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ABSTRACT

One Person, Many Votes: Divided Majority and Information Aggregation*

This paper compares the properties of three electoral systems when voters have imperfect information. Imperfect information blurs voter decisions and may divorce the electoral outcome from the true preferences of the electorate. The challenge for electoral design is therefore to translate the (sometimes contradictory) elements of information dispersed in the electorate into the most efficient aggregate outcome. We propose a novel model of multi-candidate elections in Poisson games, and show that Approval Voting produces a unique equilibrium that is fully efficient: the candidate who wins the election is the one preferred by a majority of the electorate under full information. By contrast, traditional systems such as Plurality and Runoff elections cannot cope satisfactorily with information imperfections.

JEL Classification: C72, D72, D81 and D82
Keywords: approval voting, information aggregation, multicandidate elections and Poisson games

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1 Introduction

In most electoral systems, even small divisions within the majority can have a dramatic impact on the election outcome. The history of US "first-past-the-post" elections offers many examples, two recent ones being the 1992 and 2000 presidential elections, in which the third candidate, R. Perot in 1992 and R. Nader in 2000, is regularly claimed to have deprived the majority from its victory. The impact of such divisions is almost as important in two-round systems. In 2002 in France, divisions among leftist voters led the socialist candidate, Lionel Jospin, to lose the first round by a hair's breadth to J.-M. Le Pen, an extreme-right candidate with no chance of winning the second round. Another case is Nicaragua, where the ex-Sandinista D. Ortega won the 2006 election despite being only supported by a minority. He primarily owed his victory to internal divisions among the right-wing majority.[1]

Expressing such divisions is however necessary when voters have contradictory information on the relative merits of the candidates; it is the only channel through which elections can aggregate the information dispersed in the population.[2] The above examples illustrate that Plurality and Runoff elections are ill-adapted to aggregate information in multicandidate elections. To prevent the election of their most disliked candidate, the majority are constrained to avoid divisions; they should coordinate all their votes on a single "focal" candidate.[3] The problem is that, like for any coordination problem, the selection of "the" focal candidate may itself be orthogonal to his or her intrinsic qualities: some information is necessarily lost with these electoral systems.

We propose a model of elections in which voter preferences can be affected by information about the candidates. More specifically, a majority of the electorate is unified against the minority, but divided about which of two candidates would best represent them. We compare the voting equilibria produced by three alternative voting systems and find that Approval Voting emerges as an institution that strictly dominates Plurality and Runoff elections, for at least two reasons. First, it produces a unique equilibrium, which saves voters from the risk of coordination failures. Second, despite contradictory priors

[1] Nicaragua's system is a runoff where a candidate wins in the first round if he obtains more than 35% and a 5-point lead over the nearest competitor. D. Ortega (left-wing) won with 38% because the right-wing majority divided their votes between E. Montealegre (28.3%) and J. Rizo (27.1%).

[2] This information-aggregation property is known as the *Condorcet Jury Theorem*: for 2-candidate elections, see Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1996 and 1997), Myerson (1998a). For multicandidate elections, see Piketty (2000), Martinelli (2002) and Castanheira (2003).

[3] This is known as *Duverger's Law*. See a.o. Duverger (1954) and Cox (1997).

among majority voters, the candidate preferred by the majority is the only likely winner. That is, Approval Voting produces the same outcome as the one that would be chosen if information and coordination problems were absent.

In contrast with most of the literature on voting rules, our model departs from the axiom that voter preferences must be expressed in terms of electoral outcomes.[4] That axiom implicitly assumes that voters are perfectly informed about the relative merits of the candidates. No piece of evidence, as convincing as it might be, can influence the voter's ranking of the candidates. This assumption is clearly at odds with reality. Most voters would change their preference ordering if sufficient information were produced against their candidate – the Watergate scandal and J.-M. Aznar's attempt to blame ETA for Al Qaeda's 2004 bombing in Madrid being striking examples of preference reversals in light of fresh information. Acknowledging that the voters' preferences over candidates might be sensitive to information is at the center of our results.

The analysis of Approval Voting began with the works of Weber (1977, 1995) and Brams and Fishburn (1978, 1983) – see also Myerson (2002) and Laslier (2006) for recent advances on its modeling in large electorates. The idea of Approval Voting is to allow voters to "approve of"(or vote for) as many candidates as they wish. Each approval counts as one vote, and the candidate gathering the largest number of approvals wins the election. This is actually a very natural mechanism. We spontaneously use it in our daily choices: this is for instance the way in which we organize appointments when many people are involved. This mechanism is used by the Arbitration Committee of Wikipedia to resolve disputes, and Approval polls are used to elect the Committee itself. More prominently, Approval Voting is used by many academic societies and to elect the UN Secretary-General. However, even if Brams and Fishburn hoped that it would become "the election reform of the twentieth century", Approval Voting did not pervade to the elections where more is at stake, such as a presidential position.[5]

Our results contradict two prejudices against Approval Voting, and may also help explain why it has not been implemented in large-scale elections. First, according to traditional analyses, Approval Voting would also be displaying a multiplicity of equilibria (see e.g. Myerson and Weber 1993), and may produce inferior equilibria in which the

[4]This axiom is used in essentially all the literature that compares electoral systems. The most prominent example is Arrow (1951), whose impossibility theorem rests on the premise that voter preferences must be directly expressed in terms of electoral outcomes.

[5]See also Brams (2007), as well as Laslier and Van der Straeten (2007) who ran a large-scale experiment during the 2002 presidential election in France.

Condorcet winner fails to be elected (see De Sinopoli *et al.* 2006 and Nuñez 2007). We show that these conclusions are no longer valid when the assumption of perfect information over electoral outcomes is relaxed. Second, Approval Voting is at times accused of inducing "excessive closeness" of the candidates' results. Nagel (2007) calls this the "Burr dilemma" in approval voting: voters may end up voting indiscriminatingly for all the candidates in the majority. Similarly, Myerson and Weber (1993, p106) propose an example in which all candidates obtain the same vote share in equilibrium. According to our results, the evidence produced by Nagel does not extend to large electorates: since voters have opposing *ex ante* preferences, they always have an incentive to deviate from a strategy of all voting for the same set of candidates. In contrast to Myerson and Weber, we introduce information uncertainty: some voters *believe* that the best candidate is A; others *believe* it is B. Yet, all realize that they may be wrong. Hence, each voter has an incentive to also rely on the information present in the rest of the electorate. In equilibrium, this incentive will imply that A has the highest expected vote share when she is the best, and conversely when B is actually the best. Third, our results show that the incumbency advantage no longer exists under Approval Voting: leading politicians and parties cannot foreclose entry on the political marketplace. This is because Approval Voting makes experimentation easier for the voters, which stiffens up competition and reduces the rents of the main parties. In our view, this in itself helps explain why Approval Voting did not pervade to large-scale elections.

Our modeling of large-scale elections draws on the Condorcet Jury Theorem literature. We rely on extended Poisson games to model a three-candidate election, and compare voting equilibria across electoral systems in the spirit of Myerson and Weber (1993).[6] This generalization of Poisson games was introduced by Myerson (1998a), who also shows that they simplify the analysis of the Condorcet Jury Theorem. As in Austen-Smith and Banks (1996), the goal of the electorate in Myerson (1998a) is to select the "best" alternative. Depending on the state of nature, either A or B can be the best, but voters have different prior opinions about these alternatives. One of the main results of the Condorcet Jury Theorem literature is that, at least in a two-candidate setting, there exists an equilibrium in which the best alternative is elected almost certainly. That result is robust to changes in the information structure or in the size of the majority required to win – with the notable exception of the unanimity rule (Feddersen and Pesendorfer 1997, 1998).[7]

[6]Though based on Poisson games, our results directly extend to multinomial distributions.

[7]See Kim and Fey (2007) and Bhattacharya (2007) for precise necessary conditions on voter preferences.

In our model, the majority of the electorate knows that they always prefer both A and B to a third alternative, C. However, majority-block voters hold opposing convictions as to which of A and B is the best alternative: in the absence of additional information, some prefer A and the others prefer B. They also face opposition by the minority who staunchly supports C. Hence, the majority may be forced to avoid internal divisions, to prevent C from winning the election. In this setup, we analyze the equilibrium properties of Approval Voting, Plurality and Runoff elections. Only Approval Voting produces a unique equilibrium, in which the best alternative is the sole likely winner.

The intuition is two-pronged. First, by its very design, Approval Voting allows voters to hit two birds with one stone: they can both vote for their most preferred alternative *and* lend support to their second-best alternative if they view C as a threat – this is the classical argument in favor of Approval Voting. Second, we show that the trade-off between dividing the majority and conveying information is drastically different under Approval Voting. This is the rationale behind equilibrium uniqueness: when voters know that with some (even tiny) probability, their alternative might be "bad", they want to avoid that any of the majority-backed alternatives be too much ahead of the other.[8] This would make her win even when she is bad. Hence, whenever there is an imbalance between the two alternatives, majority-group votes prefer to double vote, *i.e.* vote for both A and B. This both reduces the imbalance and ensures that C remains weak. Only when the vote shares are balanced and there is enough double-voting to drag C behind, majority voters start to single-vote for their most preferred alternative. This is the channel through which voter preferences generate the information necessary to select the best alternative.

The paper is organized as follows: Section 2 lays out the model. Section 3 identifies actions that are strictly dominated under Approval Voting and identifies pivot probabilities for the remaining actions. Section 4 analyzes equilibrium behavior under Approval Voting. Section 5 and 6 analyze equilibria under Plurality Voting and Runoff respectively. Section 7 analyzes Approval Voting with purely partisan preferences. Section 8 concludes.

2 The Model

There are three alternatives, indexed by $P \in \{A, B, C\}$, two states of nature, $\omega \in \{a, b\}$, and three types of voters, $t \in \{t_A, t_B, t_C\}$. Conditional on the state of nature, types t_A

[8] This incentive is absent when voters put a probability zero on their candidate being "bad". In that case, multiple equilibria can coexist (see Nuñez (2007) and Section 7 in this paper).

and t_B hold identical preferences: they always want to elect the best alternative, which is A in state a and B in state b:

$$
\begin{aligned}
U(P, t_A, \omega) = U(P, t_B, \omega) &= 1 \text{ if } (P, \omega) = (A, a) \text{ or } (B, b) \\
&= 0 \text{ if } (P, \omega) = (A, b) \text{ or } (B, a) \\
&= -1 \text{ if } P = C,
\end{aligned} \tag{1}
$$

where $U(P, t, \omega)$ denotes the utility of a voter with type t when alternative P is elected and the true state is ω.

Yet, from an *ex ante* vantage point, types t_A and t_B have opposite convictions regarding alternatives A and B: they hold different beliefs as to which state is most likely. As detailed below, a voter with type t believes that the true state is ω with a probability $q(\omega|t)$. We impose that:

$$
\infty > \frac{q(a|t_A)}{q(b|t_A)} > 1 > \frac{q(a|t_B)}{q(b|t_B)} > 0. \tag{2}
$$

That is, information is imperfect and divides types t_A and t_B. The former believe that A is most likely to be the best alternative, whereas the latter believe it is alternative B. Additional information on the true state of nature could nevertheless affect these convictions (more on this below).

Types t_C are pure partisans: independently of the true state of nature, they always prefer alternative C. For the sake of tractability, they are also assumed indifferent between the other two alternatives:

$$
\begin{aligned}
U(P, t_C, \omega) &= 1 \text{ if } P = C \\
&= 0 \text{ if } P \in \{A, B\}.
\end{aligned}
$$

Timing. At the beginning of the game (**time 0**), nature chooses the state ω, which remains unobserved until after the election. The probabilities of states a and b are respectively $q(a)$ and $q(b)$, with $q(a) + q(b) = 1$. At **time 1**, nature selects a random number of voters from a Poisson distribution of mean n and, conditional on the state, assigns them a type t by iid draws.[9] The conditional probability of being assigned type t is $r(t|\omega)$, with $\sum_t r(t|\omega) = 1$, $\forall \omega$. These probabilities correlate with the true state of nature:

$$
\begin{aligned}
r(t_A|a) &> r(t_A|b), \\
r(t_B|a) &< r(t_B|b), \\
r(t_C|a) &= r(t_C|b),
\end{aligned}
$$

[9]The main properties of extended Poisson games are summarized in Appendix A1 and in the next section, where we also explain why our results extend to multinomial distributions.

and, to ensure that our results cannot hinge on some type of symmetry across types t_A and t_B, we allow types t_A to be potentially more "abundant" than t_B:

$$r(t_A|a) + r(t_A|b) \geq r(t_B|a) + r(t_B|b).$$

Of course, the distribution of voters determines which type has the majority. We focus on the case:

$$r(t_C|\omega) < 1/2, \tag{3}$$

which implies that types t_C are a minority.[10] Hence, types t_A and t_B compose the *majority block*, whereas types t_C form the *minority block*. Hence, the majority's preferred alternative, A or B, depends on the state of nature, a or b, which is unknown at the time of the election.

The election is held at **time 2**. The probabilities $q(\omega)$ and $r(t|\omega)$ are common knowledge. In contrast, neither the actual state of nature nor the actual number of voters of each type are observed: voters only know their own type, t. By Bayesian updating, a voter with type t infers that the probability of state ω is $q(\omega|t)$:

$$q(\omega|t) = \frac{q(\omega)\, r(t|\omega)}{q(a)\, r(t|a) + q(b)\, r(t|b)}. \tag{4}$$

Clearly, condition (2) imposes restrictions on $q(\omega)$ and $r(t|\omega)$. As already explained, (2) implies that t_A-voters' prior preferences lean towards A and conversely for t_B-voters. However, these priors are based on only one element of information: the voter's own perception of the candidates, formally represented by her type. Through Bayesian updating, any additional element of information will affect the voter's beliefs and therefore her preferences. In particular, the information revealed by the election can have a major impact, by eliciting information about the distribution of preferences in the entire electorate.

Payoffs are realized at **time 3**: the winning alternative $W \in \{A, B, C\}$ is selected and each voter's utility $U(W, t, \omega)$ then realizes. In sections 5 and 6, we analyze Plurality and Runoff elections respectively. Here, we introduce Approval Voting.

Action set under Approval Voting. Under *Approval Voting*, each voter can cast a ballot on as many (or as few) alternatives as she wishes. Each approval counts as one vote: when a voter only approves of A, then only alternative A is credited with one vote. If the

[10]For $r(t_C|\omega) > 1/2$, a majority of the electorate prefers to have C elected, independently of ω. This case is trivial to investigate: since types t_C are a majority, they can elect C with a probability that converges to 1 when population size increases to infinity.

voter approves of both A and B, then both A and B are credited with one vote, and so on. Hence, the voters' action set is:

$$\Psi = \{A, B, C, AB, AC, BC, ABC, \varnothing\},$$

where, by an abuse of notation, action A denotes a ballot in favor of A only, action BC denotes a joint approval of B <u>and</u> C, etc. $\varnothing$ denotes abstention. The difference between approval voting and other, more common, electoral rules is that a voter can cast a single, a double or a triple approval. Single approvals ($\psi = A$, B and C) act as positive votes: an A-vote can only be pivotal in favor of A, against B or against C; a B-vote can be pivotal against A or C, etc. In our three-candidate setup, double approvals ($\psi = AB$, BC and AC) act as negative votes. For instance, if the voter plays AC, she is acting against B: her ballot can only be pivotal against that alternative, either in favor of A or of C. Finally, a triple approval (ABC) can never be pivotal: it is strategically equivalent to abstention.

Letting $x(\psi)$ denote the number of voters who played action $\psi \in \Psi$ at time 2, the *total number of approvals* received by alternatives A, B, and C are respectively:

$$\begin{aligned} X(A) &= x(A) + x(AB) + x(AC) + x(ABC), \\ X(B) &= x(B) + x(AB) + x(BC) + x(ABC), \\ X(C) &= x(C) + x(AC) + x(BC) + x(ABC). \end{aligned} \tag{5}$$

The alternative with the largest total number of approvals wins the election. Ties are resolved by the toss of a fair coin. We will see below that, given a Poisson-distributed total size of the population, each random variable $x(\psi)$ itself follows a Poisson distribution. This will imply that each voter has a strictly positive probability of being pivotal.

Strategy space and equilibrium. A type t's *strategy function* is any mapping $\sigma(t) : t \rightarrow \psi$ that specifies a probability distribution over the set of actions Ψ for each type t. $\sigma(\psi|t)$ denotes the probability that a randomly sampled voter of type t plays action ψ, and the usual constraints apply: $\sigma(\psi|t) \geq 0$ and $\sum_\psi \sigma(\psi|t) = 1$, $\forall t$. This strategy function $\sigma(t)$ reflects the fact that a voter can only condition her strategy on her type t.

Given the strategy function $\sigma(t)$ of each type t, a fraction:

$$\tau(\psi|\omega) = \sum\nolimits_t r(t|\omega)\ \ \sigma(\psi|t) \tag{6}$$

of the electorate is expected to play action ψ in state ω. We call $\tau(\psi|\omega)$ the *expected share of voters* who choose action ψ in state ω. Importantly, if types t_A and t_B play the same strategy $\sigma(t)$, then vote shares $\tau(\psi|\omega)$ are identical in the two states of nature. If instead they play different strategies, then expected shares vary with the state of nature.

We analyze symmetric Bayesian Nash equilibria of this voting game for an expected population size n that becomes infinitely large.[11] We shall say that:

Definition 1 *An equilibrium produces an* ***informational trap*** *if the expected result of the election is independent of the state of nature:*

$$\mathsf{E}_\sigma(X(P)|a) = \mathsf{E}_\sigma(X(P)|b), \ \forall P \in \{A, B, C\},$$

which only happens if $\sigma(t_A) = \sigma(t_B)$*.*

In the presence of an informational trap, the outcome of the election does not reveal anything about the actual state of nature.

3 Approval Voting: Elimination of Dominated Strategies

The action set contains eight elements. Identifying strictly dominated strategies, which players never use in equilibrium, allows us to focus on only three elements.

Denoting by $\Pr(W)$ the probability that alternative $W \in \{A, B, C\}$ wins the election, the expected utility of a majority-block voter is:

$$\mathsf{E}U(t) = q(a|t) \ \ [\Pr(A|a) - \Pr(C|a)] + q(b|t) \ \ [\Pr(B|b) - \Pr(C|b)], \ t \in \{t_A, t_B\}. \quad (7)$$

This reads as follows: having observed her type t, the voter anticipates that the true state of nature is a with probability $q(a|t)$. In that case, by (1), her utility would be 1 if A wins, 0 if B wins, and -1 if C wins. With probability $q(b|t) \equiv [1 - q(a|t)]$ the true state is b. In that case, her payoff is 0 if A wins, 1 if B wins, and -1 if C wins. The expected utility of a minority-block voter is:

$$\mathsf{E}U(t_C) = \Pr(C).$$

The value of each action depends on its probability of affecting the outcome of the election, *i.e.* on its probability of being *pivotal.* A ballot can be pivotal in two cases: when an alternative trails behind the leader by **exactly one** vote or when the leading alternatives have **the same** number of votes. It immediately follows that:

[11] Note that the equilibrium mapping $\sigma(\psi|t)$ *must* be identical for all voters of a same type t, by the very nature of population uncertainty (see Myerson 1998b, p377, for more detail). Therefore, symmetry is necessarily part of the equilibrium.

Lemma 1 *For a majority-block voter $t \in \{t_A, t_B\}$, in equilibrium:*

$$\sigma(A|t) + \sigma(B|t) + \sigma(AB|t) = 1. \tag{8}$$

For minority-block voter, action $\psi = C$ is a strictly dominant strategy:

$$\sigma(C|t_C) = 1. \tag{9}$$

The proof is straightforward: consider a majority-block voter and compare actions AB and ABC. While the latter can never be pivotal, an AB-ballot can be pivotal against C, either in favor of A or in favor of B. Both events increase a majority-type's expected utility. Hence, AB strictly dominates ABC. All other strict dominance relationships are obtained by performing similar two-by-two comparisons: AB strictly dominates ABC, $\varnothing$ and C; A strictly dominates AC; and B strictly dominates BC.

Lemma 1 tells us that we must only focus on three undominated actions. Let $G(\psi|t)$ denote the *expected gain* of these actions, $\psi = A, B, AB$:

$$\begin{aligned} G(A|t) &= q(a|t)\ \left[\Pr(piv_{AB}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) - \Pr(piv_{AB}|b)\right], \end{aligned} \tag{10}$$

$$\begin{aligned} G(B|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) - \Pr(piv_{BA}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{BA}|b) + 2\Pr(piv_{BC}|b)\right], \end{aligned} \tag{11}$$

$$\begin{aligned} \text{and } G(AB|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) + 2\Pr(piv_{BC}|b)\right]. \end{aligned} \tag{12}$$

This gain depends on the voter's initial *preference*, summarized by $q(\omega|t)$, and on the strategy function $\sigma \equiv \{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\}$ of the other majority-block voters. These strategies determine the expected number of votes received by each alternative, and thereby the pivot probabilities $\Pr(piv_{PQ}|\omega)$.

These pivot probabilities depend on the distribution of the number of voters who play each action. As shown by Myerson (1998a, 1998b, 2000), since the total number of voters follows a Poisson distribution of mean n, the actual number $x(\psi)$ of voters who play each action ψ follow mutually independent Poisson distributions: $x(\psi) \sim \mathcal{P}(n \cdot \tau(\psi|\omega))$, where $\tau(\psi|\omega)$ is the expected fraction of voters playing action ψ in state ω (see (6) above).

Under approval voting, the number of votes received by alternative A or B is the sum of two independent Poisson random variables: $X(A) = x(A) + x(AB)$ and $X(B) =$

$x(B) + x(AB)$. A pivot probability is therefore the joint probability of two events:[12]

$$\begin{aligned}\Pr(piv_{PQ}|\omega) &= \frac{1}{2}\Pr(\overbrace{X(Q) - X(P) \in \{0,1\}|\omega}^{Q \text{ is ahead of } P \text{ by 0 or 1 vote}}) \times ... \\ &\qquad ... \Pr(\underbrace{X(R) \leq X(Q)|X(Q) - X(P) \in \{0,1\}, \omega}_{3^d \text{ alternative, } R\text{, trails behind}})\end{aligned} \tag{13}$$

Property 1 summarizes some of the properties proven by Myerson (1998a, 1998b, 2000) and extends them to Approval Voting (the proofs are in Appendix A1). Denoting by P_1 P_2 and P_3 the alternatives respectively with the largest, second largest, and lowest expected vote totals, we have:

Property 1 *For a large electorate (n large), the probability that two alternatives P and $Q \in \{A, B, C\}$ have (almost) the same number of votes converges to zero at an exponential rate called the* magnitude *of the probability. We denote it $mag(PQ|\omega)$:*

$$mag(PQ|\omega) \equiv \lim_{n\to\infty} \frac{\log[\Pr(|X(P) - X(Q)| \leq 1|\omega)]}{n}. \tag{14}$$

The exact form of the different magnitudes $mag(PQ|\omega)$ are given in Property 2 in Appendix A1. It follows that:

a) *if two events have a different magnitude, then (Property 3 in Appendix A1):*

$$\lim_{n\to\infty} \frac{\Pr(X(P)=X(Q)|\omega)}{\Pr(X(P)=X(R)|\omega')} = 0 \text{ if and only if } mag(PQ|\omega) < mag(PR|\omega'), \tag{15}$$

with $P, Q, R \in \{A, B, C\}, P \neq Q \neq R$ and $\omega, \omega' \in \{a, b\}$.

b) *The magnitude of a pivot probability $\Pr(piv_{PQ})$ is such that:*

$$\begin{aligned} mag(piv_{PQ}|\omega) &= mag(PQ|\omega) \text{ if } \Pr(X(R) \leq X(Q)|X(Q) - X(P) \in \{0,1\}, \omega) \underset{n\to\infty}{\to} 1 \\ &< mag(PQ|\omega) \text{ if } \Pr(X(R) \leq X(Q)|X(Q) - X(P) \in \{0,1\}, \omega) \underset{n\to\infty}{\to} 0. \end{aligned}$$

c) *Under Approval Voting, the pivot probability with the largest magnitude need not be the one between the top two alternatives. Yet, a sufficient condition for $mag(piv_{P_1P_2}|\omega) > mag(piv_{P_1P_3}|\omega) \geq mag(piv_{P_2P_3}|\omega)$ is that C is one of the top-two contenders in state ω (Property 4 in Appendix A1).*

The result summarized by equations $(14-15)$ has been called the *magnitude theorem* by Myerson (2000). The intuition is that pivot probabilities do not converge towards zero

[12] We omit three-way ties in (13). The reason is that the probability of such an event is much smaller than the probability of a two-way tie, and can thus be disregarded.

at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity. In addition to these classical results, Property 1c tells us that the expected ranking of the pivot probabilities need not correspond to the ranking of vote shares. This is because those voters who double-vote for A and B introduce some correlation between $X(A)$ and $X(B)$ that reduces $mag(piv_{AB}|\omega)$. This correlation is taken care of by computing pivot probabilities on the $x(\psi)$.

These properties are quite general and not specific to Poisson games. For instance, Myerson (2000, Section 4) shows that pivot probabilities under multinomial distributions are simply a monotone transformation of their Poisson equivalent.[13] This is why our results extend directly to the multinomial distribution.

4 Approval Voting: Equilibrium Analysis

Classically, elections with three or more alternatives suffer from information and coordination problems: which is the best alternative is unclear, and one voter's best response depends on the action profile of the rest of the electorate. In the present setup, under full information, alternative A should win in state a and alternative B should win in state b. Yet, the voters' lack of information means that they cannot make their ballot contingent on the true state of nature. What is more, perfect information is not even sufficient to ensure that the best candidate wins. Indeed, voters could experience a coordination failure: as shown in Sections 5 and 6, all majority-block voters may be induced to vote for the same alternative in common electoral systems, even if it is not the best one.

We shall say that:

Definition 2 *Elections satisfy* ***full information and coordination equivalence*** *if, equilibrium vote shares are such that:*

$$\begin{aligned} &\tau(A|a) + \tau(AB|a) > \max\{\tau(B|a) + \tau(AB|a), \tau(C)\} \text{ in state } a, \text{ and} \\ &\tau(B|b) + \tau(AB|b) > \max\{\tau(A|b) + \tau(AB|b), \tau(C)\} \text{ in state } b. \end{aligned} \tag{16}$$

That is, alternative A's expected vote share must be the largest one in state a and conversely for alternative B in state b. Asymptotically, the winning alternative is then the

[13] Myerson (2000) shows that limits of pivot probabilities under Poisson games are such that $\lim_{n\to\infty} \log(\Pr(piv_{PQ}))/n = \mu$. In his Section 4, Myerson shows that, if the distribution is Multinomial instead of Poisson, then $\lim_{n\to\infty} \log(\Pr(piv_{PQ}))/n = \log(\mu+1)$, where μ is the limit under the Poisson distribution. Therefore, the limit likelihood ratio (15) is the same under both distributions.

one preferred by a majority of the population under full information.[14]

Typically, satisfying this constraint is not trivial in a three-candidate setting: first, as seen in the introduction, C may win the election if the majority divide their votes. Second, even if C is only supported by a small minority, there may be multiple equilibria, and hence a coordination problem. Third, the outcome cannot vary with the state of nature if the majority coordinate on exactly one alternative. Our main contribution is to show that these problems vanish under Approval Voting:

Theorem 1 *Under Approval Voting, the equilibrium is unique and satisfies* full information and coordination equivalence*: the equilibrium strategies are such that* (16) *holds.*

The fact is that the possibility of double voting, which is built into Approval Voting, deeply modifies the trade-off that is present in other systems. When majority voters can use double-voting to avoid C's victory, coordinating on one alternative is both unnecessary and undesired. Indeed, if A's victory is threatened in state a, then even types t_B will be willing to lend support to A by double voting, i.e. by playing AB. Importantly, this is not only valid when A is threatened by C. It is also true when B threatens the victory of A in state a: types t_B understand that the true state might be a. Similarly, when B is threatened in state b, then types t_A will be willing to play AB. Only when A and B's vote shares are sufficiently high compared to C's and balanced with one another, majority voters are willing to divide their votes to aggregate information. As we show below, the simple fact that majority-group voters represent more than half of the electorate is sufficient to ensure that information aggregation takes place in equilibrium.

Proving this theorem is the purpose of this section. Each of the next two subsections focuses on one aspect of the proof: first, we prove in Propositions 1 and 2 that there cannot be an informational trap under Approval Voting. Second, we derive the equilibrium strategies: Proposition 3 identifies them and shows that they are unique and induce full information and coordination equivalence. As shown in Section 7, this result would no longer hold if the majority was composed of voters whose preferences do not depend on information: equilibrium multiplicity is again a concern.

[14]This concept of *full information and coordination equivalence* is the natural extension to multicandidate elections of Feddersen and Pesendorfer's (1997) concept of *full information equivalence.*

4.1 Absence of Informational Traps

In this subsection, we prove that there cannot be informational traps in equilibrium, either in pure or in mixed strategy (remember that informational traps arise if all majority types, t_A and t_B, adopt the same strategy profile in equilibrium). We underline the main trade-off faced by majority-block voters in Proposition 1. Proposition 2 then shows that types t_A and t_B necessarily specialize in playing A and B respectively.

Proposition 1 *There cannot be an informational trap in which all majority-block voters play the same pure strategy. That is, none of the three corner strategies:*

$$\{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\} \in \{\{1,0,0\}, \{0,1,0\}, \{0,0,1\}\}$$

in which all majority block voters $t = t_A, t_B$ play the same action with probability 1 can be an equilibrium.

Proof. See Appendix A2. ■

The intuition is as follows. Imagine that all majority-block voters are expected to play A. This would generate an informational trap, in which case the election result cannot influence voter preferences. In particular, a type t_B still wants to vote for B: this is her *preference motive*. Yet, she knows that, with a vote share of 0, B has virtually no chance of winning. So, her *strategic motive* induces her to support A. Under approval voting, t_B-voters can hit these two birds with one ballot: they can combine their strategic motive (vote A) together with their preference motive (vote B) through a joint AB approval; this deviation is always profitable.[15] Hence, approval voting frees the voters from the trap of "having to" single vote for a lesser-liked candidate.

The balance between these two motives is reversed when all majority voters are expected to double-vote. If they all play AB, alternatives A and B top the polls with the same expected vote share. In this case, any type t_A or t_B would deviate and single-vote for her preferred alternative: the other majority-block voters are already taking care of the strategic motive (trailing behind, C has virtually no chance of winning), whereas a single ballot has a very high probability of making the difference between A and B.

[15] This feature is specific to Approval Voting. Consider any other voting rule, in which the voter must withdraw some "voting points" from A if she wants to also vote for B. In that case, there is a conflict between the preference and strategic motives: the probability that a ballot is pivotal in favor of B is infinitesimal compared to the pivot probability in favor of A. Hence, any "voting points" withdrawn from A has a cost that is infinitely larger than the benefit of the point(s) given to B. Like in a prisoner's dilemma, no voter then affords to express her preferences.

Hence, Proposition 1 eliminates three problematic equilibria. The first two candidate equilibria are the game theoretic materialization of Duverger's Law. In such equilibria, majority-block voters feel compelled to coordinate all their votes on only one alternative (we will see in Sections 5 and 6 that these equilibria exist under Plurality and Runoff elections). Such Duvergerian outcomes pose two problems. They prevent information aggregation and, most importantly, they erect barriers to entry: without sufficient initial support, challengers would be sure to lose the election even when a large fringe of the population perceives them as better than incumbent alternatives. Thus, Proposition 1 also shows that the incumbency advantage vanishes under Approval Voting.

The third candidate equilibrium has been termed the *Burr dilemma* by Nagel (2007). He documents the "[approval] *experiment* [that] *ended disastrously in 1800 with the infamous Electoral College tie between Jefferson and Burr*". Proposition 1 shows why such a "disaster" cannot happen in large-scale elections –the Electoral College involved few voters, whose behavior was dictated by party discipline.

Even though Proposition 1 eliminates these three candidate equilibria, it does not ensure that equilibrium vote shares are necessarily different in the two states of nature. Myerson and Weber (1993), for instance, present an example in which all candidates have the same vote share in equilibrium. This is another version of the Burr dilemma: A and B indeed end up in a tie. Our second proposition shows that this cannot happen in our setup: since majority-block voters t_A and t_B "specialize" into playing A and B respectively, there can never be an informational trap:

Proposition 2 *In equilibrium, we must have:* $\sigma(A|t_A) + \sigma(AB|t_A) = 1$ *and* $\sigma(B|t_B) + \sigma(AB|t_B) = 1$ *with* $\sigma(A|t_A) > 0$ *and* $\sigma(B|t_B) > 0$*. Hence, majority-block mix between their 'preferred alternative' and the joint* AB *approval.*

Proof. See Appendix A2. ■

The intuition for the proof is as follows: first, we show that a voter never wants to mix between actions A and B. Such a mixed strategy would imply that she is indifferent between the two alternatives. Expressed differently, the voter does not want to choose between them. However, a safer option is then to play action AB: this action has a higher probability of being pivotal against C, and can never be mistakenly pivotal, *e.g.* in favor of A against B when the true state of nature is b.

This intuition also relates to the "swing voter's curse": in Feddersen and Pesendorfer (1996), voters with imperfect information abstain, to avoid "noising" the election result.

Proposition 2 shows why this incentive to abstain is actually absent under Approval Voting: a double vote is more effective than abstention when there are more than two candidates.

It remains to see why types t_A and t_B necessarily play A and B with strictly positive probability. To understand this, imagine for a moment that no voter plays $\psi = A$. Even if we constrain $\sigma(A|t)$ to be equal to 0, the vote share of A will be larger in state a than in state b, because the best response of types t_A is always to play AB with a strictly higher probability than types t_B. This difference in vote shares in turn implies that even types t_B do not want to be pivotal against A: they prefer to play AB in pure strategy. This leads to a contradiction: by Proposition 1, it cannot be an equilibrium that all majority types play AB with probability 1. Hence, the action $\psi = A$ must be played with strictly positive probability in equilibrium. Given the preference motive, types t_A can be identified as the ones playing A with strictly positive probability (they therefore never play B), and conversely for types t_B.[16]

4.2 Equilibrium Uniqueness

From Proposition 2, we know that all majority-type voters include their *a priori* preferred alternative in their ballot: since they mix between A and AB, types t_A necessarily approve of A. Types t_B mix between B and AB, which always includes B. Hence, the strategy of a type t_A does not influences the vote count of alternative A. It only influences that of B: the more types t_A double-vote, the higher the expected vote share of B. Likewise, the strategy of a type t_B influences the expected vote count of alternative A.

The vote share of either alternative will thus increase when the incentives of types t_A and t_B become more aligned, *i.e.* when either type feels it must support the other group. We identify two cases in which their incentives are aligned: first, when there is a "major imbalance" between the expected vote shares of A and B. Second, when they need to fight alternative C.

A "major imbalance" between the two alternatives occurs when either alternative A or B is too much ahead of the other. Imagine for instance that A is expected to receive much more votes than B. In that case, t_A-voters are quite certain that A wins in state a, given alternative A's lead. Instead, they are not quite certain that B wins in state b. They thus realize that they have to lend support to B as well: this does not threaten A

[16] In a setup with fixed preferences, Brams and Fishburn (2007, Theorem 2.1) show that a voter will always include her most preferred alternative in her ballot. One aspect of Proposition 2 is to show how their Theorem extends to voters whose preference ordering is state-dependent.

in state a, but does give B a chance in state b. Hence, they will prefer to play AB if they expect a major imbalance in favor of A.

The fight against C aligns incentives in the same way. Imagine that a vote for B is much more likely to be pivotal against C than against A (this happens when A and B's vote shares are not sufficiently above C's). In that case as well, a type t_A prefers to cast a double ballot: it provides additional insurance against the election of C.

These two cases lead to the same conclusion: if B's vote share is too low, either compared to A's or to C's, the incentives of types t_A become aligned with that of types t_B –this is again the strategic motive at work– which induces them to double-vote with a higher probability. By symmetry, if A's vote share is too low, then it is types t_B who must lend support to A, and double-vote.

On the other hand, from the previous propositions, we know that the preference motive dominates when sufficiently many majority-block voters double vote. In what follows, we show that there is a unique relationship between the strategy of the types t_A and t_B that prevents major imbalances between A and B, and there is a unique "aggregate level" of double-voting that balances the preference and strategic motives. This is why the equilibrium is unique.

Formally, using the expected gain functions $(10)-(12)$, we have:

$$\begin{aligned} G(A|t_A) - G(AB|t_A) &= q(a|t_A)\ \left[\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)\right] \\ &\quad - q(b|t_A)\ \left[2\Pr(piv_{BC}|b) + \Pr(piv_{AB}|b)\right], \end{aligned} \tag{17}$$

$$\begin{aligned} G(B|t_B) - G(AB|t_B) &= q(b|t_B)\ \left[\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)\right] \\ &\quad - q(a|t_B)\ \left[2\Pr(piv_{AC}|a) + \Pr(piv_{BA}|a)\right]. \end{aligned} \tag{18}$$

From Proposition 2, types t_A and t_B must single-vote with positive probability in equilibrium. A necessary condition to have $G(A|t_A) - G(AB|t_A) \geq 0$ is that $\Pr(piv_{AB}|a)$ be sufficiently large compared to the other three pivot probabilities in (17). Similarly, a necessary condition to have $G(B|t_B) - G(AB|t_B) \geq 0$ is that $\Pr(piv_{BA}|b)$ is sufficiently large compared to the other three pivot probabilities in (18). From Property 1, this requires:

$$\begin{aligned} mag(piv_{AB}|a) &\geq \max\{mag(piv_{AB}|b), mag(piv_{BC}|a), mag(piv_{BC}|b)\}, \\ mag(piv_{BA}|b) &\geq \max\{mag(piv_{BA}|a), mag(piv_{AC}|a), mag(piv_{AC}|b)\}. \end{aligned} \tag{19}$$

Let us first focus on the constraint that appears between the vote shares of alternatives A and B. The combination of the two inequalities in (19) imposes that the magnitudes

$mag\left(piv_{AB}|a\right)$ and $mag\left(piv_{BA}|b\right)$ be equal. Since they must also be larger than all the magnitudes against C, we have by Property 4 in Appendix A1:

$$\begin{aligned}\left(\sqrt{r\left(t_A|a\right)\cdot\sigma\left(A|t_A\right)}-\sqrt{r\left(t_B|a\right)\cdot\sigma\left(B|t_B\right)}\right)^2 &= \\ \left(\sqrt{r\left(t_A|b\right)\cdot\sigma\left(A|t_A\right)}-\sqrt{r\left(t_B|b\right)\cdot\sigma\left(B|t_B\right)}\right)^2 &.\end{aligned} \tag{20}$$

This condition is seen to depend on the two strategy profiles, $\sigma\left(A|t_A\right)$ and $\sigma\left(B|t_B\right)$. Yet, defining:

$$\rho \equiv \sigma\left(A|t_A\right)/\sigma\left(B|t_B\right),$$

one readily sees that condition (20) is satisfied iff:

$$\left|\sqrt{r\left(t_A|a\right)\cdot\rho}-\sqrt{r\left(t_B|a\right)}\right| = \left|\sqrt{r\left(t_B|b\right)}-\sqrt{r\left(t_A|b\right)\cdot\rho}\right|,$$

which has a unique solution in $\mathbb{R}^+$:

$$\rho^* = \left(\frac{\sqrt{r\left(t_B|a\right)}+\sqrt{r\left(t_B|b\right)}}{\sqrt{r\left(t_A|a\right)}+\sqrt{r\left(t_A|b\right)}}\right)^2. \tag{21}$$

This solution in turn implies: $\tau\left(A|a\right) > \tau\left(B|a\right)$ and $\tau\left(A|b\right) < \tau\left(B|b\right)$.

Hence, we are now left with one unknown variable: if we find the equilibrium probability $\sigma\left(B|t_B\right)$ with which types t_B single-vote in equilibrium, the value of $\sigma\left(A|t_A\right)$ follows immediately. The following proposition shows that there is a unique solution to $\sigma\left(B|t_B\right)$. This equilibrium value of $\sigma\left(B|t_B\right)$ is the highest one that allows (19) to be satisfied:

Proposition 3 *The equilibrium is unique and such that:*
i) $\sigma\left(B|t_B\right)=1$, $\sigma\left(A|t_A\right)=\rho^*$ *iff, for this strategy profile,*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) \geq \max_{\omega}\left\{mag\left(piv_{AC}|\omega\right), mag\left(piv_{BC}|\omega\right)\right\}.$$

ii) *Otherwise,* $\sigma\left(B|t_B\right)=\bar{\sigma}$, $\sigma\left(A|t_A\right)=\rho^*\bar{\sigma}$ *with* $\bar{\sigma}\in(0,1)$ *such that:*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) = \max_{\omega}\left\{mag\left(piv_{AC}|\omega\right), mag\left(piv_{BC}|\omega\right)\right\}. \tag{22}$$

Proof. See Appendix A2. ■

Proposition 3 shows that there is a unique equilibrium value for $\sigma\left(A|t_A\right)$ and $\sigma\left(B|t_B\right)$. The reason is as follows: whenever C's vote share is sufficiently below that of A and B, the preference motive dominates: types t_B strictly prefer to single vote for B, and so do types t_A, who want to single vote for A. This increases the gap between A and B in both states of nature. The only obstacle to furthering the difference between A and B is the threat posed by C: if there exists a strategy profile for which (22) binds, then the strategic motive starts dominating again, and both types t_A and t_B prefer to double-vote with a sufficiently high probability. The equilibrium is reached when this strategic motive to beat C balances the preference motive, unless a corner solution is reached. The solution is unique because the perceived threat posed by C is monotonically decreasing in the fraction of voters who double vote.

4.3 Numerical Examples

These examples focus on symmetric priors: $q(a) = \frac{1}{2} = q(b)$ and a symmetric distribution of types: $r(t_A|a) = r(t_B|b)$. This is meant to simplify exposition: from (21) and Proposition 3, this symmetry imposes that $\sigma^*(A|t_A) = \sigma^*(B|t_B)$. We shall illustrate the effect of a variation in $r(t_C)$, the size of the minority group in the population, and of a variation in the ratio $r(t_A|a)/r(t_A|b)$, which proxies the quality of the information available to the voters.

Let $r(t_C) = 0.4$, $r(t_A|a) = 0.36$ and $r(t_A|b) = 0.24$. With these parameter values, like for the actual cases discussed in the introduction, the Condorcet loser, C, would asymptotically be sure to win the election if the majority divide their votes. Vote shares would indeed be: $\tau(C) = 0.4 > \tau(A|a) = \tau(B|b) = 0.36 > \tau(A|b) = \tau(B|a) = 0.24$. This implies that we are in case *ii* of Proposition 3, and that there must be some double-voting in equilibrium. The equilibrium strategy profile is $\sigma(AB|t_A) = 0.57 = \sigma(AB|t_B)$, which leads to the vote shares and magnitudes illustrated in Table 1.

Table 1: equilibrium vote shares (left) and magnitudes (right).[17]

Vote shares	state a	state b
A	0.497 (first)	0.445 (second)
B	0.445 (second)	0.497 (first)
C	0.4 (third)	0.4 (third)
Total	1.342	1.342

and

Magnitudes	state a	state b
$mag(piv_{AC}\|\omega)$	-0.0052	(small)
$mag(piv_{BC}\|\omega)$	(small)	-0.0052
$mag(piv_{AB}\|\omega)$	-0.0052	-0.0052

This example illustrates the effect of the double-vote: it allows the majority to "inflate" the expected vote shares of both A and B above the share of C. This is why the sum of

[17]The pivot probability between the second and third candidates is infinitely lower than the pivot probability between the first and second candidate. In the absence of a closed-form solution for these magnitudes, we cannot compute their exact value.

the three vote shares exceeds 100% of the population: majority-block voters double-vote up to the point in which the magnitude of the pivot probability between A and B is equal to the largest magnitudes against C.

The equilibrium propensity to double-vote is directly related to the size of the minority. If the fraction of types t_C is sufficiently low, majority-group voters do not actually need to double-vote: let $r(t_C) = 0.25$, $r(t_A|a) = 0.45$ and $r(t_A|b) = 0.30$. With these parameter values, the quality of information is the same as in the previous example ($r(t_A|a)/r(t_A|b) = 1.5$) but full information and coordination equivalence obtains even if majority-group voters divide their votes. Indeed, with $\sigma(AB|t_A) = 0 = \sigma(AB|t_B)$, we have: $\tau(A|a) = \tau(B|b) = 0.45 > \tau(A|b) = \tau(B|a) = 0.30 > \tau(C) = 0.25$, and $mag(piv_{AB}|\omega)$ is strictly larger than the other magnitudes. We are therefore in case i of Proposition 3. More generally, in such a symmetric setup, majority-block voters double-vote in equilibrium if and only if $r(t_C) > r(t_A|b) = r(t_B|a)$ and, the higher is $r(t_C)$, the higher is the majority's propensity to double vote (holding $r(t_A|a)/r(t_A|b)$ constant).

This shows that double-voting may vanish when $r(t_C)$ falls, and be valuable again when $r(t_C)$ increases. This observation directly links to Brams and Fishburn's (2005) case study of the Institute of Electrical and Electronics Engineers (IEEE). In 1986, because of a division within the majority, the minority-backed candidate almost won the election for the presidency. This triggered the adoption of Approval Voting by the Institute. Subsequently, both majority divisions and minority size decreased, which induced the IEEE to revert to Plurality Voting. Arguably, the latter decision overlooks the option value of a double-vote:

> *According to the IEEE executive director* [...] *'few of our members were using* [multiple voting...]'. *Brams responded in an e-mail exchange (June 2, 2002) that since "candidates now can get on the ballot with 'relative ease'* [...] *the problem of multiple candidates* [...] *might actually be exacerbated ... and come back to haunt you [IEEE] some day"* (Brams and Fishburn 2005, p16).

Returning to our numerical examples, we now analyze the effect of an improvement in information. Surprisingly, better information induces *more* double-voting in equilibrium. The rationale is as follows: increasing $r(t_A|a)$ and decreasing $r(t_A|b)$ while holding $r(t_C)$ constant implies that, *ceteris paribus*, the gap between the first and the second alternative's vote shares increases. For a given strategy profile, the probability that one vote is pivotal between A and B decreases in magnitude. In comparison, the gap between the first alternative and C does not increase as fast. Hence, the balance between the strategic and preference motives tilts again in favor of the former: the value of a double vote increases compared to that of a single vote. To illustrate this, set $r(t_A|a) = 0.48$ and $r(t_A|b) = 0.12$ and keep $r(t_C) = 0.4$ as in the first example. We find that $\sigma(AB|t_A) = 0.8580 = \sigma(AB|t_B)$ in equilibrium, and hence:

Table 2: equilibrium vote shares (left) and magnitudes (right).

Vote shares	state a	state b
A	0.583 (first)	0.532 (second)
B	0.532 (second)	0.583 (first)
C	0.4 (third)	0.4 (third)
Total	1.5144	1.5144

and

Magnitudes	state a	state b
$mag(piv_{AC})$	-0.0172	(small)
$mag(piv_{BC})$	(small)	-0.0172
$mag(piv_{AB})$	-0.0172	-0.0172

Compared to the first example, the equilibrium ranking remains the same but there is more double-voting and pivot magnitudes are lower, which means that the probability of a mistake, *i.e.* that A wins in state b or B wins in state a, decreases substantially.

5 Plurality Elections

Now that we have analyzed the properties of approval voting, we can compare them with those of other, commonly used, electoral systems. We analyze two such systems: *plurality elections* in this section, and *runoff elections* in the next one.

Under plurality, like under approval voting, the alternative receiving the most votes wins the election. The only difference is that voters can only cast a single ballot or abstain. That is, their action set is restricted to: $\Psi_{Plurality} = \{\varnothing, A, B, C\}$. Otherwise, all pivot probabilities remain the same as in Property 1, with the only difference that, by the definition of $\Psi_{Plurality}$, we have: $\sigma\left(AB|t\right) = 0$, $\forall t$ and hence:

$$\begin{aligned} X\left(A\right) &= x\left(A\right), \\ X\left(B\right) &= x\left(B\right), \\ X\left(C\right) &= x\left(C\right). \end{aligned} \tag{23}$$

Theorem 2 shows that this single difference between the two electoral procedures is sufficient to induce multiplicity of equilibria. Moreover, as already highlighted by Piketty (2000), many such equilibria fail to produce full information and coordination equivalence:

Theorem 2 *Under plurality elections, there are at least three equilibria. The first and second are self-fulfilling equilibria in which all majority types vote for A (resp. B), because they expect the other alternative, B (resp. A) to receive no vote. These equilibria produce an informational trap.*

In the third equilibrium, majority types adopt different strategies, hence there is no informational trap. Yet, for $\tau(C) > 1/[2 + r(t_A|b)/r(t_A|a)]$, *equilibrium vote shares are such that:*

$$\tau(C) > \tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0.$$

In this equilibrium, the dominated candidate C wins with a probability that converges to 1 as $n \to \infty$.

Proof. See Appendix A3. ■

6 Runoff Elections

This section analyzes the properties of another commonly used electoral system: *plurality runoff elections*, also known as *two-round elections.* In this electoral system, an alternative wins outright if it collects more than 50% of the votes in the first round. If no alternative reaches this 50%-threshold, then a runoff is organized between the two alternatives with the most votes.[18] This runoff procedure is often proposed as a solution to the coordination failures that lead to informational traps. Piketty (2000) for instance professes that runoff elections should be able to separate the "communication stage", in which voters learn which of A and B is best, from the "election stage". This intuition finds support in Martinelli (2002) who analyses the equilibrium properties of plurality runoff elections with privately informed voters. However, in his analysis, Martinelli (2002) assumes away the risks that are present in the second round: the majority-backed candidate wins with probability 1. In contrast, we let, in each round, the population follow the same Poisson distribution as under the other electoral systems, which means that the probability of winning is only *asymptotically* equal to 1. As we show here, this implies that, unless types t_C represent a very small part of the electorate, runoff elections suffer from the same informational traps as plurality elections.

[18]Note that there exists other types of two-round elections in which the threshold for first-round victory is below 50% (for instance in Argentina, Nicaragua, Costa Rica and North Carolina). For an analysis of such two-round elections in Poisson games, see Bouton (2007).

To show this, we need to check whether the first-period strategies $(\sigma(A,t), \sigma(B,t)) \in \{(1,0),(0,1)\}$ for $t = t_A, t_B$ can be an equilibrium. Solving the game backwards, we are therefore only interested in the subgames in which C reaches the second round. Let us focus on the subgame that opposes A to C: in that case, all majority-block voters play $\psi = A$, and all minority-block voters play $\psi = C$. The expected utility of a majority type $t \in \{t_A, t_B\}$ negatively depends on the probability that C wins the election, $\Pr(C)$:

$$\begin{aligned} \mathsf{E}U(t|A \text{ vs.} C \text{ in 2d round}) &= q(a|t) - \Pr(C) \\ &= q(a|t) - \left(\frac{\Pr\left[\tilde{X}(C)=\tilde{X}(A)\right]}{2} + \Pr\left[\tilde{X}(C) > \tilde{X}(A)\right] \right) \\ &< q(a|t) - \frac{\Pr\left[\tilde{X}(C)=\tilde{X}(A)\right]}{2} = q(a|t) - \Pr\left(piv^2_{AC}\right), \end{aligned}$$

where $\Pr\left(piv^2_{AC}\right)$ denotes the second-round pivot probability between A and C. By Property 2, $\Pr\left(piv^2_{AC}\right)$ is proportional to:

$$\Pr\left[\tilde{X}(C) = \tilde{X}(A)\right] \propto \exp\left[-\left(\sqrt{1-\tau(C)} - \sqrt{\tau(C)}\right)^2 n \right].$$

This (whatever small) second-round risk influences the incentives of a majority block voter in the first round: consider the first-round strategy profile $\sigma(B|t_B) \to 0$ and $\sigma(B|t_A) = 0$, for which alternative B's expected vote share is vanishingly small. What is a given t_B-voter's best response? If she plays $\psi = A$ and is pivotal to elect A in the first round, she saves herself from the second-round risk. In comparison, action $\psi = B$ is valuable if a second round is organized and if her ballot is pivotal in bringing B to that round.

Comparing the probabilities of each of these events shows that:

Theorem 3 *Under runoff elections, unless the fraction of types t_C is sufficiently small, there exist two self-fulfilling equilibria in which all majority types play $\psi = A$ (resp. B). These equilibria produce an informational trap.*

Proof. See Appendix A4. ■

The trade-off is self-explanatory. A majority voter has an incentive to abandon a trailing candidate (B in the above case) if the second-round risk is too high compared to the first-round chance of bringing the trailing candidate to the second round. Typically, the larger is C's vote share, the higher is the second-round risk, and the lower is the probability that one vote may bring B to the second round. The surprising result is that, even though we only focus on a lower bound of that risk (we only compute the probability that the two candidates tie in the second round to assess it), we find that a vote share of C as low as 6.7% is sufficient to generate such informational traps.[19]

Note still that Theorem 3 does *not* claim that there is no equilibrium with full information and coordination equivalence. Runoff elections actually feature many equilibria, and some of them do satisfy this equivalence. This is however immaterial to the analysis, for two reasons. First, the equilibrium under Approval Voting is unique. Approval Voting therefore Pareto-dominates Runoff elections. Second, organizing elections is extremely costly. Runoff elections may therefore cost about twice as much as Approval Voting elections, despite its less desirable properties.

7 Approval Voting with Purely Partisan Preferences

Traditionally, the literature compares the equilibrium properties of electoral systems when voter preferences are not state-contingent, *i.e.* when voters are pure partisans. This section proposes a numerical example to highlight that the existence of inferior equilibria under Approval Voting indeed depends on this assumption. In our setup, majority-group voters are insensitive to information if their utility is as follows:

$$\begin{aligned} U(A, t_A, \omega) = 1 \quad &\text{and} \quad U(B, t_B, \omega) = 1, \forall\omega, \\ U(B, t_A, \omega) = 0 \quad &\text{and} \quad U(A, t_B, \omega) = 0, \forall\omega, \\ U(C, t_A, \omega) = -1 \quad &\text{and} \quad U(C, t_B, \omega) = -1, \forall\omega. \end{aligned}$$

This change in specification is not innocuous: now, the three groups of voters always disagree, independently of the state of nature. No piece of evidence, as convincing as it might be, can influence the voter's perception of the candidates.

Under this specification, there is no ambiguity about the Condorcet Winner if one of the groups represents a majority of the electorate. Assume that types t_A form 55% of the electorate in state a and 51% in state b, i.e. $r(t_A|a) = 0.55$ and $r(t_A|b) = 0.51$.[20] Assume as well that $r(t_C) = 0.4$ and hence that $r(t_B|a) = 0.05$ and $r(t_B|b) = 0.09$.

Below, we show that the following strategy profile is an equilibrium: $\sigma(AB|t_A) = 1$, $\sigma(B|t_B) = 1$ and $\sigma(C|t_C) = 1$. With these strategies the Condorcet winner, A, loses the

[19] As emphasized in Section 4.1, all our results directly extend to multinomial distributions. In the case of runoff elections, results would even be stronger with such a multinomial distribution. Indeed, the share of C sufficient to generate an informational trap converges to zero as population size increases.

[20] Since preferences are not state-contingent, the results below do not actually depend on the presence of two states of nature. We maintain them only to keep the example as close as possible to the initial setup.

election: 60% of the electorate approves of B and 40% approves of C, whereas alternative A lies in between with 51% or 55%. To show that this strategy profile is an equilibrium, we apply Properties 2 and 4 in Appendix A1. They reveal that the largest magnitude is always the one between B and C:

Magnitudes	state a	state b
$mag(piv_{AC})$	-0.062	-0.097
$mag(piv_{BC})$	-0.02	
$mag(piv_{AB})$	-0.062	-0.097

Given these magnitudes, types t_A strictly prefer to play $\psi = AB$ since $G(A|t_A) < G(AB|t_A)$, the reason being that B is the only serious contender against C. In contrast, types t_B strictly prefer to play $\psi = B$. Their preferences are indeed assumed insensitive to information. Hence, they prefer to take advantage of their lead, and only approve of B. To show this, note that B's total number of approvals can never be inferior to A's, since $\tau(A) = 0$. Hence, alternative A can only win when it ties against B, i.e. when B can also win. In this setup, this implies that types t_B strictly prefer to play $\psi = B$, which shows that the strategy profile proposed above is an equilibrium.

The contrast between this result and Theorem 1 illustrates how deeply the behavior of "swing" or "independent" voters can affect the equilibrium properties of Approval Voting. The assumptions laid out in this section overlook their presence but are nonetheless common in the literature. In reality, most voters would "swing" if sufficiently overwhelming information was gathered against their candidate. Their presence can therefore not be overlooked when analyzing the equilibrium properties of electoral systems.

8 Conclusion

We have argued that one must take account of the voters' sensitiveness to information when studying the properties of electoral systems. Under imperfect information, the voters' preference ordering is bound to depend, among other things, on fresh information about candidate competence, probity or political preferences.

We proposed a model of elections that captures this information imperfection. Voters in the majority are divided about the candidate they prefer but know that they only have a fraction of the information needed to make a fully informed decision. A third candidate, backed by another part of the electorate, runs against the majority. Hence, voters in the majority bear a risk of losing the election altogether if they divide their votes.

In this setup, we studied the asymptotic equilibrium properties of three electoral systems and showed that Approval Voting is ideally suited to aggregate information: it produces a unique equilibrium, in which the candidate who wins the election is actually the one preferred by a majority of the electorate under full information. The other two systems, Plurality and Runoff, produce multiple equilibria. This gives rise to coordination problems and implies that a bad candidate may be sure to win.

The reason why Approval Voting dominates the other systems is that majority divisions need not translate into divided votes. Voters can double vote (that is: approve of their two candidates) both to fight the minority-backed candidate and to balance the support in favor of either majority alternatives. In equilibrium, there always are sufficiently many voters who single-vote for their preferred alternative to ensure that the best candidate actually wins the election.

Arguably, the model focused on a simplified baseline case but the trade-offs and strategies that emerged are quite general. First, the equilibrium strategy proves to be extremely intuitive: voters only need to understand that a multiple ballot is valuable whenever a potentially good candidate trails behind and/or when a disliked candidate gets too close. Second, these trade-offs should be robust to several extensions. Think for instance of a world with more candidates. If there are k candidates in the majority and l candidates in the minority, the trade-off remains identical. As long as their primary objective is to fight one another, both majority-block and minority-block voters will "multiple-vote" for all their candidates. Within the majority, voters will also maintain the balance between all their potentially good candidates, to make sure that the best can win. Indeed, our results have shown that, whenever a candidate trails behind, the other voters in the majority also want to support her with a multiple ballot. Hence, while the analysis would become much more cumbersome given the number of strategies to consider, the main results should remain.

Another simplifying assumption we made is that all voters of a same type have the same preferences and the same information. What would happen if voters either had more general priors or different preference intensities? In a two-candidate setup, Feddersen and Pesendorfer (1997) analyze this issue and show that full information equivalence prevails even if voters have access to different information channels and have heterogeneous preferences. If we extend our model in this direction, the same result should hold under Approval Voting: instead of adopting a common and symmetric strategy, the voters would adopt cutoff strategies, in which voters with the most intense preferences single vote, whereas the most moderate double vote. This is again quite intuitive: the incentive to double-vote increases when the voter is more doubtful regarding the relative merits of two candidates.

Finally, we have made the assumption that *all* voters in the majority value information aggregation. We have also seen (Section 7) that, if *none* of them is sensitive to fresh information, then multiple equilibria may arise under Approval Voting. These extreme cases suggest that a certain fraction of information-sensitive voters is necessary for our results to hold. Where the number of such "independent voters" is sufficient in real-world elections will be worth investigating in future research. Similarly, the properties of other voting systems, such as instant runoff, the Borda count or storable votes will be worth being analyzed.

References

[1] Arrow, Kenneth (1951). *Social Choice and Individual Values.* New York: Wiley.

[2] Austen-Smith, David and Banks, Scott (1996). "Information Aggregation, Rationality, and the Condorcet's Jury Theorem." *American Political Science Review,* 90, pp. 34-45.

[3] Bhattacharya, Sourav. (2007). "Preference Monotonicity and Information Aggregation in Elections." University of Pittsburgh mimeo.

[4] Bouton, Laurent (2007). "The Ortega Effect: On the Influence of Threshold in Runoff Elections." ECARES mimeo, Université Libre de Bruxelles.

[5] Brams, Steven and Fishburn, Peter (1978)."Approval Voting." *American Political Science Review,* 72, pp. 113-134.

[6] Brams, Steven and Fishburn, Peter (1983). *Approval Voting.* 1st ed., Boston: Birkhauser.

[7] Brams, Steven and Fishburn, Peter (2005). "Going from Theory to Practice: The Mixed Success of Approval Voting." *Social Choice and Welfare,* 25, pp. 457-474.

[8] Brams, Steven and Fishburn, Peter (1983). *Approval Voting.* 2nd ed., New York: Springer..

[9] Castanheira, Micael (2003). "Why Vote for Losers?" *Journal of the European Economic Association*, 1, pp. 1207-1238.

[10] Cox, Gary (1997). *Making Votes Count.* Cambridge: Cambridge University Press.

[11] De Sinopoli, Francesco, Dutta, Bhaskar and Laslier, Jean-François (2006) "Approval Voting: Three Examples.", *International Journal of Game Theory*, 35, pp. 27-38.

[12] Duverger, Maurice (1954). *Political Parties.* New York: Wiley.

[13] Feddersen, Timothy and Wolfgang Pesendorfer (1996). "The Swing Voter's Curse." *American Economic Review*, 86, pp. 408-424.

[14] Feddersen, Timothy and Wolfgang Pesendorfer (1997). "Voting Behavior and Information Aggregation in Elections with Private Information." *Econometrica*, 65, pp. 1029-1058.

[15] Feddersen, Timothy and Wolfgang Pesendorfer (1998). "Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting." *American Political Science Review*, 92, pp. 22-35.

[16] Kim, Jaehoon, and Mark Fey (2007). "The Swing Voter's Curse with Adversarial Preferences", *Journal of Economic Theory*, 135, pp. 236-252.

[17] Laslier, Jean-François (2006). "Strategic Approval Voting in a Large Electorate." Laboratoire d'Econometrie mimeo, Ecole Polytechnique.

[18] Laslier, Jean-François and Van der Straeten, Karine (2007). "Approval Voting in the French 2002 Presidential Election: a Live Experiment." *Experimental Economics*, forthcoming.

[19] Martinelli, Cesar (2002). "Simple Plurality Versus Plurality Runoff with Privately Informed Voters". *Social Choice and Welfare* 19, pp. 901-919.

[20] Myerson, Roger (1998a). "Extended Poisson Games and the Condorcet Jury Theorem." *Games and Economic Behavior*, 25, pp.111-131.

[21] Myerson, Roger (1998b). "Population Uncertainty and Poisson Games." *International Journal of Game Theory*, 27, pp. 375-392.

[22] Myerson, Roger (2000). "Large Poisson Games." *Journal of Economic Theory*, 94, pp. 7-45.

[23] Myerson, Roger (2002). "Comparison of Scoring Rules in Poisson Voting Games." *Journal of Economic Theory*, 103, pp. 219-251.

[24] Myerson, Roger and Weber, Robert (1993). "A Theory of Voting Equilibria." *American Political Science Review*, 87, pp. 102-114.

[25] Nagel, Jack (2007). "The Burr Dilemma in Approval Voting." *Journal of Politics*, 69, pp. 43-58.

[26] Nuñez, Matias (2007). "A Study of Approval Voting on Large Poisson Games." Laboratoire d'Econometrie mimeo, Ecole Polytechnique.

[27] Piketty, Thomas (2000). "Voting as Communicating." *Review of Economic Studies*, 67, pp. 169-191.

[28] Riker, William (1982). "The Two-Party System and Duverger's Law: An Essay on the History of Political Science." *American Political Science Review*, 76, pp. 753-766.

[29] Weber, Robert (1977). "Comparison of Voting Systems." Cowles Foundation Discussion Paper No 498. New Haven: Yale University.

[30] Weber, Robert (1995). "Approval Voting." *Journal of Economic Perspectives*, 9, pp. 29-49.

Appendix

Appendix A1 summarizes and extends to approval voting some properties of Poisson Games proven by Myerson (1998a, 1998b, 2000). Appendices A2, A3 and A4 demonstrate the claims made in Sections 4, 5 and 6 respectively.

Appendix A1: Some Properties of Poisson Voting Games

Property 2 *(Myerson 2000, Theorem 1 and extension to Approval Voting)*
Subject to $\sum_{\psi\in\{A,B,AB,C\}}\tau(\psi|\omega)=1$, *and for* $\omega\in\{a,b\}$, *given the expected numbers of votes* $n\tau(\omega)$, *the probability that the realized number of votes are* $\mathsf{x}=\{x(A),x(B),x(AB),x(C)\}$ *is:*

$$\Pr(\mathsf{x}|\tau(\omega)) \underset{n\longrightarrow\infty}{\longrightarrow} \max_{\mathsf{x}} \frac{\exp[mag\,[\mathsf{x}]]}{\prod\limits_{\psi\in\Psi}\sqrt{2\pi x(\psi)+\frac{\pi}{3}}},$$

$$\textit{where: } mag\,[\mathsf{x}]=\sum_{\psi}\frac{x(\psi)}{n}\left(1-\log(\frac{x(\psi)}{n\tau(\psi|\omega)})\right)-1 \quad (\leq 0) \tag{24}$$

For a large electorate (n large), the probability that two alternatives P *and* $Q\in\{A,B,C\}$ *have (almost) the same number of votes converges to zero at an exponential rate called the* magnitude *of the probability:*

$$mag\,(PQ|\omega)\equiv\lim_{n\to\infty}\frac{\log\left[\Pr\left(|x(P)-x(Q)|\leq 1|\omega\right)\right]}{n},$$

where the magnitudes $mag\,(PQ|\omega)$ *are given by:*

$$mag\,(AB|\omega) = -\left(\sqrt{\tau(A|\omega)}-\sqrt{\tau(B|\omega)}\right)^2, \tag{25}$$

$$mag\,(AC|\omega) = -\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2, \tag{26}$$

$$mag\,(BC|\omega) = -\left(\sqrt{\tau(B|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2. \tag{27}$$

Proof. (24) is the application of Theorem 1 in Myerson (2000). (25), (26) and (27) extend this theorem to Approval Voting. From Theorem 1 in Myerson (2000), the magnitude of the probability that alternatives A and C have (almost) the same number of votes is:

$$\lim_{n\to\infty}\tfrac{\log[\Pr(|X(C)-X(A)|\leq 1|\omega)]}{n} = \max_{\lambda} -1+\sum_{\psi}x(\psi)\left(1-\log\frac{x(\psi)}{n\tau(\psi|\omega)}\right) \tag{28}$$
$$s.t.\ x(A)+x(AB)=x(C)$$

If we denote $x(A)+x(AB)=x$, $x(A)=\alpha x$, and $x(AB)=(1-\alpha)x$, we find that this is maximized in:

$$\alpha^*_{AC} = \frac{\tau(A|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}, \tag{29}$$
$$x^*_{AC} = n\sqrt{\tau(C)\left[\tau(A|\omega)+\tau(AB|\omega)\right]},$$
$$x(B)^*_{AC} = n\tau(B|\omega).$$

Substituting for α^*_{AC}, x^*_{AC}, and $x(B)^*_{AC}$ in (28) thus yields:

$$\lim_{n\to\infty} \tfrac{\log[\Pr(|X(C)-X(A)|\leq 1|\omega)]}{n} = -\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2 .$$

By analogy:

$$\begin{aligned}
\lim_{n\to\infty} \tfrac{\log[\Pr(|X(C)-X(B)|\leq 1|\omega)]}{n} &= -\left(\sqrt{\tau(B|\omega)+\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2, \text{ and} \\
\lim_{n\to\infty} \tfrac{\log[\Pr(|x(B)-x(A)|\leq 1|\omega)]}{n} &= -\left(\sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)}\right)^2 .
\end{aligned}$$

Note the symmetry between $mag(PQ)$ and $mag(QP)$:

$$\lim_{n\to\infty} \tfrac{\log[\Pr(|X(P)-X(Q)|\leq 1|\omega)]}{n} = \lim_{n\to\infty} \tfrac{\log[\Pr(|X(Q)-X(P)|\leq 1|\omega)]}{n}.$$

■

Property 3 *(Myerson 2000, Corollary 1) The relative probability of two events x and x' converges to ∞ as population size increases to infinity when the magnitude of x is larger than that of x', and conversely:*

$$\begin{aligned}
\frac{\Pr(x|\tau(\omega))}{\Pr(x'|\tau(\omega))} &\underset{n\longrightarrow\infty}{\longrightarrow} \infty \text{ if } mag\,[x] > mag\,[x'] \\
&\underset{n\longrightarrow\infty}{\longrightarrow} 0 \text{ if } mag\,[x] < mag\,[x'] .
\end{aligned}$$

Property 4 *If C is expected to rank first in state ω, then, for $\tau(A|\omega) > \tau(B|\omega)$, we have:*

$$mag(piv_{AC}|\omega) > mag(piv_{BC}|\omega) \geq mag(piv_{AB}|\omega).$$

Conversely, for $\tau(A|\omega) < \tau(B|\omega)$, we have $mag(piv_{BC}|\omega) > mag(piv_{AC}|\omega) \geq mag(piv_{AB}|\omega)$. If C is expected to rank second in state ω, then, for $\tau(A|\omega) > \tau(B|\omega)$, we have:

$$mag(piv_{AC}|\omega) > mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega).$$

Conversely, for $\tau(A|\omega) < \tau(B|\omega)$, we have $mag(piv_{BC}|\omega) > mag(piv_{AB}|\omega) \geq mag(piv_{AC}|\omega)$. That is, whenever C is expected to rank first or second, the pivot probability between the expected top (resp. bottom) two alternatives has the largest (resp. smallest) magnitude.

Proof. As formally stated in (13), the pivot probability between P and Q is the joint probability of two events. These two events can in fact be viewed as two constraints imposed on the number of votes to make a ballot pivotal: (i) Q is ahead of P by 0 or 1 vote and (ii) the 3^d alternative, R, trails behind. To compute the magnitude of the different pivot probabilities, we use Theorem 1 in Myerson (2000) and impose these constraints. Applying this Theorem to compute the magnitude of the pivot probability between A and C gives:

$$\begin{aligned}
mag(piv_{AC}|\omega) &= \max_x \sum_\psi x(\psi)\left(1 - \log\frac{x(\psi)}{n\tau(\psi|\omega)}\right) - 1 \qquad (30) \\
&\quad s.t.\ x(A) + x(AB) = x(C) \text{ and } x(C) \geq x(B) + x(AB)
\end{aligned}$$

If we abstract from the constraint $x(C) \geq x(B) + x(AB)$, or if this constraint is not binding, from Property 2, (30) is maximized for α^*_{AC}, x^*_{AC} and $x(B)^*_{AC}$ as defined in (29). Substituting for α^*_{AC}, x^*_{AC}, and $x(B)^*_{AC}$ in (30) yields:

$$mag(piv^*_{AC}|\omega) = \lim_{n\to\infty} \tfrac{\log[\Pr(|X(C)-X(A)|\leq 1|\omega)]}{n} = -\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2.$$

We refer to this as the *unrestricted* magnitude (denoted by *).

If the constraint is binding, i.e. if $\alpha^*_{AC}x^*_{AC} \leq x(B)^*_{AC}$, the joint probability also depends on another event that has a strictly negative magnitude. Taking this constraint into account implies:

$$mag(piv_{AC}|\omega) \leq mag(piv^*_{AC}|\omega) = -\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2.$$

By analogy, it is immediate to check that:

$$\begin{aligned} mag(piv_{BC}|\omega) &\leq mag(piv^*_{BC}|\omega) = -\left(\sqrt{\tau(B|\omega)+\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2, \\ \text{and } mag(piv_{AB}|\omega) &\leq mag(piv^*_{AB}|\omega) = -\left(\sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)}\right)^2. \end{aligned}$$

Now, note that the three events piv_{AB}, piv_{AC} and piv_{BC} are identical if their respective constraints are binding. Indeed, whatever the event, a binding constraint implies: $x(A)+x(AB) = x(C) = x(B) + x(AB)$. We refer to the magnitude of this binding events as the *restricted* magnitudes (denoted by **):

$$mag(piv^{**}_{AC}|\omega) = mag(piv^{**}_{BC}|\omega) = mag(piv^{**}_{AB}|\omega),$$

which, by definition, are smaller than the lowest unrestricted magnitude:

$$mag(piv^{**}_{AC}|\omega) \leq \min_{P,Q\in\{A,B,C\}} mag\left(piv^*_{PQ}|\omega\right).$$

Having observed this, we are now in a position to prove that, if the expected ranking is $A > C > B$ in state ω, then:

$$mag(piv_{AC}|\omega) > mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega). \tag{31}$$

The proof is in 3 steps: first, we compare the *unrestricted* magnitudes and show that:

$$mag(piv^*_{AC}|\omega) > mag(piv^*_{AB}|\omega). \tag{32}$$

This amounts to showing that:

$$\tau(A|\omega) + \tau(AB|\omega) > \tau(C|\omega) > \tau(B|\omega) + \tau(AB|\omega) \tag{33}$$

implies:

$$-\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2 > -\left(\sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)}\right)^2. \tag{34}$$

Rearranging terms, we find that (34) holds iff:

$$\sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)} > \sqrt{\tau(A|\omega)+\tau(AB|\omega)} - \sqrt{\tau(C|\omega)},$$

which is necessarily true. Hence, the ranking (33) indeed implies (32).

Second, we show that $mag(piv_{AC}|\omega)$ is always equal to the unrestricted magnitude. For this, we need to prove that: $x(A) + x(AB) = x(C)$ implies $x(C) > x(B) + x(AB)$ at the optimum, that is:

$$\alpha_{AC}^{**} x_{AC}^{**} > x(B)_{AC}^{**}.$$

Using (29) and performing some manipulations, we see that the latter inequality holds iff:

$$\sqrt{\frac{\tau(C)}{\tau(A|\omega) + \tau(AB|\omega)}} > \frac{\tau(B|\omega)}{\tau(A|\omega)}, \tag{35}$$

in which both sides are smaller than one. Hence: $\frac{\tau(B|\omega)}{\tau(A|\omega)} \leq \frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)} \leq \sqrt{\frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}}$, and by (33), the last member of this inequality is smaller than $\sqrt{\frac{\tau(C)}{\tau(A|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{AC}|\omega)$ is always unrestricted. Hence $mag(piv_{AC}|\omega)$ is always larger than $mag(piv_{AB}|\omega)$, be the latter restricted or not.

Third, to complete the proof that (31) always holds under the expected ranking (33), we prove that $mag(piv_{BC}|\omega)$ is always the *restricted* magnitude $mag(piv_{BC}^{**}|\omega)$, which implies: $mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega)$.

Mutatis mutandis, the derivation of the critical values α_{BC}^{**}, x_{BC}^{**}, and $x(A)_{BC}^{**}$ is identical to that of α_{AC}^{**}, x_{AC}^{**}, and $x(B)_{AC}^{**}$ in (29):

$$\begin{aligned}
\alpha_{BC}^{**} &= \frac{\tau(B|\omega)}{\tau(B|\omega) + \tau(AB|\omega)} \\
x_{BC}^{**} &= n\sqrt{\tau(C)\left[\tau(B|\omega) + \tau(AB|\omega)\right]} \\
x(A)_{BC}^{**} &= n\tau(A|\omega)
\end{aligned}$$

and the magnitude $mag(piv_{BC}|\omega)$ would be unrestricted iff:

$$\alpha_{BC}^{**} x_{BC}^{**} > x(A)_{BC}^{**} \tag{36}$$

To show that the latter inequality can never hold, we proceed as with (35) and show that:

$$\sqrt{\frac{\tau(C)}{\tau(B|\omega) + \tau(AB|\omega)}} < \frac{\tau(A|\omega)}{\tau(B|\omega)},$$

in which both fractions are larger than one. This implies: $\frac{\tau(A|\omega)}{\tau(B|\omega)} \geq \frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)} \geq \sqrt{\frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}}$ and, by (33), the last member of this inequality is always larger than $\sqrt{\frac{\tau(C)}{\tau(B|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{BC}|\omega)$ is always restricted and completes the proof of (31).

The proof is identical for all the other possible rankings: $C > B > A$, $C > A > B$ and $B > C > A$, which proves the property. ■

Property 5 *(Myerson 2000, Theorem 2)) The probability that two alternatives, $P, Q \in \{A, B, C\}$, receive a number of votes that differs by a constant c ($c << n$) in state of the nature $\omega \in \{a, b\}$, is:*

$$\lim_{n\to\infty} \Pr\left(x(P) = x(Q) + c|\omega, \tau(P|\omega), \tau(Q|\omega)\right) = \left(\frac{\tau(P|\omega)}{\tau(Q|\omega)}\right)^{c/2} \frac{\exp[-\left(\sqrt{\tau(P|\omega)} - \sqrt{\tau(Q|\omega)}\right)^2 n]}{2\sqrt{\pi n}\left(\tau(P|\omega)\tau(Q|\omega)\right)^{1/4}}.$$

Appendix A2: Proofs for Section 4

Lemma 2

$$G(A|t) \geq G\left(AB|t\right) \Longleftrightarrow \quad \tfrac{q(b|t)}{q(a|t)} \leq \frac{1}{M_1} \equiv \tfrac{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)} \tag{37}$$

$$G(B|t) \geq G\left(AB|t\right) \Longleftrightarrow \quad \tfrac{q(a|t)}{q(b|t)} \leq M_2 \equiv \tfrac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)} \tag{38}$$

Proof. Immediate from $(10) - (12)$. ■

Proof of Proposition 1.
Conjecture the following strategy functions: $\sigma(t_A) = \sigma(t_B) = \{1, 0, 0\}$. That is, all majority types play $\psi = A$ with probability 1. These strategies imply that $\tau(\psi|a) = \tau(\psi|b)$, $\forall\psi$. Therefore: $\Pr(piv_{PQ}) \equiv \Pr(piv_{PQ}|a) = \Pr(piv_{PQ}|b)$. Now, we show that playing $\psi = AB$ is a best response to $\sigma(t)$ for a type t_B:

$$\begin{aligned} G\left(AB|t\right) - G\left(A|t\right) &= q\left(a|t\right) \ \left\{\Pr\left(piv_{BC}\right) - \Pr\left(piv_{AB}\right)\right\} \\ &\quad + q\left(b|t\right) \ \left\{2\Pr\left(piv_{BC}\right) + \Pr\left(piv_{AB}\right)\right\} \\ &= \left(1 + q\left(b|t\right)\right)\Pr\left(piv_{BC}\right) + \left(q\left(b|t\right) - q\left(a|t\right)\right)\Pr\left(piv_{AB}\right). \end{aligned} \tag{39}$$

Since $q(b|t_B) > q(a|t_B)$, all terms in (39) are strictly positive, which proves that a type t_B always wants to deviate from $\sigma(t_A) = \sigma(t_B) = \{1, 0, 0\}$. By symmetry, $\sigma(t_A) = \sigma(t_B) = \{0, 1, 0\}$ cannot be an equilibrium either.

It remains to show that $\sigma(t_A) = \sigma(t_B) = \{0, 0, 1\}$ cannot be an equilibrium. That is, all majority types will never play $\psi = AB$ with probability 1. To see this, note that, by Properties 1 and 2:

$$\lim_{n\to\infty} \frac{\Pr\left(piv_{BC}\right)}{\Pr\left(piv_{AB}\right)} = \lim_{n\to\infty} \frac{\Pr\left(piv_{AC}\right)}{\Pr\left(piv_{BA}\right)} = 0,$$

since alternatives A and B are expected to lead the election, with the same vote share.[21] Hence:

$$\begin{aligned} \lim_{n\to\infty} \tfrac{G(AB|t) - G(A|t)}{\Pr(piv_{AB})} &= q\left(b|t\right) - q\left(a|t\right), \\ \lim_{n\to\infty} \tfrac{G(AB|t) - G(B|t)}{\Pr(piv_{BA})} &= q\left(a|t\right) - q\left(b|t\right). \end{aligned}$$

[21] We have two strategies being played: minority types play C and majority types play AB. Hence: $\tau(AB|\omega) = (1 - r(t_C)) > \tau(C) = r(t_C)$ and $\tau(A|\omega) = \tau(B|\omega) = 0$. Applying Property 1 yields the result.

The former value is strictly positive for types t_A and the latter is strictly positive for types t_B. Hence, both types strictly prefer to deviate from a pure AB vote, and single-vote for their preferred alternative. ■

Proof of Proposition 2.

From Proposition 1, we know that majority-block voters never play the same action in pure strategy. It thus remains to show that majority block voters never play the same mixed strategy in equilibrium. We begin by showing that $\sigma(A|t) > 0$ implies $\sigma(B|t) = 0$ and conversely, for any $t \in \{t_A, t_B\}$. We use a proof by contradiction.

We know that equilibrium strategies lie on the simplex $\{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\}$. A necessary condition for A and B to be played with positive probability in equilibrium is that, for some $t \in \{t_A, t_B\}$:

$$G(A|t) = G(B|t) \geq G(AB|t), \tag{40}$$

and, from Lemma 2 (in this Appendix), $G(A|t), G(B|t) \geq G(AB|t)$ require $\Pr(piv_{AB}|a) > \Pr(piv_{BC}|a)$ and $\Pr(piv_{BA}|b) > \Pr(piv_{AC}|b)$.

Using (10) and (11), a necessary condition for $G(A|t) = G(B|t)$ is:

$$\frac{q(a|t)}{q(b|t)} = \frac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b) + \Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a) + \Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)}. \tag{41}$$

Now, we prove that (40) can never hold: using Lemma 2, we identify a lower bound for M_1 and an upper bound for M_2. Then, we show that this lower bound for M_1 is strictly larger than the upper bound for M_2, whereas condition (40) requires:

$$M_1 \leq M_2, \tag{42}$$

hence the contradiction.

$M_1 = \frac{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}$ is strictly increasing in $\Pr(piv_{BC}|a)$ and $\Pr(piv_{BC}|b)$. A lower bound to M_1 is thus found by setting these two pivot probabilities equal to 0. Similarly, an upper bound to M_2 is found by setting $\Pr(piv_{AC}|a)$ and $\Pr(piv_{AC}|b)$ equal to zero. This establishes that:

$$\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)} < M_1 \text{ and } M_2 < \frac{\Pr(piv_{BA}|b)}{\Pr(piv_{BA}|a)}, \tag{43}$$

and hence that a necessary condition for (42) is that:

$$\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{BA}|b)} \frac{\Pr(piv_{BA}|a)}{\Pr(piv_{AB}|a)} < 1.$$

Using Property 5 (in Appendix A1), the left-hand side of this expression is equal to:

$$\sqrt{\frac{\tau(A|a)}{\tau(A|b)} \frac{\tau(B|b)}{\tau(B|a)}},$$

which cannot be smaller than 1. Indeed, by (41), types t_A must vote for A with a higher probability than types t_B, since $\frac{q(a|t_A)}{q(b|t_A)} > \frac{q(a|t_B)}{q(b|t_B)}$. Hence, in equilibrium:

$$\frac{\tau(A|a)}{\tau(A|b)} \geq 1 \text{ and } \frac{\tau(B|b)}{\tau(B|a)} \geq 1. \tag{44}$$

It follows that $G(A|t) = G(B|t)$ implies $G(AB|t) > G(A|t)$, and therefore that a strict mixture between A and B is a strictly dominated strategy: $\sigma(A|t) > 0$ implies $\sigma(B|t) = 0$ and conversely.

It remains to prove that $\sigma(A|t_A)$ and $\sigma(B|t_B)$ are strictly positive in equilibrium. To this end, we show that:

$$\sigma(B|t_B) > 0 \text{ and } \sigma(A|t_A) = 0 \tag{45}$$

leads to a contradiction. Indeed, (45) implies $\tau(A|\omega) = 0$ in both states. Hence, by Property 2:

$$mag\,(piv_{BA}|\omega) = \tau(B|\omega)\,.$$

By (44), we have: $\tau(B|a) < \tau(B|b)$, which implies that $\lim_{n\to\infty} \Pr(piv_{BA}|b) / \Pr(piv_{BA}|a) = 0$ and therefore that $\lim_{n\to\infty} M_2 \leq 0$ in Lemma 2. Instead, $\sigma(B|t_B) > 0$ imposes that M_2 be strictly positive. This shows that $\sigma(A|t_A) = 0$ contradicts the possibility that $\sigma(B|t_B) > 0$. By symmetry, we cannot either have: $\sigma(A|t_A) > 0$ and $\sigma(B|t_B) = 0$.

Together with Proposition 1 and (44), this proves that, in equilibrium, we must have $\sigma(A|t_A) > 0$ and $\sigma(B|t_B) > 0$. From the first part of this proof, this also implies that: $\sigma(B|t_A) = 0 = \sigma(A|t_B)$. ■

Proof of Proposition 3.

To prove that there is an unique equilibrium, we proceed in two steps. First, we show that $\sigma(A|t_A) = \rho^*\sigma(B|t_B)$ is the unique best response of types t_A given the strategy of types t_B. Second, we prove that there is a unique equilibrium strategy $\sigma^*(B|t_B)$.

From (19) and (21), we must have in equilibrium:

$$\begin{aligned} mag\,(piv_{AB}|a) = mag\,(piv_{AB}|b) \geq \quad & \max\{mag\,(piv_{BC}|a)\,, mag\,(piv_{BC}|b)\,, \\ & mag\,(piv_{AC}|a)\,, mag\,(piv_{AC}|b)\}. \end{aligned} \tag{46}$$

We can check that types t_A never want to deviate from $\sigma(A|t_A) = \rho^*\sigma(B|t_B)$: for any $\sigma(A|t_A) < \rho^*\sigma(B|t_B)$, we have $\sigma(AB|t_A) > 1 - \rho^*\sigma(B|t_B)$. This implies that the expected share of alternative B increases in both states and hence that: $mag\,(piv_{AB}|a)$ increases above $mag\,(piv_{AB}|b)$, whereas $mag\,(piv_{BC}|a)$ and $mag\,(piv_{BC}|b)$ decrease.

Using Lemma 2 and (46), this implies:

$$\frac{q(b|t_A)}{q(a|t_A)} < \lim_{n\to\infty} \frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)} = \infty,$$

and hence: $G(A|t_A) > G(AB|t_A)$. Therefore, $\sigma(A|t_A) < \rho^*\sigma(B|t_B)$ cannot be true in equilibrium.

For any $\rho\sigma(B|t_B) < 1$, we also have to check that $\sigma(A|t_A) > \rho^*\sigma(B|t_B)$ cannot be either an equilibrium. Following the same procedure as above, one can check that $\sigma(A|t_A) > \rho^*\sigma(B|t_B)$ implies:

$$\frac{q(b|t_A)}{q(a|t_B)} > \lim_{n\to\infty} \frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)} \leq 0,$$

which in turn implies $G(A|t) < G\left(AB|t\right)$. Hence, $\sigma\left(A|t_A\right) > \rho^* \sigma\left(B|t_B\right)$ cannot be true in equilibrium. Therefore, when (46) holds, $\sigma^*\left(A|t_A\right) = \rho^* \sigma\left(B|t_B\right)$ is the unique best response of types t_A to $\sigma\left(B|t_B\right)$.

It remains to prove that there is a unique equilibrium strategy $\sigma^*\left(B|t_B\right)$, which will always imply (46). Two cases must be considered:

Case 1: $G(B|t_B) - G(AB|t_B) \geq 0$ in $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho$.

In that case, $\sigma(B|t_B) = 1$ is the only possible best response for types t_B. Indeed, $\sigma(B|t_B) < 1$ would imply $\sigma(AB|t_B) > 0$. This induces an increase in the expected vote share of alternative A in both states of nature and hence that: $mag(piv_{BA}|b)$ increases above $mag(piv_{BA}|a)$, whereas $mag(piv_{AC}|a)$ and $mag(piv_{AC}|b)$ decrease. Using Lemma 2 and (46), this implies:

$$\frac{q(a|t_B)}{q(b|t_B)} < \lim_{n\to\infty} M_2 \equiv \tfrac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)} = \infty,$$

and hence $G(B|t_B) > G(AB|t_B)$. Therefore, $\sigma(B|t_B) = 1$ is the unique best response to $\sigma(A|t_A) = \rho$.

It remains to show that types t_B would deviate from any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho\sigma, \sigma\}$ if $\sigma < 1$. To this end, we need to show that

$$\lim_{n\to\infty} \frac{G(B|t_B) - G(AB|t_B)}{\Pr(piv_{AB}|a)} = q(b|t_B)\frac{\Pr(piv_{BA}|b)}{\Pr(piv_{AB}|a)} - q(a|t_B)\frac{\Pr(piv_{BA}|a)}{\Pr(piv_{AB}|a)} > 0, \tag{47}$$

for any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho\sigma, \sigma\}$, $\sigma < 1$.

The strategy of the types t_A implies:

$$\begin{aligned} \lim_{n\to\infty} \frac{G(A|t_A) - G(AB|t_A)}{\Pr(piv_{AB}|a)} &= q(a|t_A) - q(b|t_A)\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)} = 0 \\ &\Longrightarrow \frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)} = \frac{q(a|t_A)}{q(b|t_A)}. \end{aligned}$$

By Myerson's offset theorem: $\Pr(piv_{BA}|\omega) = \Pr(piv_{AB}|\omega)\sqrt{\frac{\tau(A|\omega)}{\tau(B|\omega)}}$. Hence, (47) can be rewritten as:

$$\frac{q(b|t_B)}{q(a|t_B)}\frac{q(a|t_A)}{q(b|t_A)} > \sqrt{\frac{\tau(A|a)\,\tau(B|b)}{\tau(B|a)\,\tau(A|b)}}.$$

By (4), the left-hand side of this inequality is equal to: $\frac{\tau(A|a)\tau(B|b)}{\tau(B|a)\tau(A|b)} > 1$, which proves that (47) holds.

Case 2: $G(B|t_B) - G(AB|t_B) < 0$ in $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho$.

In this case, there must exist a $\bar{\sigma} \in (0,1)$ such that, for $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho\bar{\sigma}, \bar{\sigma}\}$, we have: $G(B|t_B) - G(AB|t_B) = 0$. Indeed, by Proposition 1, $G(B|t_B) - G(AB|t_B) > 0$ for $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{0, 0\}$. The existence of $\bar{\sigma}$ immediately follows from the continuity of the G function.

This value of $\bar{\sigma}$ is unique and such that:

$$\begin{aligned} mag(piv_{AB}|a) = mag(piv_{AB}|b) = \; & \max\{mag(piv_{BC}|a), mag(piv_{BC}|b), \\ & mag(piv_{AC}|a), mag(piv_{AC}|b)\}. \end{aligned} \tag{48}$$

Indeed, any $\sigma < \bar{\sigma}$ implies that the total expected vote shares of alternatives A and B increase. Since (48) implies that C is third in both states, the magnitudes $mag(piv_{PC}|\omega)$ must decrease, for any $P \in \{A, B\}$ and $\omega \in \{a, b\}$. In contrast, the magnitudes $mag\,(piv_{AB}|\omega)$ must increase, since:

$$\begin{aligned} mag\,(piv_{AB}|a) = mag\,(piv_{AB}|b) &= \left(\sqrt{r\,(t_A|a) \cdot \rho\sigma} - \sqrt{r\,(t_B|a) \cdot \sigma}\right)^2 \\ &= \left(\sqrt{r\,(t_A|a) \cdot \rho} - \sqrt{r\,(t_B|a)}\right)^2 \sigma \end{aligned}$$

is strictly increasing in σ. Hence (46) holds with a strict inequality for any $\sigma < \bar{\sigma}$. This implies that (47) holds, and hence that $G(B|t_B) - G(AB|t_B) > 0$ for any $\{\sigma\,(A|t_A)\,, \sigma\,(B|t_B)\} = \{\rho\sigma, \sigma\}$, $\sigma < \bar{\sigma}$.

Similarly, one can check that (46) is violated for any $\sigma > \bar{\sigma}$ which implies $G(B|t_B) - G(AB|t_B) < 0$ for any $\{\sigma\,(A|t_A)\,, \sigma\,(B|t_B)\} = \{\rho\sigma, \sigma\}$, $\sigma > \bar{\sigma}$. This proves that (48) must hold in $\{\sigma\,(A|t_A)\,, \sigma\,(B|t_B)\} = \{\rho\bar{\sigma}, \bar{\sigma}\}$, and that the solution to $\bar{\sigma}$ is unique. ■

Appendix A3: Proof for Section 5

Proof of Theorem 2.

1) First, we prove that, for all majority types $t \in \{t_A, t_B\}$, $G\,(A|t) - G\,(B|t)$ is strictly positive if $\tau(B|\omega) \to 0$. This proves that, if B is expected to receive too few votes, all majority types strictly prefer to vote for A. By symmetry, it also proves that all majority types vote for B if they expect A to receive too few votes.

For any strategy profile, we have:

$$\begin{aligned} G\,(A|t) - G\,(B|t) = \quad & q\,(a|t) \;\; \{2\Pr\,(piv_{AC}|a) + \Pr\,(piv_{AB}|a) + \Pr\,(piv_{BA}|a) - \Pr\,(piv_{BC}|a)\} \\ & + q\,(b|t) \;\; \{\Pr\,(piv_{AC}|b) - \Pr\,(piv_{AB}|b) - \Pr\,(piv_{BA}|b) - 2\Pr\,(piv_{BC}|b)\}\,. \end{aligned} \tag{49}$$

By (6), for $\tau(B|\omega) \to 0$ we have: $\tau(A|\omega) \to 1 - r\,(t_C)$. Hence, by Properties 1 and 2, for any given $\omega = a, b$ we have:

$$\lim_{n\to\infty} \frac{\Pr\,(piv_{BC}|\omega)}{\Pr\,(piv_{AC}|\omega)} = \lim_{n\to\infty} \frac{\Pr\,(piv_{AB}|\omega)}{\Pr\,(piv_{AC}|\omega)} = \lim_{n\to\infty} \frac{\Pr\,(piv_{BA}|\omega)}{\Pr\,(piv_{AC}|\omega)} = 0.$$

Hence:

$$\lim_{n\to\infty \tau(B|\omega)\to 0} \frac{G\,(A|t) - G\,(B|t)}{\Pr\,(piv_{AC}|a)} = 2q\,(a|t) + q\,(b|t)\,\frac{\Pr\,(piv_{AC}|b)}{\Pr\,(piv_{AC}|a)},$$

which is strictly positive. This proves the existence of the two "sunspot" equilibria.

2) Second, we show the existence of the third equilibrium. Following Theorem 2 of Myerson (1998a), if a type $t \in \{t_A, t_B\}$ adopts a strictly mixed strategy, then the other type $t' \neq t$, $t' \in \{t_A, t_B\}$ votes for "his" candidate with probability 1. The reason is that $q\,(a|t_A) > q\,(a|t_B)$, which implies $G\,(A|t_A) - G\,(B|t_A) > G\,(A|t_B) - G\,(B|t_B)$ for any expected voting profile.

Having noted this, we know that a necessary condition for majority-types voters to adopt a different strategy is that:

$$\begin{aligned} & G\,(A|t_A) - G\,(B|t_A) \geq 0, \text{ and} \\ & G\,(A|t_B) - G\,(B|t_B) \leq 0. \end{aligned} \tag{50}$$

Next, remark that: *a)* pivot probabilities are continuous in the voters' propensity to cast their ballot on A and on B, and *b)* payoffs are bounded. Therefore, the difference $G(A|t) - G(B|t)$ is continuous in the voters' propensity to vote for A, and we can apply Kakutani's fixed point theorem.

Now, consider a strategy profile $\bar{\sigma}$ such that: $\tau(A|a) = \tau(B|b) \equiv \bar{\tau}$. If voters marginally increase their propensity to vote A above $\bar{\sigma}$, we have: $\tau(A|a) > \tau(B|b) > \tau(A|b) > \tau(B|a)$. By Property 1, for any such strategy profile, we have:

$$\begin{aligned} G(A|t) - G(B|t) &> 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(C) < \bar{\tau}, \\ G(A|t) - G(B|t) &< 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(C) > \bar{\tau}, \end{aligned}$$

and the inequalities are reversed if the voters' propensity to vote for A decreases below $\bar{\tau}$. By the continuity of the payoff functions, (50) must hold in a neighborhood of $\bar{\sigma}$.

Now, we show that, for $\tau(C) > 1/[2 + r(t_A|b)/r(t_A|a)]$, the following strategy profile is an equilibrium:

$$\begin{aligned} \sigma(\varnothing|t_A) &= 0 = \sigma(\varnothing|t_B), \\ \sigma(B|t_B) &= 1, \\ \sigma(A|t_A) &\simeq \frac{r(t_B|b) + r(t_A|b)}{r(t_A|a) + r(t_A|b)}, \text{ and } \sigma(B|t_A) = 1 - \sigma(A|t_A). \end{aligned} \tag{51}$$

For that strategy profile, we have $\tau(A|a) \simeq \tau(B|b) \equiv \bar{\tau}$ and: $\tau(C) > \bar{\tau} > \tau(A|b) \simeq \tau(B|a)$. By Property 1, this implies:

$$\lim_{n\to\infty} \frac{\Pr(piv_{BC}|a)}{\Pr(piv_{AC}|a)} = \lim_{n\to\infty} \frac{\Pr(piv_{AC}|b)}{\Pr(piv_{BC}|b)} = 0.$$

Finally, since alternative A and B's vote shares are second and third in both states of nature, by Property 4 in Appendix A1, we have:

$$\lim_{n\to\infty} \frac{\max\{\Pr(piv_{AB}|a), \Pr(piv_{BA}|a)\}}{\Pr(piv_{AC}|a)} = \lim_{n\to\infty} \frac{\max\{\Pr(piv_{AB}|b), \Pr(piv_{BA}|b)\}}{\Pr(piv_{BC}|b)} = 0.$$

It results that, in $\bar{\sigma}$:

$$\lim_{n\to\infty} \frac{G(A|t) - G(B|t)}{\Pr(piv_{AC}|a)} = 2\left[q(a|t) - q(b|t)\frac{\Pr(piv_{BC}|b)}{\Pr(piv_{AC}|a)}\right],$$

and, by Kakutani's fixed point theorem, there must exist a strategy profile $\sigma(A|t_A)$ in the neighborhood of $\frac{r(t_B|b)+r(t_A|b)}{r(t_A|a)+r(t_A|b)}$ such that: $\lim_{n\to\infty} \frac{G(A|t_A)-G(B|t_A)}{\Pr(piv_{AC}|a)} = 0$. It remains to prove that abstention is strictly dominated. To this end, it can be checked that: $G(A|t_A) > 0$ and $G(B|t_B) > 0$, which can be compared to the value of abstention: zero. ■

Appendix A4: Proof for Section 6

Proof of Theorem 3. The probability that A is elected from the first round, with a majority of the votes is:

$$\Pr\left[X(A) \geq X(B) + X(C) + 1\right].$$

For $\sigma(A|t_A) = 1$ and $\sigma(A|t_B) \to 1$, we have $\tau(A|\omega) \to 1 - r(t_C)$ and $\tau(B|\omega) \to 0$. The magnitude of this probability is therefore:

$$\lim_{\tau(B|\omega)\to 0} mag(piv_{AC}^1|\omega) = -\left(\sqrt{1-r(t_C)} - \sqrt{r(t_C)}\right)^2, \forall\omega \in \{a, b\},$$

where piv_{AC}^1 denotes the event that a ballot is pivotal in electing A in the first round. In contrast, the probability that a B ballot is pivotal in bringing B to a second round is given by:

$$\tfrac{1}{2}\Pr\left[\max\{X(A), X(B), X(C)\} \leq \tfrac{X(A)+X(B)+X(C)}{2} \cap \min[X(A), X(C)] - X(B) \in \{0, 1\}\right].$$

When alternative B's vote share approaches zero, the magnitude of this joint event converges to -1.

However, if $X(A) = X(B) + X(C)$, a ballot for A would be pivotal to elect A in the first round. Similarly, if $X(A) = X(B) + X(C) + 1$, a B-ballot would be pivotal in forcing the organization of a second round. Hence, when a voter compares the two options, she values the A-ballot only in proportion to the second-round risk:

$$G(A|t) > \tfrac{1}{2}\Pr\left(piv_{AC}^1\right)\Pr\left(piv_{AC}^2\right),$$

where $\Pr\left(piv_{AC}^2\right)$ denotes the second-round pivot probability. Yet, the two probabilities, $\Pr\left(piv_{AC}^1\right)$ and $\Pr\left(piv_{AC}^2\right)$ are identical. Hence:

$$G(A|t) > \tfrac{1}{2}\Pr\left(piv_{AC}^1\right)^2.$$

Taking logarithms and dividing by n:

$$\frac{\log\left[\Pr\left(piv_{AC}^1\right)^2\right]}{n} \to -2\left(\sqrt{1-r(t_C)} - \sqrt{r(t_C)}\right)^2,$$

which must be compared to the magnitude of the probability that a B ballot is pivotal in bringing B to a second round. That magnitude is equal to -1. Hence:

$$-2\left(\sqrt{1-r(t_C)} - \sqrt{r(t_C)}\right)^2 \geq -1$$

is a sufficient condition for $G(A|t) > G(B|t)$. Solving it in $r(t_C)$ yields: $r(t_C) \geq 0.06699$. Hence, for any $r(t_C) \geq 0.06699$, there exists an informational trap equilibrium with $\sigma(A|t) = 1$, $t \in \{t_A, t_B\}$. By symmetry, there exists another equilibrium with $\sigma(B|t) = 1$, $t \in \{t_A, t_B\}$. ■