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ABSTRACT

Modelling Time and Macroeconomic Dynamics*

In this paper, we analyze the importance of the frequency of decision making for macroeconomic dynamics. We explain how the frequency of decision making (period length) and the unit of time measurement (calibration frequency) differ and study the implications of this difference for macroeconomic modelling. We construct a generic dynamic general equilibrium model that nests a wide range of macroeconomic models and which leaves the period length as an undetermined parameter. We provide a series of examples (variations of the Cass-Koopmans and the New Keynesian models) that fit into this framework and use these to do comparative dynamics with respect to the period length. In particular, we analyze local stability and how this is affected by changes in the period length. We find that in models with endogenous capital accumulation, as the period gets longer, indeterminacy occurs less often. Moreover, as economic agents become less patient and as capital depreciates more, indeterminacy also occurs less often. We also show that, in the case of the New Keynesian model, standard continuous and discrete time versions have entirely different local stability properties due to a discontinuity at zero period length.

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"*The element of Time, which is the centre of the chief difficulty of almost every economic problem, is itself absolutely continuous: Nature knows no absolute partition of time into long periods and short; but the two shade into one another by imperceptible gradations, and what is a short period for one problem, is a long period for another.*" – Marshall (1920)

1. Introduction

Modern macroeconomic theory relies on the construction, parametrization and solution of dynamic optimization problems. The interest in dynamic optimization problems is due to their close relation to dynamic general equilibrium models. In economies where the fundamental welfare theorems hold, one can find the equilibrium of an economy by focusing on the corresponding planner's optimization problem; but even when the welfare theorems fail to apply, the various agents in the model may have to solve dynamic optimization problems. As a result, the equilibrium conditions of the model are typically described by a set of differential or difference equations. To construct, parametrize, solve and intuitively interpret the results of such settings one needs to make an assumption on the frequency with which economic decisions are made, i.e. the period length. In this paper we explore and analyze the issues arising from this assumption. With our analysis, we hope to shed as much light as possible on the implications of this assumption and thus help researchers choose a sensible period length when modelling macroeconomic dynamics.

The main insights of our work are the following. First, the choice of whether to model the economy as a dynamic system in continuous or discrete time is not innocuous. Specifically, this cannot be chosen solely on the basis of technical/computational convenience, because it implies that agents live in economic environments that are different, not only quantitatively, but often also qualitatively. We explain that simply distinguishing between discrete and continuous time is only part of the story, since the actual issue is the period length; it clearly differs between a continuous and a discrete time model, but can also differ between two discrete time models. The latter point is relevant when one makes the seemingly innocuous choice of a frequency of calibration. Whereas in continuous time, the period length (zero!) and the unit of time measurement (i.e. the calibration frequency) are clearly two separate concepts, in discrete time the two can easily be confused. This can lead to the erroneous belief that the period length can be a choice of no real economic significance, just like the frequency of calibration.

Second, the paper provides a modelling framework that can be used to conduct explicit comparative dynamics analysis with respect to the period length. In particular, we set up a generic dynamic general equilibrium model where we let the period length be an undetermined parameter. This model nests as special cases the standard discrete and standard continuous time models. Subsequently, we provide a series of examples of models that can fit into this general framework and demonstrate the usefulness of the framework by analyzing the (local) dynamics of each model as the parameter of interest changes. The examples we provide are the textbook Cass-Koopmans model, the model of increasing returns of Benhabib and Farmer (1994) and Farmer and Guo (1994), the model of balanced budget rules of Schmitt-Grohé and Uribe (1997), and the New Keynesian model with endogenous capital as in Dupor (2001) and Huang and Meng (2007). The latter is the only case were we find a discontinuity in the dynamics of the equilibrium as the period length goes to zero, i.e. when moving to a continuous time setup.

Third, the examples we focus on provide a good platform for analyzing how local, real indeterminacies can arise; we show that standard discrete and continuous versions of the 'same' model can lead to different conclusions regarding local determinacy. We focus on examples that may exhibit local indeterminacies because in such models the change in the dynamics as the period length changes is not only quantitative, but may also be qualitative.[1] Our framework allows us to consider a whole range of intermediate cases

[1] We choose examples with indeterminate local dynamics because we believe they provide a neat illustration of the

and pinpoint the exact period length at which the switch from determinacy to indeterminacy occurs and whether or not there is a discontinuity between a discrete time and a continuous time version of the same model. Across different models with endogenous capital accumulation we find an interesting regularity: the smaller the period length is (i.e. the more frequently decisions are made), the larger the ranges of indeterminacy are. We believe that this pattern follows from a simple intuitive explanation: indeterminacy arises when expectations about the future affect current investment decisions in such a way as to render the expectations self-fulfilling. Specifically, the indeterminacy arises if the effect on current decisions is strong enough. The closer the future is (i.e. the shorter the period length and the sooner agents are allowed to make decisions again), the stronger the effect on today and therefore, the easier it is for expectations to be self-fulfilling. This point has previously been noted by Guo (2004) in the context of a real business cycle model with increasing returns and variable capital utilization. We provide additional examples where this is true. We conjecture that this is a quite general result in models with endogenous capital, but we have not yet been able to provide a formal proof. However this result does not necessarily hold in models with no or exogenous capital (for example, it can be shown that this is not the case in the New Keynesian model with an exogenous endowment of capital).

Fourth, we find that the sensitivity of the indeterminacy region to the period length crucially depends on two parameters: the rate of time preference and the capital depreciation rate. In particular, we find that as the rate of time preference increases (i.e. as agents become less patient), the ranges of indeterminacy decrease, i.e. indeterminacy occurs less often. This is because as agents become less patient, they value the future less; therefore expectations about the future have a smaller impact on today's decisions and are thus less likely to be self-fulfilling. Similarly, as the capital depreciation rate increases (i.e. accumulation of future capital becomes less important since capital depreciates a lot), future expectations have a smaller effect on decisions taken currently and therefore indeterminacy occurs less often. The importance of the rate of time preference and the capital depreciation rate for indeterminacy has been implicitly or explicitly pointed out by various authors in the literature (see Schmitt-Grohé, 1997, Baierl, Nishimura and Yano, 1998, Mitra, 1998, Guo 2004); here we provide a more detailed exploration of this point, in a broad context that includes many examples.

It is important to clarify what we do not do in this paper. First, although our paper is broadly related to the work of Mercenier and Michel (1994), our approach is fundamentally different from theirs. Mercenier and Michel (1994) show that one can construct a discrete time model that approximates the continuous counterpart fairly well by making sure that steady states in the two problems coincide. This approach presupposes that a continuous time formulation is better than a discrete time formulation of an economic problem. We, on the other hand, feel that there is no clear reason why a discrete time formulation is more natural or preferable to a continuous one or vice versa.[2] Following this view, we simply explore how economic dynamics may vary when the frequency of decision making changes. Second, our hybrid model is based on assumptions that are fundamentally different from the concept of *time-to-build*, famously studied in discrete time by Kydland and Prescott (1982) and more recently in the context of continuous time models with delays (e.g. Licandro and Puch, 2006). In these models, the basic premise is that decisions and actions taken today affect the economy with a delay. Our formulation captures the main ingredient of a standard discrete model without any such delay: savings are accumulated in every

potential pitfalls of taking the choice of the period length lightly. However, it should be clear that the dynamic adjustment in the same model when calibrated to, say, quarterly versus annual data, will differ even when saddle path stability is not affected. In perfect foresight models, the speed of convergence to steady state will be different across the two calibrations even after properly aggregating across time. (This is by no means an easy task. See, for example, Aadland and Huang (2004) for the issues arising in consistent time aggregation.) In stochastic economies, second moments of the variables of interest will also be different across the two calibrations. Whether these differences are important is a quantitative question that has to be addressed in any specific context.

[2] Hood (1948) provides an extended discussion of continuous versus discrete time modelling.

period and they are suddenly invested and produce capital only at the beginning of the next period. We show that this different arrangement is an implicit assumption of a standard discrete model and also provide a market (equilibrium) interpretation of this assumption.

The idea that discrete and continuous versions of the same model can lead to substantially different conclusions is not novel. In a variety of different contexts, this point has been made among others, by Telser and Graves (1968), Foley (1975), Turnovsky and Burmeister (1977), Karni (1979), Leung (1995), Li (2003), Mino, Nishimura, Shimomura and Wang (2005) and Carlstrom and Fuerst (2005).[3] We differ from those in analyzing the importance of period length as opposed to two extreme cases of discrete versus continuous time. This allows us to further refine some of the statements made in the discrete versus continuous time literature. In particular, we can distinguish cases in which differences arise due to a discontinuity from cases where the differences are simply an implication of different period lengths.

The rest of the paper is organized as follows. In Section 2 we present a generic equilibrium model that mixes discrete and continuous elements and that nests the two standard formulations as specific cases. Sections 3, 4, 5 and 6 present four examples that fit into this framework: the textbook Cass-Koopmans model, the model of increasing returns by Benhabib and Farmer (1994) and Farmer and Guo (1994), the model of balanced budget rules by Schmitt-Grohé and Uribe (1997) and the New Keynesian model by Dupor (2001) and Huang and Meng (2007). Section 7 discusses the importance of discounting and capital depreciation for our results. A final short section concludes.

2. A General Continuous Time Model with Discrete Decision Making

We begin by setting up a dynamic model that allows for different period lengths and nests discrete and continuous time models as special cases. This setup is intended to cover a wide range of standard dynamic macroeconomic models, including variants of the neoclassical growth model and of the New Keynesian model. We maintain the assumption that time evolves continuously. A discrete time model can be thought of as a continuous time model where the time line $[0,\infty)$ has been partitioned in intervals of length h: $[0,h)$, $[h,2h)$, etc. These intervals are called periods and can be indexed by $\frac{t}{h}\in\{0,1,2,...\}$ where t is the time instant at the beginning of each period, so that $t\in I=\{0,h,2h,...\}$.[4] This continuous time interpretation of discrete time models also requires the following assumptions: First, stock (state) variables can only be adjusted at the beginning of a period (at $t\in I$) and second, flow (control) variables are constant within a period, but can be changed at the beginning of each period. Generally, these assumptions have no particular economic content. There is no a priori economic intuition for why they should hold. Our formulation is intended to allow an analysis of the conditions under which such assumptions might make sense as well as an analysis of their implications. Nevertheless, we find the following interpretation of these assumptions useful and intuitive: certain markets only open up for trade at discrete points in time but some other markets open continuously. This interpretation will prove useful especially when looking at variations of a standard growth model. In our general formulation we will justify the first of these assumptions by assuming that the markets for stock variables open discretely. On the other hand, we will allow flow variables to change continuously and study under what conditions the optimal choice is to have flows constant within a period.

Our main focus is on the role of h, i.e. the period length. By explicitly writing up our general model we aim to analyze as crisply as possible the importance of the period length for dynamic behavior in standard macroeconomic models. We proceed by describing such a general model which we view as an appropriate analogue of the standard discrete time model.

[3] Other broadly related work includes Turnovsky (1977), Jovanovic (1982), McGill and Benhabib (1983), Romer (1986), Benhabib (2004) and Hintermaier (2005).

[4] In what follows we will use t to index variables even though, strictly speaking, the index is $\frac{t}{h}$.

The economy is described by an $m \times 1$ vector of state variables $x_t = \left[x_t^1, ..., x_t^m\right]^T$, where changes in the state variables accrue at the end of each period and several control variables. We distinguish between two types of control variables; an $m \times 1$ vector $\omega_t = \left[\omega_t^1, ..., \omega_t^m\right]^T$ of control variables that do not appear in the objective and an $n \times 1$ vector of other control variables $y_t = \left[y_t^1, ..., y_t^n\right]^T$. The model might also contain other variables, summarized in an $l \times 1$ vector z_t, that are not choice variables for the representative household. These can be exogenously or endogenously determined. The household faces the following dynamic maximization problem

$$\max_{\{y_t(s),\omega_t(s),x_{t+h}\}} \sum_{t=0,h,2h,...} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h e^{-\phi s} u(y_t(s), x_{t+h}, z_t(s))ds \tag{1}$$

$$s.t. \quad x_{t+h}^i - x_t^i = \int_0^h Q^i(\omega_t^i(s), x_t, z_t(s))ds \quad \text{for } t = 0, h, 2h, ... \text{ and } i = 1, ..., m \tag{2}$$

$$H^j(y_t(s), \omega_t(s), x_t, z_t(s)) = 0 \quad \text{for } s \in [0, h),\ t = 0, h, 2h, ... \text{ and } j = 1, ..., p \tag{3}$$

$$x_0 \text{ given}$$

In the above setting, $\{y_t(s)\}_{t=0,h,2h,...}$, $\{\omega_t(s)\}_{t=0,h,2h,...}$ and $\{z_t(s)\}_{t=0,h,2h,...}$ are sequences of functions. Each element $y_t(s)$ of the first sequence is a function mapping the interval $[0, h)$ to $\mathbb{R}^n$ and each element $w_t(s)$ of the second sequence is a function mapping the interval $[0, h)$ to $\mathbb{R}^m$. Note that the sizes of ω and x are the same, capturing the idea that changes in each state variable x_t^i are governed by the accumulation of a corresponding flow variable ω_t^i as in (2); e.g. net savings are the flow variable that dictates how the state variable capital accumulates in a standard growth model. Each element $z_t(s)$ of the second sequence is a function mapping the interval $[0, h)$ to $\mathbb{R}^l$. On the other hand, $\{x_t\}_{t=0,h,2h,...}$ is a sequence of vectors in $\mathbb{R}^m$. We use x_t to denote the level of the state variable that prevails at the end of period $\frac{t}{h} - 1$ (i.e. at the end of the interval $[t - h, t)$). There are two types of constraints: m dynamic constraints governing the evolution of each state variable from period to period given by (2) and p static constraints that must hold at every instant, given by (3). The functions $Q^i : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$, $H^j : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ and $u : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ will be determined by the particular problem at hand.[5]

To close the model we also need to append l other equilibrium conditions

$$G^\kappa(y_t(s), \omega_t(s), x_t, z_t(s)) = 0 \text{ for all } s \in [0, h),\ t = 0, h, 2h, ... \text{ and } \kappa = 1, ..., l, \tag{4}$$

that will determine $z_t(s)$. These could include optimality conditions for other agents in the model (e.g. firms, government), market clearing and equilibrium consistency or symmetry conditions. Again, the functions $G^\kappa : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ will be determined by the particular problem considered. The crucial aspect of these additional conditions is that they are assumed to be static.

Some explanations for our choice of discounting are in order. Given that time is assumed to be continuous, we allow for continuous discounting. The presence of both discrete and continuous elements in our setup implies that there are alternative forms of discounting which we can consider. Using the standard continuous time discount term we could write the objective as

$$\sum_{t=0,h,2h,...} \int_0^h e^{-\rho(t+s)} u(y_t(s), x_{t+h}, z_t(s))ds = \sum_{t=0,h,2h,...} e^{-\rho t} \int_0^h e^{-\rho s} u(y_t(s), x_{t+h}, z_t(s))ds. \tag{5}$$

[5] Standard monotonicity and convexity assumptions are maintained throughout for the return and constraint functions. We also assume standard Inada conditions for utility and production and thus ignore any non-negativity constraints.

Alternatively, using the standard discrete time discount term we could write

$$\begin{aligned}&\sum_{t=0,h,2h,...}\int_0^h\left(\frac{1}{1+\rho h}\right)^{\frac{t+s}{h}}u(y_t(s),x_{t+h},z_t(s))ds\\ =&\sum_{t=0,h,2h,...}\left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}}\int_0^h\left(\frac{1}{1+\rho h}\right)^{\frac{s}{h}}u(y_t(s),x_{t+h},z_t(s))ds. \end{aligned} \tag{6}$$

We follow a middle road and choose discrete style discounting across periods and continuous time discounting within periods. This seems to us to be a reasonable compromise, since it ensures that standard discrete and continuous time models are both nested in this specification (we derive the limiting cases in section 2.1). In the objective (1), we also allow for different discount rates for 'within' and 'across' period discounting. There is, of course, no substantial economic argument to support such an asymmetry so we are essentially interested only in the case $\phi=\rho$. However, maintaining this general assumption will allow us to illustrate more clearly the fundamental assumptions underlying a standard discrete time model by shutting down within-period discounting, i.e. setting $\phi=0$.

We let $[1/(1+\rho h)]^{\frac{t}{h}}\mu_t^i$, $i=1,...,m$ be multipliers corresponding to the m dynamic constraints at instant $t\in I$ and $[1/(1+\rho h)]^{\frac{t}{h}}e^{-\phi s}\lambda_t^j(s)$, $j=1,...,p$ be multipliers corresponding to the p static constraints at instant $t+s$. The Lagrangian of the household's problem is defined as

$$\begin{aligned}\mathcal{L}\equiv&\sum_{t=0,h,2h,...}\left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}}\left[\int_0^h e^{-\phi s}u(y_t(s),x_{t+h},z_t(s))ds\right.\\ &+\sum_{i=1}^m\mu_t^i\left(\int_0^h Q^i(\omega_t^i(s),x_t,z_t(s))ds-x_{t+h}^i+x_t^i\right)\\ &\left.+\sum_{j=1}^p\int_0^h e^{-\phi s}\lambda_t^j(s)H^j(y_t(s),\omega_t(s),x_t,z_t(s))ds\right].\end{aligned} \tag{7}$$

The first order conditions for the control variables are given by[6]

$$\frac{\partial u_t(s)}{\partial y_t^\nu(s)}=-\sum_{j=1}^p\lambda_t^j(s)\frac{\partial H_t^j(s)}{\partial y_t^\nu(s)}, \tag{8}$$

$$\mu_t^i\frac{\partial Q^i(s)}{\partial\omega_t^i(s)}=-e^{-\phi s}\sum_{j=1}^p\lambda_t^j(s)\frac{\partial H_t^j(s)}{\partial\omega_t^i(s)}, \tag{9}$$

for all $\nu=1,...,n$, $i=1,...,m$, $s\in[0,h)$, $t\in I$. The m first order conditions for the state variables are

$$\begin{aligned}&-\int_0^h e^{-\phi s}\frac{\partial u_t(s)}{\partial x_{t+h}^i}ds+\mu_t^i\\ =&\frac{1}{1+\rho h}\mu_{t+h}^i\left[\left(\int_0^h\frac{\partial Q_{t+h}^i(s)}{\partial x_{t+h}^i}ds+1\right)+\sum_{j=1}^p\int_0^h e^{-\phi s}\lambda_{t+h}^j(s)\frac{\partial H_{t+h}^j(s)}{\partial x_{t+h}^i}\right]\end{aligned} \tag{10}$$

for all $t=0,h,2h,...$. The latter conditions are the standard Euler equations. As usual, these intertemporal marginal conditions equate the current utility cost μ_t^i of increasing the variable x_{t+h}^i with

[6] To save on notation, we supress the arguments of functions and instead index functions by the initial instant t of a period to indicate if they are evaluated at their arguments in the current ($\frac{t}{h}$) or future ($\frac{t}{h}+1$) period.

the (current and) future utility benefits. The first term on the left hand side is the direct current utility benefit and the terms on the right hand side are discounted future returns valued at marginal utility. An equilibrium in this economy will be fully characterized by equations (2)-(4) and (8)-(10).[7]

We can obtain a more familiar version of the Euler equations by making the further assumption that $H^j((y_t(s), \omega_t(s), x_t, z_t(s)))$ depends on ω only for $j = 1$ and $\frac{\partial Q^i(s)}{\partial \omega_t^i(s)} = \frac{\partial H_t^1(s)}{\partial \omega_t^i(s)}$. This is true in all of the variations of the growth model that we consider.[8] This additional structure allows us to use (9) to obtain $\mu_t^i = e^{-\phi s}\lambda_t^1(s)$. Using this and rearranging the Euler equation, we derive

$$\mu_t^i - \int_0^h e^{-\phi s}\frac{\partial u_t(s)}{\partial x_{t+h}^i} ds = \frac{1}{1+\rho h}\mu_{t+h}^i \left[1 + \int_0^h \left(\frac{\partial Q_{t+h}^i(s)}{\partial x_{t+h}^i} + \frac{\partial H_{t+h}^1(s)}{\partial x_{t+h}^i}\right) ds\right]. \tag{11}$$

We can also rewrite the Euler as

$$\frac{\mu_{t+h}^i - \mu_t^i}{h} = \frac{-(1+\rho h)\frac{1}{h}\int_0^h e^{-\phi s}\frac{\partial u_t(s)}{\partial x_{t+h}^i} ds + \mu_t^i\left[\rho - \frac{1}{h}\int_0^h \left(\frac{\partial Q_{t+h}^i(s)}{\partial x_{t+h}^i} + \frac{\partial H_{t+h}^1(s)}{\partial x_{t+h}^i}\right) ds\right]}{1 + \int_0^h \left(\frac{\partial Q_{t+h}^i(s)}{\partial x_{t+h}^i} + \frac{\partial H_{t+h}^1(s)}{\partial x_{t+h}^i}\right) ds.}. \tag{12}$$

This formulation of the Euler equation is useful for deriving the limiting case $h \to 0$.

2.1. Limiting Cases. Standard discrete time Euler equations can be obtained by setting $h = 1$, $\phi = 0$ and assuming that all flows are constant within a period, i.e. $y_t(s) = y_t$, $w_t(s) = w_t$ and $z_t(s) = z_t$ for all $s \in [0, 1)$:

$$\mu_t^i - \frac{\partial u_t(s)}{\partial x_{t+1}^i} = \frac{\mu_{t+1}^i}{1+\rho}\left(1 + \frac{\partial Q_{t+1}^i(s)}{\partial x_{t+1}^i} + \frac{\partial H_{t+1}^1(s)}{\partial x_{t+1}^i}\right). \tag{13}$$

In some contexts, for example in the growth model of the following section, we can show that letting $\phi = 0$ implies *optimally* constant flow variables. That is, the extra assumption of constant within-period flows is not needed. In those cases, we can think of standard discrete time models as implicitly assuming a zero within period discount rate. Note, however, that this is a model-specific result and, in general, the standard discrete time model assumption of constant within period flows is even harder to justify.

It is straightforward to obtain the continuous time version by letting $h \to 0$:[9]

$$\dot{\mu}_t = -\frac{\partial u_t}{\partial x_t^i} + \mu_t^i\left[\rho - \left(\frac{\partial Q_t^i}{\partial x_t^i} + \frac{\partial H_t^1}{\partial x_t^i}\right)\right]. \tag{14}$$

These limiting cases also illustrate more clearly the asset pricing interpretation of the Euler equations, i.e. that the return on the asset x_t has to equal the rate of time preference ρ. Using the continuous time version as an example, the asset return consists of capital gains captured by the rate of change in the shadow price of the asset and dividends captured by the utility benefits arising from holding the asset.

2.2. Period length and calibration frequency. Before we proceed with the analysis of the dynamics for each example, we first make some important clarifications regarding the interpretation of the period length h. In any dynamic model, there are two important concepts that have to do with modelling time: the *unit of measurement* of time and the *frequency* with which activities take place or decisions are made. The first one relates to the calibration frequency and the second relates to the period length. Given a continuous time line, the unit of measurement of time gives meaning to the quantity $\int_0^1 y_t(s)ds$ where $y_t(s)$ is a flow. For example, if the measurement unit is years, this quantity measures the total

[7] Plus a transversality condition omitted here.

[8] For the New Keynesian model only a slight modification is required, see section 6.

[9] Strictly speaking, we should be using the notation $x(t)$, $y(t)$, $z(t)$ instead of x_t, y_t, z_t when writing the standard continuous time model. We stick to the latter to maintain conformity with the rest of the paper.

flow of y in one year, starting at t. Moreover, the choice of units (e.g. years) dictates the values for parameters such as the discount rate and the depreciation rate (e.g. ρ and δ will denote yearly rates).

The frequency with which decisions (on stock variables) are made determines the period length h. For example, if decisions are to be updated four times a year in a yearly calibrated model, then $h = 1/4$ and the corresponding parameters ρh and δh now denote quarterly rates. If decisions are made once a year in a yearly calibrated model, then $h = 1$ and the corresponding parameters ρh and δh now denote yearly rates. As the frequency of decision making increases, we retrieve the standard continuous time model in the limit. In short, the measurement unit of time and the period length are not necessarily the same, but in order to obtain the *standard* discrete time model we have to set $h = 1$. This is because in the standard discrete time framework, choosing a calibration frequency also implicitly leads to the choice of the frequency of decision making. In our general setup, we disentangle the two concepts as explained above.

We proceed with the four examples that fit into our general framework. We start by analyzing the model of Cass-Koopmans in detail and then we continue with the models of Benhabib and Farmer (1994) and Schmitt-Grohé and Uribe (1997) which are simple extensions of the former. Last, we consider a variation of the standard New Keynesian model, with endogenous capital, as in Dupor (2001) and Huang and Meng (2007). Each of the examples we present has been chosen because it has some particular feature that helps us make a specific point: The first example (Cass-Koopmans model) provides a good platform for understanding the intuition of why discrete and continuous dynamics may differ and how our general model captures these differences; the second model (i.e. the RBC model with increasing returns) is an example where for a reasonable parametrization, the discrete time model has a determinate steady state, while it has an indeterminate one in continuous time; the third example (the model of balanced budgets) illustrates that the assumption of within period discounting can make a difference; finally, the fourth example (New Keynesian model with capital accumulation) shows that there can actually be a discontinuity when taking the limit of the discrete time model as the period length becomes zero, i.e. the limiting dynamics of the discrete time model do not correspond to the dynamics of the standard continuous time model.

3. The Cass-Koopmans Model

Our first example is the Cass-Koopmans textbook model. We focus on an equilibrium interpretation of the model where all markets are perfectly competitive. Households decide on capital and labor supply as well as demand for the single good produced. Firms decide on capital and labor demand and use these inputs to produce the single good and supply it in the goods market. Thus, there are three markets in this economy, namely the goods, capital and labor markets. It is assumed that a representative household works, receives income, saves and consumes continuously over the period, at rates that are allowed to vary over time.

To capture the essence of a discrete time model, we assume that the capital market opens only at discrete points in time $t \in I$ so that within a period, no capital markets operate. Within a period, savings accrue continuously, but they are only turned into investment when the market opens and the accumulated savings can be supplied in the capital market.[10] This model fits the general formulation proposed in the previous section by making the following choices: The state variable is the capital stock, i.e. $x_t = k_t$. Control variables include consumption $c_t(s)$, labor $n_t(s)$ and net savings $S_t^k(s)$

[10] We find this interpretation of the standard discrete time model useful, but it is certainly not unique. One could also just outright assume that capital can be changed only at discrete points in time.

(net of depreciation) and $y_t(s) = [c_t(s), n_t(s)]^T$, $\omega_t(s) = S_t^k(s)$.[11] The dynamic equation describing the evolution of the state variable dictates that increases in the capital stock should equal the total accumulated flow of net savings over a period

$$k_{t+h} - k_t = \int_0^h S_t^k(s)ds, \tag{15}$$

so that the correspondence with the general model is $Q(\omega_t(s), x_t, z_t(s)) = S_t^k(s)$. The household is subject to a budget constraint

$$c_t(s) + S_t^k(s) + \delta k_t = r_t(s)k_t + w_t(s)n_t(s) + \pi_t(s), \tag{16}$$

which ensures that consumption and savings must equal capital income, labor income and income from firm profits. Here $r_t(s)$ and $w_t(s)$ are the rental price of capital and the wage rate respectively and $\pi_t(s)$ are the firm's instantaneous profits at time $t+s$. All three of these variables are taken as given by the household so they correspond to the variables $z_t(s) = [\pi_t(s), w_t(s), r_t(s)]^T$. This extra constraint is what is captured by the function $H(.)$ in the general setup of section 2, that is,

$$H(y_t(s), \omega_t(s), x_t, z_t(s)) = r_t(s)k_t + w_t(s)n_t(s) + \pi_t(s) - c_t(s) - S_t^k(s) - \delta k_t. \tag{17}$$

Note that we assume a constant rate of depreciation of the capital stock and that depreciation affects the capital stock being used in production, whereas savings not yet put into production do not depreciate. Finally, here the return function is assumed to depend on consumption and labor and it is given by[12]

$$u(c_t(s), n_t(s)) = \log c_t(s) - A\frac{n_t(s)^{1+\sigma}}{1+\sigma},\ \sigma \geq 0. \tag{18}$$

This provides a complete description of the household's problem as in section 2. Before presenting the first order conditions for household utility maximization, we introduce firms and market clearing, i.e. we define the function $G(.)$.

The firm's production inputs at an instant $t+s$ are the (operative) capital stock k_t and labor $n_t(s)$. The production function is Cobb-Douglas:

$$k_t^{s_k} n_t(s)^{s_n}, \tag{19}$$

with $s_k + s_n = 1$ and $s_k,\ s_n > 0$. When the capital market is open, the firm and the household make the following agreements. They decide on the amount of the capital stock that the household rents out to the firm. They agree that after the capital market closes, the firm will provide a continuous flow of capital income to the household at a (possibly variable) rate, consistent with capital market clearing, until the market opens again.[13]

The firm maximizes lifetime discounted profits

$$\sum_{t\in I}\left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h e^{-\phi s}\frac{u_{c,t}(s)}{u_{c,0}(0)}\pi_t(s)ds, \tag{20}$$

[11] One can think of net savings as the analogue of $i_t - \delta k_t$ in a standard growth model, where i_t is investment flow and $0 < \delta < 1$ the depreciation rate.

[12] Many of the results here can be derived under more general utility specifications. Separability between consumption and labor is needed to obtain analytical expressions for within period dynamics.

[13] We assume a competitive market for inputs to production. Thus the rate of return will be determined in equilibrium by the demand and supply for capital and, in particular, is taken as given by both firms and households.

where $\pi_t(s)$ stands for instantaneous profits, i.e.

$$\pi_t(s) = k_t^{s_k} n_t(s)^{s_n} - r_t(s)k_t - w_t(s)n_t(s). \tag{21}$$

Households are assumed to be identical, which allows us to focus on the representative household and ensures that the firm's objective is well defined. In particular, the firm is owned by the representative household, which has a unique valuation of instantaneous profits at any point in time $t+s$. From the point of view of time 0, the value of one unit of profits at time $t+s$ is simply the price of a contingent claim in terms of time 0 consumption, given by

$$\left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} e^{-\phi s} \frac{u_{c,t}(s)}{u_{c,0}(0)}. \tag{22}$$

The first order conditions for the firm's problem are

$$w_t(s) = s_n {k_t}^{s_k} n_t(s)^{-s_k}, \text{ for all } s \in [0,h) \text{ and } t \in I, \tag{23}$$

$$\int_0^h e^{-\phi s} u_{c,t}(s) r_t(s) ds = \int_0^h e^{-\phi s} u_{c,t}(s) s_k {k_t}^{s_k - 1} n_t(s)^{s_n} ds, \text{ for all } t \in I. \tag{24}$$

The first condition is the standard condition determining labor demand as a function of the wage rate. It specifies that the firm will demand labor so that it equates the marginal product of labor to the wage rate.

The second condition is less standard; given that capital markets are closed within the period, there are potentially many different within-period rental rate functions that bring the capital market in equilibrium. It is sufficient that the overall demand for capital is met, i.e. the beginning-of-period arrangement is such that the total discounted return equals the total discounted marginal product of capital, appropriately valued. In equilibrium, the weighing factors will be constant within a period and this condition can be further simplified (as shown below). It then states that the average rental rate of capital must equal the average marginal product of capital over a period. The reason that the capital rental rate is not equated to the marginal product of capital at every instant is the absence of capital markets. As long as capital markets remain closed, the marginal product of capital can fluctuate above or below the agreed-upon rental price without the market forces rectifying this. But it must be that, when the markets open, and the rental rate for the next interval is set, the average price of capital is equal to the average marginal product. Generically, there could be many $r_t(s)$ profiles that deliver the same average real rental rate.

Finally, all markets have to clear. We have implicitly imposed capital and labor market clearing by not distinguishing between capital (and labor) demand and supply. Goods market clearing dictates

$$c_t(s) + S_t^k(s) + \delta k_t = (k_t)^{s_k} (n_t(s))^{1-s_n} \quad \text{for all } s \in [0,h) \text{ and all } t \in I, \tag{25}$$

and is automatically satisfied by Walras' Law. To complete the correspondence with the model of section 2, the functions G^κ are given by (21), (23) and (24).

We find it convenient to define the average rental rate of capital over a period as

$$\tilde{r}_t \equiv \frac{1}{h} \int_0^h r_t(s) ds. \tag{26}$$

Using this definition, the first order conditions for the household's problem corresponding to (8) are

$$\lambda_t(s) = \frac{1}{c_t(s)}, \tag{27}$$
$$An_t(s)^\sigma = \lambda_t(s)w_t(s), \tag{28}$$

for all $s \in [0,h)$, $t \in I$, the one corresponding to (9) is

$$e^{-\phi s}\lambda_t(s) = \mu_t, \tag{29}$$

for all $s \in [0,h)$, $t \in I$ and the dynamic first order condition corresponding to (10) is

$$\frac{\mu_{t+h} - \mu_t}{h} = \frac{\rho - \tilde{r}_{t+h} + \delta}{1 + \tilde{r}_{t+h}h - \delta h}\mu_t, \tag{30}$$

for all $t \in I$. In addition, a transversality condition must hold

$$\lim_{T\to\infty}\left(\frac{1}{1+\rho h}\right)^{\frac{T}{h}}\frac{1}{c_T(0)}k_{T+h} = 0. \tag{31}$$

Condition (27) is the standard first order condition for consumption ensuring that, at an optimum, the marginal value of income equals the marginal utility of consumption. Condition (28) is the standard first order condition for labor supply, ensuring that the marginal utility of leisure is equalized to the marginal value of income times the wage rate. Finally, condition (29) describes optimal savings, equating marginal cost and benefit of savings. The marginal cost of savings at instant $t + s$ arises from reduced resources available for consumption. This cost depends on the specific instant $t+s$ because consumption at different points in time is valued differently. The marginal benefit comes from the fact that these savings will eventually become investment and be added to the capital stock, k_{t+h}. This will only happen the next time the capital market opens, so the specific instant within period t is irrelevant. Equations (26)-(31) together with capital accumulation (15), firm optimality conditions (23)-(24) and goods market clearing (25) fully characterize the equilibrium of this economy.[14]

Substituting the factor prices from (23) and (24), one obtains restrictions on allocations that are equivalent to those that would arise from a social planner formulation. Thus, the competitive equilibrium implements the first best and existence of an optimum implies existence of an equilibrium. Note however that, even though the wage rate that implements the equilibrium is unique, the rental rate of capital $r_t(s)$ is only restricted to satisfy (26) and is generally not unique. Nevertheless, this does not affect the uniqueness of the real allocations because the average rental rate $\tilde{r}_t$ is unique and the average is all that matters in this equilibrium. Put differently, there can potentially be many within period profiles for the rental rate, all of which are consistent with a particular value for $\tilde{r}_t$ as well as particular values for allocations.

3.1. Within-period dynamics. We first look at the behavior of flow variables within a period. That is, given a period $\frac{t}{h}$, we can describe the evolution of $c_t(s)$, $n_t(s)$ and $S_t^k(s)$ for all $s \in (0,h)$ given their values at $s = 0$. We use the first order condition for savings (29) and notice that it holds for all $s \in [0,h)$ so that

$$e^{-\phi s}\lambda_t(s) = \mu_t = \lambda_t(0). \tag{32}$$

[14] A precise definition of equilibrium is given in Appendix A.

From (27), we obtain the evolution of consumption as

$$c_t(s) = e^{-\phi s} c_t(0). \tag{33}$$

Therefore, within the period $[t, t+h)$, consumption starts at $c_t(0)$ and declines exponentially until $\lim_{s \to h} c_t(s) = e^{-\phi h} c_t(0)$. The rate of decline is naturally increasing in the discount rate ϕ.[15] We can derive a similar expression for labor:

$$n_t(s) = e^{\frac{\phi}{\sigma+1-s_n} s} n_t(0). \tag{34}$$

Labor is increasing exponentially within the period $[t, t+h)$, i.e. leisure is decreasing. The rate of increase is increasing in ϕ and in the elasticities of labor supply $1/\sigma$ and labor demand $s_n - 1$. Output, net savings and the wage rate can be easily derived given the above expressions:

$$y_t(s) = e^{\frac{\phi s_n}{\sigma+s_k} s} y_t(0), \tag{35}$$

$$S_t^k(s) = e^{\frac{\phi s_n}{\sigma+s_k} s} y_t(0) - e^{-\phi s} c_t(0) - \delta k_t, \tag{36}$$

$$w_t(s) = e^{-\frac{\phi s_k}{\sigma+s_k} s} w_t(0). \tag{37}$$

Output and savings are increasing in s and the wage rate is decreasing in s. Finally, the capital labor ratio is determined by

$$\kappa_t(s) = \kappa_t(0) e^{-\frac{\phi}{\sigma+s_k} s}, \tag{38}$$

and is decreasing since labor is increasing.

To summarize, because the operative capital cannot be changed during the period $[t, t+h)$, the household consumes more and enjoys more leisure early in the period and postpones savings for later in the period. If we were to assume no discounting within the period, i.e. $\phi = 0$, all the above flows would be constant.

3.2. The long run and between-period dynamics. In the long run, capital stock reaches a steady state as usual. However, when that steady state is reached, flow variables still exhibit the within-period dynamics described above, so that they are characterized by a steady state function, rather than the usual steady state value of the standard discrete or continuous time models. It can be shown that the steady state function for the capital labor ratio is given by[16]

$$\kappa(s) = \kappa^* q(h, \phi) e^{-\frac{\phi s}{\sigma+s_k}}, \text{ for all } s \in [0, h), \tag{39}$$

where

$$\kappa^* = \left(\frac{s_k}{\rho+\delta} \right)^{\frac{1}{s_n}}, \tag{40}$$

$$q(h, \phi) = \left(\frac{e^{\frac{s_n}{\sigma+s_k}\phi h} - 1}{\frac{s_n}{\sigma+s_k}\phi h} \right)^{\frac{1}{s_n}}. \tag{41}$$

The steady state of standard continuous (and discrete) time models is κ^* and the steady state function here has the capital labor ratio starting above that level at the beginning of the period and finishing

[15] If a power utility for consumption were used, the trade-off between the preference for early consumption and the preference for consumption smoothing would be more explicit. Overall, the consumption profile would still be decreasing but at a rate that is a function of the elasticity of intertemporal substitution.

[16] See Appendix A.

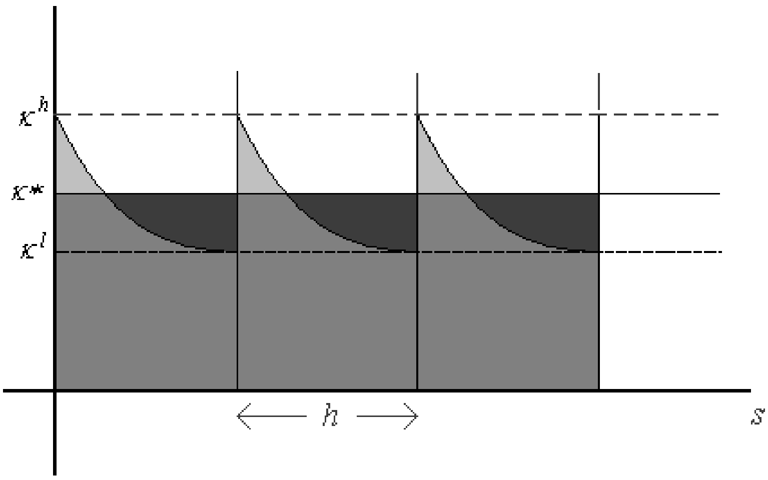

Figure 1: The steady state of the capital labor ratio, κ. The capital labor ratio in the standard discrete and continuous time model is κ^*.

below, as illustrated in Figure 1. Convergence to this long run behavior can be analyzed as usual by considering the Jacobian of the dynamic system. As shown in Appendix A, the Jacobian depends on the period length h and therefore the dynamics of the model will depend on h.

Here, saddle path stability obtains for any h so the effect of h will be quantitative, e.g. on the speed of convergence. Given that this model does not exhibit local indeterminacy, we find it more instructive to use this example to discuss the intuitive, qualitative effects of h. We embark on a detailed quantitative analysis in later sections, where we present variations of this model that can exhibit local indeterminacies.

3.3. Discussion. Our general model nests discrete and continuous time models as special cases. The important condition is the Euler equation

$$\frac{\mu_{t+h} - \mu_t}{h} = \frac{\rho - \tilde{r}_{t+h} + \delta}{1 + \tilde{r}_{t+h} h - \delta h} \mu_t. \tag{42}$$

In Appendix A, we show that taking limits as $h \to 0$ gives the standard continuous time Euler equation and choosing $h = 1$ as well as $\phi = 0$ leads to the standard discrete time Euler equation.

To understand the importance of h we rewrite the Euler equation to exploit its asset pricing interpretation

$$\rho = \frac{\frac{\mu_{t+h} - \mu_t}{h} + \mu_{t+h}(\tilde{r}_{t+h} - \delta)}{\mu_t}. \tag{43}$$

The right hand side is the total return on the asset, which in this case is physical capital. The multiplier μ corresponds to the shadow price of capital. The first term in the numerator represents capital gains. The second term is simply the dividend received, comprising of the average rental rate $\tilde{r}$ minus the depreciation rate adjusted by the price of capital μ. At an optimum, the total rate of return on capital must equal the time preference rate ρ. One way to see the importance of h is by noticing that the dividend component is dated $t + h$. As h decreases, investment decisions depend on dividends received sooner. In the limit, investment decisions depend on current dividends.

Here, capital k_t is rented out once at the beginning of the period; whatever is saved throughout the period remains inoperative in the possession of consumers. At the end of the period, the rented (depreciated) capital returns to the possession of the households and is added to the newly accumulated capital. This new capital stock k_{t+h} remains in the possession of the households until the beginning of next period, when it is rented out again. For this reason, the households are interested in the return they will get for their capital once *all* of it becomes operative. Therefore, when optimizing in the current period, they choose how much to invest so that their subjective discount rate ρ is balanced out by capital gains, plus the average rental rate over the following period. This model has an inherent delay (just like any discrete time model), since at any point in time within the period there exists capital that is not used for production. This type of delay is not the same as what is commonly known as *time-to-build* delay. The classic example of time-to-build is given in Kydland and Prescott (1982). In that model, an h-period delay implies that investment at t will only produce capital at $t+h$, where h is an integer. This still allows for new investment to take place at $t+1$, that will yield capital at $t+h+1$ and so on. The continuous time counterpart of this, studied in Licandro and Puch (2006), is one where investment can take place continuously but productive capital is only created after an interval h, where now h is a real number. In our model, savings take place continuously within a period at every instant $t+s$, $s \in [0,h)$ but the accumulated savings are suddenly invested and produce capital at $t+h$, regardless of whether they were saved at the beginning of the period or right before the end. Put differently, in our case the delay in putting capital into production varies and depends on the instant within the period at which this capital is put aside. We have shown that this different arrangement is an implicit assumption of any standard discrete model, even in the absence of a time-to-build delay. We have also provided a market (equilibrium) interpretation of this assumption.

One could also think of this model in relation to the work of Turnovsky (1977). Turnovsky interprets the discrete time model as a setting where time is continuous, but due to adjustment costs, firms can alter their capital only in a discrete manner. He then shows that the standard limiting continuous time relation between capital and investment, $i = \dot{k}$, is true only under the restrictive assumption of no adjustment costs. In a continuous time model with adjustment costs, this would not be true in general: the demand for investment i cannot be matched with a change in capital, since capital is not perfectly malleable. Thus, in our setting we can interpret the fact that the capital market is closed within a period as an infinite adjustment cost. When the model is viewed in this way, a natural question that arises is what would happen if labor was also costly to adjust within a period. It can be shown that imposing infinite within-period adjustment costs to labor as well as capital would lead to a situation where both capital and labor are constant within a period. However, in the presence of within-period discounting, consumption and savings would still exhibit within-period dynamics of the type discussed above.

4. A Model with Increasing Returns

We consider the model of Benhabib and Farmer (1994) and Farmer and Guo (1994), a straightforward variation of the Cass-Koopmans model. The household sector is identical to the model of section 3. The production side differs by including increasing returns through externalities. The production function is now:

$$y_t(s) = k_t^{s_k} n_t(s)^{s_n} \bar{k}_t^{\frac{1-\lambda}{\lambda} s_k} \bar{n}_t(s)^{\frac{1-\lambda}{\lambda} s_n}, \tag{44}$$

with $s_k + s_n = 1$ and s_k, $s_n > 0$. Here $\bar{k}$ and $\bar{n}$ indicate aggregate capital and labor, variables that the individual firm does not realize it can affect.[17] The correspondence with our general framework is very

[17] As discussed by Benhabib and Farmer (1994), the model can equivalently be interpreted as a model of monopolistically competitive firms with the parameter $\lambda \in (0,1)$ representing the degree of monopoly power. This alternative interpretation

similar to the one described in section 3. In particular, states x_t and controls $y_t(s), \omega_t(s)$ are the same as before and the functions $Q(.)$, $H(.)$ and $u(.)$ are identical. The only difference comes in the vector $z_t(s)$, which now includes also aggregate capital $\bar{k}$ and labor $\bar{n}$, and in the functions $G^{\kappa}(.)$, $\kappa = 1, ...5$. The functions $G^{\kappa}(.)$ differ because the firm's first order conditions have to take into account the increasing returns to scale of the production function and, in addition, equilibrium symmetry amounting to $k_t = \bar{k}_t$ and $n_t = \bar{n}_t$ has to be imposed. Thus we have the functions G^{κ} compactly arranged in a vector:

$$G(y_t(s), \omega_t(s), x_t, z_t(s)) = \begin{bmatrix} s_n k_t^{\frac{s_k}{\lambda}} (n_t(s))^{\frac{s_n}{\lambda}-1} - w_t(s) \\ \int_0^h e^{-\phi s} u_{c,t}(s) \left[r_t(s) - s_k k_t^{\frac{s_k}{\lambda}-1} (n_t(s))^{\frac{s_n}{\lambda}} \right] ds \\ k_t^{\frac{s_k}{\lambda}} (n_t(s))^{\frac{s_n}{\lambda}} - r_t(s) k_t - w_t(s) n_t(s) - \pi_t(s) \\ k_t - \bar{k}_t \\ n_t - \bar{n}_t \end{bmatrix}. \tag{45}$$

It can be shown (proof omitted) that taking limits as $h \to 0$ gives the continuous time model of Benhabib and Farmer (1994) and setting $h = 1$ and $\phi = 0$ gives the discrete time model of Farmer and Guo (1994).

Analyses of within-period dynamics and long run behavior of the economy can be carried out in a manner entirely analogous to the Cass-Koopmans model. We focus our attention on the dynamic adjustment in this model and specifically on the issue of local indeterminacy. The clearest intuition for why indeterminacy arises in the presence of increasing returns to scale is provided by Schmitt-Grohé (1997). If households expect higher than average production in the future, their expected future marginal utility is lower and, by the capital Euler equation, their current marginal utility has to decrease. This perception of wealth leads households to consume more and supply less labor. The crucial question is what happens to equilibrium employment as a result of this. Increasing returns implies that the demand for labor can be upward sloping. If in fact, labor demand slopes upward and more steeply than labor supply, then a reduction in labor supply implies an increase in equilibrium employment. Higher employment means more production and so, the original expectations of the households may become self-fulfilling. In particular, given that on an equilibrium trajectory output will have to decrease monotonically towards the steady state, the only way in which an expectation of an increase in future output can be self-fulfilling, is if it leads to an even larger increase in current output.

The above intuition arises from the following analytical result of Benhabib and Farmer (1994). They show in their continuous time model that a necessary condition for local indeterminacy is that $\frac{s_n}{\lambda} - 1 > \sigma$, i.e. that the labor demand slope is higher than the labor supply slope.[18] Here we use our hybrid model to analyze the dependence of the model's dynamics on h, the period length. Let

$$\xi_1 = \frac{(\sigma+1)(\frac{s_k}{\lambda} - 1) + \frac{s_n}{\lambda}}{\sigma + 1 - \frac{s_n}{\lambda}} \text{ and } \xi_2 = -\frac{\frac{s_n}{\lambda}}{\sigma + 1 - \frac{s_n}{\lambda}}. \tag{46}$$

Then, log-linearizing the equilibrium conditions at the beginning of period steady state we obtain the following 2x2 system for the dynamics of this economy:[19]

$$\begin{pmatrix} \frac{\hat{\mu}_{t+h} - \hat{\mu}_t}{h} \\ \frac{\hat{k}_{t+h} - \hat{k}_t}{h} \end{pmatrix} = C(\phi, h, \lambda) \begin{pmatrix} \hat{\mu}_t \\ \hat{k}_t \end{pmatrix}, \tag{47}$$

---

will guide the calibration.

[18] This is true because we assume diminishing marginal returns to capital $(s_k/\lambda) < 1$, i.e. that the demand for capital is downward sloping.

[19] See Appendix B.

where

$$C(\phi,h,\lambda) = \begin{pmatrix} \frac{c_{11}+hc_{12}c_{21}}{1+\rho h-hc_{11}} & \frac{c_{12}(1+hc_{22})}{1+\rho h-hc_{11}} \\ c_{21} & c_{22} \end{pmatrix}, \tag{48}$$

$$\text{and } \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} (\rho+\delta)\,\xi_2 & -(\rho+\delta)\,\xi_1 \\ \frac{(\rho+\delta)}{s_k}(-\xi_2)+\frac{(\rho+\delta(1-s_k))}{s_k} & \frac{(\rho+\delta)}{s_k}\xi_1+\frac{\rho+\delta(1-s_k)}{s_k} \end{pmatrix}. \tag{49}$$

Note that the parameter of within-period discounting ϕ does not enter the matrix that characterizes the dynamics so within-period dynamics here have no effect on between-period dynamics.[20] We examine how the indeterminacy regions vary with the period length h and the key parameter λ, i.e. the degree of monopoly power. Clearly, the indeterminacy regions will depend on the parameterization of the model. Consider, for example, the benchmark parameterization used by Benhabib and Farmer (1994). The parameters reflect the yearly calibration for the US, $\rho = 0.065$, $s_k = 0.42$, $\delta = 0.1$, $\sigma = 0.25$ and $s_n = 0.58$. Studying determinacy as λ and h change we find that, for $h = 0$, there is indeterminacy whenever $\lambda \in [0.42, 0.46]$. This is a very small range of indeterminacy, which in fact disappears for h larger than approximately 0.4 (graph omitted). In other words, indeterminacy may occur for some λs in the continuous time model, but as h increases the range of λs shrinks and for $h = 1$ the model always has a determinate steady state. Small changes in the parameterization can lead to very different conclusions.

We next consider the case where $s_n = 0.7$ instead of 0.58, and $\sigma = 0$ instead of 0.25 (this set of parameters is used in one of the experiments by Benhabib and Farmer, table IV). The regions of indeterminacy are shown in Figure 2. White regions show indeterminacy and gray areas show determinate steady states. The parameter λ on the horizontal axis varies from $s_k = 0.3$ to 1. For this parametrization, when λ is calibrated to US data so that $\lambda = 0.66$, the continuous time model yields an indeterminate steady state, while the discrete time model has a locally saddle path stable steady state. In fact, there is a very wide range of values for λ between 0.55 and 0.7, for which the discrete time model and continuous time model give different predictions.

The numerical experiments confirm our intuition that the shorter the period length h, the more possibilities for indeterminacy arise. The crucial aspect of the intuition for indeterminacy which brings into play the period length is the capital Euler equation. The expected future increase in output translates into lower current marginal utility through this intertemporal condition. The strength of the current response is what determines whether expectations are self-fulfilling. But the strength of the current response to future events depends on how far away in the future these events are. Thus, the intertemporal links are strong when h is small and are weakened as h increases. This explains why indeterminacy is more likely for smaller h, at least in setups where the capital Euler equation is central to self-fulfilling beliefs.

5. A Model with Balanced Budget Rules

Another variation of the Cass-Koopmans model comes through the addition of a government which uses distortionary labor taxation to finance exogenous government spending, as in Schmitt-Grohé and Uribe (1997). The authors show how endogenous fluctuations can arise due to local indeterminacy if the government is restricted to balance its budget every period. Here, we show how the model can fit into our general framework and how this general framework can be used to analyze the importance of time modelling. The model is found to lead to substantially different (contradictory from a policy perspective) predictions when one moves from a continuous time setup to a yearly calibrated discrete time model. It also serves as an illustration of how ignoring within period discounting can mask some important effects.

[20] This is specific to this example. Other examples, such as the model of balanced budget rules examined in Section 5, do exhibit a feedback from within period dynamics and between period dynamics.

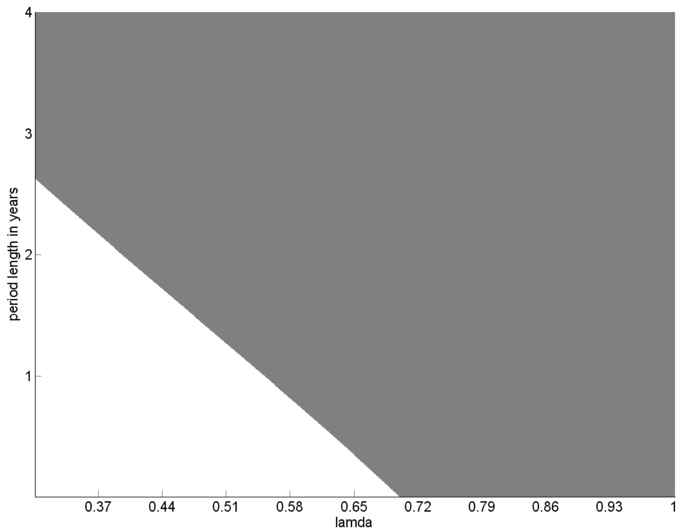

Figure 2: Stability in the model with increasing returns (Benhabib and Farmer, 1994). $s_n = 0.7$ and $\sigma = 0$. White areas show indeterminacy. Gray areas show determinate steady states.

The correspondence with our general model is again a straightforward extension of the correspondence in the Cass-Koopmans model. The function $Q(.)$ describing the dynamic adjustment of the capital stock is identical. The household's budget constraint is modified to take into account the effect of taxation on labor income, so the function $H(.)$ is now given by

$$H(y_t(s), \omega_t(s), x_t, z_t(s)) = r_t(s)k_t + (1 - \tau_t(s))\, w_t(s)n_t(s) + \pi_t(s) - c_t(s) - S_t^k(s) - \delta k_t, \tag{50}$$

where $\tau_t(s)$ is the labor tax rate at $t+s$ and is included in the vector $z_t(s)$ of variables exogenous from the point of view of the household. With regard to $G(.)$, in addition to identical firm first order conditions and profit definition, one has to include the government's budget constraint, which stipulates a balanced budget

$$g = \tau_t(s)\, w_t(s)\, n_t(s), \tag{51}$$

where g is the constant flow of government expenditures. Period utility is further simplified to be linear in labor as proposed by Hansen (1985), so

$$u(c_t(s), n_t(s)) = \log c_t(s) - An_t(s). \tag{52}$$

Compared to the model without a government, the crucial difference in the equilibrium conditions arises from the effect of labor income taxes on labor supply

$$A = \lambda_t(s)(1 - \tau_t(s))w_t(s), \tag{53}$$

where $\lambda_t(s)$ is the marginal utility of income. An increase in taxes now shifts labor supply downward and, given a downward sloping labor demand curve, leads to a reduction in equilibrium employment. The dependence of labor supply on taxes together with the requirement for balanced budgets create the possibility of self-fulfilling expectations. In particular, if households expect high future labor taxes, they reduce future labor supply. As explained, this leads to a reduction in future equilibrium labor.

The resulting fall in the marginal product of capital implies, through the Euler equation, that current marginal utility has to fall. This, in turn, leads to a combination of increased consumption and leisure. The expectation of high future taxes thus leads to less work today which, in turn, forces the government to increase current labor income taxes to maintain a balanced budget. If the resulting increase in current taxes is large enough, such a situation can be an equilibrium and expectations become self-fulfilling. Schmitt-Grohé and Uribe consider the possibility of such indeterminacy as a function of steady state tax rates.[21] We perform a similar analysis in our context and look for the dependence of indeterminacy on period length h.

The long run behavior, as well as within-period dynamics, are straightforward extensions of those in the Cass-Koopmans model. We focus instead on the dynamic adjustment. Let lower case letters with bars denote the beginning-of-period steady state levels of variables and define $s_i = \delta\bar{k}/F(\bar{k}, \bar{n})$ and $s_c = \bar{c}/F(\bar{k}, \bar{n})$. The local dynamics around the initial point of a steady state function are derived in Appendix C:

$$\begin{pmatrix} \frac{\hat{\mu}_{t+h} - \hat{\mu}_t}{h} \\ \frac{\hat{k}_{t+h} - \hat{k}_t}{h} \end{pmatrix} = \begin{pmatrix} -\frac{c_{11} + hc_{12}c_{21}}{1+\rho h + hc_{11}} & -\frac{c_{12}(1+hc_{22})}{1+\rho h + hc_{11}} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \hat{\mu}_t \\ \hat{k}_t \end{pmatrix}, \tag{54}$$

where

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \equiv \begin{pmatrix} c_{11}(\tau) & c_{12}(\tau) \\ c_{21}(\phi, h, \tau) & c_{22}(\tau) \end{pmatrix} = \begin{pmatrix} -\frac{(\rho+\delta)s_n(1-\tau)}{s_k - \tau} & -\frac{(\rho+\delta)s_n\tau}{s_k-\tau} \\ \frac{(\rho+\delta)}{s_k}\left[\frac{s_n(1-\tau)}{s_k-\tau} + \frac{\theta_c}{\theta_y}s_c\right] & \frac{(\rho+\delta)(1-\tau)}{s_k-\tau} - \delta \end{pmatrix}. \tag{55}$$

The term θ_c/θ_y is defined in Appendix C and depends on both h and ϕ. We fix parameters using the yearly calibration of Schmitt-Grohé and Uribe, i.e. $s_k = 0.3$, $s_n = 0.7$, $\delta = 0.1$ and $\rho = 0.04$. With regard to within period discounting, we examine two cases, namely $\phi = 0$ and $\phi = \rho$. The first serves as a benchmark where steady state functions are constant, while the second makes the more plausible assumption that utility is discounted at the same rate within and across periods.

Figures 3 and 4 show the stability properties of the two models, for $\tau \in [0, 1)$ and $h \in [0, 4]$. Gray areas indicate saddle path stability and white areas indicate indeterminate dynamics. Figure 3 corresponds to the case where $\phi = 0$. As h increases, the indeterminacy regions become smaller overall, despite the fact that the upper bound remains constant and equal to 0.75. For $h = 0$, the range of indeterminacy is $\tau \in (0.3, 0.75)$, just as in Schmitt-Grohé and Uribe (1997), whereas for $h = 1$ the range of indeterminacy is $\tau \in (0.38, 0.75)$. In other words, for a labor tax rate between 30% and 38%, the result of Schmitt-Grohé and Uribe is reversed when we move to a discrete time setup. The range of taxes where this contradiction arises is both wide and empirically relevant. The result of Schmitt-Grohé and Uribe is particularly important and relevant because many OECD countries' tax rates fall within or very close to the range of indeterminacy they computed. Looking at the estimated, effective labor income tax rates in Mendoza, Razin and Tesar (1994), in 1988 the U.S., the U.K., Canada and Japan had rates only just below 30%. Italy, Germany and France on the other hand, fell within the range of indeterminacy with rates at 40% or more. Of course, these rates vary over time and one can find years where the UK rate was above 30% and European rates were less than 40%. De Haan and Volkerink (2001) provide updated estimates for 18 OECD countries in 1992. The labor income tax rates reported vary between 25% and 45%. Roughly speaking, this is the range of cross-sectional variation across developed countries. The range of taxes for which a standard discrete and a standard continuous time model produce opposite results ($30\% - 38\%$) lies exactly in the middle of this and covers almost half of the interval width.

[21] Strictly speaking the labor tax rate in this model is an endogenous variable and g is an exogenous constant parameter. Due to the existence of a Laffer curve there are two steady state labor taxes for a given g. However, like Schmitt-Grohe and Uribe (1997), we choose to take the steady state labor tax rate (at the begining of the period) as a parameter and work out what the corresponding g is.

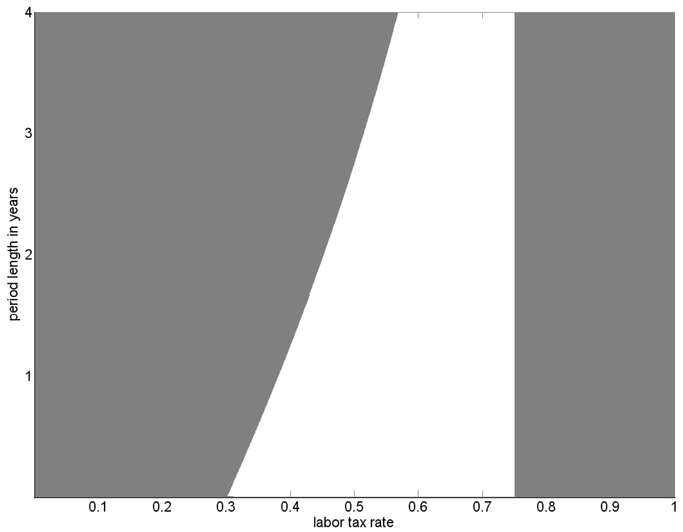

Figure 3: Stability properties for the general model with labor taxes, $\phi = 0$. Gray areas show saddle-path stability and white areas show indeterminacy.

To understand why period length h matters for local determinacy, consider the intuition for indeterminacy explained earlier. This relies on the assumption that an increase in the labor tax will increase government revenues. This is true as long as tax rates are to the left of the peak of the Laffer curve. With the current calibration, the peak of the Laffer curve is at $\tau^* = 0.75$ so indeterminacy can only arise for $\tau < 0.75$. This upper bound does not depend on h because the steady state is invariant to changes in h (as long as $\phi = 0$). For the lower bound, period length actually does matter. The intuition is as follows: for the intuitive argument for indeterminacy to work, it must be that an increase in expected future labor tax rates leads to a *large enough* increase in current labor tax rates. This is due to the fact that, on an equilibrium path, tax rates must converge monotonically to their steady state. The question is, therefore, how strongly current choices are affected by changed expectations about the future. Not surprisingly, since agents discount the future, if the future is one year ahead it has less of an impact on today's choices than if it is one quarter (or an instant) away. Put differently, as the frequency of decision making decreases (i.e. as h becomes larger), the response of current employment to higher expected future tax rates is milder. This is a direct result of the effects of the period length on optimal intertemporal decisions as described by the Euler equation.

Figure 4 shows the indeterminacies for the case $\phi = \rho$, i.e. the case where we allow for within-period discounting. Once again, the case $h = 0$ is equivalent to the standard continuous time model, as in Schmitt-Grohé and Uribe (1997), where indeterminacies occur for $\tau \in (0.3, 0.75)$. As in Figure 3, the intervals of tax rates for which indeterminacy occurs shrink as the period length becomes larger. However, compared to Figure 3, the upper bound of the interval is not a vertical line any longer. This is because when we allow for within period discounting, the steady state function for the tax rate depends on h. It can be shown that the tax rate $\tau^*(h)$ corresponding to the peak of the Laffer curve decreases in h. Thus, the upper bound of the indeterminacy region in Figure 4 traces exactly this dependence of the peak of the Laffer curve on the frequency of decision making. Regarding the lower bound of the indeterminacy area, the intuition is the same as for the case of no within-period discounting.

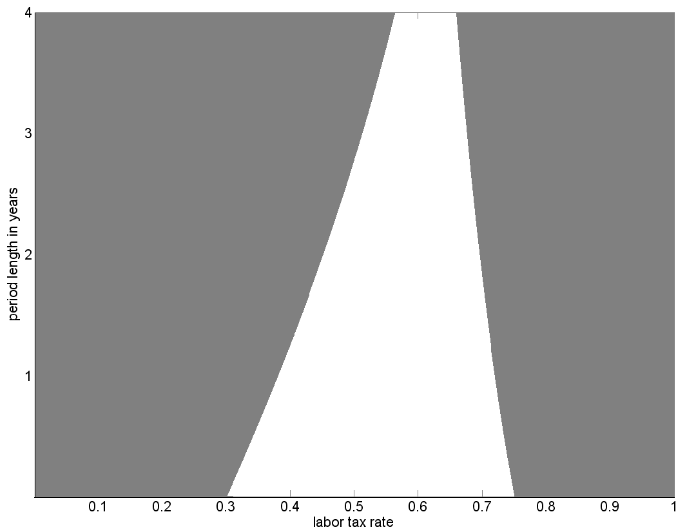

Figure 4: Stability properties for the general model with labor taxes, $\phi = \rho$. Gray areas show saddle-path stability and white areas show indeterminacy.

6. The New Keynesian Model

Our last example is a simple version of the New Keynesian model with endogenous capital accumulation. It is a production economy with monopolistic competition, quadratic price adjustment costs and a forward looking monetary policy rule whereby the interest rate responds to deviations of inflation from a target. This example has been analyzed both in continuous time (Dupor, 2001) and discrete time (Huang and Meng, 2007). Moreover, Li (2003) and Carlstrom and Fuerst (2005) study variations of this model with Calvo instead of Rotemberg pricing.

We assume that time is continuous and that stock variables such as bond holdings B, money holdings M and capital k vary discretely, while flow/control variables may vary continuously. We also assume that the prices of goods P (and thus the aggregate price index $\bar{P}$) vary discretely and define the inflation rate[22]

$$\bar{\pi}_t = \frac{\bar{P}_{t+h} - \bar{P}_t}{h\bar{P}_t}.$$

This model can be mapped into our general framework of section 2 by making the following definitions[23]

$$x_t = \left[P_t, B_{t-h}, k_t, M_t,\right]^T, \tag{56}$$

$$y_t(s) = \left[c_t(s), n_t(s), \pi_t(s), \tilde{n}_t(s), \tilde{k}_t(s)\right]^T, \tag{57}$$

$$\omega_t(s) = \left[S_t^M(s), S_t^B(s), S_t^k(s), \pi_t(s)\right]^T, \tag{58}$$

$$z_t(s) = \left[w_t(s), r_t(s), \tau_t(s), R_{t-h}(s), \bar{P}_t, \bar{\pi}_t\right]^T, \tag{59}$$

where $n_t(s)$ and $\tilde{n}_t(s)$ are labor supply and labor demand respectively and k_t and $\tilde{k}_t(s)$ are capital supply and capital demand respectively. S^B and S^M are the rates at which bond and money holdings accumulate notionally within a period (in the form of "non-operative" bonds and cash). Lump-sum taxes

[22] We do not attempt to argue here whether this is the best way of modelling how prices change or not; the assumption of discretely varying prices is made in order to capture the underlying assumption of a discrete time model.

[23] The model is presented in detail in Appendix D.

(or transfers) are denoted by $\tau_t(s)$ and $R_{t-h}(s)$ is the net nominal interest rate earned when holding a bond from $t-h$ to t. Finally, π_t is the rate of change of the price P_t defined as

$$\pi_t = \frac{P_{t+h} - P_t}{hP_t}. \tag{60}$$

We assume $\pi_t(s)$ does not vary within a period and write it as π_t. One could allow this 'own price inflation' rate to vary within a period with no significant alterations in the result that follows. The rest of the variables are the same as in previous sections.[24]

Utility is given by

$$u(y_t(s), x_{t+h}, z_t(s)) = \log c_t(s) + \psi \log \frac{M_{t+h}}{\bar{P}_t} - \nu n_t(s) - \frac{\gamma}{2} \left(\pi_t - \pi^*\right)^2, \tag{61}$$

where the parameter $\gamma > 0$ measures the size of the price adjustment cost. The household/firm faces a budget constraint

$$\begin{aligned} & \bar{P}_t \left(c_t(s) + \tau_t(s)\right) + S_t^B(s) + S_t^M(s) \\ = \; & R_{t-h}(s)B_{t-h} + P_t \tilde{k}_t^{s_k} \tilde{n}_t^{s_n}(s) - \bar{P}_t \left(r_t(s)\tilde{k}_t + w_t(s)\tilde{n}_t(s)\right) + \bar{P}_t \left(r_t(s)k_t + w_t(s)n_t(s)\right), \end{aligned} \tag{62}$$

and has to meet demand with its (Cobb-Douglas) production

$$\tilde{k}_t^{s_k} \tilde{n}_t^{s_n}(s) = Y_t^d(s) \left(\frac{P_t}{\bar{P}_t}\right)^{\eta}, \tag{63}$$

where $Y_t^d(s)$ is aggregate demand and $\eta < -1$. These two constraints give the functions $H^1(y_t(s), \omega_t(s), x_t, z_t(s))$ and $H^2(y_t(s), \omega_t(s), x_t, z_t(s))$ respectively. Stock accumulation constraints are given as usual by defining the functions $Q^i(.)$ in a compact vector form:

$$Q(y_t(s), \omega_t(s), x_t, z_t(s)) = \left[S_t^M(s), S_t^B(s), S_t^k(s), hP_t\pi_t\right]^T. \tag{64}$$

Finally, the functions $G^\kappa(.)$, $\kappa = 1, ..., 7$, are given by seven other conditions which we present in what follows. In equilibrium, factor markets have to clear, $\tilde{k}_t(s) = k_t$ and $\tilde{n}_t(s) = n_t(s)$ and so does the goods market

$$c_t(s) + \delta k_t + S_t^k(s) = k_t^\alpha n_t(s)^\beta. \tag{65}$$

We look for a symmetric equilibrium where $P_t = \bar{P}_t$, and thus $\pi_t = \bar{\pi}_t$. In addition, the government budget constraint has to hold, so

$$S_t^B(s) - R_{t-h}(s)B_{t-h} + S_t^M(s) + \tau_t(s) = 0, \tag{66}$$

and we assume a monetary policy rule

$$R_t = \psi\left(\bar{\pi}_t\right), \tag{67}$$

where ψ is strictly positive, differentiable and non-decreasing in $\bar{\pi}_t$. The derivative $\psi'\left(\bar{\pi}^*\right)$ measures the degree of activeness or passiveness of monetary policy. In what follows, the monetary rule will be a response of nominal interest rates to deviations of inflation from steady state.

[24] Note that in the correspondence with Section 2, π_t is both in y and in ω. This could easily be rectified by introducing a dummy variable so that this element of ω does not appear in the objective. We avoid introducing such notation for the purpose of clarity of exposition.

6.1. Dynamics. This example is particularly interesting and different from the rest for two reasons. First, because in this context assuming no within-period discounting (i.e. $\phi = 0$) and taking $h = 1$ does not necessarily imply that all variables are constant within a period. In particular, labor and the rental rate of capital are both time varying, and satisfy the relationship

$$\int_0^1 r_t(s)\,ds = \frac{s_k}{s_n}\frac{w_t}{\tilde{k}_t}\int_0^1 n_t(s)\,ds. \tag{68}$$

In order to revert to the standard discrete time version of the model, we need to *assume* that variables such as the rental rate $r_t(s)$ and $n_t(s)$ are constant within the period.

Second, for this example there is a discontinuity when we take the limit as $h \to 0$: the limiting system of dynamics does not correspond to the usual continuous time model. This is because the no arbitrage condition in the general discrete time model introduces an additional dynamic equation that is not present in continuous time.[25] To make this second point clearer, we abstract from within period discounting and consider a general discrete time model where all variables are assumed to be constant within a period. With all this in place, we can reduce the linearized system of period-by-period dynamics to the following[26]

$$\begin{pmatrix} \hat{\pi}_{t+h} \\ \hat{c}_{t+h} \\ \hat{r}_{t+h} \\ \hat{k}_{t+h} \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & 0 & 0 \\ g_1 & 1 & 0 & 0 \\ j_1 & 0 & 0 & 0 \\ 0 & l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{r}_t \\ \hat{k}_t \end{pmatrix}, \tag{69}$$

where

$$f_1 = (1+\rho h) + \frac{h}{\gamma(1+h\pi^*)}\frac{k^*}{c^*}\left[\frac{h}{(1+\rho h)}\frac{s_n}{s_k}(\rho+\delta)+1\right]\frac{\psi'(\pi^*)-(1+\rho h)}{(1+h\pi^*)}\eta, \tag{70}$$

$$f_2 = \frac{h\eta}{\gamma(1+h\pi^*)}\frac{s_n}{s_k}(\rho+\delta)\frac{k^*}{c^{*2}}, \quad g_1 = \left(\frac{\psi'(\pi^*)}{1+\rho h}-1\right)\frac{hc^*}{1+h\pi^*}, \tag{71}$$

$$j_1 = \frac{\psi'(\pi^*)-(1+h\rho)}{1+h\pi^*}, \quad l_1 = -\left[s_n\left(\frac{\delta k^*}{c^*}+1\right)+1\right]h, \tag{72}$$

$$l_2 = \frac{hs_n y^*}{r^*} = \frac{hs_n y^*}{(\rho+\delta)}, \quad l_3 = \left(\frac{c^*}{k^*}h+1\right). \tag{73}$$

We have three jump variables and one predetermined variable, thus a necessary and sufficient condition for determinacy is that there are three eigenvalues strictly outside the unit circle. If the eigenvalues outside of the unit circle are fewer than three, then we have indeterminacy and if they are more than three, then we have explosive dynamics.

The two eigenvalues of the bottom two rows of the matrix are 0 and $l_3 > 1$ (for any $h > 0$), so that the necessary and sufficient condition for determinacy is that the two eigenvalues of

$$A' = \begin{pmatrix} f_1 & f_2 \\ g_1 & 1 \end{pmatrix} \tag{74}$$

are strictly outside the unit circle. It is possible to show that these necessary and sufficient conditions are equivalent to requiring that the following inequalities are satisfied:

$$(1+\rho h) < \psi'(\pi^*) < (1+\rho h) - \frac{1}{\eta h}\frac{c^*}{k^*}\gamma(1+h\pi^*)^2 \min\left[h\rho, \frac{2+2(1+\rho h)}{1+\kappa}\right], \tag{75}$$

[25] A similar point is made in Carlstrom and Fuerst (2005) in the corresponding model with Calvo pricing.

[26] Detailed derivations of the linearized system as well as indeterminacy conditions subsequently presented are provided in Appendix D.

where

$$\kappa = \frac{h}{(1+\rho h)} \frac{s_n}{s_k} (\rho + \delta) + 1.$$

Note that the upper bound of inequality (75) is always strictly larger than the lower bound, so that there is always some parameter region for which the steady state is determinate. While it is not possible in general to determine $\min \left[h\rho, \frac{2+2(1+\rho h)}{1+\kappa} \right]$, we can confirm numerically that for reasonable parameter values, $h\rho$ is smaller than $[2 + 2(1 + \rho h)] / (1 + \kappa)$. Therefore for such parameters the necessary and sufficient condition for determinacy reduces to

$$(1 + \rho h) < \psi'(\pi^*) < (1 + \rho h) - \frac{\rho}{\eta} \frac{c^*}{k^*} \gamma (1 + h\pi^*)^2 .$$

How this condition varies with h depends on the exact functional form of the monetary policy rule. Suppose for example that we assume a non-linear rule

$$\mathcal{R}_t = \mathcal{R}^* \left(\frac{\Pi_t}{\Pi^*} \right)^q , \tag{76}$$

where $\mathcal{R}_t = 1 + hR_t$ and $\Pi_t = 1 + h\pi_t$ are the gross nominal interest rate and gross inflation rate respectively. Then,

$$R_t = \frac{\mathcal{R}^*}{h} \left(\frac{1 + h\pi_t}{\Pi^*} \right)^q - \frac{1}{h} \equiv \psi(\pi_t) , \tag{77}$$

and

$$\psi'(\pi^*) = q (1 + \rho h) . \tag{78}$$

Therefore, the condition for determinacy becomes

$$1 < q < 1 - -\frac{1}{\eta h} \frac{c^*}{k^*} \gamma \frac{(1 + h\pi^*)^2}{1 + \rho h} \min \left[h\rho, \frac{2 + 2(1 + \rho h)}{1 + \kappa} \right] . \tag{79}$$

The lower bound on q for determinacy yields the familiar Taylor principle result: in order to have a determinate steady state, monetary policy should be active, i.e. $q > 1$.[27] The upper bound for determinacy simply states that if monetary policy is "too" active, then the steady state may become indeterminate again. To understand the intuition behind why indeterminacy may arise in this setting, suppose that inflation is expected to rise. If monetary policy is passive, then the real interest rate goes down. This implies higher current demand for consumption goods. Moreover, from the no-arbitrage condition, a lower real interest rate implies lower future returns on capital, i.e. lower demand for investment goods. Whether overall demand for the single consumption/investment good increases or not, depends on how large this effect is on consumption and investment demand. If demand goes up, then prices will increase; for sufficiently high price increases, inflation will increase as well and the expectations will be fulfilled. Similarly, if the monetary policy is too active (i.e. when $\psi'(\pi^*)$ exceeds the upper bound in (75)), then the real interest rate and the expected return on capital will increase a lot. Even though current consumption demand will go down, the large increase in the expected return on capital may increase investment demand so much that the overall demand will go up, thereby pushing prices upwards and confirming the original expectations of higher inflation.

[27] Note that, were we to assume a linear monetary policy rule, then the standard Taylor principle would have to be modified to require $q > 1 + \rho h$. That is, our model shows that the correct discrete time counterpart to a continuous time linear monetary policy rule is to assume a log-linear rule.

6.2. The limiting case. Suppose that we now want to examine the dynamics of the limiting case $h \to 0$. We then transform the system to

$$\begin{pmatrix} \frac{\hat{\pi}_{t+h}-\hat{\pi}_t}{h} \\ \frac{\hat{c}_{t+h}-\hat{c}_t}{h} \\ \frac{\hat{r}_{t+h}-\hat{r}_t}{h} \\ \frac{\hat{k}_{t+h}-\hat{k}_t}{h} \end{pmatrix} = \begin{pmatrix} \frac{f_1-1}{h} & \frac{f_2}{h} & 0 & 0 \\ \frac{g_1}{h} & 0 & 0 & 0 \\ \frac{j_1}{h} & 0 & \frac{-1}{h} & 0 \\ 0 & \frac{l_1}{h} & \frac{l_2}{h} & \frac{l_3-1}{h} \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{r}_t \\ \hat{k}_t \end{pmatrix}. \tag{80}$$

Taking the limits as $h \to 0$, it becomes clear that the third row is not well-defined. This is because in the continuous time version of the model the rate r is a static variable. Indeed, as Dupor (2001) shows, the continuous time dynamics are given by

$$\begin{pmatrix} \dot{\pi}_t \\ \dot{c}_t \\ \dot{k}_t \end{pmatrix} = \begin{pmatrix} f_1' & f_2' & 0 \\ g_1' & 0 & 0 \\ l_1' & h_2' & l_3' \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{k}_t \end{pmatrix}, \tag{81}$$

where

$$f_1' = \lim_{h\to 0} \frac{f_1-1}{h}, \quad f_2' = \lim_{h\to 0} \frac{f_2}{h}, \quad g_1' = \lim_{h\to 0} \frac{g_1}{h}, \quad l_3' = \lim_{h\to 0} \frac{l_3-1}{h}. \tag{82}$$

Now we have two jump variables and one predetermined variable, so for determinacy we need two eigenvalues to be positive. Since $l_3' = \frac{c^*}{k^*} > 0$, we need one of the eigenvalues of the upper left block of the matrix to be positive and one to be negative (this is equivalent to requiring a passive monetary policy, as shown in Dupor, 2001). If there was continuity when going from the discrete time model to the continuous time one, then the necessary and sufficient condition for determinacy should be that both these eigenvalues are positive.

It is important to clarify that the discontinuity occurs in the *dynamics* of the model and it is not just a mere artefact of choosing the wrong continuous time model as a counterpart to the discrete model. Dupor's (2001) model *is* obtained as a result of taking limits as $h \to 0$ of all the equilibrium conditions. The discontinuity arises simply because the rental price of capital r is a dynamic variable in any discrete time model but a static variable in a continuous time model.

7. Depreciation and Discounting

A common feature across our examples is the effect on indeterminacy of two parameters: the capital depreciation rate δ and the time preference rate ρ. To analyze this, we concentrate on the last three examples, i.e. the three models that exhibit local indeterminacy. To understand how δ and ρ affect the dynamics in these three models, it is useful to reiterate on the role of h for indeterminacy. In all our examples, we can show that as h increases, the indeterminacy regions become smaller. We argue that this is due to the following reason: in models with capital accumulation, indeterminacy (i.e. self-fulfilling expectations in an environment with uncertainty), is closely related to the investment decisions of households as described by the Euler equation for capital. Whether agents' expectations are such that today's investment decisions are self-fulfilling or not, depends on how important the effect of the future is for decisions taken today. As h increases, the frequency of decision making in the dynamic problem decreases and, due to discounting, the future becomes less important, thus reducing the impact of expectations on today's decisions.

We discuss each parameter in turn. Starting with the depreciation rate, it is possible to show that as δ increases, i.e. as capital depreciates more, the range of indeterminacy becomes smaller. To understand the intuition behind why higher δ implies less indeterminacy, recall that for any calibration frequency,

δ represents the depreciation of capital over one unit of time. As δ increases, it means that capital depreciates more overall, so when households make decisions today about future investment, future capital is less attractive for them and therefore it has a smaller impact on current decisions. For all three examples, we present figures (5a-5f), that replicate the regions of indeterminacy for the extreme case of complete depreciation of capital. From these figures, it is quite clear that with $\delta = 1$, indeterminacy becomes much less likely for the discrete time model with $h = 1$. Notice that for the model with balanced budget rules (Example 3), a higher delta implies a wider range of indeterminacy for $h = 0$ compared to the benchmark case, but the slope of the lower bound for indeterminacy is so small that the ranges become very small very quickly. In that model, the reason that the indeterminacy range for $h = 0$ is larger as δ increases, is that the upper bound for indeterminacy is dictated by the peak of the Laffer curve, τ^* which increases in δ. This is because as capital depreciates more, households find it more worthwhile to hold less capital and work more, and thus the distortion due to labor taxation is present at higher labor tax rates.

A similar reasoning can be given for the role of the discount rate ρ. As ρ increases, i.e. as households become less patient, the future becomes less important to them and thus their expectations about it have a smaller effect on decisions taken today. This implies a reduced possibility of indeterminacy, since it is less likely that expectations will be self-fulfilling. Again, we present figures for all the three examples (Figures 6a-6f), where it is clearly seen that the regions of indeterminacy reduce as ρ increases. In contrast to the case of δ, for the model of balanced budget rules, we now see that a higher ρ implies a lower upper bound of indeterminacy when $h = 0$. This is because as households become less patient, current labor income is more important than future income from holding capital and therefore the labor tax distortion is bigger, moving the peak of the Laffer curve to the left.

In some sense, it is not surprising that we observe these three results relating to h, δ and ρ in all our examples. All these three parameters reflect how relevant the future is when making consumption/savings decisions today. The more irrelevant the future becomes (i.e. the larger h, δ and ρ are), the weaker is the intertemporal link that renders expectations self-fulfilling. With this discussion in mind, we conjecture that these results generally hold true in similar general equilibrium models with capital accumulation, to the extent that the same reasoning may apply whenever the Euler equation that determines the dynamics of consumption links current and future consumption through rates of return of future savings. However, we are unable to generalize this statement for other settings.

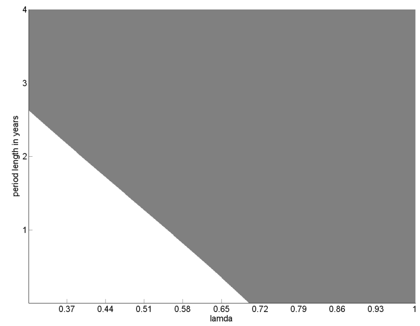

Figure 5a: Benhabib-Farmer, Benchmark

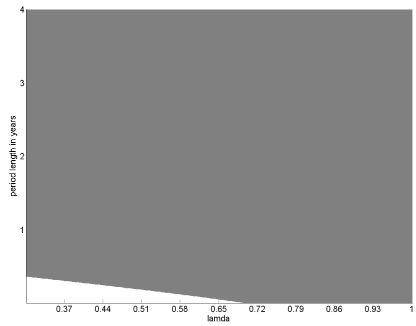

Figure 5b: Benhabib-Farmer, $\delta = 1$

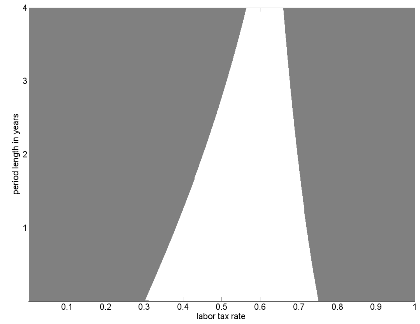

Figure 5c: Schmitt-Grohe and Uribe, Benchmark

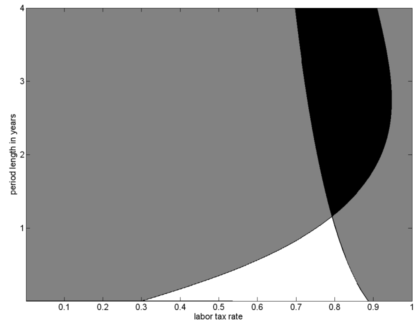

Figure 5d: Schmitt-Grohe and Uribe, $\delta = 1$

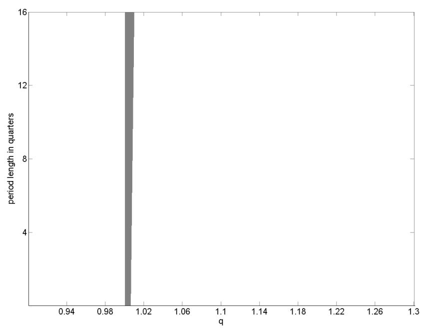

Figure 5e: New Keynesian, Benchmark

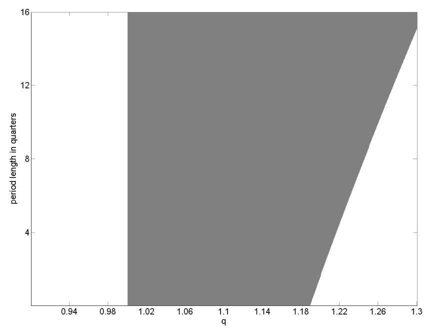

Figure 5e: New Keynesian, $\delta = 1$

Figures 5a-5f: Indeterminacy for larger δ. The left column of the panel shows the benchmark cases. For the New Keynesian model, the benchmark corresponds to the parametrization of Dupor (2001), i.e. $s_n = 0.7$, $\gamma = 350$, $\eta = -21$, $\pi^* = 2\%$, $\rho = 0.0045$, $\delta = 0.025$. Note that black areas show explosive solutions.

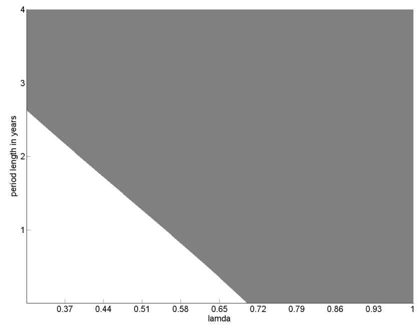

Figure 6a: Benhabib-Farmer, Benchmark

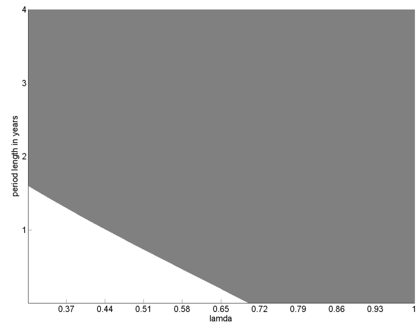

Figure 6b: Benhabib-Farmer, $\rho = 0.2$

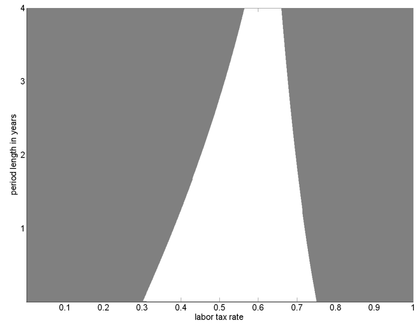

Figure 6c: Schmitt-Grohe and Uribe, Benchmark

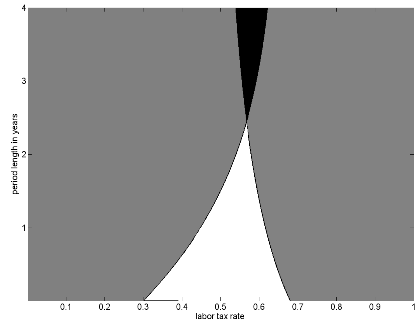

Figure 6d: Schmitt-Grohe and Uribe, $\rho = 0.2$

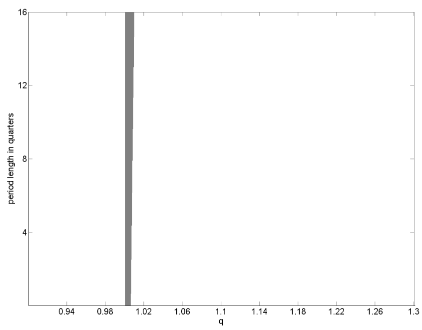

Figure 6e: New Keynesian, Benchmark

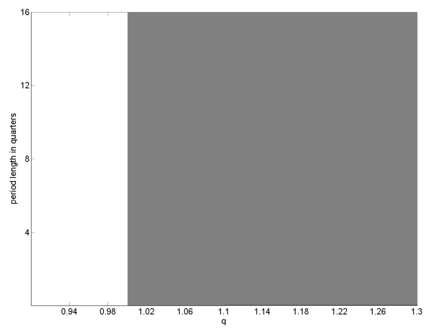

Figure 6f: New Keynesian, $\rho = 0.2$

Figures 6a-6f: Indeterminacy for larger ρ. The left column of the panel shows the benchmark cases. For the New Keynesian model, the benchmark corresponds to the parametrization of Dupor (2001), i.e. $s_n = 0.7$, $\gamma = 350$, $\eta = -21$, $\pi^* = 2\%$, $\rho = 0.0045$, $\delta = 0.025$. Note that black areas show explosive solutions.

8. Concluding Remarks

This paper has brought to the fore the underlying assumptions inherent in discrete time modelling and explored the, often hidden, consequences of such assumptions. We have shown that the choice of period length is a choice of economic significance that is separate from, although related to, the issue of calibration. We presented a general dynamic general equilibrium framework that can be used with any specific macroeconomic model in order to properly address the robustness of the model's predictions with respect to alternative choices for the period length parameter. We have used four concrete examples to illustrate the usefulness of the framework by pointing out the effects of period length in these specific cases. The different examples served different purposes: the Cass-Koopmans model allowed us to illustrate in detail how to apply our framework in the context of a widely known textbook model. The other three models were chosen because they are well known examples where indeterminacies arise. The models of Benhabib and Farmer (1994) and Schmitt-Grohé and Uribe (1997) serve to illustrate how sensitive indeterminacy is to the choice of period length. The latter also provides an example where within-period discounting does affect local dynamics. Contrary to the other examples, the model by Dupor (2001) provides an example where moving from a strictly positive period length to a continuous time model leads to completely different results because of a discontinuity in the dynamics at 0. We view our framework as a good platform within which such discontinuities can be brought to the researcher's attention. Finally, across all of our examples, we have been able to identify the effects of parameters such as the depreciation rate and the time preference rate on local dynamics.

A final comment is in order for our last example, the New Keynesian model with capital accumulation. This is an unusual case, where taking the limit of the discrete time model as $h \to 0$ does not yield the dynamics of the model with $h = 0$. Foley (1975), asserted that *"No substantive prediction or explanation in a well-defined macroeconomic [discrete time] model should depend on the real time length of the period. [...] If the results of a [discrete time] model do not depend in any important way on the period, the model can be formulated as a continuous model."* We interpret Foley's statement to mean that no *qualitative* prediction in a well-defined macroeconomic discrete time model should depend on the real time length of the period; in other words, we believe that it is not unreasonable to observe differences in the dynamics of discrete time models as h changes, as long as the qualitative structure remains the same. We have provided examples where, depending on the calibration, the dynamics change qualitatively and which, as a result, do not pass Foley's test. Our last example is even more striking because this occurs for any calibration. Moving from a discrete to a continuous time model, the qualitative features of the dynamics change entirely. We believe that in future work, it would be interesting to explore this example further and understand why these two models (discrete and continuous time versions of the New Keynesian model) are so radically different, but also try to construct an appropriate continuous time model that would yield the same dynamics as the limit of the discrete time model as the period length approaches zero.

We wish to close the paper with a word of caution to researchers that employ dynamic general equilibrium models for analysis of macroeconomic dynamics and policy design. Given our findings, that is that the stability of such systems may be quite sensitive to the period length, quantitative results based on such models should be interpreted with care. We hope that our work will aid researchers in assessing the robustness of their results to different assumptions about the period length. Ultimately, we believe that policy prescriptions arising from dynamic macroeconomic modelling would be significantly strengthened if we could carefully estimate (or at least calibrate) the period length, h, in actual economies.

Appendix

A. Analysis of the Cass-Koopmans Model

We provide a formal definition of the equilibrium in the Cass-Koopmans economy with discretely opening capital markets. Let us first describe the set of admissible paths. This will consist of sequences of continuous functions $\{c_t(\cdot)\}_{t\in I}$, $\{n_t(\cdot)\}_{t\in I}$, $\{S_t^k(\cdot)\}_{t\in I}$, $\{w_t(\cdot)\}_{t\in I}$, $\{r_t(\cdot)\}_{t\in I}$ and $\{\pi_t(\cdot)\}_{t\in I}$ for consumption, labor, net savings, wages, rental rates and profits respectively, where each element of these sequences is a continuous function with domain $[0,h)$ and range R_+. It will also contain $\{k_{t+h}\}_{t\in I}$, i.e. a sequence of real numbers for capital stock.

Definition 1. *A competitive equilibrium with sequential trade consists of sequences of price functions $\{w_t^*(\cdot)\}_{t\in I}$ and $\{r_t^*(\cdot)\}_{t\in I}$, sequences of quantity functions $\{c_t^*(\cdot)\}_{t\in I}$, $\{n_t^*(\cdot)\}_{t\in I}$, $\{S_t^{k*}(\cdot)\}_{t\in I}$, $\{\pi_t^*(\cdot)\}_{t\in I}$ and a sequence of capital stocks $\{k_{t+h}^*\}_{t\in I}$ such that*

(i) Given $\{w_t^*(\cdot)\}_{t\in I}$, $\{r_t^*(\cdot)\}_{t\in I}$ and $\{\pi_t^*(\cdot)\}_{t\in I}$, the quantities $\{c_t^*(\cdot)\}_{t\in I}$, $\{n_t^*(\cdot)\}_{t\in I}$, $\{S_t^{k*}(\cdot)\}_{t\in I}$ and $\{k_{t+h}^*\}_{t\in I}$ are optimal for the households. That is

$$\{c_t^*, n_t^*, i_t^*, k_{t+h}^*\}_{t\in I} = \underset{\{c_t, n_t, i_t, k_{t+h}\}_{t\in I}}{\arg\max} \sum_{t\in I} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h e^{-\phi s} u(c_t(s), n_t(s)) ds, \tag{83}$$

$$\begin{aligned} s.t.\ \ c_t(s) + S_t^k(s) + \delta k_t &= r_t^*(s) k_t + w_t^*(s) n_t(s) + \pi_t^*(s), && (84)\\ k_{t+h} - k_t &= \int_0^h S_t^k(s) ds, && (85)\\ c_t(s) &\geq 0, n_t(s) \geq 0, k_{t+h} \geq 0 && (86)\\ & k_0 \text{ given}, && (87) \end{aligned}$$

ii) Given $\{w_t^*(\cdot)\}_{t\in I}$ and $\{r_t^*(\cdot)\}_{t\in I}$, the quantities $\{n_t^*(\cdot)\}_{t\in I}$, $\{k_{t+h}^*\}_{t\in I}$ and $\{\pi_t^*(\cdot)\}_{t\in I}$ are optimal for the firms. That is

$$\left\{n_t^*(s), \pi_t^*(s), k_{t+h}^*\right\}_{t\in I} = \underset{\{n_t^*, k_{t+h}^*\}_{t\in I}}{\arg\max} \sum_{t\in I} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h e^{-\phi s} \frac{u_{c,t}(s)}{u_{c,0}(0)} \pi_t(s) ds, \tag{88}$$

subject to

$$\pi_t(s) = k_t^{s_k} n_t(s)^{s_n} - r_t^*(s) k_t - w_t^*(s) n_t(s) \tag{89}$$

iii) All markets clear at every instant s. The market clearing condition for the goods market is

$$c_t^*(s) + S_t^{k*}(s) + \delta k_t^* = (k_t^*)^{s_k} \left(n_t^*(s)\right)^{1-s_n} \quad \text{for all } s \in [0,h) \text{ and all } t \in I. \tag{90}$$

The labor and capital markets clear by definition of the sequences $\{n_t^*(\cdot)\}_{t\in I}$and $\{k_{t+h}^*\}_{t\in I}$.

A.1. Long run and dynamic adjustment. Here we describe the long run behavior of our economy. We argue that there is a period from which onwards the capital stock remains unchanged, since this variable never varies within a period. However, the rest of the (flow) variables will vary within each period in the way described in the previous section (i.e. exponentially decreasing or increasing), so that they are characterized by a steady state function, rather than the usual steady state value of the standard discrete or continuous time models.

The steady state for $\tilde{r}$ is easily derived from the Euler equation $\rho + \delta = \tilde{r}$. Let the steady state function for the capital labor ratio be $\kappa(s)$, $s \in [0, h)$ and

$$\kappa^h \equiv \kappa(0) \text{ and } \kappa^l \equiv \lim_{s \to h} \kappa(s). \tag{91}$$

From the definition of $\tilde{r}$, we obtain that

$$\rho + \delta = \tilde{r} = \frac{e^{\frac{\phi s_n h}{\sigma + s_k}} - 1}{\frac{\phi s_n h}{\sigma + s_k}} s_k \left(\kappa(0)\right)^{-s_n}. \tag{92}$$

Therefore,

$$\kappa^h = \frac{k}{n(0)} = \kappa^* q(h, \phi), \tag{93}$$

where

$$\kappa^* = \left(\frac{s_k}{\rho + \delta}\right)^{\frac{1}{s_n}}, \tag{94}$$

$$q(h, \phi) = \left(\frac{e^{\frac{s_n}{\sigma + s_k}\phi h} - 1}{\frac{s_n}{\sigma + s_k}\phi h}\right)^{\frac{1}{s_n}}. \tag{95}$$

In other words, at the beginning of the period the capital labor ratio is equal to the usual steady state capital labor ratio κ^*, corrected by a term that depends on the length of the period h and the within-period time preference rate ϕ. From (34), we also get the steady state function for the capital labor ratio

$$\kappa(s) = \kappa^* q(h, \phi) e^{-\frac{\phi s}{\sigma + s_k}}, \text{ for all } s \in [0, h). \tag{96}$$

We can establish the following properties for the steady state function of the capital labor ratio:

1. $\kappa(s)$ is continuous, strictly decreasing and convex, and
2. $\kappa^h > \kappa^*$ and $\kappa^l = \kappa^* q(h, \phi) e^{-\frac{\phi h}{\sigma + s_k}} < \kappa^*$.

The first property is obvious. To show the second, we first make some convenient changes of notation and establish some properties of q. Let

$$x = \frac{s_n \phi h}{\sigma + s_k} > 0. \tag{97}$$

We know that

$$e^x > 1 + x. \tag{98}$$

Since $x > 0$, this implies that

$$q(\phi, h) \equiv q(x) = \left(\frac{e^x - 1}{x}\right)^{\frac{1}{s_n}} > 1 \tag{99}$$

and therefore

$$\kappa^h = \kappa^* q(x) > \kappa^*. \tag{100}$$

Similarly,

$$\kappa^l = \kappa^* q(x) e^{-\frac{\phi h}{\sigma + s_k}} = \kappa^* \left(\frac{e^x - 1}{x}\right)^{\frac{1}{s_n}} \left(e^{-x}\right)^{\frac{1}{s_n}} = \kappa^* \left(\frac{1 - e^{-x}}{x}\right)^{\frac{1}{s_n}} < \kappa^*. \tag{101}$$

Combining the two properties, we have that $\kappa(s)$ crosses the horizontal line κ^* once and we can graphically summarize the behavior of capital labor ratio in steady state, in Figure 1. The Figure shows the steady state capital labor ratio for the standard discrete and continuous time model, as well as the capital labor ratio for our general continuous time model. In standard continuous time models, the steady state capital labor ratio is constant and equal to κ^*. In this case, for any interval of length h the total capital labor ratio is given by $h\kappa^*$, i.e. the rectangle composed by the dark and medium gray areas. If time is discrete, since the flow of labor is constant within a period and $h = 1$, the total capital labor ratio is given by $h\kappa^* = \kappa^*$ as in the standard continuous time case.

A.2. Dynamic Adjustment. We consider the dynamic adjustment of capital and the multiplier and convergence to the long run steady state functions. The period-by-period dynamics are described by the Euler equation (30) and the capital accumulation equation (15). For expositional clarity, from now on utility is taken to be linear in labor, i.e. $\sigma = 0$. We derive the Jacobian matrix describing the local dynamics around the long run beginning-of-period variables. In order to study the dynamics, we consider log-linear approximations of the variables around some point. In the standard discrete and continuous time models, this is done around the (invariable) long-run steady state level of every variable. Here, since the variables are described by steady state functions in the long run, we need to choose a specific point of these functions around which we approximate. The most natural choice is to consider the *beginning-of-period* steady state levels of variables. This is because the dynamics of capital (i.e. the state variable) change only at the beginning of each period, rather than continuously.

We denote the beginning-of-period steady state levels of variables with upper bars and the log-deviations of variables from these with hats. Also, let s_c be the share of steady state consumption in total income at the beginning of a period, which in general depends on h. Furthermore, let

$$\theta_y \equiv \frac{e^{\frac{\phi s_n}{s_k}h} - 1}{\frac{\phi s_n}{s_k}}, \tag{102}$$

$$\theta_c \equiv \frac{e^{-\phi h} - 1}{-\phi}. \tag{103}$$

We define

$$C(\phi, h) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21}(\phi, h) & c_{22} \end{pmatrix} = \begin{pmatrix} -\frac{(\rho+\delta)s_n}{s_k} & 0 \\ \frac{(\rho+\delta)}{s_k}\left[\frac{s_n}{s_k} + \frac{\theta_c}{\theta_y}s_c\right] & \frac{(\rho+\delta)}{s_k} - \delta \end{pmatrix}. \tag{104}$$

The dependence on the within-period discounting ϕ and the period length h in the bottom left element is through the factor θ_c/θ_y. The dynamics are then given by

$$\begin{pmatrix} \frac{\hat{k}_{t+h} - \hat{k}_t}{h} \\ \frac{\hat{\mu}_{t+h} - \hat{\mu}_t}{h} \end{pmatrix} = D(h, \phi)\begin{pmatrix} \hat{k}_t \\ \hat{\mu}_t \end{pmatrix}, \tag{105}$$

with

$$D(h, \phi) = \begin{pmatrix} \frac{c_{11} + hc_{12}c_{21}(\phi,h)}{1+\rho h - hc_{11}} & \frac{c_{12}(1+hc_{22})}{1+\rho h - hc_{11}} \\ c_{21}(\phi, h) & c_{22} \end{pmatrix}. \tag{106}$$

The stability properties are determined by the eigenvalues of $I_2 + hD(h, \phi)$. Since $c_{12} = 0$, the relevant eigenvalues $m_{1,2}$ are

$$m_1 = 1 + \frac{hc_{11}}{1 + \rho h - hc_{11}} = 1 - \frac{h\frac{(\rho+\delta)s_n}{s_k}}{1 + h\left(\rho + \frac{(\rho+\delta)s_n}{s_k}\right)} \tag{107}$$

$$m_2 = 1 + c_{22} = 1 + h\left[\frac{(\rho+\delta)s_n}{s_k} - \delta\right] \tag{108}$$

Given the parameter assumptions, i.e. $0 < \delta < 1$, $\rho > 0$ and $s_n, s_k \in (0,1)$, it can easily be shown that $0 < m_1 < 1$ and $m_2 > 0$, and therefore the system is saddle-path stable.

A.3. The limiting cases. We have argued that our general formulation nests standard discrete and continuous time models as special cases. The standard continuous time model can be retrieved by considering the limit as $h \to 0$. This is immediate for intratemporal conditions once we realize that flow functions become instantaneous rates. Dynamic conditions become

$$\dot{k}_t = S_t^k \tag{109}$$

$$\dot{\mu}_t = \left(\rho - \delta - s_k k_t^{s_k - 1} n_t^{s_n}\right) \mu_t, \tag{110}$$

where S_t^k is the flow of investment minus depreciation.[28] Furthermore, $\lim_{h \to 0} q(h, \phi) = 1$ and thus $\kappa^h = \kappa^l = \kappa^*$. Last, since $\lim_{h \to 0} (\theta_c / \theta_y) = 1$, by letting $h \to 0$, we get the standard continuous time dynamics

$$\lim_{h \to 0} D(h, \phi) = C, \tag{112}$$

where $C \equiv C(\phi, 0)$ is defined as in (104).

To retrieve the discrete time model, we assume that $h = 1$ and that there is no discounting within periods, i.e. that $\phi = 0$. The latter assumption, when used in conjunction with (33), (34), (35), (36) and (37), implies that all flows are constant and that $\tilde{r}$ is equal to the marginal product of capital. Thus, the Euler equation becomes

$$\mu_{t+1} - \mu_t = \frac{\rho - s_k k_{t+1}^{s_k - 1} n_{t+1}^{s_n} - \delta}{1 + \left(s_k k_{t+1}^{s_k - 1} n_{t+1}^{s_n} - \delta\right)} \mu_t \tag{113}$$

and finally,

$$\lim_{\phi \to 0} q(1, \phi) = 1, \tag{114}$$

so that

$$\kappa^h = \kappa^l = \kappa^*. \tag{115}$$

Finally, for $h = 1$ and $\phi = 0$ the matrix $D(h, \phi)$ reduces to the standard discrete time matrix.

B. Analysis of the Model of Increasing Returns

In this appendix, we show how to obtain (47)-(49) i.e. the log-linearization of the model in section 4. Define

$$\theta_y = \frac{e^{\frac{\phi \frac{s_n}{\lambda}}{\sigma + 1 - \frac{s_n}{\lambda}} h} - 1}{\frac{\phi \frac{s_n}{\lambda}}{\sigma + 1 - \frac{s_n}{\lambda}}} \text{ and } \theta_c = \frac{e^{-\phi h} - 1}{-\phi} \tag{116}$$

[28] Here we have used the fact that for any function $f(y_t(s))$ of the variable $y_t(s)$,

$$\lim_{h \to 0} \frac{\int_0^h [f(y_t(s))] \, ds}{h} = f(y_t) \tag{111}$$

As with the first two examples, we linearize the equilibrium conditions around beginning-of-period steady state levels of variables and we obtain the following

$$\hat{k}_{t+h} = \frac{\theta_y}{\bar{k}}\left[\frac{s_k}{\lambda}\bar{k}^{\frac{s_k}{\lambda}}\bar{n}^{\frac{s_n}{\lambda}}\hat{k}_t + \frac{s_n}{\lambda}\bar{k}^{\frac{s_k}{\lambda}}\bar{n}^{\frac{s_n}{\lambda}}\hat{n}_t(0) + \frac{\theta_c}{\theta_y}\bar{\mu}^{-1}\hat{\mu}_t + \frac{(1-\delta h)\,\bar{k}}{\theta_y}\hat{k}_t\right], \tag{117}$$

$$\tag{118}$$

$$\hat{\mu}_t = \hat{\mu}_{t+h} + \left(\frac{s_k}{\lambda}-1\right)\frac{\theta_y}{1+\rho h}s_k\bar{k}^{\frac{s_k}{\lambda}-1}\bar{n}^{\frac{s_n}{\lambda}}\hat{k}_{t+h} + \frac{s_n}{\lambda}\frac{\theta_y}{1+\rho h}s_k\bar{k}^{\frac{s_k}{\lambda}-1}\bar{n}^{\frac{s_n}{\lambda}}\hat{n}_{t+h}(0), \tag{119}$$

$$\hat{n}_t(0) = \frac{\frac{s_k}{\lambda}}{(\sigma+1)-\frac{s_n}{\lambda}}\hat{k}_t + \frac{1}{(\sigma+1)-\frac{s_n}{\lambda}}\hat{\mu}_t, \tag{120}$$

Next substitute out the labor variable to obtain the matrix that describes the dynamics of the system:

$$\hat{k}_{t+h} = \frac{\theta_y}{\bar{k}}\left[\left(\frac{s_k}{\lambda}\bar{k}^{\frac{s_k}{\lambda}}\bar{n}^{\frac{s_n}{\lambda}} + \frac{(1-\delta h)\,\bar{k}}{\theta_y} + \frac{\frac{s_k}{\lambda}\frac{s_n}{\lambda}\bar{k}^{\frac{s_k}{\lambda}}\bar{n}^{\frac{s_n}{\lambda}}}{(\sigma+1)-\frac{s_n}{\lambda}}\right)\hat{k}_t + \left(\frac{\frac{s_n}{\lambda}\bar{k}^{\frac{s_k}{\lambda}}\bar{n}^{\frac{s_n}{\lambda}}}{(\sigma+1)-\frac{s_n}{\lambda}} + \frac{\theta_c}{\theta_y}\bar{\mu}^{-1}\right)\hat{\mu}_t\right], \tag{121}$$

$$\hat{\mu}_t = \left[\left(\frac{s_k}{\lambda}-1\right)\frac{\theta_y}{1+\rho h}s_k\bar{k}^{\frac{s_k}{\lambda}-1}\bar{n}^{\frac{s_n}{\lambda}} + \frac{s_n}{\lambda}\frac{\theta_y}{1+\rho h}s_k\bar{k}^{\frac{s_k}{\lambda}-1}\bar{n}^{\frac{s_n}{\lambda}}\frac{\frac{s_k}{\lambda}}{(\sigma+1)-\frac{s_n}{\lambda}}\right]\hat{k}_{t+h} \tag{122}$$

$$+\left[1+\frac{s_n}{\lambda}\frac{\theta_y}{1+\rho h}s_k\bar{k}^{\frac{s_k}{\lambda}-1}\bar{n}^{\frac{s_n}{\lambda}}\frac{1}{(\sigma+1)-\frac{s_n}{\lambda}}\right]\hat{\mu}_{t+h}, \tag{123}$$

Following Benhabib and Farmer (1994), we define

$$\xi_1 = \frac{(\sigma+1)(\frac{s_k}{\lambda}-1)+\frac{s_n}{\lambda}}{\sigma+1-\frac{s_n}{\lambda}} \tag{124}$$

$$\xi_2 = -\frac{\frac{s_n}{\lambda}}{\sigma+1-\frac{s_n}{\lambda}} \tag{125}$$

Using the steady state conditions

$$(\rho+\delta)\,h = \theta_y s_k\bar{k}^{\frac{s_k}{\lambda}-1}\bar{n}^{\frac{s_n}{\lambda}} \tag{126}$$

and

$$0 = \theta_y\bar{k}^{\frac{s_k}{\lambda}}\bar{n}^{\frac{s_n}{\lambda}} - \frac{\theta_c}{\bar{\mu}} - \delta h\bar{k} \Rightarrow \frac{\theta_c}{\bar{\mu}\bar{k}} = \theta_y\bar{k}^{\frac{s_k}{\lambda}-1}\bar{n}^{\frac{s_n}{\lambda}} - \delta h = \frac{(\rho+\delta(1-s_k))\,h}{s_k} \tag{127}$$

we can simplify the equations to the discrete time system

$$P\begin{pmatrix}\hat{\mu}_{t+h}\\ \hat{k}_{t+h}\end{pmatrix} = S\begin{pmatrix}\hat{\mu}_t\\ \hat{k}_t\end{pmatrix} \tag{128}$$

where

$$P = \begin{pmatrix}1+\frac{h}{1+\rho h}(\rho+\delta)(-\xi_2) & \frac{h}{1+\rho h}(\rho+\delta)\,\xi_1\\ 0 & 1\end{pmatrix} \tag{129}$$

and

$$S = \begin{pmatrix}1 & 0\\ h\left[\frac{(\rho+\delta)}{s_k}(-\xi_2)+\frac{(\rho+\delta(1-s_k))}{s_k}\right] & 1+h\left[\frac{(\rho+\delta)}{s_k}\xi_1+\frac{\rho+\delta(1-s_k)}{s_k}\right]\end{pmatrix} \tag{130}$$

Let

$$C = \begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix} = \begin{pmatrix}(\rho+\delta)\,\xi_2 & -(\rho+\delta)\,\xi_1\\ \frac{(\rho+\delta)}{s_k}(-\xi_2)+\frac{(\rho+\delta(1-s_k))}{s_k} & \frac{(\rho+\delta)}{s_k}\xi_1+\frac{\rho+\delta(1-s_k)}{s_k}\end{pmatrix} \tag{131}$$

Which is the Jacobian of the Benhabib and Farmer model (the continuous version) if we substitute $\hat{\mu}_t = -\hat{c}_t$ and write their Jacobian with $\hat{\mu}$ first. Then

$$P = \begin{pmatrix} 1 - \frac{h}{1+\rho h} c_{11} & -\frac{h}{1+\rho h} c_{12} \\ 0 & 1 \end{pmatrix} \tag{132}$$

$$S = \begin{pmatrix} 1 & 0 \\ hc_{21} & 1 + hc_{22} \end{pmatrix} \tag{133}$$

Therefore

$$\begin{aligned} \begin{pmatrix} \frac{\hat{\mu}_{t+h} - \hat{\mu}_t}{h} \\ \frac{\hat{k}_{t+h} - \hat{k}_t}{h} \end{pmatrix} &= \frac{(P^{-1}S - I)}{h} \begin{pmatrix} \hat{\mu}_t \\ \hat{k}_t \end{pmatrix} \\ &= \begin{pmatrix} \frac{c_{11} + hc_{12}c_{21}}{1+\rho h - hc_{11}} & \frac{c_{12}(1+hc_{22})}{1+\rho h - hc_{11}} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \hat{\mu}_t \\ \hat{k}_t \end{pmatrix} \end{aligned} \tag{134}$$

C. Analysis of the Model of Balanced Budget Rules

In this appendix we show how to obtain the log-linearization for the model of section 5 presented in equations (54) - (55). Let

$$\theta_y \equiv \frac{e^{\frac{\phi s_n}{s_k} h} - 1}{\frac{\phi s_n}{s_k}} \tag{135}$$

$$\theta_c \equiv \frac{e^{-\phi h} - 1}{-\phi} \tag{136}$$

We start from the conditions describing equilibrium in the economy and reduce them to the following three relations:

$$\mu_t = \frac{\mu_{t+h}}{1+\rho h} \left[1 + \theta_y s_k k_{t+h}^{-s_n} n(t+h)^{s_n} - \delta h \right], \tag{137}$$

$$A n_t(0) = s_n \mu_t k_t^{s_k} n_t(0)^{s_n} - G\mu_t, \tag{138}$$

$$\theta_c \mu_t^{-1} + k_{t+h} - (1 - \delta h) k_t = \theta_y k_t^{s_k} n_t(0)^{s_n} - Gh. \tag{139}$$

Log-linearizing these around the beginning of period steady state values of the variables we get

$$\hat{\mu}_t = \hat{\mu}_{t+h} + (s_k - 1) \frac{\theta_y}{1+\rho h} s_k \bar{k}^{s_k - 1} \bar{n}^{s_n} \hat{k}_{t+h} + s_n \frac{\theta_y}{1+\rho h} s_k \bar{k}^{s_k - 1} \bar{n}^{s_n} \hat{n}_{t+h}(0), \tag{140}$$

$$\hat{n}_t(0) = \frac{s_k}{\psi(\sigma+1) - s_n} \hat{k}_t + \frac{\psi}{\psi(\sigma+1) - s_n} \hat{\mu}_t, \tag{141}$$

$$\hat{k}_{t+h} = \frac{\theta_y}{\bar{k}} \left[s_k \bar{k}^{s_k} \bar{n}^{s_n} \hat{k}_t + s_n \bar{k}^{s_k} \bar{n}^{s_n} \hat{n}_t(0) + \frac{\theta_c}{\theta_y} \bar{\mu}^{-1} \hat{\mu}_t + \frac{(1-\delta h)\bar{k}}{\theta_y} \hat{k}_t \right], \tag{142}$$

where

$$\psi = \frac{A\bar{n}^{s_k}}{s_n \bar{\mu} \bar{k}^{s_k}}. \tag{143}$$

We eliminate $\hat{n}(t)$ to end up with a dynamic system of equations in $\hat{\mu}_t$ and $\hat{k}_t$ given by

$$P\begin{pmatrix}\hat{\mu}_{t+h}\\ \hat{k}_{t+h}\end{pmatrix} = S\begin{pmatrix}\hat{\mu}_t\\ \hat{k}_t\end{pmatrix} \tag{144}$$

where

$$P = \begin{pmatrix} 1+s_n\frac{\theta_y}{1+\rho h}s_k\bar{k}^{s_k-1}\bar{n}^{s_n}\frac{\psi}{\psi(\sigma+1)-s_n} & s_n\frac{\theta_y}{1+\rho h}s_k\bar{k}^{s_k-1}\bar{n}^{s_n}\left[\frac{s_k}{\psi(\sigma+1)-s_n}-1\right] \\ 0 & 1\end{pmatrix}, \tag{145}$$

$$S = \begin{pmatrix} 1 & 0 \\ \left[\frac{\psi(s_n)\theta_y\bar{k}^{s_k-1}\bar{n}^{s_n}}{\psi(\sigma+1)-s_n}+\frac{\theta_c}{\bar{k}}\frac{1}{\gamma}\bar{\mu}^{-\frac{1}{\gamma}}\right] & \left[(1-\delta h)+\theta_y s_k\bar{k}^{s_k-1}\bar{n}^{s_n}+\frac{s_n\theta_y s_k\bar{k}^{s_k-1}\bar{n}^{s_n}}{\psi(\sigma+1)-s_n}\right]\end{pmatrix}. \tag{146}$$

Using the steady state relations, the elements of these matrices simplify to

$$p_{11} = 1+\frac{s_n(\rho+\delta)h}{1+\rho h}\frac{1-\tau}{s_k-\tau}, \tag{147}$$

$$p_{12} = \frac{s_n(\rho+\delta)h}{1+\rho h}\frac{\tau}{s_k-\tau}, \tag{148}$$

$$s_{21} = h\frac{\rho+\delta}{s_k}\left[\frac{(1-\tau)s_n}{s_k-\tau}+\frac{\theta_c}{\theta_y}s_c\right], \tag{149}$$

$$s_{22} = h(\rho+\delta)\frac{1-\tau}{s_k-\tau}+1-\delta h. \tag{150}$$

Next, let

$$C(\phi,h,\tau) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21}(\phi,h) & c_{22}\end{pmatrix} = \begin{pmatrix} s_n(\rho+\delta)\frac{1-\tau}{s_k-\tau} & s_n(\rho+\delta)\frac{\tau}{s_k-\tau} \\ \frac{\rho+\delta}{s_k}\left[\frac{(1-\tau)s_n}{s_k-\tau}+\frac{\theta_c}{\theta_y}s_c\right] & (\rho+\delta)\frac{1-\tau}{s_k-\tau}-\delta\end{pmatrix}, \tag{151}$$

so that

$$P = \begin{pmatrix} 1+\frac{h}{1+\rho h}c_{11} & \frac{h}{1+\rho h}c_{12} \\ 0 & 1\end{pmatrix}, \tag{152}$$

$$S = \begin{pmatrix} 1 & 0 \\ hc_{21} & 1+hc_{22}\end{pmatrix}, \tag{153}$$

and the dynamic system simplifies to

$$\begin{pmatrix}\frac{\hat{\mu}_{t+h}-\hat{\mu}_t}{h} \\ \frac{\hat{k}_{t+h}-\hat{k}_t}{h}\end{pmatrix} = \begin{pmatrix} -\frac{c_{11}+hc_{12}c_{21}}{1+\rho h+hc_{11}} & -\frac{c_{12}(1+hc_{22})}{1+\rho h+hc_{11}} \\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix}\hat{\mu}_t\\ \hat{k}_t\end{pmatrix}. \tag{154}$$

D. Analysis of New Keynesian Model

In parts D.1 to D.3 we explicitly present the model of section 6. In part D.4 we derive the log-linearization resulting in (69) - (73). In part D.5 we provide a proof for the indeterminacy condition in (75).

D.1. Household/Firm Problem. The household/firm maximizes

$$\max\sum_{t\in I}\left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}}\left\{\int_0^h e^{-\phi s}\left[\log c_t(s)+\psi\log\frac{M_{t+h}}{\bar{P}_t}-\nu n_t(s)-\frac{\gamma}{2}\left(\pi_t-\pi^*\right)^2\right]ds\right\}, \tag{155}$$

Subject to the following constraints

$$\begin{aligned}
& \bar{P}_t\left(c_t(s)+\tau_t(s)\right)+S_t^B(s)+S_t^M(s) \\
= \;& R_{t-h}(s)B_{t-h}+P_t\tilde{k}_t^{s_k}\tilde{n}_t^{s_n}(s)-\bar{P}_t\left(r_t(s)\tilde{k}_t+w_t(s)\tilde{n}_t(s)\right)+\bar{P}_t\left(r_t(s)k_t+w_t(s)n_t(s)\right), \quad (156)\\
\text{for all } t \in \;& I \text{ and } s\in[0,h)
\end{aligned}$$

$$M_{t+h}-M_t = \int_0^h S_t^M(s)ds \quad (157)$$

$$B_t-B_{t-h} = \int_0^h S_t^B(s)ds \quad (158)$$

$$k_{t+h}-k_t = \int_0^h S_t^k(s)ds \quad (159)$$

$$P_{t+h}-P_t = hP_t\pi_t \quad (160)$$

$$\tilde{k}_t^{s_k}\tilde{n}_t^{s_n}(s) = Y_t^d(s)\left(\frac{P_t}{\bar{P}_t}\right)^\eta, \quad s_k\in[0,1), \quad \eta<-1 \quad (161)$$

We introduce multipliers (adjusted by the discount factor) for each of the constraints $\lambda_t(s)$, μ_{Mt}, μ_{Bt}, μ_{kt}, μ_{Pt}, $\xi_t(s)$. The first order conditions with respect to

$$c_t(s),\; M_{t+h},\; n_t(s),\; \pi_t,\; S_t^B(s),\; S_t^M(s),\; S_t^k(s),\; B_t,\; P_{t+h},\; k_{t+h},\; \tilde{k}_t,\; \tilde{n}_t$$

are

$$\frac{1}{c_t(s)} = \bar{P}_t\lambda_t(s) \quad (162)$$

$$\psi\frac{1}{M_{t+h}}\int_0^h e^{-\phi s}ds = \mu_{Mt}-\left(\frac{1}{1+\rho h}\right)\mu_{M,t+h} \quad (163)$$

$$\nu = \bar{P}_t\lambda_t(s)w_t(s) \quad (164)$$

$$\gamma(\pi_t-\pi^*)\int_0^h e^{-\phi s}ds = \mu_{Pt}hP_t \quad (165)$$

$$e^{-\phi s}\lambda_t(s) = \mu_{Bt} \quad (166)$$

$$e^{-\phi s}\lambda_t(s) = \mu_{Mt} \quad (167)$$

$$e^{-\phi s}\lambda_t(s)\bar{P}_t = \mu_{kt} \quad (168)$$

$$\left(\frac{1}{1+\rho h}\right)\int_0^h e^{-\phi s}\lambda_{t+h}(s)R_t(s)ds = \mu_{Bt}-\left(\frac{1}{1+\rho h}\right)\mu_{Bt+h} \quad (169)$$

$$\left(\frac{1}{1+\rho h}\right)\left[\int_0^h e^{-\phi s}\lambda_{t+h}(s)\tilde{k}_{t+h}^{s_k}\tilde{n}_{t+h}^{s_n}(s)ds+\int_0^h e^{-\phi s}\xi_{t+h}(s)Y_{t+h}^d(s)\eta\left(\frac{P_{t+h}}{\bar{P}_{t+h}}\right)^{\eta-1}\frac{1}{\bar{P}_{t+h}}ds\right] \quad (170)$$

$$=\mu_{Pt}-\left(\frac{1}{1+\rho h}\right)\mu_{Pt+h}(1+h\pi_{t+h}) \quad (171)$$

$$\left(\frac{1}{1+\rho h}\right)\bar{P}_{t+h}\int_0^h e^{-\phi s}\lambda_{t+h}(s)(r_{t+h}(s)-\delta)ds = \mu_{kt}-\left(\frac{1}{1+\rho h}\right)\mu_{kt+h} \quad (172)$$

$$\int_0^h e^{-\phi s}\lambda_t(s)\left[s_kP_t\tilde{k}_t^{s_k-1}\tilde{n}_t^{s_n}(s)-\bar{P}_tr_t(s)\right]ds = \int_0^h e^{-\phi s}\xi_t(s)s_k\tilde{k}_t^{s_k-1}\tilde{n}_t^{s_n}(s)ds \quad (173)$$

$$\lambda_t(s)\left[s_nP_t\tilde{k}_t^{s_k}\tilde{n}_t^{s_n-1}(s)-\bar{P}_tw_t(s)\right] = \xi_t(s)s_n\tilde{k}_t^{s_k}\tilde{n}_t^{s_n-1}(s) \quad (174)$$

D.2. Government. The government is assumed to transfer in a lump sum fashion the proceeds from the issuance of new bonds and money

$$S_t^B(s) - R_{t-h}(s)B_{t-h} + S_t^M(s) + \tau_t(s) = 0 \tag{175}$$

Monetary policy follows a simple interest rate rule

$$R_t = \psi\left(\bar{\pi}_t\right) \tag{176}$$

D.3. Remaining Equilibrium Conditions.

$$\begin{aligned} \tilde{k}_t(s) &= k_t && (177)\\ \tilde{n}_t(s) &= n_t(s) && (178)\\ c_t(s) + \delta k_t + S_t^k(s) &= k_t^{\alpha} n_t(s)^{\beta} && (179)\\ P_t &= \bar{P}_t && (180)\\ \pi_t &= \bar{\pi}_t && (181) \end{aligned}$$

D.4. System of Dynamics. Within period dynamics are not central for making our main points. Therefore, to keep things simple, we assume all flow variables are constant within a period. Straightforward algebra allows us to reduce the equilibrium conditions to four dynamic conditions: the Phillips curve, the relationship between the real interest rate and consumption growth, an arbitrage condition and the resource constraint

$$\begin{aligned} &\frac{1}{h}\gamma\left[\left(1+h\pi_t\right)\left(\pi_t-\pi^*\right) - \frac{1}{1+\rho h}\left(1+h\pi_{t+h}\right)\left(\pi_{t+h}-\pi^*\right)\right] && (182)\\ &= \frac{1}{1+\rho h}\left[\left(1+\eta\right)\left(\frac{s_n}{v s_k}\right)^{s_n}\frac{k_{t+h}}{c_{t+h}^{1+s_n}} r_{t+h}^{s_n} - \eta\frac{k_{t+h} r_{t+h}}{s_k c_{t+h}}\right] && (183)\\ \frac{1+hR_t}{1+h\pi_t} &= \left(1+\rho h\right)\frac{c_{t+h}}{c_t} && (184)\\ 1 + h r_{t+h} - h\delta &= \frac{1+hR_t}{1+h\pi_t} && (185)\\ k_{t+h} - k_t &= h k_t^{s_k} n_t^{s_n} - h c_t - \delta h k_t && (186) \end{aligned}$$

where it is understood that the nominal interest rate is given by the monetary rule $R_t = \psi\left(\pi_t\right)$. We denote the (now always constant) steady state values of the variables with a star. Linearizing these conditions around their steady states, we obtain

$$\begin{aligned} \hat{\pi}_{t+h} &= \left\{\left(1+\rho h\right) + \frac{h}{\gamma\left(1+h\pi^*\right)}\left[\frac{h}{\left(1+\rho h\right)}\frac{s_n}{s_k}\left(\rho+\delta\right)+1\right]\frac{k^*}{c^*}\frac{\psi'\left(\pi^*\right)-\left(1+\rho h\right)}{\left(1+h\pi^*\right)}\eta\right\}\hat{\pi}_t && (187)\\ &+\frac{h\eta}{\gamma\left(1+h\pi^*\right)}\frac{s_n}{s_k}\left(\rho+\delta\right)\frac{k^*}{c^{*2}}\hat{c}_t && (188)\\ \hat{c}_{t+h} &= \left(\frac{\psi'\left(\pi^*\right)}{1+\rho h}-1\right)\frac{hc^*}{1+h\pi^*}\hat{\pi}_t + \hat{c}_t && (189)\\ \hat{r}_{t+h} &= \frac{\psi'\left(\pi^*\right)-\left(1+h\rho\right)}{1+h\pi^*}\hat{\pi}_t && (190)\\ \hat{k}_{t+h} &= -\left(\frac{s_n y^*}{c^*}+1\right)h\hat{c}_t + \frac{h s_n y^*}{r^*}\hat{r}_t + \left(\frac{c^*}{k^*}h+1\right)\hat{k}_t && (191) \end{aligned}$$

D.5. Proof of (75). We start by calculating the determinant and the trace

$$\begin{aligned}&\det A' \\ =\;& f_1 - f_2 g_1 \\ =\;& 1+\rho h + \underbrace{\frac{h\eta}{\gamma\left(1+h\pi^*\right)}\frac{\psi'\left(\pi^*\right)-\left(1+\rho h\right)}{\left(1+h\pi^*\right)}\frac{k^*}{c^*}}_{=\xi} = 1+\rho h+\xi \end{aligned} \tag{192}$$

Note that $\xi > 0$ if $\psi'\left(\pi^*\right) < \left(1+\rho h\right)$ and $\xi < 0$ if $\psi'\left(\pi^*\right) > \left(1+\rho h\right)$

$$\begin{aligned} tr\left(A'\right) &= f_1 + 1 \\ &= \left(1+\rho h\right) + \frac{h\eta}{\gamma\left(1+h\pi^*\right)}\frac{\psi'\left(\pi^*\right)-\left(1+\rho h\right)}{\left(1+h\pi^*\right)}\frac{k^*}{c^*}\underbrace{\left[\frac{h}{\left(1+\rho h\right)}\frac{s_n}{s_k}\left(\rho+\delta\right)+1\right]}_{=\varkappa>1}+1 \\ &= 1+\rho h+\xi\kappa+1 \end{aligned} \tag{193}$$

We also have

$$\begin{aligned} \det A' + trA' + 1 &= 1+\rho h+\xi+1+\rho h+\xi\kappa+2 \\ &= 2+2\left(1+\rho h\right)+\left(1+\kappa\right)\xi \end{aligned} \tag{194}$$

and

$$\begin{aligned} \det A' - trA + 1 &= \left(1-\kappa\right)\xi < 0 \text{ if } \xi > 0\text{, i.e. if } \psi'\left(\pi^*\right) < \left(1+\rho h\right) \\ &= \left(1-\kappa\right)\xi > 0 \text{ if } \xi < 0\text{, i.e. if } \psi'\left(\pi^*\right) > \left(1+\rho h\right) \end{aligned} \tag{195}$$

The necessary and sufficient conditions for both roots to lie outside the unit circle are in general: if $\det A' > 1$ then

$$\det(A') - tr(A') + 1 > 0 \tag{196}$$
$$\det(A') + tr(A') + 1 > 0 \tag{197}$$

and if $\det A' < 1$

$$\det(A') - tr(A') + 1 < 0 \tag{198}$$
$$\det(A') + tr(A') + 1 < 0 \tag{199}$$

We consider each case in turn.

1. If $\det A' > 1$ then

$$\det(A') - tr(A') + 1 > 0 \tag{200}$$
$$\det(A') + tr(A') + 1 > 0 \tag{201}$$

For this to be relevant, we check when the determinant is larger than one.

If $\psi'\left(\pi^*\right) < \left(1+\rho h\right)$, because $\eta < 0$, then $\det A' > 1$ and the first necessary and sufficient condition for indeterminacy is

$$\begin{aligned} \det(A') - tr(A') &= f_1 - f_2 g_1 - f_1 - 1 > -1 \Longleftrightarrow \\ -f_2 g_1 &> 0 \\ -\frac{h\eta}{\gamma(1+h\pi^*)}\frac{s_n}{s_k}(\rho+\delta)\frac{k^*}{c^*}\left(\frac{\psi'(\pi^*)}{1+\rho h}-1\right)\frac{h}{1+h\pi^*} &> 0 \end{aligned} \tag{202}$$

But this can never be true when $\psi'(\pi^*) < (1+\rho h)$ because $\eta < 0$. *Thus, for* $\psi'(\pi^*) < (1+\rho h)$ *(i.e. passive policy) there is never determinacy.*

If $\psi'(\pi^*) > (1+\rho h)$, we can still have that $\det A' > 1$, if

$$\rho h + \frac{h\eta}{\gamma(1+h\pi^*)}\frac{\psi'(\pi^*)-(1+\rho h)}{(1+h\pi^*)}\frac{k^*}{c^*} > 0 \tag{203}$$

$$h\rho > -\frac{\eta h}{\gamma(1+h\pi^*)}\frac{\psi'(\pi^*)-(1+\rho h)}{(1+h\pi^*)}\frac{k^*}{c^*} = -\xi \tag{204}$$

If the above is true, then the conditions are

$$\det A' - trA + 1 = (1-\kappa)\xi > 0 \tag{205}$$

which is true for $\psi'(\pi^*) > (1+\rho h)$. The second condition is

$$\det A' + trA' + 1 = 2 + 2(1+\rho h) + (1+\kappa)\xi > 0 \tag{206}$$

because $\xi < 0$ this is true when

$$-\xi < \frac{2+2(1+\rho h)}{1+\kappa} \tag{207}$$

2. If $\det A' < 1$ then

$$\det(A') - tr(A') + 1 < 0, \tag{208}$$

$$\det(A') + tr(A') + 1 < 0. \tag{209}$$

For this to be relevant, we need $\det A' = 1 + \rho h + \xi < 1$, i.e. we need $h\rho < -\xi$, in which case the first condition becomes

$$\det A' - trA + 1 = (1-\kappa)\xi < 0 \text{ if } \xi > 0, \tag{210}$$

but this cannot be true together with $h\rho < -\xi$ since it would imply $h\rho < 0$. Since the first condition cannot be satisfied when $\det A' < 1$, it means there can never be determinacy in this case. Collecting all this together, we have that there is determinacy whenever

$$\psi'(\pi^*) > (1+\rho h), \tag{211}$$

$$h\rho > -\xi, \tag{212}$$

$$\frac{2+2(1+\rho h)}{1+\kappa} > -\xi, \tag{213}$$

i.e.

$$-\xi < \min\left[h\rho, \frac{2+2(1+\rho h)}{1+\kappa}\right], \tag{214}$$

$$-\frac{h\eta}{\gamma(1+h\pi^*)}\frac{\psi'(\pi^*)-(1+\rho h)}{(1+h\pi^*)}\frac{k^*}{c^*} < \min\left[h\rho, \frac{2+2(1+\rho h)}{1+\kappa}\right], \tag{215}$$

$$\psi'(\pi^*) < (1+\rho h)-\frac{1}{\eta h}\frac{c^*}{k^*}\gamma(1+h\pi^*)^2\min\left[h\rho, \frac{2+2(1+\rho h)}{1+\kappa}\right] \tag{216}$$

References

[1] Aadland, D. and K.X.D. Huang, 2004. "Consistent high-frequency calibration." *Journal of Economic Dynamics and Control* 28, 2277 – 2295.

[2] Baierl, G., K. Nishimura and M. Yano, 1998. "The role of capital accumulation in multi-sectoral models." *Journal of Economic Behavior and Organization* 33, 467 - 479.

[3] Bambi, M. and O. Licandro, 2004. "(In)determinacy and Time-to-Build", *Working Paper, ECO no. 2004/17, European University Institute.*

[4] Benhabib, J., 2004, "Interest Rate Policy in Continuous Time with Discrete Delays", *Journal of Money, Credit and Banking* 36, 1-15.

[5] Benhabib, J. and R. E. A. Farmer, 1994. "Indeterminacy and Increasing Returns", *Journal of Economic Theory* 63, 19-41.

[6] Carlstrom, C. T. and T. S. Fuerst, 2005. "Investment and interest rate policy: a discrete time analysis". *Journal of Economic Theory* 123, 4 - 20.

[7] De Haan, J. and B. Volkerink (2001). "Tax Ratios: A Critical Survey". *OECD Tax Policy Studies,* No. 5, 2001.

[8] Dupor, B., 2001. "Investment and Interest Rate Policy" *Journal of Economic Theory* 98, 85 - 113.

[9] Farmer, R. and J.-T. Guo, 1994. "Real Business Cycles and the Animal Spirits Hypothesis", *Journal of Economic Theory* 63, 42 - 72.

[10] Foley, D. K., 1975. "On Two Specifications of Asset Equilibrium in Macroeconomic Model", *The Journal of Political Economy* 83, 303 - 324.

[11] Guo, J.-T., 2004. "Increasing Returns, Capital Utilization, and the Effects of Government Spending", *Journal of Economic Dynamics & Control* 28, 1059-1078.

[12] Hansen, G. 1985. "Indivisible Labor and the Business Cycle", *Journal of Monetary Economics* 16, 309 - 341.

[13] Hintermaier, T., 2005. "A Sunspot Paradox", *Economics Letters* 87, 285 - 290.

[14] Hood, W. C., 1948. "On Some Aspects of the Treatment of Time in Economic Theory", *The Canadian Journal of Economics and Political Science* 14, 453-468.

[15] Huang, K. X. D and Q. Meng, 2007. "Capital and macroeconomic instability in a discrete-time model with forward-looking interest rate rules", *Journal of Economic Dynamics and Control* 31, 2802 - 2826.

[16] Jovanovic, B., 1982. "Inflation and Welfare in the Steady State", *The Journal of Political Economy* 90, 561-577.

[17] Karni, E., 1979. "On the Specification of Asset Equilibrium in Macroeconomic Models: A Note", *The Journal of Political Economy* 87, 171-177.

[18] Kydland, F. E. and E. C. Prescott, 1982. "Time to Build and Aggregate Fluctuations", *Econometrica* 50, 1345-1370.

[19] Leung, S. F., 1995. "A Distinction between Continuous-time and Discrete-time Models of Uncertain Lifetime", *Economics Letters* 47, 291-296.

[20] Li, H., 2003. "Inflation determination under a Taylor rule: consequence of endogenous capital accumulation", *Mimeograph.*

[21] Licandro, O. and L. A. Puch, 2006. "Is discrete time a good representation of continuous time?", *Mimeograph.*

[22] Magill, M. J. P. and J. Benhabib, 1983. "Price Relations on Futures Markets for Storable Commodities", *Journal of Mathematical Analysis and Applications* 91, 567-591.

[23] Marshall, A., 1920. "Principles of Economics", *McMillan and Co.*, 8th Edition.

[24] Mendoza, E.G., A. Razin and L.L. Tesar, 1994. "Effective Tax Rates in Macroeconomics. Cross-Country Estimates of Tax Rates on Factor Incomes and Consumption", *Journal of Monetary Economics*, 34, 297-323.

[25] Mercenier, J. and P. Michel, 1994. "Discrete-time finite horizon approximation of infinite horizon optimization problems with steady state invariance." *Econometrica* 62, 635 - 656.

[26] Mino K., K. Nishimura, K. Shimomura and P. Wang, 2008. "Equilibrium Dynamics in Discrete-Time Endogenous Growth Models with Social Constant Returns", *Economic Theory* 34, 1-23.

[27] Mitra, T., 1998. "On the relationship between discounting and complicated behavior in dynamic optimization models." *Journal of Economic Behavior and Organization* 33, 421 - 434.

[28] Romer, D., 1986. "A Simple General Equilibrium Version of the Baumol-Tobin Model", *Quarterly Journal of Economics* 101 (November 1986): 663-685

[29] Samuelson, P. A., 1957. "Intertemporal Price Equilibrium: A Prologue to the Theory of Speculation", *The Collected Scientific Papers of Paul A. Samuelson*, MIT Press, Cambridge, Mass., 946-984.

[30] Schmitt-Grohé, S. , 1997. "Comparing Four Models of Aggregate Fluctuations due to Self-Fulfilling Expectations", *Journal of Economic theory*, 72, 96 - 147.

[31] Schmitt-Grohé, S. and M. Uribe, 1997. "Balanced-Budget Rules, Distortionary Taxes, and Aggregate Instability", *The Journal of Political Economy* 105, 976 - 1000.

[32] Telser L. G. and R. L. Graves, 1968. "Continuous and Discrete Time Approaches to a Maximization Problem", *The Review of Economic Studies* 35, 307-325.

[33] Turnovsky, S. J., 1977. "On the Formulation of Continuous Time Macroeconomic Models with Asset Accumulation", *International Economic Review* 18, 1-28.

[34] Turnovsky S. J. and Burmeister, E., 1977. "Perfect Foresight, Expectational Consistency, and Macroeconomic Equilibrium", *The Journal of Political Economy* 85, 379 - 393.