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ABSTRACT

The Comparative Statics of Collusion Models*

We develop and illustrate a methodology for obtaining robust comparative statics results for collusion models in markets with differentiated goods by analyzing the homogeneous goods limit of these models. This analysis reveals that the impact of parameter changes on the incentives to deviate from collusion and the punishment profits are often of different order of magnitude yielding comparative statics results that are robust to the functional form of the demand system. We demonstrate with numerical calculations that these limiting results predict the global comparative statics at any degree of product differentiation. We use this methodology to demonstrate the non-robustness of Nash reversion equilibria and to develop new results in the comparative statics of collusion.

JEL Classification: D43, L13 and L41
Keywords: collusion, comparative statics, cross-ownership, differentiated products and robustness

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1 Introduction

Competition and merger policy often draw heavily on the conclusions of collusion theory to determine what environments increase the likelihood and severity of collusion. Such assessments have become even more important with the increased attention given recently to the assessment of coordinated effects of mergers both in Europe and the US.[1] For policy purposes, there is a need to determine specific characteristics of a market that make it particularly at risk from collusive behavior. For this reason, there has been a strong policy interest in answering questions like: Should collusion be seen as more important a threat in homogeneous goods markets than in differentiated goods markets? Is cross-ownership between competitors a reason to be more worried about collusion? What other factors should be considered as important to assess whether collusion may be an important phenomenon in a specific market?

To answer such policy questions, theory has to provide results on the comparative statics of collusive equilibria in response to changes in exogenous parameters like product differentiation or cross-ownership structure. However, such an exercise must confront the fact that collusion theory always generates multiple equilibria. A meaningful way to deal with multiple equilibria is to study the comparative statics for the whole equilibrium value set. Such analysis can yield a clear sense in which collusion gets easier or more difficult: collusion becomes easier when, after a parameter change, the equilibrium value set strictly encompasses the equilibrium value set before the change. Such "nesting" results for the equilibrium value set have been provided in the literature with respect to the discount factor (Abreu, Pearce, Stacchetti 1990) and the degree of uncertainty in imperfect monitoring models (see Kandori 1992).

However, this approach has severe limitations. First, many of the most interesting comparative statics exercises do not lead to nested equilibrium value sets (see Kühn 2004). Second, it is impossible to even talk about the comparative statics in price because only the whole set of outcomes is studied. We can overcome these problems by explicitly modelling how firms select a specific equilibrium from the equilibrium value set (see Harrington 1991). We can think of this rule as closing the collusive model by explicitly allowing colluding firms to bargain over the equilibrium that should be played. We show that, under weak assumptions on the bargaining rule and the profit functions, the comparative statics in values and prices are independent of the specific bargaining rule in symmetric models.[2]

Obtaining analytical solutions in this framework is nonetheless a formidable task. As a result, researchers have often restricted themselves to specific equilibria or specific sets of equilibria in the hope that the comparative statics obtained from such an analysis would be driven by the same general effects that would affect the whole equilibrium value set. The most popular approach has been to analyze equilibria that could be generated by a return to a one-shot Nash equilibrium forever. More sophisticated approaches have analyzed the set of equilibria generated when all firms use the same strategy on and off the equilibrium path ("symmetric optimal punishment equilibria"). Furthermore, researchers have often made highly restrictive assumptions on functional forms, most often assuming linear demand.

[1] See Kühn (2005a) for references to the recent policy debate on coordinated effects in mergers in Europe and the US.

[2] Numerical analysis of asymmetric models also shows that rules that satisfy our assumption yield qualitatively similar comparative statics.

With so many specific arbitrary assumptions, it is difficult to determine how robust any given set of theoretical results about the comparative statics of collusion in the literature is. Are results robust to the specification of the functional form of demand? Is the choice of the specific equilibrium set studied for determining comparative statics important for the qualitative features of comparative statics? In this paper we attempt to give a systematic answer to these questions by analyzing collusion in a general differentiated products oligopoly model. We focus the analysis in the first part of the paper on the impact of a change in the degree of product differentiation on collusion. We then show how the techniques developed in that analysis can be applied to more general comparative statics exercises.

Often it is difficult to characterize the comparative statics of collusion because changing a parameter may increase the short run temptation to cheat on a collusive agreement but make the worst available punishment more severe. This leads to countervailing effects for comparative statics which have led to the conclusion in the literature that comparative statics are ambiguous or non-robust. In this paper, we suggest an analysis that allows us to compare how robust different comparative statics results are for a very general general class of models. The idea is to analyze the comparative statics of collusion in a model with product differentiation close to the limit of perfect homogeneity. We show that often the effects of a parameter change on the incentives to deviate from collusion and the severity of punishment are of different order of magnitude. This allows us to conclude which effect should generally be expected to dominate. We confirm with numerical methods that, for the usual product differentiation models used, the limiting analysis predicts the comparative statics for the whole range of our product differentiation parameter. There is therefore a clear sense in which models with unambiguous comparative statics in the limit yield more robust insights than those that have non-robust comparative statics in the limit.

Contrary to the earlier literature that has focused on linear demand, our limit analysis allows us to obtain results that are robust to specific functional forms. Our analysis also allows us to identify rigorously under what assumptions specific modelling short cuts used in the literature lead to robust comparative statics results in the sense just defined. For example, we show that the comparative statics of Nash punishment equilibria may vary greatly depending on functional form and parameterization, even when symmetric optimal punishment equilibria generate robust results. We show that the robustness of results under symmetric optimal punishment equilibria arises from the nature of optimal punishments. Hence, the comparative statics on the set of symmetric optimal punishment equilibria will generate qualitatively the same results as comparative statics on the full equilibrium set, as long as the model is symmetric. This provides a more rigorous justification for the use of symmetric optimal punishment equilibria in such settings.

In Section 3, we derive all of the above results for the comparative statics of collusive equilibria in a product differentiation parameter. The analysis of the impact of product differentiation on collusion is of some interest in the industrial organization literature in its own right. Part of the general folklore of competition policy holds that collusion is considered easier in homogeneous goods markets than in markets with differentiated goods. The degree of product "heterogeneity" appears, for example, on the European Commission checklist used to determine whether coordinated effects of mergers are likely to arise (see Kühn 2001). But it is not obvious that this claim of practitioners is supported by collusion theory. First, the effect of increased product differentiation must be that the incentives to deviate from a collusive price are reduced because any given price cut wins over less market share. This is an effect that works in the opposite direction of the policy view. However, there should also be some effect on the punishment side. As goods are more differentiated, firms have more market power and therefore will be more reluctant to participate in severe punishments. This suggests a general trade-off between the impact of product differentiation on the incentives to deviate from collusive agreements and its impact on the intensity of punishments that can be imposed.

In the theoretical literature, results about the impact of product differentiation on collusion are ambiguous. Deneckere (1983) shows in the context of a duopoly model with linear demand that there may be theoretical support for the prevailing policy view: He demonstrates that, starting from perfect homogeneity, the minimal discount factor that can support collusion at monopoly prices is initially increasing for the best Nash reversion equilibrium. However, Lambertini and Sasaki (1999) have shown that the critical discount factor for sustaining the monopoly price is monotonically decreasing in the degree of product differentiation in the best symmetric optimal punishment equilibrium. Our analysis explains why the differences between these studies arise and show that robust results on this issue are available under optimal punishment schemes.

In Section 4, we give three examples of how the framework developed in section 3 can be extended to obtain general robust comparative statics results in other parameters. This analysis generates new results in the comparative statics of collusion. The first example analyzes the consolidation of differentiated products in the hands of less firms in a symmetric version of the product line model of Kühn (2004). Standard reasoning would suggest that there may be countervailing effects from such consolidation: while firms have less incentives to deviate from a collusive price when they earn a higher proportion of varieties in the market, they also have less incentives to use severe punishments. The analysis shows that the former effect is of larger order of magnitude. More fragmentation hinders collusive activity.[3] The second example concerns another issue for which the literature has traditionally pointed to countervailing effects (see Malueg 1992): the existence of cross-ownership shares in competing firms. We show that cross-ownership shares robustly facilitate collusion since, for optimal punishment equilibria, there exists no trade-off for low degrees of product differentiation. The incentives to deviate from collusion are reduced and the punishment profits are increased at the same time. Furthermore, numerical analysis shows that the impact of small cross-shareholdings on the collusive price can be substantial. This suggests that cross-shareholdings should be an important factor in assessing the coordinated effects of mergers. From a policy point of view, it is also important to note that the use of equilibria generated from Nash reversion would have severly underestimated the collusive impact of such cross ownership. The third comparative static we analyze concerns an increase in the marginal cost for all firms. Antitrust investigations of gasoline retailing are often triggered when input price shocks lead to price increases that are more than one to one. We show that collusion is indeed facilitated when marginal costs rise, but the analysis does not support that evidence of prices rising more than one-to-one is evidence of collusion.

In section 5 we discuss the complications that arise for the analysis of comparative statics in asymmetric models. We show how a combination of theoretical analysis and numerical calculation using the Abreu-Pearce-Stacchetti (1990) algorithm can yield important insights in such models. As a first example we study cost asymmetries. We show not only that collusion is made harder, but that for fairly homogeneous goods all prices may fall dramatically as the marginal costs of one of the firm in a duopoly are increased. A second example demonstrates that not all increases in asymmetry undermine collusion. We show that increases in cross-ownership typically facilitate collusion even when such increases

[3] This is not a trivial result. For example, in a homogeneous goods market with capacity constraints, Kühn (2005) shows that fragmentation of a given amount of capacity among more firms facilitates collusion.

exacerbate the asymmetries between firms. We show in this example that our method of analyzing incentive constraints at the homogeneous goods limits still allows one to prove some analytical results on comparative statics but that the comparative statics on prices need to be determined numerically. As such this demonstrates how a combination of analytical and numerical methods can lead to deeper insights into the comparative statics of collusion even in asymmetric models. Section 6 concludes the paper.

2 A General Model of Collusion in Differentiated Products Markets

2.1 The Model

We analyze a general oligopoly model with differentiated goods in which the degree of product differentiation is well defined and captured by a one dimensional parameter $\phi \in [0,1]$. At $\phi = 0$ products are homogeneous. Product differentiation increases in ϕ. The model is sufficiently general to encompass all of the popular single parameter symmetric product differentiation models in the literature. These include the linear demand system of Shubik (1980), CES preferences as in Spence (1976) or Dixit and Stiglitz (1977), the symmetric nested logit model with an outside option (see Anderson, De Palma, and Thisse 1992), and the Salop (1979) circular road model after appropriate normalization.[4]

The demand system is symmetric. Demand for product i is given by the demand function $D(p_i, \mathbf{p}_{-i}, \phi, n)$, where p_i is own price, $\mathbf{p}_{-i}$ is the vector of prices charged for other products, and n is the number of products in the market.[5] Demand is strictly decreasing and strictly log-concave in p_i, and $\lim_{p_i \to \infty} D(p_i, \mathbf{p}_{-i}, \phi, n) = 0$. For every price vector $\mathbf{p}$ for which the demands for all products are strictly positive, demand is differentiable, $\left|\frac{\partial D(p_i,\mathbf{p}_{-i},\phi,n)}{\partial p_i}\right| > \sum_{j \neq i} \frac{\partial D(p_i,\mathbf{p}_{-i},\phi,n)}{\partial p_j} > 0$, and $\left|\frac{\partial^2 \ln D(p_i,\mathbf{p}_{-i},\phi,n)}{\partial p_i^2}\right| > \sum_{j \neq i} \frac{\partial^2 \ln D(p_i,\mathbf{p}_{-i},\phi,n)}{\partial p_i \partial p_j} > 0$ for all $\phi \in (0,1)$. The latter assumptions guarantee that goods are imperfect substitutes, prices are strategic complements, and cross-price effects dominate own price effects in demand.

In order to attribute our results to changes in product differentiation, we need to impose some more structure on the demand system. First, we ensure that market size effects do not confound the impact of product differentiation. Second, we need to assume that the residual demand function generated by the model converges to the residual demand function faced by firms selling homogeneous products as the degree of product differentiation goes to zero.

To eliminate market size effects, we adopt a normalization for the demand functions that guarantees that the degree of product differentiation does not affect monopoly profits.[6] At equal prices $p_j = p$ for all

[4] We show explicitly in Appendix A that CES preferences, the nested logit model, and a Hotelling duopoly model satisfy all the assumptions.

[5] The symmetry assumption does not exclude the possibility of different degrees of substitution between different goods. For example, symmetric spatial models are covered by our analysis. In that case, the second element in the price vector of the demand function refers to the left neighbor, the third to the right neighbor, etc. As long as these patterns are symmetric as in the Salop circular road model such a model will fit our assumptions.

[6] In the language of Shaked and Sutton (1990) we adopt a normalization that eliminates any market expansion effect of product differentiation.

$j = 1, ..., n$, $D(p_i, \mathbf{p}_{-i}, \phi, n) = \frac{1}{n}\bar{D}(p)$, where $\bar{D}(p)$ is independent of ϕ and n. $\bar{D}(p)$ can be interpreted as the aggregate demand function and has elasticity $\varepsilon(p)$.Given symmetry in demand and cost function, this assumption guarantees that the competitive and the industry profit maximizing allocations do not vary with the degree of product differentiation or the number of firms in the market. The monopoly price p^m is independent of ϕ and n. The product differentiation parameter only affects the intensity of preferences of customers for their preferred brands. Consumers become less price sensitive if there is more product differentiation, so that larger price cuts are needed to attract the same number of customers. This is captured by the assumption $\frac{\partial D(p_i,\mathbf{p}_{-i},\phi,n)}{\partial \phi} > 0$ (< 0) if $p_i < \min_{j\neq i}\ p_j$ $(p_i > \max_{j\neq i}\ p_j)$. For given prices a firm charging the lowest price will lose demand to the firms with higher prices when product differentiation increases.

To have a consistent model that has the homogeneous goods model as a proper limit for $\phi \to 0$, we also need the demand functions for individual goods to converge pointwise to the residual demand function of the homogeneous goods model. At prices strictly above any rival price, demand must converge to zero. At a price strictly below the price of rivals, a firm must face the aggregate demand function in the limit. If several firms charge the lowest price, the demand is shared equally between the firms. More formally, if $p_i \geq \min_{j\neq i} p_j$, then $\lim_{\phi\to 0} \frac{\partial D(p_i,\mathbf{p}_{-i},\phi,n)}{\partial p_i} = -\infty$, demand converges to zero if the inequality is strict and to $\frac{1}{m}\bar{D}(p)$ otherwise. (m is the number of firms charging $\min_{j\neq i} p_j$.) If $p_i < p_j$ for all $j \neq i$, then $\lim_{\phi\to 0} D(p_i, \mathbf{p}_{-i}, \phi, n) = \bar{D}(p_i)$ and $\lim_{\phi\to 0} \frac{\partial D(p_i,\mathbf{p}_{-i},\phi,n)}{\partial p_i} \frac{p_i}{D(p_i,\mathbf{p}_{-i},\phi,n)} = \varepsilon(p) > -\infty$. We also want to guarantee three more properties that are important for the consistency of the differentiated goods model and the homogeneous goods limiting model. To state them formally, let $\mathbf{p}(\phi)$ be some function that maps the degree of product differentiation into a price vector. Then we assume:

Assumption 1: If $\lim_{\phi\to 0} D(p_i(\phi), \mathbf{p}_{-i}(\phi), \phi, n) = \bar{D}(\lim_{\phi\to 0} p_i(\phi))$, then

(a) $\lim_{\phi\to 0} \frac{\partial D(p_i(\phi),\mathbf{p}_{-i}(\phi),\phi,n)}{\partial \phi} < 0$,

(b) $\lim_{\phi\to 0} \left[\frac{\partial D(p_i(\phi),\mathbf{p}_{-i}(\phi),\phi,n)}{\partial p_i} + \sum_{j\neq i} \frac{\partial D(p_i(\phi),\mathbf{p}_{-i}(\phi),\phi,n)}{\partial p_j}\right] = \bar{D}'(\lim_{\phi\to 0} p_i(\phi))$, and

(c) $\lim_{\phi\to 0} \frac{\partial^2 D(p_i(\phi),\mathbf{p}_{-i}(\phi),\phi,n)/\partial p_i\partial\phi - \sum_{j\neq i} \partial^2 D(p_i(\phi),\mathbf{p}_{-i}(\phi),\phi,n)/\partial p_j\partial\phi}{\partial D(p_i(\phi),\mathbf{p}_{-i}(\phi),\phi,n)/\partial p_i} > 0$.

Note that along the path of $\mathbf{p}(\phi)$, a firm must be strictly undercutting all other firms in the neighborhood of $\phi = 0$ in order to obtain all of the demand. Assumption 1 (a) then says that any firms that strictly undercuts all other firms but is not strictly undercut by others in the homogeneous goods limit will strictly lose demand when some product differentiation is introduced. Assumption 1 (b) asserts that at the limit the demand from the undercutting firms must increase with the slope of aggregate demand when *all* prices are increased by the same amount. Finally, Assumption 1(c) says that an increase in product differentiation at the homogeneous goods limit strictly increases the elasticity of demand of a strictly undercutting firm and more so than the cross-elasticity of demand is increased. Essentially this says that some product differentiation will make some customers stop purchasing because their ideal product is not available. These three properties are satisfied in our examples in Appendix A.[7]

[7]Note that our allow for demands with a finite choke off price for demand that depends on the prices charged by other firms. In this case there will be, for every price vector $\mathbf{p}_{-i}$, and every ϕ, some price $\hat{p}(\mathbf{p}_{-i}, \phi)$, such that $D(p_i, \mathbf{p}_{-i}, \phi, n) = 0$ for all $p_i \geq \hat{p}(\mathbf{p}_{-i}, \phi)$. The price $\hat{p}$ is increasing in the prices of competitors and decreasing in ϕ. To achieve convergence to the homogeneous goods residual demand function, we assume that $\lim_{\phi\to 0} \hat{p}(\mathbf{p}_{-i}, \phi) = \min_{j\neq i} p_j$. Note that the fact that another product may have zero output at some price vector induces a kink in the demand function of rivals at $\hat{p}(\mathbf{p}_{-i}, \phi)$.

In addition, we need some assumptions to obtain well defined comparative statics results for best response prices and Nash equilibrium prices, including assumptions for the homogeneous goods (or large n) limit: (i) The elasticity of demand at any Nash equilibrium decreases with product differentiation and increases in the number of firms: $\frac{\partial^2 \ln D(p,p,\phi,n)}{\partial p_i \partial \phi} > 0$ and $\frac{\partial^2 \ln D(p,p,\phi,n)}{\partial p_i \partial n} < 0$. (ii) To guarantee well defined comparative statics of the Nash equilibrium price in ϕ at the homogeneous goods limit we assume: $\lim_{\phi \to 0} \left| \frac{\frac{d}{dp} D_{p_i}(p,p,\phi,n)}{D_{p_i}(p,p,\phi,n)} p \right| < B_i < \infty$, $\lim_{\phi \to 0} \left| \frac{\frac{d}{dp} \sum_{j \neq i} D_{p_j}(p,p,\phi,n)}{\sum_{j \neq i} D_{p_j}(p,p,\phi,n)} p \right| < B_j < \infty$ and that for $p_i \geq \min_{j \neq i} p_j$, $\frac{\partial D(p,p,\phi,n)}{\partial p_i}$ has an asymptotic expansion around $\phi = 0$ with a leading term that is a polynomial. For every $\phi \in (0,1)$ there exists $B(\phi) < \infty$ such that $\left| \frac{\partial D(p_i, \mathbf{p}_{-i}, \phi, n)}{\partial p_i} \right| < B(\phi)$ for all n, $\mathbf{p}$.

Every product can be produced at constant marginal cost $c > 0$. The payoffs in period t from product i are given by $\pi_{it}(p_{it}, \mathbf{p}_{-it}, \phi) = (p_{it} - c) D(p_{it}, \mathbf{p}_{-it}, \phi, n)$. Future profits are discounted at rate $\delta \in (0,1)$. There are n firms. A firm i produces only variety i. Firms play an infinite horizon game. In each time period, a publicly observable signal σ with continuous distribution on $[0,1]$ is realized.[8] Then all firms simultaneously set their price p_i and receive the profits obtained in that period. Let P be the (one-dimensional) set of non-negative prices. A history of the game at time t is then defined as $H_t \in P^{n \times t-1} \times [0,1]^t$. A pure strategy $\widetilde{p}_i$ for firm i, is a sequence of functions $\{p_{it}\}_{t=0}^{\infty}$, where $p_{it} : P^{n \times t-1} \times [0,1]^t \to P$ is a mapping from t-period histories to a price p_i. Denote the set of all such strategies by $\mathbb{P}$.

We analyze the subgame perfect equilibria of this game. Let $v_i^*(\widetilde{\mathbf{p}}^*, \phi)$ denote an average equilibrium payoff for some equilibrium strategy profile $\widetilde{\mathbf{p}}^*$ and $\mathbf{v}^*(\widetilde{\mathbf{p}}^*, \phi) \in \mathsf{R}^n$ a vector of such payoffs.[9] The set $\mathbf{V}^*(\phi)$ contains all such vectors and is called the "equilibrium value set". Associated with the equilibrium value set is a set $\mathfrak{P}^*(\phi) \subset \mathsf{R}^n$ of all price vectors that can be sustained in some period of some equilibrium of the game. Given our assumptions, a Nash equilibrium of the one shot game exists, so that $\mathbf{V}^*(\phi)$ and $\mathfrak{P}^*(\phi)$ are not empty:

Proposition 1 *For every (ϕ, n), there exists a unique Nash equilibrium of the one shot game with price vector $\mathbf{p}^N(\phi, n)$. It is symmetric. Nash equilibrium prices increase in ϕ, decrease in n, and $\lim_{\phi \to 0} p_i^N(\phi, n) = c$ for all i.*

Proof. See Appendix B. ■

The proposition follows directly from the assumptions by standard arguments. Besides establishing the existence of a subgame perfect equilibrium in our game, it also establishes some properties of the

In particular, at $(\hat{p}(\mathbf{p}_{-i}, \phi), \mathbf{p}_{-i})$, a competitor will face a strictly less elastic residual demand function when lowering the price than when raising the price. This is the only non-differentiability that we allow in our model.ular non-differentiability. Our Hotelling example in section 3.2 has this property. However, to simplify the proofs, we ignore this case for most of our arguments on the comparative statics of collusion. All proofs can be extended to cover this case.

[8] The public signal is a purely technical device that allows firms to correlate their strategies and thereby serves to convexify the equilibrium value set. This assumption is not necessary at all, when the most severe punishment can be implemented through a low price in the first period of punishments and a return to thew Pareto frontier of the equilibrium value set after wards. The assumption does nothing more than convexify the equilibrium value set, which permits the application of the Abreu, Pearce, Stacchetti (1990) algorithm for computing equilibrium value sets in infinitely repeated games.

[9] "Average payoffs" here refer to the present value of profits in the equilibrium, multiplied by $1 - \delta$. This convention is standard in the theory of repeated games and is convenient to keep mathematical expressions simple.

Nash equilibrium prices that are useful for the later analysis: In the one-shot Nash equilibrium, an increase in product differentiation must increase prices, while an increase in the number of competitors decreases prices. The Nash equilibrium prices converge to marginal cost as $\phi \to 0$. This follows directly from the fact that the residual demand functions converge to the homogeneous goods limit. Note, that proposition 1 does not include a convergence result for large numbers of firms. The reason is that we allow for models of true monopolistic competition for which price remains above marginal cost even as $n \to \infty$.[10]

2.2 The Comparative Statics of Collusion

In this paper we are interested in predicting changes in collusive profits and prices as the environment changes. However, this poses a serious conceptual problem because the set of equilibria is always large. There is, of course, no sense in which it is meaningful to study the comparative statics of a specific equilibrium. Instead, rigorous work on comparative statics in collusion models has focused on the comparative statics of the whole equilibrium value set in, for example, the discount factor (see Abreu, Pearce, Stacchetti 1990) or the degree of uncertainty (see Kandori 1992). These studies can often make a clear claim that collusion "gets easier" when the equilibrium value sets are nested as the parameter changes. However, this does not allow for a prediction for the comparative statics of profits and prices but only predictions for the sets $V^*(\phi)$ and $\mathfrak{P}^*(\phi)$. Furthermore, value sets will often not even be nested when a relevant parameter changes in some of the most interesting applications of collusion theory (see Kühn 2004). This problem arises fundamentally because the collusive model is not closed. It is silent about the way that firms coordinate on the collusive equilibrium. To close the collusion model we specifically model how firms determine the equilibrium they play. We assume that firms bargain over the possible set of incentive compatible strategy profiles in the spirit of Harrington (1991). Given a fixed bargaining rule, the comparative statics of equilibrium profits in any parameter that is changed are well defined.

More formally let a bargaining rule be represented by a function $B(\mathbf{V}^*(\phi)) : \mathcal{V} \to \mathbb{P}^n$, where $\mathcal{V}$ is the set of compact, convex subsets of R^n. The bargaining rule maps from equilibrium value sets to a unique equilibrium strategy profile.[11] We make the following assumption on B:

Assumption 2: *For any set value set V, and any $\pi \in V$, $B(V)$ satisfies the following properties:*

(a) It selects a strategy profile that generates a Pareto optimal outcome in V

(b) If V is symmetric, it selects a strategy profile with a symmetric outcome in V

Clearly, the Nash bargaining solution over the equilibrium value set falls into the class of solutions considered.[12] But there are many others that are feasible like the Kalai egalitarian solution or the Kalai-

[10] Of the one-parameter product differentiation models cited earlier, only the Salop circular road model has a competitive limit. The CES, the nested logit, and the linear demand models are examples that converge to monopolistic competition. There are many utility functions that do give rise to competitive limits. Kühn and Vives (1998) give examples in the Spence-Dixit-Stiglitz framework. However, there is no widely used single parameter model in this class other than the CES formulation.

[11] We may allow this mapping to depend on the static Nash equilibrium profits $\pi^N(\phi)$ to model a threat point. We suppress this potential dependence to save on notation.

[12] There can, of course, be several equilibria that support the outcome selected by the standard bargaining rules. We

Smorodinski bargaining solution. We could also use rules that are based on the ownership of assets. For example a concept often invoked in real collusive agreement is that the equilibrium values should be proportional to the ratio of capacities (see Vasconcelos 2004). Note that the precise bargaining solution will not matter in our symmetric model of product differentiation because any bargaining solution in this class will select the Pareto efficient symmetric outcome in the equilibrium value set. We call the equilibrium selected by the bargaining solution the "best collusive equilibrium".

For practical purposes of comparative statics in prices, this selection procedure still poses potential problems. The reason is that, in general, strategies supporting the selected value on the equilibrium value set need not be stationary on the equilibrium path. To derive the change of non-stationary strategies due to changes in parameters of the model appears an impossible task either analytically or numerically. For this reason we specialize our selection rule further, by requiring firms to only select equilibria which are stationary on the equilibrium path. Let $\mathbf{V}^s(\phi)$ be the set of all equilibrium values that can be generated from an equilibrium strategy, which is stationary on the equilibrium path. Note that these are equilibria that may be supported by the most severe punishments from the full equilibrium set. Our comparative statics analysis will focus on $B(\mathbf{V}^s(\phi))$, which we will call the "best stationary collusive equilibrium". We will show below that for symmetric collusion games $B(\mathbf{V}^s(\phi)) = B(\mathbf{V}^*(\phi))$ under fairly weak conditions on payoff functions. Furthermore, numerical results for the comparative statics of selected values are virtually indistinguishable numerically even for the asymmetric games we analyze in this paper.

Even with a restriction to stationary collusive equilibria, the comparative statics are typically very difficult, if not impossible, to derive analytically. This is true particularly for the comparative statics of asymmetric models (see Compte, Jenny, and Rey 2002, Kühn 2004 for examples). For this reason, a large body of literature has analyzed subsets of the set of subgame perfect equilibria for which strategy profiles on and off the equilibrium path are restricted to a subset of $\mathbb{P}$, denoted $\mathbb{P}^r$. Such restrictions on equilibrium strategies generate restricted equilibrium value and price sets, $\mathbf{V}^r(\phi)$ and $\mathfrak{P}^r(\phi)$ that are strict subsets of $\mathbf{V}^*(\phi)$ and $\mathfrak{P}^*(\phi)$. For example the set of "symmetric optimal punishment equilibria" is generated by the restriction that only symmetric strategies are used on and off the equilibrium path.[13] Abreu (1988) has shown that the best symmetric optimal punishment equilibrium is typically less collusive than the best collusive equilibrium. The reason is that symmetric optimal punishment equilibria exclude the possibility of asymmetric continuation equilibria after deviations. Such asymmetric continuation equilibria typically allow for more severe punishments. The best symmetric optimal punishment equilibria will, for the same reason, typically be less collusive than the best stationary collusive equilibrium. Much of the literature has, however, been even more restrictive by focusing on symmetric subgame perfect equilibria that are supported by threats to switch to one shot Nash behavior forever ("Nash reversion equilibrium"). This imposes an even more severe restriction on the equilibrium value set and the set of supportable prices. Since it limits the punishments relative to symmetric optimal punishment equilibria, the best Nash reversion equilibrium will again be less collusive than the best symmetric optimal punishment equilibrium.

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do not elaborate on this issue since the strategy profile supporting the selected proves to be unique in symmetric models. In asymmetric models we restrict the analysis to equilibria stationary on the equilibrium path, which again generates uniqueness of the selected equilibrium.

[13] For asymmetric games, researchers have sometimes applied quasi-symmetric equilibria for the same reason (see Compte, Jenny and Rey 2002).

Studying equilibria based on restricted equilibrium strategies would be very useful if the qualitative comparative statics of the best equilibria in the restricted set were the same as for the best collusive equilbirium. This would allow researchers useful shortcuts to derive the qualitative comparative statics for symmetric collusion models. The purpose of the next section is to explore this question systematically for the example of comparative statics in the degree of product differentiation. In section 4 we apply the techniques developed in section 3 to the comparative statics in other parameters.

3 The Impact of Product Differentiation on Collusion

In this section, we use our base model to analyze the comparative statics of collusion in the degree of product differentiation. We study the robustness of the comparative statics conclusions to the restrictions on the strategies permitted. We start the analysis with a global comparison of the best equilibria under different restrictions on equilibrium strategies. However, global comparative statics are essentially impossible to derive for a general model such as ours. To obtain more insights, we study the comparative statics in product differentiation at the homogeneous goods limit. We start with the most restrictive, but most popular solution, the best Nash reversion equilibrium. We show that the comparative statics results are highly non-robust for this equilibrium. In contrast, we find that the best symmetric optimal punishment equilibrium generates robust results. We complete the section by demonstrating that the qualitative comparative statics of best symmetric optimal punishment equilibrium are the same as those of the the best stationary collusive equilibrium. Finally, we show that under relatively weak conditions on profit functions, the best stationary collusive equilibrium coincides with the best collusive equilibrium.

3.1 Some Preliminary Results

For symmetric models it is clear that our bargaining rule requires that firms share the expected net present value of total profits equally. It is then clear that the best stationary collusive equilibrium must be symmetric. Furthermore, it is easy to show that the best symmetric optimal punishment equilibrium and the best Nash reversion equilibrium must both be stationary on the equilibrium path. Hence, independently of which restriction we apply, the best equilibrium will be obtained by charging the highest sustainable price in every period. In order for this price $p^c \in (p^N(\phi), p^m]$ to be supported in an equilibrium, the net present value of obtaining a share $\frac{1}{n}$ of the industry profits at p^c forever must exceed the benefits obtained from deviating from collusion in one period and reverting to the worst feasible equilibrium in the set of permitted equilibrium strategies. Hence, in all cases the incentive compatibility constraint on p^c for given (ϕ, δ, n) can be written as:

$$\pi(p^c, p^c, n) \geq (1-\delta)\pi(p^*(p^c, \phi), p^c, \phi, n) + \delta \underline{v}(\phi, \delta, n), \tag{1}$$

where $\underline{v}(\phi, \delta, n)$ is the worst possible average continuation value in any equilibrium given the permitted strategies. The essential difference from the study of different subsets of equilibria arises because of differences in the continuation equilibria considered, i.e. in $\underline{v}(\phi, \delta, n)$. In the best Nash reversion equilibrium, this value is given by the static Nash equilibrium profits, i.e. $\underline{v}(\phi, \delta, n) = \pi(p^N(\phi), p^N(\phi))$. In contrast, in the best symmetric optimal punishment equilibrium and the best stationary collusive equilibrium, $\underline{v}(\phi, \delta, n)$ is determined by an incentive compatibility condition. The difference in the comparative statics of the best equilibria under the different restrictions on equilibrium strategies therefore arise because the comparative statics of $\underline{v}(\phi, \delta, n)$ in ϕ differ.

We are primarily interested in deriving the comparative statics of profits and price in ϕ. However, in the literature authors have also frequently asked a question that is slightly easier to analyze: What is the lowest discount factor $\delta(\phi)$ for which each firm can obtain $1/n$ of the total monopoly profit in equilibrium (i.e. each firm earns $\pi(p^m, p^m, n)$)? We show a close relationship between the comparative static of the critical discount factor $\delta(\phi)$ and the comparative static of the price in the best equilibrium in this section. Using the superscript N for the best Nash reversion equilibrium, O for the best optimal symmetric punishment equilibrium, and S for best stationary collusive equilibrium, we can immediately state some simple results on the comparison of these equilibria:

Lemma 1 *For any $\phi \in (0, 1)$, the set of equilibria generated from Nash punishments is a strict subset of the set generated from optimal symmetric punishments. In the limit as $\phi \to 0$, the two sets coincide. In particular, for all $\phi \in (0, 1)$*

(a) $\underline{v}^N(\phi, \delta, n) > \underline{v}^O(\phi, \delta, n) \geq \underline{v}^S(\phi, \delta, n) \geq 0.$

(b) $p^{cN}(\phi, \delta, n) \leq p^{cO}(\phi, \delta, n) \leq p^{cS}(\phi, \delta, n) \leq p^m$

(c) $1 > \delta^N(\phi, n) \geq \delta^O(\phi, n) \geq \delta^S(\phi, n) > 0$

(d) $\lim_{\phi\to 0} \underline{v}^N(\phi, \delta, n) = \lim_{\phi\to 0} \underline{v}^O(\phi, \delta, n) = \lim_{\phi\to 0} \underline{v}^S(\phi, \delta, n) = 0,$

$\lim_{\phi\to 0} p^{cN}(\phi, \delta, n) = \lim_{\phi\to 0} p^{cO}(\phi, \delta, n) = \lim_{\phi\to 0} p^{cS}(\phi, \delta, n) = p^c(0, \delta, n),$

and $\lim_{\phi\to 0} \delta^N(\phi, n) = \delta^O(\phi, n) = \delta^S(\phi, n) = \frac{n-1}{n}$

Proof. See Appendix B. ■

The intuition for this result is simple. Punishments for deviation in optimal symmetric punishment equilibria are always strictly harsher than Nash reversion. The reason is that the initial punishment price is allowed to fall below the Nash price. Punishments in the unrestricted equilibrium set can be even harsher because the punishments of symmetric optimal punishment equilibria are available, but even harsher asymmetric punishments can be used. This directly implies that prices will be highest in the best stationary collusive equilibrium and lowest in the best Nash reversion equilibrium. The opposite ranking follows for the critical discount factors. However, note that continuation profits are always bounded from below by zero since a firm can always guarantee such profits by setting price equal to marginal cost. Equilibrium punishments and equilibrium prices must then coincide in the homogeneous goods limit.

It is important to note that some of the qualitative features of the different equilibria always coincide. These are summarized in proposition 2:

Proposition 2 *(a) For any $(\phi; \delta, n)$ with $\phi > 0$, the highest sustainable collusive price strictly exceeds the Nash equilibrium price, i.e. $p^c(\phi; \delta, n) > p^N(\phi, n)$.*

(b) If $\delta \geq \frac{n-1}{n}$*, then* $\lim_{\phi\to 0} p^c(\phi;\delta,n) = p^m$*. If* $\delta < \frac{n-1}{n}$*, then* $\lim_{\phi\to 0} p^c(\phi;\delta,n) = c$*.*

Proof. See Appendix B. ■

Proposition 2 has some strong implications, which are independent of the equilibrium set studied. First, some collusion is always sustainable in models with product differentiation in contrast to the homogeneous goods limit model in which collusion is either sustainable at any price or not at all. In product differentiation models, a less favorable environment for collusion will be accommodated by a lower maximal collusive price p^c. Second, behavior of the firms sharply divides for discount factors above and below the critical discount factor of the homogeneous goods limit model. Most importantly, proposition 2 imposes some significant restrictions on the qualitative comparative statics of collusion around the homogeneous goods limit - independent of the equilibrium set studied. For $\delta < \frac{n-1}{n}$, it implies that the price in the best equilibrium must be increasing faster than the Nash equilibrium price close to the homogeneous goods limit. The only difference between the different equilibria considered is that in the equilibrium that has higher prices for a given ϕ, the price will rise faster above the Nash equilibrium price around the homogeneous goods limit. For $\delta \geq \frac{n-1}{n}$, proposition 2 also gives us a clean classification of the potential comparative statics for the most collusive price. If $\delta \geq \frac{n-1}{n}$ and $\delta(\phi,n) \leq \delta$ globally, the most collusive price is always p^m, independent of ϕ. On the other hand, if there exists some interval (ϕ_L, ϕ_H) on which $\delta(\phi,n) > \delta$ it must be the case that $p^c(\phi;\delta,n)$ is decreasing in ϕ on some interval since $\lim_{\phi\to 0} p^c(\phi;\delta,n) = p^m$ and $p^c(\phi;\delta,n) < p^m$ for $\phi \in (\phi_L, \phi_H)$ by definition of $\delta(\phi,n)$. Note, that these results imply that $p^c(\phi;\delta,n)$ can only be non-monotonic in ϕ if $\delta(\phi;n)$ is as well.

To determine the comparative statics in ϕ define ξ as the the degree of slackness in the incentive compatibility constraint (1):

$$\xi \equiv \pi(p^c,p^c,n) - (1-\delta)\pi(p^*(p^c,\phi),p^c,\phi,n) - \delta\underline{v}(\phi,\delta,n) \tag{2}$$

Let $\delta(\phi,n,p^c)$ be the lowest discount factor such that p^c can be sustained (i.e. $\delta(\phi,n) = \delta(\phi,n,p^m)$), i.e. the discount factor that makes $\xi = 0$ at (ϕ,n,p^c). We now show that the comparative statics of $p^c(\phi,\delta,n)$ and of $\delta(\phi,n,p^c)$ are completely determined by:

$$\xi_\phi(p^c,\delta,\phi,n) = -(1-\delta)\frac{d\pi(p^*(p^c,\phi),p^c,\phi)}{d\phi} - \delta\cdot\frac{\partial\underline{v}(\phi,\delta,n)}{\partial\phi} \tag{3}$$

Expression (3) represents the marginal trade-off between the deviation and the punishment effect induced by a small change in ϕ. The first term is positive since the profit from an optimal deviation decreases as products become more differentiated. The second term is negative in the case of Nash reversion since prices rise as a response to increased product differentiation. It is less obvious to see that the same conclusion also holds in the case of symmetric optimal punishment equilibria. The reason is that it become less credible to punish harshly, when products become more differentiated because each firm has more market power. Collusion is facilitated by a small increase in product differentiation at (p^c,δ,ϕ,n) if and only if $\xi_\phi(p^c,\delta,\phi,n) > 0$. In this case, either the highest sustainable price increases when product differentiation is slightly increased or the lowest feasible discount factor for sustaining a collusive price p^c decreases:

Lemma 2 *The highest sustainable collusive price* $p^c(\phi,\delta,n)$ *increases in* ϕ *at* (p^c,δ,ϕ,n) *and the lowest feasible discount factor for collusion at* p^c*,* $\delta(\phi,n,p^c)$ *decreases in* ϕ *at* (p^c,δ,ϕ,n) *if and only if* $\xi_\phi(p^c,\delta,\phi,n) > 0$*.*

Proof. See Appendix B. ■

The central problem in pushing the analysis beyond propositions 2 and Lemma 2 arises because it is generally not possible to sign ξ_ϕ for the whole domain. Instead we analyze (3) at the limit of large numbers of firms and at the limit of complete homogeneity of goods. This gives important insights as to the relative magnitude of the two effects, which allows us to distinguish between robust and non-robust results that can be derived for Nash reversion or optimal symmetric punishments. Furthermore, it allows us to pinpoint the reason why comparative statics of the best Nash reversion equilibrium has sometimes given different results to comparative statics of the best symmetric optimal punishment equilibrium. For the rest of this section we proceed in two steps. We first shows the general non-robustness of the Nash reversion approach. We then demonstrate why the comparative statics of the best symmetric optimal punishments leads to robust results, which are qualitatively the same as for the best collusive equilibrium.

3.2 Are there Robust Comparative Statics for the Best Nash Reversion Equilibrium?

We first show that one can obtain robust results about the comparative statics of the best Nash reversion equilibrium when the number of firms in the market is large enough. Then we show that robustness fails otherwise: For the small numbers case, conclusions can dramatically change with functional form.

For the large number case, we distinguish between models in which behavior converges to perfect competition as numbers get large and those models that have a monopolistically competitive limit. In the latter, Nash equilibrium prices will not converge to marginal cost as the number of firms in the industry becomes arbitrarily large. One needs slightly stricter conditions to obtain a robust result for the monopolistically competitive case, but all models with a monopolistically competitive limit that we have analyzed have the required property:

Proposition 3 *Suppose the model has a competitive limit or that* $\lim_{n\to\infty} D_\phi(p^*(p^m,\phi),\mathbf{p}^m,\phi) = -\infty$. *Then, for every* $\phi \in (0,1)$, *there exists* $\bar{n} < \infty$ *such that for all* $n > \bar{n}$,

(a) $\delta^N(\phi, n)$ *is strictly decreasing in* ϕ *and*

(b) $p^{cN}(\phi,\delta,n)$ *is either strictly increasing in* ϕ *or constant at* p^m.

Proof. See appendix A ■

The intuition for this result is fairly simple. First consider the effect on the incentives to deviate from the collusive price when ϕ is slightly increased. For any reduction in price a firm wins less consumers and the incentive to undercut is reduced. Even as n gets large there must be a strict reduction in the demand the firm can gain when product differentiation is increased. This effect is therefore of first order. However, the weight on the profits from deviation in total profits is given by $(1-\delta)$. As n goes up the critical discount factor converges to 1 at rate $1/n$. Hence, the total effect of an increase of product differentiation on the incentive to deviate from collusion is negative and of order $1/n$.

Now consider the impact on the continuation profits after deviation. If the model has a competitive limit, price will converge to marginal cost , whatever the value of the product differentiation parameter. Essentially, product space becomes so tightly packed that for any degree of product differentiation there is still an arbitrarily close competitor available. At the same time, output in the market goes to zero at rate $1/n$. Hence, in the limit, as n becomes arbitrarily large, the impact of an increase on product differentiation on punishments goes to zero at a rate strictly faster than $1/n$. Hence, for large numbers of firms the effect on the incentives to deviate from collusion dominates and an increase in product differentiation unambiguously facilitates collusion.

In contrast, for monopolistically competitive models the effect on punishments is only of order $1/n$. However, in that case we assume that $\lim_{n\to\infty} D_\phi(p^*(p^m,\phi),\mathbf{p}^m,\phi) = -\infty$, a property of all models with a monopolistically competitive limit that we are aware of. Since the marginal effect of ϕ on the Nash equilibrium price remains finite, it again follows that collusion is facilitated by an increase in product differentiation when the number of firms is large enough. It should be noted that our results extend globally to all ϕ for all examples of demand functions we have considered numerically.

Proposition 3 generalizes an observation in Mollegard and Overgaard (2002) that, in the linear model, the results of Deneckere (1983) are reversed for $n > 5$. Our proposition reveals the general underlying reason why such a result would come about for large enough n.

Note that this robust result runs counter to the general policy wisdom on the impact of product differentiation on the ability to collude. The general policy view might still have some merit, if one could robustly show that for small numbers of firms, the opposite comparative static to propositions 3 were to hold. However, we now demonstrate with a number of examples that there is no such robust result. The following characterization result facilitates this discussion:

Proposition 4 *At the homogeneous goods limit the largest discount factor that supports the monopoly price in the best Nash reversion equilibrium is decreasing in* ϕ *if and only if*

$$\lim_{\phi\to 0} -n\xi_\phi = (p^m - c)\lim_{\phi\to 0} D_\phi(p^*(p^m,\phi),\phi) + \frac{n-1}{n}D(c)\lim_{\phi\to 0}\frac{dp^N}{d\phi} < 0.$$

where $\lim_{\phi\to 0} D_\phi(p^*(p^m,\phi),\phi) < 0$ *and* $\lim_{\phi\to 0}\frac{dp^N}{d\phi} > 0$.

Proof. See Appendix B. ■

There is a corresponding result for the most profitable sustainable collusive price, which we suppress here for reasons of space. Note that, just as in the limiting case for n, it remains true that the sign of ξ_ϕ depends on the relative magnitudes of D_ϕ and $\frac{dp^N}{d\phi}$. Since both effects are of first order, this implies that under Nash reversion there is a general tradeoff between the degree to which an increase in product differentiation decreases the demand obtained at an optimal deviation, and the degree to which an increase in product differentiation increases the static Nash equilibrium price. We will now show that the direction in which this trade-off is resolved depends critically on the functional form and the parameters assumed.

The following results focus on functional forms for the demand system that have not previously been analyzed in the literature: Spence-Dixit-Stiglitz CES preferences and the symmetric nested logit model. We demonstrate how limiting analysis close to complete homogeneity makes general comparative statics possible. We confirm numerically that the qualitative results extend to the whole range of product differentiation parameters. We start with the case of CES preferences:

Corollary 1 *With CES preferences $\delta(\phi, n)$ is strictly increasing at $\phi = 0$ if and only if $\frac{D(p^m)}{D(c)} \ln(n) < 1$. This inequality holds for all κ if $n \leq 15$. For $n \geq 16$, there exists $\underline{\kappa}(n) \in (0, \infty)$, which is strictly increasing in n, such that for all $\kappa \in (0, \underline{\kappa}(n)]$, $\delta(\phi, n)$ is strictly decreasing.*

Proof. See Appendix B. ■

Corollary 1 shows that with CES preferences it is very difficult to obtain anything else but a facilitation of collusion by introducing product differentiation. Indeed, Figure 1 below shows that for $n = 2$, collusion is always more difficult under any finite degree of product differentiation than under homogeneous products. Over a wide range of product differentiation parameters $\delta(\phi, 2)$ remains increasing in ϕ. Later it declines and converges to .5.

[Insert Figure 1]

However, note from Figure 2 that when collusion is just sustainable for homogeneous goods, there is only a small range close to homogeneity in which the highest sustainable collusive price dramatically falls. Indeed, this drop gets more and more dramatic as the elasticity of aggregate demand, κ, increases. When no collusion is sustainable under homogeneity, the collusive price increases monotonically and initially faster than the monopoly price as claimed in the propositions.

[Insert Figure 2]

When collusion is difficult to establish then collusion will be facilitated by product differentiation in any model. However, the above result is also suggestive that for cases in which markets are concentrated and collusion is easy, product differentiation will make collusion more difficult. While the results from SDS preferences make it tempting to conclude that product differentiation facilitates collusion when there are small numbers of firms, we now show that this is not a robust result. For example, in the case of the nested logit model, the conditions for increasing $\delta(\phi, n)$ around homogeneous products are much more restrictive:

Corollary 2 *With nested logit preferences $\delta(\phi, n)$ is increasing at $\phi = 0$ if and only if $\frac{D(p^m)}{D(c)} \ln(n) < 1$. In particular, if $n = 2$ then this condition always holds. If $n \geq 3$, then there exists $\underline{A}(n) > -\infty$, such that $\delta(\phi, n)$ is increasing at $\phi = 0$ if and only if $A \in (-\infty, \underline{A}(n))$. $\underline{A}(n)$ is strictly decreasing in n.*

Proof. See Appendix B. ■

Note that the limiting condition for the qualitative comparative statics in the SDS and the nested logit model are the same. However, in the nested logit model the demand reduction due to monopoly pricing tends to be smaller, which makes it much less likely that collusion becomes easier for differentiated products than for undifferentiated products. However, these models and the linear model of Deneckere (1983) coincide in the result that collusion is hindered by product differentiation in the two firm case. Again this is not a robust result. For a simple Hotelling model we obtain:

Corollary 3 *Consider a two firm Hotelling model with firms located at the endpoints of a line. Then an increase of product differentiation measured by transport costs around the homogeneous goods limit facilitates collusion.*

Proof. See Appendix B. ■

These examples demonstrate that in highly concentrated markets, the scope for collusion under Nash reversion punishment schemes can go up or down when product differentiation is slightly increased at the limit of perfect homogeneity. The only clear result emerging from this analysis is that the comparative statics of collusion will tend to go in the opposite direction of suggesting that collusion is facilitated under homogeneous goods when the number of firms is large enough. We now show that in contrast to this finding, results on the set of symmetric optimal punishment strategies are robust to functional form and we explain why this is the case.

3.3 Comparative Statics for the Best Symmetric Optimal Punishment Equilibrium are Robust

The reason why symmetric optimal punishments generate robust results on the comparative statics of collusion arises because punishments are much more severe in the short run. Optimal punishments allow punishment prices to drop significantly below Nash equilibrium prices. Any short run optimal deviation from such punishments will therefore lead to an upward deviation to a price below the Nash equilibrium price. This entails both a smaller margin and a smaller output than Nash punishments. Indeed, as goods become more and more homogeneous, both the price-cost margin earned from a deviation, as well as the overall demand when deviating, converge to 0. This is in contrast to the Nash punishment case, where demand remains $\frac{1}{n}$ of total demand at Nash prices and only the price-cost margin converges to zero. To see this formally, note that both in the Nash reversion and the symmetric optimal punishment cases we can write the marginal impact of product differentiation on the value of the worst punishment as:

$$\begin{aligned}\frac{d\pi(p^*(p_L,\phi,n),p_L,\phi,n)}{d\phi} &= \frac{\partial\pi(p^*(p_L,\phi,n),p_L,\phi,n)}{\partial p_j}\frac{\partial p^L}{\partial\phi}+\pi_\phi(p^*(p_L,\phi,n),p_L,\phi,n) \\ &= D(p^*(p_L,\phi,n),p_L,\phi,n)\left(\frac{-D_{p_j}}{D_{p_i}}\right)\frac{\partial p_L}{\partial\phi} \\ &\quad +D_\phi(p^*(p_L,\phi,n),p_L,\phi,n)\left[p^*(p_L,\phi,n)-c\right],\end{aligned} \tag{4}$$

where p^L is the static Nash equilibrium price in a Nash reversion equilibrium and the most severe punishment price in symmetric optimal punishment equilibria. Note that for both types of equilibrium sets, the term in the third line must converge to zero. Similarly, in both cases, punishments will be less severe when there is more product differentiation, i.e. $\frac{\partial p^L}{\partial\phi}>0$, even in the limit as $\phi\to 0$. This effect of increased punishment prices is applied to $\frac{1}{n}$ of demand in the Nash punishment case. However, under optimal punishments the best response price remains strictly above marginal costs, while the punishment price falls strictly below. This means that demand in the homogeneous goods limit must converge to zero. Hence, even when an increase in product differentiation from perfect homogeneity leads to a slightly positive price-cost margin, this is applied to an arbitrarily small demand. This leads to a qualitative difference in the comparative statics of collusion depending on the strategy set permitted in equilibrium. Under symmetric optimal punishments, an increase of product differentiation around the homogeneous goods limit will lead to only a second order effect on the punishment profits. Note that this result is analogous to our result for the case of Nash reversion with large n when the model has a competitive limit. In that case both the price cost margin and demand converge to zero and make the impact of product differentiation on punishment profits of second order in the limit.

Note, however, that the impact of product differentiation on the incentives to deviate from a collusive price remain unchanged and, hence, of first order. It follows, that for symmetric optimal punishment equilibria it is a robust result that the reduction in the incentives to deviate due to an increase in product differentiation dominates the decrease in the intensity of punishments at the homogeneous goods limit. Formally we can state:

Proposition 5 *For each n, there exists $\hat{\phi} \in (0,1)$ such that $\delta^O(\phi, n)$ is strictly decreasing for all $\phi \in (0, \hat{\phi})$.*

Proof. See Appendix B. ■

This proposition together with proposition 2 perfectly determine the comparative statics of the highest sustainable collusive price. If $\delta \geq \frac{n-1}{n}$, the limit price is p^m. Price will be constant at p^m for ϕ close to zero, since $\delta(\phi, n)$ falls below $\frac{n-1}{n}$ and p^m can be sustained for all $\delta > \delta(\phi, n)$. If $\delta < \frac{n-1}{n}$, $\lim_{\phi \to 0} p^{Oc} = c$. By proposition 2 and Lemma 1, the best collusive price, p^{cO}, strictly increases around $\phi = 0$. It increases at a faster rate than $p^N(\phi, n)$, since $p^{Oc} > p^N(\phi, n)$. When $\delta(\phi, n)$ is monotonic, as in all of our examples, p^{Oc} will also be monotonic, although it will eventually rise by less than the static Nash equilibrium price.

Because of the different order of magnitude of the effects on the incentive to deviate and the punishment close to homogeneity, it would be extremely difficult for the punishment effect ever to dominate. For this reason, we generate monotonic comparative statics in ϕ for all the demand specifications we have analyzed numerically over the whole range $\phi \in (0, 1)$. These cover the typical models of product differentiation used in the literature. It follows from this analysis that product differentiation robustly facilitates collusion relative to the perfectly homogenous case.

3.4 The Best Collusive Equilibrium and Robustness

We now show that the qualitative comparative statics for the best collusive equilibrium is captured by the analysis of the best symmetric optimal punishment equilibrium. To see this we proceed in two steps. We first show that the best stationary collusive equilibrium qualitatively has the same comparative statics properties as the best symmetric optimal punishment equilibrium. To see this, simply note that the worst continuation equilibrium $\underline{v}^S(\phi)$ has a value not exceeding $\underline{v}^O(\phi)$. Since both converge to zero profits as $\phi \to 0$, it follows that $\underline{v}^S(\phi)$ cannot grow faster in ϕ than $\underline{v}^O(\phi)$ close to $\phi = 0$. Hence, punishment profits will be at most of second order as in the case of symmetric optimal punishment equilibria:

Proposition 6 *For each n, there exists $\hat{\phi} \in (0, 1)$ such that $\delta^S(\phi, n)$ is strictly decreasing for all $\phi \in (0, \hat{\phi})$.*

Proof. See Appendix B. ■

Obviously, the argument on comparative statics of price is exactly the same as in the case of the best symmetric optimal punishment equilibrium. It should only be added that $p^S(\phi, n)$ increases faster than $p^O(\phi, n)$ close to $\phi = 0$, since $p^S(\phi, n) \geq p^O(\phi, n)$.

We will now show that under weak assumptions on the demand function this result carries over to the Best Collusive Equilibrium. To do so, we make two assumptions:

Assumption B: $\pi(\mathbf{p}, \phi, n)$ is concave in $\mathbf{p}$.

Assumption C: $\pi_i(p_i^*(\mathbf{p}_{-i}), \mathbf{p}_{-i}, \phi, n)$ is strictly convex in $\mathbf{p}_{-i}$.

Given these assumptions it is the straightforward to prove:

Proposition 7 *Under assumptions B and C, the Best Stationary Collusive Equilibrium is the Best Collusive Equlibrium.*

Proof. See Appendix B. ■

The proof proceeds by showing that under the assumptions above it is always easier to sustain a symmetric value pair with a symmetric price charged forever than a non-stationary price path. The concavity assumption ensures that for every asymmetric price vector charged in the first period of a Best Collusive Equilibrium there exists a uniform price below the average price in that price vector, that would yield exactly the same profits if repeated forever. It is then shown that a stationary price can be sustained as a subgame perfect equilibrium given the value of the lowest equilibrium, if the greatest one period profits from deviating from this price are no larger than the average profits from an optimal one period deviation across all firms in the hypothesized Best Collusive Equilibrium. The convexity condition then guarantees that this inequality is strict for the average price, showing that the original equilibrium cannot be the Best Collusive Equilibrium.

It should be noted that the qualitative equivalence of the comparative statics for Best Collusive Equilibrium and the Best Symmetric Optimal Punishment Equilibrium arise basically because punishment profits are of the same order close to the homogeneous goods limit. This comes from the property that maximal punishments for both concepts rely on maximal frontloading of the punishments. This is the feature that drives the robustness of the results. We will now discuss how these insights can be carried over to more general comparative statics problems in the context of differentiated products models.

4 General Implications for Comparative Statics in Symmetric Models

In the previous section, we introduced a general technique for deriving robust comparative statics for collusion models that rely on the analysis of the model around the homogeneous goods limit. This analysis generated intuition about the different orders of magnitude in the degree of product differentiation of the incentives to deviate from collusive prices and of punishment profits. We showed that our local results, which were based on the different order of magnitude of effects around the homogeneous goods limit, extended globally by numerical analysis. In this section we demonstrate that this technique can be applied with minor modifications to the comparative statics in any parameter θ. Again we analyze the sign of the marginal change of the slack in the incentive constraint, ξ, in the parameter θ as the degree of product differentiation goes to zero, i.e. $\lim_{\phi \to 0} \xi_\theta$. The qualitative comparative statics can be gleaned directly from the homogeneous goods limiting model when $\lim_{\phi \to 0} \xi_\theta \neq 0$. When $\lim_{\phi \to 0} \xi_\theta = 0$ the homogeneous goods limit model has degenerate comparative statics in price and the critical discount factor. In that case an analysis of $\lim_{\phi \to 0} \xi_{\theta\phi}$ still provides a comparative statics result for the differentiated goods model.

One complication arises for the comparative statics in price. As we have seen the comparative statics in prices sharply differ whether the δ exceeds or is below the critical discount factor for sustaining collusion in the homogeneous goods limiting model. If δ exceeds that discount factor price is always at the monopoly price, otherwise it converges to marginal costs as $\phi \to 0$. Interesting comparative statics can therefore only be obtained for δ lower than that critical discount factor. However at the limit as $\phi \to 0$ price cost margins under collusion, under optimal punishments and under Nash punishments converge to zero. As a consequence it proves practically impossible to determine the relative order of magnitude of deviation and punishments effects of a change in a parameter θ at that limit. To get insight into the comparative statics of collusion we fix a collusive price p^c and consider the critical discount factor $\delta(\phi, \theta, p^c)$ that can sustain this price. Then we look at the comparative statics at p^c when taking the limit in ϕ, adjusting the discount factor along $\delta(\phi, \theta, p^c)$. This leads to some limitation for the analysis: while $\delta(\phi, \theta, p^c) < \delta(\phi, \theta)$ for all $\phi \in (0, 1)$, we have $\lim_{\phi \to 0} \delta(\phi, \theta, p^c) = \delta(0, \theta)$. This means that we may not capture the comparative statics in price at the homogeneous goods limit when δ is small. However, our numerical analysis confirms in each case that the analytical comparative statics results obtained in this way hold for numerical analysis of the small δ case.

We discuss three examples in this section. First, we given an example in which the effects on profits from deviation and profits in the punishment equilibrium are of different order of magnitude in ϕ. We continue with an example (on cross-shareholdings), where both are of the same order at the limit but reinforce each other. It turns out that in this case the best Nash reversion equilibrium is robust. However, we show that it greatly underestimates the potential effect of small cross-share holdings on collusion. We then show that we can derive new results even when there is no comarative statics effect in the homogeneous goods limit model. Specifically we show that collusion is facilitated when the common marginal costs of firms rise, but that this effect cannot explain price cost margins rising more than one to one with marginal cost as a result of collusion. These examples demonstrate how our techniques can be used to derive new and robust results on the comparative statics of collusion even when the comparative statics in the limiting model is degenerate.

4.1 Example 1: The Impact of Consolidating Varieties

In this example we consider a variant of our basic model in which each of $k < n$ firms owns $\frac{n}{k}$ varieties. This is a symmetric version of the model of Kühn (2004). We ask the question whether collusion is facilitated or hindered if less firms own the same number of total varieties. As Kühn (2004, 2005b) has stressed, this is very different from decreasing the number of varieties. The latter would unambiguously lead to facilitated collusion. But redistributing a given number of varieties (symmetrically) among less firms has countervailing effects: On one hand, it becomes easier to collude because the short run benefits of cheating on a collusive agreement are reduced. On the other hand, owning a greater proportion of the varieties should make punishments less severe. Kühn (2005b) has shown in a model with fixed total capacity that fragmentation of capacity facilitates collusion.

It is easy with our methodology to show that this result does not hold for the redistribution of varieties. The reason is that, as in the base model, the effects on the profits in the punishment equilibrium are of smaller order than the impact on the incentives to deviate from the collusive price. To see this formally, note that for any k the incentive compatibility condition for deviation from a stationary collusive price p^c is given by:

$$\frac{n}{k}\pi(p^{ck}, p^{ck}, n) \geq (1-\delta)\frac{n}{k}\pi^k(p^*(p^{ck}), p^{ck}, \phi, n) + \delta\frac{n}{k}\underline{v}(\phi, k) \tag{5}$$

where $\pi^k(p^*(p^c), p^c)$ refers to the profit function for one variety in which the first $\frac{n}{k}$ prices are set at $p^*(p^c)$ and $\underline{v}(\phi, k)$ is the average value per variety in the worst equilibrium. Let $\overline{k} > \underline{k}$. Then an increase of the number of firms from $\underline{k}$ to $\overline{k}$ has a marginal effect on the incentive condition for given p^c and given δ of:

$$\xi_k = (1-\delta)\left[\pi^{\underline{k}}(p^*(p^{c\underline{k}}), p^{c\underline{k}}, \phi, n) - \pi^{\overline{k}}(p^*(p^{c\underline{k}}), p^{c\underline{k}}, \phi, n)\right] + \delta[\underline{v}(\phi, \underline{k}) - \underline{v}(\phi, \overline{k})] \tag{6}$$

It is immediately clear that $\pi^{\underline{k}}(p^*(p^{c\underline{k}}), p^{c\underline{k}}, \phi, n) - \pi^{\overline{k}}(p^*(p^{c\underline{k}}), p^{c\underline{k}}, \phi, n) < 0$ for all $\phi \in (0,1)$. Suppose that $\delta \geq \frac{\underline{k}-1}{\underline{k}}$. Then full collusion can be sustained close to the homogeneous goods limit when the number of firms is small, i.e. $\underline{k}$. This means that $\pi^{\underline{k}}(p^*(p^{c\underline{k}}), p^{c\underline{k}}, \phi, n) - \pi^{\overline{k}}(p^*(p^{c\underline{k}}(\phi)), p^{c\underline{k}}(\phi), \phi, n)$ converges to a number no larger than $\frac{[\underline{k}-\bar{k}]}{n}(p^c - c)\bar{D}(p^c) < 0$ as $\phi \to 0$. By the arguments made in the previous section $\lim_{\phi\to 0}\underline{v}(\phi, k) = 0$ for all k, so that $\lim_{\phi\to 0}[\underline{v}(\phi, \underline{k}) - \underline{v}(\phi, \overline{k})] = 0$. Hence, the impact on the incentives to deviate from the collusive price dominates and $\xi_k < 0$. It follows that $\delta(k, \phi)$ increases in k. For ϕ close to zero, the highest collusive price $p^{ck}(\phi)$ strictly falls in k if $\frac{\bar{k}-1}{\bar{k}} > \delta \geq \frac{\underline{k}-1}{\underline{k}}$ or if $\delta < \frac{\underline{k}-1}{\underline{k}}$ and $\delta = \delta(\underline{k}, \phi, p^{c\underline{k}}(\phi))$:

Proposition 8 *For any $\bar{k} > \underline{k}$ there exists $\hat{\phi}(p^c) > 0$, such that for all $\phi \in (0, \hat{\phi}(p^c))$*

(a) $\delta(\bar{k}, \phi, p^c) > \delta(\underline{k}, \phi, p^c)$.

(b) $p^{c\bar{k}}(\phi, \delta) \leq p^{c\underline{k}}(\phi, \delta) = p^m$, where the inequality is strict if $\frac{\underline{k}-1}{\underline{k}} \leq \delta < \frac{\bar{k}-1}{\bar{k}}$ or if $\delta = \delta(\underline{k}\phi, p^{c\underline{k}}) < \frac{\underline{k}-1}{\underline{k}}$.

Proof. See Appendix B. ■

As in the basic product differentiation analysis, the comparative statics is robustly determined because close to homogeneity the impact of the number of firms on the incentives to deviate is of higher order than the impact on the punishment profits. The proof shows that the dominating effect is that with less firms in the market the market share gain from undercutting is smaller. This is the dominating effect that drives the comparative statics.

As we have indicated, there are some limitations in providing comparative statics for arbitrary discount factors δ. The problem is that for $\delta < \frac{k-1}{\underline{k}}$, all collusive equilibria converge to marginal cost pricing as $\phi \to 0$. Also, the most severe punishment price converges to marginal cost and the elasticity at the best response price to the collusive price becomes infinite. At the limit of competitive pricing it becomes practially impossible to determine the relative order of magnitude of the effects of a parameter change on punishment profits and the benefits from one shot deviations. To understand the limit on the prediction for the price comparative statics note that, for any p^c, $\lim_{\phi \to 0} \delta(\phi, \underline{k}, p^c) = \frac{k-1}{\underline{k}}$. However, by our results on product differentiation we know that $\delta(\phi, \underline{k}, p^c)$ is strictly increasing in some neighborhood of $\phi = 0$. This means that we show that for some $\delta(\phi, \underline{k}, p^c) < \frac{k-1}{\underline{k}}$ and so for some region of ϕ and some δ we have shown that the price falls when k is increased. Since $\delta(\phi, \underline{k}, p^c)$ falls more dramatically, the lower p^c this generates a strong presumption that the comparative statics can only fail for extremely low discount factors. In our numerical analysis we have not been able to generate examples in which the comparative statics suggested by our limiting results are violated.

4.2 Example 2: Symmetric Cross Ownership Shares

In this section we study the impact of increasing symmetric cross-shareholdings between all firms in the market on the ability to collude. The current literature on the subject (see Malueg 1992) suggests that there are countervailing effects from such shareholdings: The incentives to deviate from collusion are reduced, but punishments are also weakened because the profits in a one shot Nash equilibrium increase in cross-shareholdings. Malueg (1992) therefore finds that cross-ownership has ambiguous effects on collusion. We show in this subsection that in a precise sense it is a robust result that collusion is facilitated by cross-ownership - even under Nash reversion. But we also show that the best Nash reversion equilibrium will predict a much smaller order of magnitude of the effects of small cross-shareholdings on collusion. Even when the best Nash reversion equilibrium is robust, it still does not reflect the correct order of magnitude of the impact of a parameter change on collusion.

To remain in the framework of symmetric models, we analyze a special case of cross-ownership in which ownership shares are symmetric and each firm i controls the production of only one good, good i. We assume that each firm owns a share α in each rival $j \neq i$, with $0 \leq \alpha < \frac{1}{n}$. The share of firm i in its own product i is given by $1 - (n-1)\alpha$. Let $\hat{\alpha} \equiv (n-1)\alpha$. Then total profits of firm i, $\Pi_i(\mathbf{p}, \hat{\alpha})$, can be written as the weighted average of the profits from variety i and the average profits from all other varieties $j \neq i$, with $(1-\hat{\alpha})$ and $\hat{\alpha}$ as the respective weights. Notice that, in the limit, as $\alpha \to \frac{1}{n}$, $\hat{\alpha} \to \frac{n-1}{n}$ and the profits of all varieties have equal weights in the one period profit function of firm i.

Consider first the limiting homogeneous goods model. We characterize the critical discount factor at which the monopoly price p^m is just sustainable. The incentive compatibility constraint (**??**) then becomes

$$\Pi_i(p,\hat{\alpha}) = (1-\hat{\alpha})\pi_i(p^m) + \hat{\alpha}\frac{1}{n-1}\sum_{j\neq i}\pi_j(p^m) \geq (1-\delta)(1-\hat{\alpha})n\pi_i(p^m) + \delta\underline{v}, \tag{7}$$

where $\pi_i(p) = \frac{1}{n}(p-c)\bar{D}(p)$. Consider first the best Nash reversion equilibrium. It is easy to see that in the limit model, the Nash equilibrium in prices for any $\hat{\alpha} \in [0, \frac{n-1}{n}]$ requires marginal cost pricing. Hence $\underline{v}_i^N = 0$ and the critical discount factor under Nash reversion is given by $\delta^N = 1 - \frac{1}{n(1-\hat{\alpha})}$. Note, that this becomes $\frac{n-1}{n}$ when there is no cross-ownership and converges to 0 as $\hat{\alpha} \to \frac{n-1}{n}$.

The central insight from looking at symmetric optimal punishment equilibria is that punishments will not be limited to a zero continuation profit. Even if the firm deviates from a low punishment price, it can never escape the losses from its ownership in rivals that price below marginal cost. But then punishments can be increased, by increasing the stakes in rivals. To see this write the incentive constraint for an optimal symmetric punishment as:

$$\underline{v} = (1-\delta)\pi_i(p_L) + \delta\pi_i(\hat{p}) \geq (1-\delta)\hat{\alpha}n\pi_i(p_L) + \delta\underline{v}, \tag{8}$$

which reduces to

$$\delta\pi_i(\hat{p}) \geq (\hat{\alpha}n - 1 + \delta)\pi_i(p_L) \tag{9}$$

To simplify the exposition assume that $\lim_{p\to 0}\bar{D}(p) = \infty$. Then $\hat{p} = p^m$. Furthermore, if $\delta > 1 - \hat{\alpha}n$, then the right hand side is negative and a further decrease in p_L would always be incentive compatible. But then (7) can always be made to hold with strict inequality. Hence $\delta^O < 1 - \hat{\alpha}n$, and (9) strictly binds. Substituting from (14) into (12) and rearranging terms yields:

$$\delta^O = 1 - \frac{1}{n(1-\hat{\alpha})} - \hat{\alpha}[n - \frac{1}{1-\hat{\alpha}}] < \delta^N.$$

Note that the critical discount factor under symmetric optimal punishments is strictly below that of Nash reversion. Indeed, the difference is substantial. At a 5% cross shareholding and 2 firms in the market $\delta^O = 0.426\,32$ while $\delta^N = 0.473\,68$, i.e. the critical discount factor under Nash reversion is 10% higher. At a 10% cross-ownership we have $\delta^O = 0.355\,55$ and $\delta^N = 0.444\,44$. This shows that even small cross-shareholdings can have large effects on the critical discount factor in the best optimal symmetric punishment equilibrium. However, to give any indication what this means for a change in the exercise of market power, we need a model in which the comparative statics of price is not degenerate as in the homogeneous goods model. For this purpose it is useful to see that the comparative statics of the critical discount factor and of price in the differentiated products model is qualitatively the same as the comparative statics in the discount factor for the homogeneous goods limit model. Furthermore, we will show numerically that the large quantitative effect of optimal symmetric punishments carries over to the price effect.

As in the example of product differentiation, we can write the incentive condition determining the lowest δ at which collusion at p^c can be sustained under threat of Nash reversion analogously to (1) as:

$$\Pi(p^c,p^c) = (1-\delta^N(\hat{\alpha},\phi,p^c))\Pi_i(p^*(p^c,\hat{\alpha},\phi),p^c,\phi) + \delta^N(\hat{\alpha},\phi;p^c)\Pi(p^N(\hat{\alpha},\phi),p^N(\hat{\alpha},\phi)) \tag{10}$$

The comparative statics of the critical discount factor and the collusive price is determined by sign of:

$$\begin{aligned}\xi_{\hat{\alpha}} \quad = \quad & (1-\delta^N(\hat{\alpha},\phi,p))[\pi(p^*(p,\hat{\alpha},\phi),p,\phi) - \pi(p,p^*(p,\hat{\alpha},\phi),p,\phi)] \\ & -\delta^N(\hat{\alpha},\phi,p)\pi_{p_j}(p^N(\hat{\alpha},\phi),p^N(\hat{\alpha},\phi))\frac{dp^N}{d\hat{\alpha}},\end{aligned} \tag{11}$$

where we differentiated the right hand side of (10) with respect to $\hat{\alpha}$, using (39). The expression then follows by the envelope theorem. The first term in (11) represents the fact that a symmetric increase in cross-share holdings will reduce the weight of own product profits in the overall profit calculation of the firm and increase the weight of the average profits of others. Hence, there is less of an incentive to deviate because less profits are won by undercutting. On the other hand, the Nash equilibrium price is increased, reducing the intensity of punishments that can be imposed for higher levels of cross share holdings. This effect will make collusion more difficult to achieve. This apparent trade-off has been at the heart of a small literature on the effect of cross-shareholdings on collusion (see Malueg 1992).

However, using limiting results we can obtain the relative order of magnitude of the two effects at the homogeneous goods limit (or at the limit of high market fragmentation). In this case we find that the impact of an increase in shareholdings on Nash prices is not of first order at the limit. The reason is that for any cross-share holdings price competition at the homogeneous goods limit forces prices down to marginal cost. This is a strong version of the general insight that small share holdings only have small impact on static equilibrium in the market. Indeed, when goods are almost homogeneous, all share holdings below those that induce monopoly pricing can be considered small, i.e. will have almost no effect.

This insight directly leads to the two main comparative statics results for collusion under Nash reversion: Close to perfect homogeneity the effect on the incentives to deviate from a change in cross-ownership shares will dominate, yielding the unambiguous result that collusion is facilitated by cross-share holdings:

Proposition 9 *There exists $\hat{\phi}(p^c) > 0$ such that for all $\phi \in (0, \hat{\phi}(p^c))$,*

(a) $\delta^N(\hat{\alpha}, \phi, p^c)$ is strictly decreasing in $\hat{\alpha}$.

(b) $p^c(\hat{\alpha}, \phi, \delta)$ is strictly increasing at $(\phi, \delta^N(\hat{\alpha}, \phi, p^c))$ if $\delta^N(\hat{\alpha}, \phi, p^c) < \delta(\hat{\alpha}, \phi)$ and constant at p^m otherwise.

Proof. See Appendix B. ■

Note, that close to perfect homogeneity only the incentive to deviate is of first order. Consequently, only the impact of an increase in $\hat{\alpha}$ on the incentive to deviate from the collusive price matters for the comparative statics of collusion. Consequently, all impediments for collusion disappear at the homogeneous good limit when $\hat{\alpha}$ goes to its maximum at $\frac{n-1}{n}$. The critical discount factor for collusion at the monopoly price converges smoothly to 0. When full collusion is possible under homogeneous goods and no cross-shareholdings, it will still be possible with increased shareholdings. This simply comes from the fact that $\lim_{\phi \to 0} \delta(\hat{\alpha}, \phi) < \frac{n-1}{n}$ and decreases in $\hat{\alpha}$. Hence, for $\delta < \frac{n-1}{n}$ there will exist ϕ close to zero, such that $\delta(0, \phi) < \frac{n-1}{n}$ and full collusion can be sustained. But then it can be sustained for all $\hat{\alpha} > 0$ as well. More interesting is the case at which collusion cannot be sustained for perfect homogeneity without cross-shareholdings. As in the pure product differentiation model, the highest collusive price then collapses to the short run Nash equilibrium price, i.e. marginal cost. However, slightly away from the homogeneous good limit, the highest sustainable price strictly exceeds the Nash equilibrium price if the discount factor is high enough. Then an increase in $\hat{\alpha}$ will strictly increase the highest sustainable collusive price and will increase it all the way to the fully collusive price at some

$\hat{\alpha} < \frac{n-1}{n}$. This is the $\hat{\alpha}$, where cross-ownership just compensates for the low discount factor.

We now show that for symmetric optimal punishment strategies the result becomes much stronger. Instead of punishments being of second order in $\hat{\alpha}$ at the homogeneous goods limit, there will be an interval of ϕ in which punishments are strictly enhanced by the existence of cross-shareholdings. This then leads to much larger effects than under Nash punishments. Similarly to the product differentiation example we have shown here that the order of magnitude of the punishments in the relevant parameter is different under symmetric optimal punishments than under Nash reversion. As a result we obtain unambiguous comparative statics for changes in the degree of cross-ownership:

Proposition 10 *There exists $\hat{\phi}(p^{cO}) > 0$, such that, for all $\phi \in (0, \hat{\phi}(p^{cO}))$,*

(a) $\delta^O(\hat{\alpha}, \phi, p^{cO})$ is strictly decreasing in $\hat{\alpha}$ and

(b) $p^{cO}(\hat{\alpha}, \phi, \delta)$ is strictly increasing in $\hat{\alpha}$ at $(\phi, \delta^O(\hat{\alpha}, \phi, p^{cO}))$ for $\delta^O(\hat{\alpha}, \phi, p^{cO}) < \delta(\hat{\alpha}, \phi)$.

Proof. See Appendix B ■

The intuition for this result is simple. Consider the range of ϕ for which $p_L < c$ when $\hat{\alpha} = 0$. From our earlier analysis it is clear that such a range exists. Now suppose $\hat{\alpha}$ is slightly increased. Consider what happens when the previous strategies are maintained. Then the incentive compatibility constraint for deviations from punishments become slack, because the firm now cannot fully avoid losses. Hence, the most severe punishment price can be slightly reduced leaving this incentive constraint slack and slackening the constraint of deviation from the highest feasible collusive price. At the same time the cross-share holdings increases the weight of the rival's profits in own profits and reduces the incentives to deviate from the collusive price directly. Note, that the key insight is that increases in cross-share holdings will enhance punishments over at least a significant range of the parameter space.

The interesting question is what this tells us for the order of magnitude of effects of small cross-share holdings on the ability to collude in markets. The following graph shows that this can be substantial:

[Insert Figure 3]

The figure contrasts the price effects of going from a 5% cross-shareholding to a 10% cross share-holding in a duopoly for three different assumptions on the equilibirium played and for the whole range of levels of product differentiation. The discount factor is at a level such that in all cases the solution collapses to perfect competition at the homogeneous goods limit. The lowest two lines correspond to the impact of cross-shareholdings on Nash equilibria. As is well known, such cross-shareholdings have virtually no impact on the static Nash equilibrium. For the best Nash reversion equilibrium prices are higher but the marginal effect of an increase in cross shareholdings from 5% to 10% are still very small. However, under optimal symmetric punishments price effects of such small cross-shareholding changes can have an impact of up to 5% in the price level. This analysis thus suggests that, for example, in mergers that raise coordinated effects concerns small cross-shareholdings should be of significant importance for remedies while this would not be the case in unilateral effects cases. It should be noted how important it is to analyze optimal punishment equilibria in order to get a proper insight into the order of magnitude of this effect.

4.3 Example 3: The Impact of Marginal Cost on Collusion

In some markets, in particular gasoline retailing, increases in input prices often lead to increased margins. In public debate this is often attributed to increased input prices facilitating downstream collusion. However, our basic homogeneous goods collusive model does not conform to that prediction. The incentive condition to sustain some collusive price $p^C > c$ is given by:

$$(p^C - c)\frac{1}{n}D(p^C) \geq (1-\delta)(p^C - c)D(p^C) \tag{12}$$

This means that the incentive compatibility constraint depends neither on p^c nor on c. But this result in the homogeneous goods model depends on the very special property that deviation profits are proportional to collusive profits. This is not the case as soon as we introduce some product differentiation. We will now show that generally an increase in marginal costs does facilitate collusion, but that it is unlikely for price cost margins to increase as a result of increased marginal costs.

In all our earlier examples the monopoly price was indpendent of the parameter for which we analyzed the comparative statics of collusion. However, this is different when we study the comparative statics with respect to c because the monopoly price must increase when c increases. We therefore interpret $\delta(c,\phi)$ as the lowest discount factor for which $p^m(c)$, the monopoly price when marginal cost is c, can be sustained when the degree of product differentiation is ϕ. We will be interested in the comparative statics of $\delta(c,\phi)$ in c. The incentive compatibility constraint for any δ and $p^c$is given by:

$$\begin{aligned} & (p^c(c) - c)\frac{1}{n}\bar{D}(p^c(c)) \\ \geq\ & (1-\delta)[p^*(p^c(c),\phi) - c]D_i(p^*(p^c(c),\phi), p^c(c),\phi) + \delta\underline{v}(c,\phi) \end{aligned} \tag{13}$$

where the constraint is binding if $p^c(c) < p^m(c)$ or if $\delta = \delta^*(c,\phi)$ and $p^c(c,\phi) = p^m(c)$. Note that for any given c, $p^*(p^m(c),\phi) \to p^m(c)$ for $\phi \to 0$ and $\delta(c,\phi) \to \frac{n-1}{n}$ for $\phi \to 0$. We now want to show that for small degrees of product differentiation collusion is strictly facilitated in the sense that $\delta(c,\phi)$ is strictly decreasing when ϕ is close to zero. We discuss the implicatios for the comparative statics in price below. As in previous analyses we consider the marginal change in the tightness of the incentive constraint as c changes, by looking at:

$$\begin{aligned} \xi_c = & -\frac{1}{n}\bar{D}(p^m(c)) + (1-\delta(c,\phi))D_i(p^*(p^m(c),\phi), p^m(c),\phi) \\ & -(1-\delta(c,\phi))(p^*(p^m(c),\phi) - c)\sum_{j\neq i}\frac{\partial D_i}{\partial p_j}\frac{\partial p^m}{\partial c} \\ & -\delta\frac{\partial \underline{v}(c,\phi)}{\partial c} \end{aligned} \tag{14}$$

The first line captures the fundamental effect that dominates the comparative statics in the model. An increase in c will decrease the profits from collusion, which tightens the constraint. This is represented by the first term. On the other hand, the profits from deviation are reduced. These two terms cancel as $\phi \to 0$ since $\lim_{\phi\to 0}\delta(c,\phi) = \frac{n-1}{n}$ and $\lim_{\phi\to 0} D_i(p^*(p^m(c),\phi), p^m(c),\phi) = \bar{D}(p^m(c))$. Slightly away from zero this term becomes strictly positive: There is a direct effect that demand at the optimal deviation declines when product differentiation is introduced. Close to $\phi = 0$, this effect is exactly compensated by the decline in the critical discount factor that arises when products become more differentiated. The effect that remains is that the best response price declines, leading to a relative increase in demand, making the first term positive. A countervailing effect arises from the second line. It is generated because the monopoly price increases in c. Since all other firms raise the monopoly price, a best response to collusion becomes more profitable at given costs. Hence, the incentive constraint is tightened. However, since $\lim_{\phi \to 0} D_{p_j}(p^*(p^m, \phi), p^m, \phi) = 0$, the effects in the first and second line are of the same order. We show in the proof of proposition 11 in Appendix B that the first effect dominates the second effect and the incentive constraint is strictly relaxed. The third line captures the impact on the continuation value in the worst continuation equilibrium, $\frac{\partial \underline{v}(c,\phi)}{\partial c}$. This effect is positive but converges to zero at the homogeneous goods limit. To see this note that there are two things affecting this continuation profit. First, costs go up, reducing the profits from punishments. However, the punishment price that is set will also increase, increasing punishment profits. It can be shown that the net effect is positive. These observations together lead to:

Proposition 11 *(a) There exists $\hat{\phi} > 0$ such that for all $\phi \in (0, \hat{\phi})$ the critical discount factor $\delta(c, \hat{\phi})$ is strictly decreasing in c.*

Proof. See Appendix B. ■

We omit a statement and proof of a related result on the comparative statics of price, because it is relatively complicated. It is analogous to the proof of comparative statics in price of the first two examples and works along the same lines as the proof of the proposition on the critical discount factor. Intuitively, the proposition about the discount factor suggests that at a price close to the collusive price, the collusive price should rise at least as fast as the monopoly price if the monopoly price increases less than one to one in costs (this is the case when demand is log-concave). Simulations on a wide range of parameter values for the nested logit model confirms that this is the case over a wide range of parameters. When demand is not log-concave as in the CES case, prices always rise more than one-to-one in the cost parameter, but less than under full monopoly.

This conclusion is in contrast to a model of transitory cost shocks. In a version of the Rotemberg and Saloner model with cost, instead of demand, uncertainty Rimler (2005) shows that price rises more than one to one in transitory cost shocks. In that model the anticipation of lower costs in the future (and with it higher profitability) makes collusion easier in a high cost state. Our analysis combined with this result suggests that one to one increases in costs are only likely to be related to collusion if they arise from cost shocks that are perceived to be transitory.

This example also demonstrates that our method allows us to assess the relative order of magnitude of different comparative statics effects on collusion. The impact of increases in marginal cost on collusion turn out to be very small. This is unsurprising since there is no effect in the homogeneous goods limit. A similar result has been obtained by Schultz (2005) for the impact of consumer market transparency on collusion. In his model there is no impact of consumer market transparency in the limit model as product differentiation goes to zero. Schultz's result is derived for a simple Hotelling model but could be greatly generalized to other demand structures using our methods. For policy purposes such results give important insights to the type of variables an antitrust authority should look at: the effect of consumer market transparency or marginal cost changes on collusion is small relative to the impact of, for example, cross-ownership. This is thus another example of how the analysis of a change in the incentive constraints close to homogeneity can lead to a rich set of new results that allow us to assess both the robustness and order of magnitude different variables on the ability to collude.

5 Comparative Statics in Asymmetric Games

Some of the important policy issues in collusion theory concern the impact of asymmetries on collusion in a market. An example is the analysis of the coordinated effects of mergers in Compte, Jenny, and Rey (2002) or Kühn (2004). Clearly in such games we can identify a best collusive equilibrium outcome on the Pareto frontier of the equilibrium value set for any bargaining solution we select. However, it is exceedingly difficult to establish any analytical comparative statics results. Typically value sets will not be nested as the relevant parameter is changed in such models - even at the homogeneous goods limit. However, this is not the most important obstacle for comparative statics given a well defined selection rule from the equilibrium value set. The essential problem for comparative statics in equilibrium values is that changes in parameters make the equilibrium value set more asymmetric. This means information about how the slope of the Pareto frontier of the equilibrium value set changes is crucial in determining the selection of equilibrium. This is typically impossible to determine analytically. An easier task is the comparative statics of the critical discount factor at which the bargaining selection from the unconstrained Pareto frontier can just be sustained. This will be possible whenever the Pareto frontier (or an appropriate tranformation) is invariant to the parameter change. However, even that is often not the case. Because of these features we will typically have to rely primarily on numerical methods to gain insights into the comparative statics of collusion in asymmetric games.

However, there remains an important limitation to deriving comparative statics in prices even when the analysis is done numerically. We have not been able to find sufficient conditions that guarantee that the best collusive equilibrium under some bargaining rule is stationary on the equilibrium path. However, our numerical methods only allow us to derive comparative statics in prices for the best stationary equilbirium. Fortunately, this does not seem to be a very serious restriction given that the equilibrium value set is calculated in an unconstrained way. The selection of the best stationary equilibrium makes practical sense because it is extremely simple, but does not rely om arbitrary assumptions on the payoffs obtained under punishments. The punishments remain exactly the same as the most severe punishment for the best collusive equilibrium. We have also confirmed for all the models we have solved numerically, that the comparative statics of values in the best stationary equilibrium follows those of the best collusive equilibrium very closely. We are therefore confident that this is a restriction on the equilibria considered that is fairly innocuous.

In the rest of this section we first continue with our discussion of the impact of the comparative statics on costs. Here we demonstrate numerically the importance of cost asymmetries in preventing collusion. This is similar to previous work on asymmetries in collusion in Kühn (2004). However, the view that asymmetries always make collusion more difficult is shown to be incorrect in our second example, where we look at asymmetric increases in cross-ownership. We show that such increases always facilitate collusion.

5.1 Example 1: Cost Asymmetries

In this section we continue our discussion of the effects of cost changes. In contrast to a common permanent change in costs we now look at cost asymmetries and their impact on collusion. However, the analysis in the symmetric setting allows us to generate some intuition about the impact of asymmetries. Suppose we start from a position of symmetric costs and increase the marginal costs of firm i slightly. We want to show in this section that all collusive prices may fall as a result of this cost increase for one firm.

Note that none of the incentive constraints of other firms j are directly affected when firm i's costs are increased. Consider the possiblity that the collusive price p^m before this cost changes can be maintained after wards and that the same punishment prices are incentive compatible for the firms not experiencing the price change. Then the change in the tightness of the incenive constraint for firm i can be written analogously to (14) as:

$$\xi_c = -\frac{1}{n}\bar{D}(p^m(c)) + (1-\delta(c,\phi))D_i(p^*(p^m(c),\phi), p^m(c), \phi) - \delta\frac{\partial \underline{v}(c,\phi)}{\partial c} \tag{15}$$

Note that the second term in (14) now does not appear since the other firms are not increasing the collusive price. Using our previous analysis it is straightforward that $\lim_{\phi\to 0}\xi_c = 0$. Differentiating in ϕ yields

$$\xi_{c\phi} = (1-\delta(c,\phi))D_{p_i}\frac{\partial p^*}{\partial \phi} - \delta\frac{\partial^2 \underline{v}(c,\phi)}{\partial c \partial \phi}. \tag{16}$$

The first term is clearly strictly positive and the effect on the continuation value in punishment should be of smaller order. Hence, it appears that the price would still be sustainable. However, this overlooks that the incentive constraint for i is tightened when it comes to punishing other firms. To see this write this constraint as:

$$(1-\delta)(p_{i_L} - c)D(\mathbf{p}_L, \phi) + \delta v^c = (p_i^*(\mathbf{p}_{-L}, \phi) - c)D(p_i^*, \mathbf{p}_{-L}, \phi), \tag{17}$$

where v^c is the average value in the continuation equilibrium, which is not changed by assumption. The marginal impact on the tightness of the incentive constraint is given by:

$$\xi_c^L = -(1-\delta)D(\mathbf{p}_L, \phi) + D(p_i^*, \mathbf{p}_{-L}, \phi),$$

where $D(p_i^*, \mathbf{p}_{-L}, \phi)$ becomes arbitrarily close to zero as $\phi \to 0$. Hence, close to perfect homogeneity the incentive constraint for conforming with punishments for another firms's deviation is strictly tightened. The only way to satisfy this incentive constraint is to raise p_{iL}. But then the punishment profits of firms j must strictly increase, violating the incentive compatibility condition whenever the incentive constraints for collusion binds. This contradicts the claim that the price p^m can be sustained.

In fact, the impact on the punishment price p_{iL} of an increase in marginal cost c is a first order effect, while the impact on the collusion incentive constraint of firm i is a second order effect. For this reason an increase in the costs of firm i makes collusion overall more difficult. This is dramatically demonstrated in our numerical example of Figure 4.

[Insert Figure 4]

Figure 4 shows the price comparative statics obtained when using the Nash bargaining solution with a $(0,0)$ threat point.[14] Close to complete symmetry we obtain the expected comparative statics that prices strictly decline as long as the incentive constraint is initially strictly binding. Eventually prices have to rise in line with the increasing price in the one shot Nash equilibrium. Note, that the prices in this example are often identical which occurs because ϕ is small and the price grid used in the calculations limits the ability of firms to set slightly different prices. What the example shows is that with fairly homogeneous goods the ability to collude dramatically declines with some cost heterogeneity and then remains low. The analysis therefore justifies the importance that the antitrust literature has put in recent years on cost asymmetries for evaluating the likelihood of collusion in a market.

This comparative statics of prices is quite distinct from that in Kühn (2004) where the comparative statics was in length of product line, not differences in marginal cost. In Kühn (2004) the price of the company losing market share declines while the firm increasing market share increases the price. The difference arises because in Kühn (2004) the the incentive constraint at the collusive price is relaxed while it is tightened for the punishment price and these effects are of the same order.

5.2 Example 2: Asymmetric Changes in Cross-Ownership

We now show with the example of cross-shareholdings that increased asymmetry does not necessarily imply a lower ability to collude. For that purpose we extend the symmetric cross-shareholding model of section 4.2 to allow for any cross shareholdings with the property that firm i controls its own variety i and does not control varieties j in which it has a minority share holding. We denote the share of firm i in the profits of variety j as α_{ij}. Consider an arbitrary increase of the share held by firm l in variety k, α_{lk}. Suppose that $\mathbf{p}^C$ is the price vector in the best stationary equilibrium, which is sustained by reverting to the lowest continuation value $\underline{v}_i$ if i deviates from p_i^c. Fix the strategy profile that sustains this outcome. The incentive compatibility constraint for a firm i in the best stationary equilibrium is then given by:

$$\sum_{j=1}^{n} \alpha_{ij}\pi_j(\mathbf{p}^c) \geq (1-\delta)\left[\alpha_{ii}\pi_i(p_i^*(\mathbf{p}_{-i}^c,\phi),\mathbf{p}_{-i}^c,\phi) + \sum_{j\neq i}\alpha_{ij}\pi_j(\mathbf{p}_{-i}^c, p_i^*(\mathbf{p}_{-i}^c,\phi),\phi)\right] + \delta\underline{v}_i \tag{18}$$

We can write $\underline{v}_i$ in terms of profits as $\underline{v}_i = \sum_{j=1}^{n}\alpha_{ij}\pi_j^*(\mathbf{p}_{-i}^c, p_i^*(\mathbf{p}_{iL},\phi),\phi)$, where $\mathbf{p}_{iL}$ is the price vector to be charged after i deviates from a collusive agreement. The expression $\pi_j^*(\mathbf{p}_{iL},\phi)$ represents the profit of variety j when firm i optimally deviates from p_{iL}.

We will show that, for ϕ close enough to zero, a strategy profile that sustains $\mathbf{p}^c$ for the initial shareholding distribution can still be sustained if a firm l increases its ownership share in a variety k.

Fix $\mathbf{p}^c > 0$ and let $\delta(\boldsymbol{\alpha},\phi,\mathbf{p}^c)$ be the lowest dissount factor at which the price vector $\mathbf{p}^c$ can be made incentive compatible for all i. Let $\mathbf{p}_L^i(\phi)$ be the associated vector of punishment prices in the first period after firm i deviated. We want to show that if α_{lk} is increased for some pair (l,k), the incentive compatibility constraint will be weakly relaxed at the old strategies for all firms. First note, that incentives for firms $i \neq l,k$ are not affected directly affected by the asset transaction. Even firm k's

[14] This is identical to the Rubinstein bargaining solution over the value set. We do not use the Nash equilibrium threatpoint here to avoid having asymmetry effects enter purely through the threat point.

incentive constraints will not be affected if it does not have any cross-shareholdings in other varieties. In that case a change in α_{lk} alters profits on the left and right hand side of the incentive constraint of firm k proportionally, so that the incentive constraint remains unchanged. One can show that otherwise the incentive constraints are relaxed for firm k when ϕ is close to zero. Here we focus on the incentive constraint for firm l. Again consider the marginal impact on the incentive constraint (18) for given $\mathbf{p}^C$ and p_L:

$$\begin{aligned}\xi_{\alpha_{lk}} &= \pi_k(\mathbf{p}^c) - (1-\delta(\boldsymbol{\alpha},\phi,\mathbf{p}^c))\pi_k(\mathbf{p}^c_{-l}, p^*(\mathbf{p}^c_{-l},\phi),\phi) \\ &\quad -\delta(\boldsymbol{\alpha},\phi,\mathbf{p}^c)\pi_k(\mathbf{p}^l_{-lL}, p^*(\mathbf{p}^l_{-lL},\phi),\phi)\end{aligned} \tag{19}$$

First, note that $\pi_k(\mathbf{p}^c_{-l}, p, \phi)$ is increasing in p. Since $p^*(\mathbf{p}^c_{-l},\phi) \leq p^c_k$, we have $\pi_k(\mathbf{p}^c) \geq \pi_k(\mathbf{p}^c_{-l}, p^*(\mathbf{p}^c_{-l},\phi),\phi)$, so that the sum of the first line in (**??**) is strictly positive for all ϕ. For ϕ close to 0, we also have $\pi_k(\mathbf{p}_{-lkL}, p^*(\mathbf{p}^C_{-lkL},\phi),\phi) < 0$. The reason for this is that in the homogeneous goods limit model a one shot Nash equilibrium will attain zero profits. This means that in the limit model optimal punishment prices are strictly below marginal costs, which will also hold for the differentiated goods model for a low enough degree of product differentiation. Hence, $\lim_{\phi\to 0}\xi_{\alpha_{lk}} > 0$.

Now consider the incentive constraint for p_{lL}:

$$(1-\delta)\sum_{j=1}^{n}\alpha_{lj}\pi_j(\mathbf{p}_L) + \delta v_l^C \geq \left[\alpha_{ii}\pi_i(p_i^*(\mathbf{p}_{-iL},\phi),\mathbf{p}_{-iL},\phi) + \sum_{j\neq i}\alpha_{ij}\pi_j(\mathbf{p}_{-iL}, p_i^*(\mathbf{p}_{-iL},\phi),\phi)\right] \tag{20}$$

For a given strategy profile represented by $(\mathbf{p}^C, \mathbf{p}_L)$, the marginal impact on the incentive constraint is given by:

$$\begin{aligned}\xi^L_{\alpha_{lk}} &= (1-\delta)\pi_k(\mathbf{p}_L) - \pi_k(\mathbf{p}_{-iL}, p_i^*(\mathbf{p}_{-iL},\phi),\phi) \\ &= (p_{kL}-c)[(1-\delta)D(\mathbf{p}_L) - D(\mathbf{p}_{-iL}, p_i^*(\mathbf{p}_{-iL},\phi),\phi)]\end{aligned}$$

Note that $D(\mathbf{p}_{-iL}, p_i^*(\mathbf{p}_{-iL},\phi),\phi) > D(\mathbf{p}_L)$, so that $\xi^L_{\alpha_{lk}} > 0$ if and only if $p_{kL} - c < 0$. Since this is the case for ϕ close to zero, any strategy profile that can be sustained at an ownership distribution $\boldsymbol{\alpha}$ can also be sustained at a new ownership distribution $\hat{\boldsymbol{\alpha}}$ that differs from $\boldsymbol{\alpha}$ because $\hat{\alpha}_{lk} > \alpha_{lk}$ for some $l \neq k$. As a result, an increase in cross-ownership shares must strictly increase the set of sustainable price vectors. In this sense one could prove a result that collusion becomes easier. Such a proposition would not allow one to infer the comparative statics of prices. Although the set of prices that is sustainable in a stationary collusive equilibrium is increased by the change in ownership share, it cannot be concluded that all prices are increased at the equilibrium selected by any specific bargaining solution.[15]

However, the analytical argument is suggestive for the type of comparative statics results that we should expect in numerical analysis. Consider first the case in which the firm that sells a share holding to the acquiring firm does not possess any ownership in other varieties itself. Then only the incentive constraint for the firm acquiring the cross shareholding is relaxed from the increase in the

[15] In this example there is a comparative statics result available for the critical discount factor when the bargaining solution selects the industry profit maximizaing equilibrium. This is the case because the joint profit maximizaing price is p^m, independently of the share distribution. However, such a result is not available for any other bargaining solution selected, because the price vector selected from the unconstrained Pareto frontier will typically vary in $\boldsymbol{\alpha}$. For this reason we omit this result.

cross-shareholding. This suggests that in any bargaining solution we should expect the price of firm l to rise relative to the price of all other firms. This feature is confirmed in numerical calculations for the CES and nested logit models with duopoly and under the assumption of Nash bargaining (with a $(0, 0)$ threatpoint). Both prices rise but the price of the share acquiring firm rises faster. This is clearly seen in the lowest graph in Figure 5. Both prices rise as the unilateral share holding is increased but the price of the firm that increases the shareholding is increased more.

[Figure 5]

For pre-existing cross-share holdings this insight also applies. Consider the second lowest price path, which is generated when firm 2 has a cross-share holding of 24% in firm 1 and we increase the share of firm 1 in firm 2 from 0 to 50%. Note, that initially the increased cross- shareholding dramatically increases all prices. However the price of firm 2 is initially higher and the difference in the prices becomes smaller as α_{12} approaches 24%. Note also that the price of firm 2 overshoots the monopoly price in order to give incentives to the firm with the lower cross-share holding to charge a higher price. This is an effect that has been demonstrated in Kühn (2004) for asymmetric product lines as well. Once firm 1 has a larger cross-shareholding than firm 2 the relative prices invert. In fact the price of firm 2 starts falling below the monopoly price while the price for firm 1 rises above the monopoly price. The reason for this is not that collusion at the monopoly price is not sustainable. In fact, the monopoly price is sustainable for a higher cross shareholding if it is sustainable for a lower cross-share holding. The reason for the difference in price is a result of bargaining over an equilibrium value set that has become very asymmetric. Essentially, acquiring a bigger share holding weakens the bargaining position of the firm with the larger share holding, which has to accept an unfavorable price differential in order to shift market share to the firm with the smaller cross-share holding. This effect is not specific to the bargaining solution used in this calculation. Any symmetric solution applied to an increasingly asymmetric equilibrium value set will shift profits to the firm with the tighter incentive constraints. This effect completely drives the comparative statics in the last example in which firm 2 already owns almost 50% of variety 1. Here firm 2 already sets a price above the monopoly price to give incentives to firm 1 to set a high price. With an increased cross-shareholding of firm 1 both prices go up. However, as the cross-shareholding of firm 1 in firm 2 increases, firm 2 can keep firm 1at the monopoly price even when charging a lower price. The main effect of further increases in cross-shareholdings is to bring the overall price level down. [16]

This observation shows that in asymmetric models two different effects impact on the comarative statics of prices. One is the usual effect of relaxing or tightening incentive constraints. The second comes in through the impact of bargaining. Bargaining under asymmetric conditions can move firms away from the industry profit maximizing solution although that solution would be incentive compatible. The impact on consumers may, however be even worse because price may be raised even above the monopoly price to induce redistribution required by bargaining.

This example shows how sometimes analytical results can help to understand asymmetric models

[16] This effect echoes a result in Kühn (2004). In his model prices were higher in a very asymmetric market than in a completely monopolized market precisely because in an asymmetric market the larger firm sets prices above the monopoly price to give the smaller firm incentives. Similarly, in this model large cross-shareholdings can be worse than outright mergers when collusion is possible.

using the limiting arguments of this paper. However, to obtain comparative statics for prices we need to complement this analysis with numerical methods.

6 Conclusions

In this paper we have presented a framework to study the robustness and order of magnitude of results in collusion theory. On the basis of a general product differentiation model we have been able to contrast the robustness of symmetric optimal punishments relative to equilibria obtained under Nash reversion in symmetric models. We have also identified why symmetric optimal punishments can be legitimately used as a modelling shortcut. We have used these results to prove new results on the impact of product differentiation, the fragmentation of asset ownership, cross-shareholdings, and cost changes on the ability to collude. We have also shown that the methods are useful for exploring the comparative statics of collusion in asymmetric models and illustrated how a combination of theoretical tools using the behavior at the homogeneous goods limit and numerical methods can be applied to obtain insight into the robustness and order of magnitude of the comparative statics of collusion.

7 Appendix A: Examples for the General model

In this appendix we show that some commonly used specifications of product differentiation models fall in the class of our general formulation.

Example 1: Spence-Dixit-Stiglitz preferences with Constant Elasticity of Substitution

The first example is the widely used CES version of the Spence-Dixit-Stiglitz model. There is a representative consumer, whose utility function is given by $U(x)=\frac{n^{1-\kappa}}{1-\kappa}\left(\frac{1}{n}\sum x_i^{1-\phi\kappa}\right)^{\frac{1-\kappa}{1-\phi\kappa}}+m$, where $\mathbf{x}$ is the vector of consumption of the n differentiated products, m is expenditure on outside goods, and $\kappa\in[0,1]$ and $\phi\in[0,1]$ are parameters. The direct demand function for product i generated from this utility function is given by:

$$D(p_i,\mathbf{p}_{-i},\phi,n)=\frac{1}{n}p_i^{-\frac{1}{\kappa}}\left(n\mathbb{Q}_i\right)^{\frac{1-\phi}{1-\phi\kappa}} \tag{21}$$

where $\mathbb{Q}_i=\left(\sum_j\left(\frac{p_i}{p_j}\right)^{\frac{1-\phi\kappa}{\phi\kappa}}\right)^{-1}$. We now show that this demand function has the desired properties. First, note that

$$\lim_{\phi\to 0}\mathbb{Q}_i=\begin{cases}1 & \text{if} \quad p_i<\min_{j\neq i}p_j\\ \frac{1}{m} & \text{if} \quad p_i=\min_{j\neq i}p_j \text{ and } m-1 \text{ other firms charge } p_i\\ 0 & \text{if} \quad p_i>\min_{j\neq i}p_j\end{cases} \tag{22}$$

It follows immediately from (21) and (22) that demand is independent of the product differentiation parameter if all products are offered at the same price. Furthermore, total demand at equal prices is given by $p^{-\frac{1}{\kappa}}$, i.e. "aggregate demand" is independent of the product differentiation parameter and the number of firms. If a firm has the strictly lowest price its demand converges to the aggregate demand as products become homogeneous, as required. Furthermore, demand converges to zero in the homogeneous goods limit if a firm charges a price strictly above the lowest price other firms charge.

We now verify that the assumptions on the slope of the demand functions made in our paper are also satisfied:

$$\varepsilon_i(p_i, \mathbf{p}_{-i}, \phi)) = -\frac{\partial D_i(p_i, \mathbf{p}_{-i}, \phi)}{\partial p_i} \frac{p_i}{D_i(p_i, \mathbf{p}_{-i}, \phi)} = \frac{1}{\kappa}\left\{1 + \frac{1-\phi}{\phi}[1 - \mathbb{Q}_i]\right\} \tag{23}$$

Unless firm i has all the demand, i.e. $\mathbb{Q}_i = 1$, the elasticity of a product i's residual demand is greater than that of the aggregate demand function. If all varieties set the same price, the term in curly brackets becomes $1 + \frac{1-\phi}{\phi}\frac{n-1}{n}$ and the elasticity converges to ∞ as $\phi \to 0$ as required. If firm i's price is higher than the lowest price $\mathbb{Q}_i \to 0$ as $\phi \to 0$ and again demand becomes infinitely elastic. However, when $p_i < \min_{j\neq i} p_j$, $\lim_{\phi\to 0} \frac{1-\mathbb{Q}_i}{\phi} = -\lim_{\phi\to 0}\frac{\partial \mathbb{Q}_i}{\partial\phi} = 0$ and $\lim_{\phi\to 0}\varepsilon_i \to \frac{1}{\kappa}$, the market elasticity of demand. To see the latter point, note that

$$\begin{aligned} \frac{\partial \mathbb{Q}_i}{\partial \phi}\frac{1}{\mathbb{Q}_i} &= -\mathbb{Q}_i \sum_{j\neq i} \frac{\partial}{\partial\phi}\left(\frac{p_i}{p_j}\right)^{\frac{1-\phi\kappa}{\phi\kappa}} & (24)\\ &= \mathbb{Q}_i \sum_{j\neq i}\left(\frac{p_i}{p_j}\right)^{\frac{1-\phi\kappa}{\phi\kappa}} \frac{1}{\phi^2\kappa}\ln\left(\frac{p_i}{p_j}\right) & (25) \end{aligned}$$

For $\frac{p_i}{p_j} < 1$, $\left(\frac{p_i}{p_j}\right)^{\frac{1-\phi\kappa}{\phi\kappa}} \frac{1}{\phi^2\kappa} \to 0$ if $\phi \to 0$. Hence, we confirm that in the limit residual demand has the same properties as the residual demand function of the homogeneous goods model. From (23) we can immediately determine the monopoly price and the static Nash equilibrium price as $p^m = \frac{c}{1-\kappa}$ and $p^N(\phi, n) = \frac{c}{(1-\kappa\left[1+\frac{1-\phi}{\phi}\frac{n-1}{n}\right]^{-1})}$ respectively.

Furthermore, we can show that demand has the properties assumed in the paper. First, we have:

$$\frac{\partial D_i(p_i, p_j, \phi)}{\partial\phi}\frac{1}{D_i} = \frac{\partial \ln D_i}{\partial \phi} = -\frac{1-\kappa}{(1-\phi\kappa)^2}\ln(n\mathbb{Q}_i) + \frac{1-\phi}{(1-\phi\kappa)}\frac{\partial\mathbb{Q}_i}{\partial\phi}\frac{1}{\mathbb{Q}_i} \tag{26}$$

When $p_i = p_j$ for all j, the first term in (26) is zero since $\ln(n\mathbb{Q}_i) = 0$, and the second term in (26) is also zero, since, from (24), $\frac{\partial\mathbb{Q}_i}{\partial\phi} = 0$ at $p_i = p_j$ for all j. For $p_i < \min_{j\neq i} p_j$, $\mathbb{Q}_i > \frac{1}{n}$ and hence $-\ln(n\mathbb{Q}_i) < 0$. Furthermore $\frac{\partial\mathbb{Q}_i}{\partial\phi} < 0$ since $\ln\left(\frac{p_i}{p_j}\right) < 0$. Hence, demand goes down as products get more differentiated for a given price ratio at which firm i has the lower price. The sign is reversed for $p_i > \max_{j\neq i} p_j$. Note that (26) also implies that our limiting condition is satisfied: If $\lim_{\phi\to 0} D(p_i(\phi), \mathbf{p}_{-i}(\phi), \phi, n) = \bar{D}(\lim_{\phi\to 0} p_i(\phi))$, then $\lim_{\phi\to 0}\mathbb{Q}_i = 1$. Hence,

$$\lim_{\phi\to 0}\frac{\partial D_i(p_i(\phi), \mathbf{p}_{-i}(\phi), \phi)}{\partial\phi} \leq -\frac{1}{1-\kappa}\ln(n) < 0.$$

where the first inequality follows from the fact that $\frac{\partial\mathbb{Q}_i}{\partial\phi} \leq 0$.

Example 2: A Nested Logit Model

We now show how a simple nested choice model falls into the class of models covered by our analysis. We consider a model in which consumers first have to decide whether to purchase or not and then decide, which product to purchase. Conditional on having made the decision to purchase, the probability of choosing brand i is given by:

$$\mathbb{Q}_i = \frac{\exp\{-p_i/\phi\kappa\}}{\sum_j \exp\{(-p_j)/\phi\kappa\}} = \frac{1}{\sum_j \exp\{-(p_j - p_i)/\phi\kappa\}} \tag{27}$$

This can be interpreted as the market share of product i. The expected value to the customer from purchasing in the market is then by standard reasoning given by (up to a constant):

$$\begin{aligned} A_{PC} &= \phi\kappa \ln\left[\frac{1}{n}\sum_i \exp\{(-p_i)/\phi\kappa\}\right] \\ &= -p_i + \phi\kappa\ln\left[\frac{1}{n}\sum_j \exp\{-(p_j - p_i)/\phi\kappa\}\right] \\ &= -p_i + \phi\kappa\ln[\frac{1}{n\mathbb{Q}_i}] = -p_i - \phi\kappa\ln[n\mathbb{Q}_i] \end{aligned} \tag{28}$$

The choice between purchasing and not purchasing is then again given by a simple logit choice:

$$\hat{\mathbb{Q}}(\mathbb{Q}_i) = \frac{\exp\{A_{PC}/\kappa\}}{\exp\{A/\kappa\} + \exp\{A_{PC}/\kappa\}} = \frac{1}{\exp\{(p_i + A)/\kappa\}\,(n\mathbb{Q}_i)^{\phi} + 1} \tag{29}$$

where $A > c$ is a constant and κ is a parameter determining the aggregate demand elasticity. Demand for variety i is then given by:

$$D(p_i, \mathbf{p}_{-i}, \phi, n) = \hat{\mathbb{Q}}\mathbb{Q}_i = \frac{1}{n}\frac{1}{\exp\{(p_i + A)/\kappa\}\,(n\mathbb{Q}_i)^{\phi} + 1}\,(n\ \mathbb{Q}_i) \tag{30}$$

When all prices are the same, $\mathbb{Q}_i = \frac{1}{n}$, and the demand function reduces to:

$$D(p, p, \phi, n) = \frac{1}{n}\frac{1}{\exp\{(p + A)/\kappa\} + 1} \tag{31}$$

This is independent of ϕ. Furthermore, for $p_i < p_j$, demand converges to $\bar{D}(p_i) = \frac{1}{\exp\{(p_i+A)/\kappa\}+1}$ as $\phi \to 0$, i.e. as goods become homogeneous the lowest price firm faces total demand. Conversely for $p_i > p_j$, demand converges to zero. To check the other properties of demand first consider:

$$\begin{aligned} \frac{D_{p_i}(p_i, p_j)}{D(p_i, p_j, \phi)} &= \frac{d\hat{\mathbb{Q}}}{dp_i}\frac{1}{\hat{\mathbb{Q}}} + \frac{\partial \mathbb{Q}_i}{\partial p_i}\frac{1}{\mathbb{Q}_i} \\ &= -\left[\frac{1}{\kappa}(1 - \hat{\mathbb{Q}})\mathbb{Q}_i + \frac{1}{\phi\kappa}(1 - \mathbb{Q}_i)\right], \end{aligned} \tag{32}$$

which is a weighted average of the aggregate semi-elasticity $\frac{1}{\kappa}(1-\hat{\mathbb{Q}})$ and the semi-elasticity of a monopolistically competitive firm $\frac{1}{\phi\kappa}$. We also have that :

$$\begin{aligned} D_\phi(p_i, p_j, \phi) &= -\frac{p-p_i}{\phi^2\kappa}(1-\mathbb{Q}_i)D(p_i, \mathbf{p}_{-i}, \phi) \\ &+ \left[\frac{1}{\kappa}\ln\left(\frac{1}{n}\sum_j \exp\left\{-(p-p_i)/\phi\right\}\right) - \frac{\phi}{\kappa}\frac{p-p_i}{\phi^2}(1-\mathbb{Q}_i)\right](1-\hat{\mathbb{Q}})D(p_i, p, \phi) \end{aligned} \tag{33}$$

At $p_i = p$ this collapses to 0 for any ϕ.

Example 3: A normalized Hotelling Duopoly

In the standard Hotelling model the monopoly price changes as the product differentiation parameter changes. We here describe a simple modification of a Hotelling duopoly model for which all the assumptions of our model hold. There are two firms each located at the end of a line of length 1. A buyer x is located a distance x away from the left end of the line. The utility from buying good 1 at the left end of the line is $v + \frac{\phi}{4} - \phi x^2 - p_1$. The utility of buying from firm 2 at the right end of the line is $v + \frac{\phi}{4} - \phi(1-x)^2 - p_2$. Note that our only modification from the standard Hotelling model is that the reservation price changes in ϕ. We assume that $\phi < (v-c)/3$. This condition is sufficient for a monopolist with two outlets to cover the market in equilibrium. Clearly, if the market is covered, the monopoly price is $p^m = v$, and monopoly profit $(v-c)\,\frac{1}{2}$. The elasticity of demand for a given variety at equal prices is $\frac{p}{\phi}$. Hence, the Nash equilibrium between two competing firms is $p^N = \phi + c$, which converges to marginal cost as $\phi \to 0$. Nash equilibrium profits are given by $\phi\frac{1}{2}$.

The condition $\phi < (v-c)/3$ also guarantees that the optimal deviation from a common price $p^m = v$ is to the highest price at which the deviating firm just obtains the full market, i.e. $p^*(v) = v - \phi$. Hence, deviation profits are given by $(v - \phi - c)$. It follows that $\frac{d\Pi(p^*(p^m), p^m, \phi)}{d\phi} = -1 < 0$, a property that we have generally assumed in the paper.

8 Appendix B: Proofs

Proof of Proposition 1: (*Existence and Comparative Statics of the Nash Equilibrium*)

Proof: The assumptions in the model section guarantee that demand is log-supermodular in prices and that best responses are contraction mappings that implies that there exists a unique Nash equilibrium in prices. Furthermore, assuming $\frac{\partial^2 \ln D(p_i, \mathbf{p}_{-i}, \phi, n)}{\partial p_i \partial \phi} > 0$ and $\frac{\partial^2 \ln D(p_i, \mathbf{p}_{-i}, \phi, n)}{\partial p_i \partial n} < 0$ guarantees that equilibrium prices are increasing in ϕ and decreasing in n. QED.

Lemma B1: For any function $p(\phi)$ with $p(\phi) \in [p^N(\phi), p^m]$ for all ϕ, the price

$$p^*(p, \phi) = \max_{p_i}\left[(p_i - c)D(p_i, p, \phi)\right],$$

converges to $\lim_{\phi\to 0} p(\phi)$ for $\phi \to 0$. Furthermore,

$$\lim_{\phi\to 0} -\frac{D_{p_i}(p^*(p(\phi), \phi), \phi)}{D(p^*(p(\phi), \phi), \phi)} = \lim_{\phi\to 0}\frac{1}{p(\phi) - c} \geq \lim_{\phi\to 0} -\frac{D'(p(\phi))}{D(p(\phi))},$$

where the last holds with equality if and only if $\lim_{\phi\to 0} p(\phi) = p^m$.

Proof: In the limit $p^* > p$ is impossible since the firm would make zero profits and it could make strictly positive profits and face the market demand function by setting $p_i < p$. We now show that $\lim_{\phi\to 0} p^* < p$ is also not possible. Suppose that would be the case. Then $\lim_{\phi\to 0} p^* - c < p - c \leq -\frac{D(p)}{D'(p)}$, where the last inequality follows by the fact that $\frac{D'(p)}{D(p)}$ is decreasing in p, due to log-concavity of demand. Hence, by quasi-concavity of the limit industry profit function in p, the firm could, for ϕ close to zero, increase price and increase profits. Since the model is smooth for every ϕ the elasticity has to converge to the inverse of $\frac{p-c}{p}$QED.

Lemma B2: Under the assumptions of our model $\lim_{\phi\to 0} D(p^*(p,\phi), p, \phi) = \bar{D}(p)$

and $\lim_{\phi\to 0} D_\phi(p^*(p,\phi), p, \phi) < 0$.

Proof: Suppose $\lim_{\phi\to 0} D(p^*(p,\phi), p, \phi) = \lambda\bar{D}(p) < \bar{D}(p)$. Now consider the alternative strategy $p^*(p,\phi) - \varepsilon$. Since $p^*(p,\phi) \to p$ it follows from our assumptions that $\lim_{\phi\to 0} D(p^*(p,\phi) - \varepsilon, p, \phi) = \bar{D}(p - \varepsilon)$. Clearly for ε small enough $(p - \varepsilon - c)\bar{D}(p - \varepsilon) > (p - c)\lambda\bar{D}(p)$. Hence, for some $\hat{\phi} > 0$ zero this strategy give strictly higher profits for all $\phi \in (0, \hat{\phi})$, contradicting that $p^*(p,\phi)$ is a best response on this range. Hence, $\lim_{\phi\to 0} D(p^*(p,\phi), p, \phi) = \bar{D}(p)$. Then it follows directly from assumption 1 that $\lim_{\phi\to 0} D_\phi(p^*(p,\phi), p, \phi) < 0$. QED

We now derive a Lemma that allows us to characterize the pay offs from symmetric optimal punishment equilibria. To do this let p_L be the price set at the start of a punishment period and let $\hat{v}$ be the continuation value after the first punishment period, i.e. $\underline{v}^O(\phi, \delta, p^c, n) = (1 - \delta)\pi(p_L, p_L, n) + \delta\hat{v}$. Then we have:

Lemma B3: The incentive constraint for optimal punishments (**??**) is always binding, so that $\underline{v}^O(\phi, \delta, p^c, n) = \pi(p^*(p_L, \phi, n), p_L, \phi, n)$. Either $p_L = 0$ and $\hat{v} < \pi(p^c, p^c, n)$, or the optimal punishment price solves

$$\pi(p^*(p_L, \phi, n), p_L, \phi, n) = (1 - \delta)\pi(p_L, p_L, n) + \delta\pi(p^c, p^c, n) \tag{34}$$

In this case $p_L(\phi, \delta, p^c, n)$ *is increasing in* ϕ *and* n *and decreasing in* δ *and* p^c.

Proof of Lemma B3: Suppose that (**??**) were slack, i.e.$\underline{v}^O(\phi, \delta, p^c, n) = (1 - \delta)\pi(p_L, p_L, n) + \delta\hat{v} > \pi(p^*(p_L, \phi, n), p_L, \phi, n)$. Then $\pi(p^c, p^c, n) \geq \hat{v} > \underline{v}^O$ since $\pi(p_L, p_L, n) \leq \pi(p^*(p_L, \phi, n), p_L, \phi, n)$. By convexity of the value set $\hat{v}$ could be strictly reduced and the constraint would still be slack and the constraint on the highest sustainable price would be relaxed. This contradicts the definition of $\underline{v}^O(\phi, \delta, p^c, n)$. Hence, $\underline{v}^O(\phi, \delta, p^c, n) = \pi(p^*(p_L, \phi, n), p_L, \phi, n)$. Now suppose $p_L > 0$ and $\hat{v} < \pi(p^c, p^c, n)$. Then p_L could be lowered slightly and $\hat{v}$ increased to leave $\underline{v}^O(\phi, \delta, p^c, n)$ unchanged. But then the incentive constraint is slack since $\pi(p^*(p_L, \phi, n), p_L, \phi, n)$ strictly increase in p_L. It follows that either $p_L = 0$ and $\hat{v} < \pi(p^c, p^c, n)$ or $p_L > 0$ and $\hat{v} = \pi(p^c, p^c, n)$. In the first case p_L does not vary with small changes in ϕ, δ, or p^c. In the second case $p_L(\phi, \delta, p^c, n)$ solves (34). First note, that at any optimal punishment price p_L, $(1 - \delta)\frac{d\pi(p_L, p_L, n)}{dp_L} - \frac{d\pi(p^*(p_L, \phi, n), p_L, \phi, n)}{dp_L} > 0$. Otherwise, p_L could be slightly reduced and the incentive constraint be relaxed. By totally differentiating (**??**) we find that $\frac{dp_L}{d\phi}$ has the same sign as $\pi_\phi(p^*(p_L, \phi, n), p_L, \phi, n) > 0$, $\frac{dp_L}{d\delta}$ has the same sign as $-[\pi(p^c, p^c, n) - \pi(p_L, p_L, \phi)] < 0$, $\frac{dp_L}{dp^c}$ has the same sign as $-\delta\frac{d\pi(p^c, p^c, n)}{dp^c}$ and $\frac{dp_L}{dn}$ has the same sign as $\frac{1}{n}\pi(p^*(p_L, \phi, n), p_L, \phi, n) + \pi_n(p^*(p_L, \phi, n), p_L, \phi, n) > 0$. QED.

Proof of Lemma 1: We begin by proving that the highest sustainable collusive price under Nash reversion always exceeds the Nash price for $\phi \in (0,1)$. Consider the incentive condition on p^c given by (1). Differentiating $\pi(p^c, p^c, n) - (1-\delta)\pi(p^*(p^c, \phi, n), p^c, \phi, n)$ with respect to p^c and evaluating this at $p^c = p^N(\phi, n)$ yields $\delta \frac{\partial \pi(p^N, p^N, n)}{\partial p_j} > 0$, which implies that the incentive constraint is strictly relaxed. Hence, $p^c > p^N$ and $\pi(p^{Nc}, p^{Nc}, \phi, n) > \pi(p^N(\phi, n), p^N(\phi, n), \phi, n) > 0$. Since the Nash punishment strategy is available when considering the set of fully optimal punishments, $\underline{v}^O(\phi, \delta, p^c, n) \leq \underline{v}^N(\phi, \delta, p^c, n)$ and $\pi(p^{Nc}, p^{Nc}, n)$ can be sustained as average profits under optimal punishment equilibria. Furthermore, by the existence of a public signal any $v \in [\underline{v}^N(\phi, \delta, p^c, n), \pi(p^{Nc}, p^{Nc}, n)]$ can be supported as an average payoff in some equilibrium belonging to the set of optimal symmetric punishment equilibria. Now choose $p_L = p^N(\phi, n) - \varepsilon$ so that $\pi(p^N(\phi, n), p^N(\phi, n), \phi, n) - \pi(p^*(p_L, \phi, n), p_L, \phi, n) = \eta(\varepsilon)$ and $\pi(p^N(\phi, n), p^N(\phi, n), \phi, n) - \pi(p_L, p_L, \phi, n) = \gamma(\varepsilon)$, where clearly $\frac{\eta(\varepsilon)}{\gamma(\varepsilon)} < 1$ for all $\varepsilon > 0$, increasing in ε in the neighborhood of $\varepsilon = 0$, and $\lim_{\varepsilon \to 0} \frac{\eta(\varepsilon)}{\gamma(\varepsilon)} = 1$, as well as $\lim_{\varepsilon \to 0} \eta(\varepsilon) = \lim_{\varepsilon \to 0} \gamma(\varepsilon) = 0$. Then there exists $\hat{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \hat{\varepsilon})$, $\eta(\varepsilon) - (1-\delta)\gamma(\varepsilon) > 0$. Then one can choose $\varepsilon \in (0, \hat{\varepsilon})$, such that $\hat{v}(\varepsilon) = \pi(p^N(\phi, n), p^N(\phi, n), \phi, n) + \frac{1}{\delta}[\eta(\varepsilon) - (1-\delta)\gamma(\varepsilon)]$ and $\hat{v} \in (\underline{v}^N(\phi, \delta, p^c, n), \pi(p^{Nc}, p^{Nc}, n))$. By construction this satisfies $(1-\delta)\pi(p_L, p_L, n) + \delta\hat{v} = \pi(p^*(p_L, \phi, n), p_L, \phi, n)$. Hence, $p_L < p^N(\phi, n)$ is sustainable as a punishment equilibrium and $\underline{v}^O(\phi, \delta, p^c, n) < \underline{v}^N(\phi, \delta, p^c, n)$. This immediately implies property (2). To see this note that at p^{Nc} the incentive constraint for the most profitable collusive price is binding under Nash punishments unless $p^{Nc} = p^m$. Hence, it is slack under symmetric optimal punishments at p^{Nc} and $p^{Oc} > p^{Nc}$ unless $p^{Nc} = p^m$. To see property (3), first note that at $\delta > 1$ the constraint is strictly slack at $p^c = p^m$. If $\delta = 0$, it is violated. Property (3) follows by noting that $\delta(\phi, n)$ is strictly increasing in $\underline{v}$. Property (4) almost directly follows from the previous results: Average profits are bounded from below by 0 since a firm can always deviate to a strategy of marginal cost pricing in every period. Since $\underline{v}^N(\phi, \delta, p^c, n) \to 0$ as $\phi \to 0$, the same holds for $\underline{v}^O(\phi, \delta, p^c, n)$ by property 1. Note that at the limit the same incentive condition has to be satisfied for p^c whether Nash reversion punishments are used or symmetric optimal punishments. Hence $\lim_{\phi \to 0} p^{Nc} = \lim_{\phi \to 0} p^{Oc}$. Finally, note that by Lemma B1, $\lim_{\phi \to 0} p^*(p^m, \phi, n) = p^m$. We first show that Hence, $\lim_{\phi \to 0} \pi(p^*(p^{Nc}(\phi), \phi, n), \phi, n) = (p^m - c)D(p^m)$. Consider a strategy where the undercutting firm sets $p^*(p^m, \phi, n) - \varepsilon$ for every ϕ. By Lemma B1, $\lim_{\phi \to 0} p^*(p^m, \phi, n) - \varepsilon = p^m - \varepsilon$. Since $p^m - \varepsilon < p^m$, the firm would serve the whole market at $p^m - \varepsilon$ in the limit. Hence, in the limit, the firm would make a profit of $(p^m - \varepsilon - c)D(p^m - \varepsilon)$. Since the firm can use such a strategy for every $\varepsilon > 0$, in the limit, profits cannot fall below $(p^m - c)D(p^m)$, the maximal attainable profits in the market. Hence,

$$\begin{aligned}
\lim_{\phi \to 0} \delta(\phi, n) &= \frac{\lim_{\phi \to 0} \pi(p^*(p^{Nc}(\phi), \phi, n), \phi, n) - \lim_{\phi \to 0} \pi(p^{Nc}, p^{Nc}, n)}{\lim_{\phi \to 0} \pi(p^*(p^{Nc}(\phi), \phi, n), \phi, n) - \lim_{\phi \to 0} \underline{v}(\phi, \delta, p^m, n)} \\
&= \frac{\frac{n-1}{n}(p^m - c)D(p^m)}{(p^m - c)D(p^m)} = \frac{n-1}{n}
\end{aligned}$$

where we have used the fact that $\lim_{\phi \to 0} \underline{v}(\phi, \delta, p^m, n) = 0$ independently of the punishment strategies allowed. This completes the proof.QED.

Proof of Proposition 2: The fact that $p^c > p^N$ in part (a) follows directly from Lemma 1. Now suppose that $p^{Nc} \to p^m$ for $\phi \to 0$ if $\delta \geq \frac{n-1}{n}$. By Lemma A1, $\lim_{\phi \to 0} p^*(p^m, \phi, n) = p^m$. Then the right hand side of (1) converges to

$$(1-\delta) \lim_{\phi \to 0} \pi(p^*(p^m, \phi, n), p^m, \phi, n) \leq \frac{1}{n}(p^m - c)D(p^m) = \pi(p^m, p^m, n)$$

and the incentive compatibility constraint is satisfied. It follows that p^m can be sustained and therefore p^{Nc} and by Lemma 1 also p^{Oc} must converge to p^m. Now assume for contradiction that $p(0) = \lim_{\phi\to 0} p^{Oc} > c$ if $\delta < \frac{n-1}{n}$. Consider a strategy where the undercutting firm sets $p^*(p^{Oc}(\phi), \phi, n) - \varepsilon$ for every ϕ. By Lemma A1, $\lim_{\phi\to 0} p^*(p^{Oc}(\phi), \phi, n) - \varepsilon = p(0) - \varepsilon$. Since $p(0) - \varepsilon < p(0)$, the firm would serve the whole market at $p(0) - \varepsilon$ in the limit. Hence, in the limit, the firm would make a profit of $(p(0) - \varepsilon - c)D(p(0) - \varepsilon)$. Since the firm can use such a strategy for every $\varepsilon > 0$, in the limit, average profits from deviation have to exceed $(1 - \delta)(p(0) - c)D(p(0)) > \frac{1}{n}(p(0) - c)D(p(0))$. Hence, the firm would have a strict incentive to deviate close to $\phi = 0$. It follows that $\lim_{\phi\to 0} p^{Oc} = \lim_{\phi\to 0} p^{Nc} = c$, where the first equality follows from Lemma 1. To prove part (b) note that $\delta(\phi, n)$ is defined as the lowest discount factor for which p^m is sustainable. Hence, it cannot be sustained by lower discount factors by definition. For $\delta > \delta(\phi, n)$ incentive constraints are always strictly relaxed. Hence, p^m is still sustainable. This completes the proof.QED.

Proof of Lemma 2: The comparative statics of collusion can be calculated from (1) by totally differentiating. This yields

$$\frac{\partial p^c(\phi, \delta, n)}{\partial \phi} = \frac{\xi_\phi(\phi, p^c(\phi, \delta, n), \delta, n)}{\frac{d\pi(p^c, p^c, n)}{dp^c} - \xi_{p^c}(\phi, p^c(\phi, \delta, n), \delta, n)}$$

and

$$\frac{\partial \delta(\phi, n)}{\partial \phi} = -\frac{\xi_\phi(\phi, p^m, \delta(\phi, n), n)}{\xi_\delta(\phi, p^m, \delta(\phi, n), n)}$$

Now note that $\frac{d\pi(p^c, p^c, n)}{dp^c} - \xi_{p^c}(\phi, p^c(\phi, \delta, n), \delta, n) < 0$. Otherwise the incentive constraint could be relaxed by increasing p^c, contradicting the definition of p^c. Furthermore,

$$\xi_\delta(\phi, p^m, \delta(\phi, n), n) = -(\pi(p^*(p^m, \phi, n), p^m, \phi, n) - \underline{v}(\phi, p^m, \delta(\phi, n), n)) + \delta\frac{\partial \underline{v}(\phi, p^m, \delta(\phi, n), n))}{\partial \delta}.$$

In the case of Nash punishments $\frac{\partial \underline{v}(\phi, p^m, \delta(\phi, n), n))}{\partial \delta} = 0$ and $\xi_\delta < 0$. For the case of symmetric optimal punishment equilibria,

$$\frac{\partial \underline{v}(\phi, p^m, \delta(\phi, n), n))}{\partial \delta} = \frac{d\pi(p^*(p_L, \phi, n), p_L(\phi, \delta, p^m, n), \phi, n)}{dp_L}\frac{\partial p_L}{\partial \delta} \leq 0,$$

where the last inequality follows from Lemma B2. Hence, $\frac{\partial p^c(\phi, \delta, n)}{\partial \phi} > 0$ and $\frac{\partial \delta(\phi, n)}{\partial \phi} < 0$ if and only if $\xi_\phi < 0$.QED.

Lemma B4: Suppose $D_{p_i}(p^N, p^N, \phi)$ has an asymptotic expansion in ϕ with leading term $\phi^{-\xi}$, then $\lim_{\phi\to 0} \frac{dp^N}{d\phi} = c\frac{\lim_{\phi\to\infty} \varepsilon_\phi^{D_{p_i}}(p^N, \phi)}{\lim_{\phi\to\infty} \phi\varepsilon_i(p^N, \phi)}$

Proof of Lemma B4: Totally differentiating the first order condition at a Nash equilibrium given by $\left(p^N - c\right) D_{p_i}(p^N, p^N, \phi) + D(p^N, p^N, \phi) = 0$ with respect to ϕ and p^N yields:

$$\begin{aligned}
\frac{dp^N}{d\phi} &= -\frac{\left(p^N - c\right) D_{p_i\phi}(p^N, p^N, \phi) + D_\phi(p^N, p^N, \phi)}{D_{p_i}(p^N, p^N, \phi) + \frac{1}{n}D'(p^N) + (p^N - c)\frac{d}{dp^N}D_{p_i}(p^N, p^N, \phi)} \\
&= -p^N \frac{-\frac{D_{p_i\phi}(p^N, p^N, \phi)}{D_{p_i}(p^N, p^N, \phi)}\phi}{\phi\frac{D_{p_i}(p^N, p^N, \phi)}{D(p^N, p^N, \phi)}p^N + \phi\{\varepsilon(p^N) + \frac{\frac{d}{dp^N}D_{p_i}(p^N, p^N, \phi)}{D_{p_i}(p^N, p^N, \phi)}p^N\}} \\
&= p^N \frac{\varepsilon_\phi^{D_{p_i}}(p^N, \phi)}{\phi\varepsilon_i(p^N, \phi) + \phi\left\{\varepsilon(p^N) + \varepsilon_{p^N}^{D_{p_i}}(p^N, \phi)\right\}}
\end{aligned}$$

Since we assume $D_{p_i}(p^N, p^N, \phi)$ has an asymptotic expansion in ϕ with leading term $\phi^{-\xi}$ and since we can normalize the parameter ϕ in such a way that $\xi = 1$, we have that $D_{p_i\phi}(p^N, p^N, \phi)$ is of order ϕ^{-2}. Then $\varepsilon_\phi^{D_{p_i}}(p^N, \phi)$ and $\phi\varepsilon_i(p^N, \phi)$ both converge to some positive number. Since $\varepsilon(c)$ is finite and we assume that $\lim_{\phi\to 0}\left|\frac{\frac{d}{dp^N}D_{p_i}(p^N, p^N, \phi)}{D_{p_i}(p^N, p^N, \phi)}p^N\right| < B < \infty$, the Lemma follows.

Proof of Proposition 3: Part (a): We will show that for sufficiently large $n, \frac{\partial\delta(\phi,n)}{\partial\phi} > 0$. It is straightforward to show that $\frac{\partial\delta(\phi,n)}{\partial\phi}$ has the same sign as $n\xi_\phi(p^m, \delta(\phi, n), \phi, n)$. To show that the limit of this expression is strictly negative, we take the limits of some of the component parts of this expression. First, we show that $\lim_{n\to\infty} n(1 - \delta^N(\phi, n)) > 0$. We have:

$$\lim_{n\to\infty} n\frac{\pi(p^m, p^m, n) - \pi(p^N, p^N, n)}{\pi(p^*(p^m), p^m, n) - \pi(p^N, p^N, n)} = \frac{(p^m - c)D(p^m) - \lim_{n\to\infty}(p^N - c)D(p^N)}{\lim_{n\to\infty}(p^*(p^m) - c)D(p^*(p^m), p^m, n)}$$

Clearly, the sign of $\lim_{n\to\infty}(p^*(p, n) - c)$ is bounded from above by our assumptions on individual elasticities of demand. Now note that with $p_i = p^*(p, n)$ and $p_j = p$ for all $j \neq i$, we have $D(p_i, \mathbf{p}_{-i}, n) \leq \sum_j D(p_i, \mathbf{p}_{-i}, n) < D(p^*)$. Hence, $\lim_{n\to\infty}(p^*(p, n) - c)\, D(p^*(p, n), p, n) \leq \lim_{n\to\infty}(p^*(p, n) - c)\, D(p^*) < \infty$. Since $(p^m - c)\, D(p^m) - \lim_{n\to\infty}(p^N(n) - c)D(p^N(n)) > 0$ by quasi-concavity of $(p - c)D(p)$ in p it follows that $\lim_{n\to\infty} n(1 - \delta^N(\phi, n)) > 0$.

Second, we show that $\lim_{n\to\infty}\frac{d\pi(p^*(p,\phi),p,\phi)}{d\phi} < 0$. By the envelope theorem $\frac{d\pi(p^*(p,\phi),p,\phi)}{d\phi} = (p^*(p, n) - c)$ $D_\phi(p^*(p^m, \phi), p^m, \phi)$. Then $\lim_{n\to\infty}\frac{d\pi(p^*(p^m,\phi),p^m,\phi)}{d\phi} < 0$ immediately follows from our assumptions on demand. Indeed, the expression must go to $-\infty$ when $D_\phi(p^*(p^m, \phi), p^m, \phi) \to -\infty$ for $n \to \infty$. Finally, we can write $n\delta^N(\phi, n)\frac{\partial v}{\partial\phi}$ as $n\delta^N(\phi, n)D(p^N, p^N, n)\frac{D_{p_j}(p^N, p^N, n)}{-D_{pi}(p^N, p^N, n)}\frac{dp^N}{d\phi}$.Since $\delta^N(\phi, n; p) \leq 1$, $D(p^N(n)) \leq D(c)$, and $\left(-\frac{\sum_{j\neq i} D_{p_j}(p^N(n), p^N(n), n)}{D_{pi}(p^N(n), p^N(n), n)}\right) \leq 1$ by our assumptions on demand, this expression is positive and bounded from above by $D(c)\lim_{n\to\infty}\frac{dp^N}{d\phi}$. By Lemma B3, $\frac{dp^N}{d\phi}$ is bounded and converges to zero for $n \to \infty$ if the model has a competitive limit, i.e. if $\lim_{n\to\infty}\varepsilon_i(p, n, \phi) = \infty$. Hence, if the model has a competitive limit, the term $n\delta^N(\phi, n)\frac{\partial v}{\partial\phi}$ vanishes as $n \to \infty$, while the first term in $n\xi_\phi$ converges to a strictly negative number. If the model has a monopolistically competitive limit, $n\delta^N(\phi, n)\frac{\partial v}{\partial\phi}$ is bounded from above in the limit. But if $\lim_{n\to\infty}\frac{d\pi(p^*(p,\phi),p,\phi)}{d\phi} = -\infty$ as assumed in the proposition, $n\xi_\phi$ is still strictly negative in the limit. This completes the proof.

Part (b): The proof of part (b) is similar to the proof of part (a). First, we show that $\lim_{n\to\infty} n(1-\delta)\frac{d\pi(p^*(p^c,\phi),p^c,\phi)}{d\phi} < 0$. By the envelope theorem $\frac{d\pi(p^*(p,\phi),p,\phi)}{d\phi} = (p^*(p^c,\phi)-c)\, D_\phi(p^*(p^c,\phi),p^c,\phi)$. Since $\left|\frac{\partial D(p_i,\mathbf{p}_{-i},\phi)}{\partial p_i}\right| < B(\phi) < \infty$, it follows that $\lim_{n\to\infty} n(p^N-c) = \lim_{n\to\infty} \frac{\bar{D}(p^N(\phi,n))}{D_{p_i}(p^N(\phi,n))} > \frac{\bar{D}(p^N(\phi,n))}{B(\phi)}$. But then

$$\begin{aligned}
\lim_{n\to\infty} n\frac{d\pi(p^*(p,\phi),p,\phi)}{d\phi} &= \lim_{n\to\infty} n\,(p^*(p^c,n)-c)\, D_\phi(p^*(p^c,\phi),p^c,\phi) \\
&\leq \lim_{n\to\infty} n\left(p^N(\phi,n)-c\right) D_\phi(p^*(p^c,\phi),p^c,\phi) \\
&< \frac{\bar{D}(p^N(\phi,n))}{B(\phi)} \lim_{n\to\infty} D_\phi(p^*(p^c,\phi),p^c,\phi)
\end{aligned}$$

The first inequality follows because $\lim_{n\to\infty} D_\phi$ is strictly negative by assumption. The second inequality directly follows by the limiting assumption in the slope of demand. The limiting argument for $n\delta\frac{\partial \underline{v}}{\partial\phi}$ is essentially the same as in proposition 3. The result then follows.QED.

Proof of Proposition 4: From (3) we have:

$$\begin{aligned}
\lim_{\phi\to 0} n\xi_\phi(p^m,\phi,\delta(\phi,n),n) &= \xi_\phi(p^c,\delta,\phi,n) = \lim_{\phi\to 0} n\left\{(1-\delta)\frac{d\pi(p^*(p^c,\phi),p^c,\phi)}{d\phi} + \delta\cdot\frac{\partial \underline{v}(\phi,\delta,p^m,n)}{\partial\phi}\right\} \\
&= n\lim_{\phi\to 0}(1-\delta(\phi,n))\lim_{\phi\to 0}\frac{d\pi(p^*(p^c,\phi),p^c,\phi)}{d\phi} \\
&\quad +n\lim_{\phi\to 0}\delta(\phi,n)\cdot\lim_{\phi\to 0}(p^N-c)\sum_{j\neq i}\frac{\partial D(p^N,p^N,n)}{\partial p_j}\frac{dp^N}{d\phi} \\
&= \lim_{\phi\to 0}\left\{(p^*(p^m,\phi)-c)\frac{\partial D(p^*(p^m,\phi),p^m,\phi)}{\partial\phi}\right\} \\
&\quad +(n-1)\lim_{\phi\to 0}\left\{\frac{-D(p^N,p^N,\phi)}{D_1(p^N,p^N,\phi)}\sum_{j\neq i}\frac{\partial D(p^N,p^N,n)}{\partial p_j}\frac{dp^N}{d\phi}\right\} \\
&= (p^m-c)\lim_{\phi\to 0} D_\phi(p^*(p^m,\phi),p^m,\phi) \\
&\quad +\frac{n-1}{n}\bar{D}(c)\lim_{\phi\to 0}\left\{\frac{-\sum_{j\neq i}\frac{\partial D(p^N,p^N,n)}{\partial p_j}}{D_1(p^N,p^N,\phi)}\right\}\lim_{\phi\to 0}\frac{dp^N}{d\phi}
\end{aligned}$$

where the first equality follows from the envelope theorem and the fact that $D_\phi = 0$ at equal prices. The second equality follows from the fact that $\lim_{\phi\to 0}\delta(\phi,n) = \frac{n-1}{n}$ and by substituting for $p^N - c$ from the first order condition of the firm's maximization problem. The last equality follows because $\lim_{\phi\to 0} p^*(p^m,\phi) = p^m$ and because $\lim_{\phi\to 0} D(p^N,p^N,\phi) = \lim_{\phi\to 0}\frac{\bar{D}(p^N)}{n} = \frac{\bar{D}(c)}{n}$. The proposition then follows by showing that

$$\lim_{\phi\to 0}\left\{\frac{-\sum_{j\neq i}\frac{\partial D(p^N,p^N,n)}{\partial p_j}}{D_{p_i}(p^N,p^N,\phi)}\right\} = 1$$

To see this note that

$$\frac{\frac{\partial D(p^N,p^N,n)}{\partial p_j} + \sum_{j\neq i}\frac{\partial D(p^N,p^N,n)}{\partial p_j}}{\bar{D}(p^N)}p^N = \varepsilon(p^N)$$

Hence,

$$n\frac{\frac{\partial D(p^N,p^N,n)}{\partial p_j}p^N}{D(p^N,p^N,n)}\left[1+\frac{\sum_{j\neq i}\frac{\partial D(p^N,p^N,n)}{\partial p_j}}{D_1(p^N,p^N,\phi)}\right]=\varepsilon(p^N)$$

But $\lim_{\phi\to 0}\frac{\frac{\partial D(p^N,p^N,n)}{\partial p_j}p^N}{D(p^N,p^N,n)}=-\infty$ and hence, $\lim_{\phi\to 0}\frac{\sum_{j\neq i}\frac{\partial D(p^N,p^N,n)}{\partial p_j}}{D_1(p^N,p^N,\phi)}=-1$ in order for the left hand side of the expression to satisfy $\varepsilon(c)$ in the limit.QED.

Lemma B4: *For SDS preferences* $\lim_{\phi\to 0}D_\phi(p^*(p,\phi),p,\phi)=-(p)^{-\frac{1}{\kappa}}\left[\frac{1-\kappa}{\kappa}\ln(n)+\left(\frac{p}{p-c}-\frac{1}{\kappa}\right)\right]$

Proof: By Lemma A1, $\lim_{\phi\to 0}\varepsilon_i(p^*(p^m,\phi),\phi)=\frac{p}{p-c}$. Hence,

$$\lim_{\phi\to 0}\varepsilon_i(p^*(p^m,\phi),\phi)=\lim_{\phi\to 0}\frac{1}{\kappa}\left\{1+\left(\frac{1}{\phi}-1\right)(1-\mathbb{Q}_i)\right\}=\frac{p}{p-c}$$

rearranging terms solving the last equation for $\lim_{\phi\to 0}\frac{1}{\phi}(1-\mathbb{Q}_i)$, yields

$$\lim_{\phi\to 0}\frac{1}{\phi}(1-\mathbb{Q}_i)=\lim_{\phi\to 0}\left\{\frac{1}{1-\phi}\left[\frac{p}{p-c}-\frac{1}{\kappa}\right]\right\}=\frac{p}{p-c}-\frac{1}{\kappa},$$

It follows, by L'Hôpital's rule, that $\lim_{\phi\to 0}-\frac{d\mathbb{Q}_i}{d\phi}=\frac{p}{p-c}-\frac{1}{\kappa}$. Taking limits in (26) above and using Lemma A1 then yields:

$$\lim_{\phi\to 0}\frac{\partial D_i(p_i,p_j,\phi)}{\partial\phi}=D(p)\left\{-(1-\kappa)\ln(n)-\left(\frac{p}{p-c}-\frac{1}{\kappa}\right)\right\}$$

Lemma B5: For SDS preferences $\lim_{\phi\to 0}\frac{dp^N(\phi,n)}{d\phi}=\kappa\frac{n}{n-1}c$

Proof: *Since* $p^N(\rho,n)=\frac{c}{(1-\kappa\left[1+\frac{1-\phi}{\phi}\frac{n-1}{n}\right]^{-1})}$, we have:

$$\lim_{\rho\to 0}\frac{\partial p^N}{\partial\phi}=c\lim_{\rho\to 0}\left[\kappa\frac{\frac{1}{\phi^2}\frac{n-1}{n}}{\left[1+\frac{1-\phi}{\phi}\frac{n-1}{n}-\kappa\right]^2}\right]=c\kappa\frac{n}{n-1}$$

Proof of Corollary 1: By Lemmas B4 and B5 and proposition 4 above, the sign of the comparative statics under CES preferences is the same sign as:

$$\begin{aligned} & -(p^m-c)\,D(p^m)\frac{1-\kappa}{\kappa}\ln(n)+D(c)c && (35)\\ = & -p^m D(p^m)(1-\kappa)\ln(n)+D(c)c \\ = & D(c)c\left[-\frac{D(p^m)}{D(c)}\ln(n)+1\right] \end{aligned}$$

Now note that $D(p^m)=D(c)(1-\kappa)^{\frac{1}{\kappa}}$. The term $(1-\kappa)^{\frac{1}{\kappa}}$ is strictly decreasing in κ and $\lim_{\kappa\to 0}(1-\kappa)^{\frac{1}{\kappa}}=\frac{1}{e}$. Hence, if $\ln(n)<e$ the expression in brackets in the last line of is strictly positive for all κ. Since

$15 < e^e < 16$, the first part of the proposition follows. If $n \geq 16$ there then exists some $\kappa > 0$ such that the expression is strictly negative. Furthermore, for $\kappa \to 1$, $(1-\kappa)^{\frac{1}{\kappa}} \to 0$, so that there must exist some $\underline{\kappa}(n) > 0$ beyond which the sign switches. Clearly, by monotonicity of $(1-\kappa)^{\frac{1}{\kappa}}$ in κ, the higher n the higher κ can be for δ to decrease in ϕ. QED.

Lemma B6: For the nested logit model $\lim_{\phi\to 0} D_\phi(p^*(p,\phi),p,\phi) = -\varepsilon(p)\frac{1}{p}\ln(n)\,D(p)$.

Proof: Note that aggregate demand is given by $D(p) = \frac{1}{\exp\{(p+A)/\kappa\}+1}$. By Lemma B1, we have that

$$\lim_{\phi\to 0} -\frac{D_{p_i}(p^*(p^m,\phi),p^m,\phi)}{D(p^*(p^m,\phi),p^m,\phi)} = -\frac{D'(p^m)}{D(p^m)} = \frac{1}{\kappa}[1-D(p^m)]$$

Furthermore, note that $\lim_{\phi\to 0}\hat{\mathbb{Q}}(p^*(p^m),p^m,\phi) = D(p)$.This follows because $p^*(p^m) < p^m$ implies that $\lim_{\phi\to 0}\mathbb{Q}_i(p^*(p^m,\phi),p^m,\phi) = 1$. We can now show that $\lim_{\phi\to 0}\frac{\partial\mathbb{Q}_i(p^*(p^m,\phi),p^m,\phi)}{\partial\phi} = -\lim_{\phi\to 0}\frac{1}{\phi}[1-\mathbb{Q}_i(p^*(p^m,\phi),p^m,\phi)] = 0$. By Lemma A1, the expression in (**??**), converges to $\frac{D'(p^m)}{D(p^m)}$. The first term on the right hand side of (**??**) therefore must converge to 0. Then:

$$\begin{aligned}
\lim_{\phi\to 0} D_\phi(p^*(p^m,\phi),p^m,\phi) &= \lim_{\phi\to 0}\left\{\hat{\mathbb{Q}}\frac{d\mathbb{Q}_i}{d\phi} - \hat{\mathbb{Q}}(1-\hat{\mathbb{Q}})\frac{d\mathbb{Q}_i}{d\phi}\frac{1}{\mathbb{Q}_i} - \hat{\mathbb{Q}}\mathbb{Q}_i(1-\hat{\mathbb{Q}})\frac{1}{\kappa}\ln(n\mathbb{Q}_i)\right\} \\
&= -\frac{1-\lim_{\phi\to 0}\hat{\mathbb{Q}}}{\kappa}\ln(n)\,D(p^m) = -\frac{1}{\kappa}[1-D(p^m)]\ln(n)D(p^m) \\
&= -\varepsilon(p^m)\frac{1}{p^m}\ln(n)\,D(p^m),
\end{aligned}$$

where the first equality in the second line comes from $\lim_{\phi\to 0}\frac{\partial\mathbb{Q}_i(p^*(p^m,\phi),p^m,\phi)}{\partial\phi} = 0$ and $\lim_{\phi\to 0}\mathbb{Q}_i(p^*(p^m,\phi),p^m,\phi$ 1.

Lemma B7: For the nested logit model, $\lim_{\phi\to 0}\frac{dp^N(\phi,n)}{d\phi} = \frac{n}{n-1}$

Proof: Taking the ratio of (**??**) and (**??**) and multiplying it yields:

$$\lim_{\phi\to\infty}\varepsilon_\phi^{D_{p_i}}(p^N,\phi) = \lim_{\phi\to\infty}\frac{\frac{1}{\phi}\frac{n-1}{n}}{\frac{1}{\phi}\frac{n-1}{n}+\frac{1}{\kappa}(1-\hat{\mathbb{Q}})\frac{1}{n}} = 1$$

Furthermore, from (**??**), $\lim_{\phi\to\infty}\phi\varepsilon_i(p^N,\phi) = \frac{n-1}{n}c$. The proposition follows by Lemma A4.

Proof of Corollary 2:

$$\frac{D(p^m)}{D(c)} = \frac{\exp\left\{\frac{c+A}{\kappa}\right\}+1}{\exp\left\{\frac{1}{1-\hat{\mathbb{Q}}^m(\frac{c+A}{\kappa})}+\frac{c+A}{\kappa}\right\}+1} \tag{36}$$

By our arguments before, this expression converges to 1 when $\frac{c+A}{\kappa}$ converges to $-\infty$. Hence, if $\ln(n) > 1$, then, by continuity, there must exist $A > -\infty$ such that $\frac{D(p^m)}{D(c)}\ln(n) > 1$. Clearly $\ln(2) < 1$. However, $\ln(3) = 1.0986 > 1$. Hence, for all $n \geq 3$, there exists $\underline{A}(n)$ such that $\delta(\phi,n)$ is decreasing at $\phi = 0$ for all $A \in (-\infty, \underline{A}(n))$. The remainder of the result is implied by (36) increasing in $\frac{c+A}{\kappa}$, which is easily confirmed knowing that $\hat{\mathbb{Q}}^m$ is decreasing in $\frac{c+A}{\kappa}$. QED

Proof of Corrollary 3: From the exposition of the Hotelling model in Appendix A it follows that

$$\begin{aligned} & \lim_{\phi \to 0} \; (1-\delta(\phi))\frac{d\Pi(p^*(p^m,\phi),p^m,\phi)}{d\phi} + \delta(\phi)\frac{d\Pi(p^N,p^N,\phi)}{d\phi} \\ = & \lim_{\phi \to 0} \; -(1-\delta(\phi)) + \delta(\phi)\frac{1}{2} = -\frac{1}{4} < 0 \end{aligned}$$

where the first equality comes from the profit expressions derived in Appendix A and the last inequality derives from the fact that $\lim_{\phi \to 0} \delta(\phi) = \frac{1}{2}$. It follows that a small amount of product differentiation strictly reduces the critical discount factor at which collusion at the monopoly price is feasible. QED

Proof of Proposition 5: We show that $\lim_{\phi \to 0} \frac{\partial \underline{v}^O(\phi,\delta,p^c,n)}{\partial \phi} = 0$, implying that $\lim_{\phi \to 0} \xi_\phi < 0$. Then by continuity the proposition follows. By Lemma B2 we have $\underline{v}^O(\phi,\delta,p^c,n) = \pi(p^*(p_L,\phi,n),p_L,\phi,n)$. By Lemma 2, $\lim_{\phi \to 0} \pi(p^*(p_L,\phi,n),p_L,\phi,n) = 0$, and hence, from(34),

$$\frac{1}{n}\lim_{\phi \to 0} \pi(p_L,p_L,n) + \frac{n-1}{n}\pi(p^m,p^m,n) = 0$$

It follows $\lim_{\phi \to 0} p_L < c$. Since $\pi(p^*(p_L,\phi,n),p_L,\phi,n) \geq 0$, it follows that $p^*(p_L,\phi,n) \geq 0$. Hence, $\lim_{\phi \to 0} D(p^*(p_L,\phi,n),p_L,\phi,n) = 0$, and, consequently, $\lim_{\phi \to 0} p^*(p_L,\phi,n) = c$. Now note that:

$$\begin{aligned} \frac{\partial \underline{v}^O(\phi,\delta,p^c,n)}{\partial \phi} &= \pi_{p_j}(p^*(p_L,\phi,n),p_L,\phi,n)\frac{\partial p_L}{\partial \phi} + \pi_\phi(p^*(p_L,\phi,n),p_L,\phi,n) \\ &= [p^*(p_L,\phi,n)-c]\left[D_{p_j}(p^*(p_L,\phi,n),p_L,\phi,n)\frac{\partial p_L}{\partial \phi} + D_\phi(p^*(p_L,\phi,n),p_L,\phi,n)\right. \\ &= -\frac{D(p^*(p_L,\phi,n),p_L,\phi,n)}{D_{p_i}(p^*(p_L,\phi,n),p_L,\phi,n)}D_{p_j}\frac{\partial p_L}{\partial \phi} + D_\phi(p^*(p_L,\phi,n),p_L,\phi,n)\,[p^*(p_L,\phi,n)-c] \\ &\leq D(p^*(p_L,\phi,n),p_L,\phi,n)\frac{\partial p_L}{\partial \phi} + D_\phi(p^*(p_L,\phi,n),p_L,\phi,n)\,[p^*(p_L,\phi,n)-c] \\ &= \left[\frac{D(p^*(p_L,\phi,n),p_L,\phi,n)}{(1-\delta)\frac{d\pi(p_L,p_L,n)}{dp_L} - \frac{d\pi(p^*(p_L,\phi,n),p_L,\phi,n)}{dp_L}} + [p^*(p_L,\phi,n)-c]\right]D_\phi(p^*(p_L,\phi,n),p_L,\phi,n) \end{aligned}$$

Now $0 \leq \lim_\phi (1-\delta)\frac{d\pi(p_L,p_L,n)}{dp_L} - \frac{d\pi(p^*(p_L,\phi,n),p_L,\phi,n)}{dp_L} = \frac{1}{n}\lim_\phi \frac{d\pi(p_L,p_L,n)}{dp_L} < \infty$ and $0 \leq \lim_{\phi \to 0} D_\phi(p^*(p_L,\phi,n),p_L,$ ∞. But since demand and price cost margin converge to zero $\lim_{\phi \to 0} \frac{\partial \underline{v}^O(\phi,\delta,p^c,n)}{\partial \phi} = 0$. QED.

Proof of Proposition 6: Since for all $\phi \in (0,1)$, $\underline{v}^O(\phi,\delta,p^c,n) \geq \underline{v}^S(\phi,\delta,p^c,n) \geq 0$ and $\lim_{\phi \to 0} \underline{v}^O(\phi,\delta,p^c,n) = \lim_{\phi \to 0} \underline{v}^S(\phi,\delta,p^c,n) = 0$ (by Lemma 1) it follows directly that $\lim_{\phi \to 0} \frac{\partial \underline{v}^O(\phi,\delta,p^c,n)}{\partial \phi} \geq \lim_{\phi \to 0} \frac{\partial \underline{v}^O(\phi,\delta,p^c,n)}{\partial \phi}$. The proposition then follows from the proof of proposition 5. QED.

Proof of Proposition 7: Since the game is symmetric the bargaining solution over the equilibrium value set selects a symmetric value vector on the Pareto frontier of the equilibrium value set. We will now show that under assumptions B and C strategies in this equilbirium must be symmetric on the equilibrium path. Suppose at the best collusive equilibrium firms charge a price vector $\mathbf{p}$ in period 1 and have an average continuation profit of $\hat{v}_i$, $i = 1, ...n$. For contradiction assume that firms do not charge the same price and have different continuation profits. For every i we have

$$(1-\delta)\pi_i(\mathbf{p},\phi) + \delta\hat{v}_i \geq (1-\delta)\pi_i(p^*(\mathbf{p}_{-i}), \mathbf{p}_{-i}, \phi) + \delta\underline{v}$$

where by symmetry of the equilibrium value set the punishment value for each i must be the same. It follows that

$$(1-\delta)\frac{1}{n}\sum_i \pi_i(\mathbf{p},\phi) + \delta\frac{1}{n}\sum_i \hat{v}_i \geq (1-\delta)\frac{1}{n}\sum_i \pi_i(p^*(\mathbf{p}_{-i}), \mathbf{p}_{-i}, \phi) + \delta\underline{v},$$

where $(1-\delta)\pi_i(\mathbf{p},\phi) + \delta\hat{v}_i = (1-\delta)\frac{1}{n}\sum_i \pi_i(\mathbf{p},\phi) + \delta\frac{1}{n}\sum_i \hat{v}_i$ by symmetry of the selected equilibrium value. Let $\bar{p} = \frac{1}{n}\sum_i p_i$. By strict concavity of $\frac{1}{n}\sum_i \pi_i(\mathbf{p},\phi)$ in $\mathbf{p}$, $\frac{1}{n}\sum_i \pi_i(\mathbf{p},\phi) < \pi_i(\bar{p},\bar{p})$. Furthermore, by convexity of $\frac{1}{n}\sum_i \pi_i(p^*(\mathbf{p}_{-i}), \mathbf{p}_{-i}, \phi)$ in $\mathbf{p}_{-i}$ we have $\frac{1}{n}\sum_i \pi_i(p^*(\mathbf{p}_{-i}), \mathbf{p}_{-i}, \phi) \geq \pi_i(p^*(\bar{p}), \bar{p}, \phi)$. Furthermore, by convexity of the equilibrium value set, there exists an equilibrium with a continuation value in which each player obtains an average continuation value of $\bar{v} = \frac{1}{n}\sum_i \hat{v}_i$. But then a price $\bar{p}$ for every palyer in period 1 with a continuation value of $\bar{v}$ satisfies the incentive constraint for all players. Furthermore the value of the equilibrium is higher, contradicting the assumption that the previous equilibrium was the best collusive equilibrium. Hence, the best collsuive equilibrium is symmetric on the equilibrium path. But then the equilibrium must also be stationary. Suppose otherwise. Then either $\pi_i(p^c, p^c) > \bar{v}$ or $\pi_i(p^c, p^c) < \bar{v}$. Consider the first case. Since p^c is sustainable, it can be played forever and will satisfy incentive compatibility. Hence, there would be a better equilibrium contradicting that we are at the best collusive equilibrium. Consider $\pi_i(p^c, p^c) < \bar{v}$. If $\bar{v}$ is available as an continuation equilibrium outcome it can also be played at the beginning. But $\bar{v} > (1-\delta)\pi_i(p^c, p^c) + \delta\bar{v}$, contradicting that we are at the best collusive equilibrium. Hence, the equilibrium is stationary on the equilibrium path. QED.

Proof of Proposition 8: Fix $p^{c\underline{k}}$ and let $\delta(\underline{k}, \phi, p^{c\underline{k}})$ be the lowest discount faactor at which $p^{c\underline{k}}$ can just be sustained. Note that the incentive constraint is just binding when there are fewer firms. We show that the incentive constraint for the market with the strictly larger number of firms is strictly slack for ϕ close to zero. This implies that $\delta(\underline{k}, \phi, p^{c\underline{k}}) < \delta(\bar{k}, \phi, p^{c\underline{k}})$ and $p^{c\underline{k}}(\phi, \delta(\underline{k}, \phi, p^{c\underline{k}})) > p^{c\bar{k}}(\phi, \delta(\underline{k}, \phi, p^{c\underline{k}}))$.To show the result it is sufficient to show that

$$\begin{aligned}\xi_k(\phi) \quad = \quad & (1-\delta(\underline{k}, \phi, p^{c\underline{k}}))\left[\pi^{\underline{k}}(p^{*\underline{k}}(p^{c\underline{k}}, \phi), p^{c\underline{k}}, \phi, n) - \pi^{\bar{k}}(p^{*\bar{k}}(p^{c\underline{k}}, \phi)), p^{c\underline{k}}, \phi, n)\right] \\ & +\delta(\underline{k}, \phi, p^{c\underline{k}})[\underline{v}(\phi, \underline{k}) - \underline{v}(\phi, \bar{k})]\end{aligned} \tag{37}$$

is strictly positive for ϕ close enough to zero. Since $\lim_{\phi\to 0}(p^{*k}(p,\phi) - c) = p - c > 0$ and $\lim_{\phi\to 0}\underline{v}(\phi, k) = 0$ for all k,

$$\lim_{\phi\to 0}\xi_k(\phi) < \frac{1}{\underline{k}}\left(p^{c\bar{k}} - c\right)\left[\frac{\underline{k}}{n} - \frac{\bar{k}}{n}\right]\bar{D}(c) < 0. \tag{38}$$

It follows directly that $\delta(\underline{k}, \phi, p^{c\underline{k}}) < \delta(\bar{k}, \phi, p^{c\underline{k}})$. Furthermore, if ξ increases in p^c at $p^{c\underline{k}}$, then $p^{c\underline{k}}$ is not the highest sustainable collusive price. Hence, if $p^{c\underline{k}}$ is the highest sustainable collusive price at ϕ the highest collusive price for $\bar{k}$ at the same ϕ and the same δ must be strictly lower, i.e. $p^{c\underline{k}}(\phi, \delta(\underline{k}, \phi, p^{c\underline{k}})) > p^{c\bar{k}}(\phi, \delta(\underline{k}, \phi, p^{c\underline{k}}))$ QED.

We now derive a result for the comparative statics of the Nash equilibrium when there are corss-shareholdings that we will use for the proof of proposition 9. The short run best response is determined from the first order condition:

$$D(p^*(p,\alpha,\phi),p,\phi)+(p^*(p,\alpha,\phi)-c)\,D_{p_i}(p^*(p,\alpha,\phi),p,\phi)+\frac{\hat{\alpha}}{1-\hat{\alpha}}\frac{1}{n-1}(p-c)D_{p_j}(p,p^*(p,\alpha,\phi),p,\phi)=0 \tag{39}$$

At a Nash equilibrium (39) is satisfied for all firms. By an argument analogous to that in proposition 1 this equilibrium exists and is unique. Nash equilibrium prices have the following limiting properties when goods become perfectly homogeneous:

Lemma B8: Nash equilibrium prices converge to marginal cost for every $\hat{\alpha}\in[0,\frac{n-1}{n})$. Furthermore, Nash equilibrium prices are constant at $\phi=0$, i.e. $\lim_{\phi\to 0}\frac{dp^N}{d\hat{\alpha}}=0$.

Proof. The first order condition for Nash equilibrium prices can be written as:

$$\begin{aligned}\frac{p^N(\hat{\alpha},\phi)-c}{p^N(\hat{\alpha},\phi)} &= -\frac{D(p^N(\alpha,\phi),p^N(\alpha,\phi))}{D_{p_i}(p^N(\alpha,\phi),p^N(\alpha,\phi),\phi)p^N(\alpha,\phi)}\left[1+\frac{\hat{\alpha}}{1-\hat{\alpha}}\frac{1}{n-1}\frac{D_{p_j}}{D_{p_i}}\right]^{-1}\\ &= \frac{1}{\varepsilon_i(p^N(\alpha,\phi),\phi)}\left[1+\frac{\hat{\alpha}}{1-\hat{\alpha}}\frac{1}{n-1}\frac{D_{p_j}}{D_{p_i}}\right]^{-1}\end{aligned}$$

From the proof of proposition 4 we know that $\lim_{\phi\to 0}\frac{D_{p_j}}{D_{p_i}}=-1$. Hence, $\lim_{\phi\to 0}\left[1+\frac{\hat{\alpha}}{1-\hat{\alpha}}\frac{D_{p_j}}{D_{p_i}}\right]=1-\frac{\hat{\alpha}}{1-\hat{\alpha}}\frac{1}{n-1}>0$ for all $\hat{\alpha}<\frac{n-1}{n}$. By our assumptions on demand, $\lim_{\phi\to 0}\varepsilon_i(p^N(\alpha,\phi),\phi)\to-\infty$. Hence, $\lim_{\phi\to 0}p^N(\hat{\alpha},\phi)=c$ for all $\hat{\alpha}$. It follows that $\lim_{\phi\to 0}\frac{\partial p^N(\hat{\alpha},\phi,n)}{\partial\hat{\alpha}}=0$. ■

Proof of Proposition 9:

We first show the limit result on $\delta(\hat{\alpha},\phi,p^c)$ as $\phi\to 0$.

$$\begin{aligned}\lim_{\phi\to 0}1-\delta(\hat{\alpha},\phi,p^c) &= \lim_{\phi\to 0}\frac{\Pi(p^c,p^c)-\Pi(p^N,p^N,\hat{\alpha},\phi)}{\Pi(p^*(p^c,\hat{\alpha},\phi),\hat{\alpha},\phi)-\Pi(p^N,p^N,\hat{\alpha},\phi)}\\ &= \frac{\pi(p^c,p^c)}{(1-\hat{\alpha})n\pi(p^c,p^c)}=\frac{1}{(1-\hat{\alpha})\,n}\end{aligned}$$

Then

$$\begin{aligned}\lim_{\phi\to 0}\xi_{\hat{\alpha}} &= -\frac{1}{(1-\hat{\alpha})n}\lim_{\phi\to 0}[\pi(p^*(p^c,\hat{\alpha},\phi),p^c,\phi)-\pi(p^c,p^*(p^c,\hat{\alpha},\phi),p^c,\phi)]\\ &\quad+\left(1-\frac{1}{(1-\hat{\alpha})n}\right)\lim_{\phi\to 0}\left[\pi_{p_j}(p^N(\hat{\alpha},\phi),p^N(\hat{\alpha},\phi))\frac{dp^N}{d\hat{\alpha}}\right]\end{aligned}$$

The term on the first line in brackets converges to $(p^c-c)\bar{D}(p^c)>0$. In the second term $\lim_{\phi\to 0}\pi_{p_j}(p^N(\hat{\alpha},\phi),p^N($
$\frac{1}{n}\bar{D}(c)$ and $\lim_{\phi\to 0}\frac{dp^N}{d\hat{\alpha}}=0$.

Hence, $\lim_{\phi\to 0}\xi_{\hat{\alpha}}=-\frac{1}{(1-\hat{\alpha})n}(p^m-c)\bar{D}(p^m)<0$. The comarative statics results on δ and p^c follow directly. QED.

Proof of Proposition 10:

Fix a collusive price p^c and let $\delta(\hat{\alpha}, \phi, p^c)$ be the lowest discount factor that supports that price at cross-shareholding parameter $\hat{\alpha}$. The incentive compatibility constraint for deviating from the collusive price is given by:

$$\begin{aligned} & \pi(p^m, p^m) \\ = \ & (1-\delta(\hat{\alpha}, \phi, p^c))[(1-\hat{\alpha})\pi(p^*(p^c, \hat{\alpha}, \phi), p^c, \phi) + \hat{\alpha}\pi(p^c, p^*(p^c, \alpha, \phi)] \\ & +\delta(\hat{\alpha}, \phi, p^c)[(1-\hat{\alpha})\pi(p^*(p_L, \hat{\alpha}, \phi), p_L, \phi) + \hat{\alpha}\pi(p_L, p^*(p^c, \alpha, \phi)] \end{aligned} \tag{40}$$

Hence,

$$\begin{aligned} \xi_{\hat{\alpha}} = \ & (1-\delta(\hat{\alpha}, \phi, p^c))[\pi(p^*(p^c, \hat{\alpha}, \phi), p^c, \phi) - \pi(p^c, p^*(p^c, \alpha, \phi)] \\ & +\delta(\hat{\alpha}, \phi, p^c)[\pi(p^*(p_L, \hat{\alpha}, \phi), p_L, \phi) - \pi(p_L, p^*(p^c, \alpha, \phi)] \end{aligned} \tag{41}$$

Now note that

$$\pi(p^*(p^c, \hat{\alpha}, \phi), p^c, \phi) - \pi(p^c, p^*(p^c, \hat{\alpha}, \phi), \phi) > 0 \tag{42}$$

because $p^*(p^c, \hat{\alpha}, \phi) < p^c$ and thus

$$\pi(p^*(p^m, \hat{\alpha}, \phi), p^m, \phi) > \pi(p^m, p^m) > \pi(p^m, p^*(p^m, \hat{\alpha}, \phi), \phi).$$

Further, note that the second line in (41) is strictly positive whenever $p_L - c \leq 0$, since $\pi(p^*(p_L, \hat{\alpha}, \phi), p_L, \phi) > 0$ for all $\phi > 0$. Hence, to prove the proposition we only need to show that $\lim_{\phi\to 0} p_L(\hat{\alpha}, \phi, \delta(\hat{\alpha}, \phi, p^c)) < c$. First, note that $\lim_{\phi\to 0} p_L(\hat{\alpha}, \phi, \delta(\hat{\alpha}, \phi, p^c)) > 0$ is impossible, since the $\lim_{\phi\to 0} \pi(p^*(p_L, \hat{\alpha}, \phi), p_L, \phi) > \lim_{\phi\to 0} \pi(p^N(\hat{\alpha}, \phi), p^N(\hat{\alpha}, \phi)) = 0$ contradicting the definition of p_L as resulting from the most severe punishment scheme. Now suppose for contradicition that $\lim_{\phi\to 0} p_L(\hat{\alpha}, \phi, \delta(\hat{\alpha}, \phi, p^c)) = 0$. Then, in any optimal punishment scheme p_L is followed by charging p^c at ϕ close to zero and the incentive constraint onpunishments can be written as:

$$(1-\delta)\pi(p_L, p_L) + \delta\pi(p^c, p^c) = [(1-\hat{\alpha})\pi(p^*(p_L, \hat{\alpha}, \phi), p_L, \phi) + \hat{\alpha}\pi(p_L, p^*(p^c, \alpha, \phi)]$$

Since the right hand side cannot exceed the punishment profits under Nash reversion we have:

$$\lim_{\phi\to 0}(1-\delta)\pi(p_L(\hat{\alpha}, \phi, \delta(\hat{\alpha}, \phi, p^c)), p_L(\hat{\alpha}, \phi, \delta(\hat{\alpha}, \phi, p^c))) + \delta\pi(p^c, p^c) = 0$$

Hence, $\lim_{\phi\to 0} p_L(\hat{\alpha}, \phi, \delta(\hat{\alpha}, \phi, p^c) < c$. It follows that there exists $\phi_L(p^c, \hat{\alpha}) > 0$ such that for all $\phi \in (0, \hat{\phi}_L(p^c, \hat{\alpha}))$, $p_L(\hat{\alpha}, \phi, \delta(\hat{\alpha}, \phi, p^c) < c$. But then it follows that for all $\phi \in (0, \hat{\phi}_L(p^c, \hat{\alpha}))$, $\xi_{\hat{\alpha}} > 0$. The comparative statics results in the proposition follow directly from this result. QED.

Proof of proposition 11: First note that $\lim_{\phi\to 0} \xi_c = 0$. To find out the sign of ξ_c close to zero, we analyze $\lim_{\phi\to 0} \xi_{c\phi}$, where we are changing the discount factor δ, so that $p^m(c)$ remains just sustainable as $\phi \to 0$. We have:

$$\begin{aligned} \xi_{c\phi} = \ & (1-\delta)D_\phi(p^*(p^m(c), \phi), p^m(c), \phi) - \frac{\partial\delta^*(c,\phi)}{\partial\phi} D(p^*(p^m(c)), p^m(c), \phi) \\ & +\frac{\partial\delta^*(c,\phi)}{\partial\phi}(p^* - c)D_{p_j}\frac{\partial p^m(c)}{\partial c} \\ & +(1-\delta)D_{p_1}\frac{\partial p^*}{\partial\phi} - (1-\delta)(p^*-c)\frac{\partial^2 D}{\partial p_j \partial\phi}\frac{\partial p^m(c)}{\partial c} \\ & -(1-\delta)\left[\frac{\partial p^*}{\partial\phi}D_{p_j} + (p^*-c)\frac{\partial^2 D}{\partial p_j \partial p_i}\frac{\partial p^*}{\partial\phi}\right]\frac{\partial p^m(c)}{\partial c} \\ & -\delta\underline{v}_{c\phi} \end{aligned} \tag{43}$$

Suppose that for a given c the incentive constraint is just binding for $\delta^*(c,\phi)$. If $\lim_{\phi\to 0}\xi_{c\phi} > 0$ or $\lim_{\phi\to 0}[\xi_{c\phi}/D_\phi(p^*(p^m(c),c,\phi),p^m,\phi)] < 0$ then it follows that for small ϕ the incentive constraint is strictly relaxed if marginal costs rises to $c+\varepsilon$ and $\delta^*(c+\varepsilon,\phi) < \delta^*(c,\phi)$. Since we know that $\lim_{\phi\to 0} D_\phi(p^*(p^m(c),\phi),p^m(c),\phi) < 0$, but cannot rule out that this expression converges to $-\infty$,we proceed by analyzing $\lim_{\phi\to 0}\frac{\xi_{c\phi}}{D_\phi}$.

First, it is immediate from (13) that

$$\frac{\partial\delta^*(c,\phi)}{\partial\phi} = \frac{(1-\delta^*(c,\phi))(p^*-c)D_\phi + \delta\underline{v}_\phi(c,\phi)}{(1-\delta^*(c,\phi))(p^*-c)D - \underline{v}(c,\phi)}$$

From our analysis in section 3 it follows that $\lim_{\phi\to 0}\underline{v}(c,\phi) = \lim_{\phi\to 0}\underline{v}_\phi(c,\phi) = 0$, so that

$$\lim_{\phi\to 0}\frac{\frac{\partial\delta^*(c,\phi)}{\partial\phi}}{D_\phi} = \frac{1-\delta}{\bar{D}}.$$

It follows that the first term is zero for ϕ close to zero. Similarly, the second line converges to zero, since $\lim_{\phi\to 0}\frac{\frac{\partial\delta^*(c,\phi)}{\partial\phi}}{D_\phi} = \frac{1-\delta}{\bar{D}}$ and $\lim_{\phi\to 0} D_{p_j} = 0$. It is also easy to see that the last term coming from the continuation value under optimal punishments must converges to zero: Since $\underline{v}_{c\phi} = \underline{v}_{\phi c}$ and $\lim_{\phi\to 0}\underline{v}_\phi = 0$ for all ϕ, it follows that $\lim_{\phi\to 0}\underline{v}_{\phi c} = \lim_{\phi\to 0}(\underline{v}_{\phi c}/D_\phi) = 0$.

To deal with the remaining two lines in $\xi_{c\phi}$, first note that

$$\lim_{\phi\to 0}\left[\frac{\partial p^*}{\partial\phi}/D_\phi\right] = \lim_{\phi\to 0}\frac{1}{D_{p_i}}\frac{dp^m(c)}{dc}\lim_{\phi\to 0}\frac{-\left[(p^*-c)D_{p_i\phi}+D_\phi\right]}{D_\phi}$$

Furthermore, the elasticity of $D(p^*,p_j,\phi)$ in p_i is given by $\varepsilon_i = -\frac{D_{p_i}p_i}{D}$. We have:

$$\frac{\frac{\partial\varepsilon_i}{\partial\phi}}{\varepsilon_i}D(p_i,p_j,\phi) = \frac{D}{D_{p_i}}D_{p_i\phi} - D_\phi = -\left[(p^*-c)D_{p_i\phi}+D_\phi\right]$$

From our assumption $\lim_{\phi\to 0}\frac{\partial\varepsilon_i}{\partial\phi} > 0$ and the fact that $[(p^*-c)D_{p_i\phi}+D_\phi]$ can therefore go no slower to $-\infty$ than D_ϕ it follows that $\lim_{\phi\to 0}\frac{-\left[(p^*-c)D_{p_i\phi}+D_\phi\right]}{D_\phi} < 0$ and thus $\lim_{\phi\to 0}\left[\frac{\partial p^*}{\partial\phi}/D_\phi\right] > 0$. If $\lim_{\phi\to 0}\left[\frac{\partial p^*}{\partial\phi}/D_\phi\right] < \infty$, the fourth line in (43) converges to zero as $\phi\to 0$, since $\lim_{\phi\to 0}D_{p_j} = \lim_{\phi\to 0}D_{p_jp_i} = 0$. In the case $\lim_{\phi\to 0}\left[\frac{\partial p^*}{\partial\phi}/D_\phi\right] = \infty$ it is easy to show that all our results go through by analyzing $\lim_{\phi\to 0}(\xi_{c\phi}/\left[(p^*-c)D_{p_i\phi}+D_\phi\right])$ instead.

To derive the sign of line 3 in (43), note that

$$(p^*-c)D_{p_j\phi} = \frac{\frac{\partial\varepsilon_j}{\partial\phi}}{\varepsilon_i}\frac{p_i}{p_j}D(p_i,p_j,\phi) - \frac{D_{p_j}}{D_{p_i}}D_\phi$$

where $\varepsilon_j = \frac{D_{p_j} p_j}{D}$. It follows that

$$
\begin{aligned}
& \lim_{\phi \to 0} \frac{(1-\delta) D_{p_1} \frac{\partial p^*}{\partial \phi} - (1-\delta)(p^* - c) \frac{\partial^2 D}{\partial p_j \partial \phi} \frac{\partial p^m(c)}{\partial c}}{D_\phi} \\
= \quad & (1-\delta) \left[\lim_{\phi \to 0} \frac{-\frac{\partial \varepsilon_i}{\partial \phi} \frac{1}{\varepsilon_i} D(p_i, p_j, \phi) + \frac{p_i}{p_j} \frac{\partial \varepsilon_j}{\partial \phi} \frac{1}{\varepsilon_i} D(p_i, p_j, \phi) - \frac{D_{p_j}}{D_{p_i}} D_\phi}{D_\phi} \right] \frac{\partial p^m(c)}{\partial c} \\
= \quad & (1-\delta) \lim_{\phi \to 0} - \left[\frac{\left(\frac{\partial \varepsilon_i}{\partial \phi} \frac{1}{\varepsilon_i} - \frac{p_i}{p_j} \frac{\partial \varepsilon_j}{\partial \phi} \frac{1}{\varepsilon_i} \right) D(p_i, p_j, \phi)}{D_\phi} + \frac{D_{p_j}}{D_{p_i}} \right] \frac{\partial p^m(c)}{\partial c} \\
= \quad & -(1-\delta) \left[\lim_{\phi \to 0} \frac{\left(\frac{\partial \varepsilon_i}{\partial \phi} \frac{1}{\varepsilon_i} - \frac{\partial \varepsilon_j}{\partial \phi} \frac{1}{\varepsilon_i} \right)}{D_\phi} \right] \bar{D}(p_i^m) \frac{\partial p^m(c)}{\partial c} < 0.
\end{aligned}
$$

where the last inequality follows from the fact that $\lim_{\phi \to 0} \left(\frac{\partial \varepsilon_i}{\partial \phi} \frac{1}{\varepsilon_i} - \frac{\partial \varepsilon_j}{\partial \phi} \frac{1}{\varepsilon_i} \right) > 0$ by assumption 1c and since $\frac{\partial \varepsilon_i}{\partial \phi} \frac{1}{\varepsilon_i} = -\frac{1}{D} [(p^* - c) D_{p_i \phi} + D_\phi]$, so that $\frac{\partial \varepsilon_i}{\partial \phi} \frac{1}{\varepsilon_i}$ is of the same order in ϕ as D_ϕ. It follows that, for ϕ close to zero, the incentive constraint becomes strictly slack when c is increased. Hence, there exists $\hat{\phi} > 0$ such that $\delta^*(c, \phi)$ is strictly decreasing in c for all $\phi \in (0, \hat{\phi})$. QED.

9 References

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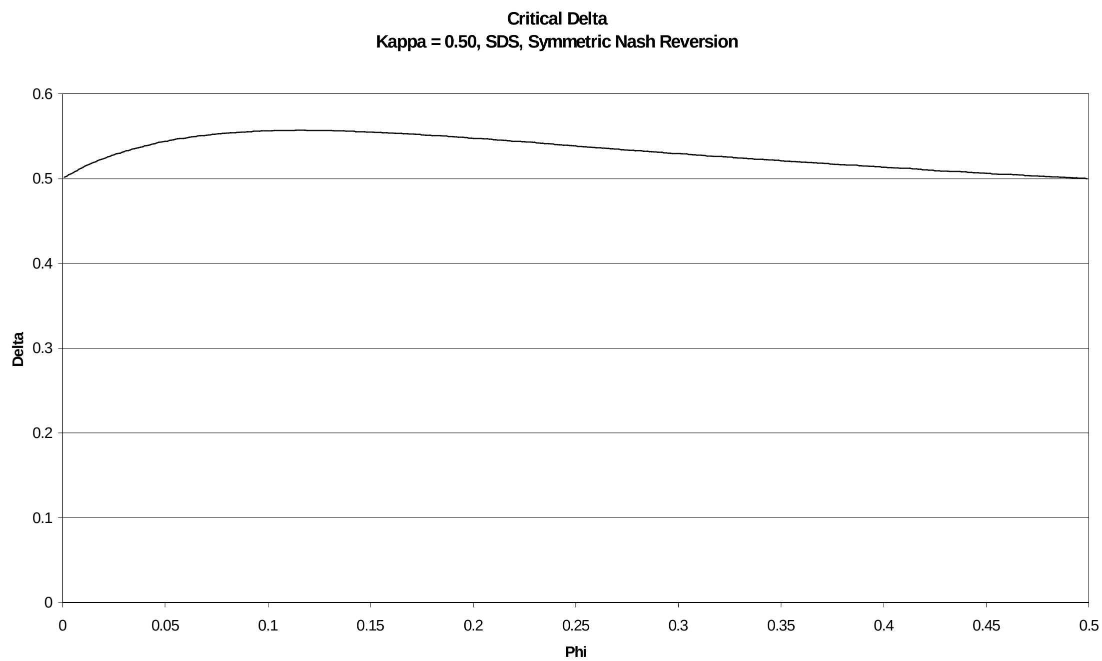

Figure 1:

10 Figures

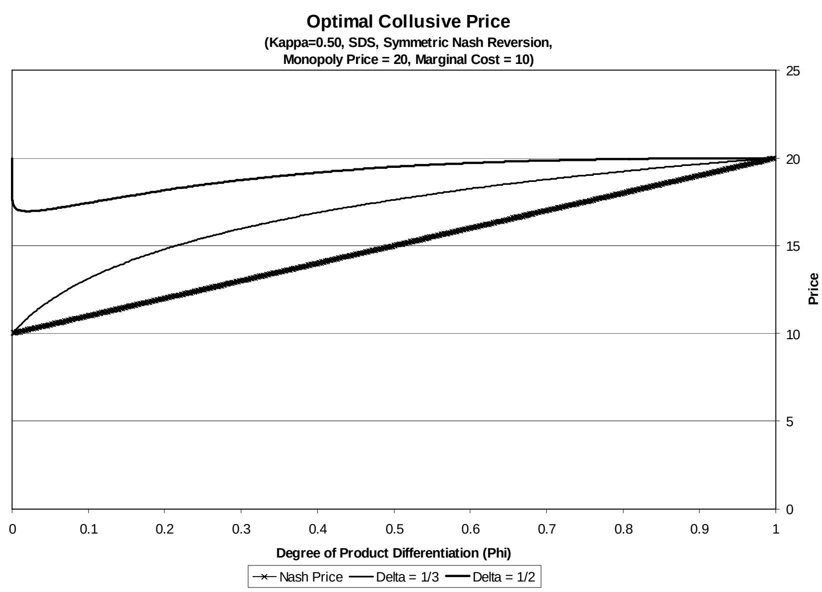
Optimal Collusive Price
(Kappa=0.50, SDS, Symmetric Nash Reversion,
Monopoly Price = 20, Marginal Cost = 10)
25
20
15
10
5
0
Price
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Degree of Product Differentiation (Phi)
Nash Price
Delta = 1/3
Delta = 1/2

Figure 2:

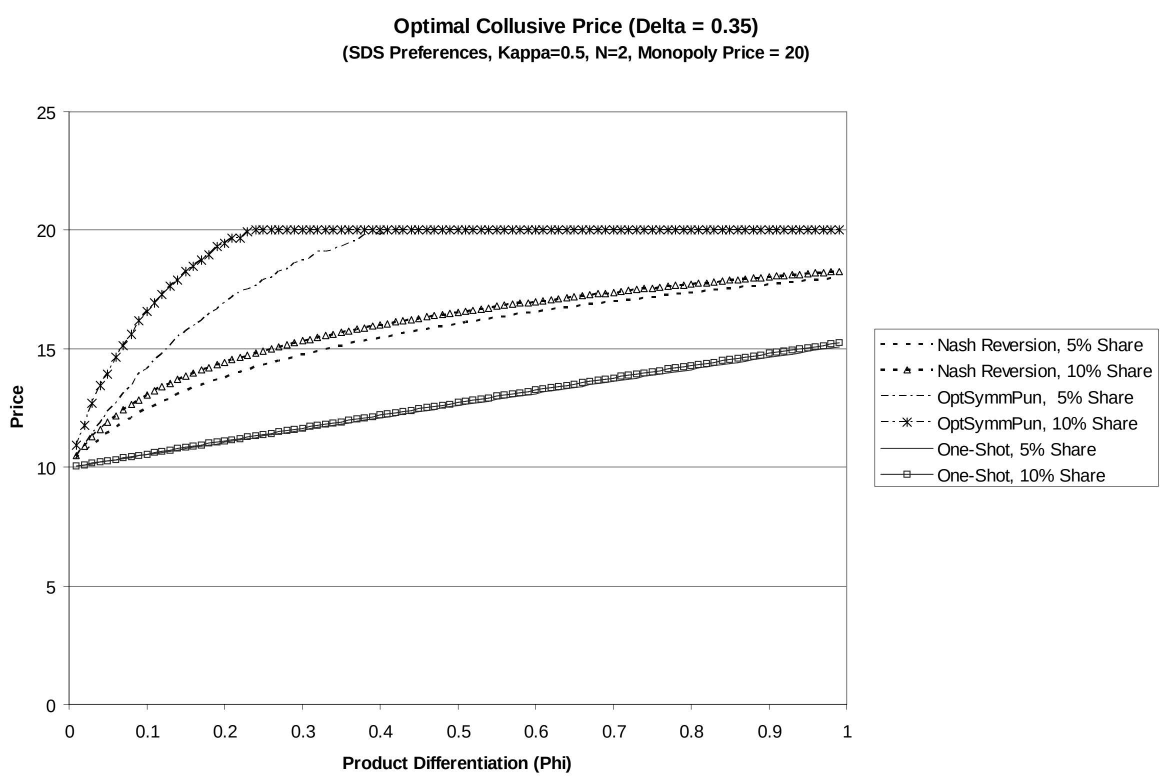
Optimal Collusive Price (Delta = 0.35)
(SDS Preferences, Kappa=0.5, N=2, Monopoly Price = 20)
Price
25
20
15
10
5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Product Differentiation (Phi)
Nash Reversion, 5% Share
Nash Reversion, 10% Share
OptSymmPun, 5% Share
OptSymmPun, 10% Share
One-Shot, 5% Share
One-Shot, 10% Share

Figure 3:

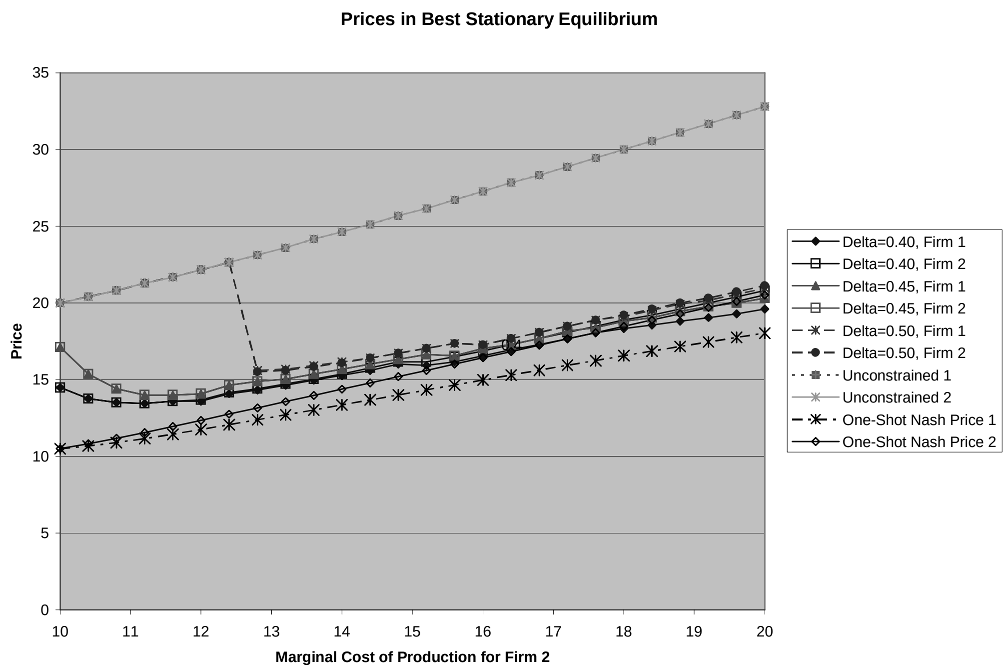

Figure 4:

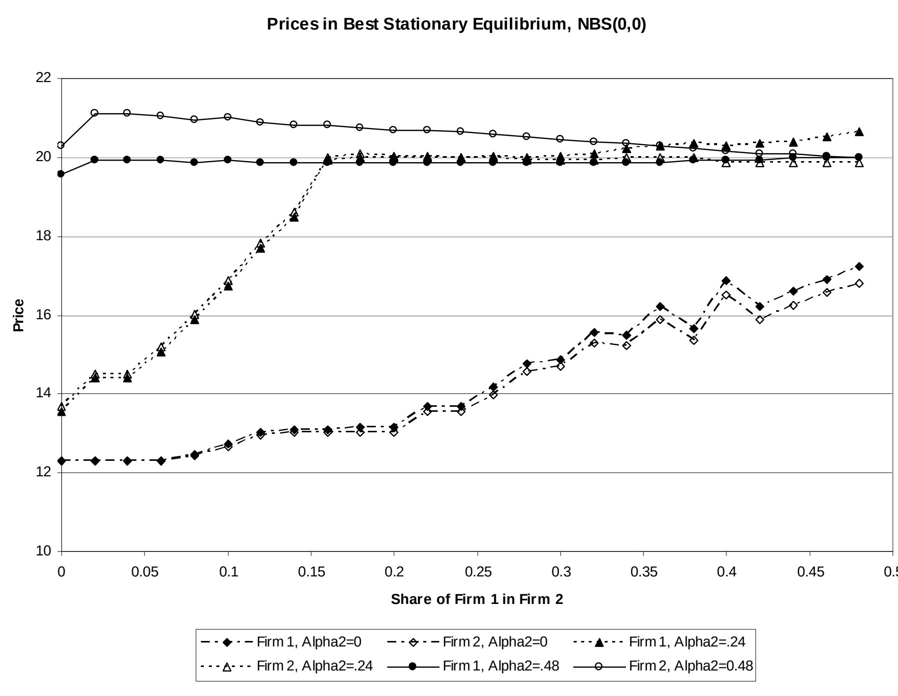

Figure 5: