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No. 6470

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Discussion Paper No. 6470  
September 2007

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## ABSTRACT

### Recursive Global Games\*

The present paper contributes to the literature on dynamic games with strategic complementarities, in two interrelated ways. First, we identify a class of dynamic complete information games in which intertemporal complementarities and multiple equilibria can be fruitfully analyzed. Second, we extend the analysis to an incomplete information framework, where results from the literature on global games can be applied to select a unique Markov perfect equilibrium in monotone strategies.

JEL Classification: C73, D43 and E32

Keywords: dynamic global games, dynamic supermodular games and endogenous cycles

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\* We wish to thank Jonathan Levin, David Frankel, Chris Harris, Mich Tvede, Mario Pascoa, Sergiu Hart, Alessandro Pavan, Stephen Morris, Frank Heinemann and Michael Peters for useful comments. We are also pleased to acknowledge suggestions and comments from participants at the 2003 Winter Meetings of the Econometric Society, Washington, D. C., the 2003 Summer Meetings of the Econometric Society, Evanston, IL., the 2003 Money, Macro and Finance meetings in Cambridge, the 2004 Econometric Society European Meetings, Madrid, the Coordination Failures Workshop in Paris, 2004, the 2005 ESSET meetings in Gerzensee, the 2007 Winter Meetings of the Econometric Society, Chicago, IL and seminar participants at the Hebrew University of Jerusalem and the Universidade Nova de Lisboa.

Submitted 31 August 2007

## 1. INTRODUCTION

Over the last decade, it has been widely recognized that complementarities may be at the heart of a satisfactory explanation of why there are large shifts and fluctuations in economic activity. A variety of specific channels, such as search externalities, thick market externalities and increasing returns to scale in specific activities or sectors, yield a tendency to cluster and agglomerate. Models with multiple equilibria have been offered as descriptions of how such shifts may come about, arguing e.g. that when an economy slides into recession, it is nothing but a transition to a low activity equilibrium. While these types of explanations clearly have merit, they leave important issues unexplained. In particular, they are silent about the important question of how these equilibria are reached and why there are shifts between them.<sup>1</sup> Partly as a response to this type of reservations about multiple equilibrium explanations, recent years have seen an increasing interest in equilibrium selection in games with strategic complementarities. Building on insights from Carlsson and van Damme (1993), a fast-growing literature has evolved in which many coordination games have been analyzed within the so-called global games framework.<sup>2</sup> The powerful results of the global games literature has allowed applied modelers to pin down equilibria in games that notoriously have very rich equilibrium sets and extending the success to dynamic settings is now an active research field.

The present paper contributes to literature of dynamic games with strategic complementarities, in two interrelated ways. First, it identifies a class of fully dynamic games in which, under the assumption of complete information, contemporaneous and intertemporal complementarities and multiple equilibria can be fruitfully analyzed. Second, it extends the analysis to an incomplete information framework, where results from the literature on global games can be applied to identify a unique Markov perfect equilibrium in monotone strategies. The analysis is carried out based on a model in which a continuum of players choose actions simultaneously over a finite number of periods. We show that under certain assumptions on payoff functions and the evolution of the stochastic variable, expected (current and future) payoffs are characterized by strategic complementarities. This approach allows us to invoke the technology of the static global games literature to each stage of the dynamic game. It has the advantage that the relation to the static global games methodology become very transparent.

The class of games we analyze is characterized by two distinct types of complementarities, contemporaneous and intertemporal. Higher actions of other players make a higher action for the remaining player more desirable. Furthermore, intertemporal complementarities are achieved by the introduction of state variables, one characterizing the

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<sup>1</sup>An exponent of this critique is Durlauf (1991) who states succinctly (but harshly) that ‘Most of these models describe multiple steady states in economies rather than multiple non-degenerate time-series paths, and consequently cannot address issues of aggregate fluctuations. Further, this literature has not shown how economies can shift across equilibria, inducing periods of boom and depression.’

<sup>2</sup>See Morris and Shin (2003) for a survey of the literature and its relation to other strands of research.

players idiosyncratically and one characterizing a quantity common to all players. Assuming complementarities between these state variables and assuming that the transitions of these are monotone in players' actions, we show that these intertemporal links reinforce the contemporaneous complementarities. It is worth noting that while most models of dynamic complementarities circumvent the dynamic nature of the problem by assuming myopic behavior, we explicitly deal with the problem of how actions at both the individual and aggregate level influence the future evolution of aggregate and individual play. These two types of complementarities are interesting because they can produce multiple equilibria. Furthermore, contemporaneous complementarities can magnify shocks to the economy while intertemporal complementarities can propagate them. Specifically, within an infinite horizon version of the model, we show a momentum theorem which implies that under certain conditions, growth in the economy can be self-sustaining, making future growth more likely. After an unexpectedly low realization of the economic shocks, this momentum is broken and the economy may shift to a path with self-enforcing decline. Thus the model may explain paths of the economy where growth is followed by decline and so on, even when shocks to the underlying economy are small in magnitude.

As noted above, strategic complementarities can create multiple equilibria, even in static settings. Not surprisingly, dynamics do nothing but add to this multiplicity. If one assumes, as we do in our model, that a player's action in any given period influences this player's future prospects in a non-trivial way, multiplicity of equilibria may be increased because optimal play in a given period will depend on future play. Thus multiplicity in future stages of the game may create multiplicity now if there is not consensus about which equilibrium will be played in the future. To address this problem, we extend results from the literature on global games and find conditions under which there is a unique Markov perfect Bayesian equilibrium. We do this by employing a specific, but reasonable, information technology. In particular, we assume that common knowledge of fundamentals occurs only with a lag and that in each period there is asymmetric information on the current realization.<sup>3</sup>

Our paper relates closely to two strands of literature. First, the basic model of complete information in which complementarities are analyzed shares features with an existing literature on supermodular games. Most closely related is work by Curtat (1996) which considers stochastic games with strategic complementarities. In his setup, there are a finite set of players who can influence the evolution of a state variable probabilistically. In contrast, we will work within a continuum player setup and thus our results complement his analysis. Moreover, our analysis has similarities with the work of Milgrom, Qian and Roberts (1991). They consider an up-stream down-stream model in which knowledge in the two industries is accumulated over time and where actions in each period are com-

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<sup>3</sup>While this assumption effectively sidesteps issues of learning from the observation of past behavior, learning is not within the scope of this paper.

plementary with the state of knowledge. They prove a momentum theorem which shows that when knowledge accumulation is increasing in actions and actions are complementary with the state of knowledge, the system may enter a self-enforcing dynamic path where actions and knowledge increase over time. Other related work includes Sleet (2001).

Second, the analysis of the model of incomplete information is directly related to the literature on dynamic global games. Four somewhat different approaches to dynamic ‘global’ equilibrium selection have previously been developed. The first, exemplified by the work of Burdzy, Frankel and Pauzner (2001) is to consider players that are randomly matched over a sequence of periods to play a  $2 \times 2$  coordination game, but where the payoff matrix is parametrized by a stochastically evolving economic fundamental. The second approach, exemplified by Frankel and Pauzner (2000), is to consider continuous time games in which the stochastic economic fundamental is governed by a Brownian motion and where players receive revision opportunities according to a Poisson process. The third approach is that of Levin (2001) who considers players who only live for a single period, but whose payoffs may depend on actions of players living before or after them. The fourth approach is that of Toxvaerd (2007) who considers a timing game in which players decide when to move and where players at each point in time trade off current payoffs with expected future payoffs. Although formally not a supermodular game, the recursive payoff structure allows for the static global games techniques to be employed at each stage of the game and thus the full dynamic game can be analyzed as a sequence of (interrelated) static games. The present analysis is most closely related to this fourth approach. Recent work in this spirit includes Steiner (2005, 2007).

Throughout the paper, we have based the analysis on first principles, in order to present the results in a self-contained and accessible way. For this reason, Appendix A provides all the definitions and relevant results from lattice programming which we will be using. In most cases, these can be generalized to sets other than  $\mathbb{R}^n$  and we refer the interested reader to Topkis (1998) for a thorough exposition. Section 2 provides the setup of the class of models that can be studied with our approach and analysis, under the assumption of complete information. Section 3 analyzes the incomplete information case. Finally, Section 4 offers some concluding remarks.

## 2. COMPLETE INFORMATION

Assume that time is discrete and that there is a finite horizon, so  $t = 1, \dots, T$ . In each period, a continuum of players of measure one simultaneously choose an action from a binary action set,  $A = \{0, 1\}$ . Let  $i \in I$  be an arbitrary player and  $z$  be the measure of players choosing high action  $a = 1$  in a given period.

To make the dynamics interesting, we assume that there are intertemporal links such that play at each point in time will be influenced by past play and will influence the game to be played subsequently. Specifically, we introduce two such possible links. First, we assume that a randomly evolving economic fundamental  $\theta$  influences players’ payoffs

directly. This variable is exogenous. Second, we assume that each player is characterized by a personal state variable  $\tau$  which depends on the specific player's past play and is independent of aggregate play. We shall refer to a player's personal state  $\tau$  as the player's *type*. A *strategy* for a player is a mapping from the state space into the action space (i.e. the strategy is a mapping that for every triple  $(z, \tau, \theta)$  yields an action  $a$ ), while a *strategy profile* is a collection of strategies, one for each player. A *policy* for a player is a sequence of strategies, one for each period, while a *policy profile* is a collection of such policies, again one for each player. We restrict attention to Markov policies. This restriction is crucial, since strategic complementarities are very difficult to obtain in dynamic games when fully history dependent strategies are allowed.

In what follows, we begin by analyzing the decision problem of a representative player under the assumption of a fixed profile of policies of all other players and common knowledge of both the current realization of the economic fundamental  $\theta$  and the distribution of player types  $\tau$ . Given such a policy profile, the remaining player essentially faces a Markov decision process, in which other players' actions and the distribution of player types can be treated as elements of an exogenous state vector. Specifically, suppose that we fix a policy profile; with this knowledge, a player can perfectly forecast the future evolution of aggregate actions and the distribution of player types. Thus the player may view these as state variables.<sup>4</sup>

To summarize the above, let  $\chi = (z, \tau, \theta)'$  and  $a$  denote the state and action respectively in any given period, where in any period  $z \in [0, 1]$  is the measure of players choosing action 1,  $\tau \in \{1, 2, \dots, k\}$  is the personal state variable for the player (his type) and  $\theta \in \mathbb{R}$  is the exogenous economic fundamental. The collection of all possible states in any period is  $\mathcal{X} = [0, 1] \times \{1, 2, \dots, k\} \times \mathbb{R} \subseteq \mathbb{R}^3$  and  $a \in A = \{0, 1\}$ . For later use, note that the set  $\mathcal{X} \times A$  is a sublattice of  $\mathbb{R}^5$ .

We assume that the evolution of types is determined by  $\tau' = \phi(\tau, a)$  where primes denote next period's variables and that  $z = v(\theta)$ . These formulations mean that the type of a player depends on his type and action in the previous period and that the measure of players choosing  $a = 1$  depends on the exogenous fundamental state variable  $\theta$ .<sup>5</sup> The economic fundamental  $\theta$  evolves according to a first order Markov process. Finally, we assume that the only source of uncertainty is  $\theta$  (i.e.  $\theta$  is the only stochastic variable).

Denote by  $r(a, z, \tau, \theta)$  the one-stage return for an arbitrary player choosing action  $a \in A = \{0, 1\}$  in period  $t$  given a vector of state variables  $\chi = (z, \tau, \theta)$ . Let  $0 < \delta < 1$  denote the common discount factor and assume that  $r(a, z, \tau, \theta)$  is bounded in all arguments and

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<sup>4</sup>For a discussion of the relationship between stochastic games and Markov decision problems, see e.g. Heyman and Sobel (1984).

<sup>5</sup>We suppress any dependence on the type distribution in all notation.

continuous in  $z$  and  $\theta$ . Next, define the following recursion:

$$\begin{aligned} W_t(a, z, \tau, \theta) &= r(a, z, \tau, \theta) + \delta \int V_{t+1}(z', \tau', \theta') dF_t(\theta'|\theta) \\ V_t(z, \tau, \theta) &= \max_a W_t(a, z, \tau, \theta) \\ W_T(a, z, \tau, \theta) &= r(a, z, \tau, \theta) \end{aligned}$$

where the expectation is taken with respect to the distribution function  $F_t(\theta'|\theta)$  of next period's  $\theta'$  conditional on  $\theta$ . The interpretation of these expressions is as follows: The function  $W_t$  denotes the expected discounted future payoffs from playing action  $a$  in period  $t$  and then playing optimally thereafter. The function  $V_t$  simply expresses the maximum possible expected discounted future payoffs from periods  $t$  through  $T$ . By definition, last period's  $W_t$  is just the one period return, as there is no further play thereafter.<sup>6</sup>

Next, we summarize the assumptions required for the proceeding results. Let

$$\Omega = \{(a, z), (a, \theta), (z, \tau), (z, \theta), (\tau, \theta), (z, \tau, \theta)\}$$

**(A1)** The functions  $\phi$  and  $v$  are increasing in all their arguments.

**(A2)** For a fixed policy profile and for all  $t = 1, \dots, T$ ,  $r(a, z, \tau, \theta)$  is strictly super-modular in all the elements of  $\Omega$ .

**(A3)**  $F_t(\theta'|\theta)$  is stochastically increasing in  $\theta$ .

Assumption (A1) means that higher actions increase the agent's type and furthermore implicitly considers strategies that are monotone in the economic fundamental. Assumption (A2) expounds the basic notions of complementarities between the different variables of interest while assumption (A3) simply means that there is persistence in the economic fundamental.

Moreover, we state the following lemma (proof trivial), which will be useful for proving the main result of the paper.

**Lemma 1.** *Suppose  $x_1, x_2 \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing function. Let  $\bar{x} = x_1 \vee x_2 = \max\{x_1, x_2\}$  and  $\underline{x} = x_1 \wedge x_2 = \min\{x_1, x_2\}$ . Then,*

$$\begin{aligned} \min\{f(x_1), f(x_2)\} &= f(\underline{x}) \\ \max\{f(x_1), f(x_2)\} &= f(\bar{x}) \end{aligned}$$

Also,

$$\max\{x_i, \bar{x}\} = \bar{x}, \quad \min\{x_i, \underline{x}\} = \underline{x}$$

With assumptions (A1)-(A3) and Lemma 1 in place, we can state the first main result of the paper. Roughly, the result is that under the maintained assumptions, complemen-

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<sup>6</sup>Note that for clarity of exposition, we have omitted the dependence on the horizon  $T$  in the notation.

tarities between players' actions and between actions and the economic fundamental are reinforced from one period to the next.

**Theorem 2.** *Under (A1) - (A3),  $W_t(a, z, \tau, \theta)$  is strictly supermodular in all the elements of  $\Omega$  for all  $t = 1, \dots, T$ .*

**Proof.** The proof is by induction. In the last period  $t = T$ ,  $W_T(a, z, \tau, \theta) = r(a, z, \tau, \theta)$  thus the result holds by (A2). Next, assume that in period  $t = k + 1$  the result holds, i.e. that  $W_{k+1}(a', z', \tau', \theta')$  is strictly supermodular in all elements of  $\Omega$ . Under this assumption, consider the function in period  $t = k$ :

$$\begin{aligned} W_k(a, z, \tau, \theta) &= r(a, z, \tau, \theta) + \delta \int V_{k+1}(z', \tau', \theta') dF_k(\theta'|\theta) \\ &= r(a, z, \tau, \theta) + \delta \int V_{k+1}(v(\theta'), \phi(\tau, a), \theta') dF_k(\theta'|\theta) \end{aligned}$$

First note that  $r(a, z, \tau, \theta)$  is strictly supermodular in all elements of  $\Omega$  by (A2). Thus, since  $z$  is not contained in the integral and  $r$  is supermodular in all elements of  $\Omega$ , it follows that  $W_k(a, z, \tau, \theta)$  is supermodular in all elements that contain  $z$ , i.e.  $(a, z), (z, \tau), (z, \theta)$  and  $(z, \tau, \theta)$ .

It remains to be shown that  $W_k(a, z, \tau, \theta)$  is supermodular in  $(a, \theta)$  and  $(\tau, \theta)$ . Let

$$h_k(a, \tau, \theta) \equiv \int V_{k+1}(z', \tau', \theta') dF_k(\theta'|\theta) = \int V_{k+1}(v(\theta'), \phi(\tau, a), \theta') dF_k(\theta'|\theta)$$

Recall that

$$V_{k+1}(z', \tau', \theta') = \max_{a'} W_{k+1}(a', z', \tau', \theta')$$

Since  $W_{k+1}(a', z', \tau', \theta')$  is strictly supermodular in all elements of  $\Omega'$ , it follows that it is also supermodular and therefore  $V_{k+1}(z', \tau', \theta')$  is supermodular in  $(z', \tau')$ ,  $(z', \theta')$ ,  $(\tau', \theta')$  and  $(z', \tau', \theta')$ . Using this, it is possible to show that  $h_k(a, \tau, \theta)$  is supermodular in  $(a, \theta)$  and  $(\tau, \theta)$ . The proof for both is the same, so we show it only for supermodularity in  $(a, \theta)$ . The result for  $(\tau, \theta)$  follows with analogous steps.

Consider  $(a_1, \theta_1)$  and  $(a_2, \theta_2)$  and assume without loss of generality that  $\theta_1 \leq \theta_2$ , i.e. that  $\bar{\theta} = \theta_2$  and  $\underline{\theta} = \theta_1$ . Then

$$\begin{aligned} & h_k(a_1, \tau, \theta_1) - h_k(\underline{a}, \tau, \underline{\theta}) \\ &= \int [V_{k+1}(v(\theta'), \phi(\tau, a_1), \theta') - V_{k+1}(v(\theta'), \phi(\tau, \underline{a}), \theta')] dF_k(\theta'|\theta_1) \\ &\leq \int [V_{k+1}(v(\theta'), \phi(\tau, \bar{a}), \theta') - V_{k+1}(v(\theta'), \phi(\tau, a_2), \theta')] dF_k(\theta'|\theta_1) \\ &\leq \int [V_{k+1}(v(\theta'), \phi(\tau, \bar{a}), \theta') - V_{k+1}(v(\theta'), \phi(\tau, a_2), \theta')] dF_k(\theta'|\theta_2) \\ &= h_k(\bar{a}, \tau, \bar{\theta}) - h_k(a_2, \tau, \theta_2) \implies \end{aligned}$$

$$h_k(a_1, \tau, \theta_1) + h_k(a_2, \tau, \theta_2) \leq h_k(\bar{a}, \tau, \bar{\theta}) + h_k(\underline{a}, \tau, \underline{\theta})$$

The first inequality follows from (A1) and from the fact that  $V_{k+1}$  is trivially supermodular in  $\tau'$  (in fact it holds with equality). The second inequality follows from the fact that the function

$$p(\theta') \equiv V_{k+1}(v(\theta'), \phi(\tau, \bar{a}), \theta') - V_{k+1}(v(\theta'), \phi(\tau, a_2), \theta')$$

is increasing in  $\theta'$ , Proposition 13 in Appendix A and Lemma 1. To see why  $p$  is increasing, consider  $\theta'_1 \leq \theta'_2$ . We want to show that

$$\begin{aligned} p(\theta'_1) &= V_{k+1}(v(\theta'_1), \phi(\tau, \bar{a}), \theta'_1) - V_{k+1}(v(\theta'_1), \phi(\tau, a_2), \theta'_1) \\ &\leq V_{k+1}(v(\theta'_2), \phi(\tau, \bar{a}), \theta'_2) - V_{k+1}(v(\theta'_2), \phi(\tau, a_2), \theta'_2) = p(\theta'_2) \end{aligned}$$

But the inequality is true from the assumption that  $\theta'_1 \leq \theta'_2$  and the assumption of supermodularity of  $V_{k+1}$  in  $(z', \tau', \theta')$ .

To conclude, since  $r$  is strictly supermodular and  $h_k$  is supermodular in all elements of  $\Omega$  and  $\delta > 0$ , from Lemma 9 in Appendix A it follows that  $W_k(a, z, \tau, \theta)$  is strictly supermodular in all elements of  $\Omega$ . This concludes the induction ■

An immediate consequence of Theorem 2 is the following:

**Corollary 3.** *Under (A1) - (A3),  $W_t(a, z, \tau, \theta)$  has strictly increasing differences in  $(a, z)$  and  $(a, \theta)$  for all  $t = 1, \dots, T$ .*

**Proof.** Follows immediately from Theorem 2 and Proposition 11 in Appendix A ■

Note that by relaxing (A2) so that  $r$  is supermodular in all elements of  $\Omega$ , we can obtain an analogous result to Theorem 2 stating that  $W_t$  is supermodular in all elements of  $\Omega$  and thus restate Corollary 3 so that  $W_t$  has increasing differences in  $(a, z)$  and  $(a, \theta)$ . Nevertheless, we state the theorem in terms of strict supermodularity so that we can apply results by Frankel, Morris and Pauzner (2002), henceforth FMP, in the incomplete information case.

Next, consider the role of the horizon  $T$ . The above analysis was carried out for arbitrary, but finite horizon. We now extend the analysis to infinite horizon. Let  $W_t^T$  denote the value of  $W$  at period  $t$ , when the length of the horizon is  $T$ .

**Proposition 4.** *Under (A1) - (A3), for all  $t = 1, 2, \dots$  the limit  $\bar{W}_t = \lim_{T \rightarrow \infty} W_t^T(a, z, \tau, \theta)$  is strictly supermodular in all the elements of  $\Omega$ .*

**Proof.** Consider the constituent parts of the function  $W_t^T(a, z, \tau, \theta)$ . The first part, namely  $r(a, z, \tau, \theta)$  is independent of the horizon length and thus remains strictly supermodular when the limit is taken, i.e. as  $T \rightarrow \infty$ . The second part,  $\delta \int V_{t+1}(z', \tau', \theta') dF_t(\theta'|\theta)$ ,

is supermodular in all elements of  $\Omega$  for all  $t$ , for each horizon length  $T$  (as shown in Theorem 2). But since supermodularity is preserved when taking limits,  $W_t^T(a, z, \tau, \theta)$  converges to the sum of a strictly supermodular function and a supermodular function and this limit is thus itself strictly supermodular by Lemma 9 ■

Next, assume that the horizon is infinite and suppose that the players use stationary Markov policies. Then, the limiting  $\bar{W}$  is time invariant. The following result is a momentum theorem, for the infinite horizon case.

**Theorem 5. Momentum.** *Suppose that*

$$(\theta_t, \tau_t) \geq (\theta_{t-1}, \tau_{t-1})$$

for some period  $t$ . Furthermore, assume that the representative player conjectures that  $z_t \geq z_{t-1}$ . Then

(i)  $a_t \geq a_{t-1}$ .

(ii) the conjecture that  $z_t \geq z_{t-1}$  is confirmed.

(iii)  $\tau_{t+1} \geq \tau_t$ .

**Proof.** Part (i) follows from Theorem 2 and Topkis' monotonicity theorem. To see this, let  $S(\chi)$  be the set of maximizers of  $\bar{W}$  with respect to  $a$ , i.e.

$$S(\chi) = \arg \max_a [\bar{W}(a, \chi), a \in \{0, 1\}]$$

Then, by Theorem 2,  $a$  is complementary with each component of  $\chi = (\tau, z, \theta)$ . Therefore, from Theorems 2.8.1 and 2.6.1 and Corollary 2.6.1 of Topkis (1998), the set of maximizers  $S(\chi)$  is increasing in  $\chi$ . Thus, since  $(\theta_t, \tau_t) \geq (\theta_{t-1}, \tau_{t-1})$  and the player believes that  $z_t \geq z_{t-1}$ , it follows that

$$S(\chi_t) \geq S(\chi_{t-1})$$

If the set of maximizers consists of a unique element, then  $a = S(\chi)$  and it follows that  $a_t \geq a_{t-1}$ . If there is more than one maximizer, we can assume that players use a consistent way of choosing among those maximizers in order to take their action, e.g.  $a_t = \min S(\chi_t)$ , in which case again the result holds.

To show part (ii), since  $a_t \geq a_{t-1}$  for the representative player, it follows trivially that  $z_t \geq z_{t-1}$ . Last, to show for part (iii), we use parts (i) and (ii) and A1:

$$\tau_{t+1} = \phi(\tau_t, a_t) \geq \phi(\tau_{t-1}, a_{t-1}) = \tau_t$$

■

**2.1. Discussion.** The results derived so far merit some further comments. Under the maintained assumptions, an increase in aggregate activity  $z$  induces any agent to increase his own action  $a$ , for two separate reasons. First, there is a direct contemporaneous effect, captured by the notion of complementarity in the pair  $(a, z)$ . Second, there is an intertemporal effect through the state variables. An increase in the economic fundamental induces an increase in the current action through the contemporaneous complementarity. But because an increase in  $\theta$  causes an upward shift in the distribution of tomorrow's realization and there are complementarities in the pair  $(\tau, \theta)$ , the agent has a further incentive to increase the current action to benefit from a higher future realization of  $\theta$  by having a higher type.

As for the momentum theorem, it shows the possibility of multiple self-confirming equilibria, in that an initial conjecture about high activity may indeed bring about high activity as an equilibrium outcome. This momentum theorem comes with more qualifications than do those of Milgrom, Qian and Roberts (1991) and Topkis (1998) and for good reasons. First, their results are derived within deterministic frameworks, allowing them to predict perpetual growth in the state variables. For example, Milgrom, Qian and Roberts (1991) state that '[...] once the system begins along a path of growth of the core variables, it will continue forever along that path or, more realistically, until unmodeled forces disturb the system'.

Apart from our conditions on the distribution of types and the common state variable, which closely mirrors those in Milgrom, Qian and Roberts (1991) and Topkis (1998), our result hinges on the fact that the economic fundamental does not decrease from one period to the next. In a sense, this is an instance of the unmodeled forces mentioned in the quote. A second and more crucial difference is that we, in contrast to their results, derive a momentum theorem with a game and not in a decision theoretic framework. Thus, in our setup a player cannot determine (or even influence) some of the key variables of the model. Thus we are forced to state our result under the conjecture that all players believe that aggregate activity will be high, a conjecture which is indeed confirmed in equilibrium.

Before turning to the incomplete information case, consider the following special case of the model. Assume that  $\tau' = \phi(\tau, a) = \tau$  for all  $a$ . That is, a model in which players' types are exogenous. Under this assumption, the statement of Corollary 3 holds under assumptions (A1)-(A2) alone, i.e. also for distributions  $F_t(\theta'|\theta)$  that are not stochastically increasing. In this case, since the players cannot influence their own future prospects by changing current actions, the future plays no role in current decisions. A model with such features is that of Morris and Shin (1999) in which speculators can attack a currency in each period, but where neither the speculators' ability to attack, nor the monetary authority's ability to defend the peg is influenced by past play.

### 3. INCOMPLETE INFORMATION FRAMEWORK

We now extend the model to one of incomplete information in order to invoke the global games methodology. The basic idea is as follows. In FMP, a very general but static game is considered in which there are strategic complementarities, each player is characterized by a specific type and where higher levels of the economic fundamental raises the relative desirability of higher actions. What we will do is to reinterpret their results so that instead of receiving a level of utility once and for all, the players receive expected discounted payoffs. FMP's results cannot be applied directly though, for the following reason. In FMP, the payoffs are uniquely determined, given a strategy profile and a realization of the economic fundamental. But recall that in the present paper, all the analysis has been carried out given a fixed profile of policies. Thus, even if a player knows what other players will do in the current period, there may still be multiple equilibria in any future period. In turn, this implies that the expected future payoffs (i.e. the value functions) are not well defined. But if players do not agree on the continuation values, equilibrium in the current period cannot be unique. The uniqueness proof below shows by an inductive argument that FMP's results yield well defined value functions and so their results may be invoked at each stage of the game.

Assume that all assumptions and definitions of the complete information framework are maintained, but consider the following change in the informational structure. In each period, rather than observing the economic fundamental directly, a player obtains a noisy signal  $s$  where  $s = \theta + \sigma\eta$  with  $\sigma > 0$  and where  $\eta$  is distributed according to  $G$  with density  $g$  (signals may vary across types and across individuals within the same type and the realizations of the noise terms  $\eta$  will be independent across players). We also assume that the noise  $\eta$  is independent of the economic fundamental  $\theta$ . Last period's realization is common knowledge and provides a prior  $F_t(\theta'|\theta)$  over the current period's realization. After a signal is received the player forms a posterior of  $\theta$ . Define

$$\Delta_t(a_1, a_2, z, \tau, \theta) \equiv W_t(a_1, z, \tau, \theta) - W_t(a_2, z, \tau, \theta)$$

where  $W_t$  is defined as in the previous section. In addition to (A1) - (A3), we make the following assumptions.<sup>7</sup>

**(A4)** For all  $\tau$  and  $z$  there exist realizations  $\underline{\theta}, \bar{\theta} \in \mathbb{R}$  with  $\underline{\theta} < \bar{\theta}$  such that  $\Delta(a_1, a_2, z, \tau, \theta) > 0$  for  $\theta > \bar{\theta}$  and  $\Delta(a_1, a_2, z, \tau, \theta) < 0$  for  $\theta < \bar{\theta}$ .

**(A5)** The return function  $r(a, z, \tau, \theta)$  is continuous in  $z$  and  $\theta$ .

**3.1. Uniqueness of Equilibrium.** With assumptions (A1) - (A5), the present setup satisfies the conditions in FMP in each period. We can thus state the following result:

---

<sup>7</sup>Note that in the case of incomplete information, each player faces what is known as a partially observed Markov decision process. Such a process is easily obtained from the underlying observed (core) process by suitably redefining the system's state variables. See e.g. Bertsekas (2005) for details.

**Theorem 6.** *Under (A1) - (A5), as  $\sigma \rightarrow 0$  the game in any period  $t \leq T$  has an essentially unique strategy profile surviving iterative strict dominance.*

**Proof.** We prove the statement by induction, using Theorem 5 of FMP. First note that, to apply this theorem, we need to ensure that the required assumptions of FMP are satisfied.<sup>8</sup> FMP1 (strategic complementarities) is satisfied by Corollary 3, under (A1) - (A3). FMP2 (dominance regions), is assumption (A4). FMP3 (state monotonicity) is ensured by Corollary 3, under (A1) - (A3). FMP4 is assumption (A5). Finally, we do not need to check FMP5, because the action space is finite.

At  $t = T$ , uniqueness follows trivially from Theorem 5 of FMP. Now assume that for some period  $k + 1 < T$  there is a unique equilibrium and consider the problem in period  $k$ . Since the continuation values are well defined as a function of the current period's data by the inductive hypothesis that there is a unique equilibrium in period  $k + 1$ , uniqueness in period  $k$  follows ■

As the analysis above has shown, it is possible to identify a unique equilibrium for any finite horizon version of the model. It is interesting to know if uniqueness carries over to the case of an infinite horizon. As shown in the section on the complete information model, strict supermodularity at each stage continues to hold when the horizon recedes. Whether uniqueness also obtains is still an open question.

#### 4. CONCLUSION

In this paper we have set out a dynamic model in which a continuum of players simultaneously choose actions in an uncertain environment. By assuming complementarity between players' actions and components of a state vector, we have shown that complementarities at each point in time spill over across periods. Using these results, methodology from the static global games literature has been shown to yield a unique equilibrium.

While still at an early stage, this approach to dynamic global games seems promising, in particular from an applied perspective. There are also a number of interesting features of the model that remain to be studied. For example, empirical studies have shown convincing evidence of dynamic complementarities, whereby higher activity in a given period raises productivity in subsequent periods. Such evidence may be rationalized through models like the one presented here. Last, there are a plethora of intertemporal links in the industrial organization literature that merit further study within a dynamic framework of strategic complementarities. For example, time to build, learning by doing, irreversibility of investment decisions, consumer switching costs, capital accumulation and contractual obligations spanning several periods are but some of the many features that may link a firm's current decisions and future prospects and influence the strategic interaction between industry participants.

<sup>8</sup>In what follows, we denote assumptions A1 - A5 in FMP, by FMP1 - FMP5 correspondingly, in order to avoid confusion with the assumptions (A1) - (A5) we make here.

The model has purposely been kept as parsimonious as possible, but we will now briefly discuss the generality of the results. We start with the restriction to finite (binary) action spaces. This assumption is in fact not essential and most proofs go through unchanged with continuum action spaces, as long as the variable  $z$  is reinterpreted as a one-dimensional summary statistic which is increasing in each player's strategy. This added generality comes at the cost of further assumptions needed for existence of optimal strategies. In particular, the return function  $r$  must be bounded and continuous in the action  $a$  for an optimal policy to exist. The proof is straightforward and available from the authors upon request.

Next, we turn to the uniqueness result based on Frankel, Morris and Pauzner (2002). In their work, they impose Lipschitz continuity assumptions on the payoff functions that are vacuous when action spaces are discrete. In the current setting, things are slightly more involved as we would require Lipschitz continuity of value functions rather than of the one stage return function. Ensuring this is possible, but again at the cost of further assumptions. A promising avenue for ensuring this is to use recent results due to Hinderer (2005).

Moreover, we may want to relax the assumption of a one-dimensional summary statistic of aggregate play and suppose that each player's reward is a function of the cross-population distribution of actions. In this case, the mathematics become considerably more complicated and equilibrium existence may become an issue of concern. With this assumption, the game becomes what the literature has termed a *sequential anonymous game*. Relevant results for studying such games are found in Jovanovic and Rosenthal (1988) and Bergin and Bernhardt (1992, 1995). While we have not pursued this avenue, we believe it to be worthwhile.

Last, one may consider the introduction of aggregate state variables that depend in some way on the (lagged) distribution of actions. Such variables makes the analysis considerably more difficult and properties such as supermodularity seem only to be feasible under very strong additional assumptions.

## APPENDIX

### A. MATHEMATICAL CONCEPTS

We present definitions and known results that will be used throughout the rest of the paper. For more general definitions and results see Topkis (1998).

Throughout the paper, we endow  $\mathbb{R}^n$  with the element-wise order, i.e. for  $x^1 = (x_1^1, \dots, x_n^1)$  and  $x^2 = (x_1^2, \dots, x_n^2)$  we write  $x^1 \leq x^2$  if  $x_i^1 \leq x_i^2$  for all  $i = 1, \dots, n$ . Let

$$\begin{aligned} x^1 \wedge x^2 &= (\min(x_1^1, x_1^2), \dots, \min(x_n^1, x_n^2)) \\ x^1 \vee x^2 &= (\max(x_1^1, x_1^2), \dots, \max(x_n^1, x_n^2)) \end{aligned}$$

**Definition 7.** A set  $X \subseteq \mathbb{R}^n$  is called a **sublattice of  $\mathbb{R}^n$** , if  $x^1 \wedge x^2 \in X$  and  $x^1 \vee x^2 \in X$  for all  $x^1, x^2 \in X$ .

Consider a vector  $x = (x_1, \dots, x_n) \in X$ . Let  $\hat{x} = (x_{m_1}, \dots, x_{m_r})$ , where  $\{m_1, \dots, m_r\} \subseteq \{1, \dots, n\}$  and  $x^c = (x_{n_1}, \dots, x_{n_q})$ , where  $\{n_1, \dots, n_q\} = \{1, \dots, n\} \setminus \{m_1, \dots, m_r\}$ . I.e., the notation  $\hat{x}$  is used to denote a tuple of  $x$  constructed by picking out specific coordinates of interest from  $x$ . With this notation and by treating the rest of the coordinates of  $x$  as constant (fixed), we can write  $x = (\hat{x}; x^c)$ . The following gives the definition of supermodularity of a function in all its arguments, or specific tuples of its arguments.

**Definition 8.** Let  $X \subseteq \mathbb{R}^n$  be a sublattice of  $\mathbb{R}^n$ .

(a) A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is **supermodular on  $X$** , if for all  $x^1, x^2 \in X$

$$f(x^1) + f(x^2) \leq f(x^1 \wedge x^2) + f(x^1 \vee x^2)$$

(b) A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is **supermodular in  $\hat{x}$** , if for all  $x^1 = (\hat{x}^1; x^c)$  and  $x^2 = (\hat{x}^2; x^c)$

$$f(\hat{x}^1; x^c) + f(\hat{x}^2; x^c) \leq f(\hat{x}^1 \wedge \hat{x}^2; x^c) + f(\hat{x}^1 \vee \hat{x}^2; x^c)$$

for fixed  $x^c$ .

(c) A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is **strictly supermodular on  $X$** , if for all unordered  $x^1, x^2 \in X$

$$f(x^1) + f(x^2) < f(x^1 \wedge x^2) + f(x^1 \vee x^2)$$

(d) A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is **strictly supermodular in  $\hat{x}$** , if for all  $x^1 = (\hat{x}^1; x^c)$  and  $x^2 = (\hat{x}^2; x^c)$ , such that  $\hat{x}^1$  and  $\hat{x}^2$  are unordered

$$f(\hat{x}^1; x^c) + f(\hat{x}^2; x^c) < f(\hat{x}^1 \wedge \hat{x}^2; x^c) + f(\hat{x}^1 \vee \hat{x}^2; x^c)$$

for fixed  $x^c$ .

Given these definitions, it follows trivially that a function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is (strictly) supermodular on  $X$ , if it is (strictly) supermodular in  $x \in X$ . Also trivially, if  $f$  is strictly supermodular then it is supermodular as well. Furthermore, if  $f$  and  $g$  are (strictly) supermodular and  $\delta > 0$ , then  $f + \delta g$  is (strictly) supermodular. Moreover, if  $f$  is (strictly) supermodular, then taking the max of  $f$  over one of its arguments yields a (strictly) supermodular function. Next, the limit of a sequence of supermodular functions (if it exists) is also supermodular.<sup>9</sup> Finally, the following holds:

<sup>9</sup>For detailed statements, proofs and discussion of these results, see Topkis (1998), sections 2.6.2 and 2.7.2.

**Lemma 9.** *Let  $X \subseteq \mathbb{R}^n$  be a sublattice of  $\mathbb{R}^n$ . If  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is strictly supermodular in  $\hat{x}$  and  $g : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is supermodular in  $\hat{x}$ , then the sum  $f + g$  is strictly supermodular in  $\hat{x}$ .*

**Proof.** Consider  $x^1 = (\hat{x}^1; x^c) \in X$  and  $x^2 = (\hat{x}^2; x^c) \in X$ , such that  $\hat{x}^1$  and  $\hat{x}^2$  are unordered and let  $h(x) = f(x) + g(x)$ . Then

$$\begin{aligned} h(\hat{x}^1; x^c) + h(\hat{x}^2; x^c) &= f(\hat{x}^1; x^c) + f(\hat{x}^2; x^c) + g(\hat{x}^1; x^c) + g(\hat{x}^2; x^c) \\ &\leq f(\hat{x}^1; x^c) + f(\hat{x}^2; x^c) + g(\hat{x}^1 \wedge \hat{x}^2; x^c) + g(\hat{x}^1 \vee \hat{x}^2; x^c) \\ &< f(\hat{x}^1 \wedge \hat{x}^2; x^c) + f(\hat{x}^1 \vee \hat{x}^2; x^c) + g(\hat{x}^1 \wedge \hat{x}^2; x^c) + g(\hat{x}^1 \vee \hat{x}^2; x^c) \\ &= h(\hat{x}^1 \wedge \hat{x}^2; x^c) + h(\hat{x}^1 \vee \hat{x}^2; x^c) \end{aligned}$$

where the first inequality follows from supermodularity of  $f$  in  $\hat{x}$  and the second inequality follows from strict supermodularity of  $g$  in  $\hat{x}$ . Thus,  $h$  is strictly supermodular in  $\hat{x}$ . ■

Next we give the definition of increasing differences and a result that relates it to supermodularity.

**Definition 10.** *Let  $X \subseteq \mathbb{R}^n$  be a sublattice of  $\mathbb{R}^n$ .*

(a) *Consider a vector  $x = (x_1, \dots, x_n)$ . A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  has **increasing differences in  $(x_i, x_j)$**  if for all  $x'_i \leq x''_i$  and  $x'_j \leq x''_j$*

$$\begin{aligned} f((x_1, \dots, x''_i, \dots, x''_j, \dots, x_n)) - f((x_1, \dots, x'_i, \dots, x'_j, \dots, x_n)) &\geq \\ f((x_1, \dots, x'_i, \dots, x''_j, \dots, x_n)) - f((x_1, \dots, x'_i, \dots, x'_j, \dots, x_n)) & \end{aligned}$$

(b) *Consider a vector  $x = (x_1, \dots, x_n)$ . A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  has **increasing differences on  $X$** , if it has increasing differences in  $(x_i, x_j)$  for all  $i \neq j$ ,  $i, j = 1, \dots, n$ .*

(c) *Consider a vector  $x = (x_1, \dots, x_n)$ . A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  has **strictly increasing differences in  $(x_i, x_j)$**  if for all  $x'_i < x''_i$  and  $x'_j < x''_j$*

$$\begin{aligned} f((x_1, \dots, x''_i, \dots, x''_j, \dots, x_n)) - f((x_1, \dots, x'_i, \dots, x'_j, \dots, x_n)) &> \\ f((x_1, \dots, x'_i, \dots, x''_j, \dots, x_n)) - f((x_1, \dots, x'_i, \dots, x'_j, \dots, x_n)) & \end{aligned}$$

(d) *Consider a vector  $x = (x_1, \dots, x_n)$ . A function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  has **strictly increasing differences on  $X$** , if it has strictly increasing differences in  $(x_i, x_j)$  for all  $i \neq j$ ,  $i, j = 1, \dots, n$ .*

**Proposition 11.** *Let  $X \subseteq \mathbb{R}^n$  be a sublattice of  $\mathbb{R}^n$ . If a function  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is (strictly) supermodular in  $\hat{x} = (x_{m_1}, \dots, x_{m_r})$  then it has (strictly) increasing differences in  $(x_{m_i}, x_{m_j})$ , for all  $m_i \neq m_j$ ,  $m_i, m_j \in \{m_1, \dots, m_r\} \subseteq \{1, \dots, n\}$ .*

**Proof.** We first show that supermodularity implies increasing differences. Consider two arbitrary  $m_i \neq m_j$  and suppose  $x'_{m_i} \leq x''_{m_i}$  and  $x'_{m_j} \leq x''_{m_j}$ . Let

$$\begin{aligned}\widehat{x}^1 &= (x_{m_1}, \dots, x''_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}) \\ \widehat{x}^2 &= (x_{m_1}, \dots, x'_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r})\end{aligned}$$

Then

$$\begin{aligned}\widehat{x}^1 \vee \widehat{x}^2 &= (x_{m_1}, \dots, x''_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r}) \\ \widehat{x}^1 \wedge \widehat{x}^2 &= (x_{m_1}, \dots, x'_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r})\end{aligned}$$

Since  $f$  is supermodular in  $\widehat{x}$ , it follows from the definition of supermodularity that

$$\begin{aligned}f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) &\leq f(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + f(\widehat{x}^1 \vee \widehat{x}^2; x^c) \implies \\ f((x_{m_1}, \dots, x''_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}; x^c) &+ f(x_{m_1}, \dots, x'_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r}; x^c) \leq \\ f((x_{m_1}, \dots, x''_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r}; x^c) &+ f(x_{m_1}, \dots, x'_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}; x^c)\end{aligned}$$

Therefore

$$\begin{aligned}f((x_{m_1}, \dots, x''_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r}; x^c) &- f((x_{m_1}, \dots, x''_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}; x^c) \geq \\ f(x_{m_1}, \dots, x'_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r}; x^c) &- f(x_{m_1}, \dots, x'_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}; x^c)\end{aligned}$$

which means that  $f$  has increasing differences in  $(x_{m_i}, x_{m_j})$ , for all  $m_i \neq m_j$ .

To show that strict supermodularity implies strict increasing differences, consider two arbitrary  $m_i \neq m_j$  and suppose now that  $x'_{m_i} < x''_{m_i}$  and  $x'_{m_j} < x''_{m_j}$  and notice that the two vectors

$$\begin{aligned}\widehat{x}^1 &= (x_{m_1}, \dots, x''_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}) \\ \widehat{x}^2 &= (x_{m_1}, \dots, x'_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r})\end{aligned}$$

are unordered. By the definition of strict supermodularity it follows that

$$f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) < f(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + f(\widehat{x}^1 \vee \widehat{x}^2; x^c)$$

thus

$$\begin{aligned}f((x_{m_1}, \dots, x''_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r}; x^c) &- f((x_{m_1}, \dots, x''_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}; x^c) > \\ f(x_{m_1}, \dots, x'_{m_i}, \dots, x''_{m_j}, \dots, x_{m_r}; x^c) &- f(x_{m_1}, \dots, x'_{m_i}, \dots, x'_{m_j}, \dots, x_{m_r}; x^c)\end{aligned}$$

i.e.  $f$  has strictly increasing differences in  $(x_{m_i}, x_{m_j})$ , for all  $m_i \neq m_j$ . ■

Finally, let  $\{F(x; \theta), \theta \in \mathbb{R}\}$  be a family of distribution functions on  $\mathbb{R}^n$ , parametrized by  $\theta$  and let  $\int_S dF(x; \theta)$  be the probability measure of  $S \subseteq \mathbb{R}^n$  with respect to the distribution  $F(x; \theta)$ . The following introduces the concept of stochastically increasing distribution and is followed by a necessary and sufficient condition for it to hold.

**Definition 12.**  $F(x; \theta)$  is **stochastically increasing** in  $\theta$  if  $\int_S dF(x; \theta)$  is an increasing function of  $\theta$  for each increasing set  $S \subseteq \mathbb{R}^n$ .

**Proposition 13.**  $F(x; \theta)$  is stochastically increasing in  $\theta$  if and only if the expectation

$$\int h(x) dF(x; \theta)$$

is an increasing function of  $\theta$ , for all increasing real valued functions  $h(x)$  on  $\mathbb{R}^n$ .

**Proof.** For a proof of this last proposition, see Topkis (1998), section 3.9.1. ■

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