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ABSTRACT

Tax Rate Variability and Public Spending as Sources of Indeterminacy*

We consider a constant returns to scale, one sector economy with segmented asset markets, encompassing both the Woodford (1986) and overlapping generations models. We analyze the role of public spending, financed by (labour or capital) income and consumption taxation, on the emergence of indeterminacy. We find that what is relevant for indeterminacy is the variability of the distortion introduced by government intervention. We further discuss the results in terms of the level of the tax rate, its variability with respect to the tax base and the degree of externalities in preferences due to the existence of a public good. We show that the degree of public spending externalities affects the combinations between the tax rate and its variability under which indeterminacy occurs. Moreover, in contrast to previous results, we find that consumption taxes can lead to local indeterminacy when asset markets are segmented.

JEL Classification: E32, E63 and H23

Keywords: indeterminacy, public spending, segmented asset markets and taxation

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1 Introduction

The effects of government spending and taxes on income distribution, economic growth and welfare have been thoroughly discussed in the literature. More recently some papers have also stressed that fiscal policy may create indeterminacy, and thereby have destabilizing effects on the economy by triggering expectations driven cycles, i.e. sunspots. The pioneer work of Schmitt-Grohé and Uribe (1997) shows that, in a Ramsey model with a pre-set level of government expenditures, when the labor income tax is determined by a balanced budget rule, the economy can exhibit an indeterminate steady state and a continuum of stationary sunspot equilibria. Gokan (2005) and Pintus (2003), considering the same fiscal policy rule, show that similar results are obtained in a segmented asset market economy of the Woodford (1986) type.

In this paper we extend this analysis, studying the role of public spending financed by taxation on the emergence of indeterminacy, within segmented asset markets economies, for a wider class of fiscal policy rules. We consider a general unified structure that encompasses two types of segmented asset markets economies with a constant returns to scale Cobb-Douglas technology, the Woodford (1986) and the overlapping generations (OLG) economy as developed in Reichlin (1986). We further show that a Woodford economy with government spending financed by labor income and/or consumption taxation is isomorphic to a Reichlin framework with government spending financed by capital income and/or consumption taxes. In both cases the government intervention only introduces a multiplicative (endogenously variable) distortion in the equilibrium intertemporal choice of agents between consumption and leisure, which does not affect the capital accumulation equation. The main difference between these two frameworks lies on the values assumed for the depreciation rate: high in the overlapping generations model and small in the Woodford (1986) framework.¹

It is well known that constant income tax rates can not be *per se* a source of local indeterminacy in a Ramsey model.² The same happens in a segmented asset market economy as proved below. Therefore, we focus on variable tax rates, i.e. tax rates that vary (negatively or positively) with the tax base, while government spending adjusts in order to balance the budget.

¹This is due to the fact that in the OLG model agents live for two periods, whereas in the Woodford (1986) framework they are infinitely lived.

²See Schmitt-Grohé and Uribe (1997) and Guo and Harrison (2004).

The tax rate rule considered is characterized by two parameters: the level of the tax rate and its variability with respect to the tax base.³ Our fiscal policy rule covers as particular cases those considered in Schmitt-Grohé and Uribe (1997), Pintus (2003), Dromel and Pintus (2004), Giannitsarou (2005) and Gokan (2005), and also the case of a constant tax rate.

Moreover, we also introduce the possibility of government spending externalities in preferences.⁴ We study how the degree of government spending externalities affects the combinations between the tax rate and its variability, under which indeterminacy occurs. For example, we find that in the absence of government spending externalities, for a fixed positive response of the tax rate to the tax base, a sufficiently low tax rate ensures determinacy. However, this is no longer true for higher values of public spending externalities. Also, with constant income tax rates, indeterminacy becomes possible when public spending externalities on preferences are sufficiently strong in the Woodford framework, or even with arbitrarily small externalities in the OLG model. Therefore, our results show that taking into account (and accurately measuring) the degree of government spending externalities is crucial in order to correctly evaluate the dynamic implications of different fiscal policy rules.

Another novel feature of our work is that we study the effects of consumption taxation on indeterminacy in segmented asset markets economies. Giannitsarou (2005), using a Ramsey model, addressed only the case where a fixed stream of government spending is financed by consumption taxes, and found that indeterminacy was not possible in this case. Here, we consider a more general fiscal policy rule that encompasses the one analyzed in Giannitsarou (2005). We find that, in a segmented asset markets framework, indeterminacy is possible with a consumption tax rate that responds negatively to the tax base. In particular, in the case analyzed in Giannitsarou (2005) indeterminacy may emerge, even without public spending externalities on preferences. This shows that taxes have different effects on indeterminacy in segmented asset markets and Ramsey models. Being aware of the markets' distortions (e.g. financial imperfections) existing in the pre intervention sit-

³The functional form considered for the tax rate is similar the one used by Guo and Lansing (1998), although the tax bases are different.

⁴Seegmuller (2003) analyzes the role of this type of externalities in OLG economies, but he only considers constant tax rates. Cazzavillan (1996), using an optimal growth model again with constant tax rates, introduces not only public spending externalities in preferences but also in production. Note however that the latter also distort the capital accumulation equation.

uation is therefore also crucial to address correctly the dynamic implications of fiscal policy rules.

Finally let us remark that one advantage of considering an unified framework to analyze the role of public spending financed by taxation on indeterminacy in segmented asset markets is that, by doing so, we are able to see very clearly that quite different fiscal policies in sufficiently different models share the same indeterminacy mechanism. Indeed, we find that, both in the Woodford and OLG frameworks, and for both income and consumption taxation, indeterminacy depends on the variability of the distortion introduced by government intervention. In the Woodford framework, indeterminacy requires a sufficiently strong variability of the distortion, whereas in the OLG framework, this variability can be arbitrarily small.

The rest of the paper is organized as follows. In the next section, we present a general unified framework of a segmented asset markets economy that admits as particular cases both the Woodford (1986) model and the OLG economy studied by Reichlin (1986) with a public sector. We then analyze the steady state and derive the conditions for the emergence of indeterminacy in this framework. In section 3, we extend the Woodford (1986) model to allow for public spending, that may affect preferences and is financed by labor and consumption taxes. Then, in section 4, we present and discuss in detail the indeterminacy results in the Woodford model considering labor income and consumption taxes. In sections 5 and 6, we respectively present the OLG model with government spending and discuss its stability properties in the cases of capital income and consumption taxes. Finally, in the last section, we provide some concluding remarks.

2 The General Framework

We consider a one sector perfectly competitive economy with segmented asset markets and discrete time $t = 1, 2, \dots, \infty$, where both capital k_t and labour l_t are used to produce the final good, y_t , under a Cobb-Douglas technology with constant returns to scale:

$$y_t = k_t^s l_t^{1-s} \tag{1}$$

where $s \in (0, 1)$ represents the capital share in total income. Producers maximize their profits. Since all markets are perfectly competitive, we obtain

the following expressions for the real wage and the real interest rate:

$$\omega_t = (1 - s)k_t^s l_t^{1-s} \equiv \omega(k_t, l_t) \quad (2)$$

$$\rho_t = s k_t^{s-1} l_t^{1-s} \equiv \rho(k_t, l_t) \quad (3)$$

In what follows, instead of developing a particular model with segmented asset markets based on the behavior of consumers, we just present two dynamic equations without a priori microeconomic foundations. As we will see later in Section 3 and Section 5, this reduced form admits as particular cases the Woodford (1986) model as well as the overlapping generations framework developed by Reichlin (1986). At this stage, we just clarify that the first equation corresponds to the intertemporal choice of workers between future consumption and leisure, whereas the second one describes capital accumulation.

Hence, we assume in this section that the perfect foresight intertemporal equilibrium of this economy is a sequence $(k_t, l_t) \in \mathbb{R}_{++} \times (0, \tilde{l})$, $t = 1, 2, \dots, \infty$ that, for a given $k_1 > 0$, solves the two-dimensional dynamic system:

$$(1 - \mu)y_{t+1}\varphi(k_{t+1}, l_{t+1})/B = \gamma(l_t) \quad (4)$$

$$k_{t+1} = \beta[(1 - \delta)k_t + \mu y_t] \quad (5)$$

where $B > 0$ is a scaling parameter, $\mu \in (0, 1)$ is a constant and y_t is given by (1). Moreover, both parameters β and δ belong to $(0, 1]$, and $\gamma(l)$ is a positive continuous function, differentiable as many times as needed for $l \in [0, \tilde{l}]$,⁵ such that $\varepsilon_\gamma(l) \equiv \gamma'(l)l/\gamma(l) \geq 1$. Finally, the function $\varphi(k, l)$ summarizes the effect of government intervention on this economy. On the function $\varphi(k, l)$ we further assume that it is continuous for $(k, l) \in \mathbb{R}_+ \times [0, \tilde{l}]$, positively valued and differentiable as many times as needed for $(k, l) \in \mathbb{R}_{++} \times (0, \tilde{l})$. Moreover, $\varepsilon_{\varphi x}(k, l) \equiv \frac{\partial \varphi(k, l)}{\partial x} \frac{x}{\varphi(k, l)} \in \mathbb{R}$, with $x \in \{k, l\}$. In the absence of government intervention $\varphi(k, l) = 1$, so that $\varepsilon_{\varphi x}(k, l) = 0$ for $x \in \{k, l\}$.

One can also notice that while k_t is a variable determined by past actions, the value of l_t , on the contrary, is affected by expectations of future events. Therefore, equations (4) and (5) determine the dynamics of the economy, through a two-dimensional dynamic system with one predetermined variable, the capital stock k_t .

⁵The labor endowment \tilde{l} may be finite or infinite.

As we will explain in Sections 3 and 5, equations (4) and (5) describe the dynamics of a Woodford economy for $\mu = s$ and $\delta < 1$, whereas they summarize the overlapping generations behavior for $\mu = 1 - s$, $\delta = 1$ and $\beta = 1$.

2.1 Steady State Analysis

In this section, we establish conditions for the existence of a unique steady state (k, l) of the dynamic system (4) and (5).

Proposition 1 *Existence and uniqueness of the steady state:*

Defining $H(l) \equiv l\varphi(a^*l, l)/\gamma(l)$, $a^* = (\mu\beta/\theta)^{1/(1-s)}$ and $\theta \equiv 1 - \beta(1 - \delta) \in (0, 1]$, and assuming that $(1 - \mu)(a^*)^s \max\{\lim_{l \rightarrow 0} H(l), \lim_{l \rightarrow \tilde{l}} H(l)\} < B < (1 - \mu)(a^*)^s \min\{\lim_{l \rightarrow 0} H(l), \lim_{l \rightarrow \tilde{l}} H(l)\}$, the dynamic system (4)-(5) has a unique steady state (k, l) if one of the two following conditions is satisfied:

1. Either $1 + \varepsilon_{\varphi k}(a^*l, l) + \varepsilon_{\varphi l}(a^*l, l) > \varepsilon_{\gamma}(l)$ for $l \in (0, \tilde{l})$
2. or $1 + \varepsilon_{\varphi k}(a^*l, l) + \varepsilon_{\varphi l}(a^*l, l) < \varepsilon_{\gamma}(l)$ for $l \in (0, \tilde{l})$

Proof. Studying the existence and uniqueness of the steady state (k, l) of the dynamic system (4) and (5) is equivalent to analyze the existence and uniqueness of a stationary solution (a, l) , with $a \equiv k/l$, of the two following equations:

$$\mu a^{s-1} = \theta/\beta \tag{6}$$

$$(1 - \mu)a^s l\varphi(al, l)/B = \gamma(l) \tag{7}$$

We can easily see that a^* is the unique solution to equation (6). Since $H(l)$ is a continuous function, under the boundary assumptions on B stated in Proposition 1, there exists a value l^* solving (7) with $a = a^*$. Uniqueness of l^* is ensured by inequalities 1 or 2, because in these cases $H(l)$ is a monotonic function. ■

2.2 Indeterminacy

Assuming that Proposition 1 is verified, we now study the local indeterminacy of the steady state. To do so, we differentiate the two-dimensional dynamic system (4) and (5) in the neighborhood of the steady state. The trace T and the determinant D of the associated Jacobian matrix are given by:

$$T = 1 + \frac{\varepsilon_\gamma - \theta(1-s)(1 + \varepsilon_{\varphi l} + \varepsilon_{\varphi k})}{\varepsilon_{\varphi l} + 1 - s} \quad (8)$$

$$D = \frac{\varepsilon_\gamma[1 - \theta(1-s)]}{\varepsilon_{\varphi l} + 1 - s} \quad (9)$$

where $\varepsilon_\gamma \geq 1$, $\varepsilon_{\varphi k}$ and $\varepsilon_{\varphi l}$ denote respectively the elasticities $\varepsilon_\gamma(l)$, $\varepsilon_{\varphi k}(k, l)$ and $\varepsilon_{\varphi l}(k, l)$ evaluated at the steady state.

We summarize our results in Proposition 2 below.

Proposition 2 Indeterminacy

The steady state will be indeterminate if and only if one of the following conditions is satisfied:

- (i) $\varepsilon_{\varphi l} > \max \{ \varepsilon_{\varphi l}^T, \varepsilon_{\varphi l}^H, \varepsilon_{\varphi l}^F \};$
- (ii) $\varepsilon_{\varphi l} < \min \{ s - 1, \varepsilon_{\varphi l}^T, \varepsilon_{\varphi l}^F \};$

where:

$$\begin{aligned} \varepsilon_{\varphi l}^T &= -\varepsilon_{\varphi k} + (\varepsilon_\gamma - 1) \\ \varepsilon_{\varphi l}^H &= [s - \theta(1-s)] + [1 - \theta(1-s)](\varepsilon_\gamma - 1) \\ \varepsilon_{\varphi l}^F &= -\frac{[4 - 2s - \theta(1-s)(2 + \varepsilon_{\varphi k})]}{2 - \theta(1-s)} - (\varepsilon_\gamma - 1). \end{aligned}$$

Proof. Since we only have one predetermined variable (capital) the steady-state is locally indeterminate when it is a sink (both eigenvalues with modulus lower than one). Therefore indeterminacy requires that $D < 1$, $1 - T + D > 0$ and $1 + T + D > 0$. When the denominator of both the trace and the determinant is positive ($\varepsilon_{\varphi l} > s - 1$) these conditions can be written respectively as $\varepsilon_{\varphi l} > \varepsilon_{\varphi l}^H$, $\varepsilon_{\varphi l} > \varepsilon_{\varphi l}^T$, and $\varepsilon_{\varphi l} > \varepsilon_{\varphi l}^F$ where $\varepsilon_{\varphi l}^H$, $\varepsilon_{\varphi l}^T$ and $\varepsilon_{\varphi l}^F$ are given in Proposition 2. When the denominator of both the trace and the determinant is negative ($\varepsilon_{\varphi l} < s - 1$) these conditions can be written respectively as $\varepsilon_{\varphi l} < \varepsilon_{\varphi l}^H$, $\varepsilon_{\varphi l} < \varepsilon_{\varphi l}^T$, and $\varepsilon_{\varphi l} < \varepsilon_{\varphi l}^F$. Combining these results, and noticing that $\varepsilon_{\varphi l}^H > s - 1$, Proposition 2 immediately follows. ■

Note that, in this framework, it is not the level of the distortion introduced by government intervention *per se* that is responsible for the emergence of indeterminacy, but its variability ($\varepsilon_{\varphi k} \neq 0, \varepsilon_{\varphi l} \neq 0$). Indeed when $\varepsilon_{\varphi k} = \varepsilon_{\varphi l} = 0$, we can see from Proposition 2 that the steady state can never be indeterminate. For instance, when $\varepsilon_{\varphi k} = 0$, from (i) of Proposition 2 we have that $\varepsilon_{\varphi l} > (\varepsilon_{\gamma} - 1) \geq 0$ is a necessary condition for indeterminacy, and from (ii) it is necessary that $\varepsilon_{\varphi l} < s - 1 < 0$ for indeterminacy to emerge. However, when $\theta \in (0, 1]$ is sufficiently high, indeterminacy is possible with arbitrarily small values for the elasticities $\varepsilon_{\varphi k}$ and $\varepsilon_{\varphi l}$. Consider that $\varepsilon_{\gamma} = 1$. In this case, when θ is sufficiently close to 1, such that $\theta(1 - s) > s$, $\varepsilon_{\varphi l}^H$ is negative. Moreover, with $\varepsilon_{\varphi k}$ arbitrarily small, $\varepsilon_{\varphi l}^F$ is also negative. Therefore, condition (i) of Proposition 2 becomes in this case $\varepsilon_{\varphi k} + \varepsilon_{\varphi l} > 0$, which can be satisfied with arbitrarily small positive values for $\varepsilon_{\varphi l}$ and $\varepsilon_{\varphi k}$. On the contrary, when θ is small, i.e. $\theta(1 - s) < s$, $\varepsilon_{\varphi l}^H$ is positive, so that condition (i) requires a positive value for $\varepsilon_{\varphi l}$ bounded away from zero. Since from condition (ii) $\varepsilon_{\varphi l}$ must take a negative value bounded away from zero ($\varepsilon_{\varphi l} < s - 1 < 0$) we conclude that indeterminacy is not possible for arbitrarily small values of $\varepsilon_{\varphi k}$ and $\varepsilon_{\varphi l}$ when θ is small. For future reference we summarize this result in Corollary 1. Let us define

$$\epsilon_{\varphi} \equiv \varepsilon_{\varphi k} + \varepsilon_{\varphi l} \quad (10)$$

Then we can state:

Corollary 1

Assuming $\varepsilon_{\gamma} = 1$ and $\varepsilon_{\varphi k}$ arbitrarily small, indeterminacy can arise for any $\epsilon_{\varphi} > 0$ if and only if $\theta(1 - s) > s$.

3 The Woodford (1986) Model with Government Spending

In this section we apply the general framework developed above to the case of a Woodford (1986) model extended to allow for public spending financed by taxation. We consider a perfectly competitive monetary economy with discrete time $t = 1, 2, \dots, \infty$ and heterogeneous infinite lived agents. Indeed, following Woodford (1986), we assume that there are two types of agents, workers and capitalists. While workers and capitalists consume both the final good, only workers supply labour. Moreover, there is a financial market

imperfection that prevent workers from borrowing against their wage income and workers are more impatient than capitalists, i.e. they discount the future more than the latter. So, in a neighborhood of a monetary steady state, capitalists hold the whole capital stock and no money, whereas workers save their wage earnings through money balances. As before the final good is produced by firms under a Cobb-Douglas technology characterized by constant returns to scale. Finally, we introduce government policy in this framework, and assume that public spending, that may affect workers utility, is financed by labour and/or consumption taxes. The detailed description of the model is provided below. We further show that this economic environment is nothing more than a special case of the general framework presented before and, therefore, we apply our previous results summarized in Proposition 2 to the discussion of indeterminacy in this case.

3.1 The Government

The government chooses the tax policy and balances its budget at each period in time. Therefore, real public spending in goods and services in period t , $G_t \geq 0$ is given by:

$$G_t = \tau(\omega_t l_t) \omega_t l_t + \tau(c_t) c_t \quad (11)$$

where $\tau(\omega_t l_t)$ represents the labour tax rate determined as a function of aggregate labour income in the economy. Also $c_t = c_t^w + c_t^c$, where c_t^w denotes consumption of workers in period t , c_t^c denotes consumption of capitalists in period t and $\tau(c_t)$ represents the consumption tax rate determined as a function of aggregate consumption. We assume that:

$$\tau(\omega_t l_t) = \alpha_l \left(\frac{\omega_t l_t}{\omega l} \right)^{\phi_l} \quad (12)$$

$$\tau(c_t) = \alpha_c \left(\frac{c_t}{c} \right)^{\phi_c} \quad (13)$$

where ωl is the steady state value of the wage bill and c is the steady state level of total consumption. The parameter $0 \leq \alpha_l < 1$ determines the level of the tax rate on labour income at the steady state. Similarly $\alpha_c \geq 0$ determines the level of the consumption tax rate at the steady state. The parameters ϕ_j with $j = l, c$ denote the elasticities of the tax rates with respect to the tax bases. When $\phi_j < 0$ the tax rate decreases when the tax base expands. $\phi_j > 0$ corresponds to the cases where the tax rate increases

with the tax base, and for $\phi_j = 0$ the tax rate is constant. The specification considered for the fiscal policy rule given by (11), (12) and (13) is quite general, and nests most of the cases considered in the literature. Indeed it is similar to the specification considered by Guo and Lansing (1998). It also nests the case considered in Schmitt-Grohé and Uribe (1997), Pintus (2003) and Gokan (2005) where a constant amount of public expenditures is financed by taxes on labour income, so that. $\tau_t = G/\omega_t l_t$. Hence we recover their specification when $\phi_l = -1$ and $\alpha_c = 0$. Moreover, we admit as a particular case Giannitsarou (2005) for $\alpha_l = 0$ and $\phi_c = -1$, which means that constant public expenditures are only financed by consumption taxes.

3.2 Workers

We assume a continuum of identical workers of mass one. Preferences of each worker may be affected by public expenditures and are represented by the following utility function:

$$\sum_{t=1}^{\infty} \lambda^{t-1} \left[U \left(\frac{G_t^\eta c_t^w}{B} \right) - \lambda V(l_t) \right] \quad (14)$$

where l_t is labor supply, c_t^w is his consumption, and $\lambda \in (0, 1)$ is the discount factor. Public spending externalities in preferences are given by G^η where $\eta \geq 0$ represents the degree of these externalities. Moreover, we make the following assumptions on the utility functions U and V :

Assumption 1 *The functions $U(x)$ and $V(l)$ are continuous for all $x \geq 0$ and $0 \leq l \leq \tilde{l}$, where the labor endowment $\tilde{l} > 0$ may be finite or infinite. They are C^n for $x > 0$, $0 < l < \tilde{l}$ and n large enough, with $U'(x) > 0$, $U''(x) \leq 0$, $V'(l) > 0$ and $V''(l) \geq 0$. Moreover, $\lim_{l \rightarrow \tilde{l}} V'(l) = +\infty$ and consumption and leisure are gross substitutes, i.e. $-xU''(x)/U'(x) < 1$.*

In the following m_{t+1}^w and k_{t+1}^w denote respectively the money balances and the capital stock held by the representative worker at the end of period t , $\delta \in (0, 1)$ is the depreciation rate of capital, r_t the nominal interest rate, w_t the nominal wage, and p_t the price of the final good. At each period, a worker faces the two following constraints:

$$p_t c_t^w (1 + \tau(c_t)) + p_t (k_{t+1}^w - (1 - \delta)k_t^w) + m_{t+1}^w = m_t^w + r_t k_t^w + (1 - \tau(\omega_t l_t)) w_t l_t \quad (15)$$

$$p_t c_t^w (1 + \tau(c_t)) + p_t (k_{t+1}^w - (1 - \delta)k_t^w) \leq m_t^w + r_t k_t^w \quad (16)$$

where (15) represents the budget constraint and (16) the liquidity constraint. The representative worker maximizes his utility function (14) under the constraints (15) and (16). Workers know the policy rule followed by the government. However, since there is a continuum of agents, each worker, being atomistic, does not take into account the influence of its actions on aggregate variables. This means that workers take G_t , $\tau(c_t)$ and $\tau(\omega_t l_t)$ as given when solving their maximization problem. The equilibria considered here are such that:

$$G_t^\eta U' \left(\frac{G_t^\eta c_t^w}{B} \right) > \lambda G_{t+1}^\eta U' \left(\frac{G_{t+1}^\eta c_{t+1}^w}{B} \right) [(1 - \delta) + r_{t+1}/p_{t+1}] \quad (17)$$

$$(1 - \delta)p_{t+1} + r_{t+1} > p_t \quad (18)$$

Then, workers always choose $k_t^w = 0$ and the finance constraint is binding. Therefore, we obtain the following equations:

$$u \left(\frac{G_{t+1}^\eta c_{t+1}^w}{B} \right) = v(l_t) \quad (19)$$

$$m_{t+1}^w = (1 - \tau(\omega_t l_t)) w_t l_t \quad (20)$$

$$p_t c_t^w (1 + \tau(c_t)) = m_t^w \quad (21)$$

where $u(x) = xU'(x)$ and $v(z) = zV'(z)$. We can further notice that under Assumption 1, it exists a function $\gamma \equiv u^{-1} \circ v$, such that $G_{t+1}^\eta c_{t+1}^w / B = \gamma(l_t)$ and $\varepsilon_\gamma(l) \equiv \gamma'(l)l / \gamma(l) \geq 1$.

3.3 Capitalists

Capitalists behave like a representative agent who maximizes his lifetime utility function:

$$\sum_{t=1}^{\infty} \beta^t \ln c_t^c \quad (22)$$

where $\beta \in (\lambda, 1)$ is his discount factor and c_t^c his consumption.⁶ At period t , the representative agent faces the following budget constraint:

$$p_t c_t^c (1 + \tau(c_t)) + p_t (k_{t+1}^c - (1 - \delta)k_t^c) + m_{t+1}^c = m_t^c + r_t k_t^c \quad (23)$$

where m_{t+1}^c and k_{t+1}^c are respectively the money balances and the capital stock held at the end of period t by capitalists. Since we focus on equilibria satisfying (18), capitalists do not hold money ($m_t^c = 0$) because it has a lower return than capital. We obtain then, the following optimal solution:

$$c_t^c = \frac{(1 - \beta)R_t k_t}{1 + \tau(c_t)} \quad (24)$$

$$k_{t+1} = \beta R_t k_t \quad (25)$$

where $R_t \equiv 1 - \delta + r_t/p_t$ is the real gross return on capital.⁷

3.4 The Production Sector

As before we consider a constant returns to scale Cobb-Douglas technology and perfectly competitive input markets. Producers maximize profits so that equations (2) and (3) apply.

3.5 Equilibrium

Equilibrium on labor and capital markets requires $\omega_t = w_t/p_t$, $\rho_t = r_t/p_t$. Considering that $m > 0$ is the constant money supply, and using (20) and (21), equilibrium in the money market at each period implies the following:

$$m/p_t = (1 + \tau(c_t))c_t^w = (1 - \tau(\omega_t l_t))\omega_t l_t \quad (26)$$

Therefore, at equilibrium (19) and (25) become:

$$\omega_{t+1} l_{t+1} \frac{G_{t+1}^n (1 - \tau(\omega_{t+1} l_{t+1}))}{B(1 + \tau(c_{t+1}))} = \gamma(l_t) \quad (27)$$

⁶We do not introduce public expenditures externalities into capitalists preferences because, since they have a log-linear utility function, such externalities would not affect the dynamics.

⁷The superscript on k_t^c is dropped because as we have seen before, workers do not hold capital.

$$k_{t+1} = \beta [\rho_t + 1 - \delta] k_t \quad (28)$$

where ω_t and ρ_t satisfy respectively (2) and (3), G_t , $\tau(\omega_t l_t)$ and $\tau(c_t)$ are given respectively by (11), (12) and (13), and $c_t = c_t^w + c_t^c$, where c_t^w and c_t^c satisfy respectively (26) and (24).

Equations (27) and (28) determine the dynamics of this economy, through a two-dimensional dynamic system with one predetermined variable, the capital stock k_t . Comparing these two last equations with (4) and (5) we can see that the Woodford (1986) economy with public spending, financed by labour and consumption taxes, is a special case of the general framework studied in section 2, with $\mu = s$, $\delta < 1$ and

$$\varphi(k_{t+1}, l_{t+1}) = [G_{t+1}^\eta (1 - \tau(\omega_{t+1} l_{t+1}))] / (1 + \tau(c_{t+1})) \quad (29)$$

Therefore, to study indeterminacy in this economy we can merely apply Proposition 2. That is precisely what we do in the next section, where, for simplicity of exposition, we consider separately the cases of labour and consumption taxes.

Remark that the distortion function φ can be decomposed in two factors. The distortion due to government spending externalities is given by G^η , while $(1 - \tau(\omega l)) / (1 + \tau(c))$ represents the gap between real wages relevant to consumers and producers, due to taxation (the tax wedge). Of course the functional form of the distortion function $\varphi(k, l)$ and its variability depend on the chosen fiscal policy rule and its parameters. However, as we shall see, for the class of policy rules here considered, and in both the Woodford and OLG frameworks, we are able to summarize in one single parameter the influence of the policy rule on the distortion variability.

Finally, note that in the absence of government intervention, i.e. when $\varphi(k, l) = 1$, (27) and (28) describe the dynamics of the standard Woodford (1986) model studied in Grandmont et al. (1998).

4 Indeterminacy in the Woodford Model with Government Spending

In this section, we analyze indeterminacy in the Woodford model assuming that θ is small, i.e. $s - \theta(1 - s) > 0$. Note that this assumption is frequently used in the literature (see Cazzavillan et al. (1998)) and covers most

commonly used parameterizations for the Woodford model.⁸

4.1 Indeterminacy with Labour Taxation

In this section we discuss in detail the case where only labour taxation is used to finance public spending, i.e. where $\alpha_c = 0$ so that $\tau(c_t) = 0$, see (13). In this case (11) and (29) become

$$G_{t+1} = \tau(\omega_{t+1}l_{t+1})\omega_{t+1}l_{t+1} \quad (30)$$

$$\varphi(k_{t+1}, l_{t+1}) = G_{t+1}^\eta(1 - \tau(\omega_{t+1}l_{t+1})) \quad (31)$$

where $\tau(\omega l)$ satisfies (12). Therefore we obtain

$$\varepsilon_{\varphi l} = \left[\eta(1 + \phi_l) - \phi_l \frac{\alpha_l}{1 - \alpha_l} \right] (1 - s) \quad (32)$$

$$\varepsilon_{\varphi k} = \left[\eta(1 + \phi_l) - \phi_l \frac{\alpha_l}{1 - \alpha_l} \right] s \quad (33)$$

so that ε_φ (see (10)) is given by:⁹

$$\varepsilon_\varphi = \left[\eta(1 + \phi_l) - \phi_l \frac{\alpha_l}{1 - \alpha_l} \right] \quad (34)$$

Government policy affects the economy through the parameters η , ϕ_l and α_l : $\eta(1 + \phi_l)$ represents the variability of the public spending externality on preferences,¹⁰ while $\phi_l \frac{\alpha_l}{1 - \alpha_l}$ represents the variability of the tax wedge. These parameters only influence $\varepsilon_{\varphi l}$ and $\varepsilon_{\varphi k}$ through a single factor ε_φ . Hence ε_φ summarizes the variability of the distortion introduced by fiscal policy.

We now derive the indeterminacy conditions corresponding to the labour taxation case. In order to make the analysis comparable to the existing literature we set $\varepsilon_\gamma = 1$.¹¹ We summarize our results in Proposition 3 below.

⁸Indeed most quarterly parameterizations assume θ close to 0.025.

⁹Note that here ε_φ represents also the elasticity of the function φ with respect to the tax base (the wage bill), evaluated at the steady state.

¹⁰When $\phi_l = -1$ the variability of the public spending externality disappears because G is constant.

¹¹Note that this corresponds to the case of an infinitely elastic labour supply curve.

Proposition 3 *Indeterminacy under labour taxation and public expenditures externalities in the Woodford model:*

Assuming that $\theta(1 - s) < s$ and $\epsilon_\gamma = 1$, the steady state will be indeterminate if and only if:

- (i) for $\alpha_l < \eta/(1 + \eta)$ either $\phi_l > \phi_l^H$ or $\phi_l < \phi_l^F$
- (ii) for $\alpha_l > \eta/(1 + \eta)$ either $\phi_l < \phi_l^H$ or $\phi_l > \phi_l^F$

where:

$$\phi_l^H = \frac{(1 - \alpha_l) [s - \theta(1 - s) - \eta(1 - s)]}{(1 - s) [\eta - \alpha_l(1 + \eta)]}$$

$$\phi_l^F = \frac{-(1 - \alpha_l) \{2 [2 - s - \theta(1 - s)] + \eta(1 - s)(2 - \theta)\}}{(1 - s)(2 - \theta) [\eta - \alpha_l(1 + \eta)]}$$

Proof. Since this is a special case of the general one presented in section 2, Proposition 2 applies. Using expressions (32) to (34), it is easy to show that, for $\epsilon_\gamma = 1$, conditions (i) and (ii) of Proposition 2 can be written as $\epsilon_\varphi > \max \left\{ 0, \frac{[s - \theta(1 - s)]}{(1 - s)}, \frac{-2[2 - s - \theta(1 - s)]}{(2 - \theta)(1 - s)} \right\}$ and $\epsilon_\varphi < \min \left\{ -1, 0, \frac{-2[2 - s - \theta(1 - s)]}{(2 - \theta)(1 - s)} \right\}$. Since, for $\theta(1 - s) < s$, $\frac{[s - \theta(1 - s)]}{(1 - s)} = \max \left\{ 0, \frac{[s - \theta(1 - s)]}{(1 - s)}, \frac{-2[2 - s - \theta(1 - s)]}{(2 - \theta)(1 - s)} \right\}$ and $\frac{-2[2 - s - \theta(1 - s)]}{(2 - \theta)(1 - s)} = \min \left\{ -1, 0, \frac{-2[2 - s - \theta(1 - s)]}{(2 - \theta)(1 - s)} \right\}$, these two conditions become respectively $\epsilon_\varphi > \frac{[s - \theta(1 - s)]}{(1 - s)}$ and $\epsilon_\varphi < \frac{-2[2 - s - \theta(1 - s)]}{(2 - \theta)(1 - s)}$. Then, using expression (34) to substitute ϵ_φ in these two last inequalities, it is straightforward to obtain Proposition 3. ■

From this proof we can see that indeterminacy is more likely the higher the (absolute) value of ϵ_φ , i.e. the stronger the variability of the distortion.

4.1.1 Economic Intuition

In our framework the distortion introduced by government intervention is the mechanism responsible for the emergence of indeterminacy, which requires a sufficiently strong variability of the function φ . Indeed, in the absence of government spending financed by taxation, i.e. when $\epsilon_\varphi = 0$ so that $\varepsilon_{\varphi l} = 0$ and $\varepsilon_{\varphi k} = 0$, we recover the case considered in Grandmont et al. (1998), where the steady state is never indeterminate in the Cobb-Douglas case. In order to understand why, consider the following intuitive argument that, since local indeterminacy is closely related to the existence of deterministic

and/or stochastic fluctuations, is based on the existence of a mechanism generating cyclical equilibrium trajectories. Consider for instance that in period t , starting from the steady state, there is an instantaneous increase in the capital stock k_t . Agents anticipate that this implies an increase in k_{t+1} (see (28)), and therefore in the future wage bill. In the absence of public spending financed by labour taxation this expected increase in the future wage bill must be sustained by an increase in current employment, see (27), which in turn implies an increase in the current interest rate (see (3)), that reinforces the initial increase in k_{t+1} . This in turn implies a decrease in the rental rate of capital at $t + 1$. In order to obtain a cyclical trajectory a sufficiently important decrease in the rental rate of capital at $t + 1$ should be observed, so that k_{t+2} decreases. However, in the Cobb-Douglas case this can never happen so that indeterminacy never emerges.

Consider now the case where $\epsilon_\varphi > 0$, i.e. where we have tax rates that respond negatively to the tax base ($\phi < 0$) and/or public spending externalities ($\eta > 0$). See equation (34). Now, the anticipated increase in the future wage bill implies a stronger increase in current employment (see (27)), which reinforces all the events described above, leading, if ϵ_φ is sufficiently big, to a sufficiently fall in r_{t+1} , that reverses the trajectory of capital.

Assume now that $\epsilon_\varphi < 0$, i.e. taxes vary positively with the tax base ($\phi > 0$). Then, if ϵ_φ is sufficiently big in absolute value, the anticipated increase in the future wage bill may, in contrast to what happened in the two previous cases, lead to a decrease in current employment. See equation (27). This last effect has a negative impact on the current interest rate (see (3)), which reduces k_{t+1} restoring the path to equilibrium.

We can therefore understand why the emergence of indeterminacy requires sufficiently high (in absolute terms) values for ϵ_φ , i.e. a sufficiently elastic response of the function φ with respect to capital and labor.

4.1.2 Discussion of the Results

In this section we discuss and compare our indeterminacy results with previous papers that have addressed the same issue. To ease the discussion we have plotted, in figure 1, ϕ_t^H and ϕ_t^F as functions of $0 < \alpha_l < 1$ for different values of $\eta \geq 0$, assuming that $\theta(1 - s) < s$.

(insert figures 1a, 1b and 1c here)

The case of $\eta = 0$, where public spending does not affect workers' utility,

falls into case (ii) of Proposition 3 and is depicted in figure 1a. We can see that, in this case, indeterminacy is not possible with a constant tax rate ($\phi_l = 0$).¹² The same result was obtained in a Ramsey model by Schmitt-Grohé and Uribe (1997) and Guo and Harrison (2004). However, for any given level of α_l , a sufficiently (positively or negatively) elastic tax rule implies the existence of indeterminacy. In this respect our results are in contrast with those obtained in Guo and Lansing (1998) or Dromel and Pintus (2004). Indeed, in the Benhabib and Farmer (1994) framework, Guo and Lansing (1998) find that saddle path stability is more likely when the tax schedule becomes more progressive. More recently, Dromel and Pintus (2004), in a Woodford model without government spending externalities, also find that a sufficiently progressive tax rate on labor income promotes determinacy. Note that both Guo and Lansing (1998) and Dromel and Pintus (2004) restrict their analysis to the case of weak progressivity. In fact, since in their case agents take into account how the tax rate affects their earnings, they have to exclude parameter configurations where after tax income decreases with income, i.e., where $\epsilon_\varphi < -1$ or $\phi_l > (1 - \alpha_l)/\alpha_l (< \phi_l^F)$ in our notation. On the contrary, in our case, since agents take the tax rate as given, these parameter configurations are possible.

From figure 1a we can also see that, for a given level of $\phi_l \neq 0$, indeterminacy requires a sufficiently high level of α_l . In particular, when a constant level of G is assumed ($\phi_l = -1$), local indeterminacy emerges when $\alpha_l > [s - \theta(1 - s)] / [1 - \theta(1 - s)]$.¹³ This result is in accordance with previous works considering labor taxation and no public expenditures externalities. See for instance Schmitt-Grohé and Uribe (1997), Pintus (2003) and Gokan (2005) who assume a constant level of G and all find that indeterminacy requires a lower bound for the tax rate.

We can therefore conclude that for a given value of ϕ_l , a sufficiently low level of the tax rate stabilizes economic fluctuations, whereas for a given level of the tax rate, α_l , tax schedules sufficiently elastic destabilize the economy. This means that there is a trade-off between the values of these two parameters, ϕ_l , the tax response to the business cycle and α_l , the level of tax rate, needed for determinacy: the higher the elasticity of the tax schedule, the lower the tax rate needed to ensure determinacy of the steady state.

¹²Note that, when $\eta = 0$, the case of a constant tax rate is equivalent to the case where $\epsilon_{\varphi l} = \epsilon_{\varphi k} = 0$ (see (32) to (34)), so that, as shown before, indeterminacy is not possible.

¹³When $\phi_l = -1$, we obtain this same indeterminacy condition on α_l for any value of $\eta \geq 0$, because the effect of the public spending externality disappears when G is constant.

When $\eta > 0$ two different configurations are possible depending on whether $\eta \geq [s - \theta(1 - s)] / (1 - s)$. In figure 1b we represent the case where $\eta < [s - \theta(1 - s)] / (1 - s)$. We can see that, in this case, indeterminacy is still not possible with a constant tax rate ($\phi_l = 0$). However, for low values of α_l , a new type of configuration, that reverses the trade-off between α_l and ϕ_l needed for determinacy, is obtained: the higher ϕ_l , the higher the value of α_l required for determinacy.

Finally the case of higher public spending externalities, $\eta > [s - \theta(1 - s)] / (1 - s)$, is represented in figure 1c. We can see that, in this case, indeterminacy prevails for $\phi_l \geq 0$ and $\alpha_l < \eta / (1 + \eta)$, and for $\phi_l \leq 0$ and $\alpha_l > \eta / (1 + \eta)$. In particular, and in contrast to what happened in the other two previous cases, indeterminacy is now always obtained with a constant tax rate ($\phi_l = 0$).¹⁴ Cazzavillan (1996), in an optimal growth model with public spending externalities on preferences and production and a constant tax rate, also found that a minimum bound for the elasticity of the public spending externalities was required for indeterminacy to emerge.¹⁵ These results suggest that public expenditures externalities on preferences constitute an important channel for the emergence of indeterminacy.

The results here obtained show that fiscal policy rules may be responsible for indeterminacy, and thereby their design should take into account their possible destabilizing effects through this channel. This is an issue already emphasized in several other works, but our results further show that the way ϕ_l and α_l interact in order to create local indeterminacy strongly depends on the presence and strength of public spending externalities. Therefore, in order to design the 'correct' policy rule we must estimate the existing degree of public spending externalities in preferences, i.e. the value of η .

4.2 Indeterminacy with Consumption Taxes

We consider now that the government only uses consumption taxes, i.e. that $\alpha_l = 0$ so that $\tau(\omega_t l_t) = 0$ (see (12)). To simplify the analysis, we ignore

¹⁴Indeed, when $\phi_l = 0$, indeterminacy requires a minimum degree for public spending externalities in preferences, precisely $\eta > [s - \theta(1 - s)] / (1 - s)$. See Proposition 3.

¹⁵In his case this required minimum bound is higher than ours. Note, however, that the two models are considerably different. For related results in an OLG model, see Seegmuller (2003) and Utaka (2003).

public spending externalities in preferences, i.e. $\eta = 0$, so that (29) becomes:

$$\varphi(k_{t+1}, l_{t+1}) = \frac{1}{1 + \tau(c_{t+1})} \quad (35)$$

where $\tau(c_t)$ is given by (13), and using (2), (3), (24) and (26) c_t satisfies the following equation:

$$\begin{aligned} (1 + \tau(c_t))c_t &= \omega_t l_t + (1 - \beta)R_t k_t \\ &= k_t[(1 - \beta)(1 - \delta) + (1 - s\beta)k_t^{s-1}l_t^{1-s}] \end{aligned} \quad (36)$$

Accordingly we have that

$$\varepsilon_{\varphi l} = \frac{-\alpha_c \phi_c}{1 + \alpha_c(1 + \phi_c)} \left[\frac{(1 - s)(1 - \beta s)\theta}{(1 - s)\theta + (1 - \beta)s} \right] \quad (37)$$

$$\varepsilon_{\varphi k} = \frac{-\alpha_c \phi_c}{1 + \alpha_c(1 + \phi_c)} \left[1 - \frac{(1 - s)(1 - \beta s)\theta}{(1 - s)\theta + (1 - \beta)s} \right] \quad (38)$$

so that ϵ_φ (see (10)) is given by:¹⁶

$$\epsilon_\varphi = \frac{-\alpha_c \phi_c}{1 + \alpha_c(1 + \phi_c)} \quad (39)$$

As in the case of labour taxation ϵ_φ summarizes the variability of the distortion due to fiscal policy. Since $\eta = 0$, ϵ_φ has only a term which represents the variability of the tax wedge.

We now derive the indeterminacy conditions corresponding to the consumption taxation case that we summarize in Proposition 4 below. Again, in order to make the analysis comparable to the existing literature, we have set $\epsilon_\gamma = 1$.

Proposition 4 *Indeterminacy under consumption taxation in the Woodford model:*

¹⁶Note that here ϵ_φ represents also the elasticity of the function φ with respect to income used for consumption expenditures, evaluated at the steady state. One can also easily determine the elasticity of the distortion with respect to the tax base, which is equal to $-\alpha_c \phi_c / (1 + \alpha_c)$.

Assuming that $\theta(1-s) < s$ and that $\epsilon_\gamma = 1$, the steady state is indeterminate if and only if $\phi_c^F < \phi_c < \phi_c^H$ for $\phi_c \neq -(1 + \alpha_c)/\alpha_c$ where:¹⁷

$$\begin{aligned}\phi_c^H &= -\frac{v}{(1+v)} \frac{(1+\alpha_c)}{\alpha_c} < 0 \\ \phi_c^F &= -\frac{\rho}{(\rho-1)} \frac{(1+\alpha_c)}{\alpha_c} < 0 \\ \nu &= \frac{[s - \theta(1-s)] [(1-s)\theta + (1-\beta)s]}{(1-s)(1-\beta s)\theta} > 0 \\ \rho &= \frac{2[2-s-\theta(1-s)] [(1-s)\theta + (1-\beta)s]}{\theta(1-s)[2-s-(1-s)\theta-\beta s]} > 1\end{aligned}$$

Proof. The model of this section is a special case of the general one presented in section 2, so that Proposition 2 applies. Using expressions (37) to (39), it is easy to show that, for $\epsilon_\gamma = 1$, conditions (i) and (ii) of Proposition 2 become respectively $\epsilon_\varphi > \max\{0, \nu, -\rho\}$ and $\epsilon_\varphi < \min\left\{-\frac{[(1-s)\theta+(1-\beta)s]}{(1-\beta s)\theta}, 0, -\rho\right\}$. Since for $\theta(1-s) < s$ $\nu = \max\{0, \nu, -\rho\}$ and $-\rho = \min\left\{-\frac{[(1-s)\theta+(1-\beta)s]}{(1-\beta s)\theta}, 0, -\rho\right\}$, those conditions become respectively $\epsilon_\varphi > \nu$ and $\epsilon_\varphi < -\rho$. Then, substituting expression (39) in these last two inequalities, it is straightforward to obtain Proposition 4. ■

From the proof of Proposition 4 we can immediately see that, as in the case of labour taxation, indeterminacy requires a sufficiently strong (in absolute value) variability of the function φ . Indeed, with both types of taxation the mechanisms operating are the same, so that both the requirements and the intuition for the emergence of indeterminacy are similar. Therefore, the economic intuition provided in the previous section also applies in this case. There remains however a difference worth emphasizing. In the case of labour taxation, and ignoring for the sake of simplicity public spending externalities, positive (negative) values for ϵ_φ are always associated with negative (positive) values for ϕ , whereas in the case of consumption taxes this is no longer true. See equation (39).¹⁸ This explains why the indeterminacy re-

¹⁷Note that for $\phi_c = -(1 + \alpha_c)/\alpha_c$ steady state consumption is not well defined (see (36)).

¹⁸Indeed a positive ϵ_φ implies that $-(1 + \alpha_c)/\alpha_c < \phi_c < 0$, and a negative ϵ_φ means

sults expressed in terms of the tax rate and its variability are so different in the cases of labor and consumption taxes.

(insert figure 2 here)

In figure 2 we have represented the indeterminacy region (shaded area) in the plane (α_c, ϕ_c) , by plotting the functions $-\frac{1+\alpha_c}{\alpha_c}$, ϕ_c^F , and ϕ_c^H as functions of α_c considering, as before, $s > \theta(1-s)$. We can see that indeterminacy with consumption taxes is only possible for $\phi_c < 0$, so that a consumption tax rate that is increasing in the tax base implies local determinacy. Also, when the tax rate is constant ($\phi_c = 0$) the steady state is never locally indeterminate, as in the case of labor taxation without government spending externalities.¹⁹ In the context of a Ramsey model, Giannitsarou (2005) found that indeterminacy is not possible when a fixed stream of government spending is financed by consumption taxes. On the contrary, in our set up, indeterminacy is possible when government spending is constant. Indeed, this case is here recovered assuming that $\phi_c = -1$, and, from figure 2, we can see that in this case, and as in the case of labour taxation, indeterminacy emerges when α_c is sufficiently high ($\alpha_c > \nu$).

The sharp difference between our results and those of Giannitsarou (2005) shows that the effects of policy rules on indeterminacy crucially depend on the finance market imperfections existing in the economy.

5 The OLG Model with Government Spending

In this section, we show that, when the technology is Cobb-Douglas, the dynamics of an overlapping generations economy with public spending financed through capital or consumption taxation are still described by equations (4) and (5) with $\mu = 1-s$, $\delta = 1$ and $\beta = 1$, so that our general results on indeterminacy presented in Proposition 2 apply in the particular case where $\theta \equiv 1 - \beta(1 - \delta) = 1$.

that either $\phi_c < -(1 + \alpha_c)/\alpha_c$ or $\phi_c > 0$. However, when $\phi_c > 0$, $\epsilon_\varphi > -1$, so that ϵ_φ is never below $-\rho < -1$. Therefore indeterminacy never emerges when $\phi_c > 0$. See figure 2.

¹⁹Indeed when $\phi_c = 0$ we have that $\epsilon_{\varphi l} = \epsilon_{\varphi k} = 0$, so that indeterminacy does not emerge.

We consider an OLG model with two-periods lived consumers that supply labor in the first period, save through capital, and consume only in the second period of their life (c_{t+1}). As before we introduce variable public expenditures, now financed by capital income and/or consumption taxation according to the following rule:

$$G_t = \tau(\rho_t k_t) \rho_t k_t + \tau(c_t) c_t \quad (40)$$

with $\tau(c_t)$ given by (13,) and $\tau(\rho_t k_t)$ given by:

$$\tau(\rho_t k_t) = \alpha_k \left(\frac{\rho_t k_t}{\rho k} \right)^{\phi_k} \quad (41)$$

where ρk represents capital income at the steady state, $\alpha_k \in [0, 1)$ is the steady state capital income tax rate, $\phi_k > 0 (< 0)$ means that the tax rate increases (decreases) with the tax base, whereas $\phi_k = 0$ corresponds to a constant tax rate.

Consumers' preferences are represented by the following utility function:

$$U(G_{t+1}^\eta c_{t+1}/B) - V(l_t) \quad (42)$$

where the functions $U(x)$ and $V(l)$ satisfy Assumption 1. Assuming that capital totally depreciates after one period of use ($\delta = 1$), each consumer, taking the tax policy of the government, $\tau(\rho_t k_t)$ and $\tau(c_t)$, and the public good externality as given, maximizes his utility under the two budget constraints:

$$k_{t+1} = \omega_t l_t \quad (43)$$

$$(1 + \tau(c_{t+1})) c_{t+1} = (1 - \tau(\rho_{t+1} k_{t+1})) \rho_{t+1} k_{t+1} \quad (44)$$

The solution to this problem is given by (43) and:

$$\frac{G_{t+1}^\eta (1 - \tau(\rho_{t+1} k_{t+1})) \rho_{t+1} k_{t+1}}{(1 + \tau(c_{t+1})) B} = \gamma(l_t) \quad (45)$$

where c_{t+1} satisfies (44).²⁰

²⁰Note that the OLG model presented above is equivalent to an infinite horizon model where consumption is financially constrained by capital income. Indeed if agents maximize $\sum_{t=1}^{\infty} \beta^{t-1} [U(G_t^\eta c_t/B) - \beta V(l_t)]$ subject to the usual resources constraint $(1 + \tau(c_t)) c_t + k_{t+1} = \omega_t l_t + (1 - \tau(\rho_t k_t)) \rho_t k_t$ and additionally to the following finance constraint $(1 + \tau(c_t)) c_t \leq (1 - \tau(\rho_t k_t)) \rho_t k_t$, the solution is still given by (43) and (45), if the finance constraint is binding. This provides an additional explanation for the segmented asset markets interpretation of the OLG model. For further details see Seegmuller (2005).

Using (1), (2), (3), (13), (40), (41) and (44), we see that (43) and (45) describe a dynamic system which is a particular case of (4)-(5) with $\mu = 1 - s$, $\delta = 1$, $\beta = 1$, and $\varphi(k_{t+1}, l_{t+1})$ given by:

$$\varphi(k_{t+1}, l_{t+1}) = \frac{G_{t+1}^\eta (1 - \tau(\rho_{t+1} k_{t+1}))}{1 + \tau(c_{t+1})} \quad (46)$$

As a consequence, the general conditions for indeterminacy summarized in Proposition 2, considering $\theta \equiv 1 - \beta(1 - \delta) = 1$, apply to the overlapping generations model with capital and/or consumption taxation, and public spending externalities. Recall that the same general indeterminacy conditions also apply in the Woodford model. The only difference concerns the parameter θ that in the Woodford framework is assumed to be sufficiently small, while it takes the value one in the OLG framework considered.

As in the Woodford (1986) case the distortion function φ (see (46)) can be decomposed in two factors: G^η the distortion due to government spending externalities and $(1 - \tau(\rho k))/(1 + \tau(c))$ the tax wedge between the real rental rates relevant to consumers and producers.

Finally, note that in the absence of government intervention, i.e. when $\varphi(k, l) = 1$, (43) and (45) describe the dynamics of a standard OLG model as in Reichlin (1986) and Cazzavillan (2001).

6 Indeterminacy in OLG Economies with Public Spending

For simplicity of exposition, we will consider separately the cases of capital and consumption taxes.

6.1 Indeterminacy with capital taxation

In this section we discuss the case where only capital income taxation is used to finance public expenditures in an OLG economy, i.e. where $\alpha_c = 0$ so that $\tau(c_t) = 0$. (see (13)). In this case, (46) becomes $\varphi(k_{t+1}, l_{t+1}) = G_{t+1}^\eta (1 - \tau(\rho_{t+1} k_{t+1}))$ so that the elasticities of $\varphi(k, l)$ with respect to capital

and labor are given by:

$$\varepsilon_{\varphi l} = \left[\eta(1 + \phi_k) - \phi_k \frac{\alpha_k}{1 - \alpha_k} \right] (1 - s) \quad (47)$$

$$\varepsilon_{\varphi k} = \left[\eta(1 + \phi_k) - \phi_k \frac{\alpha_k}{1 - \alpha_k} \right] s \quad (48)$$

so that ε_{φ} (see (10)) is given by:²¹

$$\varepsilon_{\varphi} \equiv \left[\eta(1 + \phi_k) - \phi_k \frac{\alpha_k}{1 - \alpha_k} \right] \quad (49)$$

As in the Woodford model ε_{φ} summarizes the variability of the distortion function φ due to government intervention.²²

Using Proposition 2 for $\theta = 1$, and assuming as before that $\varepsilon_{\gamma} = 1$, the conditions for local indeterminacy in the overlapping generations model with capital income taxation, a balanced-budget rule and public spending externalities can be summarized as follows:

Proposition 5 *Indeterminacy under capital taxation and public expenditures externalities in the OLG model ($\theta = 1$):*

Assuming that $s < 1/2$ and $\varepsilon_{\gamma} = 1$, the steady state will be locally indeterminate if and only if:

- (i) for $\alpha_k < \eta/(1 + \eta)$ either $\phi_k > \phi_k^T$ or $\phi_k < \phi_k^F$
 - (ii) for $\alpha_k > \eta/(1 + \eta)$ either $\phi_k < \phi_k^T$ or $\phi_k > \phi_k^F$
- where:

$$\phi_k^T = \frac{-\eta(1 - \alpha_k)}{[\eta - \alpha_k(1 + \eta)]}$$

$$\phi_k^F = -\frac{(1 - \alpha_k)[2 + \eta(1 - s)]}{(1 - s)[\eta - \alpha_k(1 + \eta)]}$$

²¹Note that here ε_{φ} represents also the elasticity of the function φ with respect to the tax base (capital income), evaluated at the steady state.

²²Since the technology is Cobb-Douglas and all markets are competitive, using (2) and (3), we obtain $\rho(k, l)k = \frac{s}{1-s}w(k, l)l$. Therefore we have that $\varphi(k_{t+1}, l_{t+1}) = G_{t+1}^{\eta}(1 - \tau(\rho_{t+1}k_{t+1})) = G_{t+1}^{\eta}(1 - \tau(\frac{s}{1-s}w_{t+1}l_{t+1}))$. Hence expressions (47) to (49) are identical to expressions (32) to (34) with α_l and ϕ_l replaced respectively by α_k and ϕ_k .

Proof. Applying the results established in Proposition 2 for $\theta = 1$ and $\epsilon_\gamma = 1$, and using expressions (47) to (49), we immediately have that the steady state is locally indeterminate if one of the two following conditions is satisfied: $\epsilon_\varphi > \max\{0, -\frac{1-2s}{1-s}, -\frac{2}{1-s}\}$ and $\epsilon_\varphi < \min\{-1, 0, -\frac{2}{1-s}\}$. These conditions can be rewritten as $\epsilon_\varphi > 0$ and $\epsilon_\varphi < -\frac{2}{1-s}$ since, when $s < 1/2$, we have that $0 = \max\{0, -\frac{1-2s}{1-s}, -\frac{2}{1-s}\}$ and $-\frac{2}{1-s} = \min\{-1, 0, -\frac{2}{1-s}\}$. Then, using (49) to substitute for ϵ_φ in these two inequalities, Proposition 5 follows. ■

From the proof of Proposition 5 we can see that the indeterminacy mechanism is the same as in the Woodford model, i.e. the emergence of indeterminacy depends on the variability of the function φ . Therefore, a economic intuition similar to the one described in section 4.1 applies in the OLG case. However, in the Woodford framework, a sufficiently strong, either positive or negative ϵ_φ was required for indeterminacy. In the OLG case indeterminacy still emerges for a sufficiently negative ϵ_φ , but now, in contrast to the Woodford model, it also emerges for any positive ϵ_φ , and therefore, in particular for arbitrarily small (positive) values for $\epsilon_{\varphi k}$ and $\epsilon_{\varphi l}$. This difference is due to the different values for the depreciation rate, and therefore for θ , considered in these two frameworks. This last result constitutes a direct application of Corollary 1.

A similar difference between the Woodford and OLG economies is obtained when the distortion comes from externalities in the productive process. In fact Cazzavillan (2001), shows that indeterminacy can occur in an OLG model with an arbitrarily small level of (capital) externalities in the productive process, provided the elasticity of labor supply becomes infinite. On the contrary, in the Woodford framework a positive minimum level of (labour) externalities is required for indeterminacy. See Barinci and Chéron (2001) and Lloyd-Braga and Modesto (2004) for more details.

To ease the discussion of the results in terms of α_k and ϕ_k we have represented in figure 3 the indeterminacy region, by plotting the functions ϕ_k^T and ϕ_k^F as functions of $0 < \alpha_k < 1$, for different values of $\eta \geq 0$, considering $s < 1/2$.

(insert figures 3a and 3b here)

The case where $\eta = 0$, i.e., where there are no public spending externalities in consumption, falls into case (ii) of Proposition 5 and is represented

in figure 3a. We can see that, in this case indeterminacy is possible when the capital tax rate is increasing in the tax base, provided its elasticity is sufficiently high ($\phi_k > \phi_k^F > 0$). Moreover, indeterminacy always emerges when the tax rate decreases with the tax base ($\phi_k < 0$), no matter what the strength of this effect. In particular, whatever the level of the tax rate, indeterminacy is possible for an arbitrarily small degree of $\phi_k < 0$, i.e. for an almost constant capital tax rate.²³ However, with a constant tax rate ($\phi_k = 0$) indeterminacy is not possible.

In the presence of public spending externalities a new result emerges: an arbitrarily small value for η is enough to guarantee that indeterminacy becomes pervasive with $\phi_k = 0$.²⁴ Indeed, as soon as we allow for the presence of positive effects of public spending on agent's utility ($\eta > 0$) the indeterminacy region is of the type depicted in figure 3b.

Comparing figures 1 and 3, we see that the indeterminacy results here obtained for capital taxation in a OLG model are similar to those obtained with labor taxation in the Woodford framework.²⁵ The only difference is the following. In the OLG economy with capital income taxes both an arbitrarily small (negative) ϕ_k when η is zero, and an arbitrarily small (positive) η when ϕ_k is zero, are sufficient conditions for indeterminacy to emerge. On the contrary, in the Woodford framework with labor income taxation, indeterminacy requires either a value of ϕ_l bounded away from zero when $\eta = 0$, or a value $\eta > 0$ bounded away from zero when $\phi_l = 0$. This difference is due to the different values of the depreciation rate assumed in the two models, δ small in the Woodford model and $\delta = 1$ in the OLG framework.

6.2 Indeterminacy with consumption taxes

In this section we discuss the case where only consumption taxes are used to finance public spending in an OLG economy and, for simplicity, we do not consider public spending externalities i.e. $\alpha_k = \eta = 0$. In this case, (46) becomes $\varphi(k_{t+1}, l_{t+1}) = 1/(1 + \tau(c_{t+1}))$, where c_{t+1} must satisfy (44), so that

²³Note that, when $\eta = 0$, the condition $\epsilon_\varphi > 0$ is equivalent to $\phi_k < 0$. See equation (49).

²⁴Note that, when $\phi_k = 0$, the condition $\epsilon_\varphi > 0$ is equivalent to $\eta > 0$. See (49) again.

²⁵This conclusion can be related to Dromel and Pintus (2004) who also remark that capital taxation in OLG models and labor taxation in the Woodford framework have similar effects for the occurrence of indeterminacy. In fact, their remark is not so surprising since, as we have shown, both cases are particular cases of the same general unified framework.

the elasticities of $\varphi(k, l)$ with respect to capital and labor are given by the following expressions:

$$\varepsilon_{\varphi l} = \left[\frac{-\alpha_c \phi_c}{1 + \alpha_c(1 + \phi_c)} \right] (1 - s) \quad (50)$$

$$\varepsilon_{\varphi k} = \left[\frac{-\alpha_c \phi_c}{1 + \alpha_c(1 + \phi_c)} \right] s \quad (51)$$

so that ϵ_φ (see (10)) is given by:²⁶

$$\epsilon_\varphi = \frac{-\alpha_c \phi_c}{1 + \alpha_c(1 + \phi_c)} \quad (52)$$

As before ϵ_φ summarizes the variability of the distortion due to fiscal policy.²⁷

Assuming as before that $\epsilon_\gamma = 1$, and using (50)-(52) and Proposition 2, the conditions for local indeterminacy in the overlapping generations model with consumption taxation and a balanced-budget rule can be summarized as follows:

Proposition 6 *Indeterminacy under consumption taxation in the OLG model ($\theta = 1$):*

Assuming that $s < 1/2$ and $\epsilon_\gamma = 1$, and defining $\phi_{c,olg}^F \equiv -\frac{2}{(1+s)} \frac{(1+\alpha_c)}{\alpha_c}$, the steady state is indeterminate if and only if $\phi_{c,olg}^F < \phi_c < 0$ for $\phi_c \neq -(1 + \alpha_c)/\alpha_c$.²⁸

Proof. Applying the results established in Proposition 2 for $\theta = 1$ and $\epsilon_\gamma = 1$, and using expressions (50) to (52), we immediately have that the steady state is locally indeterminate if one of the two following conditions is satisfied: $\epsilon_\varphi > \max\{0, -\frac{1-2s}{1-s}, -\frac{2}{1-s}\}$ and $\epsilon_\varphi < \min\{-1, 0, -\frac{2}{1-s}\}$. These

²⁶Note that here ϵ_φ represents also the elasticity of the function φ with respect to income used for consumption expenditures, evaluated at the steady state.

²⁷Note that expressions (50) and (51) are identical to the ones obtained in the case of the Woodford model with consumption taxes, (37) and (38), with $\beta = 1$ and $\delta = 1$ i.e. $\theta = 1$. Indeed in the Woodford model, when $\beta = 1$, capitalists do not consume so that consumption in the economy is given by $(1 + \tau(c))c = \omega(k, l)l$ (see (26)). In the OLG framework consumption is given by $(1 + \tau(c))c = \rho(k, l)k$ (see (44)). Since we have that $\rho(k, l)k = \frac{s}{1-s}\omega(k, l)l$, we can easily see that in both cases the expressions for the elasticities must be identical when $\theta = 1$.

²⁸Note that for $\phi_c = -(1 + \alpha_c)/\alpha_c$ steady state consumption is not well defined (see (44))

conditions can be rewritten as $\epsilon_\varphi > 0$ and $\epsilon_\varphi < -\frac{2}{1-s}$ since, when $s < 1/2$, we have that $0 = \max\{0, -\frac{1-2s}{1-s}, -\frac{2}{1-s}\}$ and $-\frac{2}{1-s} = \min\{-1, 0, -\frac{2}{1-s}\}$. Then, using (52) to substitute for ϵ_φ in these two inequalities, Proposition 6 follows. ■

Once more, from the proof of Proposition 6, we can see that the indeterminacy conditions can be expressed in terms of the single parameter, ϵ_φ , so that the same indeterminacy mechanism applies.

To ease the discussion of our results in terms of the policy parameters, we have represented in figure 4 the indeterminacy region in the plane (α_c, ϕ_c) , by plotting $\phi_{c,olg}^F$ as a function of α_c . We can see that in an OLG economy with public expenditures financed by consumption taxes, and similarly to what we obtained in the Woodford framework, indeterminacy is not possible when the elasticity of the consumption tax schedule is positive or too negative. Indeed, as we have seen, the indeterminacy mechanism is the same in the Woodford and OLG frameworks. However, in an OLG economy and in contrast to what happens in the Woodford framework, indeterminacy always prevails for an almost constant consumption tax rate, provided it responds negatively to the tax base.²⁹ Moreover, when in an OLG economy, a constant flow of government spending is financed by consumption taxes ($\phi_c = -1$) the steady state is *always* indeterminate.³⁰ This result, together with our finding of section 4.2 imply that, in segmented market economies, in this case, consumption taxes promote indeterminacy, in contrast to what happens in non-segmented asset markets economies.³¹

(insert figure 4 here)

7 Concluding Remarks

In this paper we studied the role of public spending externalities and variable tax rates on local indeterminacy. As it is well-known, indeterminacy of equilibria is associated with the occurrence of fluctuations driven by self

²⁹This constitutes a direct application of Corollary 1.

³⁰Indeed, from Proposition 6 and figure 4 we can easily see that, assuming that $\epsilon_\gamma = 1$ and $s < 1/2$, for $-\frac{2}{(1+s)} < \phi_c < 0$, (provided $\phi_c \neq -(1+\alpha_c)/\alpha_c$) the steady state is always indeterminate.

³¹See again Giannitsarou (2005).

fulfilling prophecies (sunspots). Therefore, as it has already been stressed in the literature, fiscal policy rules that promote indeterminacy may have destabilizing effects on the economy. In this paper we contribute to this debate by showing that the destabilizing effects of different fiscal rules can not be correctly assessed without taking into consideration the possible existence (and degree) of public spending externalities and/or the existence of financial markets imperfections. Indeed we have shown that the same policy rule can have completely different effects on stability depending on the impact of public spending on consumers' welfare. Moreover, we have established that the presence of imperfect financial markets can clearly reverse the (de)stabilizing outcomes of some fiscal policy rules (see for instance our discussion on consumption taxes).

References

- [1] Barinci, J.P. and A. Chéron (2001), "Sunspots and the Business Cycle in a Finance Constrained Economy", *Journal of Economic Theory*, 97, 30-49.
- [2] Benhabib, J. and R. Farmer, (1994), "Indeterminacy and Increasing Returns", *Journal of Economic Theory*, 63, 19-41.
- [3] Cazzavillan, G. (1996), "Public Spending, Endogenous Growth, and Endogenous Fluctuations", *Journal of Economic Theory*, 71, 394-415.
- [4] Cazzavillan, G. (2001), "Indeterminacy and Endogenous Fluctuations with Arbitrarily Small Externalities", *Journal of Economic Theory*, 101, 133-157.
- [5] Cazzavillan, G., T. Lloyd-Braga and P. Pintus, (1998), "Multiple Steady States and Endogenous Fluctuations with Increasing Returns to Scale in Production", *Journal of Economic Theory*, 80, 60-107.
- [6] Dromel, N. and P. Pintus (2004), "Progressive Income Taxes as Built-in Stabilizers", Working Paper GREQAM, Aix-Marseille.
- [7] Giannitsarou, C. (2005), "Balanced Budget Rules and Aggregate Instability: The Role of Consumption Taxes", miméo.

- [8] Gokan, Y. (2005), "Dynamic Effects of Government Expenditure in a Finance Constrained Economy", *Journal of Economic Theory*, forthcoming.
- [9] Grandmont, J. M., P. Pintus and R. de Vilder (1998), "Capital-Labour Substitution and Competitive Nonlinear Endogenous Business Cycles", *Journal of Economic Theory*, 80, 14-59.
- [10] Guo, J.T. and S. G. Harrison (2004), "Balanced-Budget Rules and Macroeconomic (In)stability", *Journal of Economic Theory*, 119, 357-363.
- [11] Guo, J.T. and K. Lansing, (1998), "Indeterminacy and Stabilization Policy", *Journal of Economic Theory*, 82,481-490.
- [12] Lloyd-Braga, T. and L. Modesto (2004), "Indeterminacy in a Finance Constrained Unionized Economy", CEPR, DP4679, <http://www.cepr.org/pubs/dps/DP4679.asp>.
- [13] Pintus, P. (2003), "Aggregate Instability in the Fixed-Cost Approach to Public Spending", mimeo, Aix-Marseille.
- [14] Reichlin, P. (1986), "Equilibrium Cycles in an Overlapping Generations Economy with Production", *Journal of Economic Theory*, 40, 89-102.
- [15] Schmitt-Grohé, S. and M. Uribe (1997), "Balanced- Budget Rules, Distortionary Taxes, and Aggregate Instability", *Journal of Political Economy*, 105, 976-1000.
- [16] Seegmuller, T. (2003), "Endogenous Fluctuations and Public Services in a Simple OLG economy", *Economics Bulletin*, Vol. 5, No. 10, 1-7.
- [17] Seegmuller, T. (2005), "Steady State Analysis and Endogenous Fluctuations in a Finance Constrained Model", Cahier de la MSE, Université Paris 1.
- [18] Utaka, A. (2003), "Income Tax and Endogenous Business Cycles", *Journal of Public Economic Theory*, 5, 135-145.
- [19] Woodford, M. (1986), "Stationary Sunspot Equilibria in a Finance Constrained Economy", *Journal of Economic Theory*, 40, 128-137.

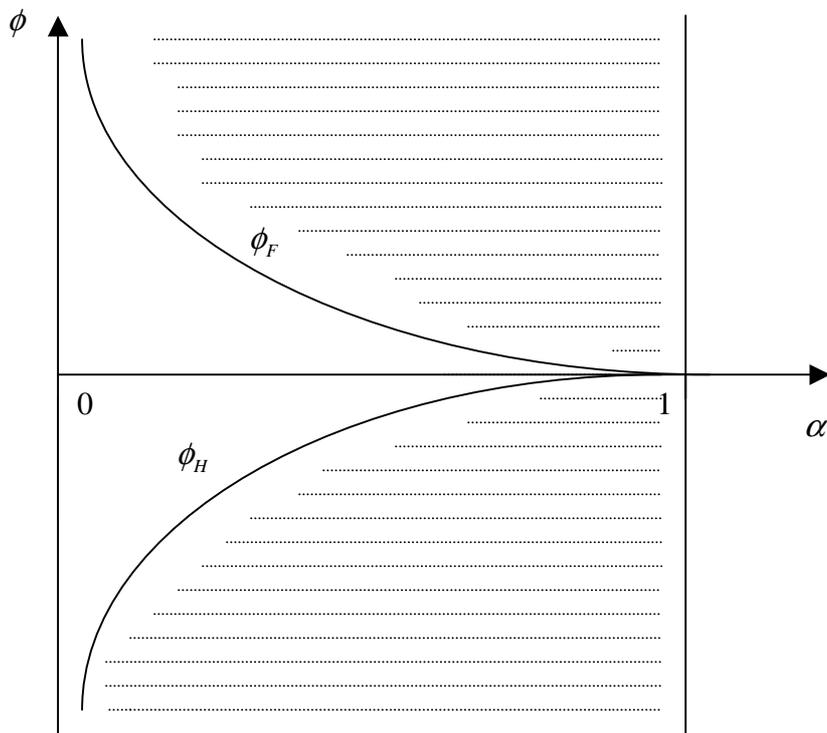


Figure 1a
 Labor income taxation in the Woodford model ($\theta(1-s) < s$)
 $\eta = 0$

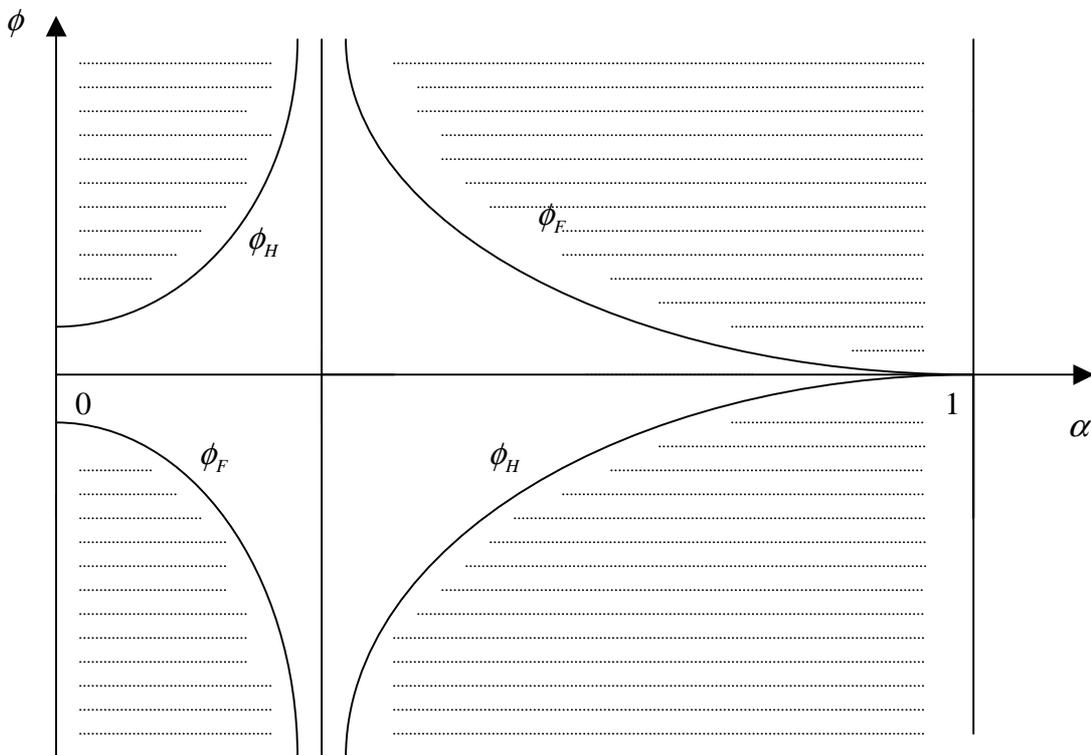


Figure 1b

Labor income taxation in the Woodford model ($\theta(1-s) < s$)

$$0 < \eta < \frac{s - \theta(1-s)}{(1-s)}$$

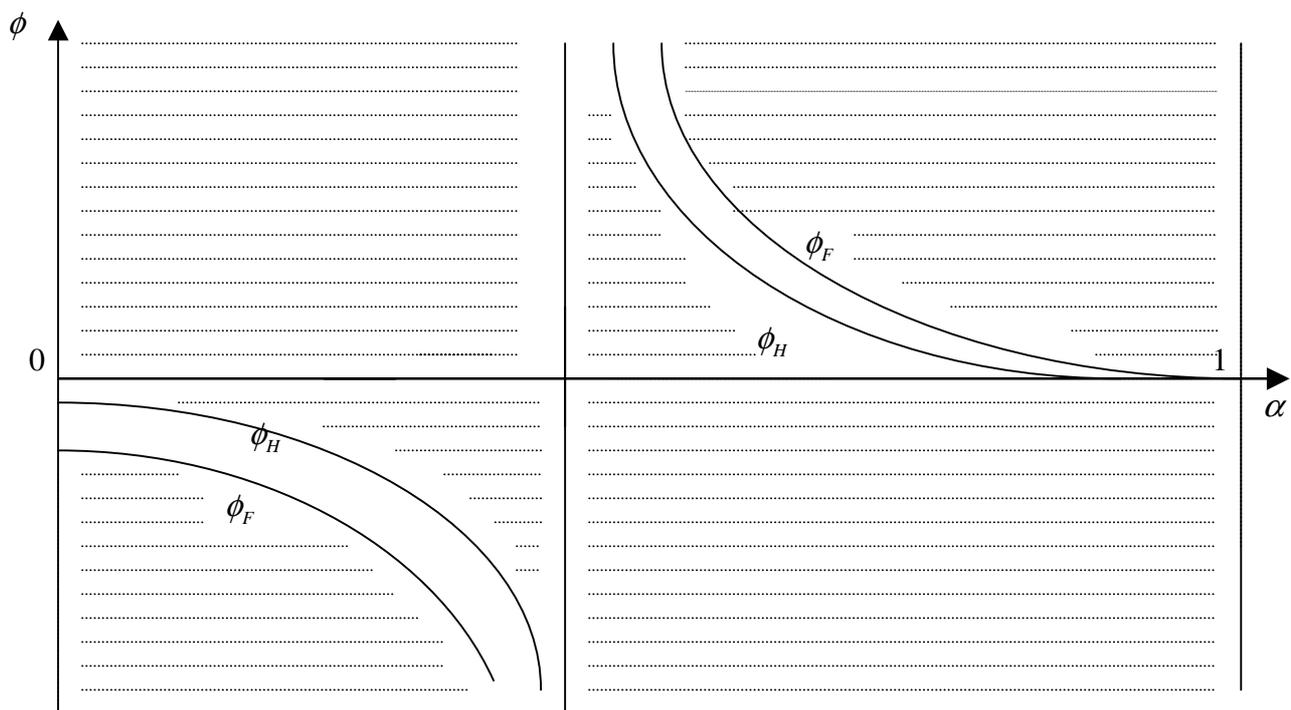


Figure 1c
 Labor income taxation in the Woodford model ($\theta(1-s) < s$)

$$\eta > \frac{s - \theta(1-s)}{1-s}$$

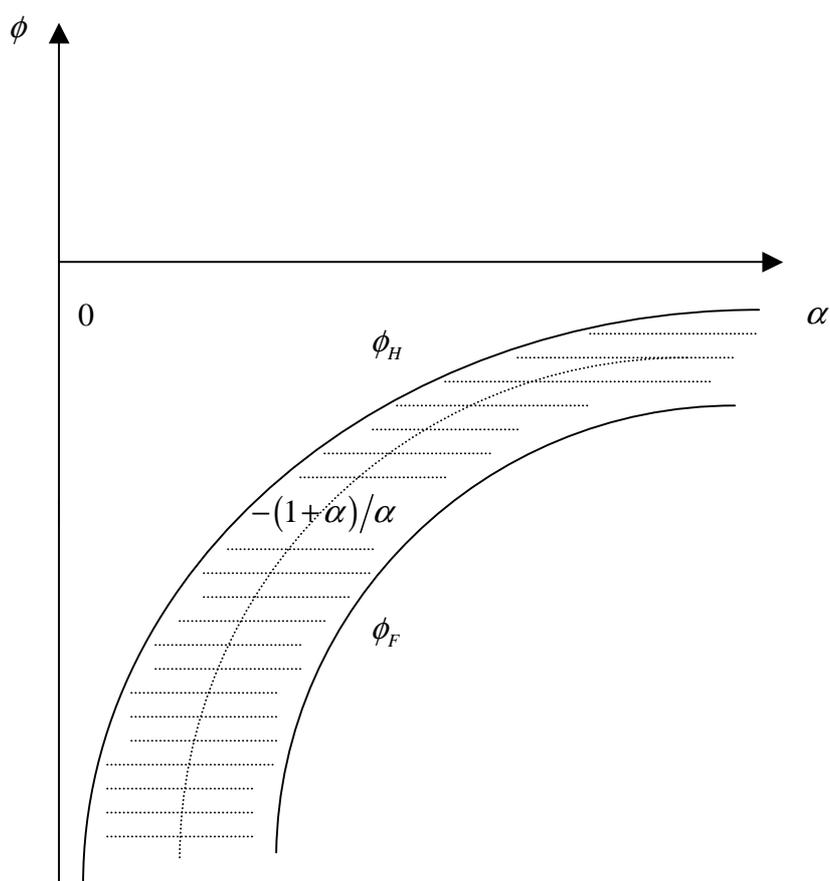


Figure 2
 Consumption taxation in the Woodford model, $\theta(1-s) < s$, $\eta = 0$

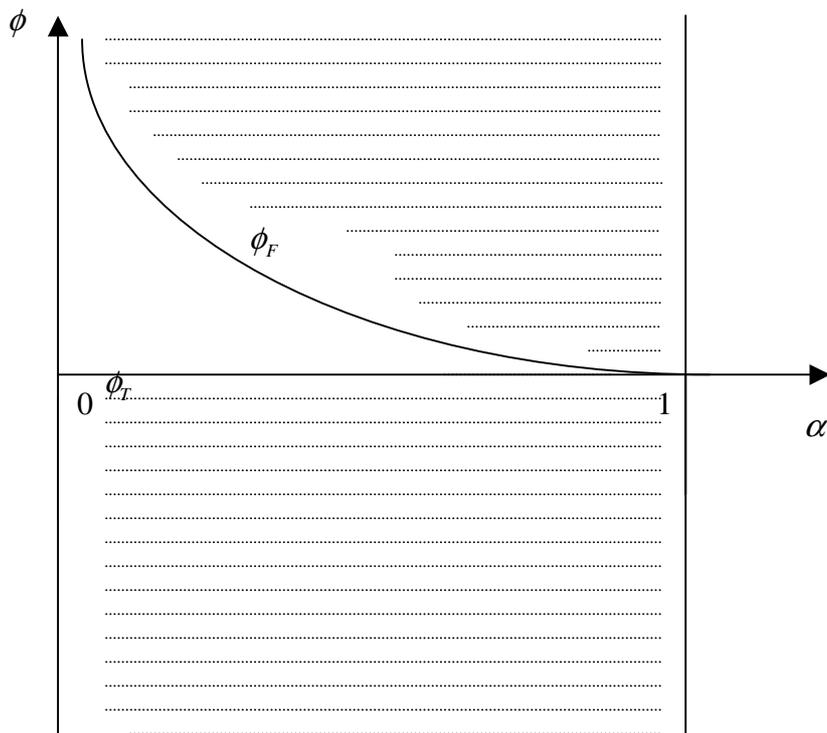


Figure 3a
 Capital income taxation in the OLG model, $\theta = 1$
 $\eta = 0$

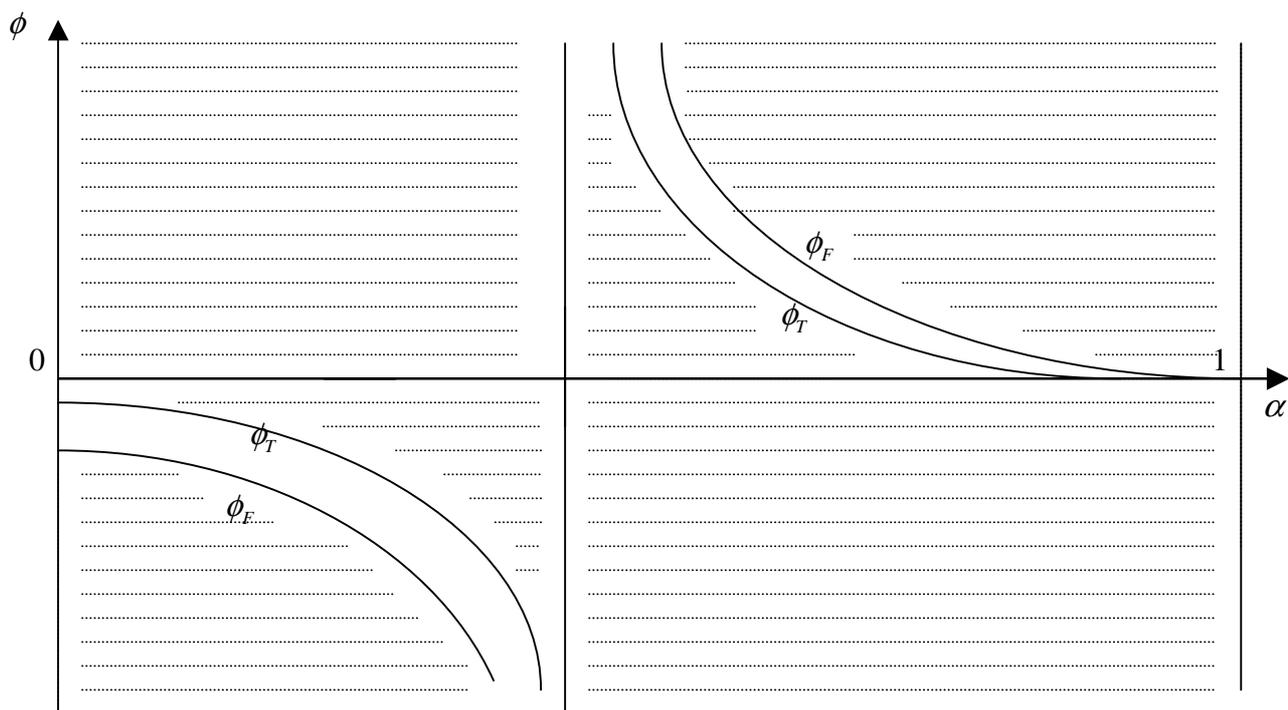


Figure 3b
 Capital income taxation in the OLG model ($\theta = 1$)
 $\eta > 0$

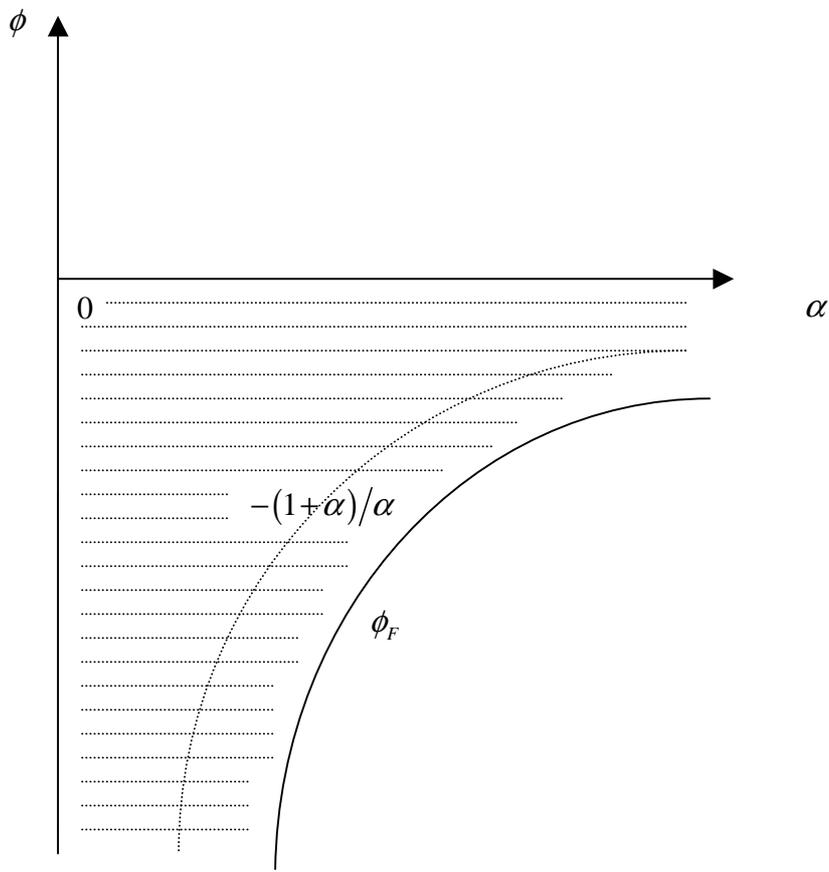


Figure 4
 Consumption taxation in the OLG model ($\theta = 1$)
 $\eta = 0$