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## LEARNING IN FINANCIAL MARKETS

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# LEARNING IN FINANCIAL MARKETS

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## ABSTRACT

### Learning in Financial Markets\*

We survey the recent literature on learning in financial markets. Our main theme is that many financial market phenomena that appear puzzling at first sight are easier to understand once we recognize that parameters in financial models are uncertain and subject to learning. We discuss phenomena related to the volatility and predictability of asset returns, stock price bubbles, portfolio choice, mutual fund flows, trading volume, and firm profitability, among others.

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# 1. Introduction

Parameter uncertainty is ubiquitous in finance. Agents are uncertain about many of the parameters characterizing financial markets, and they learn about these parameters by observing data. This learning is facilitated by the existence of vast quantities of financial data, but it is also hampered by the large amount of randomness pervading financial markets.

This survey reviews selected recent work on learning in finance. The overarching theme is that learning helps us better understand a variety of phenomena observed in financial markets. Many facts that appear baffling at first sight seem less puzzling once we recognize that parameters are uncertain and subject to learning. We ask questions such as: Why are stock returns so volatile? Why are they predictable? Why do investors trade so much? Why do stocks of young firms exhibit high valuations and high volatility? Why are technological revolutions accompanied by stock price “bubbles”? Why do fund flows react strongly to fund performance? Why do firms become less profitable after they go public? We show that learning helps us answer all of these questions, as well as many others.

Our quest for the answers is guided by the principle of parsimony. We always seek the simplest explanation, one that makes as few assumptions as possible. For example, a single-agent model is more parsimonious than a multi-agent model, symmetric information is simpler than asymmetric information, and rationality has fewer degrees of freedom than irrationality. If a fact can be explained in a rational single-agent model, then it can surely be explained in more complicated models as well. Of course, many facts cannot be explained with few assumptions. But the world appears a lot more parsimonious once parameter uncertainty is acknowledged.

# 2. Bayesian Updating

The cornerstone of learning is Bayes’ rule, which describes how rational agents update their beliefs after receiving new information. To illustrate the updating process, consider the following example of an agent who is uncertain about the parameter  $\theta$ . Before observing any signals, the agent’s prior beliefs about  $\theta$  are normally distributed with mean  $\theta_0$  and variance  $\sigma_0^2$ . The agent observes  $T$  independent signals about  $\theta$ ,  $s_t = \theta + \epsilon_t$ , where each  $\epsilon_t$  is normal with zero mean and known variance  $\sigma^2$ . According to Bayes’ rule, the agent’s posterior (i.e.,

revised) beliefs about  $\theta$  are normally distributed with mean  $\tilde{\theta}_T$  and variance  $\tilde{\sigma}_T^2$ , where

$$\tilde{\theta}_T = \theta_0 \frac{\frac{1}{\sigma_0^2}}{\frac{1}{\sigma_0^2} + \frac{T}{\sigma^2}} + \bar{s} \frac{\frac{T}{\sigma^2}}{\frac{1}{\sigma_0^2} + \frac{T}{\sigma^2}} \quad (1)$$

$$\tilde{\sigma}_T^2 = \frac{1}{\frac{1}{\sigma_0^2} + \frac{T}{\sigma^2}}, \quad (2)$$

and  $\bar{s}$  is the average signal value,  $\bar{s} = (1/T) \sum_{t=1}^T s_t$ . The posterior mean  $\tilde{\theta}_T$  is a precision-weighted average of the prior mean and the average signal. Unlike  $\tilde{\theta}_T$ , the posterior variance  $\tilde{\sigma}_T^2$  does not depend on the realizations of the signals. This variance, which we also refer to as uncertainty about  $\theta$ , decreases as the number of signals  $T$  increases (learning reduces uncertainty). The posterior variance is always smaller than the prior variance,  $\tilde{\sigma}_T^2 < \sigma_0^2$ .

Bayesian updating can also be formulated recursively. Denoting  $\Delta\tilde{\theta}_t = \tilde{\theta}_t - \tilde{\theta}_{t-1}$ , we have

$$\Delta\tilde{\theta}_t = m_t (s_t - \tilde{\theta}_{t-1}) \quad \text{with} \quad m_t = \frac{1}{1 + \sigma^2/\tilde{\sigma}_{t-1}^2}. \quad (3)$$

Intuitively, observing a higher-than-expected signal,  $s_t > \tilde{\theta}_{t-1}$ , leads the agent to revise the expectation upward,  $\tilde{\theta}_t > \tilde{\theta}_{t-1}$ , and vice versa. This revision is large when the multiplier  $m_{t-1}$  is large, which happens when the ratio of uncertainty  $\tilde{\sigma}_{t-1}^2$  to signal variance  $\sigma^2$  is large.

If time is viewed as continuous rather than discrete, the signal takes the differential form,  $ds_t = \theta dt + \sigma dW_t$ , where  $dW_t$  denotes a Brownian motion. The updating formula is then

$$d\tilde{\theta}_t = m_t (ds_t - \tilde{\theta}_t dt) \quad \text{with} \quad m_t = \frac{\tilde{\sigma}_t^2}{\sigma^2}, \quad (4)$$

which is analogous to (3). Note that  $m_t$  in (3) can also be written as  $\tilde{\sigma}_t^2/\sigma^2$ . Even in continuous time, uncertainty  $\tilde{\sigma}_t^2$  declines over time according to the same formula (2).

### 3. Stock Valuation

When the discount rate  $r$  and dividend growth  $g$  are constant, the stock price is given by

$$P = \frac{D}{r - g}, \quad (5)$$

where  $D$  is the next period's dividend. This well-known Gordon growth formula holds not only when dividend growth is constant, but also when it follows the process

$$\frac{dD_t}{D_t} = g dt + \sigma dW_t, \quad (6)$$

in which case  $g$  represents average dividend growth. See the Appendix for proof.

Interesting things happen when  $g$  in (6) is unknown. Pástor and Veronesi (2003, 2006) argue that uncertainty about  $g$  increases the stock price. The Appendix shows that for any probability density  $f(g)$  such that  $r > g$  with probability one,

$$P = E \left\{ \frac{D}{r - g} \right\} > \frac{D}{r - E \{g\}}, \quad (7)$$

where  $E \{.\}$  denotes an expectation with respect to  $f(g)$ . The inequality in (7) follows from Jensen's inequality, since  $1/(r - g)$  is convex in  $g$ . For the same reason, the price-to-dividend ( $P/D$ ) ratio increases with the dispersion of  $f(g)$ . Intuitively, uncertainty about  $g$  makes the distribution of future dividends right-skewed, thereby increasing expected future dividends. Loosely speaking, a firm with some probability of failing (a very low  $g$ ) and some probability of becoming the next Google (a very high  $g$ ) is very valuable. When  $r$  is endogenously determined in equilibrium with a power-utility representative agent, uncertainty about  $g$  may increase or decrease  $r$ , but its overall effect on  $P/D$  is positive (Pástor and Veronesi (2006)). Instead of focusing on  $P/D$ , which does not exist for non-dividend-paying firms, Pástor and Veronesi focus on the market-to-book ratio ( $M/B$ ). This ratio increases with uncertainty about the firm's average profitability, which can be interpreted as uncertainty about the average growth rate of book value.

Since uncertainty declines over time due to learning (see (2)), the Pástor and Veronesi (2003) model predicts that  $M/B$  declines over a typical firm's lifetime, so that younger firms should have higher  $M/B$ 's than otherwise identical older firms. This prediction is confirmed in U.S. stock data: the median  $M/B$  falls monotonically from 2.25 for 1-year-old firms to 1.25 for 10-year-old firms, and the cross-sectional relation between firm age and  $M/B$  is reliably negative. The model also implies that the effect of age on  $M/B$  should be stronger for younger firms and non-dividend-paying firms. Besides,  $M/B$  should decrease with expected return and increase with both the level and the volatility of profitability. All of these predictions are confirmed empirically.

### 3.1. Stock Price “Bubbles”

Pástor and Veronesi (2006) extend their 2003 model and calibrate it to match the observed stock valuations at the peak of the Nasdaq “bubble”. They argue that stocks were not necessarily overvalued in the late 1990s because uncertainty about  $g$  was unusually high. The higher the uncertainty about  $g$ , the higher the stock price in (7). The authors compute the level of uncertainty that allows their model to match the Nasdaq valuations at the peak

in March 2000. This uncertainty, which they call “implied uncertainty” for its similarity to implied volatility in option pricing, seems plausible because it matches not only the level but also the volatility of Nasdaq stock prices. These prices in the late 1990s were not only high but also highly volatile, and both facts are consistent with high uncertainty about  $g$ . (We show later that uncertainty about  $g$  increases return volatility.) Moreover, cross-sectionally, stocks with high  $M/B$ ’s also had highly volatile returns, suggesting that these stocks had highly uncertain future growth rates. In general, the authors argue that the level and volatility of stock prices are positively linked through firm-specific uncertainty about  $g$ .

The same learning model also seems capable of explaining the bursting of the Nasdaq bubble. Nasdaq’s profitability plummeted in 2000 and 2001. As a result, investors revised their expectations of Nasdaq’s future profitability downward, pushing prices down. Since the investors’ prior uncertainty was large, their expectation revision was also large (see (3)). Starting with prior beliefs that match Nasdaq’s level and volatility in March 2000, the model predicts a post-peak Nasdaq price decline that is comparable to that observed in the data.

The Nasdaq bubble, which developed during the Internet boom, is an example of a more general pattern. Technological revolutions tend to be accompanied by bubbles in the stock prices of innovative firms. This evidence is typically attributed to market irrationality, but Pástor and Veronesi (2009) argue that it is also consistent with a rational general equilibrium model of learning. They argue that new technologies are characterized by high uncertainty about their future productivity, and that the time-varying nature of this uncertainty can produce the observed bubbles. In their model, a representative agent is learning about a new technology’s productivity. If the agent learns that the technology is sufficiently productive, he adopts it on a large scale, creating a technological revolution. Most new technologies do not cause revolutions, but those that do exert two opposing effects on stock prices: a positive cash flow effect and a negative discount rate effect. On the one hand, the new technology must surprise the agent with high realized productivity (otherwise he would not adopt it), and this positive cash flow news pushes stock prices up. On the other hand, the risk associated with the new technology gradually changes from idiosyncratic to systematic, thereby pushing up discount rates and thus depressing stock prices. The cash flow effect prevails initially, but the discount rate effect prevails eventually, producing an apparent bubble in stock prices. Importantly, these bubbles are observable only in hindsight—they are unexpected by investors in real time but we observe them *ex post* when we focus only on technologies that eventually led to technological revolutions.

The Pástor-Veronesi model makes numerous additional predictions, which are supported by the evidence from 1830–1861 and 1992–2005 when the railroad and Internet technologies

spread in the U.S. A key prediction is that the market beta of innovative firms—a measure of systematic risk—should increase during technological revolutions. Indeed, the beta of the technology-loaded Nasdaq index doubled between 1997 and 2002, and the beta of railroad stocks increased sharply in the 1850s. Since stories based on irrationality do not predict increases in systematic risk during revolutions, this evidence suggests that rational learning about new technologies is useful in explaining the bubble-like patterns in stock prices.

Technological revolutions exhibit not only stock price bubbles but also apparent overinvestment. This fact is also consistent with rational learning, as shown by Johnson (2007). Johnson develops an equilibrium model of investment in a new industry whose production function has an unknown return to scale. The model implies that the most efficient way to learn about returns to scale is by overinvestment relative to the full-information case. This overinvestment is accompanied by high stock prices and low expected returns.

Other models that link stock price bubbles to learning include Scheinkman and Xiong (2003) and Hong, Scheinkman and Xiong (2006, 2008). Unlike the models discussed above, these models feature heterogeneous beliefs, and they produce bubbles with the help of additional assumptions such as short-sale constraints and behavioral biases. Battalio and Schultz (2006) argue that short-sale constraints were not responsible for the Nasdaq bubble. Li and Xue (2008) argue that this bubble can be explained by uncertainty about a possible structural break in the economy's productivity. Finally, Donaldson and Kamstra (1996) argue against a bubble in the 1920s based on a neural network model of dividend expectations.

## 4. Stock Return Volatility

The volatility of stock returns exhibits interesting empirical features. For example, it is high relative to the volatility of the underlying dividends, and it varies over time in a persistent fashion. Learning helps us understand these facts.

### 4.1. The Level of Volatility

When the discount rate  $r$  and the average dividend growth  $g$  are both constant and known, the stock price is given by (5), and return volatility equals the volatility of dividend growth. In reality, though, the post-war volatility of market returns has averaged 17% per year, whereas the dividend growth volatility has been only 5%. To reconcile this difference, it helps to view  $g$  in (6) as unknown (Timmermann (1993)). Agents learn about  $g$  by observing

realized dividends. Unexpectedly high dividends increase the stock price not only through current dividends, but also by raising expectations of future dividends. This “double kick” to the stock price increases return volatility compared to the case in which  $g$  is known.

To formalize Timmermann’s intuition, let  $r$  be constant and known, let  $g$  have a truncated normal distribution that assigns zero probability to  $g \geq r$ , and let  $\tilde{g}_t$  and  $\tilde{\sigma}_t^2$  denote the mean and variance of  $g$  as perceived at time  $t$ . Extending Timmermann’s work, we show in the Appendix that the standard deviation of returns is approximately equal to

$$\text{Return Volatility} \approx \text{Dividend Growth Volatility} \times \left[ 1 + \left( \frac{\partial \log(P/D)_t}{\partial \tilde{g}_t} \right) m_t \right], \quad (8)$$

where “dividend growth volatility” stands for  $\sigma$  in (6),  $\partial \log(P/D)_t / \partial \tilde{g}_t > 0$ , and  $m_t > 0$  is given in (4). This formula shows that return volatility exceeds the volatility of dividend growth. The difference can be substantial. For example, let  $\sigma = 5\%$ ,  $r = 10\%$ ,  $\tilde{g}_t = 3\%$  and  $\tilde{\sigma}_t = 2\%$ . Return volatility is then about 20%, four times higher than the 5% volatility of dividend growth. Equation (8) also shows that return volatility increases with  $\sigma$ , and it also increases with uncertainty  $\tilde{\sigma}_t^2$ , through  $m_t$ . If  $\tilde{\sigma}_t^2 \rightarrow 0$ , then  $m_t \rightarrow 0$ , and return volatility converges to  $\sigma$ . Finally, return volatility increases with the sensitivity of  $\log P/D$  to  $\tilde{g}_t$ . This sensitivity is higher when the discount rate is lower because distant future dividends then matter more for today’s stock price.

The key implications of the simple model used above carry over to more sophisticated models. Brennan and Xia (2001a), for example, consider a general equilibrium model with a representative agent who learns about time-varying  $g_t$ . They obtain results similar to ours in a model successfully calibrated to aggregate consumption and dividend data.

Uncertainty  $\tilde{\sigma}_t^2$  declines over time as investors learn about  $g$  (see (2)), so return volatility should decline over time as well (see (8)). One might therefore expect stocks of younger firms to have more volatile returns than stocks of older firms. Indeed, Pástor and Veronesi (2003) find a negative cross-sectional relation between volatility and firm age. The median return volatility of U.S. stocks falls monotonically from 14% per month for 1-year-old firms to 11% per month for 10-year-old firms. The authors’ model predicts higher stock volatility for firms with more volatile profitability, firms with more uncertain average profitability, and firms that pay no dividends. These predictions are confirmed empirically.

## 4.2. Time Variation in Volatility

Stock return volatility varies dramatically over time—it has been as low as 10% per year in the mid-1990s and as high as 70% in October 2008. Moreover, volatility is persistent, as there are extended periods of sustained high or low volatility. Learning helps us understand the variation in volatility. The models of Timmermann (1993) and Pástor and Veronesi (2003) cannot generate increases in volatility because they feature a constant  $g$ , and uncertainty about a constant  $g$  declines deterministically to zero (in (2),  $\tilde{\sigma}_T^2 \rightarrow 0$  as  $T \rightarrow \infty$ ). Even when  $g$  varies over time in a smooth manner, as in Brennan and Xia (2001a), the posterior uncertainty about  $g$  converges deterministically to a constant. However, if  $g$  follows a process with unobservable regime shifts, then uncertainty about  $g$  can fluctuate stochastically, and return volatility can rise. For example, if a dividend growth realization is far from the current estimate of  $g$ , the probability of a regime shift in  $g$  increases. The posterior uncertainty about  $g$  then increases because after a regime shift, past data become less useful for forecasting. The higher uncertainty pushes up return volatility through a mechanism similar to that in (8): investors' expectations react more swiftly to news when uncertainty is higher. Moreover, volatility is persistent because perceptions of regime shifts change slowly.

David (1997) develops a model with unobservable regime shifts in the average productivities of linear technologies, which are subject to learning by a representative agent. Learning induces time-varying allocations to these technologies, resulting in persistent stochastic variation in return volatility. Veronesi (1999) uses similar means to show that even if dividends display low constant volatility, stock returns may possess high volatility with persistent variation. He also shows that learning about a regime-shifting  $g$  generates stock price “over-reaction” to bad news in good times. Such news increases uncertainty about  $g$ , which might have shifted from a high- $g$  to a low- $g$  regime. This increase in uncertainty increases not only volatility but also the equilibrium discount rate, thereby amplifying the stock price drop. In a similar setting, Veronesi (2004) shows that a small probability of a long recession can induce volatility to cluster at high levels during recessions. Johnson (2001) shows that learning about the degree of persistence of fundamental shocks generates time-varying return volatility, as well as a novel relation between volatility and momentum. David and Veronesi (2002) employ unobservable regime shifts to explain the dynamics of option-implied volatility and skewness spreads. David and Veronesi (2008) develop a structural model for volatility forecasting that exploits learning-induced relations between volatility and price multiples. This model improves upon regression-based volatility forecasts.

## 5. Return Predictability

Stock returns are somewhat predictable. When the aggregate  $P/D$  ratio is low, future stock market returns tend to be high. Timmermann (1993, 1996) explains that such predictability can arise due to learning about  $g$ . When the current estimate of  $g$ ,  $\tilde{g}_t$ , is below the “true” value of  $g$ , investors are pessimistic about future dividends, so  $P/D$  is low. The future returns are likely to be high, though, because  $\tilde{g}_t$  is likely to be revised upward. As a result, low  $P/D$  forecasts high future returns.

This learning-induced predictability is observable only in hindsight, as explained by Lewellen and Shanken (2002). Returns appear predictable to econometricians analyzing historical data, but real-time investors cannot exploit this predictability. Learning drives a wedge between the distribution perceived by investors and the “true” distribution estimated by empirical tests. Lewellen and Shanken show that learning can also induce cross-sectional predictability. For example, econometricians may observe violations of the Capital Asset Pricing Model (CAPM) even if all real-time investors believe this model holds. Coles and Loewenstein (1988) argue that the CAPM should hold even with estimation risk, but that is true only for the perceived, not the empirical, distribution of returns.

Learning can also generate risk-driven predictability that is detectable by real-time investors. Veronesi (1999, 2000) shows how learning induces time-varying expected returns that are correlated with  $P/D$ . Massa and Simonov (2005) and Ozoguz (2009) argue that uncertainty is a priced risk factor in the cross-section of stock returns. Croce, Lettau and Ludvigson (2006) show that learning helps explain the cross-sectional value effect. In their model, consumption growth has a small but persistent “long-run” component (see Bansal and Yaron (2004)), as well as a transitory “short-run” component. Stocks that are more exposed to the long-run component command higher risk premia. Even though value stocks tend to have shorter-duration cash flows, they can exhibit more long-run risk and therefore higher risk premia than growth stocks. The reason is that when the long-run component of consumption is unobservable, its optimal forecasts covary with short-run consumption shocks. Learning induces positive correlation between the long-run and short-run consumption risks.

Another cross-sectional puzzle that can be understood via learning is the negative relation between stock returns and the dispersion of analysts’ earnings forecasts, documented by Diether, Malloy and Scherbina (2002). The authors interpret their result as evidence of market frictions that preclude investors with pessimistic views from shorting stocks, which are then temporarily overvalued. Johnson (2004) delivers the same result in a frictionless rational learning model. He interprets dispersion as a proxy for uncertainty about asset

value. After adding leverage to a model similar to Pástor and Veronesi (2003), Johnson shows that expected stock return decreases with this uncertainty. Equity is a call option on the levered firm's assets. More idiosyncratic uncertainty raises the option value, which lowers the stock's exposure to priced risk, thereby reducing the expected return. The model also predicts that the negative relation found by Diether et al should be stronger for firms with more leverage. Johnson finds empirical support for this prediction.

Uncertainty about the value of a firm's assets also helps us understand credit spreads on corporate bonds. In structural models of corporate bond valuation à la Merton (1974), the firm's value follows an observable diffusion process. These models imply counterfactually small credit spreads for short-term bonds because they imply that the default probability over a short period is small. Uncertainty about firm value increases short-term credit spreads, Duffie and Lando (2001) explain, because investors are uncertain about the nearness of current assets to the default-triggering level. Supporting this explanation, Yu (2005) finds empirically that firms with more accounting disclosure (and so less uncertainty about firm value) tend to have lower credit spreads, especially on short-term bonds. Finally, David (2008a) uses learning about unobservable regime shifts in the fundamentals to explain why the observed credit spreads are higher than spreads produced by Merton-like models calibrated to the observed default frequencies.

Empirical Bayesian studies that analyze return predictability include Kandel and Stambaugh (1996), Stambaugh (1999), Avramov (2002), Cremers (2002), Avramov and Chordia (2006), Pástor and Stambaugh (2009), Wachter and Warusawitharana (2009), and others.

## 6. The Equity Premium

Learning can help us understand the equity premium puzzle. Note, however, that uncertainty about average dividend growth  $g$  can increase or decrease the equity premium. In Veronesi (2000), this uncertainty decreases the equity premium. Veronesi considers an endowment economy with a power-utility representative agent whose elasticity of intertemporal substitution (EIS) is below one. The agent consumes aggregate dividends. Bad news about dividends decreases not only current consumption but also expected future consumption, as the agent revises  $\tilde{g}_t$  downward. The agent's desire to smooth consumption leads him to save more today and demand more stock, which cushions the decline in the stock price. Therefore, learning about  $g$  decreases the covariance between stock returns and consumption growth, compared to the case with known  $g$ . As a result, the equity premium is lower as well.

The opposite result obtains under different preferences. When EIS exceeds one, downward revisions in  $\tilde{g}_t$  lead the agent to save less and demand less stock, resulting in a positive relation between uncertainty and the equity premium (e.g., Brandt, Zeng, and Zhang (2004) and Ai (2007)). The relation is positive also when the agent has exponential utility (Veronesi (1999)) and when dividends and consumption follow separate processes with correlated unobservable drift rates (Li (2005)). Finally, if the agent learns about average consumption growth, the expected consumption growth varies over time. Such variation increases the equity premium under Epstein-Zin preferences (e.g., Bansal and Yaron (2004)).

The equity premium is also affected by uncertainty about the volatility of consumption growth. Weitzman (2007) considers an endowment economy with unknown consumption volatility. He shows that the posterior distribution of consumption growth is fat-tailed, which induces a power-utility representative agent to demand a substantially higher equity premium compared to the case of known volatility. Lettau, Ludvigson and Wachter (2008) also assume that consumption volatility is unobservable, but they allow it to jump between two states. They find empirically that the posterior probability of the low-volatility state increased in the 1990s, helping justify the stock price run-up in that period.

Learning can also generate higher equity premia when investors are averse to ambiguity (e.g., Cagetti et al. (2002), Leippold et al. (2008), Epstein and Schneider (2008)). When investors worry about model misspecification, their learning must take into account the set of possible alternative models. Model uncertainty is penalized and investors maximize utility over worst-case beliefs. This cautious behavior increases the risk premia in equilibrium.

## 7. Learning About the Conditional Mean Return

The studies discussed in the previous section let investors learn about fundamentals and analyze the equilibrium implications for expected returns. Another way of relating learning to expected returns is to let investors-econometricians, who do not necessarily set prices, learn about expected returns by observing realized returns and other information.

Let  $r_{t+1}$  denote a return from time  $t$  to time  $t + 1$ . This return can be decomposed as

$$r_{t+1} = \mu_t + u_{t+1}, \tag{9}$$

where  $\mu_t$  is the conditional expected return and  $u_{t+1}$  is the unexpected return with mean zero, conditional on all information at time  $t$ . In reality, investors observe only a subset of all information, so they do not observe the true value of  $\mu_t$ . How do rational investors learn

about  $\mu_t$  from realized returns?

If the conditional mean is constant,  $\mu_t = \mu$ , the updating formula (3) applies: unexpectedly high returns increase the posterior mean  $\tilde{\mu}_t$ , and vice versa. Under noninformative prior beliefs about  $\mu$  ( $\sigma_0 = \infty$  in (1)),  $\tilde{\mu}_t$  is simply the historical sample mean.

If  $\mu_t$  varies over time, though, the sample mean is no longer the best estimate of  $\mu_t$ . Given a process for the unobservable  $\mu_t$ , we obtain the best estimate of  $\mu_t$  by optimal filtering. For example, if  $\mu_t$  follows an AR(1) process with normal shocks,

$$\mu_{t+1} = (1 - \beta)\bar{\mu} + \beta\mu_t + w_{t+1}, \quad (10)$$

then the Kalman filter implies that the best estimate of  $\mu_t$  (and hence also the best forecast of  $r_{t+1}$ ) is a weighted average of all past returns,

$$E(r_{t+1}|F_t) = \sum_{s=0}^{t-1} \kappa_s r_{t-s},$$

where  $F_t$  contains the full history of returns up to time  $t$  (see Pástor and Stambaugh (2009)). The weights in this average,  $\kappa_s$ , crucially depend on  $\rho_{uw}$ , the correlation between unexpected returns,  $u_{t+1}$  in (9), and innovations in expected returns,  $w_{t+1}$  in (10). This correlation is likely to be negative because unexpected increases in discount rates tend to push prices down. If this correlation is sufficiently negative, then recent returns receive negative weights and more distant returns receive positive weights in computing the average.

To understand this result, suppose recent returns have been unusually high. On the one hand, one might think the expected return has risen, since a high mean is more likely to generate high realized returns, and  $\mu_t$  is persistent. On the other hand, one might think the expected return has declined, since declines in expected returns tend to be accompanied by high realized returns. When  $\rho_{uw}$  is sufficiently negative, the latter effect outweighs the former and recent returns enter negatively when estimating the conditional expected return. At the same time, more distant past returns enter positively because they are more informative about the unconditional mean  $\bar{\mu}$  than about recent changes in the conditional mean  $\mu_t$ .

The above analysis assumes that the information set  $F_t$  consists only of past returns. However, investors might use more information to forecast returns. For example, investors might believe that  $\mu_t$  is given by a linear combination of observable predictors  $x_t$ :

$$\mu_t = \alpha + \beta(x_t - \bar{x}), \quad (11)$$

where  $\bar{x}$  is the unconditional mean of  $x_t$ . Viewing  $\beta$  as unobservable, Xia (2001) uses continuous-time filtering to derive an updating rule for  $\tilde{\beta}_t = E[\beta|F_t]$ . This rule features

time-varying covariance between updates to  $\tilde{\beta}_t$  and realized returns. The sign of the covariance depends on whether  $x_t$  is above or below  $\bar{x}$ . When  $x_t$  exceeds  $\bar{x}$ , an unusually high return implies that  $\tilde{\beta}_t$  is revised upward, and vice versa.

The assumption (11) is unlikely to hold exactly. If the true mean  $\mu_t$  is not a linear function of  $x_t$ , the updating rule for  $\mu_t$  involves not only  $x_t$  but also past returns (Pástor and Stambaugh (2009)). Past dividends can also be useful in estimating  $\mu_t$ , as shown by van Binsbergen and Koijen (2008) and Rytchkov (2008). These studies exploit present value relations to estimate not only  $\mu_t$  but also expected dividend growth rates.

Learning about  $\mu_t$  also affects long-horizon return volatility. Let  $r_{t,t+k} = r_{t+1} + r_{t+2} + \dots + r_{t+k}$  denote the return in periods  $t+1$  through  $t+k$ . The variance of  $r_{t,t+k}$  conditional on data available at time  $t$ ,  $\text{Var}(r_{t,t+k}|F_t)$ , depends on uncertainty about  $\mu_{t+j}$ . Consider the following example from Pástor and Stambaugh (2008). Suppose  $r_t$ 's are independently and identically distributed with known variance  $\sigma^2$  and unknown constant mean  $\mu$ . Conditional on  $\mu$ , the mean and variance of  $r_{t,t+k}$  are  $k\mu$  and  $k\sigma^2$ , respectively. An investor who knows  $\mu$  faces the same per-period variance,  $\sigma^2$ , regardless of  $k$ . However, an investor who does not know  $\mu$  faces variance that increases with  $k$ . Applying the variance decomposition,

$$\text{Var}(r_{t,t+k}|F_t) = \text{E}\{k\sigma^2|F_t\} + \text{Var}\{k\mu|F_t\} = k\sigma^2 + k^2\text{Var}\{\mu|F_t\}.$$

Since  $\mu$  remains uncertain after seeing the data,  $(1/k)\text{Var}(r_{t,t+k}|F_t)$  increases with  $k$ . Thus, an investor who believes that stock prices follow a random walk but who is uncertain about  $\mu$  views stocks as riskier in the long run. When  $\mu_t$  is time-varying, predictability induces both mean reversion, which reduces long-run variance, and additional uncertainty, which increases long-run variance. The overall effect, according to Pástor and Stambaugh (2008), is a higher per-period variance at longer horizons, contrary to conventional wisdom.

## 8. Portfolio Choice

Investors appear to invest too little in stocks. Consider an investor with risk aversion  $\gamma$  who can invest in risky stocks and riskless T-bills. If the mean  $\mu$  and variance  $\sigma$  of excess stock returns are both constant and known, the investor's optimal stock allocation is given by

$$\text{Myopic Demand} = \frac{\mu}{\gamma\sigma^2} . \tag{12}$$

Based on the historical estimates,  $\mu = 7\%$  and  $\sigma = 16\%$  per year, the optimal stock allocation is 273% for  $\gamma = 1$ , 91% for  $\gamma = 3$ , and 55% for  $\gamma = 5$ , but households typically invest much less in stocks. This fact could in part be due to learning, as explained below.

If  $\mu$  is unobservable, investors learn about it by observing realized returns. Even though  $\mu$  is constant, its posterior mean  $\tilde{\mu}_t$  is not, and investors wish to hedge against learning that  $\mu$  is low (Williams (1977), Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986)). Gennotte (1986) shows that uncertainty about  $\mu$  reduces the stock allocation, as the variation in  $\tilde{\mu}_t$  generates a negative hedging demand. Following Merton (1971), investors tilt their portfolios to hedge against fluctuations in marginal utility induced by changes in the state variable  $\tilde{\mu}_t$ . The size of the hedging demand is

$$\text{Hedging Demand} = -\rho_{\tilde{\mu},r} \left( \frac{\sigma_{\tilde{\mu}}}{\sigma} \right) \frac{\partial U_W / \partial \tilde{\mu}}{W \partial U_W / \partial W}, \quad (13)$$

where  $\rho_{\tilde{\mu},r}$  is the correlation between  $d\tilde{\mu}_t$  (revisions in  $\tilde{\mu}_t$ ) and instantaneous returns,  $\sigma_{\tilde{\mu}}$  is the volatility of  $d\tilde{\mu}_t$ ,  $\sigma$  is return volatility, and  $U_W$  is marginal utility with respect to wealth  $W$ . The sign of the hedging demand depends on  $\gamma$ . Under power utility with  $\gamma > 1$ , the hedging demand is negative since  $\partial U_W / \partial W < 0$ ,  $\partial U_W / \partial \tilde{\mu} < 0$ , and learning about a constant  $\mu$  induces  $\rho_{\tilde{\mu},r} > 0$  (see (4)). Intuitively, a negative stock position (relative to the myopic demand) is a good hedge because it profits from unexpectedly low stock returns, which are accompanied by decreases in  $\tilde{\mu}_t$  that increase marginal utility. The higher the uncertainty about  $\mu$ , the higher the value of  $\sigma_{\tilde{\mu}}$ , and the more negative is the hedging demand.

Brennan (1998) shows that the learning-induced hedging demand can be large. For example, with a  $\mu$  estimate of 8.5%, prior uncertainty about  $\mu$  of 4.5%, and volatility of  $\sigma = 14\%$ , an investor with  $\gamma = 4$  and a 20-year investment horizon invests only 56% in the stock market, down from 102% when only the myopic demand is considered. Whereas Brennan assumes that  $\mu$  is constant, Xia (2001) considers time-varying  $\mu_t$ , with investors learning about the slope  $\beta$  in the predictive relation (11). The hedging demand now has two components. The first one, which is well understood outside the learning literature, stems from time variation in the predictor  $x_t$ . The second component stems from learning about  $\beta$ , and it involves the covariance between returns and  $\tilde{\beta}_t$ , as discussed earlier. Xia shows that both hedging demands are economically important.

The portfolio literature under learning has been extended to multiple assets. Brennan and Xia (2001b) assess the importance of the value and size anomalies from the perspective of an investor who is uncertain whether the anomalies are genuine. They find the value anomaly attractive even after incorporating parameter uncertainty. Pástor (2000) provides similar evidence in a single-period context, and also finds the home bias anomaly significant from the investment perspective. Cvitanic, Lazrak, Martellini, and Zapatero (2006) analyze how optimal allocations depend on the correlation between the assets' expected returns. This correlation reduces uncertainty by allowing learning across assets, but it also makes estimation risk more difficult to diversify. Another extension incorporates non-linear dynamics

of  $\mu_t$ . David (1997) and Honda (2003) solve for optimal allocations when  $\mu_t$  undergoes unobservable regime-shifts. Guidolin and Timmermann (2007) study asset allocation when regime-shifts affect not only the mean but the whole return distribution. They empirically identify four regimes and solve for the optimal allocation among four asset classes. They find that unobservable regimes have a large impact on asset allocation.

The learning models discussed above are set in continuous time. There is also a growing discrete-time portfolio literature that relies on Bayesian econometric techniques. This literature typically does not estimate learning-induced hedging demands, but it integrates portfolio choice with empirical estimation of the parameters of the return-generating process. Parameter uncertainty is incorporated by focusing on the “predictive distribution” of asset returns. Letting  $\theta$  denote the unknown parameters and  $F_t$  denote the data available at time  $t$ , the predictive distribution of returns at time  $t + k$  is given by

$$p(R_{t+k}|F_t) = \int p(R_{t+k}|\theta, F_t) p(\theta|F_t)d\theta, \quad (14)$$

where  $p(\theta|F_t)$  is the posterior distribution of  $\theta$ . Investors maximize expected utility computed with respect to the predictive distribution. Early contributions to this literature include Zellner and Chetty (1965), Brown (1976), Klein and Bawa (1976), and Bawa, Brown, and Klein (1979). Recent contributions include Kandel and Stambaugh (1996), Barberis (2000), Pástor (2000), Pástor and Stambaugh (2000, 2002b), Tu and Zhou (2004, 2008), Avramov (2004), Brandt et al (2005), Avramov and Chordia (2006), Avramov and Wermers (2006), Kan and Zhou (2007), and Wachter and Warusawitharana (2009). Whereas the early contributions used noninformative prior distributions, recent contributions increasingly emphasize informative priors motivated by economic theory. Other recent studies analyze portfolio choice of ambiguity-averse investors (e.g., Uppal and Wang (2003), Wang (2005), Garlappi, Uppal, and Wang (2007)).

## 9. Investor Behavior

### 9.1. Mutual Fund Flows

The way investors allocate their capital to mutual funds might seem puzzling. For example, net capital flows into mutual funds respond positively to past fund performance, even though there is little persistence in performance. Also, the performance-flow relation is convex and stronger for younger funds.

Berk and Green (2004) show that these facts are consistent with rational learning. Their

model makes three key assumptions. First, the fund managers' ability is unobservable, and investors learn about it by observing fund returns. Second, this ability exhibits decreasing returns to scale. Third, rational investors compete for superior returns. To illustrate the model's implications, suppose that a given fund achieved higher-than-expected returns recently. From these returns, investors infer that the fund manager's ability is higher than they previously thought, and they allocate more capital to this fund. This additional capital reduces the fund's ability to generate abnormal returns, due to decreasing returns to scale. Given perfect competition in the provision of capital, investors pour capital into the fund until its abnormal performance disappears. As a result, a fund that outperformed in the past will attract new money, but it will not outperform in the future.

The positive performance-flow relation is stronger for younger funds because recent returns of a younger fund represent a bigger portion of the fund's track record, and so they are more informative about the fund manager's ability. Put differently, investors are more uncertain about the ability of funds with shorter track records, so any signal about ability has a bigger impact on the investors' beliefs. The performance-flow relation is convex at least in part because investors expect underperforming funds to change their strategies (Lynch and Musto, 2003). Therefore, poor past performance contains less information about future performance than good past performance does. As a result, fund flows are less sensitive to past performance when that performance is poor.

Dangl, Wu, and Zechner (2008) extend the Berk-Green model to allow the management company to replace portfolio managers. They derive the optimal replacement strategy and examine fund flows and portfolio risk around manager replacements. Their model rationalizes several empirical facts: (i) managers are more likely to be fired after poor performance; (ii) manager turnover is more performance-sensitive for younger managers; (iii) managers with longer tenure tend to manage larger funds and are more likely to retain their jobs; and (iv) manager replacement is generally preceded by capital outflows and increases in portfolio risk, then followed by inflows and decreases in risk. Taylor (2008) develops a related model in which a board of directors learns about CEO skill and repeatedly decides whether to keep or fire the CEO. Taylor estimates his model and finds that very high turnover costs are needed to rationalize the observed rate of forced CEO turnover.

The above studies assume learning by agents in theoretical models, but a learning perspective also seems useful in empirical work. Examples of studies that use Bayesian empirical techniques to analyze the performance of money managers include Baks, Metrick, and Wachter (2001), Pástor and Stambaugh (2002a), Jones and Shanken (2005), Busse and Irvine (2006), and Kosowski, Naik, and Teo (2007).

## 9.2. Individual Investor Trading

The trading behavior of individual investors exhibits interesting regularities. Individuals lose money by trading, on average, but they trade frequently nonetheless. Individuals' trading intensity depends on their past performance. Poor performance is often followed by exit. More active traders outperform less active ones. Performance exhibits persistence. Explanations offered for these facts range from overconfidence to utility from gambling.

Mahani and Bernhardt (2007) and Linnainmaa (2008) show that these facts are also consistent with rational learning. When individuals are uncertain about their own trading ability, they can learn by trading and observing their profits. Individuals can find it optimal to trade even if they expect to lose money, as long as the expected short-term loss from trading is offset by the expected gain from learning. Individuals increase their trade sizes after successful trades and decrease them after unsuccessful trades, since successful (unsuccessful) trades lead to upward (downward) revisions of perceived ability. More active traders perform better because good news about one's ability leads one to trade more. Linnainmaa finds empirically that the above-mentioned empirical regularities can be explained with moderate uncertainty about trading ability. In contrast, alternative explanations such as overconfidence and risk-seeking seem unable to explain all of the regularities.

How do investors learn from their trading experience? Is their ability a constant subject to learning, as in the models described above, or does it improve as a result of more trading, as in the "learning-by-doing" literature? Seru, Shumway, and Stoffman (2008) find evidence of both types of learning. They find that poorly-performing households are more likely to cease trading, consistent with the former type of learning, and they estimate this type of learning to be quantitatively more important than learning by doing.

## 9.3. Trading Volume

Why do investors trade so much? Why is trading volume correlated with volatility? Learning combined with information asymmetry can shed light on these questions. Note, however, that heterogenous information alone cannot induce trading; given the no-trade theorem, trading requires additional motives, such as liquidity (e.g., Kyle (1985), Admati and Pfleiderer (1988), Wang (1993)), hedging (e.g., Wang (1994)), different prior beliefs (e.g., Detemple and Murthy (1994), Zapatero (1998), Basak (2000), Buraschi and Jiltsov (2006)) or different interpretation of common signals (e.g., Harrison and Kreps (1978), Harris and Raviv (1993), Kandel and Pearson (1995), Scheinkman and Xiong (2003), David (2008b)).

Wang (1994) helps us understand the correlation between trading volume and return volatility. His model features informed agents, who trade for both informational (speculative) and noninformational (hedging) reasons, and uninformed agents, who trade for noninformational reasons only. When the informed agents sell stocks, the stock price must drop to induce the uninformed agents to buy. As information asymmetry increases, the uninformed agents demand a larger discount to cover the risk of trading against private information. Therefore, trading volume is positively correlated with return volatility, and the correlation increases with information asymmetry. Wang's model also implies that hedging-motivated trading induces return reversals, whereas speculative trading induces return continuations. Llorente et al. (2002) find empirical support for these predictions.

Another way of modeling trading relies on differences in beliefs. In Scheinkman and Xiong (2003), heterogeneous beliefs arise from the presence of overconfident agents who believe their information is more accurate than it really is. These agents observe the same signals but, due to their behavioral bias, they interpret the signals differently. The resulting fluctuations in the differences of beliefs induce trading. The amount of trading in this model can be large, even infinite.

## 10. Entrepreneurial Finance

Firm profitability tends to rise before the firm's initial public offering (IPO) and fall after the IPO. Common explanations for these facts include irrationality and asymmetric information. Pástor, Taylor, and Veronesi (2009) show that these facts are also consistent with a rational symmetric-information model of learning. This model features two types of agents: investors, who are well diversified, and an entrepreneur, whose wealth is tied up in a private firm. All agents learn about the average profitability of the private firm by observing realized profits. The entrepreneur solves for the optimal time to go public, trading off diversification benefits of going public against benefits of private control. The model produces a cutoff rule whereby going public is optimal when the firm's expected future profitability is sufficiently high. Therefore, expected profitability must go up before the IPO. According to Bayes' rule, agents revise their expectations upward only if they observe realizations higher than expected. As a result, realized profitability exceeds expected profitability at the time of the IPO, and thus profitability is expected to drop after the IPO.

The model also predicts that the post-IPO drop in profitability is larger for firms with more volatile profitability and firms with less uncertain average profitability. These predictions also follow from Bayesian updating. Agents revise their expectations by less if their

prior uncertainty is lower (because prior beliefs are stronger) and if signal volatility is higher (because signals are less precise). In both cases, realized profitability must rise more sharply to pull expected profitability above the IPO cutoff. As a result, the expected post-IPO drop in profitability is larger when volatility is higher and when uncertainty is lower. These predictions are supported empirically. Volatility and uncertainty can be separated by estimating the stock price reaction to earnings announcements, which is strong when uncertainty is high and volatility is low. Firms with weaker stock price reactions experience larger post-IPO drops in profitability, as predicted by the model. Since the volatility and uncertainty predictions seem unique to learning, this evidence suggests that learning is at least partly responsible for the observed profitability patterns around IPOs.

Sorensen (2008) develops a model of learning by investing, extending the multi-armed bandit model literature (e.g., Gittins, 1989). In his model, each investment brings not only a monetary payoff but also more information, which helps improve future investment decisions. Investors learn from their own investment returns. Their optimal strategy trades off exploiting investments with known high payoffs and exploring investments with uncertain payoffs but a higher option value of learning. Sorensen estimates his model on U.S. data from venture capital (VC) investments. He finds that VCs' investment decisions are affected not only by immediate returns but also by the option value of learning. He also finds that VCs who engage in more learning are more successful.

Empirically, the performance of VC funds managed by the same general partner (GP) exhibits high persistence (unlike the performance of mutual funds). This fact raises the question why successful GPs do not raise their fees or fund size to capture all the surplus, as in Berk and Green (2004). Hochberg, Ljungqvist and Vissing-Jorgensen (2008) rationalize VC performance persistence in a learning model in which investors learn about a GP's skill over time. The idea is that limited partners (LPs) who invest in a GP's fund learn more about the GP's skill than do other investors. This asymmetric learning enables incumbent LPs to hold up the highly-skilled GP when he raises his next fund, because other potential investors would interpret incumbent LPs' failure to reinvest as a negative signal about the GP's skill. Thanks to their hold-up power, incumbent LPs continue to earn high net-of-fee returns in their investments in the follow-on funds of the same GP. In contrast, performance persistence is weaker for mutual funds where asymmetric information between the incumbent investors and outsiders is smaller. Hochberg et al also predict that LPs should earn higher returns in follow-on funds than in first-time funds, and there should be persistence in the LP composition across the funds run by the same GP. These predictions are supported empirically in a large sample of U.S. VC funds.

## 11. Future Issues

Much work on the role of learning in finance still lies ahead. Some promising directions are evident in recent work that is not examined in this survey. For example, in most existing learning models, agents learn by observing cash flows or asset returns, but they could also learn from the prices of derivative securities (e.g., Dubinsky and Johannes (2006), Beber and Brandt (2009), Johannes, Polson, and Stroud (2009)). Other interesting topics not covered here include endogenous information acquisition (e.g., Veldkamp (2006), Peng and Xiong (2006), Van Nieuwerburgh and Veldkamp (2008, 2009)), non-Bayesian learning (e.g., Gervais and Odean (2001), Brav and Heaton (2002), Piazzesi and Schneider (2007)), learning-by-doing (e.g., Arrow (1962), Berk, Green, and Naik (2004)), informational cascades (e.g., Welch (1992)), incomplete information equilibria (e.g., Feldman (2007)), and higher-order beliefs (e.g., Allen, Morris, and Shin (2006), Banerjee, Kaniel, and Kremer (2009)).

Another promising direction is to separate systematic and idiosyncratic uncertainty, which have different implications for asset prices. While idiosyncratic uncertainty increases both return volatility and asset valuations, systematic uncertainty increases volatility but decreases valuations. Time variation in the two types of uncertainty produces dynamic relations between prices, expected returns, and volatility. Separating the two types of uncertainty, perhaps with the help of option prices, could shed new light on the asset price dynamics.

Future work can also analyze strategic information generation. We have discussed learning from exogenously specified signals, but what agents observe may depend on the actions of other agents whose objectives are different. For example, corporate insiders may manipulate earnings, which are used by outside investors as signals about average profitability. It seems interesting to analyze dynamic agency models with asymmetric information. More generally, we need more dynamic learning models in corporate finance.

New learning models should be held to high standards. For each model, one should identify testable predictions that are unique to learning, so the model can be empirically distinguished from alternatives. It is also important to assess the magnitude of the learning-induced effects, either by calibration or by structural estimation. Examples of the latter approach include Linnainmaa (2008), Sorensen (2008), and Taylor (2008). We expect to see more structural estimation of learning models down the road.

## 12. Appendix

Let  $f_t(g)$  denote the probability density function of  $g$  at time  $t$ , with  $\Pr(r > g) = 1$ . The stock price is given by

$$P_t = E_t \left[ \int_t^\infty e^{-r(\tau-t)} D_\tau d\tau \right] = \int_{-\infty}^r E \left[ \int_t^\infty e^{-r(\tau-t)} D_\tau d\tau \mid g \right] f_t(g) dg.$$

Conditional on  $g$ ,  $D_\tau = D_t e^{(g-\sigma^2/2)(\tau-t) + \sigma(W_\tau - W_t)}$ , with  $W_\tau - W_t \sim N(0, \tau - t)$ . Therefore,

$$\begin{aligned} P_t &= \int_{-\infty}^r \int_t^\infty E \left[ e^{-r(\tau-t)} D_t e^{(g-\sigma^2/2)(\tau-t) + \sigma(W_\tau - W_t)} \mid g \right] d\tau f_t(g) dg \\ &= D_t \int_{-\infty}^r \int_t^\infty e^{-(r-g)(\tau-t)} d\tau f_t(g) dg = D_t \int_{-\infty}^r \frac{1}{r-g} f_t(g) dg, \end{aligned} \quad (15)$$

which is (7). When  $g$  is observable,  $f_t(g)$  is degenerate and we obtain (5).

The volatility in (8) obtains from (15) as follows. Let  $f_t(g)$  represent the normal distribution with mean  $\tilde{g}_t$  and variance  $\tilde{\sigma}_t^2$ , except for the truncation  $g < r$ . Approximate the dynamics of  $\tilde{g}_t$  and  $\tilde{\sigma}_t^2$  by (4) with  $ds_t = dD_t/D_t$ , so that  $d\tilde{g}_t \approx m_t (dD_t/D_t - \tilde{g}_t dt)$ . This is an approximation because (4) holds exactly only when  $f_t(g)$  is non-truncated normal. Let  $F(\tilde{g}_t, \tilde{\sigma}_t^2) \equiv \log(P_t/D_t)$ . From Ito's Lemma,

$$\frac{dP_t}{P_t} = \frac{dD_t}{D_t} + \left( \frac{\partial F(\tilde{g}_t, \tilde{\sigma}_t^2)}{\partial \tilde{g}_t} \right) d\tilde{g}_t + o(dt),$$

where  $o(dt)$  denotes deterministic terms of order  $dt$ . Substituting for  $d\tilde{g}_t$  and rearranging,

$$\frac{dP_t}{P_t} \approx \frac{dD_t}{D_t} \times \left[ 1 + \left( \frac{\partial F(\tilde{g}_t, \tilde{\sigma}_t^2)}{\partial \tilde{g}_t} \right) m_t \right] + o(dt).$$

Taking standard deviations of both sides, we obtain return volatility in (8).

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