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ABSTRACT

The Effect of Introducing a Non-redundant Derivative on the Volatility of Stock-Market Returns*

We study the effect of introducing a new security, such as a non-redundant derivative, on the volatility of stock-market returns. Our analysis uses a standard, continuous time, dynamic, general-equilibrium, full-information, frictionless, Lucas endowment economy where there are two classes of agents who have time-additive power utility functions and differ only in their risk aversion. We solve for equilibrium in two versions of this economy. In the first version, risk-sharing opportunities are limited because investors can trade in only the market portfolio, which is a claim on the aggregate endowment. In the second version, agents can trade in both the market portfolio and a new zero-net-supply derivative. We show analytically that for a sufficiently small precautionary-savings effect, the introduction of a non-redundant derivative on the market increases the volatility of stock-market returns.

JEL Classification: G12 and G13
Keywords: general equilibrium, options, risk-sharing and volatility

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1 Introduction and motivation

According to Alan Greenspan, "By far the most significant event in finance during the past decade has been the extraordinary development and expansion of financial derivatives."[1] Could such a major change in financial markets affect the volatility of stock-market returns? We address this question in this paper.

One line of thought suggests that the introduction of new derivatives should *decrease* stock-market return volatility. For instance, Shiller (1993, 2003) advocates opening macro-markets where derivatives on macroeconomic variables, such as GDP are traded. According to Shiller (2003), "These markets would be potentially more important in the risks they deal with than any financial markets today, and they would remove pressures and volatility from our overheated stock market." The popular press also espouses a similar view. *The Economist* (Nov 25th 1999) writes: "Derivatives are really only an efficient way of transferring risk from those who do not want it to those who do. If risk is in safer hands, the net effect ought to be to reduce volatility." But Fischer Black disagrees. Black (1995, p. 76) says: "As a country grows, it invests in more and more elaborate forms of financial intermediation. In effect, its markets become more complete ... asset volatility *increases* (our emphasis) ...."

To resolve this debate and to understand how exactly the introduction of non-redundant derivatives affects the volatility of stock-market returns, we study a model of a dynamic, continuous time, general-equilibrium, full-information, frictionless, Lucas exchange economy with a stochastic aggregate endowment, which is also equal to aggregate dividends. In this economy, the aggregate stock market is modeled as a claim on this aggregate dividend. The economy has two classes of agents who have standard time-additive power utility functions and differ only in their coefficient of relative risk aversion, so that trading is driven by their desire to share risk. Given that the only source of risk is the stochastic aggregate dividend, two securities are needed to complete the market and enable perfect risksharing. We solve for equilibrium in two versions of this economy. In the first version, risk-sharing opportunities are limited because investors can trade in only the stock market. But in the

[1] This quote is from an address by Greenspan to the Futures Industry Association in Boca Raton, Florida on March 19, 1999. According to the Office of the Comptroller of the Currency in the U.S., the total notional value of derivatives held by U.S. banks has increased from less than 10 trillion dollars in 1991 to over 80 trillion dollars in 2004—implying a compound growth rate of over 17% per annum. Over the same period, the S&P 500 has grown at a compound growth rate of just over 8% per annum.

second version, agents can trade in both the stock market and a zero-net-supply contingent claim, which can be interpreted as a non-redundant derivative on the market portfolio. We then compare the volatility in these two versions of the economy.

Our main result is to show that introducing a new derivative security that improves risk sharing may lead to an *increase* in the volatility of the stock market.[2] In particular, the volatility of stock returns is *higher* in the economy with improved risk sharing if the discount rate is countercyclical, because then the change in the discount rate magnifies the effect on stock returns of a shock to dividends. This condition is satisfied if in the economy the average prudence, which drives demand for precautionary savings, is not too large.[3]

We now explain the intuition behind our results. The current stock price is equal to the expected value of the sum of future dividends, appropriately discounted. Consequently, the volatility of stock returns depends on the size of shocks to future dividend growth rates (fundamental volatility), which is constant in our model, and shocks to the discount rate (excess volatility). So, we need to compare the behavior of the discount rate in the economy where the investor can trade only the stock and in the economy where the investor can trade the stock and also the non-redundant derivative. In the economy without the derivative, agents cannot share risk at all, and so the distribution of wealth across agents changes only deterministically. Consequently, the discount rate[4] is also deterministic, and so there is no excess volatility over and above fundamental volatility in the economy without the derivative.

The introduction of a non-redundant derivative, however, makes it possible to share risk. As a result, the wealth-distribution between agents is no longer deterministic, and the

[2]We do not focus on the effect on welfare, though in our model the welfare of both agents increases with the improvement in risk sharing. See Cass and Citanna (1998) and Elul (1995) for a discussion of the effects of financial innovation on welfare.

[3]For instance, if the growth rate and volatility of aggregate endowments are both assumed to be 2%, then in order for stock market return volatility to increase upon the introduction of a non-redundant derivative, the average prudence in the economy needs to be less than 51, or the risk aversion of the more risk averse investor needs to be less than 50.

[4]The discount rate is given by the following expression, where $\mathbf{R}$ is the average risk aversion in the economy, $\mathbf{P}$ is the average prudence, and σ_Y and μ_Y are, respectively, the volatility and expected value of the growth rate of aggregate endowments:

$$\text{Discount rate} = r + \mathbf{R}\,\sigma_Y^2 = (\rho + \mathbf{R}\,\mu_Y - \frac{1}{2}\mathbf{P}\,\mathbf{R}\,\sigma_Y^2) + \mathbf{R}\,\sigma_Y^2.$$

Observe that the discount rate consists of the riskless rate, r, plus the risk premium, $\mathbf{R}\,\sigma_Y^2$. The riskless rate, r, itself can be further decomposed into ρ, the subjective time-preference parameter, plus $\mathbf{R}\,\mu_Y$, which reflects the desire to smooth consumption over time, minus $\frac{1}{2}\mathbf{P}\mathbf{R}\,\sigma_Y^2$, which captures the effect of precautionary savings.

stochastic wealth distribution makes the discount rate stochastic. When this discount rate is countercyclical (positive shocks to future dividend growth rates lead to a decline in the discount rate), stock return volatility is magnified and so it exceeds fundamental volatility.

The discount rate will be countercyclical if the average prudence in the economy is not too large. This is because when the non-redundant derivative is available, agents will engage in risk sharing; in particular, the less risk averse agent will invest more in the stock market. Thus, after an unexpected increase in the aggregate dividend growth rate, the consumption share of the agent with lower risk aversion increases. Hence, the consumption-share-weighted average risk aversion decreases in the economy. This has three effects on the discount rate, given in Footnote 4: (i) A decrease in risk aversion decreases the risk premium, which reduces the discount rate; (ii) A decrease in risk aversion implies an increase in the elasticity of intertemporal substitution for the time-additive preferences we are using, which increases savings and decreases the riskless interest rate, and hence, the discount rate; and, (iii) The decrease in average risk aversion reduces precautionary savings, which increases the interest rate, and hence, the discount rate. So, while the first two effects reduce the discount rate, the third effect is in the opposite direction. The desire for precautionary savings is determined by the average prudence in the economy. If average prudence is not too high, then the net effect of the reduction in average risk aversion will be a decrease in the discount rate, in which case, the discount rate will be countercyclical.

Of course, in a dynamic setting, the effect of a shock to aggregate endowments today will influence not just the current discount rate but also future discount rates. We use Malliavin calculus to show that the above intuition applies even in this more general setting.

We conclude this introduction by discussing the relation of our work to the existing literature. The paper that is closest to ours is Citanna and Schmedders (2005), which considers the effect of financial innovation on asset-price volatility. They show in a three-date model that introducing new financial securities never increases price volatility if there is a single consumption good and one goes from incomplete to fully complete financial markets in an economy with *no aggregate risk*. But Levine and Zame (2001) show that the presence of aggregate risk can have a significant effect on the conclusions drawn from a model with incomplete markets. When Citanna and Schmedders study the case of aggregate risk, they show that price volatility can go either way. In contrast to Citanna and Schmedders, the focus of our analysis is an economy where there is aggregate endowment risk. More importantly, rather than analyzing a specialized three-date model, we study a continuous-time, general-equilibrium economy, which allows us to go beyond Citanna and Schmedders and derive analytically the sufficient condition for the volatility of stock market returns to increase, in terms of the parameter values for the aggregate dividend process and the preference parameters of the two agents. This then allows us to understand the exact economic forces underlying the change in the volatility of stock-market returns.

Our paper also contributes to the literature studying the effect of risk sharing on the behavior of stock returns. There is a large literature on the impact of improved risk sharing on the returns of risky and riskfree assets; see for instance, Telmer (1993), Lucas (1994), Heaton and Lucas (1996), and Basak and Cuoco (1998). While all these papers investigate the impact of risk sharing on expected stock market returns, the equity risk premium and the riskfree rate, only Basak and Cuoco (1998) and Gallmeyer and Hollifield (2004) investigate the impact of risk sharing on aggregate stock-market volatility. Basak and Cuoco (1998) examine the impact on stock market volatility of non-participation, that is, prohibiting one agent from investing in stocks. But they can analyze this for only a special case of their model—when all agents have logarithmic utility. They find that stock market volatility is unchanged. In our more general analysis, both agents have power utility with differing relative risk aversion parameters. We find that changing the level of risk sharing does impact stock-return volatility. Gallmeyer and Hollifield (2004) use Malliavin calculus to study the effect of short-sale constraints on asset returns in a more general model where agents have differences in beliefs and one agent has logarithmic utility and the other agent has power utility; however, even for this special case, their analysis of volatility is numerical.

From a modelling perspective, our paper is related to papers that study economies with heterogeneous agents, where the source of the heterogeneity is differences in risk aversion across agents.[5] These papers differ from ours in focus—they do not study the issue of stock-return volatility—and they also differ in the level of generality with which the main results

[5]Of course, heterogeneity across agents may arise not just because of differences in risk aversion but also because of differences in beliefs or differences in information. For models that consider the role of financial innovation in the presence of these sources of heterogeneity, see for instance, Leland (1980), Stein (1987), Grossman (1988, 1989), Detemple (1990), Gennotte and Leland (1990), Detemple and Selden (1991), Back (1993), Huang and Wang (1997), Zapatero (1998), and Gallmeyer and Hollifield (2004). The role of options has also been considered when background risk is present. Single-period models with background risk have been studied by Franke, Stapleton, and Subrahmanyam (1998) and Calvet, Gonzalez-Eiras, and Sodini (2004). In an economy where all agents have CARA preferences, with the same level of risk aversion, but idiosyncratic income processes, Calvet (2001) shows that with constant aggregate income, deterministic movements in the precautionary savings term of the riskfree rate decline in size as markets become more complete. Consequently, deterministic changes in asset prices become less pronounced as markets become complete.

can be characterized. For instance, Dumas (1989) studies portfolio choice and the interest rate in a production economy with two agents, one with log utility and the other with power utility; in this model, stock-return volatility is a constant that is specified exogenously and only the case of complete markets is considered, and even that admits only a numerical solution. Wang (1996) examines the term structure of interest rates and risk premium in an exchange economy similar to ours but where one agent has log utility and the other has square-root utility; again, only the case where financial markets are complete is considered. Kogan, Makarov, and Uppal (2003) also examine an exchange economy with borrowing constraints to understand their effect on the equity risk premium, but again are limited to studying the case where one of the agents has log utility; for other levels of risk aversion, they have to resort to numerical methods. In contrast to these papers, in our model agents are not restricted to have log utility and we are able to obtain all our results analytically for both the case of complete and incomplete financial markets.

Our work is complementary also to the literature studying the effect of demand for portfolio insurance (that is, payoffs with a lower bound) on the volatility of stock-market returns; see, for instance, Brennan and Schwartz (1989), Basak (1995) and Grossman and Zhou (1996). In these models, financial markets are assumed to be complete and the question asked is how does the volatility of stock-market returns change when *new agents* with a demand for portfolio insurance enter the economy. In contrast to these papers, we consider an economy where financial markets are incomplete, and the question we ask is how does the volatility of stock-market returns change when a *new security* is introduced to the market.

On the methodological side, our paper addresses three challenges and makes the following contributions. One, we solve for equilibrium in a model where agents are heterogeneous with respect to their risk aversion, without restricting any agent's utility to be logarithmic; this is in contrast to models in the existing literature with heterogeneous agents discussed above, where the utility of at least one agent is restricted to logarithmic utility. Two, we determine equilibrium in an economy where financial markets are incomplete; while this has been done for a single-agent partial-equilibrium economy in Cvitanić and Karatzas (1992), for a general-equilibrium economy results exist only for the case where agents have logarithmic utility (see Basak and Cuoco (1998)). Three, we characterize the conditional volatility of stock-market returns, which depends not just on the current stock price but also on the future evolution of the stock price. We show how to do this using Malliavin calculus, for the case of both incomplete and complete financial market structures, with heterogeneous agents, and without restricting the preferences of one agent to be logarithmic.

To characterize the equilibrium in the two versions of the economy we wish to consider, we use insights from two streams of the asset pricing literature. In the economy where financial markets are complete, we solve for equilibrium using the approach developed in Negishi (1960), and significantly extended by Magill (1981) and Mas-Colell (1986), where the optimization problem of maximizing the lifetime utility of the multiple agents is formulated as a central-planner's problem, with the weights on the agents being constant over time because risk-sharing is perfect. In the economy where only the stock is available for trading and markets are incomplete, we first derive the optimal consumption policies of individual agents by exploiting the duality approach of Cvitanić and Karatzas (1992). The main insight of this approach is that one can obtain the optimal policies of the constrained agents by fictitious completion of financial markets, accompanied by a change in asset prices, so that agents do not wish to hold the fictitious securities. For instance, if agents can trade only the stock, then the fictitious completion is accomplished by allowing the agents to trade also a riskless bond, but changing the interest rate on the bond sufficiently so that the investor chooses not to hold the bond in equilibrium. To obtain the equilibrium consumption policies, we then use the central-planner's approach of Cuoco and He (2001) for incomplete markets, where the weights assigned to agents are stochastic rather than constant because now risk-sharing is imperfect.

The rest of the paper is organized as follows. In Section 2, we describe the general features of the dynamic general equilibrium endowment economy that we are considering. In Section 3, we characterize equilibrium in the economy when financial markets are not complete and risk sharing is imperfect. In Section 4, we describe the equilibrium in the economy where financial markets are complete and risk sharing is perfect, and compare the volatility of stock returns in this economy to that in the economy with imperfect risk sharing. We conclude in Section 5. Detailed proofs for all the propositions are presented in Appendix A and an introductory guide to Malliavin calculus is presented in Appendix B. Finally, a technical appendix containing the detailed derivation of the equilibrium in the economy with incomplete financial markets can be downloaded from our web site.

2 The model

In this section, we describe the features of the model of the economy we are considering. Below, we explain our assumptions about the information structure, the endowment process, the financial assets in the economy, the preferences of agents, and conclude by stating the optimization problem agents face.

We consider a continuous-time, pure-exchange economy with a finite time horizon $[0, T]$. There is a single consumption good that serves as the numeraire. There are two types of investors, $k \in \{1, 2\}$, who have power utility and are identical in all respects except for their risk aversion, γ_k. Without loss of generality, we assume that Agent 1's relative risk aversion is less than that of Agent 2: $\gamma_1 < \gamma_2$. We adopt the convention of subscripting by k the quantities related to Agent k, where $k \in \{1, 2\}$.

2.1 The information structure and endowment process

The uncertainty in the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ on which is defined a one-dimensional Brownian motion Z.[6] The economy is modeled as being endowed with a single non-storable consumption good. The evolution of aggregate endowment (dividends), $Y_t > 0$, is:

$$dY_t = \mu_Y \, Y_t \, dt + \sigma_Y \, Y_t \, dZ_t, \tag{1}$$

where μ_Y and σ_Y are constants and Z_t is a one-dimensional standard Brownian motion.

2.2 Financial assets

The first financial asset in the economy is a risky asset (stock) with one share outstanding. The stock is a claim on the aggregate endowment (dividend). The price of the stock, which can be interpreted as the market portfolio, is denoted S_t, and its *cumulative* return, R_t, that consists of capital gains plus dividends, is described by the process:

$$\frac{dS_t + Y_t dt}{S_t} = dR_t = \mu_{R,t} dt + \sigma_{R,t} dZ_t. \tag{2}$$

[6]Given our main focus is on characterization of the stock-return volatility, we state only the most relevant technical conditions and otherwise assume that all processes and expectations are well-defined without stating explicitly the required regularity conditions.

The expected return on the stock, $\mu_{R,t}$, and the volatility of stock returns, $\sigma_{R,t}$, will be determined endogenously in equilibrium.

In the first version of the economy we consider, the risky stock described above is the only financial asset that is available to investors so risk sharing is imperfect. In the second version of the economy that we consider, we allow for perfect risk sharing by introducing another financial security. The second financial security in the economy is some zero-net-supply security. As long as the payoff on this security is not perfectly correlated with the stock, it is the only security that is needed to complete the market in our model, and thus, the particular payoff structure of this security will not affect the results.

2.3 Preferences of the two agents

The consumption of Agent k at instant u is denoted by $C_{k,u}$ and the instantaneous utility from consumption is assumed to be time additive and given by a power function:

$$U(C_{k,u}) = e^{-\rho u} \frac{C_{k,u}^{1-\gamma_k}}{1-\gamma_k}, \tag{3}$$

where ρ is the constant subjective discount rate and γ_k is the degree of relative risk aversion. Lifetime expected utility of Agent k from consuming $C_{k,u}$ is given by

$$\mathbb{E}_0 \left[\int_0^T e^{-\rho u} \frac{C_{k,u}^{1-\gamma_k}}{1-\gamma_k} du \right]. \tag{4}$$

2.4 The optimization problem of each agent

Each agent k is assumed to have an initial endowment of a_k shares of the stock, where $a_1 + a_2 = 1$. Thus, the value of the initial endowment of agent k is $a_k S_0$.

The problem of each agent is to maximize lifetime utility in (4) subject to a budget constraint. The budget constraint requires that the present value of all future consumption is no more than the initial wealth with which each agent is endowed:

$$\mathbb{E}_0 \left[\int_0^T \frac{\pi_{k,u}}{\pi_{k,0}} C_{k,u} du \right] \leq a_k S_0, \tag{5}$$

where $\pi_{k,u}$ is the marginal utility of investor k at date u (referred to by an array of names such as state-price density, stochastic discount factor, and present-value operator):

$$\kappa_k \frac{\pi_{k,u}}{\pi_{k,0}} = \frac{\partial U(C_{k,u})}{\partial C_{k,u}} = e^{-\rho u} C_{k,u}^{-\gamma_k}, \tag{6}$$

and κ_k is the Lagrange multiplier on the static budget constraint. The process for $\pi_{k,u}$ is given by (see Duffie (2001, Section 6.D, p. 106)):

$$\frac{d\pi_{k,t}}{\pi_{k,t}} = -r_{k,t}dt - \theta_{k,t}dZ_t, \tag{7}$$

where $r_{k,t}$ is the riskless interest rate and $\theta_{k,t} = \frac{\mu_{R,t} - r_{k,t}}{\sigma_{R,t}}$ is the market price of risk.

Observe that when markets are incomplete, the state-price-density process will not necessarily be the same for the two agents, and this is reflected in the above equation by using the subscript k for r, θ and π. Because the two agents can trade the stock in both versions of the economy we consider, they will always agree on the expected return and volatility of equity, and therefore, we do not subscript by k the expected return and volatility of the stock. However, when only the stock is available for trading, then investors cannot borrow or lend, and thus, will not necessarily agree on the riskless interest rate. Therefore, we subscript the interest rate by k, and $r_{k,t}$ represents the shadow interest rate that ensures demand for borrowing or lending by Agent k is zero.

2.5 The general expression for the stock price and stock-return volatility

In this section, we derive a general expression for conditional stock market return volatility. Note that the current value of the stock market (S_t) can be written as

$$S_t(Y_t, \nu_{1,t}) \equiv Y_t \; p_t(\nu_{1,t}), \tag{8}$$

where Y_t is the current level of aggregate dividends and $p_t(\nu_{1,t})$ is the current price-dividend ratio, which is a function of Agent 1's share of aggregate consumption, defined by $\nu_{1,t} = C_{1,t}/Y_t$, where $C_{1,t}$ is Agent 1's optimal consumption. Using Ito's Lemma, then gives:

$$\frac{dS_t}{S_t} = \frac{dY_t}{Y_t} + \frac{dp_t}{p_t} + \frac{dY_t}{Y_t}\frac{dp_t}{p_t}. \tag{9}$$

Equating the diffusion terms on the left and right-hand sides of the above equation leads to:

$$\sigma_{R,t} = \sigma_Y + \sigma_{p,t}, \tag{10}$$

where $\sigma_{R,t}$ is the stock-market return volatility, σ_Y is the volatility of the aggregate dividend growth rate, and $\sigma_{p,t}$ is the volatility of percentage changes in the price-dividend ratio. The above expression says that stock market return volatility consists of two components. The first component in (10), σ_Y, is the *fundamental* component. The second component, $\sigma_{p,t}$, is the *excess* volatility component.

In the next two sections, we consider two versions of the general economy described here. In the first version, markets are incomplete because we assume that agents can trade only the stock. In the second version, we introduce a zero-net-supply derivative so that financial markets are complete. For these two versions of the general economy, we derive and compare the expression for stock-return volatility corresponding to the general expression in equation (10).

3 Volatility in the economy with imperfect risk sharing

In the first version of the economy that we consider, we assume that only the stock is available for trading. Consequently, financial markets are incomplete and perfect risk sharing is not possible.[7]

The main result of this section is that if agents can invest in only the stock, then the conditional volatility of stock-market returns is equal to just the volatility of the growth rate of aggregate dividends, that is, fundamental volatility.

Proposition 1 *In the economy with imperfect risk sharing, the volatility of stock returns is equal to just the fundamental volatility:*

$$\sigma_{R,t} = \sigma_Y. \tag{11}$$

[7] One could consider a more complicated setup where the endowment process was driven by multiple shocks rather than a single one, and therefore, markets were incomplete even if agents could trade in both a riskless bond and the stock but not in other derivative securities needed to complete the market. We did consider such a setup in an earlier version of the paper. Because the insights from such a model are similar to the ones from the simple model described above, we decided to present only the simpler model.

The intuition for this result is the following. The only asset available for trading in the economy is the stock, which is a claim on the aggregate endowment process Y_t. Because both investors hold only the risky stock (that is, they have the same portfolio strategy with hundred percent of their wealth invested in the stock), it is not possible for the investors to share risk, even though they differ in risk aversion and so would very much like to do so. The only trade investors can execute is to sell some of the stock that they own (which is a claim on future consumption), in exchange for current consumption. Hence, consumption can be smoothed only over time but not across states. Therefore, the optimal consumption policy of each agent is a deterministic proportion of the aggregate endowment, implying that $\sigma_{p,t} = 0$, and so stock-return volatility is equal to just the volatility of the aggregate endowments, σ_Y. In other words, because both agents can invest in only the stock, the evolution of their wealth is perfectly correlated. That is, the volatility of the cross-sectional wealth distribution is zero, and this is shown formally in the proof for Proposition 1. It follows from this that there is no "excess" volatility generated by the fluctuation of the wealth distribution across the two agents. Consequently, stock market return volatility depends only on the fundamental volatility of the aggregate dividends.

4 Volatility in the economy with perfect risk sharing

Financial markets were incomplete in the economy analyzed above. Now, we introduce a zero-net-supply derivative, so that agents can share perfectly the risks that they face. As long as the payoff to this derivative is not perfectly correlated to the stock, it will lead to complete financial markets. And, because a single claim is sufficient for agents to achieve perfect risk sharing in the economy we are considering, it makes the choice of the particular claim that we introduce irrelevant for its effect on stock-return volatility.[8]

With complete financial markets, agents can share risk perfectly; that is, agents will trade in the stock and the bond in such a way that consumption at all dates and states is allocated optimally across the two agents (the optimality condition being the standard one: the marginal rate of substitution of consumption is the same across agents). Therefore, the sharing rule $\nu_{k,t} = C_{k,t}/Y_t$ is stochastic and in equation (10) the volatility of stock returns may differ from fundamental volatility, σ_Y. We now identify the condition that determines

[8] While the choice of the derivative introduced does not affect the behavior of asset returns, it will have an affect on the trading strategy of the two investors.

whether stock-return volatility will be larger or smaller than fundamental volatility. This is the main result of the paper. The derivation uses Malliavin calculus. More importantly, the economics underlying the result is non-trivial and, to the best of our knowledge, new in the literature.

Proposition 2 *In the economy with perfect risk sharing, stock-return volatility is greater than fundamental volatility, that is*

$$\sigma_{R,t} > \sigma_Y, \tag{12}$$

if the discount rate is countercyclical, that is

$$\frac{\partial}{\partial \nu_{1,t}} \left(r_t + \theta_t \sigma_Y\right) < 0, \tag{13}$$

and a sufficient condition for (13) to hold is given by

$$\mu_Y > (\mathbf{P}_t - 1)\sigma_Y^2, \tag{14}$$

where $\mathbf{P}_t$ *is the average prudence in the economy.*

To see that the condition in (14) is satisfied for a broad range of reasonable parameter values, note that $\mathbf{P}_t$ lies between the relative prudence levels of the two agents, that is: $1 + \gamma_1 \leq \mathbf{P}_t \leq 1 + \gamma_2$, where γ_k is the risk aversion of agent k. Therefore, if one assumes that the expected growth rate in the economy is 4% with the volatility being 2%, then the above condition is satisfied if average prudence is smaller than 101 or the risk aversion of the more risk averse investor is smaller than 100. If the growth rate is only 2% and volatility is also 2%, then it suffices for the average prudence to be less than 51 or the risk aversion of the more risk averse investor to be smaller than 50.

We now explain the intuition underlying Proposition 2 about the effect of a positive shock to aggregate dividends on the behavior of the riskless rate, r_t, and the risk premium, $\theta_t \sigma_Y = \mathbf{R}_t \sigma_Y^2$, whose sum gives the discount rate:

$$r_t + \mathbf{R}_t \, \sigma_Y^2 = \left(\rho + \mathbf{R} \, \mu_Y - \frac{1}{2} \mathbf{P} \, \mathbf{R} \, \sigma_Y^2\right) + \mathbf{R} \, \sigma_Y^2. \tag{15}$$

It is clear from the Gordon Growth Model, that stock-return volatility is greater than dividend volatility if the discount rate is countercyclical, that is, if (13) holds. In the next paragraph, we relate this condition to the average prudence in the economy.

A positive shock to aggregate dividend growth, increases the consumption of both agents. But market clearing implies that one agent will consume a greater proportion of aggregate dividends. This agent will be the one with the larger elasticity of intertemporal substitution (in our time-separable, power utility framework, this is also the agent with the smaller risk aversion). Therefore, the consumption-share-weighted average elasticity of intertemporal substitution in the economy increases, or equivalently, the average risk aversion, $\mathbf{R}$, decreases. As explained in the introduction, and as can be seen from equation (15), a decrease in $\mathbf{R}$ affects the discount rate through three channels. One, the premium required for bearing risk, $\mathbf{R}\,\sigma_Y^2$, declines, which leads to a decrease in the discount rate. Two, because the average elasticity of intertemporal substitution has increased, the demand for saving increases, which lowers the riskless rate through a drop in the $\mathbf{R}\,\mu_Y$ term in (15), and so this further reduces the discount rate. But, the decrease in risk aversion also reduces demand for precautionary savings, given in (15) by the term $-\frac{1}{2}\mathbf{P}\,\mathbf{R}\,\sigma_Y^2$, which increases the interest rate, and hence, the discount rate; the magnitude of this effect will depend on average prudence, $\mathbf{P}$. Therefore, the discount rate will be countercyclical when average prudence is not too large. Condition (14) states this precisely. Thus, when this condition is satisfied, then the introduction of a non-redundant security will lead to an increase in stock market return volatility.

5 Conclusions

There is an active debate in the academic literature and in the business press about the effect on the volatility of stock markets of introducing new derivative securities in order to improve the ability of agents to share risks. Intuition perhaps suggests that the introduction of non-redundant derivatives that improve risk sharing may also lead to a reduction in stock market volatility. In this paper, we develop a model that shows that this is not necessarily true.

The model we consider is that of a dynamic, general-equilibrium, full-information, endowment economy where there are two types of agents who differ only in risk aversion. Our experiment consists of comparing two versions of this economy: in the first version, risk sharing is not possible at all, and in the second, agents can share risk perfectly. We find that an improvement in risk sharing will lead to an *increase* in the volatility of the stock market as long as average prudence is not too large.

While we consider a model that has been specialized so that we can obtain results in closed form, the intuition underlying our results is quite general. Essentially, the current stock price is equal to the expected value of the discounted sum of future dividends, which can be written as current dividends, divided by the discount rate minus the expected growth rate of future dividends. Consequently, the volatility of stock-market returns depends on the size of shocks to future dividend growth rates (fundamental volatility, which in our model is constant) and shocks to the discount rate (excess volatility). In the economy with imperfect risk sharing, the discount rate is deterministic, so there is no excess volatility. But when risk sharing is improved via the introduction of a non-redundant derivative, the discount rate is stochastic. When this discount rate is countercyclical (positive shocks to future dividend growth rates lead to a decline in the discount rate), stock-return volatility will be greater than fundamental volatility. The only procyclical component of the discount rate is the precautionary savings term in the riskless rate. If average prudence is not too large, the precautionary savings effect is small and stock-return volatility will be higher than fundamental volatility. Hence, stock-return volatility will be higher when risk sharing is improved by the introduction of new derivative securities.

A Appendix: Proofs for propositions in the text

Proof of Proposition 1

With heterogeneity, the price-dividend ratio, p_t, depends on a state variable, which can be any quantity related to how either aggregate consumption or aggregate wealth is shared between agents. In contrast to the rest of the paper, where it is more convenient to use as the state variable the consumption-sharing rule, $\nu_{1,t} = C_{1,t}/Y_t$, in this proof we shall use as the state variable the cross-sectional wealth distribution, $\omega_t = W_{1,t}/(W_{1,t} + W_{2,t})$, where $W_{k,t}$ is the wealth of Agent k.[9] We will now show that, if only the stock is available for trading, then $\sigma_{R,t} = \sigma_Y$ or, equivalently, $\sigma_{p,t} = 0$. For convenience, we omit arguments and time subscripts on stochastic processes in the rest of this proof.

Applying Ito's Lemma to $p = p(\omega)$ and considering only the diffusion term, we obtain

$$\sigma_p = \frac{p_\omega}{p}\sigma_\omega, \tag{A1}$$

where p_ω is the partial derivative of p with respect to ω, and σ_ω is the diffusion term of $d\omega$, given explicitly in (A4). Note that the dynamic budget constraint for Agent k is

$$\frac{dW_k}{W_k} = \left[\phi_k\left(\mu_R - r\right) + r - \frac{C_k}{W_k}\right]dt + \phi_k\sigma_R dZ, \tag{A2}$$

where ϕ_k is the proportion of Agent k's wealth invested in the stock market. Applying Ito's Lemma to ω, it follows that

$$d\omega = \mu_\omega dt + \sigma_\omega dZ, \tag{A3}$$

where the diffusion coefficient, σ_ω, is given by

$$\sigma_\omega \quad = \quad \omega\left(1-\omega\right)\left(\phi_1 - \phi_2\right)\sigma_R. \tag{A4}$$

When only the stock is available for trading, then both agents invest all their wealth in the stock market, that is, $\phi_1 = 1 = \phi_2$, and it then follows from (A4) that $\sigma_\omega = 0$. Consequently, from (A1), $\sigma_p = 0$, which then gives the result in the proposition.

[9] The proof where instead of using the wealth distribution as the state variable we use the consumption-sharing rule, $\nu_{1,t} = C_{1,t}/Y_t$, is included in the technical appendix that also provides the full details of equilibrium in the economy where financial markets are incomplete. This appendix is available on our web site.

Proof of Proposition 2

Given that markets are complete, agents face a common state-price density. Therefore, we omit the subscript k for π, r, and θ. Equation (7) then has the following integral representation for $u \geq t$:

$$\pi_u = \pi_t \exp\left\{ -\int_t^u r_s ds - \frac{1}{2}\int_t^u \theta_s^2 ds - \int_t^u \theta_s dZ_s \right\}. \tag{A5}$$

Consequently, the price-dividend ratio p_t is equal to

$$p_t = \mathbb{E}_t \int_t^T \frac{\pi_u Y_u}{\pi_t Y_t} du = \mathbb{E}_t \int_t^T \frac{M_u}{M_t} e^{-\int_t^u f_s ds} du, \tag{A6}$$

where f_s is the discount rate adjusted for growth in dividends, μ_Y,

$$f_s = (r_s + \theta_s \sigma_Y) - \mu_Y, \tag{A7}$$

and

$$M_t = \exp\left\{ -\frac{1}{2}\int_0^t (\theta_s - \sigma_Y)^2 ds - \int_0^t (\theta_s - \sigma_Y) dZ_s \right\}. \tag{A8}$$

The process M_t is a martingale under the natural measure $\mathbb{P}$, if the process θ_t is bounded.[10] We can define the new probability measure $\mathbb{P}'$ on $(\Omega, \mathcal{F})$ via

$$\mathbb{P}'(A) = \mathbb{E}(1_A M_T). \tag{A9}$$

The advantage of the new measure is that it will be easier to take the Malliavin derivative under $\mathbb{P}'$ than under $\mathbb{P}$, because changing the measure eliminates M from (A6). Girsanov's Theorem implies that the process, $Z_t^{\mathbb{P}'}$, defined by

$$Z_t^{\mathbb{P}'} = Z_t + \int_0^t (\theta_s - \sigma_Y) ds \tag{A10}$$

is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P}')$. Therefore, we can rewrite (A6) as

$$p_t = \mathbb{E}_t^{\mathbb{P}'} \int_t^T e^{-\int_t^u f_s ds} du. \tag{A11}$$

[10] We know that θ_t/σ_Y is equal to the average relative risk aversion ($\mathbf{R}_t$) in the economy, given in (A18). It follows from (A18) that $\mathbf{R}_t \in [\gamma_1, \gamma_2]$ for all $t \in [0, T]$, so θ_t is indeed bounded.

The following condition

$$\frac{1}{(u-t)}\int_t^u (r_s + \theta_s \sigma_Y) ds > \mu_Y \tag{A12}$$

is sufficient to guarantee that the integral in (A11) is well-defined. Note that if $r_t + \theta_t \sigma_Y$ is a constant and T tends to infinity, then (A11) simplifies to give

$$p_t = \frac{1}{r_t + \theta_t \sigma_Y - \mu_Y}, \tag{A13}$$

which implies that the Gordon growth formula holds, with $r_t + \theta_t \sigma_Y$ being the discount rate. By analogy, we shall still refer to $r_t + \theta_t \sigma_Y$ as the "discount rate", even when it is a stochastic process and the time horizon is finite. With heterogeneity, the price-dividend ratio need not be constant. It will depend on an additional state variable, and we choose this state variable to be the share of aggregate consumption consumed by Agent 1, $\nu_{1,t} = C_{1,t}/Y_t$. The share of aggregate consumption consumed by Agent 2, ν_2, is defined analogously.

Using the central-planner's formulation of the optimization problem of the lifetime utility of the two agents, with weights λ and $1-\lambda$, leads to the following expression for the sharing rule in complete markets:

$$\frac{u_1'(C_{1,t})}{u_2'(C_{2,t})} = \frac{1-\lambda}{\lambda}. \tag{A14}$$

Since now $C_{k,t} = \nu_{k,t} Y_t$, it is immediate that

$$\frac{\nu_{2,t}^{\gamma_2}}{\nu_{1,t}^{\gamma_1}} Y_t^{\gamma_2 - \gamma_1} = \frac{1-\lambda}{\lambda}. \tag{A15}$$

By differentiating equation (A15) implicitly with respect to Y_t and taking into account the market clearing condition $\nu_{1,t} + \nu_{2,t} = 1$, we can show that

$$Y_t \frac{d\nu_{1,t}}{dY_t} = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) \nu_{1,t} (1 - \nu_{1,t}) \mathbf{R}_t, \tag{A16}$$

$$Y_t^2 \frac{d^2 \nu_{1,t}}{dY_t^2} = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) \nu_{1,t} (1 - \nu_{1,t}) \mathbf{R}_t \left(\frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2\right), \tag{A17}$$

where

$$\mathbf{R}_t = \frac{1}{\frac{1}{\gamma_1}\nu_{1,t} + \frac{1}{\gamma_2}(1 - \nu_{1,t})}, \tag{A18}$$

is the harmonic-weighted-average relative risk aversion of the economy (or the reciprocal of the weighted average risk tolerances of the two agents) with the weights being the consumption shares, $\nu_{1,t}$ and $(1-\nu_{1,t})$.

Applying Ito's lemma to $\nu_{1,t} = \nu_{1,t}(Y_t)$, it follows that the stochastic process for the consumption-sharing rule, $\nu_{1,t}$, is

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_{1,t}} dt + \sigma_{\nu_{1,t}} dZ_t, \tag{A19}$$

with the drift and diffusion terms given by

$$\mu_{\nu_{1,t}} = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)(1-\nu_{1,t})\,\mathbf{R}_t\left[\mu_Y + \frac{1}{2}\left(\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2} - 2\right)\sigma_Y^2\right], \tag{A20}$$

$$\sigma_{\nu_{1,t}} = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)(1-\nu_{1,t})\,\mathbf{R}_t\,\sigma_Y. \tag{A21}$$

We now derive the explicit expression for $\sigma_{p,t}$, which depends on the discount rate and the sharing rule. Then we use that expression as well as equations (A20) and (A21) to derive a sufficient condition for the excess volatility $\sigma_{p,t}$ to be strictly positive.

Using the chain rule of Malliavin calculus (the necessary background for understanding Malliavin calculus is provided in Appendix B), the Malliavin derivative of p_t at t, denoted by $\mathcal{D}_t p_t$ is equal to:

$$\mathcal{D}_t p_t = \frac{\partial p_t}{\partial \nu_{1,t}} \mathcal{D}_t \nu_{1,t} = \frac{\partial p_t}{\partial \nu_{1,t}} \nu_{1,t} \sigma_{\nu_{1,t}}. \tag{A22}$$

Note that this Malliavin derivative is defined relative to the probability space $(\Omega, \mathcal{F}, \mathbb{P}')$. From Ito's Lemma, we know that the diffusion coefficient of dp_t/p_t is given by $\sigma_{p,t}$, which is equal to $(\partial p_t/\partial \nu_{1,t})(\nu_{1,t}/p_t)\sigma_{\nu_{1,t}}$. Therefore, from (A11),

$$\begin{aligned}
\sigma_{p,t} &= \frac{\mathcal{D} p_t}{p_t} = \frac{1}{p_t}\mathcal{D}_t \mathbb{E}_t^{\mathbb{P}'} \int_t^T e^{-\int_t^u f_s ds} du = \frac{1}{p_t}\mathbb{E}_t^{\mathbb{P}'} \int_t^T \mathcal{D}_t e^{-\int_t^u f_s ds} du \\
&= -\frac{1}{p_t}\mathbb{E}_t^{\mathbb{P}'} \int_t^T e^{-\int_t^u f_s ds}\left(\mathcal{D}_t \int_t^u f_s ds\right) du = -\frac{1}{p_t}\mathbb{E}_t^{\mathbb{P}'} \int_t^T e^{-\int_t^u f_s ds}\left(\int_t^u \mathcal{D}_t f_s ds\right) du \\
&= -\frac{1}{p_t}\mathbb{E}_t^{\mathbb{P}'} \int_t^T e^{-\int_t^u f_s ds}\left(\int_t^u \frac{\partial f_s}{\partial \nu_{1,s}} \mathcal{D}_t \nu_{1,s} ds\right) du,
\end{aligned} \tag{A23}$$

where in the last step we applied the chain rule of Malliavin calculus to f_s.

Expression (A23) for $\sigma_{p,t}$ implies that $\sigma_{p,t} > 0$ if

$$\frac{\partial f_s}{\partial \nu_{1,s}} < 0, \tag{A24}$$

and

$$\mathcal{D}_t \nu_{1,s} > 0. \tag{A25}$$

We first prove that (A25) holds. Our argument is based on the result that a process which follows a geometric Brownian motion with bounded drift and diffusion will always be of the same sign. Since from (A21), $\mathcal{D}_t \nu_{1,t} = \sigma_{\nu_{1,t}} > 0$, all we have to do is to show that the drift and diffusion terms of the Ito process $d(\mathcal{D}_t \nu_{1,s})/\mathcal{D}_t \nu_{1,s}$ are bounded.

From (A10), process (A19) may be rewritten under the measure $\mathbb{P}^{'}$ as

$$\nu_{1,s} = \nu_{1,t} + \int_t^s \nu_{1,u} \mu^{\mathbb{P}'}_{\nu_{1,u}} du + \int_t^s \nu_{1,u} \sigma_{\nu_{1,u}} dZ^{\mathbb{P}'}_u, \tag{A26}$$

where

$$\mu^{\mathbb{P}'}_{\nu_{1,u}} = \mu_{\nu_{1,u}} - (\theta_u - \sigma_Y)\, \sigma_{\nu_{1,u}}. \tag{A27}$$

Applying the Malliavin operator to $\nu_{1,s}$ gives

$$\begin{aligned} \mathcal{D}_t \nu_{1,s} &= \mathcal{D}_t \nu_{1,t} + \int_t^s \mathcal{D}_t [\nu_{1,u} \mu^{\mathbb{P}'}_{\nu_{1,u}}] du + \int_t^s \mathcal{D}_t [\nu_{1,u} \sigma_{\nu_{1,u}}] dZ^{\mathbb{P}'}_u \\ &= \mathcal{D}_t \nu_{1,t} + \int_t^s \frac{\partial}{\partial \nu_{1,u}} [\nu_{1,u} \mu^{\mathbb{P}'}_{\nu_{1,u}}] \mathcal{D}_t \nu_{1,u} du \\ &\quad + \int_t^s \frac{\partial}{\partial \nu_{1,u}} [\nu_{1,u} \sigma_{\nu_{1,u}}] \mathcal{D}_t \nu_{1,u} dZ^{\mathbb{P}'}_u, \end{aligned} \tag{A28}$$

where the last step follows using the chain rule of Malliavin calculus. Therefore,

$$\frac{d\,[\mathcal{D}_t \nu_{1,s}]}{\mathcal{D}_t \nu_{1,s}} = \frac{\partial}{\partial \nu_{1,s}} [\nu_{1,s} \mu^{\mathbb{P}'}_{\nu_{1,s}}] ds + \frac{\partial}{\partial \nu_{1,s}} [\nu_{1,s} \sigma_{\nu_{1,s}}] dZ^{\mathbb{P}'}_s. \tag{A29}$$

From (A20) and (A21) we find

$$\begin{aligned} \frac{\partial}{\partial \nu_{1,s}} [\nu_{1,s} \mu^{\mathbb{P}'}_{\nu_{1,s}}] &= \frac{\partial}{\partial \nu_{1,s}} [\nu_{1\,s} \sigma_{\nu_{1,s}}] \left[\mu_Y + \frac{1}{2} \left(\frac{\mathbf{R}_s^2}{\gamma_1 \gamma_2} - 2 \right) \sigma_Y^2 - (\mathbf{R}_s - 1)\, \sigma_Y^2 \right] \frac{1}{\sigma_Y} \\ &\quad - \frac{\nu_{1,s} \sigma_{\nu_{1,s}}}{\sigma_Y} \left[\frac{\mathbf{R}_s}{\gamma_1 \gamma_2} - \sigma_Y^2 \right] \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_s^2, \end{aligned} \tag{A30}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \nu_{1,s}}[\nu_{1,s}\sigma_{\nu_{1,s}}] &= \left(\frac{1}{\gamma_1}-\frac{1}{\gamma_2}\right)\frac{\partial}{\partial \nu_{1,s}}\left[\nu_{1,s}\left(1-\nu_{1,s}\right)\mathbf{R}_s\right]\sigma_Y \qquad \text{(A31)}\\
&= \left(\frac{1}{\gamma_1}-\frac{1}{\gamma_2}\right)\left[\left(1-2\nu_{1,s}\right)\mathbf{R}_s-\nu_{1,s}\left(1-\nu_{1,s}\right)\left(\frac{1}{\gamma_1}-\frac{1}{\gamma_2}\right)\mathbf{R}_s^2\right]\sigma_Y.
\end{aligned}
$$

Consequently, taking also into account that $\nu_{1,s}\in[0,1]$ and $\mathbf{R}_s\in[\gamma_1,\gamma_2]$, we find that the drift and diffusion terms in the right-hand side of equation (A29) are bounded, and thus, $\mathcal{D}_t\nu_{1,s}>0$.

Therefore, the excess volatility term is positive, $\sigma_{p,t}>0$, as long as

$$
\frac{\partial f_t}{\partial \nu_{1,t}}=\frac{\partial}{\partial \nu_{1,t}}\left(r_t+\theta_t\sigma_Y\right)<0. \qquad \text{(A32)}
$$

We now find a sufficient condition for (A32) to hold. To do that we derive an expression for the discount rate, $r+\theta\sigma_Y$, in terms of exogenous parameters and the consumption sharing rule. Agent 1's first-order condition implies that

$$
\pi_t=\lambda e^{-\rho t}(\nu_{1,t}Y_t)^{-\gamma_1}. \qquad \text{(A33)}
$$

Applying Ito's Lemma to (A33), and matching the drift and diffusion coefficients with those in (7), we find that the riskless interest rate is

$$
r_t=\rho+\gamma_1\left(\mu_Y+\mu_{\nu_{1,t}}+\sigma_Y\sigma_{\nu_{1,t}}\right)-\frac{1}{2}\gamma_1\left(1+\gamma_1\right)\left(\sigma_Y+\sigma_{\nu_{1,t}}\right)^2, \qquad \text{(A34)}
$$

and that the market price of risk is

$$
\theta_t=\gamma_1\left(\sigma_Y+\sigma_{\nu_{1,t}}\right). \qquad \text{(A35)}
$$

Substituting the expressions for $\mu_{\nu_{1,t}}$ and $\sigma_{\nu_{1,t}}$ from (A20) and (A21) into (A34) and (A35) gives

$$
r_t=\rho+\mathbf{R}_t\,\mu_Y-\frac{1}{2}\mathbf{R}_t\,\mathbf{P}_t\sigma_Y^2, \qquad \text{(A36)}
$$

and

$$
\theta_t=\mathbf{R}_t\sigma_Y, \qquad \text{(A37)}
$$

where, $\mathbf{P}_t$ is the average prudence in the economy:

$$
\mathbf{P}_t=\nu_{1,t}\left(1+\gamma_1\right)\left(\frac{\mathbf{R}_t}{\gamma_1}\right)^2+\left(1-\nu_{1,t}\right)\left(1+\gamma_2\right)\left(\frac{\mathbf{R}_t}{\gamma_2}\right)^2. \qquad \text{(A38)}
$$

We can now write the discount-rate process as[11]

$$r_t + \theta_t \sigma_Y = \rho + \mathbf{R}_t\, \mu_Y - \frac{1}{2}\mathbf{R}_t\, \mathbf{P}_t \sigma_Y^2 + \mathbf{R}_t \sigma_Y^2. \tag{A39}$$

Because $\partial \nu_{1,t}/\partial \mathbf{R}_t < 0$, condition (13) is equivalent to

$$\frac{\partial}{\partial \mathbf{R}_t}\left(r_t + \theta_t \sigma_Y\right) > 0. \tag{A40}$$

From (A39) it follows that (A40) can be written as

$$\mu_Y > \frac{1}{2}\left[\frac{\partial}{\partial \mathbf{R}_t}(\mathbf{R}_t\, \mathbf{P}_t) - 2\right]\sigma_Y^2 = \frac{1}{2}\left[\mathbf{P}_t\left(1 + \frac{\partial \ln \mathbf{P}_t}{\partial \ln \mathbf{R}_t}\right) - 2\right]\sigma_Y^2. \tag{A41}$$

We want to find an upper bound for

$$\frac{\partial \ln \mathbf{P}_t}{\partial \ln \mathbf{R}_t} = 2 - \frac{\mathbf{R}_t}{\mathbf{P}_t}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + 1\right). \tag{A42}$$

Differentiating the expression in (A42), after substituting the definition of $\mathbf{P}_t$ from equation (A38) into the expression above, and using equation (A18) to substitute $\mathbf{R}_t$ for ν_t, gives:

$$\frac{\partial_t}{\partial \mathbf{R}_t}\frac{\partial \ln \mathbf{P}_t}{\partial \ln \mathbf{R}_t} = -\frac{\gamma_1\left(1+\gamma_2\right)+\gamma_2}{\left(\gamma_1\left(1+\gamma_2\right)+\gamma_2 - \mathbf{R}_t\right)^2} < 0, \tag{A43}$$

and thus, $\frac{\partial \ln \mathbf{P}_t}{\partial \ln \mathbf{R}_t}$ is monotonically decreasing in $\mathbf{R}_t$. Therefore,

$$\frac{\partial \ln \mathbf{P}_t}{\partial \ln \mathbf{R}_t} \leq \left.\frac{\partial \ln \mathbf{P}_t}{\partial \ln \mathbf{R}_t}\right|_{\mathbf{R}_t = \boldsymbol{\gamma}_1} = 1 - \frac{\gamma_1}{\gamma_2\left(1+\gamma_1\right)} < 1. \tag{A44}$$

Finally,

$$\frac{1}{2}\left[\mathbf{P}_t\left(1 + \frac{\partial \ln \mathbf{P}_t}{\partial \ln \mathbf{R}_t}\right) - 2\right]\sigma_Y^2 < \frac{1}{2}\left(2\,\mathbf{P}_t - 2\right)\sigma_Y^2 = \left(\mathbf{P}_t - 1\right)\sigma_Y^2, \tag{A45}$$

and inequality (14) is a sufficient condition for (A41), and thus, for (12) to hold.

[11] Note that the discount-rate process, $r_t + \theta_t \sigma_Y$, is a function of $\nu_{1,t}$ alone—any dependence on Y_t is through $\nu_{1,t}$.

B Appendix: An introductory guide to Malliavin calculus

This short appendix contains all that the reader needs in order to understand the applications of Malliavin calculus in this paper. In order to simplify the exposition, we describe the basic facts about Malliavin calculus with an absolute minimum of mathematical rigor. For a mathematical exposition of Malliavin calculus, see Nualart (1995).

Consider a random variable of the form

$$F \equiv f(Z_{t_1}, \ldots, Z_{t_n}), \tag{B1}$$

where f is a suitably well-behaved function (continuously differentiable) and Z is a standard Brownian motion with respect to the measure $\mathbb{P}$. F depends on the values taken by Z at the times $t_1, \ldots, t_n$, where $0 < t_1 < \ldots < t_n < T$.[12] Suppose that t lies between time-points t_{k-1} and t_k, that is, $t \in (t_{k-1}, t_k)$, for some $k \in \{2, \ldots, n\}$. Consider a small change to Z for all $s \geq t$, so that Z_s is changed to $Z_s + \epsilon$ for all $s \geq t$. For $s < t$, Z is unchanged. The Malliavin derivative of F at t, denoted by $\mathcal{D}_t F$ is defined as

$$\mathcal{D}_t F = \lim_{\epsilon \to 0} \frac{f(Z_{t_1}, \ldots, Z_{t_{k-1}}, Z_{t_k} + \epsilon, \ldots, Z_{t_n} + \epsilon) - f(Z_{t_1}, \ldots, Z_{t_n})}{\epsilon}. \tag{B2}$$

In words, we can say that the Malliavin derivative of F at t is the change in F when the path of the Brownian motion Z is shifted by the same small amount at all times after and including time t.

From standard calculus we know

$$\mathcal{D}_t F = \sum_{i=k}^{n} \frac{\partial f\left(Z_{t_1}, \ldots, Z_{t_n}\right)}{\partial Z_i}. \tag{B3}$$

Hence, for

$$F = Z_{t_1} + \ldots + Z_{t_n}, \tag{B4}$$

we have

$$\mathcal{D}_t F = n - (k-1). \tag{B5}$$

Consider now the particular stochastic process we use to describe aggregate dividends:

$$Y_t = Y_0 e^{(\mu_Y - \frac{1}{2}\sigma_Y^2)t + \sigma_Y Z_t}, \tag{B6}$$

[12] Such a function that depends on the values taken by a Brownian motion at a finite number of time-points on its sample path is a smooth Brownian functional.

where Z is a standard Brownian motion with respect to the measure $\mathbb{P}$. In this case

$$D_s Y_t = \frac{\partial Y_t}{\partial Z_t} = Y_0 e^{(\mu_Y - \frac{1}{2}\sigma_Y^2)t + \sigma_Y Z_t} \sigma_Y = Y_t \sigma_Y. \tag{B7}$$

We can extend the definition (B2) to random variables which depend on the path of a Brownian motion over the continuous interval $[0, T]$,[13] not just at a discrete number of time-points.[14] For example, when

$$F = \int_0^T t^2 dZ_t, \tag{B8}$$

we have

$$\mathcal{D}_t F = t^2, \tag{B9}$$

which follows naturally from the definitions (B2) and (B3) if one recalls that in standard calculus

$$\frac{d}{dx} \int_0^a x^2 dx = x^2, \tag{B10}$$

where x is a real variable and a a constant.

We often use the Chain Rule of Malliavin calculus. When $\boldsymbol{F} = (F_1, F_2)$ is a vector of random variables and ϕ is a well-behaved function of $\boldsymbol{F}$ (differentiable with bounded derivatives), then

$$\mathcal{D}_t \phi(\boldsymbol{F}) = \frac{\partial \phi(\boldsymbol{F})}{\partial F_1} \mathcal{D}_t F_1 + \frac{\partial \phi(\boldsymbol{F})}{\partial F_2} \mathcal{D}_t F_2. \tag{B11}$$

For example, when the price-dividend ratio at time t, p_t is a function of the consumption share, $\nu_{1,t}$, we can use the Chain Rule of Malliavin calculus to show that

$$\mathcal{D}_t p_t = \frac{\partial p_t}{\partial \nu_{1,t}} \mathcal{D}_t \nu_{1,t}. \tag{B12}$$

If $\nu_{1,t}$ follows an Ito process, then

$$\nu_{1,t} = \nu_{1,0} + \int_0^t \mu_{\nu_{1,u}} du + \int_0^t \sigma_{\nu_{1,u}} dZ_u. \tag{B13}$$

[13]It is not possible to define a Malliavin derivative for random variables that depend on the path of a Brownian motion over an infinite horizon. That is why this paper uses a finite horizon economy.

[14]The space of random variable over which Malliavin derivatives can be defined is $\mathbb{D}^{1,2}$, which is the completion of the set of smooth Brownian functionals with respect to the norm $\|F\|_{1,2} = E\left(F^2\right)^{1/2} + E\left(\int_0^T \|\mathcal{D}_t F\|^2 dt\right)^{1/2}$.

Taking the Malliavin derivative of $\nu_{1,t}$ at time gives

$$\mathcal{D}_t \nu_{1,s} = \mathcal{D}_t \int_0^s \mu_{\nu_{1,u}} du + \mathcal{D}_t \int_0^s \sigma_{\nu_{1,u}} dZ_u. \tag{B14}$$

When f is a suitably behaved stochastic process (progressively measurable and bounded),

$$\mathcal{D}_t \int_0^T f_s ds = \mathcal{D}_t \left(\int_0^t f_s ds + \int_t^T f_s ds \right) = \mathcal{D}_t \int_t^T f_s ds = \int_t^T \mathcal{D}_t f_s ds. \tag{B15}$$

Note that

$$\mathcal{D}_t \int_0^t f_s ds = 0, \tag{B16}$$

because the integral is unaffected by shifts to the path of Z which occur before time t. We can also use the definition of the Malliavin derivative to show that

$$\mathcal{D}_t \int_0^T f_s dZ_s = f_t + \mathcal{D}_t \int_t^T f_s dZ_s = f_t + \int_t^T \mathcal{D}_t f_s dZ_s. \tag{B17}$$

We then use (B15) and (B17) to show that

$$\mathcal{D}_t \nu_{1,s} = \int_t^s \mathcal{D}_t \mu_{\nu_{1,u}} du + \int_t^s \mathcal{D}_t \sigma_{\nu_{1,u}} dZ_u + \sigma_{\nu_{1,t}}. \tag{B18}$$

If $\mu_{\nu_{1,u}}$ and $\sigma_{\nu_{1,u}}$ are functions of $\nu_{1,u}$, then using the Chain Rule of Malliavin calculus, we find that

$$\mathcal{D}_t \nu_{1,s} = \int_t^s \frac{\partial \mu_{\nu_{1,u}}}{\partial \nu_{1,u}} \mathcal{D}_t \nu_{1,u} du + \int_t^s \frac{\partial \sigma_{\nu_{1,u}}}{\partial \nu_{1,u}} \mathcal{D}_t \nu_{1,u} dZ_u + \sigma_{\nu_{1,t}}, \tag{B19}$$

which implies that

$$d\left(\mathcal{D}_t \nu_{1,s}\right) = \frac{\partial \mu_{\nu_{1,u}}}{\partial \nu_{1,s}} \left(\mathcal{D}_t \nu_{1,s}\right) ds + \frac{\partial \sigma_{\nu_{1,s}}}{\partial \nu_{1,s}} \left(\mathcal{D}_t \nu_{1,s}\right) dZ_s, \tag{B20}$$

where

$$\mathcal{D}_t \nu_{1,t} = \sigma_{\nu_{1,t}}. \tag{B21}$$

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