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# OPTIMAL PORTFOLIO ALLOCATION FOR CORPORATE PENSION FUNDS

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## ABSTRACT

### Optimal Portfolio Allocation for Corporate Pension Funds\*

We model the asset allocation decision of a defined benefit pension fund using a stochastic dynamic programming approach. Our model recognizes the fact that asset allocation decisions are made by trustees who are mandated to act in the best interests of beneficiaries - not by sponsoring employers - and that trustees face payoffs that are linked in an indirect way to the value of the underlying assets. This is because of the presence of pension insurance - which may cover a portion of deficits in the event of a sponsor default - and a sponsoring employer who may make good any shortfall in assets, and who may reclaim some pension surplus. Our model includes an allowance for uncertainty both of the future value of assets (because of uncertain investment returns) and liabilities (because of uncertainty in future longevity and in future interest rates). We find that we are able to substantially replicate observed DB pension asset allocations in the UK and conclude that institutional details - in particular asymmetries in payoffs to pension trustees - are crucial in understanding pension asset allocation.

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## 1. Introduction

Standard portfolio allocation tools - certainly those based on mean variance optimisation - are not well suited to help with the investment decisions facing those running corporate pension funds with defined benefit pension obligations. The payoffs to the members of schemes - those to whom pensions are due - are not linked in a straightforward way to the value of the fund of assets. If the corporate sponsor is certain to be able and willing to make good any shortfall in asset values from the cost of pension liabilities then underperformance of assets is not a risk that is passed through to scheme members. But neither are the interests of the shareholders in the company sponsoring the pension scheme linked in a clear and straightforward way to the return on the assets in the pension fund. The possibility that scheme members will have their pensions increased somewhat if the value of the fund rises above the assessed present value of the promised pension payout means that not all the upside of fund performance will necessarily accrue to shareholders. The possibility of default means that shareholders in companies with limited liability do not face all of the downside risk either. The existence of forms of state insurance of pension promises - often not complete insurance - adds further complexity.

More fundamentally, it is not entirely clear on whose behalf those with the power to decide on asset allocation in the pension fund are acting.

A further reason why the asset allocation decisions of pension funds are not readily amenable to standard portfolio allocation tools is that longevity risk is particularly important; this is something that is difficult to hedge and it generates significant risks of a mismatch between fund value and the cost of pension liabilities. The link between the value of liabilities and of unexpected movements in life expectancy is highly non-linear so the distribution of overall risks to pension funds is not standard.

As a result of all this there is no widely accepted answer to a set of fundamental questions facing those who make portfolio allocation decisions for pension funds: does it make sense to try to match as closely as possible the sensitivity of the value of assets to interest rates risk to the sensitivity of the assessed present value of liabilities? Can it make sense to look for such immunisation against interest rate risk when other large risks - most clearly longevity risk - are very hard to hedge? Should the portfolio allocation decisions be different depending on the maturity of the scheme and should it depend on the value of assets relative to an assessment of existing liabilities (ie upon today's level of funding)? How are the answers to these questions affected - if at all - by the asymmetry in payoffs to scheme members arising from a given degree of over-funding and under-funding (assets differing from the value of liabilities)? How sensitive to the strength of the corporate sponsor should asset allocation decisions be?

Because of the size of pension funds in many countries (particularly in the US and the UK) these issues are important for financial markets generally.

We aim to throw light on all these questions by developing a dynamic, stochastic portfolio allocation model that takes account of the features that are important for pension

funds, many of which make asset allocation decisions quite different from those taken by other investors.

Much of the theoretical literature on optimal portfolio decisions for corporate pensions places the decisions in the broader context of the overall balance sheet of the sponsoring company. (For a very clear and recent statement of this position see Merton (2006)). This way of framing the issue really makes the pension fund asset allocation question a part of the standard corporate finance issue of optimal balance sheet structure for a company which happens to have a particular type of debt obligation. Work based on this premise typically concludes that pension assets should be invested entirely in bonds to maximise the value of the tax shield granted to pension assets. (See, for example, Black (1980), Tepper (1977) and Exley, Mehta and Smith (1997)). This prediction is strikingly at odds with observed pension asset allocations in the UK and the US. In the UK, for instance, a typical UK defined benefit pension scheme invests approximately 60% of its assets in equities (McCarthy and Neuberger (2005)), and the amount is similar in the US (Rauh (2007)). We argue that the institutional reality for many pension schemes explains much of this divergence: pension schemes in many countries are entities distinct from the corporations which sponsor them. In the UK, for instance, the trustees of a scheme - who are separate from the management of the sponsoring company and who have an explicit responsibility to the members of the pension scheme - are the ones with the ultimate decision making power over asset allocation<sup>1</sup>. This means that it is not appropriate to assume that those who make decisions over pension fund policy act in the interests of the shareholders of the sponsoring company. The way we model the pension fund portfolio decisions takes this into account. Our model is able to explain the observed asset allocation of defined benefit pension schemes much more satisfactorily than shareholder-based models, such as those examined by Rauh (2007), arguing in favour of the importance of institutional constraints in understanding the investment strategy of defined benefit pension funds.

## 2. The Pension Model and Sources of Uncertainty

### 2.1 Overview

The way we analyse the investment decision is based on a simple but fundamental assumption: those with the responsibility for asset allocation consider what the implications are for the possible future deficit/surplus of the fund.<sup>2</sup> We focus on the decisions to be taken now about the appropriate allocation of funds between two broad classes of assets. We refer to these asset classes as ‘bonds’ and ‘equities’.

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<sup>1</sup> Cocco and Volpin (2005) present evidence which illustrates the effect which corporate insiders have on fund contributions and investment policy. The fact that some trustees have responsibility to both the company and the members merely strengthens our argument that payoffs to trustees are a complex function of assets and liabilities.

<sup>2</sup> This does not mean that those responsible for investment decisions ignore the interests of the sponsoring company and its shareholders. In our framework, the prospect that asset out-performance can generate a fund surplus can be valued, even though most (or all) of that upside benefit may accrue to the sponsoring company.

We want to focus on allocating existing assets, so we abstract from future pension contributions and from the accrual of new pension liabilities. In the UK and US many defined benefit corporate pension schemes are closed to new members and an increasing proportion are closed to additional accruals of pension claims of existing members. So for a great many schemes this is a realistic assumption.

To be more specific, the pension problem we analyse is one where decisions are made at regular intervals about allocating a portfolio of assets. The payoff (or utility) to those making the decision depends upon the value of the fund relative to the value of pension obligations. The size of the liabilities are assumed to become known at some point several years in the future. We think of this as the point when the scale of pension obligations becomes clear - a point at which annuities are bought to pay pensions and so all remaining risk is passed on from the pension fund to an annuity provider. Those annuities will have prices that reflect life expectancy and the yield on bonds at the time the pension obligations are crystallised. At that time, the value of pension fund assets will be judged against the value of the pension liabilities. It is the relative size of assets in the pension fund to the cost of paying pensions in full at this maturity date that is the payoff which enters the preference function of those who decide pension fund asset allocation. We will allow for the fact that when assets exceed liabilities not all of that benefit will feed through to members (indeed only a small part might); since trustees cannot be assumed to identify completely with shareholders it is therefore not appropriate that all of the excess of asset values over the cost of pensions should feed into the payoffs of the decision makers.

We also allow for the fact that a corporate sponsor might make good any shortfall between asset values and pension costs, but with less than certainty. Finally, we permit the possibility of (less than full) insurance - for instance from the Pension Protection Fund in the UK or the Pension Benefit Guaranty Corporation in the US - should the corporate sponsor not be able to make good a deficit in the pension scheme.

Both asset values and pension liabilities in the model are uncertain. Some of the factors that influence them are common and affect asset prices (e.g., bond yields) while others are assumed to be quite distinct (e.g., longevity has a major effect upon the value of pension liabilities but we assume that it does not impact the value of any assets).

We think of the relevant investment horizon as the period up to the point at which the value of pension liabilities, on average, gets realised. For the pension liability of an individual worker, one might think of that as the period until they retire.<sup>3</sup> We will value the fund's liabilities at this point by reference to the cost of an annuity, given bond yields and life expectancy at that date. Of course, in practice pension funds have a wide range of types of liability (with many pensions already in payment as well as accrued liabilities to current active and deferred members of various ages which, in the UK at least, have different linkages to inflation). In our calibrated model, we set a single horizon (which one might think of as the weighted average time from now until pensions are first taken). We set a plausible value for that time by reference to both the age structure of actual members of UK

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<sup>3</sup>At which point the liability could be extinguished and an annuity bought. Even if it is not extinguished, its value at that point becomes much more certain.

pension schemes and also by reference to the duration of the liabilities which it generates (which also depends on life expectancy and bond yields).

## 2.2 The model

We assume that pension fund assets can be invested in two types of bonds (short and long dated) and alternative assets (which we call ‘equities’ and whose characteristics will be set to match those of stocks).

The liabilities are commitments to pay pensions in the future. We define a horizon ( $T$ ) which we think of as the average time to when those pensions are first paid. At horizon  $T$  a commitment to pay a pension which gives a constant real flow of income has a value that we model as proportional to a level annuity. The value of that annuity at time  $T$ , and hence the value of the liabilities, depends on two variables that are highly uncertain — real yields on bonds and life expectancy (both at time  $T^4$ ).

In the model we assume that liabilities are commitments to pay real (inflation-adjusted) pensions. This is why the relevant returns are always real returns — hence our focus on the real yield on bonds, and uncertainty about it, and the real yield on other assets.

We use a Vasicek model of the evolution of the yield curve and a model of life expectancy developed by Lee and Carter. The return on ‘equities’ is also uncertain. It depends on an expected annual rate of return plus a random element (where log returns and the random element are assumed to be normally distributed).<sup>5</sup>

The pension fund makes decisions at regular intervals on their holdings of the three asset classes (short bonds, long bonds with a duration roughly matching that of the liabilities, and equities). A key question is by what criterion are those decisions judged.

We assume that what matters to a pension fund (or to its trustees and advisors) is the balance between assets and liabilities at horizon  $T$ . Crucial to the optimal choices is how those responsible for managing the fund (who we will from now on call the ‘trustees’) assess the costs of having a deficit at date  $T$ , and what the benefits (if any) of assets exceeding liabilities at  $T$  are. Those costs, and potential benefits, are likely to be asymmetric. In other words, the cost of assets falling 1% short of liabilities at date  $T$  is likely to be greater than the benefit of assets exceeding liabilities by 1%. Indeed, if pensioners cannot gain any of the upside benefits of asset out-performance, *and if those managing the pension fund identify completely with the interests of the pensioners*, there

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<sup>4</sup>The duration of the liability at time  $T$  depends upon both real yields and life expectancy at time  $T$ . If real yields are low, that duration is approximately half life expectancy. So one natural asset to hold today is a bond with a duration of approximately  $T + \text{half the expected post-retirement life expectancy at time } T$ . We assume that there is a zero-coupon bond with a duration equal to  $T$  plus half the anticipated post-retirement life expectancy. When we calibrate the model, we set  $T$  equal to 10 (years). In the developed world life expectancy at age 65 now is a little under 20 years. In 10 years’ time a reasonable central expectation is that it will be around 20 years. If the expected value of post-retirement life expectancy 10 years ahead is 20 then  $T$  plus half anticipated life expectancy is also 20 years. In fact 20 is about in line with the typical estimate of the average duration of the liabilities of UK DB pension schemes. This is why we chose  $T=10$ .

<sup>5</sup> We will assume that the random element of equity returns over  $T$  years is normally distributed with standard deviation of  $\sqrt{T} \times \sigma$  (where  $\sigma$  is the *annual* standard deviation of equity returns).

may be no value in having a surplus at all. Of course, the trustees may also give weight to the interests of the sponsoring company and so attach significant value to the chance that assets exceed pension liabilities even if the entire excess may, effectively, accrue back to the sponsoring company and its shareholders.

Unless there is complete certainty that the sponsoring company will make good any shortfall at time  $T$ , and that any out-performance of assets over liabilities is of no value, then these risk preferences of those acting on behalf of members are central to the asset allocation decision.<sup>6</sup> Neither of those conditions - complete certainty that the corporate sponsor will stand fully behind the scheme at time  $T$  and that if assets exceed statutory pension obligations all the excess accrues back to the company - is plausible. Since it is highly unlikely that for any DB pension fund these conditions hold, the risk preferences of those making decisions ('the trustees') about the way fund assets are invested do matter.

We describe the risk preferences of the trustees with a utility (or payoff) function that allows for risk aversion. We also allow for the payoffs that ultimately matter to the decision makers to differ from the unadjusted balance between asset values and pension costs at time  $T$ . If asset values fall short of pension costs the sponsoring company and/or a pension fund insurer may step in; when assets exceed pension costs those making decisions over the pension scheme may not get the full value of that because part of the gain - indeed maybe most of it - is channeled back to the sponsoring company and those returns are not valued in the same way as money flowing to scheme members.

### 2.3 The model in detail:

We denote the value of the assets of the pension fund at time  $T$  (when the value of liabilities is crystallised and the scheme reaches a level of maturity such that pension obligations are effectively settled) by  $A_T$ . The value of the pension obligations at that date is  $L_T$ . At time  $T$  we assume that those to whom pensions are due have a remaining life that can stretch as far ahead as 35 years (we think of them as 65 year olds who have a maximum age at death of 100). The pricing of annuities, and therefore the value of pension obligations, will depend on the pattern of anticipated future mortality rates at that time.

The *expected* mortality rate *at time T* for this cohort for  $j$  periods ahead, at which point they are aged  $65+j$ , is denoted:

$$m(65+j, j, k_T)$$

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<sup>6</sup>If the sponsoring company were 100% certain to make good any shortfall of assets over liabilities in the future, and also could appropriate any out-performance, then pension scheme members — and trustees acting on their behalf — should be completely indifferent to asset allocation. The shareholders in the sponsoring company might care a lot about the asset allocation. But their interests are really nothing to do with the details of the pension scheme. If shareholders can offset any asset allocation decisions in the DB fund by re-arranging their own investment portfolios, then the asset allocation of the pension fund is totally irrelevant (ignoring tax effects).



This mortality rate depends on a state variable,  $k_T$ , whose evolution is governed by a stochastic process to be described below. It is this trending, stochastic variable  $k$  that makes life expectancy at time  $T$ , conditional on information at some earlier date  $T-t$ , uncertain.

The level of the short-term interest rate at time  $T$  is denoted  $r_T$ . The whole of the yield curve at any time is pinned down by the value of the short rate in a way determined by a version of the Vasicek model described below. The evolution of the short rate (and so the evolution of the whole yield curve) is driven by a stochastic process. Equity prices also follow a stochastic process. Given the previous period portfolio allocation decision on the shares of bonds and equities in the portfolio, the realisation of the stochastic processes for equity and bond prices determines the value of assets,  $A$ . The three state variables at any period  $t$  are therefore:  $A_t, r_t, k_t$ .

We denote the present value at time  $T$  of payment of one unit at some period  $T+i$  by  $P(i, r_T)$ . We let the value of pension liabilities at time  $T$  be equal to the price of an annuity sold at an actuarially fair rate which pays a level real amount of one per period and until death for someone aged 65 at period  $T$ , conditional on all the relevant information about life expectancy at time  $T$  (*i.e.* conditional on  $k_T$ ). This is:

$$L_T = \sum_{i=0}^{35} P(i+1, r_T) \left( \prod_{j=0}^i (1 - m(65+j, j, k_T)) \right)$$

### Utility function for trustees

We assume that the utility function, or payoff function, for those with responsibility for the pension fund - who we call trustees - depends on the funding ratio at time  $T$ . The funding ratio is  $A_T/L_T$ . But we make adjustments to the funding ratio to allow for the fact that if  $A_T > L_T$  not all the benefit accrues to members. We apply a discount factor, a sort of tax rate, of  $t$  to any surplus. We also allow for a probability,  $p$ , of the sponsoring company being willing and able to make good any deficit. If the sponsoring company does not make good any deficit then an insurer covers a proportion,  $s$ , of any shortfall.

Thus, the effective stock of assets at  $T$ , which we denote  $A_T^*$  is defined as follows:

$$A_T^* = \begin{cases} L_T + (1-t)(A_T - L_T) & \text{if } A_T > L_T \\ A_T & \text{if } A_T = L_T \\ \begin{cases} L_T & \text{with probability } p \\ A_T + s(L_T - A_T) & \text{with probability } 1-p \end{cases} & \text{if } A_T < L_T \end{cases}$$

We use a power utility function which allows for risk aversion with respect to the adjusted, or effective, outcomes. The argument of the utility function is the ratio of adjusted assets to liabilities.

$$U(k_T, r_T, A_T) = Eu(A_T^* / L_T)$$

$$L_T = \sum_{i=0}^{35} P(i+1, r_T) \left( \prod_{j=0}^i (1 - m(65+j, j, k_T)) \right)$$

$$m(65+j, j, k_T) = \exp(a_{65+j} + b_{65+j}(k_T + j\mu_k))$$

The function  $u$  is given by a standard power utility function, so that the trustees are assumed to have constant risk aversion over the adjusted funding ratio. At time  $T$ , once asset price and longevity are known but before the status of the corporate sponsor is known, the expectation is over the random variable contained in the definition of  $A_T^*$  which reflects whether the sponsor bails an under-funded pension scheme out or not.

### Interest rate calibration

The *risk-neutral* instantaneous short rate is assumed to follow the Vasicek model, so:

$$dr_t = a(b - r_t)dt + \sigma_r dB_t$$

We assume that the rate faced by the pension fund trustees in the market is equal to this risk neutral rate, so implicitly we assume there are some risk neutral agents in the economy who operate in the bond market and have deep enough pockets to drive bond pricing.

This implies that the conditional distribution of  $r_{t+1}$  given  $r_t$  is:

$$r_{t+1} | r_t \sim N(r_t e^{-a} + b(1 - e^{-a}), \frac{\sigma_r^2}{2a} (1 - e^{-2a}))$$

The current shape of the yield curve that is consistent with no arbitrage can be derived from the above relation, allowing all riskless bonds to be priced. We denote by  $P(i, r_t)$  the price of a zero-coupon bond that matures in  $i$  years time (in other words at time  $t+i$ ) with a maturity value of 1, conditional on a current spot rate of  $r_t$ . The Vasicek model then implies:

$$P(i, r_t) = C(i)e^{-B(i)r_t}, B(s) = \frac{1}{a}(1 - e^{-as}), \text{ and } \log(C(s)) = (b - \frac{\sigma_r^2}{2a^2})(B(s) - s) - \frac{\sigma_r^2}{2a} B(s)^2$$

## Mortality calibration

We use the Lee-Carter model of life expectancy, or mortality (see Lee and Carter, 1992), which is the canonical model of stochastic mortality used in demography. The Lee Carter model assumes that:

$$\log(m_{x,t}) = a_x + b_x k_t,$$

where  $m_{x,t}$  is the crude death rate of individuals aged  $x$  in year  $t$  ( $m_{x,t}$  is the number of deaths of people aged  $x$  last birthday in year  $t$  divided by the total number aged  $x$  last birthday). The state variable  $k_t$  follows a random walk with drift.

$$k_{t+1} = k_t + \mu_k + \varepsilon_t^k, \text{ where } \varepsilon_t^k \sim N(0, \sigma_k^2)$$

The degree of uncertainty about the evolution of life expectancy is reflected by  $\sigma_k$ . The expected rate of improvement of life expectancy over time is reflected in  $\mu_k$ .

## The portfolio problem:

There are three assets to choose from: equities (with portfolio weight  $\alpha$ ), long bonds (with portfolio weight  $\beta$ ) and short bonds (weight  $1-\alpha-\beta$ ). By long bonds we mean bonds that mature ten years after the terminal horizon of the problem (which have a duration close to a plausible estimate of the duration of a pension liability to pay a level real pension to a 65 year old at current rates of life expectancy).

Log stock returns are denoted  $\varepsilon^Z$ . The price of the long bond is  $P_L$  and of the short bond is  $P_S$ . Then the equation governing the development of the assets between times  $t$  and  $t+1$  is:

$$A_{t+1} = A_t \left( \alpha_t e^{\varepsilon_t^Z} + \beta_t \frac{P_{t+1}^L(r_{t+1})}{P_t^L(r_t)} + (1 - \alpha_t - \beta_t) \frac{1}{P_t^S(r_t)} \right),$$

$$\text{where } \varepsilon_t^Z \sim N(\mu_e - \frac{1}{2}\sigma_e^2, \sigma_e^2), P_t^L(r_t) = P(20-t, r_t), \text{ and } P_t^S(r_t) = P(1, r_t).$$

$\alpha_t$  (proportion of fund invested in equities at time  $t$ ) and  $\beta_t$  (proportion of fund invested in the long bond at time  $t$ ) are decision variables, chosen optimally at each time period to maximise the expected value of the trustee payoff function in the last period, conditional on all future decisions being made optimally. These will be evaluated backwards in the usual way using standard numerical dynamic programming techniques.

The trustee's decision problem at time period  $t$  can be written as:

$\max_{\alpha_t, \beta_t} E_t[V_{t+1}(A_{t+1}, r_{t+1}, k_{t+1}) | A_t, r_t, k_t]$ , subject to:

$$k_{t+1} = k_t + \mu_k + \varepsilon_t^k, \text{ where } \varepsilon_t^k \sim N(0, \sigma_k^2)$$

$$r_{t+1} = r_t e^{-a} + b(1 - e^{-a}) + \varepsilon_t^r, \text{ where } \varepsilon_t^r \sim N(0, \frac{\sigma_r^2}{2a}(1 - e^{-2a}))$$

$$A_{t+1} = A_t(\alpha_t e^{\varepsilon_t^z} + \beta_t \frac{P_{t+1}^L(r_{t+1})}{P_t^L(r_t)} + (1 - \alpha_t - \beta_t) \frac{1}{P_t^S(r_t)}),$$

where  $\varepsilon_t^z \sim N(\mu_e - \frac{1}{2}\sigma_e^2, \sigma_e^2)$ ,  $P_t^L(r_t) = P(20 - t, r_t)$ , and  $P_t^S(r_t) = P(1, r_t)$

In the final time period,  $V(k_T, r_T, A_T) = U(k_T, r_T, A_T) = Eu(A_T^* / L_T)$ , where

$$L_T = \sum_{i=0}^{35} P(i+1, r_T) \left( \prod_{j=0}^i (1 - m(65 + j, j, k_T)) \right)$$

$$m(65 + j, j, k_T) = \exp(a_{65+j} + b_{65+j}(k_T + j\mu_k))$$

$$A_T^* = \begin{cases} L_T + (1-t)(A_T - L_T) & \text{if } A_T > L_T \\ A_T & \text{if } A_T = L_T \\ \begin{cases} L_T & \text{with probability } p \\ A_T + s(L_T - A_T) & \text{with probability } 1-p \end{cases} & \text{if } A_T < L_T \end{cases}$$

## 2.4 Paramaterisation:

For the Vasicek model of the yield curve we used:  $a = 0.3$ ,  $b = 0.04$  and  $\sigma_r = 0.01$ .

This implies that the short term (real) interest rate has an unconditional mean of 4% and with a conditional standard deviation of 1% over 1 period (which we take to be 1 year). This is roughly calibrated to the UK case. In the base case simulations we look at results when we start from a position with the short rate at its conditional mean.

For the process driving equity returns we used values broadly consistent with the properties of equity returns in the UK over the past hundred years (see Dimson, Marsh and Staunton (2001)). We set the expected annual return on equities at 7%, and the standard deviation is 20%.

In the base case we set the trustees risk aversion parameter,  $\gamma$ , equal to 5; this represents a substantial degree of risk aversion that we assume trustees exhibit over the funding ratio. For values of  $p$ ,  $s$  and  $t$  we examine four different cases:

**Case 1:  $s = t = p = 1$ .**

In this case, the beneficiaries of the fund are entirely protected from the effects of assets returns in the fund because the sponsor makes good any shortfall with certainty at the terminal horizon, but also reclaims any surplus.

**Case 2:  $s = t = p = 0$ .**

This corresponds to the opposite extreme to case 1 so that the beneficiaries of the fund bear the full effect of any over or under-funding at the terminal date. If we plot the adjusted value of the funding ratio (vertical axis) against the unadjusted value (horizontal axis), we have the following graph:

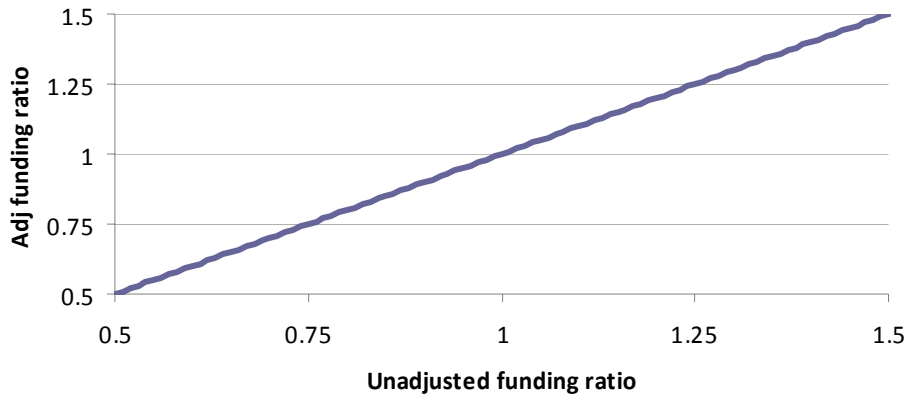


Figure 1: Payoffs to trustees in case 2:  $s = t = p = 0$

**Case 3:  $s = t = 0.5, p = 0$**

In this case, the trustees of the fund benefit from 50% of any over-funding, and suffer 50% of any under-funding. The sponsor is assumed not to be able (or willing) to intervene if the scheme is under-funded. Plotting the adjusted value of the funding ratio against the unadjusted value yields the following graph:

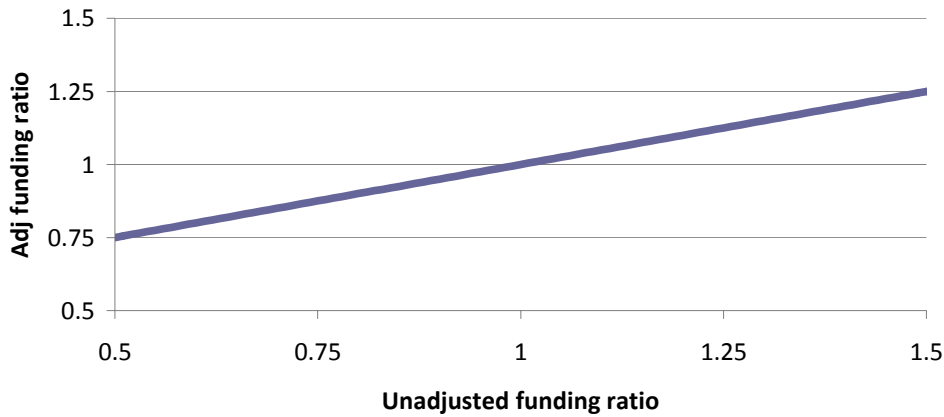


Figure 2: Payoffs to trustees in case 3:  $s = t = 0.5, p = 0$

**Case 4:  $s = 0.7, t = 0.5, p = 0.0$**

In this case, the pension insurer makes good 70% of any deficit with certainty. This corresponds very roughly to the current situation in the UK where the Pension Protection Fund covers a substantial proportion – but less than 100% - of the liabilities of a pension fund that is less than fully funded and where the corporate sponsor is unable to make good the shortfall (through insolvency, which is the only way to avoid making good the pension promise). If there is a surplus, 50% of the surplus reverts to the sponsor. The adjusted funding ratio plotted against the raw funding ratio looks like this:

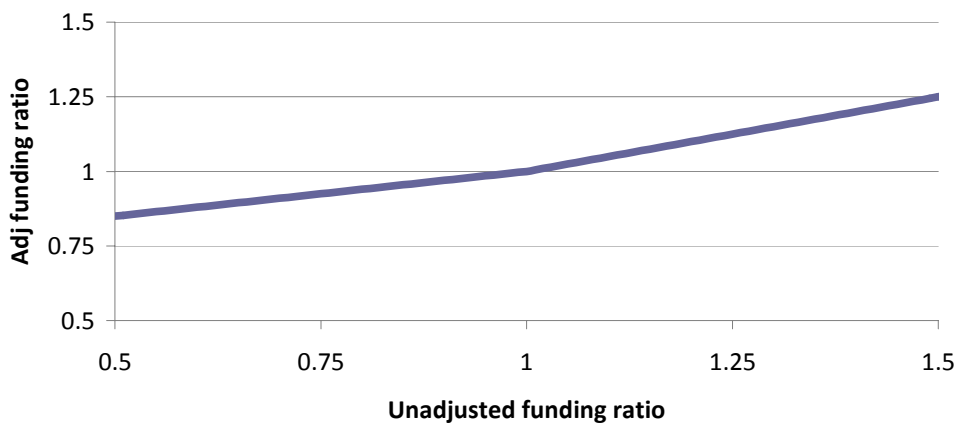


Figure 3: Payoffs to trustees in case 4:  $s = 0.7, t = 0.5, p = 0$

**Case 5:  $s = 0.7, t = 0.5, p = 0.5$**

In this case, the pension insurer makes good 70% of the deficit should the corporate sponsor be unable to make good any deficit. There is a 50% chance that the sponsor will

be willing and able to make up a deficit. If there is a surplus, 50% of the surplus reverts to the sponsor (or at least does not get reflected in the payoffs that matter to the scheme trustees). The *expected* adjusted funding ratio plotted against the raw funding ratio look like this:

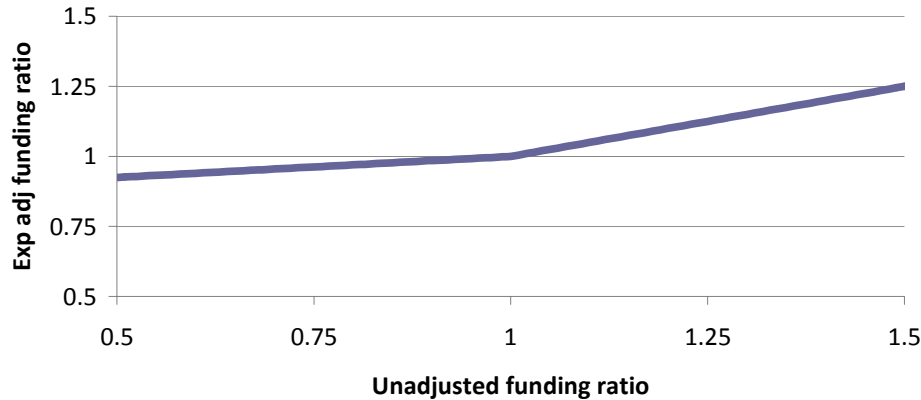


Figure 4: Expected payoffs to trustees in case 5:  $s = 0.7$ ,  $t = 0.5$ ,  $p = 0.5$

## 2.5 Mortality calibration

To calibrate the Lee-Carter model of mortality we used UK data on the death rates of civilians of ages 60+ from the period 1900 to 2003. Figure 5 shows the (log) death rate by age at each year in this period.

The Lee-Carter model assumes that:

$$\log(m_{x,t}) = a_x + b_x k_t,$$

where  $m_{x,t}$  is the crude death rate of individuals aged  $x$  in year  $t$ .

We fitted the model by maximum likelihood, assuming that the number of deaths at each age in each year has a Poisson distribution with parameter equal to  $m_{x,t}n_{x,t}$ , where  $n_{x,t}$  is the number of individuals aged  $x$  in year  $t$ . The variance of the maximum likelihood estimates is obtained using standard results. Figures 6, 7 and 8 show the resulting estimates of  $a_x$ ,  $b_x$  and the time series for  $k_t$ . Figure 8 shows that  $k_t$  clearly is strongly trended and provides some justification for our assumption that it follows a random walk with drift. We use the average slope of the series as a guide to the drift term,  $\mu_k$ . How to set the volatility of the noise term in the random walk process is less obvious. We try various values in the simulations. A natural yardstick is to calculate the implied degree of uncertainty about the life expectancy for a 65 year old some years ahead. The relation between the volatility of the underlying shock to the mortality process,  $\sigma_k^2$ , and the life expectancy of a 65 year old ten years ahead is shown in figure 9.

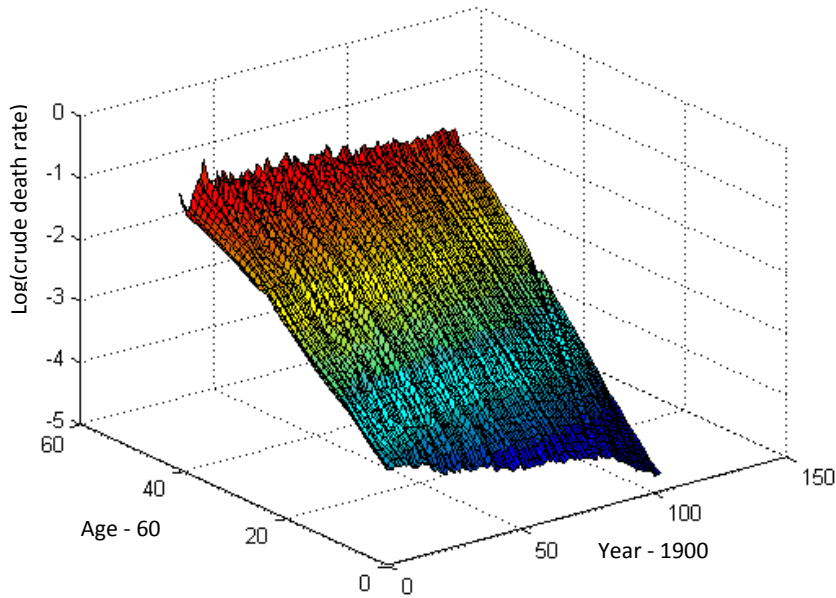


Figure 5: Log(Crude death rates) by age (60-100) and year (1900-2003) for UK civilians. Age running from 60 to 100 on left-hand axis; time running from 1900 to 2003 on right-hand axis. Source: [www.demography.org](http://www.demography.org), Berkeley Mortality Database

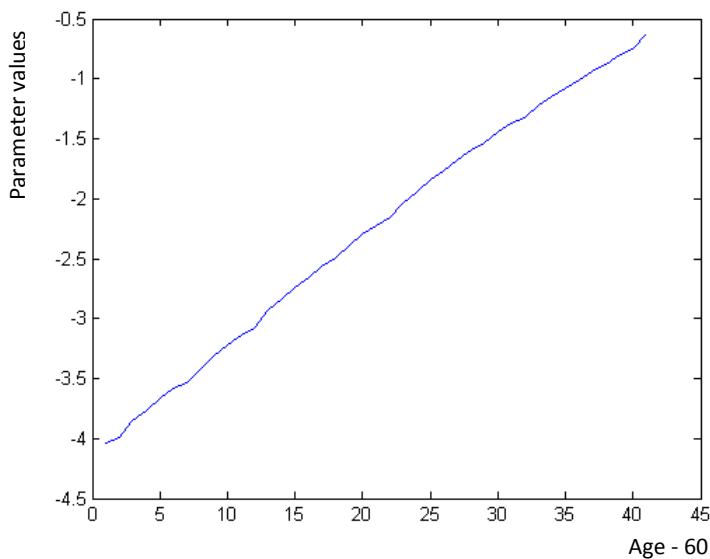


Figure 6: Estimates of  $\alpha_x$  (average of  $\log(m_{x,t})$ ) over time subscript) for each age Age, running from 60 to 100 on horizontal axis, parameter values on vertical axis



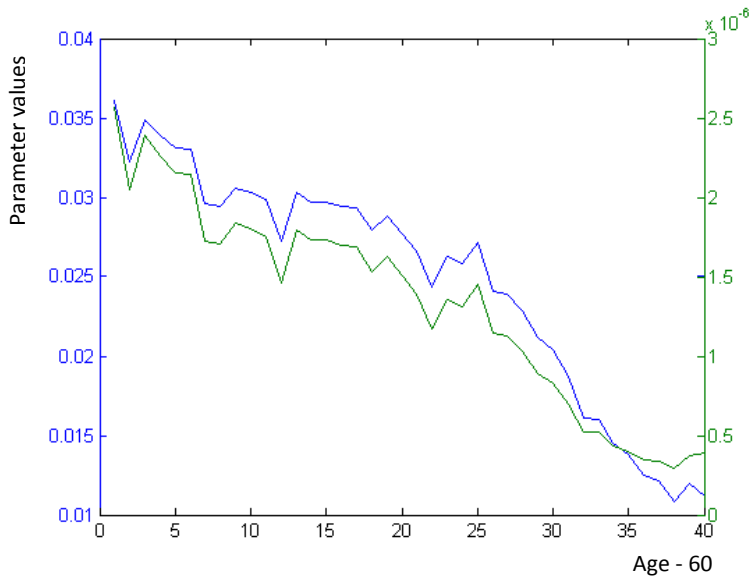


Figure 7: Estimates of  $b_x$  (left axis, mortality improvements at each age wrt index parameter  $k_x$ , and variance of parameter estimates, right axis, age running from 60 to 100, horizontal axis)

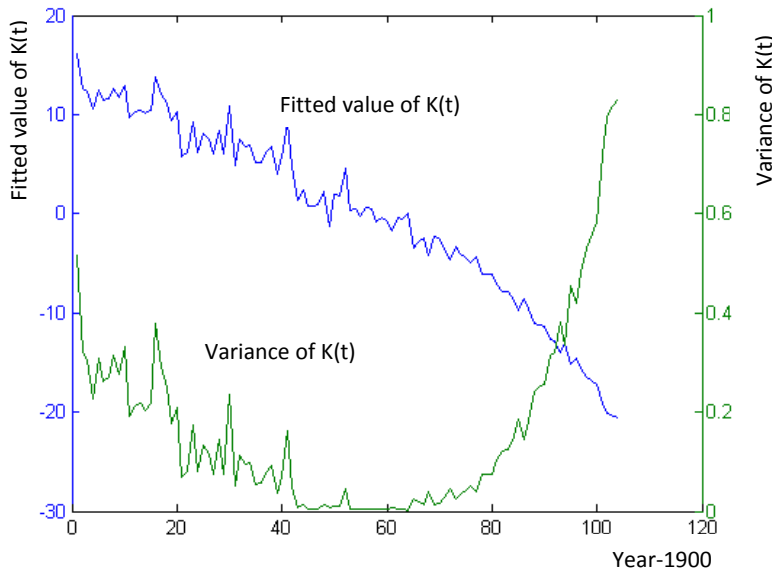


Figure 8: Estimates of mortality index  $k(t)$  (left axis: level of mortality index over time, right axis: variance of parameter estimates, age running from 60 to 100, horizontal axis)

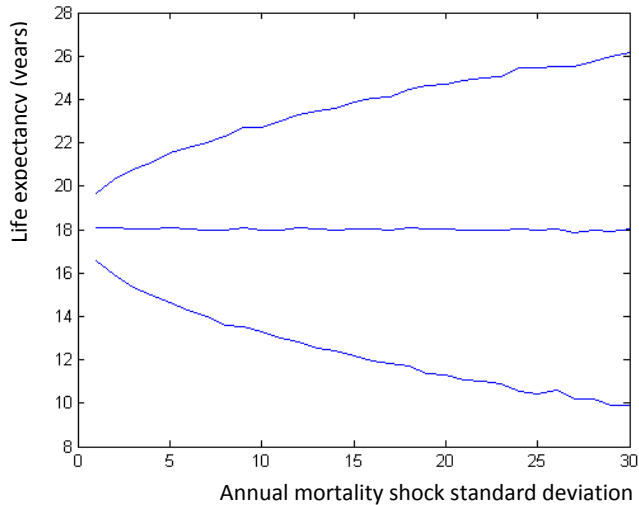


Figure 9: Annual shock standard deviation in  $k(t)$  (horizontal axis) versus 95% confidence interval in lifespan of males at age 65 in 2013 (10-year horizon from 2003).

Solving the model is difficult because of the significant non-convexities in the problem. This necessitated an extremely precise Gauss-Hermite integration and optimisation by grid search rather than hill-climbing or derivative-based methods. We ran the models on a large cluster of computers maintained by Imperial College’s High Performance Computing Centre.

### 3. Results:

We organise the discussion of results in the following way. First we consider the polar cases where the pension scheme bears no risk (case 1) or faces the entire investment risk itself (case 2 above). Next we examine a situation where only half of the investment risk is borne by the scheme (case 3), and then look at the impact of asymmetry between the amount of investment risk borne by the scheme when it is in surplus as opposed to in deficit (case 4). The final case we examine is when the sponsor might bail the fund out if it is in deficit while an insurer also stands behind the corporate sponsor (case 5). In all cases we considered three levels of mortality uncertainty – low (an annual mortality index shock standard deviation of 1, corresponding to a 95% confidence interval in life expectancy of approximately 2 years at a 10-year horizon), medium (standard deviation of 6, corresponding to a confidence interval of roughly 6 years), and high (standard deviation of 11, corresponding to a 95% confidence interval of around 10 years).

Initially we focus on results with what we call low mortality risk.

#### **Case 1: $s = t = p = 1$ .**

In this case, the pension fund is entirely a part of the sponsoring company. The sponsoring company makes up the deficit to full funding and takes all the surplus if any arises. There is no chance that the sponsoring company will default.

It is clear that our framework provides no optimal asset allocation in this case: there is no reward and no cost to the trustees of following any one asset allocation plan rather than any other. In all cases, regardless of investment policy, the liabilities of the fund are fully met and hence the duties of the trustees are fully discharged.

This case is examined by Exley, Mehta and Smith (1997). They argue that the asset allocation of the pension fund is irrelevant because shareholders can alter their own portfolios to take account of any asset allocation in the pension fund, though there may be tax factors (which we do not model) that mean there is an advantage to holding bonds inside the pension fund rather than outside it. This reflects arguments first set out in earlier papers by Black (1980) and Traynor (1977).

In practice, it is extremely unlikely that  $s = t = p = 1$ . It is routine, at least in the UK, that a surplus in a pension fund is to some extent shared between members and the plan sponsor, and no company has such a strong covenant that the chance of default is zero at every point in the future. We therefore turn to our other cases.

***Case 2:  $s = t = p = 0$ .***

This is the other end of the spectrum from case 1. Here, the pension fund stands entirely alone without any recourse either to a public pension insurer ( $s = 0$ ) or to the possibility of being bailed out by the corporate sponsor ( $p = 0$ ). In addition, any surplus earned by the fund is kept by the fund, so  $t = 0$ .

In this case, given that we have adopted a power utility function (CARA preferences), our model reduces to a Merton model of asset allocation. Because we assume that our sources of uncertainty (longevity shocks; interest rate shocks, and equity returns) are all uncorrelated, and the expected return on equities is unchanging, the inter-temporal hedging demand is zero, leaving the plain Merton model. At every point in time, the fund invests a constant proportion of its assets in equities, and this proportion depends only on the current equity risk premium, the coefficient of risk aversion and the standard deviation of equity returns.

The portfolio allocation to equities, derived by Merton (1969), is

$$\frac{\mu - r_f}{\gamma \sigma^2},$$

where  $\mu$  is the expected return on equities,  $r_f$  is the expected return on bonds with one year's duration (which depends on the current instantaneous short rate),  $\gamma$  is the trustee's coefficient of risk aversion and  $\sigma$  is the annual standard deviation of equity returns.

We ran our computer code (which generates a numerical solution based on the solution to a dynamic grid in the state space and which involves a numerical optimisation routine and numerical integration) with these parameters and confirmed that these answers held. This is a useful check on the numerical solution technique. The level of mortality uncertainty has no effect on the answers because it is orthogonal to the other risks and

cannot be hedged. In this case, mortality uncertainty has no effect on the optimal investment strategy of the fund. The funding ratio is also irrelevant – in other words the optimal portfolio is independent of either life expectancy, uncertainty about life expectancy or the level of assets. Because of this a graph of the optimal allocation to equities (risky assets) and bonds against the anticipated unadjusted funding ratio at the maturity of the scheme is flat. Figure 10 shows that graph<sup>7</sup>. The horizontal axis is the anticipated funding ratio when we discount liabilities by the current bond yield. We allow the level of assets to vary – holding constant anticipated life expectancy and the bond yields at their base case levels and calculating the anticipated funding rate at each asset level. This is roughly equivalent to an FRS17 funding ratio. In the Merton model, the time horizon is also irrelevant to the investment strategy, so figure 10 shows the optimal portfolio shares for all periods, and not just one period from the end (at T-1). In all other cases the optimal portfolios mapped against the funding position are time varying and we will show results one and ten years from the payment of pensions. Note that in case 2 the Merton formula gives an optimal equity proportion of 0.15, exactly equal to our numerical results.

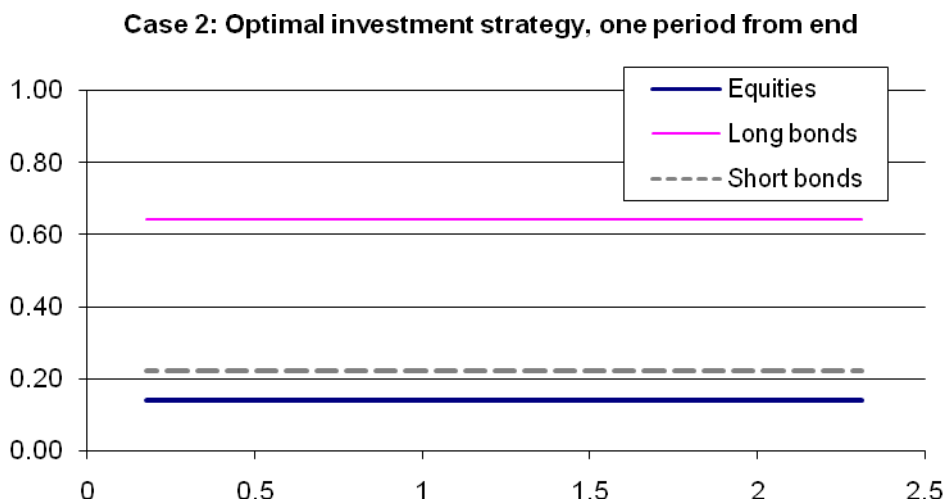


Figure 10: Optimal portfolio allocation – case 2. Horizontal axis shows current (FRS17) funding ratio.

<sup>7</sup> Results in all figures show the optimal investment policy when the instantaneous short rate equals its long-run mean of 0.04 and the life expectancy state variable,  $k$ , equal to -20 in the first period and -22 one period before the terminal date, both approximately equivalent to a life expectancy of 18 years at the terminal date. The two values are necessary to allow for the annual downward trend in the value of  $k$  of 0.2237.

**Case 3:  $s = t = 0.5, p = 0$**

Figure 11 shows the optimal portfolio one year from maturity. Now if the pension fund is under-funded, the equity proportion is higher than if plans are better funded. This is a departure from the Merton Model of Case 2 because the pension scheme shares only 50% of the deficit and surplus – the equivalent of a proportional tax and a lump sum transfer (which is why figure 2 differs from a 45 degree through the origin by a shift in slope *and* in the intercept). The proportional tax would preserve the Merton Model, but the effect of the lump sum transfer is to decrease risk aversion at lower levels of funding.

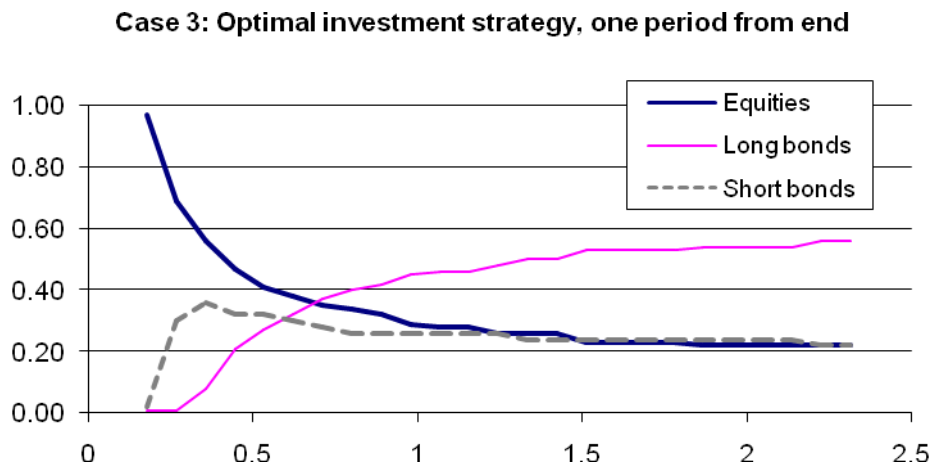


Figure 11 – optimal portfolios for case 3; one period from the end.

Figure 12 shows the optimal investment ten years from maturity. The same broad pattern emerges for the optimal portfolio allocation and its dependence on the funding status of the pension plan. For seriously under-funded schemes – a funding ratio of 0.5 or less – the allocation to equities is high (over 40%), even though trustees are risk averse and long bonds are a hedge against interest rate risk while equities are not. The existence of insurance means that risk averse trustees are willing to take much more risk when the funding status is poor. But as the funding status improves the share of equities falls sharply.

**Case 3: Optimal investment strategy, ten periods from end**

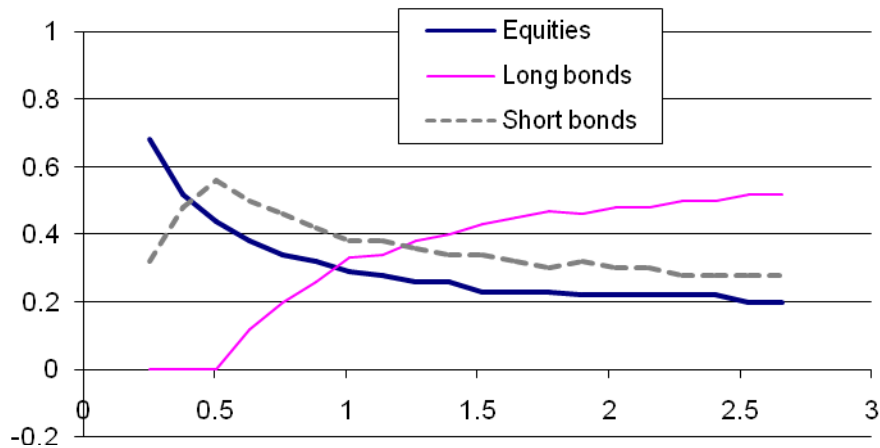


Figure 12 - optimal portfolios for case 3; ten periods from the end.

**Case 4:  $s = 0.7, t = 0.5, p = 0.0$**

As we increase the amount of insurance against the risks of under-funding it is not surprising that the optimal holding of risky assets is higher. Figures 13 and 14 show that this is true both one year from maturity and 10 years from maturity. But increasing insurance – while preserving the degree to which a surplus at maturity benefits the trustees – also creates an asymmetry in payoffs either side of full funding.

**Case 4: Optimal investment strategy, one period from end**

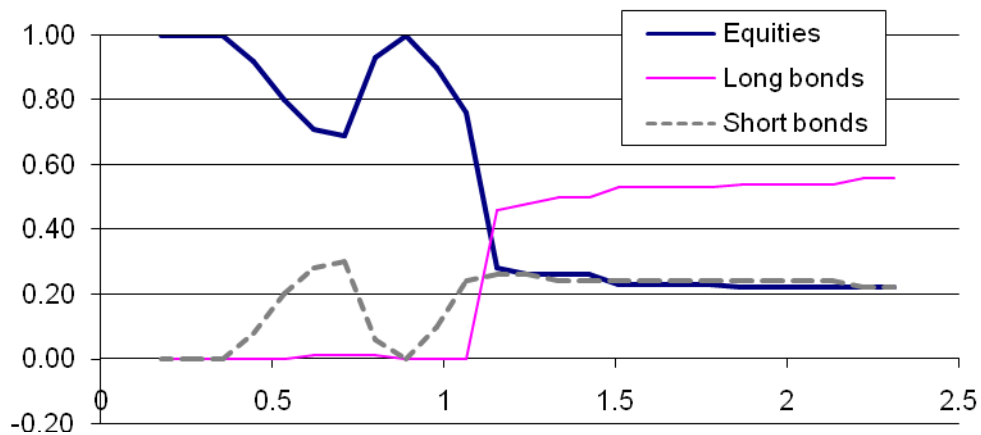


Figure 13 – optimal portfolios for case 4; one period from the end.

This asymmetry generates an interesting and non-regular strategy for optimal portfolio allocation: the optimal share held in risky equities at periods close to maturity is no longer monotonically declining as the funding position improves. If the pension fund is virtually guaranteed to be under-funded (a funding ratio one year from maturity less than

0.5), then the pension fund has an incentive to follow a risky strategy, and indeed a riskier investment strategy if 70% of the losses are made up (case 4) than if only 50% are made up (case 3). The interesting case occurs if there is a significant chance that the fund could end up fully funded but also a significant chance that it could be under-funded. If there is a kink (or asymmetry) in the payoff function, as in case 4, then if there are significant chances of being either side of that kink (at full funding) there is an incentive to increase the equity investment significantly. This is because the fund effectively gets 50% of the reward if their investments turn out well, but pay only 30% of the cost if they don't. As assets increase, and the current funding ratio improves and the chance of being fully funded increases, the strength of this effect falls and the optimal investment strategy returns to "normal". The same occurs as assets decrease.

As one moves back through time, the strength of this effect attenuates as the probability cone of the state space region likely to eventually include the kink expands. So ten years from maturity the relation between the current funding status and the share of assets held in equities is much smoother than one year from the end. However, a comparison of the optimal investment strategy in cases 3 and 4 illustrates how significant this kink is in determining investment strategy: figures 12 and 14 show that there is a significant increase in the optimal investment in equities across most values of the funding ratio. The fact that this is still true ten years before the liabilities crystallise illustrates that this effect is highly persistent through time.

**Case 4: Optimal investment strategy, ten periods from end**

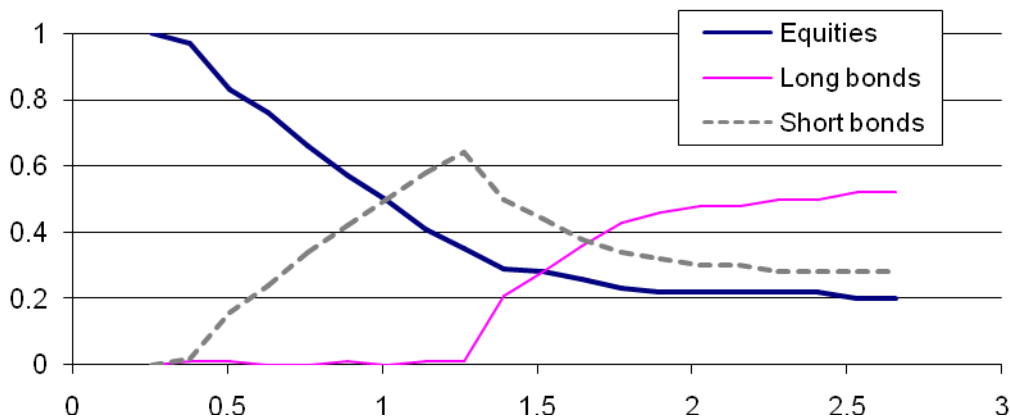


Figure 14 – optimal portfolios for case 4; ten periods from the end.

**Case 5:  $t = 0.5$ ,  $s = 0.7$ ,  $p = 0.5$**

Once we introduce a significant probability that the corporate sponsor will fully make good any shortfall in assets over liabilities then the incentives for trustees to take risk when they gain some benefit from asset out-performance is increased. Figures 15 and 16 show that the optimal allocation to equities is now significantly higher than in other cases. Indeed one year from maturity then if the scheme is under-funded on an FRS17 basis the trustees would want a close to maximum allocation to equities. In effect they are

taking advantage both of the corporate sponsor and the insurance provider to gamble in an environment when they gain some of the upside.

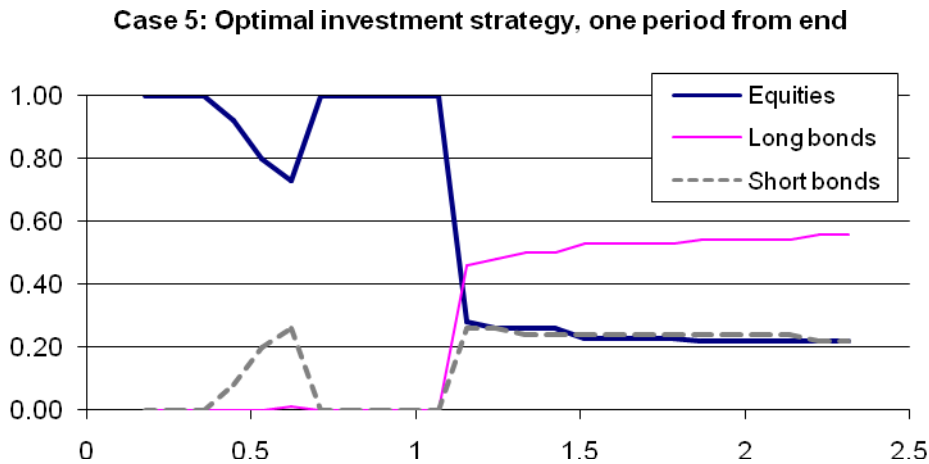


Figure 15 – optimal portfolios for case 5; one period from the end.

There is still a significantly non-monotonic relation between funding and optimal portfolio allocation created by the different slopes of the payoff function either side of full funding. At very low or very high levels of funding it is overwhelmingly likely that one year ahead the position of the pension fund will either be in deficit or surplus, so the kink in payoffs either side of full funding does not really come into play. But as the current funding situation is nearer values of 100% (1 in the figure) the kink in payoffs one period ahead plays more of a role.

If the fund is virtually certain to be under-funded, then cases 4 and 5 have the same optimal equity portfolio. At first sight, this seems counter-intuitive, because in case 5, the fund sponsor has a 50% chance of making up the deficit entirely, while in case 4, the deficit is borne by the pension fund (though partly insured) without any possibility of bailout. However, if the fund is bailed out, then the asset allocation is irrelevant, because all asset choices return the same payoff to the trustees. It is only the case when the fund is not bailed out which causes trustees to have any preference for one asset mix over another.



**Case 5: Optimal investment strategy, ten periods from end**

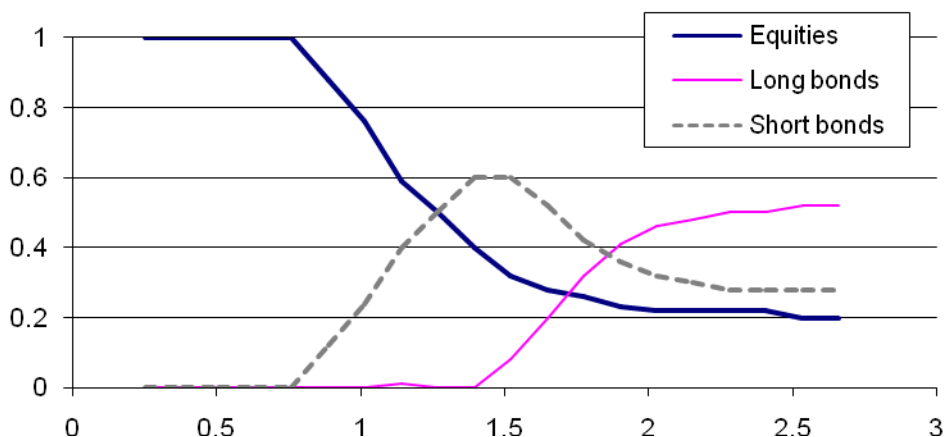


Figure 16 – optimal portfolios for case 5; ten period from the end.

Given the results for case 4, we would expect that as we move backwards in time, the results on optimal portfolio allocation for case 5 will also become smoother as the zone in the state space where there are significant chances of being either side of full funding at maturity widens. However, the possibility that the pension fund is bailed out by the sponsor increases the optimal investment in equities in the first time period significantly - as in case 4 the impact of the kink in payoffs is highly significant for most funding ratios, and highly persistent across time.

Figures 17 and 18 bring out the impact of variations in the strength of the corporate sponsor and of insurance on the optimal exposure to equity risk clearly.

**Equity proportion, all cases, one period from end**

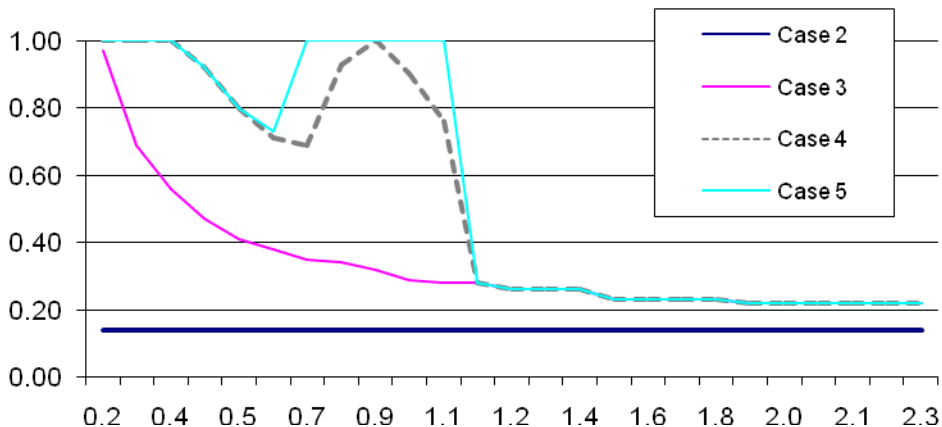


Figure 17- optimal equity shares – all cases one period from the end.

**Equity proportion, all cases, ten periods from end**

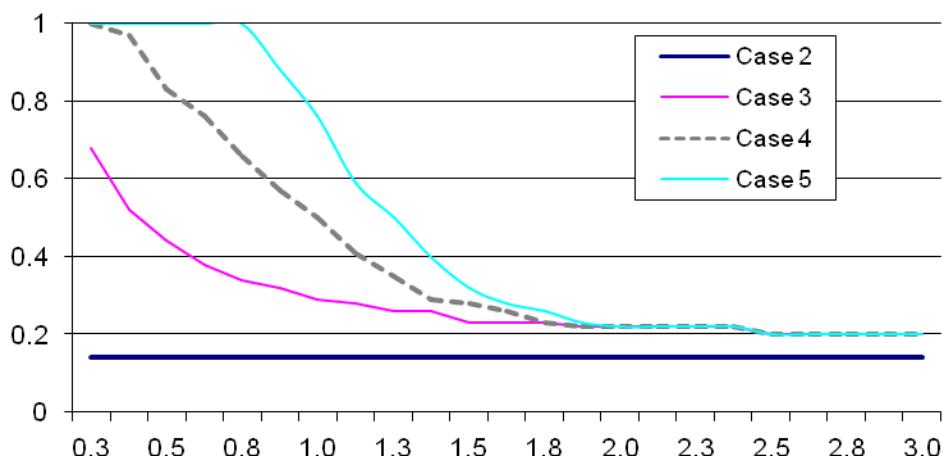


Figure 18- optimal equity shares – all cases ten periods period from the end.

**The impact of varying longevity risks:**

Figures 19 and 20 show how optimal portfolio allocation varies as we change the degree of uncertainty about longevity. Here we focus on cases 4 and 5, and look at results ten years from maturity. We find that the scale of uncertainty over how life expectancy evolves has a fairly limited effect upon portfolio allocation. But the direction is consistent across cases – more longevity risk slightly increases the optimal proportion of risky assets (equity) in optimal portfolios. The effects are most marked the more non-linear are the payoffs to trustees – the greater the degree of support from the corporate sponsor and the insurance scheme and the greater the extent to which trustees feel they benefit from any upside returns. The impact of variations in longevity risks is much less marked for case 3 and is absent altogether for case 2. In all cases where there is any effect of variations in longevity risk they have a much greater impact on portfolio allocations 10 periods from the end than they do nearer the point at which pensions are paid.

**Case 4, equity proportion, varying mortality risk, ten periods from end**

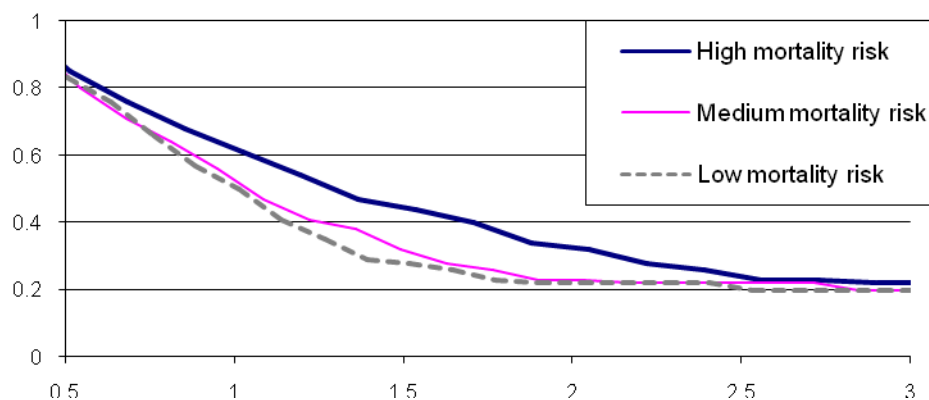


Figure 19: Optimal allocation to equities (risky assets) ten years from maturity of the fund – Case 4. Horizontal axis shows current (FRS17) funding ratio.

**Case 5, equity proportion, varying mortality risk, ten periods from end**

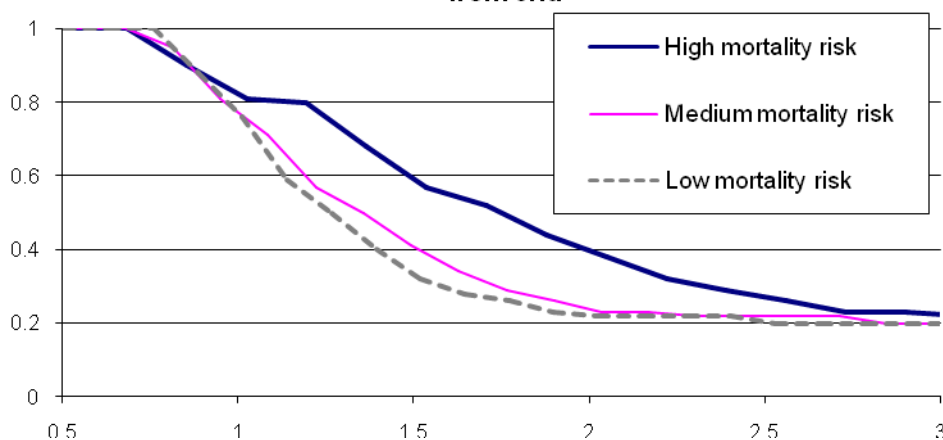


Figure 20: Optimal allocation to equities (risky assets) ten years from maturity of the fund – Case 5. Horizontal axis shows current (FRS17) funding ratio.

**Monte Carlo simulations:**

In the previous section we focused on the optimal portfolio rule – the relation between the state of the funds at a point in time and its best choices between equities, short dated bonds and long dated bonds. Here we analyse how the distribution of outcomes can evolve over time. To look at how the probability of various outcomes evolves we perform Monte Carlo simulations. The results are based on 10,000 simulations, each of which uses a series of random numbers to generate a specific path for the stochastic variables that create uncertainty for trustees – equity values, mortality rates and bond yields. We describe how the outcomes for portfolio allocation and funding status spread out if we

start from a specific position (a given funding position and values for bond yields). We summarise results by showing the percentiles of the outcome of the variable in question for the chosen parameter values. We look at 1%, 5%, 25%, 50%, 75%, 95% and 99% values.<sup>8</sup>

Tables 1 and 2 look at case 3 – a situation where trustees share in 50% of the upside and downside outcomes. In all cases we consider pension funds that 10 periods from the end have 80% FRS17 funding ratios; longevity risk is set at a relatively low level and bond yields are initially at their unconditional mean. Initially 34% of the portfolio is allocated to equity. Five years later the spread of outcomes is significant – the 25% percentile is an equity allocation of 24% while the 75<sup>th</sup> percentile is 40%. Funding ratios also get substantially more variable as time passes. At the outset the FRS17 ratio is 80%; by year 10, 25% of the time the funding ratio would be under 76% but 25% of the time it would be above 100%.

**Table 1**

The distribution of outcomes for equity allocations – case 3; initial 80% FRS17 ratio

Period	0	1	2	3	4	5	6	7	8	9	10
Percentile of outcomes											
0.01	0.34	0.13	0.09	0.07	0.05	0.04	0.04	0.04	0.04	0.04	0.04
0.05	0.34	0.18	0.16	0.14	0.12	0.12	0.12	0.12	0.12	0.12	0.11
0.25	0.34	0.28	0.25	0.25	0.24	0.24	0.23	0.23	0.23	0.23	0.23
0.5	0.34	0.33	0.33	0.33	0.33	0.32	0.32	0.32	0.32	0.32	0.31
0.75	0.34	0.39	0.40	0.41	0.41	0.40	0.41	0.41	0.41	0.41	0.40
0.95	0.34	0.48	0.53	0.54	0.55	0.55	0.55	0.56	0.56	0.56	0.56
0.99	0.34	0.55	0.61	0.63	0.65	0.66	0.68	0.68	0.69	0.68	0.69

**Table 2**

The distribution of outcomes for funding ratios – case 3; initial 80% FRS17 ratio

Period	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
Percentile of outcomes											
0.01	0.80	0.69	0.64	0.60	0.58	0.55	0.53	0.52	0.50	0.49	0.48
0.05	0.80	0.72	0.69	0.67	0.65	0.64	0.62	0.61	0.60	0.60	0.60
0.25	0.80	0.77	0.76	0.75	0.76	0.76	0.76	0.76	0.76	0.76	0.76
0.50	0.80	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.75	0.80	0.84	0.87	0.89	0.91	0.93	0.95	0.96	0.98	1.00	1.01
0.95	0.80	0.91	0.96	1.00	1.04	1.08	1.11	1.14	1.18	1.21	1.23
0.99	0.80	0.97	1.04	1.10	1.15	1.21	1.26	1.30	1.34	1.39	1.44

Tables 3 and 4 show the results for case 4 – where we now allow for an insurance company to make good 70% of any shortfall in assets at maturity of the scheme. The range of outcomes now becomes substantially more diverse than in case 3 where the absence of insurance made trustees less willing to take risks. It is not surprising that in case 5, where we also allow for a significant probability that the corporate sponsor can make good any shortfall in assets relative to liabilities, the spread of outcomes is wider

<sup>8</sup> We use a series of random numbers designed to improve convergence properties in finance applications generated by a new method of Karavias (2007).

again. Now the initial portfolio allocation is overwhelmingly to equities, assuming we start with a 20% deficit on an FRS17 basis. But after 10 years in 25% of cases the equity weight is beneath 26%. Funding outcomes – before any corporate top-ups and before an insurer steps in – are very diverse. In the worst 25% of cases the funding ratio on an FRS17 basis has fallen below 60% after 10 years, having started at 80%. But in the best 25% of cases the pension fund has a surplus of at least 40%.

**Table 3**

The distribution of outcomes for equity allocations – case 4; initial 80% FRS17 ratio

Period	0	1	2	3	4	5	6	7	8	9	10
Percentile of outcomes											
0.01	0.63	0.21	0.13	0.09	0.06	0.06	0.05	0.06	0.05	0.05	0.05
0.05	0.63	0.35	0.29	0.25	0.21	0.20	0.17	0.17	0.16	0.15	0.13
0.25	0.63	0.52	0.49	0.47	0.45	0.44	0.42	0.40	0.38	0.33	0.28
0.50	0.63	0.63	0.63	0.63	0.64	0.63	0.64	0.64	0.64	0.64	0.59
0.75	0.63	0.76	0.80	0.82	0.83	0.83	0.84	0.85	0.87	0.90	0.92
0.95	0.63	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.99	0.63	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

**Table 4**

The distribution of outcomes for funding ratios – case 4; initial 80% FRS17 ratio

Period	0	1	2	3	4	5	6	7	8	9	10
Percentile of outcomes											
0.01	0.80	0.60	0.52	0.45	0.41	0.38	0.35	0.32	0.31	0.29	0.28
0.05	0.80	0.66	0.60	0.55	0.52	0.50	0.47	0.46	0.44	0.42	0.42
0.25	0.80	0.74	0.72	0.71	0.71	0.71	0.70	0.70	0.70	0.69	0.69
0.50	0.80	0.80	0.82	0.84	0.85	0.87	0.89	0.91	0.93	0.95	0.98
0.75	0.80	0.88	0.93	0.97	1.01	1.05	1.09	1.12	1.17	1.21	1.25
0.95	0.80	1.01	1.10	1.18	1.26	1.33	1.38	1.44	1.47	1.51	1.55
0.99	0.80	1.11	1.24	1.35	1.43	1.52	1.60	1.65	1.70	1.75	1.80

**Table 5**

The distribution of outcomes for equity allocations – case 5; initial 80% FRS17 ratio

Period	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
Percentile of outcomes											
0.01	0.99	0.32	0.19	0.12	0.09	0.07	0.07	0.08	0.06	0.05	0.05
0.05	0.99	0.51	0.39	0.31	0.26	0.22	0.19	0.17	0.15	0.14	0.12
0.25	0.99	0.78	0.72	0.69	0.65	0.61	0.55	0.47	0.38	0.30	0.26
0.50	0.99	0.95	0.95	0.95	0.96	0.96	0.97	0.97	0.96	0.90	0.66
0.75	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.95	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

**Table 6**

The distribution of outcomes for funding ratios – case 5; initial 80% FRS17 ratio

Period	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
Percentile of outcomes											
0.01	0.80	0.50	0.43	0.37	0.34	0.31	0.28	0.26	0.25	0.23	0.22
0.05	0.80	0.58	0.52	0.48	0.44	0.42	0.40	0.38	0.36	0.35	0.34
0.25	0.80	0.70	0.68	0.66	0.65	0.64	0.64	0.63	0.62	0.61	0.60
0.50	0.80	0.80	0.82	0.84	0.85	0.87	0.89	0.91	0.93	0.97	0.99
0.75	0.80	0.92	0.98	1.04	1.09	1.14	1.20	1.25	1.32	1.37	1.40
0.95	0.80	1.12	1.24	1.35	1.44	1.52	1.58	1.63	1.67	1.71	1.75
0.99	0.80	1.28	1.44	1.57	1.66	1.76	1.82	1.88	1.93	1.98	2.04

Starting from a more fully funded level consistently creates a lower allocation to equities and so the range of outcomes is less spread out than shown in the tables 1-6. But higher longevity risks create yet more spread in the range of outcomes.

Overall what the Monte Carlo results reveal is that if insurance exists – either in the form of an explicit insurance scheme or because a corporate sponsor has a substantial probability of making good any shortfall - it creates incentives for trustees to make portfolio allocation decisions that generate very significant variability about future outcomes. That variability becomes greater the less strong is the initial funding position of the scheme.

### Testable implications:

From the discussion in the section above, it would be tempting to conclude that our model has strikingly different implications than standard corporate finance models of pension finance. For instance, while standard corporate finance models predict a negative relationship between the strength of the sponsor covenant and investment risk in the pension scheme, figure 18 suggests that our model has exactly the opposite conclusion. Case 5 in that model – where the sponsor has a 50% chance of stepping in to meet any shortfall at the terminal horizon – has a much higher equity allocation than case 4, where there is no sponsor at all. However, this implication is not actually correct, as can be seen from figure 17, which shows the effect of the cusp on investment policy in the final period. Figure 17 shows that the higher equity allocation in figure 18 is driven by the effect of the cusp around full funding. At all other points, the two cases result in identical asset allocations in the final period. The higher equity allocation in figure 18 is caused by increased uncertainty about the more distant final funding ratio of the scheme interacting with the effect of the cusp.

The fundamental driver of asset allocation in our model is therefore the size and direction – convex or concave – of the cusp in the payoffs to trustees around the point of full funding. If the adjusted funding rate profile is convex, our model clearly predicts that there will be a large increase in investment risk for fully funded and under-funded schemes. Similarly, although we have not demonstrated this, if trustees believe that members will not benefit at all from surpluses, the adjusted funding ratio profile could be *concave* around the fully funded point and therefore result in a large *reduction* in investment risk in the scheme for fully funded and underfunded schemes.

An implication of our model is therefore that pension schemes which are likely to share investment surpluses between the sponsor and the beneficiaries are likely to follow a riskier investment strategy than funds which have no history of sharing investment surpluses with members.

Although it is difficult to measure the attitudes of trustees, it is likely that their expectations are partly shaped by the history of the scheme. Schemes which have shared investment surpluses between sponsors and members in the past are more likely to do so in the future. Therefore our model could be tested by examining the relationship between past benefit improvements and the current investment strategy of pension funds. In performing the test, we would need to be careful about issues of endogeneity: it might be that funds which invested heavily in equities generated large surpluses at some point and then granted benefit improvements, while funds which invested in bonds never generated surpluses and then never granted benefit improvements.

Note that no corporate-finance-type theory of the asset allocation of pension schemes, such as those examined by Rauh (2007), has this implication; all these explanations ignore the relationship between investment surpluses and benefit improvements. Furthermore, if corporations do control the investment strategy of pension funds, then the prospect of investment surpluses being shared between members and sponsors should cause firms to adopt more *cautious* investment strategies. The size of the effect would depend on the extent to which firms could adjust wages to compensate workers for changes in pension benefits. If a portion of the membership of the fund were pensioners or no longer worked for the company then we would expect the effect to be stronger than if the fund were made up mostly of active members who currently worked for the company. Therefore, the relationship between benefit improvements and investment strategy is a crucial – and unique – testable implication of our model.

A further implication of our model is that underfunded pension schemes should follow riskier investment strategies than better-funded schemes because underfunded plans in our model have a greater incentive to shift risk onto the insurer than better funded plans. Our model shares this implication with standard corporate finance explanations of pension fund investment. It is interesting to note that Rauh (2007) convincingly demonstrates that the opposite is true in US data: underfunded pensions actually follow safer investment strategies than fully funded pensions. In the UK, the picture is not so clear: McCarthy and Neuberger (2005) show that underfunded plans in the UK follow slightly riskier investment strategies, although the significance of the effect is not clear.

#### **4. Conclusions:**

We have assumed that portfolio allocation decisions for defined benefit pension funds are made by rational, risk averse trustees who focus on the balance between assets and liabilities in the future and who take account of the probability of bail-outs by the corporate sponsor or by an insurer. They also factor in the prospect of sharing some of the benefits of assets exceeding liabilities. Based on this key assumption, and using the model calibrated to reflect asset price and mortality volatility, we find the following:

1. The degree of asymmetry of effective payoffs either side of full funding is a very important driver of pension fund portfolio allocation. The asymmetries reflect the possibility of the corporate sponsor bailing out a fund, the presence of an insurer and the possibility that only part of upside benefits of asset out-performance accrue to the scheme members and trustees. The more asymmetric are the payoffs, then the greater is the extent to which funds invest in equities. This has the potential to increase equity investment over large regions of the state space many years before the actual payoffs are realised.
2. Longevity risk has a relatively small impact on optimal portfolio allocation. But its impact consistently goes in one direction – more longevity risk somewhat increases the optimal allocation to riskier assets. The effect is more pronounced the greater is the degree of protection created by the corporate sponsor and an insurance scheme.
3. The equity risk premium has a powerful impact on portfolios, and the existence of forms of insurance - either from the corporate sponsor or from an insurer - makes portfolio allocation more sensitive to the expected excess return on equities than in a world without insurance.
4. The stronger the sponsor is perceived to be, the greater the proportion of equities funds will invest in. This is because the asymmetry around the point of full funding becomes larger as the sponsor becomes stronger, and hence the incentives to invest in equities become greater.
5. Even very partial insurance from an independent body (which we think of as a quasi government agency) has a big impact on portfolio choice and even when trustees of pension funds are highly risk averse.
6. When trustees (and ultimately the members of the scheme) share in some upside gains, but have substantial protection from the implications of under-funding, it creates incentives to make portfolio allocation decisions that generate very significant variability about future outcomes. That variability becomes greater the less strong is the initial funding position of the scheme.
7. Our model provides optimal equity allocations which are much closer to observed allocations in the UK than a pure corporate finance model of pensions would suggest. This indicates that institutional factors that we have modelled such as pension insurance, sponsor bailout and sharing of surplus between sponsor and beneficiaries are important drivers of pension asset allocation.



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