prices are*

$$p_i = \sum_{j=1}^{n} \alpha_j c_{ij}.$$

2. *Equilibrium fixed fees and profits are*

$$\begin{aligned} F &= f - Q + \left(\hat{R} - R\right) \alpha, \\ \pi_i &= \alpha_i^2 \hat{R}_{ii}, \end{aligned}$$

where

$$\hat{R}_{ii} = \frac{1}{\sigma (n-1)} + (a_i - \bar{a}_{-i})\, q_i + (a_i - c_{ti})\, (\bar{q}_{-i} - q_i)\,. \tag{22}$$

Proof. With uniform prices, the call externality terms in (19) cancel, and network i maximizes over p_i:

$$\sum_{j=1}^{n} \alpha_j \left((p_i - c_{ij}) q_i + v_i\right) = (p_i - \sum_{j=1}^{n} \alpha_j c_{ij}) q_i + v_i.$$

The equilibrium prices follow immediately.

As concerns fixed fees and profits, (18) and (20) still apply, but we can simplify the expression for $\hat{R}_{ii}$. Remember from above that under uniform pricing $H_{ii} = n - 1$ and $H_{ij} = -1$. Thus

$$\begin{aligned}\hat{R}_{ii} &= \frac{1}{\sigma(n-1)} - R_{ii} + \frac{1}{n-1}\sum_{j\neq i} R_{ij} \\ &= \frac{1}{\sigma(n-1)} + (a_i - \bar{a}_{-i})\, q_i + (a_i - c_{ti})(\bar{q}_{-i} - q_i)\end{aligned}$$

where $\bar{a}_{-i} = \frac{1}{n-1}\sum_{j\neq i} a_j$, $\bar{q}_{-i} = \frac{1}{n-1}\sum_{j\neq i} q_j$ as defined above. ■

Thus as with linear tariffs, under uniform pricing call externalities do not influence equilibrium outcomes. In this case call prices are set equal to average perceived marginal cost. Expression (22) now is the generalization as much of the equilibrium profits $\pi_i = 1/4\sigma$ in LRT98a (p. 21) to many asymmetric networks as of (6) in Andersson and Hansen (2009) to elastic demand and multi-part tariffs. See Section 6 for a discussion the effect MTRs under uniform pricing.

4 Theory: More Results for Symmetric Networks

4.1 Preliminaries

In this section we will revisit the equilibrium outcomes under the assumption of symmetry and present the new results that arise. First of all, though, we will consider how the size of the preference space $S(n)$ affects equilibrium outcomes as the number of networks becomes large.

Under symmetry, market shares are $\alpha_i = 1/n$. Denote perceived marginal costs as c_{on} and c_{of} for on- and off-net calls, respectively. The equilibrium surplus from on- and off-net calls is h_{on} and h_{of}, with $H_{on} = H_{ii}$ and $H_{of} = H_{ij}$. Let $R(p) = (p - c_{on})\, q$, $R_{on} = R(p_{on})$, $R_{of} = R(p_{of})$,[14] and F_0, Q_0 and f_0 be equilibrium fixed fees, FTM termination profits and fixed cost per subscriber, respectively. Monopoly call prices are $p^m_{on} = \eta c_{on}/(\eta - 1)$ and $p^m_{of} = \eta c_{of}/(\eta - 1)$. Thus profits from (6) become

$$\pi = \frac{1}{n}\left(\frac{1}{n}R_{on} + \frac{n-1}{n}R_{of} + F_0 + Q_0 - f_0\right). \tag{23}$$

Our first result is of technical nature and applies to both linear and multi-part tariffs:

Lemma 5 *With n symmetric networks, we have*

$$H_{on} = \frac{n-1}{1 - n\sigma(h_{on} - h_{of})}, \; H_{of} = -\frac{1}{1 - n\sigma(h_{on} - h_{of})}. \tag{24}$$

[14] The occurrence of c_{on} in R_{of} is not a typo — it is due the cancelling-out of mobile-to-mobile interconnection payments under symmetry.

Proof. Remember that $BB = nB$ and that with symmetric networks and prices $Bh = (h_{on} - h_{of})\,B$. Then

$$\begin{aligned} B &= (I - \sigma Bh)^{-1}(I - \sigma Bh)\,B = (I - \sigma Bh)^{-1}\left[B - \sigma\,(h_{on} - h_{of})\,BB\right] \\ &= \left[1 - \sigma n\,(h_{on} - h_{of})\right](I - \sigma Bh)^{-1}B = \left[1 - \sigma n\,(h_{on} - h_{of})\right]H. \end{aligned}$$

Thus $H = B/\left[1 - \sigma n\,(h_{on} - h_{of})\right]$. ■

It now follows from both Lemma (2) and Proposition (1) that if the symmetric equilibrium is stable in customer expectations then $H_{on} > 0$ and $H_{of} < 0$.

4.2 The Size of the Preference Space

An important characteristic of any model involving many firms is the limiting behaviour of equilibrium outcomes as the number of firms becomes large. In models with a fixed mass of consumers, such as ours, clearly each firm's market share and profits converge to zero as long as profits per consumer remain bounded. A wholly different and more interesting question is whether networks maintain some local market power, or put differently, whether *industry* profits are positive in the limit. As Chen and Riordan (2007, p. 898) have pointed out, for example in the Salop circular city model (1979) each firm's demand elasticity converges to infinity as the number of networks grows, and as a result industry profits converge to zero. Chen and Riordan's model, on the other hand, is constructed with the explicit intention of keeping a degree of local market power. It converges to monopolistic competition as the number of firms becomes large, and industry profits remain bounded away from zero.

We will now show that whether one or the other happens in our modelling framework depends on the distance between networks $l(n)$ in preference space. For simplicity, consider multi-part tariffs. From (5), in our model demand elasticity with symmetric tariffs is

$$\varepsilon^H = -\frac{d\alpha_i}{dF_i}\frac{F_i}{\alpha_i} = n\sigma H_{on}F_0 = \frac{1}{tl(n) - (h_{on} - h_{of})\,/(n-1)}F_0.$$

First of all, as indicated in Proposition 1, if there are tariff-mediated network externalities, i.e. $h_{on} > h_{of}$, then the symmetric equilibrium becomes unstable for large n if $S(n)/n$ converges to zero.[15] On the other hand, if the same expression is assumed to increase then stability is not only maintained by strengthened. In the limiting case $S(n)/n \to k$ for some positive constant k stability is maintained if $k > (h_{on} - h_{of})\,/2t$.

Abstracting now from the issue of stability by assuming $h_{on} \leq h_{of}$ or $\lim_{n\to\infty}\frac{(n-1)l(n)}{h_{on}-h_{of}} > \frac{1}{2t}$, we can now look at the demand elasticity itself. Clearly it becomes infinite if $l(n)$ converges to zero, which is the case in particular if $S(n)$ grows linearly with n as in Armstrong and Wright (2009) or if the size of the preference space is constant. Thus under the latter assumptions industry profits will converge to zero.

[15] This argument applies both to given tariffs and when $h_{on} > h_{of}$ in the limit for a sequence of equilibrium tariffs for each n.

On the other hand, if $l(n)$ converges to a positive constant then the size of the preference space is of the order of n^2, as in von Ungern-Sternberg (1991) and Chen and Riordan (2007). In this case the demand elasticity remains finite and industry profits are positive in the limit.

These considerations also allow us to draw an interesting parallel with the logit model, as for example employed by Calzada and Valletti (2008). Assuming for simplicity $h_{on} = h_{off}$, and denoting the differentiation parameter $\mu(n)$, market shares are given by

$$\alpha_i = \frac{\exp\left(-\mu(n) F_i\right)}{\sum_{j=1}^{n} \exp\left(-\mu(n) F_j\right)}, \tag{25}$$

with demand elasticity

$$\varepsilon^L = \mu(n)\alpha_i \left(1 - \alpha_i\right) \frac{F_0}{\alpha_i} = \frac{(n-1)\,\mu(n)}{n} F_0 \sim \mu(n) F_0.$$

Thus in the logit model, as the number of networks becomes large, demand elasticity remains finite and the model becomes one of monopolistic competition, if and only if it is assumed that $\mu(n) \to \bar{\mu} < \infty$. In particular, for a large number of networks, the logit model with a constant differentiation parameter μ should lead to similar predictions as our model with $l(n)$ constant, or, equivalently, if $\sigma(n)n^2$ converges to μ.

4.3 Symmetric Equilibrium with Linear Tariffs

We will now consider how equilibrium outcomes change under linear and multi-part tariffs, respectively. With linear tariffs, the condition describing symmetric off-net prices becomes, with $L_{on} = L_{ii}$ and $L_{of} = L_{ij} = \bar{L}_{ij}$,

$$L_{of} = \frac{1}{\eta} + \frac{n\left(1 + \gamma\eta\right)^{-1} - 1}{n - 1} \left(L_{on} - \frac{1}{\eta}\right). \tag{26}$$

As for the case with two networks (see Hoernig 2007), this relation between off-net and on-net Lerner indices is given by a straight line that passes through the monopoly point $\left(\frac{1}{\eta}, \frac{1}{\eta}\right)$. With $\gamma = 0$ both Lerner indices are equal, while $L_{of} > L_{on}$ for $\gamma > 0$ as long as the on-net price is smaller than the monopoly price. As a result, we have $p_{of} > p_{on}$ whenever either $\gamma > 0$ and $c_{of} \geq c_{on}$. Actually, we have $L_{of} > 1/\eta$, i.e. the off-net price is above the monopoly price based on the perceived off-net cost, whenever $\gamma\eta > n - 1$ (generalizing Berger (2004) where $n = 2$).

We can now state how the number of networks influences the equilibrium prices. Since it makes performing comparative statics significantly easier we assume a constant demand elasticity.

Proposition 8 *Assume that demand elasticity is constant at $\eta > 1$ and consider stable Nash equilibria in linear tariffs.*

1. *If t is large enough and $R\left(p_{of}^m\right) + Q_0 > f_0$, or if $\gamma \approx 0$ and $a \approx c_t$, both on- and off-net prices decrease in n if $l(n)$ decreases.*

2. *If $\gamma \approx 0$, and either τ large enough or $a \approx c_t$, then the on/off-net differential decreases in n if mobile-to-mobile termination rates are above cost and $l(n)$ decreases. If $\gamma > 0$ or $a > c_t$, the off-net price remains bounded away from the on-net price.*

3. *As the number of networks becomes large, the off-net price converges to the (Ramsey) break-even price, and industry profits converge to zero, if and only if $l(n)$ converges to zero.*

Proof. In the following assume that the stability condition holds for the n considered, i.e. that $(h_{on} - h_{of}) < 2tS(n)/n$.

1. With symmetric networks and constant-elasticity demand, the Nash equilibrium is defined by the two conditions (8) and (26), or with $\tau \equiv 1/t$,

$$\begin{aligned} \tau \left[2R_{on} + (n-2)\,R_{of} + n\,(Q_0 - f_0)\right] + \frac{n\tau\,(h_{on} - h_{of}) - 2S(n)}{n-1}\frac{1 - \eta L_{on}}{1 + \gamma\eta} &= 0, \quad (27) \\ \frac{p_{of} - c_{of}}{p_{of}} - \frac{1}{\eta} - \frac{n\,(1+\gamma\eta)^{-1} - 1}{n-1}\left(\frac{p_{on} - c_{on}}{p_{on}} - \frac{1}{\eta}\right) &= 0. \quad (28) \end{aligned}$$

We will now determine how p_{on} and p_{of} look like close to $\tau = 0$, i.e. we will express them as functions of τ for small τ. From the above first-order conditions it follows that, for given n, in the limit $\tau = 0$ networks set monopoly prices $p^m_{on} = \eta c_{on}/(\eta - 1)$ and $p^m_{of} = \eta c_{of}/(\eta - 1)$, with corresponding call profits $R^m_{on} = R\,(p^m_{on})$ and $R^m_{of} = R\left(p^m_{of}\right)$. Letting $p_{on} = p^m_{on} - \delta\tau$ and expanding (27) around $\tau = 0$ results in

$$\phi_n - \delta\frac{2S(n)\,(\eta - 1)}{(n-1)\,(1+\gamma\eta)\,p^m_{on}} + O\,(\tau) = 0,$$

where $\phi_n = 2\left(R^m_{on} - R^m_{of}\right) + n\left(R^m_{of} + Q_0 - f_0\right)$ is positive by $R^m_{on} \geq R^m_{of} > Q_0 - f_0$. Solving for δ implies that for small τ we have

$$p_{on} = p^m_{on} - \frac{\phi_n}{nl(n)}\frac{p^m_{on}\,(1+\gamma\eta)}{(\eta - 1)}\tau + O\left(\tau^2\right). \tag{29}$$

Clearly, $\phi_n/nl(n)$ increases in n if $l(n)$ decreases.

Solving (28) for p_{of} and expanding p_{of} about $\tau = 0$ leads to

$$\begin{aligned} p_{of} &= p^m_{of} - \frac{n - 1 - \gamma\eta}{(1+\gamma\eta)\,(n-1)}\frac{p^m_{of}}{p^m_{on}}\delta\tau + O\left(\tau^2\right) \\ &= p^m_{of} - \frac{(n-1-\gamma\eta)\,\phi_n}{n(n-1)l(n)}\frac{p^m_{of}}{\eta - 1}\tau + O\left(\tau^2\right) \end{aligned} \tag{30}$$

Now $(n - 1 - \gamma\eta)\,\phi_n/n(n-1)l(n)$ increases with n if $l(n)$ decreases while $n - 1 \geq \gamma\eta$.

On the other hand, at $\gamma = 0$ and $a = c_t$ condition (28) implies $p_{of} = p_{on}$. Applying the implicit function theorem to (27)

$$R_{on} + Q_0 - f_0 - tl(n)\,(1 - \eta L_{on}) = 0$$

we obtain

$$\frac{dp_{on}}{dn} = \frac{dp_{of}}{dn} = \frac{1 - \eta L_{on}}{\tau R'_{on} + l(n)\eta \frac{c_{on}}{p_{on}^2}} l'(n).$$

Since the leading factor is positive, both derivatives are negative if $l(n)$ decreases.

In both cases, the result then follows by continuity.

2. If $\gamma = 0$ then $p_{of} = p_{on}c_{of}/c_{on}$, and we have $p_{of} - p_{on} = (c_{of}/c_{on} - 1)\, p_{on}$. This decreases with p_{on} but remains positive in the limit because p_{of} will not go below the break-even price.

If $\gamma > 0$, note that $\left(n\,(1+\gamma\eta)^{-1} - 1\right)/(n-1)$ increases with n towards the limiting value $1/(1+\gamma\eta)$. Thus as n increases the difference in Lerner indices becomes smaller and converges to a limit value. Even if termination is priced at cost the off-net price will remain above the on-net price by the factor δ given by

$$\frac{p_{on}\delta - c_{on}}{p_{on}\delta} - \frac{1}{\eta} = \frac{1}{1+\gamma\eta}\left(\frac{p_{on} - c_{on}}{p_{on}} - \frac{1}{\eta}\right),$$

or $\delta = \frac{(1+\gamma\eta)c_{on}}{(\eta-1)\gamma p_{on} + c_{on}} > 1$ since p_{on} is below the monopoly price $\eta c_{on}/(\eta - 1)$.

3. As $n \to \infty$, (27) becomes

$$R_{of} + Q_0 - f_0 = \frac{t\,(1 - \eta L_{on})}{1+\gamma\eta} \lim_{n\to\infty} l(n). \ \blacksquare$$

While it is not surprising that call prices decrease as the number of networks increases and the preference space does not extend too fast, it is not obvious that the off-net price decreases faster than the on-net price. The latter does occur because off-net calls come to make up a larger and larger portion of calls on each network, increasing the cost of setting high off-net prices for strategic reasons. As we have seen, this strategic incentive does not disappear in the limit in the sense that the on-net price will remain below the off-net price, but is weakened since in the limit all call revenue is brought in through off-net calls.

Finally, competition drives the price for these off-net calls towards the break-even level if the preference space does not expand fast enough. Indeed, the last point of the Preposition confirms our findings about call elasticities: Industry profits are positive (and finite) in the limit if $l(n)$ converges to a constant. If $l(n)$ decreases to zero then industry profits converge to zero.

4.4 Symmetric Equilibrium with Multi-Part Tariffs

Since Nash equilibria with multi-part tariffs are more amenable to analysis than under linear tariffs, the case of symmetric equilibria with many networks has already been considered by several authors, such as Calzada and Valletti (2008) under logit demand and Armstrong and Wright (2007). The latter also considered call externalities, but of a different functional form. Thus our results complement both previous papers.

With multi-part tariffs and symmetric networks, the on-net price remains equal to $p_{on} = c_{on}/(1+\gamma)$. The off-net price becomes

$$p_{iu} = p_{iK} = p_{ij} = \frac{n-1}{n-1-\gamma} c_{of}. \tag{31}$$

This off-net price decreases with n and converges to perceived marginal cost c_{of} as the number of networks becomes large. As with linear tariffs (at least under certain conditions on model parameters), the on/off-net differential decreases if more networks are present since the on-net price remains constant at the efficient level.

Fixed fees are

$$F_0 = f_0 - Q_0 + tl(n) - \frac{h_{on} - h_{of}}{n-1} - \frac{2}{n}R_{on} - \frac{n-2}{n}R_{of},$$

and equilibrium network profits can be written as

$$\pi = \frac{1}{n}\left(tl(n) - \frac{h_{on} - h_{of}}{n-1} + \frac{1}{n}\left(R_{of} - R_{on}\right)\right). \tag{32}$$

Clearly, industry profits converge to $t \lim_{n\to\infty} l(n)$, which as under linear tariffs converge to zero if and only if $l(n)$ converges to zero.

Armstrong and Wright (2007), assuming call externalities of the form bq_{of}, rather than $\gamma u\left(q_{of}\right)$, obtain equilibrium call prices of (in our notation):

$$p_{on} = c_{on} - b,\ p_{of} = c_{of} + \frac{b}{n-1}.$$

They do not state the equilibrium value of fixed fees, while individual networks' profits are (their Π/K from (21), in our notation)

$$\pi = \frac{1}{n^2}\left(2t - \frac{n}{n-1}\left(h_{on} - h_{of}\right) - R_{on} + R_{of}\right).$$

This is formally equal to (32) with $S(t) = (n-1)$, as mentioned before, and therefore profits per subscriber converge to zero.

The corresponding expressions for equilibrium fixed fees and profits in Calzada and Valletti (2008) are (using our notation of (25), with $R_{on} = 0$ and $R_{of} = mq_{of}$ due to the absence of call externalities)

$$\begin{aligned} F_0 &= f_0 - Q_0 + \frac{n - \mu\left(v_{on} - v_{of}\right)}{\mu\left(n-1\right)} - \frac{n-2}{n}mq_{of} \\ \pi &= \frac{1}{n}\left(\frac{n - \mu\left(v_{on} - v_{of}\right)}{\mu\left(n-1\right)} + mq_{of}\right) \end{aligned}$$

These expressions are equal to ours for $\sigma(n) = \mu/n^2$. As we have pointed out above, when the number of networks becomes large one should expect industry profits to converge to a finite value in the logit model, and this is indeed what we find since $n/(n-1)\mu \to 1/\mu > 0$.

We have noted above that as the number of networks grows most calls become off-net calls. Thus one should expect that consumers would benefit from lower off-net prices. On the other hand, it has been shown in the duopoly models such as Gans and King (2001) that tariff-mediated network externalities caused by high off-net prices more than compensate consumers because they make networks compete harder through lower fixed fees. We now show that this is no longer true with more networks, i.e. the direct effect on consumers of higher off-net prices will outweigh the competition-enhancing network effects.

Proposition 9 *In symmetric equilibrium in multi-part tariffs, consumer surplus decreases in the off-net price if the number of networks is large enough, i.e.* $n > \tilde{n}$ *where* $\tilde{n} > 2$ *and is defined by*

$$(\tilde{n}-2)(\tilde{n}-1) = \frac{1+\gamma\eta}{\eta(c_{of}-c_{on})/c_{of}+\gamma\eta}.$$

Proof. In symmetric equilibrium with multi-part tariffs, consumer surplus becomes

$$\begin{aligned} CS &= A + \frac{1}{n}h_{on} + \frac{n-1}{n}h_{of} - F_0 - \frac{t}{4}l(n) \\ &= \frac{n-2}{n}(R_{of}+h_{of}) - \frac{1}{n(n-1)}h_{of} + const, \end{aligned} \tag{33}$$

where the last constant does not depend on p_{of}. Then

$$\begin{aligned} \frac{dCS}{dp_{of}} &= \frac{n-2}{n}\left(\frac{p_{of}-c_{on}}{p_{of}}+\gamma\right)p_{of}q'_{of} + \frac{1}{n(n-1)}\left(q_{of}-\gamma p_{of}q'_{of}\right) \\ &= \left[-\frac{n-2}{n}\eta\left(\frac{p_{of}-c_{on}}{p_{of}}+\gamma\right)+\frac{1+\gamma\eta}{n(n-1)}\right]q_{of} \end{aligned}$$

Since $p_{of} \geq c_{of}$ for all $n \geq 2$ and $\gamma \geq 0$, an upper bound $\tilde{n}$ on the number of networks that are necessary for consumer surplus to be decreasing in p_{of} is given by

$$-\frac{\tilde{n}-2}{\tilde{n}}\eta\left(\frac{c_{of}-c_{on}}{c_{of}}+\gamma\right)+\frac{1+\gamma\eta}{\tilde{n}(\tilde{n}-1)} = 0$$

or

$$(\tilde{n}-2)(\tilde{n}-1) = \frac{1+\gamma\eta}{\frac{c_{of}-c_{on}}{c_{of}}\eta+\gamma\eta}$$

Since the term on the left is increasing in $n \geq 2$, for any $n > \tilde{n}$ we have $dCS/dp_{of} < 0$. ■

Note that the first term in (33), which translates total welfare created by one off-net call, does not exist in duopoly models since there $n = 2$. Furthermore, as the number of networks increases this term becomes rapidly much more important (by a factor of n^2) than the second one which indicates tariff-mediated network effects. Thus even for markets with few more than two networks high off-net prices will lower consumer surplus even if they make networks compete more fiercely at the same time. Indeed, Harbord and Hoernig (2010) found exactly this effect in their calibration of the market for mobile telephony in the United Kingdom.

5 Application 1: Fixed-To-Mobile Termination and the Waterbed effect

In this section we will explore what our previous results imply for the fixed-to-mobile "waterbed effect", i.e. the statement according to which termination profits accruing from interconnection to the fixed network lead to reductions in prices for mobile retail customers. We will not consider the individual incentives of networks to set higher fixed-to-mobile termination rates, as in Gans and King (2000) and by Wright (2002). Rather, we are interested in the equilibrium effect of changes in the (regulated) termination rate.

Linear Tariffs. With linear tariffs, by (11) and (13), $\partial^2\pi_i/\partial p_{ii}\partial Q_i$ and $\partial^2\pi_i/\partial p_{ij}\partial Q_i$ both have the sign of $-H_{ii} < 0$. Thus higher fixed-to-mobile profits Q_i on network i lowers both p_{ii} and p_{ij}, and its market share increases. Thus each network on its own prefers being able to charge a high termination charge for fixed-to-mobile calls. As long as prices are strategic complements, all equilibrium prices will fall in response.

A different issue is to what extent all mobile networks and their subscribers benefit from higher fixed-to-mobile termination charges. The following proposition gives the answer:

Proposition 10 (rewrite) *Assume that demand elasticity η is constant. If networks are sufficiently symmetric, and call externalities and mobile-to-mobile termination rates are small, then under linear tariffs equilibrium call prices decrease and industry profits increase in fixed-to-mobile termination profits, with networks keeping the share of*

$$\omega \approx \frac{tl(n)}{tl(n) + R'_{on}p_{on}^2/\eta c_{on}}.$$

Proof. We show that the result holds for symmetric networks, zero call externalities and cost-based mobile-to-mobile termination. The general case then follows by continuity. Assume $Q_i = Q_0$ for all i, and consider a change in Q_0. With n symmetric networks, we have $p_{of} = p_{on}$ and the on-net price is given by condition (27) through

$$R_{on} + Q_0 - f_0 - tl(n)\,(1 - \eta L_{on}) = 0.$$

With $R'_{on} = dR_{on}/dp_{on} = q_{on}\,(1 - \eta L_{on})$, we have

$$\frac{dp_{on}}{dQ_0} = -\frac{1}{R'_{on} + t\eta l(n)c_{on}/p_{on}^2} < 0,$$

and the effect on industry profits is given by

$$\begin{aligned} \frac{dn\pi}{dQ_0} &= \frac{d}{dQ_0}\left(R_{on} + Q_0 - f_0\right) = R'_{on}\frac{dp_{on}}{dQ_0} + 1 \\ &= \frac{tl(n)}{tl(n) + R'_{on}p_{on}^2/\eta c_{on}} > 0. \ \blacksquare \end{aligned}$$

Thus of each additional cent of fixed-to-mobile termination profits the networks keep the share ω and pass the rest $1 - \omega$ on to subscribers through lower call prices. The extent of the waterbed effect under linear tariffs depends on the size of $tl(n)$, that is: Lower transport cost t, or smaller distance between networks in preference space, leads to more intense competition between networks and therefore networks retain a smaller share ω of these termination profits. This waterbed effect is full, and even then only in the limit, if and only if $l(n)$ converges to zero.

Multi-part tariffs. With multi-part tariffs the outcome is much easier to establish: Remember from (18) that equilibrium fixed fees are given by

$$F = f - Q + \left(\hat{R} - R\right)\alpha,$$

where Q is the vector of per-customer profits from fixed-to-mobile termination, and that call prices do not depend on Q. Thus with multi-part tariffs all termination profits are handed over to mobile consumers through lower fixed fees, i.e. there is a full waterbed effect on each individual network, even in the case of a Nash equilibrium with many asymmetric networks. As a consequence, equilibrium profits do not depend on the common level of fixed-to-mobile termination profits. On the other hand, profits do respond to changes in one's own Q_i, as can be seen from (21).

While the result of a full waterbed is clear in this context, it is not robust to modelling changes. As we have seen above, changing the type of tariff already leads to a less than full waterbed effect. Wright (2002), who showed that the waterbed effect is full for n symmetric networks, also made clear that this result is an artefact of the Hotelling model where market-wide (fixed) cost increases do not feed through into lower profits. If costs do feed through, for example because subscription demand is elastic, then networks retain some termination revenue and the waterbed effect again is not full. Genakos and Valletti (2009) show this directly for a model with a logit demand structure.

Finally, while in the present model the waterbed effect is full in aggregate and at the level of each single network, Hoernig (2008) showed in a dynamic model of network competition that, while the waterbed effect is still full on aggregate, networks with high (low) fixed-to-mobile termination profits retain (overcompensate) a part of the latter. Essentially, what is at work here is a dynamic version of the "fat cat effect", while networks with small Q_i subsidize customer acquisition from other sources.

6 Application 2: Mobile-to-mobile termination

In this section we consider the effects of mobile-to-mobile termination on prices and profits. For simplicity, we assume that networks are symmetric.

Linear tariffs. Under linear tariffs, we obtain the following results. Point 1 in particular is new and surprising at first sight.

Proposition 11 *Assume demand elasticity $\eta > 1$ is constant and that networks are symmetric.*

1. *For large t and $a > c_t$ both on- and off-net prices increase in a if $n > 2$.*

2. *If γ is small and $a \approx c_t$, then the on-net price decreases and the off-net price increases in a.*

3. *If γ is small and t is large then profits are increasing in a at $a = c_t$.*

Proof. 1. For large t, i.e. small $\tau = 1/t$, we have from (29) and (30) that

$$\begin{aligned}
\frac{dp_{on}}{da} &= (n-2)\left[-R'(p_{of}^m)\right] \frac{p_{on}^m (1+\gamma\eta)}{nl(n)(\eta-1)} \frac{dp_{of}}{da}\tau + o(\tau) \\
\frac{dp_{of}}{da} &= \frac{\eta}{\eta-1} + o(\tau)
\end{aligned}$$

The off-net price increases with the MTR to first-order, while the on-net price increases if $n > 2$ and $a > c_t$ because then $R'(p_{of}^m) < 0$.

2. Assume now $\gamma = 0$, with $p_{of} = \frac{c_{on}+a-c_t}{c_{on}} p_{on}$. The first-order condition for the on-net price becomes

$$\frac{2}{n} R_{on} + \frac{n-2}{n} R_{of} + \left(\frac{v_{on} - v_{of}}{n-1} - tl(n)\right)(1 - \eta L_{on}) = n(f_0 - Q_0)$$

implying the effect on the on-net price at $a = c_t$ of

$$\begin{aligned}
\left.\frac{dp_{on}}{da}\right|_{a=c_t} &= -\frac{\frac{n-2}{n} R'_{on} \frac{p_{on}}{c_{on}} + \frac{q_{on}(1-\eta L_{on})}{n-1} \frac{p_{on}}{c_{on}}}{\frac{2}{n} R'_{on} + \frac{n-2}{n} R'_{on} + tl(n)\eta \frac{c_{on}}{p_{on}^2}} \\
&= -\frac{\left((n-1)^2 + 1\right) R'_{on} p_{on}/c_{on}}{n(n-1) R'_{on} + tS(n)\eta c_{on}/2p_{on}^2} < 0,
\end{aligned}$$

and on the off-net price of

$$\begin{aligned}
\left.\frac{dp_{of}}{da}\right|_{a=c_t} &= \frac{1}{c_{on}} p_{on} + \left.\frac{dp_{on}}{da}\right|_{a=c_t} \\
&= \frac{(n-2) R'_{on} + tS(n)\eta c_{on}/2p_{on}^2}{n(n-1) R'_{on} + tS(n)\eta c_{on}/2p_{on}^2} \frac{p_{on}}{c_{on}} > 0.
\end{aligned}$$

The results for small but positive γ follow by continuity.

3. Symmetric equilibrium profits are

$$\pi = \frac{1}{n}\left(\frac{1}{n} R_{on} + \frac{n-1}{n} R_{of} + Q_0 - f_0\right),$$

and we have at $\gamma = 0$

$$\begin{aligned}
\left.\frac{d(n\pi)}{da}\right|_{a=c_t} &= \frac{1}{n} R'_{on} \left.\frac{dp_{on}}{da}\right|_{a=c_t} + \frac{n-1}{n} R'_{on} \left.\frac{dp_{of}}{da}\right|_{a=c_t} \\
&= \left(\frac{-nR'_{on} + (n-1)\, tS(n)\eta c_{on}/2p_{on}^2}{n(n-1) R'_{on} + tS(n)\eta c_{on}/2p_{on}^2}\right) \frac{R'_{on}}{n} \frac{p_{on}}{c_{on}} \\
&= \frac{-R'_{on} + (n-1)\, tS(n)\eta c_{on}/2np_{on}^2}{n(n-1) R'_{on} + tS(n)\eta c_{on}/2p_{on}^2} \frac{R'_{on} p_{on}}{c_{on}}.
\end{aligned}$$

The latter is positive if t is large enough. ■

Points 2 and 3 generalize corresponding results in the duopoly models of Laffont, Rey and Tirole (1998b) and Berger (2004), the latter with call externalities, to an arbitrary number of symmetric networks. Off-net prices increase in a since it directly raises off-net cost. Profits increase due to these higher off-net prices, even though at $a = c_t$ on-net prices decrease.

Still, contrary to what was "visible" in the existing literature on duopoly models, point 1 shows that the decrease in on-net prices no longer occurs if $a > c_t$ and there are more than two networks in the market. As the proof of the Proposition indicates, the fact that for large transport cost and $a > c_t$ the off-net price is set above p_{on}^m implies that networks make smaller profits from reciprocal off-net calls as compared to $p_{off} = p_{on}^m$. These smaller profits then lead to lessening of competition for subscribers and therefore higher on-net prices. Note that the corresponding term in dp_{on}/da disappears if $n = 2$.

Multi-part tariffs. As concerns multi-part tariffs, for simplicity we again consider the symmetric equilibrium. We derive a generalization of the result of Gans and King (2001) to n networks, and compare our results with those of Calzada and Valletti (2008) who made a similar analysis for symmetric networks under a logit demand specification. Industry profits are

$$n\pi = tl(n) + \frac{h_{of} - h_{on}}{n-1} + \frac{R_{of} - R_{on}}{n}. \tag{34}$$

The effect of the mobile-to-mobile MTR a on profits is indirect, through the effect of the off-net price p_{of} on h_{of} and R_{of}. As we have seen in the proof of point 1 of Proposition 5, if both h_{of} and R_{of} had the same relative weight in profits then p_{of} would be set efficiently. As it happens, though, with n networks h_{of} has larger weight $n/(n-1)$ relative to R_{of}, which implies that networks want to set an off-net price that is lower than the socially optimal value. This is what Gans and King (2001) have shown for $n = 2$. On the other hand, our result implies that this effect becomes less strong as n becomes large since $n/(n-1) \to 1$. Formally, when choosing their jointly profit-maximizing off-net price, networks maximize

$$\frac{n}{n-1}\left(v_{of} + \gamma u_{of}\right) + \left(p_{of} - c_{on}\right) q_{of}.$$

The maximum is obtained at

$$p_{of} = \frac{(n-1)\, c_{on}}{n\gamma + n - 1 + 1/\eta} < \frac{c_{on}}{1+\gamma} = p_{on}.$$

The above expression for p_{of} and (31) then imply

$$a = c_t - \frac{(n+1)\,\gamma\eta + 1}{n\gamma\eta + (n-1)\,\eta + 1} c_{on}. \tag{35}$$

The above discussion is summed up in the following Proposition:

Proposition 12 *If networks compete in multi-part tariffs, the jointly profit-maximizing MTM termination rate is set below the efficient level. It decreases in γ and increases in η and n.*

Thus a will be lower in the presence of call externalities. This is intuitive, as the aim of setting a low MTR is to reduce network effects which make networks compete harder. These network effects are stronger in the presence of call externalities.

As for related results in the literature, Armstrong and Wright's (2007) profit-maximizing off-net price and MTR can be stated as

$$\begin{aligned} p_{of} &= \frac{(n-1)\,c_{on} - nb}{n-1+1/\eta} \\ a &= c_t - \frac{c_{on}}{(n-1)\,\eta + 1} - \frac{(n^2-1)\,\eta + 1}{\eta n - \eta + 1}\frac{b}{n-1}. \end{aligned}$$

While the expressions differ somewhat from those in our model, the implied economic effects are the same. For $\gamma = b = 0$, (35) and the latter result on MTRs can be rewritten as

$$\frac{a - c_t}{c_{on}} = -\frac{1}{(n-1)\,\eta + 1}, \tag{36}$$

which is identical to Calzada and Valletti (2008, p. 1231). Thus this result seems fairly robust to different specifications of demand which assume full coverage.

As the number of networks increases, the joint profit-maximizing MTR converges towards $a = c_t - \frac{\gamma}{\gamma+1} c_{on}$ while p_{of} converges to the efficient call prices $c_{on}/(1+\gamma)$ (in Armstrong and Wright, similarly $p_{of} \to c_{on} - b$ and $a \to c_t - b$). The MTR remains below cost because the jointly profit-maximizing off-net price converges to the efficient price. Therefore the MTR converges to cost only in the absence of call externalities.

Together with our result from Proposition 9 this implies that for the case of multiple networks there is no trade-off between increasing welfare and profits on the one hand, and lower consumer surplus on the other, due to lower termination rates. Indeed, consumers and networks will share the benefits from bringing termination rates down closer to their efficient level.

Uniform tariffs. As a final point we consider termination rates under both linear and multi-part uniform tariffs. Remember that profits under uniform tariffs are

$$\pi_i = \alpha_i^2 \left(\frac{1 - \eta L_i}{\sigma\,(n-1)} + (a_i - \bar{a}_{-i})\, q_i + (a_i - c_{ti})\,(\bar{q}_{-i} - q_i) \right),$$

where $L_i = 0$ with a uniform two-part tariff. In symmetric equilibrium, industry profits become

$$n\pi = tl(n)\,(1 - \eta L),$$

where L increases in a under linear tariffs and continues equal to zero under two-part tariffs. Thus the result that profits increase in the MTR under linear tariffs holds true even with many networks, and continues to be true if they are asymmetric.

On the other hand, it is clear that while industry profits are independent of the MTR under symmetry if firms charge multi-part tariffs, departures from the symmetric setting immediately show that this “profit neutrality result” is not robust, as for example mentioned in Andersson and Hansen (2009). In general, whether a higher or lower MTR increases profits depends on networks’ relative size. Call prices are equal to average cost and the latter will be lower on larger networks due to their larger share of on-net calls. Thus if all networks set the same MTR a large networks set a lower call price and originate longer calls. The above expression for profits shows that in this case larger networks would prefer smaller MTRs than small networks, who would become net receivers of traffic.

7 Conclusions

In the preceding sections, we have presented an extension of network competition models with tariff-mediated externalities to an arbitrary number of networks with asymmetric costs and market shares, developing the equilibrium theory and presenting applications to the effects of mobile termination rates. Apart from demonstrating that this extension is tractable and generates new insights on the theory side, we have demonstrated that in going beyond the duopoly case some acquired policy conclusions from duopoly models must be reconsidered in the more realistic case of at least three competing networks.

We believe that our modelling work on the theory side opens the doors to more practical and policy-relevant exercises, such as Harbord and Hoernig (2010) which is based on the present work, while the practical insights are immediately relevant for the ongoing policy debate on mobile termination regulation.

Issues that we have not dealt with here are variable subscription demand and non-uniform calling patterns.[16] While their incorporation into the present modelling framework is straightforward, for reasons of space they will be analyzed in separate papers.

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