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Catherine Doz, Domenico Giannone and Lucrezia Reichlin

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Catherine Doz, Université Cergy-Pontoise
Domenico Giannone, Université Libre de Bruxelles and ECARES
Lucrezia Reichlin, European Central Bank and ECARES and CEPR

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Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR, UK
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: www.cepr.org

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ABSTRACT

A Quasi Maximum Likelihood Approach for Large Approximate Dynamic Factor Models*

This paper considers quasi-maximum likelihood estimations of a dynamic approximate factor model when the panel of time series is large. Maximum likelihood is analyzed under different sources of misspecification: omitted serial correlation of the observations and cross-sectional correlation of the idiosyncratic components. It is shown that the effects of misspecification on the estimation of the common factors is negligible for large sample size (T) and the cross-sectional dimension (n). The estimator is feasible when n is large and easily implementable using the Kalman smoother and the EM algorithm as in traditional factor analysis. Simulation results illustrate what are the empirical conditions in which we can expect improvement with respect to simple principle components considered by Bai (2003), Bai and Ng (2002), Forni, Hallin, Lippi, and Reichlin (2000, 2005b), Stock and Watson (2002a,b).

JEL Classification: C32, C33 and C51
Keywords: factor model, large cross-sections and Quasi Maximum Likelihood

Catherine Doz
Directrice de l'UFR Economie Gestion
Universite de Cergy-Pontoise
33 Boulevard du Port
95011 Cergy-Pontoise
FRANCE
Tel: (33 1) 34 25 60 53
Fax: (33 1) 34 25 60 52
Email: catherine.doz@eco.u-cergy.fr

For further Discussion Papers by this author see:
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Domenico Giannone
ECARES
Université Libre de Bruselles
Av. F.D. Roosevelt, 50 - CP 114
1050 Bruselles
BELGIUM
Tel: (32 2) 650 4221
Fax: (32 2) 650 4475
Email: dgiannon@ulb.ac.be

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=154334

Lucrezia Reichlin
Director General Research
European Central Bank
Kaiserstrasse 29
60311 Frankfurt
GERMANY
Tel: (49 69) 1344 7200
Fax: (49 69) 1344 6575
Email: lucrezia.reichlin@ecb.int

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1 Introduction

The idea that the dynamics of large panels of time series can be characterized as being driven by few common factors and a variable specific-idiosyncratic component is appealing for macroeconomic and finance applications where data are strongly collinear. Applications in macroeconomics date back to the seventies (Geweke, 1977; Sargent and Sims, 1977; Geweke and Singleton, 1980). In finance, the factor model has also a long tradition since it relates closely to the CAPM model of asset prices.

In traditional factor analysis, for a given size of the cross-section n, the model can be consistently estimated by maximum likelihood. The literature has proposed both frequency domain (Geweke, 1977; Sargent and Sims, 1977; Geweke and Singleton, 1980) and time domain (Engle and Watson, 1981; Stock and Watson, 1989; Quah and Sargent, 1992) methods.

Identification is achieved by assuming that, for each series, the component driven by the common factors (common component) is orthogonal to the idiosyncratic component and the common component has cross-sectionally orthogonal elements. A factor model with orthogonal idiosyncratic elements is called an exact factor model.

Although the idea of factor analysis is appealing, the traditional approach presents some limitations. First of all, the assumption of an exact factor structure is an excessive straightjacket on the data, leading to potentially harmful misspecification. In particular, with large panels, containing sectoral variables for example, the assumption of orthogonal idiosyncratic elements is likely to be less adequate than with panels with a small number of aggregate variables. Second, although the coefficients of the factors loadings can be consistently estimated for T large via maximum likelihood, the factors are indeterminate and one can only obtain their expected value (on this point, see Steiger, 1979). Third, many empirically interesting economic applications require the study of large panels, situation in which the properties of the maximum likelihood estimates are unknown and where maximum likelihood is generally considered not feasible (Bai, 2003; Bai and Ng, 2002).

As a response to this limitations, recent literature has generalized the idea of factor analysis to handle less strict assumptions on the covariance of the idiosyncratic elements (approximate factor structure) and proposed non-parametric estimators of the common factors based on principal components, which are feasible for n large (Forni, Hallin, Lippi, and Reichlin, 2000; Stock and Watson, 2002a,b).

Key feature of this approach is that identification and consistency are analysed as n, as well as T, go to infinity. It is shown that, under suitable assumptions, if the cross-sectional dimension n tends to infinity, the principal components of the observations become increasingly collinear with the common factors and identification is achieved asymptotically for n (Chamberlain, 1983; Chamberlain and Rothschild, 1983; Forni, Hallin, Lippi, and Reichlin, 2000; Forni and Lippi, 2001). Principal components are also proved to be n, T consistent estimators of the factor space (Bai, 2003; Bai and Ng, 2002; Forni, Hallin, Lippi, and Reichlin, 2000, 2005b; Stock and Watson, 2002a,b; Forni, Giannone, Lippi, and Reichlin, 2005a).

The approximate factor model presents several advantages with respect to the exact model. It is very flexible and suitable under general assumptions on measurement

error, geographical clustering and, in general, local cross correlation. However, maximum likelihood estimator has never been analyzed for this model. The reason is that, in order to estimate the model by maximum likelihood, it is necessary to impose a parametrization while retaining parsimony. Parsimony is achieved in the exact factor model by restricting the cross-correlation among idiosyncratic components to be zero. Once this restriction is relaxed, there is no obvious way to model the cross-sectional correlation among idiosyncratic terms since, in the cross-section, there is no natural order.

This paper studies maximum likelihood estimation for the approximate factor model in large panels. The central idea is to treat the exact factor model as a misspecified approximating model and analyze the properties, for n and T going to infinity, of the maximum likelihood estimator of the factors under misspecification, that is when the true probabilistic model is approximated by a more restricted model. This is a quasi maximum likelihood estimator (QML) in the sense of White (1982). We derive the n, T rates of convergence for it and show its feasibility when n is large. We show that traditional factor analysis in large cross-section is feasible and that consistency is achieved even if the underlying data generating process is an approximate factor model rather than an exact one. More precisely, our consistency result shows that the expected value of the common factors converges to the true factors as $n, T \to \infty$ along any path (we also provide the consistency rates).

This result tells us that the misspecification error due to the approximate structure of the idiosyncratic component vanishes asymptotically for n and T large, provided that the cross-correlation of the idiosyncratic processes is limited and that of the common components is pervasive throughout the cross section as n increases. These are conditions that have been introduced by Chamberlain and Rothschild (1983) and used, reinterpreted and extended by, respectively, (Connor and Korajczyk, 1986, 1988, 1993; Forni et al., 2000; Forni and Lippi, 2001; Stock and Watson, 2002a,b).

Our result should be interpreted as a reconciliation of the classical factor analysis approach with the new generation of dynamic factor models with n large in which the common factors are estimated by principal components. We show that these two approaches are related in the sense that principal components estimators can be reinterpreted as quasi-maximum likelihood estimators, i.e. maximum likelihood under a misspecified model where data are supposed to be generated by a factor model with spherical idiosyncratic components and non serially correlated observations.

From the practical point of view we show that, unlike what sometime claimed in the literature, classical likelihood based methods are feasible in the large n case. Under standard parameterizations, the factor model can in fact be cast in a state space form and the likelihood can be maximized via the EM algorithm which requires at each iteration only one run of the Kalman smoother (Engle and Watson, 1981). Under the exact factor structure restriction on the approximating model, the computational complexity of the smoother depends essentially on the number of common factors r which is typically small. The intuition of why this works was first provided in the literature by Quah and Sargent (1992) who estimated a model with $n = 60$ already in early 90s. Moreover, since principal components provide a good approximation of the common factors in a large cross-section, they can be used to get a good initial estimate of the

parameters for initializing the numerical algorithm for maximum likelihood estimation.

There are many reasons why our result is a useful contribution to the literature of factor models in large panels. First, maximum likelihood estimation is particularly attractive for economic applications since it provides a framework for incorporating restrictions deriving from economic theory in the statistical models. Indeed, an increasing number of studies in macroeconomics have used likelihood based, Bayesian, methods for extracting the common factors from a large panel of time series (Kose, Otrok, and Whiteman, 2003; Boivin and Giannoni, 2005; Bernanke, Boivin, and Eliasz, 2005). However, the model is estimated under the assumption that the data follow an exact factor structure and it is not clear what is the price one pays for this kind of misspecification. Moreover, even assuming that data factor structure is exact, the asymptotic properties of the estimates when both the sample size and the cross-sectional dimension are large have not been studied. Second, if the true data generating process (DGP) and the approximating model coincide, then maximum likelihood estimates are the most efficient. Finally, once we have a parametric model estimated by likelihood based methods, it is possible to handle missing data and enlarge the range of interesting empirical applications. In particular, missing data at the end of the sample due to unsynchronized data releases, is a typical problem for real time estimation of the common factors (Giannone, Reichlin, and Sala, 2004; Giannone, Reichlin, and Small, 2005).

The paper is organized as follows. Section two states the assumptions for the model generating the model and those for the approximating model we will use in estimation. Section three states the basic proposition showing consistency and rates for the quasi maximum likelihood estimator. Section four illustrates the empirical results and Section five concludes.

2 Models

2.1 Notation

For any positive definite square matrix M, we will denote by $\lambda_{max}(M)$ ($\lambda_{min}(M)$) its largest (smallest) eigenvalue. Moreover, for any matrix M we will denote by $\|M\|$ the spectral norm defined as $\|M\| = \sqrt{\lambda_{max}(M'M)}$. Given a stochastic process $\{X_{n,T}; T \in \mathbb{Z}, n \in \mathbb{Z}\}$, and a real sequence $\{a_{n,T}; T \in \mathbb{Z}, n \in \mathbb{Z}\}$ we will say that $X_{n,T} = O_P\left(\frac{1}{a_{nT}}\right)$ as $n, T \to \infty$, if the probability that $a_{n,T}X_{n,T}$ is bounded tends to one.

2.2 The approximate dynamic factor model

We suppose that an n-dimensional zero-mean stationary process $\mathbf{x}_t$ is the sum of two unobservable components:

$$\mathbf{x}_t = \Lambda_0 \mathbf{f}_t + \mathbf{e}_t \tag{2.1}$$

where $\mathbf{f}_t = (f_{1t}, ..., f_{rt})'$, the common factors, is an r-dimensional stationary process with mean zero; Λ_0, the factor loadings is an $n \times r$ matrix; $\mathbf{e}_t = (e_{1t}, ..., e_{nt})'$, the idiosyncratic components, is an n-dimensional stationary process with mean zero and

covariance matrix $\mathrm{E}(\mathbf{e}_t\mathbf{e}_t') = \Psi_0$, whose entries will be denoted by $\mathrm{E}(e_{it}e_{jt}) = \psi_{0,ij}$. The common factors $\mathbf{f}_t$ and the idiosyncratic component $\mathbf{e}_t$ are also assumed to be uncorrelated at all leads and lags, that is $\mathrm{E}(f_{jt}e_{is}) = 0$ for all $j = 1, ..., r$, $i = 1, ..., n$ and $t, s \in \mathbb{Z}$. The number of common factors, r, is typically much smaller than the cross-sectional dimension, n.

Notice that this model is quite general since it does not impose cross-sectional orthogonality of the e_i's and can accomodate dynamic effects of the common factors.

Given a sample of size T, we will denote by capital cases the matrices collecting all the variables, that is $\mathbf{X} = (\mathbf{x}_1, ..., \mathbf{x}_T)'$ is the $T \times n$ matrix of observables, $\mathbf{F} = (\mathbf{f}_1, ..., \mathbf{f}_T)'$ is the $T \times r$ matrix of common factors and $\mathbf{E} = (\mathbf{e}_1, ..., \mathbf{e}_T)'$. All these quantities depend on the size of the cross-section and on the sample size. For notational convenience we will not index them by n, T.

The following assumptions define an approximate factor model for large-cross-section.

Assumptions A (Approximate factor model)

A1 $0 < \underline{\lambda} < \liminf_{n\to\infty} \frac{1}{n}\lambda_{min}\left(\Lambda_0'\Lambda_0\right) \leq \limsup_{n\to\infty} \lambda_{max}\frac{1}{n}\left(\Lambda_0'\Lambda_0\right) < \bar{\lambda} < \infty$

A2 $0 < \underline{\psi} < \liminf_{n\to\infty} \lambda_{min}\left(\Psi_0\right) \leq \limsup_{n\to\infty} \lambda_{max}\left(\Psi_0\right) < \bar{\psi} < \infty$

A2 limits the cross-correlation of the idiosyncratic components. While it includes the case in which they are mutually orthogonal, it allows for a more general structure with an upper bound for the variances as a special case. A1 entails that, for n sufficiently large, $\Lambda_0'\Lambda_0/n$ has full rank r (pervasiveness of the common factors). Consequently, by regressing the observations $\boldsymbol{x}_t$ on the factor loadings Λ_0, it is possible to extract the r common factors $\boldsymbol{f}_t$.

Moreover, since $\lambda_{min}(\Psi_0) \leq \psi_{0,ii} \leq \lambda_{max}(\Psi_0)$, a consequence of A2 is that the variance of the idiosyncratic component is uniformly bounded and greater than zero. The assumption that the variance of the idiosyncratic component is uniformly greater than zero implies that the factor extraction is not trivial, i.e. that there is no variable which has no idiosyncratic component.

We will also assume that the common factors and the idiosyncratic component processes are ergodic, so that for large sample size, $T \to \infty$, the sample covariances converge to their population counterpart, uniformly with respect to the cross-sectional dimension.

Assumptions B
There exists a positive constant M such that for all $i, j \in \mathbb{N}$ and for all $T \in \mathbb{Z}$

i) $\mathrm{E}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(e_{it}e_{jt}-\psi_{0,ij})\right)^2 < M$

ii) $\mathrm{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{f}_t e_{jt}\right\|^2 < M$

iii) $\mathrm{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\mathbf{f}_t\mathbf{f}_t'-I_r)\right\|^2 < M$

Under the assumptions above, the common factors can be estimated consistently by mean of principal components. Denote by $\mathcal{D}$ the $r \times r$ diagonal matrix containing the r largest eigenvalues of sample covariance matrix $S = \frac{1}{T}\mathbf{X}'\mathbf{X}$ and and by $\mathcal{V}$ the $n \times r$ matrix whose columns are the corresponding normalized eigenvalues ($\mathcal{V}'\mathcal{V} = I_r$), that is $S\mathcal{V} = \mathcal{V}\mathcal{D}$. The common factors can be estimated as:

$$\hat{\mathbf{F}}_t = \mathbf{X}\mathcal{V}\mathcal{D}^{-1/2}$$

where $\hat{\mathbf{F}}_t$ are the principal components normalized to have sample covariance equal to the identity matrix: $\frac{1}{T}\sum_{t=1}^{T}\hat{\mathbf{f}}_t\hat{\mathbf{f}}_t' = I_r$. Recent literature has shown that the principal components estimator of the common factors provides a good approximation of the common factors for large cross-section and sample size, that is the principal components consistently estimate the space spanned by the true common factors as $n, T \to \infty$ (Forni, Hallin, Lippi, and Reichlin, 2000, 2005b; Stock and Watson, 2002a,b; Bai and Ng, 2002; Bai, 2003; Forni, Giannone, Lippi, and Reichlin, 2005a; Doz, Giannone, and Reichlin, 2005).

In order to develop our alternative maximum likelihood estimator, next section will introduce different approximating models to the data generating process defined here.

2.3 The approximating factor models

An approximating model is a possibly misspecified model that we will use to define the likelihood.

Let us first consider the approximating model: $\mathbf{f}_t$ i.i.d. $\mathcal{N}(0, I_r)$, $\mathbf{e_t}$ i.i.d. $\mathcal{N}(0, \sigma^2 I_n)$. This approximating model is parameterized by Λ and σ^2 which will be collected into θ. In this case, the log likelihood takes the form:

$$\mathcal{L}_{\mathbf{X}}(\mathbf{X};\theta) = -\frac{nT}{2}\log 2\pi - \frac{T}{2}|\Lambda\Lambda' + \sigma^2 I_n| - \frac{T}{2}\mathrm{Tr}\left(\Lambda\Lambda' + \sigma^2 I_n\right)^{-1} S$$

The following normalization is typically made to identify the model for any given cross-sectional size: $\Lambda'\Lambda$ is a diagonal matrix with diagonal entries in decreasing order of magnitude [1]. The maximum likelihood solution is given by:

$$\hat{\Lambda} = \mathcal{V}(\mathcal{D} - \hat{\sigma}^2 I_r)^{1/2} \text{ and } \hat{\sigma}^2 = \frac{1}{n}\mathrm{Trace}(S - \hat{\Lambda}\hat{\Lambda}')$$

[1] Further, if Λ_0 is the true value of the matrix Λ, it is usually assumed that $\Lambda_0'\Lambda_0$ has distinct eigenvalues.

(see, for instance, Lawley and Maxwell (1970), chap.4).

The common factors can be approximated by their expected value under the estimated parameters $\hat{\Lambda}$ and $\hat{\sigma}^2$:

$$\hat{\mathbf{F}}_{\hat{\Lambda},\hat{\sigma}^2} = \mathrm{E}_{\hat{\Lambda},\hat{\sigma}^2}\left[\mathbf{F}|\mathbf{X}\right] = \mathbf{X}\left(\hat{\Lambda}\hat{\Lambda}' + \hat{\sigma}^2 I_n\right)^{-1}\hat{\Lambda} = \mathbf{X}\hat{\Lambda}\left(\hat{\Lambda}'\hat{\Lambda}' + \hat{\sigma}^2 I_n\right)^{-1} = \mathbf{X}\mathcal{V}(\mathcal{D}-\hat{\sigma}^2 I_r)^{1/2}\mathcal{D}^{-1}$$

which are proportional to principal components.

Hence, principal components can be seen as Maximum Likelihood estimator for the factor loadings of the approximate factor model, in a situation in which the approximating probability model is not correctly specified: the true model satisfies condition A1 and A2 while we use an approximating model that restricts the data to be not serially correlated and the idiosyncratic component to be spherical. This is what White (1982) called a Quasi Maximum Likelihood (QML) estimator. Consistency results for principal components can be reinterpreted in the following way. The bias arising from this misspecification of the approximating model is negligible if the cross-sectional dimension is large enough.

The approximating model implicit in principal components has three sources of misspecification since it does not take into account: a) the serial correlation of the common factors and idiosyncratic components; b) the cross-sectional heteroscedasticity of the idiosyncratic components; c) the cross-sectional correlation of the idiosyncratic components.

We know that the misspecification implied by the principal components estimator does not compromise the consistency of the common factors for $n, T \to \infty$. Is this also true when the approximating model is less tight, i.e. when some of the assumptions above are relaxed?

We will now consider a model that is more general and that can potentially allow for efficiency improvements. A natural candidate is the model that has been typically used in traditional exact factor analysis for small cross-section (see, for example, Stock and Watson, 1991).

Approximating parametric model

R1 the common factors follow a finite order gaussian VAR: $A(L)\mathbf{f}_t = \mathbf{u}_t$, with $A(L) = I - A_1 L - ... - A_p L^p$ an $r \times r$ filter of finite length p with roots outside the unit circle, and $\mathbf{u}_t$ an r dimensional gaussian white noise, $\mathbf{u}_t \sim$ i.i.d $\mathcal{N}(0, H)$.

R2 the idiosyncratic components are cross-sectionally independent gaussian white noises: $\mathbf{e}_t \sim$ i.i.d $\mathcal{N}(0, \Psi_d)$ where Ψ_d is a diagonal matrix.

The idiosyncratic component is modelled as a cross-sectionally independent and non serially correlated gaussian processes. The orthogonality restriction among id-

iosyncratic component is key to maintain parsimony in the estimation[2]. The model defined by R1 and R2 is more general than the one under which principal components is the maximum likelihood estimator of the factors since it allows for dynamics of the factors and non sphericity of the idiosyncratic variance.

Let us characterize it by the quadruplet $\Lambda, \Psi_d, A(L), H$. All the parameters will be collected into $\theta \in \Theta$, where Θ is the parameter space defined by R1 and R2.

Notice that the approximating model under which principal components is the maximum likelihood estimator is a particular case of R1 and R2 with non serially correlated factors, $A(L) = I_r, H = I_r$, and spherical idiosyncratic component, $\Psi_d = \sigma^2 I_n$.

Under assumptions R1 and R2, the model can be cast in a state space form with the number of states equal the number of common factors r. For any set of parameters the likelihood can then be evaluated using the Kalman filter.

Given the quasi maximum likelihood estimates of the parameters θ, the common factors can be approximated by their expected value, which can be computed using the Kalman smoother:

$$\hat{\mathbf{F}}_{\hat{\theta}} = \mathrm{E}_{\hat{\theta}}\left[\mathbf{F}|\mathbf{X}\right]$$

In the next section we study the properties of the estimated common factors as n and T go to infinity. Assuming that the true model is approximated, we will consider the effects of misspecification on the estimates.

3 The asymptotic properties of the QML estimator of the common factors

We will now study the properties of a maximum likelihood estimator in which the data follow a factor model that is dynamic and approximate (Assumptions A), while we restrict the approximating model to be exact, with non serially correlated idiosyncratic component and autoregressive common factors (R1 and R2). The proposition below proves consistency of this QML estimator.

To avoid degenerate solutions for the maximum likelihood problem, we will impose the following constraints in the maximization of the likelihood:

Constraints in the maximization of the likelihood

i) $\underline{c} \leq \hat{\psi}_{ii} \leq \bar{c}$ for all $i \in \mathbb{N}$.

ii) $|\hat{A}(z)| < 1$

The constraints (i) and (ii) define a new parameter space $\Theta^c \subseteq \Theta$.

[2]We could also take into account serial correlation of the idiosyncratic components without compromising the parsimony of the model by modelling it as cross-sectionally orthogonal autoregressive process. We do not consider this case in order not to compromise the expositional simplicity.

Remark 0. The constraint is necessary to avoid situations in which estimated parameters imply non-stationarity of the common factors and/or trivial situation in which the variance of the idiosyncratic noise is either zero or infinite.

Assumption C below insures that the constraint on the size of the idiosyncratic component is never binding.

Assumption C
There exists $\delta > 0$ such that $\underline{c} \leq \underline{\psi} - \delta \leq \bar{\psi} + \delta \leq \bar{c}$ where $\underline{c}$ and $\bar{c}$ are the constant in Assumption A (ii).

Proposition 1 Under assumptions A, B and C we have:

$$\text{trace}\left(\frac{1}{T}(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})\right) = O_p\left(\frac{1}{\Delta_{nT}}\right) \text{ as } n, T \to \infty$$

where $\hat{H} = \left(\hat{\mathbf{F}}'_{\hat{\theta}}\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}\hat{\mathbf{F}}'_{\hat{\theta}}\mathbf{F}$ is the coefficient of the OLS projection of $\mathbf{F}$ on $\hat{\mathbf{F}}_{\hat{\theta}}$. The result holds under any restriction on $A(L)$ and H.

Proof See the appendix.

Remark 1 The result of Proposition 1 still holds if the approximating model has more than r common factors. The proof of this remark is in the appendix.

Remark 2 The result of Proposition 1 still holds if the approximating model has spherical idiosyncratic component, that is $\Psi = \sigma^2 I_n$. Consistency of the principal components estimates is a particular case of Proposition 1 which provides an alternative proof of the result in Bai and Ng (2002) under a different set of assumptions. The proof of this remark is in the appendix.

Remark 3 Traditional factor analysis with non serially correlated data correponds to the case $A(L) = I_r$, $H = I_r$.

4 Monte Carlo study

In this section we run a simulation study to asses the performances of our estimator.

The model from which we simulate is standard in the literature. A similar model has been used, for example, in Stock and Watson (2002a).

Let us define it below.

$$\mathbf{x}_t = \Lambda \mathbf{f}_t + \mathbf{E}_t$$

$$A(L)\mathbf{f}_t = \mathbf{u}_t, \text{ with } \mathbf{u}_t \text{ i.i.d. } \mathcal{N}(0, I_r);$$

$$D(L)\mathbf{E}_t = \mathbf{v}_t \text{ with } \mathbf{v}_t \text{ i.i.d. } \mathcal{N}(0, \mathcal{T})$$

$$A_{ij}(L) = \begin{cases} 1 - \rho L & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} ; i, j = 1, ..., r$$

$$D_{ij}(L) = \begin{cases} \sqrt{\alpha_i}(1 - dL) & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} ; i, j = 1, ..., n$$

$$\Lambda_{ij} \text{ i.i.d. } \mathcal{N}(0, 1), i = 1, ..., n; j = 1, .., r$$

$$\alpha_i = \frac{\beta_i}{1-\beta_i} \frac{1}{T} \sum_{t=1}^{T} \left(\sum_{j=1}^{r} \Lambda_{ij} f_{jt} \right)^2 \text{ with } \beta_i \text{ i.i.d. } \mathcal{U}([u, 1-u])$$

$$\mathcal{T}_{ij} = \tau^{|i-j|} \frac{1}{1-d^2}, i, j = 1, ..., n$$

Notice that we allow for cross-correlation between idiosyncratic elements. Since $\mathcal{T}$ is a Toeplitz matrix the cross-correlation among idiosyncratic elements is limited and it is easily seen that Assumption A (ii) is satisfied. The coefficient τ controls for the amount of cross-correlation. The exact factor model correspond to $\tau = 0$.

The coefficient β_i is the ratio between the variance of the idiosyncratic component, e_{it}, and the variance of the common component, $\sum_{j=1}^{r} \Lambda_{ij} f_{jt}$. The is also known as the noise to signal ratio. In our simulation this ratio is uniformly distributed with an average of 50%. If $u = .5$ then the standardized observations have cross-sectionally homoscedastic idiosyncratic components.

Notice that if $\tau = 0, d = 0$, our approximating model is well specified and hence Maximum Likelihood provides the most efficient estimates. If $\tau = 0, d = 0, \rho = 0$, we have a static exact factor model and iteratively reweighed principal components provide the most efficient estimates. Finally, if $\tau = 0, d = 0, u = 1/2$, then on standardized variables we have a static factor models with spherical idiosyncratic components, situation in which principal components on standardized variables provide the most efficient estimates.

We generate the model for different sizes of the cross-section: $n = 5, 10, 25, 50, 100$, and for sample size $T = 50, 100$.

Maximum likelihood estimates are computed using the EM algorithm as in Engle and Watson (1981) and Quah and Sargent (1992).

This algorithm has the advantage of requiring only one run of the Kalman smoother at each iteration. The computational complexity of the Kalman smoother depends mainly on the number of states which in our approximating model corresponds to the number of factors, r, and hence is independent of the size of the cross-section n.

To initialize the algorithm, we compute the first r sample principal components, $\mathbf{f}_{pc,t}$, and estimate the parameters $\hat{\Lambda}^{(0)}\hat{A}^{(0)}(L), \hat{\Psi}_d^{(0)}$ by OLS, treating the principal components as if they were the true common factors. Since these estimates have been proved to be consistent for large cross-sections (Bai, 2003; Forni et al., 2005a; Doz et al., 2005), the initialization is quite good if the cross-section dimension is large. We hence expect the number of iterations required for consistency to decrease as the cross-sectional dimension increases.

The two features highlighted above – small number of state variables and good initialization – make the algorithm feasible in a large cross-section.

To get the intuition of the EM algorithm, let us collect the initial values of the parameters in $\hat{\theta}^{(0)}$. We obtain a new value of the common factors by applying the Kalman smoother:

$$\hat{\mathbf{f}}_{\theta^{(1)},t} = \mathrm{E}_{\hat{\theta}^{(0)}}(\mathbf{f}_t|\mathbf{x}_1, ..., \mathbf{x}_T).$$

If we stop here we have the two-step estimates of the common factors proposed by (Giannone et al., 2004, 2005; Doz et al., 2005).

A new estimate of the parameters, to be collected in $\hat{\theta}^{(2)}$, can then be computed by OLS regression treating $\hat{\mathbf{f}}_{\theta^{(1)},t}$ as if they were the true common factors. If the OLS regressions are modified in order to take into account the fact that the common factors are estimated[3], then we have the EM algorithm that converges to the local maximum of the likelihood[4].

We control convergence by looking at $c_m = \frac{\mathcal{L}_{\mathbf{X}}(\mathbf{X};\hat{\theta}^{(m)}) - \mathcal{L}_{\mathbf{X}}(\mathbf{X};\hat{\theta}^{(m-1)})}{(\mathcal{L}_{\mathbf{X}}(\mathbf{X};\hat{\theta}^{(m)}) + \mathcal{L}_{\mathbf{X}}(\mathbf{X};\hat{\theta}^{(m-1)})/2}$. We stop after M iterations if $c_M < 10^{-4}$.

We simulate the model 500 times and, at each repetition, we apply the algorithm to standardized data since the principal components used for initialization are not scale invariant.

We compute the following estimates of the common factors:

- principal components: $\hat{\mathbf{f}}_{pc,t}$;
- two-steps estimates: $\hat{\mathbf{f}}_{\hat{\theta}^{(0)},t}$
- maximum likelihood estimates: $\hat{\mathbf{f}}_{\theta^{(M)},t} := \hat{\mathbf{f}}_{\hat{\theta},t}$

We measure the performance of the different estimators by means of the following trace statistics:

[3]This requires the computation of $\mathrm{E}_{\theta^{(m)}}(\hat{\mathbf{f}}_{\theta^{(m)},t} - \mathbf{f}_t)(\hat{\mathbf{f}}_{\theta^{(m)},t-k} - \mathbf{f}_{t-k})', k = 0, ..., p$, which are also computed by the Kalman smoother. See for example Engle and Watson (1981).

[4]A detailed derivation of the EM algorithm for dynamic factor model is provided by Ghahramani and Hinton (1996)

$$\frac{\text{Tr}\left(\mathbf{F}'\hat{\mathbf{F}}(\hat{\mathbf{F}}'\hat{\mathbf{F}})^{-1}\hat{\mathbf{F}}'\mathbf{F}\right)}{\text{Tr}\left(\mathbf{F}'\mathbf{F}\right)}$$

where $\hat{\mathbf{F}} = \left(\hat{\mathbf{f}}_1, ..., \hat{\mathbf{f}}_T\right)'$, and $\hat{\mathbf{f}}_t$ is any of the three estimates of the common factors. This statistics is a multivariate version of the R^2 of the regression of the observed factors on the estimated factors, and the reason why we use it is that the common factors are identified only up to a rotation. This statistics is also closely related to the empirical canonical correlation between the true factors and their estimates. A number close to one indicates a good approximation of the true common factors. Denoting by TR_{pc}, TR_{2s} TR_{ml} the trace statistics for principal component, two steps and maximum likelihood estimates of the common factors, we compute the relative trace statistics TR_{ml}/TR_{pc} and TR_{ml}/TR_{2s}. Numbers higher than one indicates that Maximum Likelihood estimates of the common factors are more accurate than principal components and two-steps estimates.

Table 1: Simulation results for the model: $\rho = .9$, $d = .5$, $\tau = .5$, $u = .1$, $r = 1$

			TR_{ml}		
	$n = 5$	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	0.52	0.68	0.74	0.75	0.76
$T = 100$	0.64	0.78	0.84	0.85	0.86
			Number of iterations		
	$n = 5$	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	13	9	5	4	3
$T = 100$	13	7	4	4	3
			Computation time: seconds		
	$n = 5$	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	0.53	0.25	0.20	0.33	1.07
$T = 100$	0.66	0.37	0.33	0.61	2.13
			TR_{ml}/TR_{pc}		
	$n = 5$	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	1.11	1.04	1.00	1.00	1.00
$T = 100$	1.09	1.02	1.01	1.00	1.00
			TR_{ml}/TR_{2s}		
	$n = 5$	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	1.03	1.01	1.00	1.00	1.00
$T = 100$	1.02	1.00	1.00	1.00	1.00

Table 1 reports the results of the Montecarlo experiment for one common factor, $r = 1$, with serial correlation in both common factors, $\rho = .9$, and idiosyncratic components, $d = .5$. The model is approximated because of the weak cross-sectional correlation among idiosyncratic components, $\tau = .5$. Finally the idiosyncratic component is cross-sectionally heteroscedastic, $u = .1$. The numbers in the table refer to the average across experiments. We would like to stress the following results:

1. The precision of the common factors estimated by Maximum Likelihood increases with the size of the cross-section n.

2. The number of iterations required for convergence is small and decreases with the size of the cross-section. As remarked above this is explained by the fact that, as n increases, the initialization provided by principal components are increasingly accurate and hence the computation time for convergence does not increase too much with the cross-sectional dimension.

3. The Maximum Likelihood estimates always dominate simple principal components and to a less extend the two-step procedure. As both n, T become large, the precision of the estimated common factors increases and all methods tend to perform similarly. This is not surprising given that both methods provide consistent estimates for n and T large. Improvement of the ML estimates are significant for $n = 5$ and the improvement is of the order of 10% with respect to principal components and less than 5% for the two-step estimates. This suggests that the two step kalman smoother estimates already take appropriately into account the dynamics of the common factors and the cross-sectional heteroscedasticity of the idiosyncratic component and hence the gains from further iterations are small.

Table 2 reports the results for $r = 3$ while the remaining parameters are the same as those used the Table 1: $\rho = .9$, $d = .5$, $\tau = .5$, $u = .1$. The simulations have been run for $n \geq 10$ only, because an exact factor model with $n = 5$ and $r = 3$ would not be identifiable. Notice that, although the main features outlined above are still present, as expected, the estimates of the common factors are less precise with respect to the case of only one common factors (given the same a set of data, it is more difficult to extract additional factors). Improvements by the maximum likelihood are more sizable in this case which just indicates that efficiency improvements are larger, the harder is the factor extraction.
We finally study a case in which our approximating model is well specified, that is the idiosyncratic components is neither serially nor cross-sectionally correlated $d = 0, \tau = 0$. The remaining parameters are set as for the experiments reported in Table 1 and 2. In this case, as one can see from table 3 below, the efficiency gains of ML estimates over the principal components and two-steps estimates are more relevant.

Summarizing, QML estimates of approximate factor models work well in finite sample. Because of the explicit modelling of the dynamics and the cross-sectional heteroscedasticity, the maximum likelihood estimates dominate the principal components and, to a less extent, the two two-step procedure. Efficiency improvements are relevant when the factor extraction is difficult, that is, when there are more common factors to estimate.

Table 2: Simulation results for the model: $\rho = .9$, $d = .5$, $\tau = .5$, $u = .1$, $r = 3$

	TR_{ml}			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	0.48	0.59	0.65	0.67
$T = 100$	0.58	0.75	0.80	0.82
	Number of iterations			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	26	12	7	5
$T = 100$	20	9	5	4
	Computation time: seconds			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	0.72	0.46	0.56	1.44
$T = 100$	1.08	0.68	0.87	2.31
	TR_{ml}/TR_{pc}			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	1.08	1.05	1.03	1.01
$T = 100$	1.10	1.06	1.02	1.01
	TR_{ml}/TR_{2s}			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	1.05	1.02	1.01	1.00
$T = 100$	1.07	1.03	1.00	1.00

Table 3: Simulation results for the model: $\rho = .9$, $d = 0$, $\tau = 0$, $u = .1$, $r = 3$

	TR_{ml}			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	0.54	0.65	0.68	0.70
$T = 100$	0.66	0.78	0.81	0.82
	Number of iterations			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	21	9	6	5
$T = 100$	15	7	5	4
	Computation time: seconds			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	0.58	0.36	0.49	1.30
$T = 100$	0.83	0.54	0.84	2.29
	TR_{ml}/TR_{pc}			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	1.14	1.06	1.03	1.01
$T = 100$	1.19	1.06	1.02	1.01
	TR_{ml}/TR_{2s}			
	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$T = 50$	1.07	1.02	1.01	1.00
$T = 100$	1.10	1.01	1.00	1.00

5 Summary and conclusions

The paper has studied quasi maximum likelihood estimation of the factors for an approximate factor model. Consistency under different source of miss-specification is shown for n and T going to infinity. The results of this paper show that the effects of misspecification of the approximating model goes to zero asymptotically.

The estimator is then a valid parametric alternative to principal components which can potentially produce efficiency improvements due to the exploitation of the factor dynamics and the non sphericity of the idiosyncratic components. The estimator is feasible when n is large and easily implementable using the Kalman smoother and the EM algorithm as in traditional factor analysis.

Simulation results illustrate in what empirical conditions we can expect improvement with respect to simple principle components.

There are three desirable characteristics of the parametric approach.

First, as mentioned, it may produce efficiency improvements.

Second, it provides a natural framework for structural analysis since it allows to impose restrictions on the loadings (as done, for example, in Bernanke, Boivin, and Eliasz (2005); Boivin and Giannoni (2005); Kose, Otrok, and Whiteman (2003); Forni and Reichlin (2001)) and to extract shocks. These features are not studied in this paper but they are natural extensions to explore in further work.

Finally, once we have a parametric model estimated by likelihood based methods, it is possible to handle missing data and enlarge the range of interesting empirical applications for large factor models. Missing data at the end of the sample due to unsynchronized data releases, is a typical problem for real time estimation of macro variables (see Giannone, Reichlin, and Sala, 2004; Giannone, Reichlin, and Small, 2005 for applications based on parametric factor models).

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6 Appendix

We adopt the following notations to define the pseudo likelihood under the approximating model which is completely characterized by the parameter θ:

- $f_{(\mathbf{X},\mathbf{F})}(X, F; \theta)$ is the joint density of the common factors and the observables, depending on the parameter θ,
- $f_{\mathbf{X}}(X; \theta)$ and $f_{\mathbf{F}}(F; \theta)$ are the corresponding marginal densities,
- $f_{\mathbf{X}|\mathbf{F}=F}(X; \theta)$ and $f_{\mathbf{F}|\mathbf{X}=X}(F; \theta)$ are the corresponding conditional densities

where $F \in \mathbb{R}^{(T\times r)}$ and $X \in \mathbb{R}^{(T\times n)}$. We know that, for any (X, F):

$$\begin{aligned} f_{(\mathbf{X},\mathbf{F})}(X, F; \theta) &= f_{\mathbf{X}|\mathbf{F}=F}(X; \theta) f_{\mathbf{F}}(F; \theta) \\ &= f_{\mathbf{F}|\mathbf{X}=X}(F; \theta) f_{\mathbf{X}}(X; \theta) \end{aligned}$$

so that:

$$f_{\mathbf{X}}(X; \theta) = \frac{f_{\mathbf{X}|\mathbf{F}=F}(X; \theta) f_{\mathbf{F}}(F; \theta)}{f_{\mathbf{F}|\mathbf{X}=X}(F; \theta)}.$$

The log-likelihood of the data $\mathcal{L}_{\mathbf{X}}(X; \theta) = \log f_{\mathbf{X}}(X; \theta)$ can then be decomposed in the following way:

$$\mathcal{L}_{\mathbf{X}}(X; \theta) = \mathcal{L}_{\mathbf{X}|\mathbf{F}}(X|F; \theta) + \mathcal{L}_{\mathbf{F}}(F; \theta) - \mathcal{L}_{\mathbf{F}|\mathbf{X}}(F|X; \theta)$$

where $\mathcal{L}_{\mathbf{X}|\mathbf{F}}(X|F; \theta) = \log f_{\mathbf{X}|\mathbf{F}=F}(X; \theta)$, $\mathcal{L}_{\mathbf{F}}(F; \theta) = \log f_{\mathbf{F}}(F; \theta)$ and $\mathcal{L}_{\mathbf{F}|\mathbf{X}}(F|X; \theta) = \log f_{\mathbf{F}|\mathbf{X}=X}(F; \theta)$.

Under our gaussian restriction, and denoting by $\mathbf{X}$ the actual observed values of the underlying process, we can write, for any value of F:

$$\mathcal{L}_{\mathbf{X}|\mathbf{F}}(\mathbf{X}|F; \theta) = -\tfrac{nT}{2}\log(2\pi) - \tfrac{T}{2}\log|\Psi_d| - \tfrac{1}{2}\mathrm{Tr}(\mathbf{X} - F\Lambda')\Psi_d^{-1}(\mathbf{X} - F\Lambda')'$$

$$\mathcal{L}_{\mathbf{F}}(F; \theta) = -\tfrac{rT}{2}\log(2\pi) - \tfrac{1}{2}\log|\mathbf{\Phi}_\theta| - \tfrac{1}{2}(\mathrm{vec}F')'\mathbf{\Phi}_\theta^{-1}(\mathrm{vec}F')$$

$$\mathcal{L}_{\mathbf{F}|\mathbf{X}}(F|\mathbf{X}; \theta) = -\tfrac{rT}{2}\log(2\pi) - \tfrac{1}{2}\log|\mathbf{\Omega}_\theta| - \tfrac{1}{2}(\mathrm{vec}(F - \hat{\mathbf{F}}_\theta)')'\mathbf{\Omega}_\theta^{-1}(\mathrm{vec}(F - \hat{\mathbf{F}}_\theta)')$$

with
$\mathbf{\Phi}_\theta = \mathrm{E}_\theta\left[(\mathrm{vec}\mathbf{F}')(\mathrm{vec}\mathbf{F}')'\right]$,
$\hat{\mathbf{F}}_\theta = \mathrm{E}_\theta\left[\mathbf{F}|\mathbf{X}\right] = (\hat{\mathbf{f}}_{\theta,1}, ..., \hat{\mathbf{f}}_{\theta,T})'$
and
$\mathbf{\Omega}_\theta = \mathrm{E}_\theta\left[(\mathrm{vec}(\mathbf{F} - \hat{\mathbf{F}}_\theta)')(\mathrm{vec}(\mathbf{F} - \hat{\mathbf{F}}_\theta)')'\right]$.

We hence have, for any value of F:

$$\begin{aligned}\mathcal{L}_{\mathbf{X}}(\mathbf{X};\theta) = &-\tfrac{nT}{2}\log(2\pi) - \tfrac{T}{2}\log|\Psi_d| - \tfrac{1}{2}\mathrm{Tr}(\mathbf{X}-F\Lambda')\Psi_d^{-1}(\mathbf{X}-F\Lambda')' \\ &-\tfrac{1}{2}(\mathrm{vec}F')'\mathbf{\Phi}_\theta^{-1}(\mathrm{vec}F') - \tfrac{1}{2}\log|\mathbf{\Omega}_\theta| + \tfrac{1}{2}(\mathrm{vec}(F-\hat{\mathbf{F}}_\theta)')'\mathbf{\Omega}_\theta^{-1}(\mathrm{vec}(F-\hat{\mathbf{F}}_\theta)')\end{aligned} \tag{6.2}$$

If we consider the likelihood computed by using $F = \hat{\mathbf{F}}_\theta$, (6.2) the above expression becomes:

$$\begin{aligned}\mathcal{L}(\mathbf{X};\theta) = \quad &-\tfrac{nT}{2}\log(2\pi) - \tfrac{T}{2}\log|\Psi_d| - \tfrac{1}{2}\mathrm{vec}(\hat{\mathbf{F}}_\theta')'\mathbf{\Phi}_\theta^{-1}\mathrm{vec}(\hat{\mathbf{F}}_\theta') - \tfrac{1}{2}\log|\mathbf{\Omega}_\theta| \\ &-\tfrac{1}{2}\mathrm{Tr}(\mathbf{F}\Lambda_0' - \hat{\mathbf{F}}_\theta\Lambda' + \mathbf{E})\Psi_d^{-1}(\mathbf{F}\Lambda_0' - \hat{\mathbf{F}}_\theta\Lambda' + \mathbf{E})'\end{aligned} \tag{6.3}$$

Let us now evaluate the likelihood at the following set of parameters:

$$\theta_0^c := \{A(L) = I_r; H = I_r; \Lambda = \Lambda_0; \Psi = \Psi_{0,d}\}$$

where $\Psi_{0,d}$ is the diagonal matrix obtained by setting equal to zero all the out of diagonal elements of Ψ_0.

For $\theta = \theta_0^c$, we have $\mathbf{\Phi}_{\theta_0^c} = I_{rT}$ and $\mathbf{\Omega}_{\theta_0^c} = I_T \otimes \left(I_r - \Lambda_0'\left(\Lambda_0\Lambda_0' + \Psi_{0,d}\right)^{-1}\Lambda_0\right)$.

It can be easily checked that

$$\left(\Lambda_0\Lambda_0' + \Psi_{0,d}\right)^{-1} = \Psi_{0,d}^{-1} - \Psi_{0,d}^{-1}\Lambda_0\left(I_r + \Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\right)^{-1}\Lambda_0'\Psi_{0,d}^{-1} \tag{6.4}$$

so that: $\mathbf{\Omega}_{\theta_0^c} = I_T \otimes \left(I_r + \Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\right)^{-1}$.

We then have:

$$\begin{aligned}\mathcal{L}(\mathbf{X};\theta_0^c) = \quad &-\tfrac{nT}{2}\log(2\pi) - \tfrac{T}{2}\log|\Psi_{0,d}| - \tfrac{1}{2}\mathrm{Tr}\hat{\mathbf{F}}'_{\theta_0^c}\hat{\mathbf{F}}_{\theta_0^c} - \tfrac{T}{2}\log\left|I_r + \Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\right| \\ &-\tfrac{1}{2}\mathrm{Tr}\left((\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c})\Lambda_0' + \mathbf{E}\right)\Psi_{0,d}^{-1}\left((\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c})\Lambda_0' + \mathbf{E}\right)'.\end{aligned} \tag{6.5}$$

As n and T go to infinity (6.5) simplifies drastically since some of the terms are asymptotically negligible. This is shown as a corollary of the following Lemma.

Lemma 1 Under assumptions A, B, we have

1. $\left\|\frac{\mathbf{E}'\mathbf{E}}{nT}\right\| = O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$ as $n, T \to \infty$
2. $\frac{1}{T}\mathrm{Tr}(\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c})'(\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c}) = O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$ as $n, T \to \infty$
3. $\frac{1}{nT}\mathrm{Tr}(\mathbf{E}'\Psi_{0,d}^{-1}\mathbf{E}) = 1 + O_p\left(\frac{1}{\sqrt{T}}\right)$ as $n, T \to \infty$
4. $\frac{1}{nT}\mathrm{Tr}\hat{\mathbf{F}}'_{\theta_0^c}\hat{\mathbf{F}}_{\theta_0^c} = O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$ as $n, T \to \infty$
5. $\frac{1}{n}\log\left|I_r + \Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\right| = \left(\frac{\log(n)}{n}\right)$ as $n \to \infty$

Proof

We have:

$$\left\|\frac{\mathbf{E}'\mathbf{E}}{nT}\right\| \leq \frac{1}{n}\|\Psi_0\| + \frac{1}{n}\left\|\frac{\mathbf{E}'\mathbf{E}}{T} - \Psi_0\right\|$$

$$\left\|\frac{1}{n}\left(\frac{\mathbf{E}'\mathbf{E}}{T} - \Psi_0\right)\right\|^2 \leq \frac{1}{n^2}\mathrm{trace}\left[\left(\frac{\mathbf{E}'\mathbf{E}}{T} - \Psi_0\right)'\left(\frac{\mathbf{E}'\mathbf{E}}{T} - \Psi_0\right)\right] = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\left(\frac{1}{T}\sum_{t=1}^T e_{it}e_{jt} - \psi_{0,ij}\right)^2$$

Taking expectations, from assumption B we obtain:

$$\frac{1}{n^2}\mathrm{E}\left[\sum_{i=1}^n\sum_{j=1}^n\left(\frac{1}{T}\sum_{t=1}^T e_{it}e_{jt} - \psi_{0,ij}\right)^2\right] = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\mathrm{E}\left[\left(\frac{1}{T}\sum_{t=1}^T e_{it}e_{jt} - \psi_{0,ij}\right)^2\right] \leq \frac{M}{T}$$

Result 1 hence follows from the Markov inequality.

Let us turn now to result 2. First, we have: $\mathrm{Tr}(\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c})'(\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c}) \leq r\|\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c}\|^2$.
Then, using (6.4), we have:

$$\hat{\mathbf{F}_{\theta_0^c}} = \mathbf{X}\Psi_{0,d}^{-1}\Lambda_0(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0 + I_r)^{-1} = \mathbf{F}\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0 + I_r)^{-1} + \mathbf{E}\Psi_{0,d}^{-1}\Lambda_0(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0 + I_r)^{-1}$$

so that:

$$\frac{1}{\sqrt{T}}\|\mathbf{F} - \hat{\mathbf{F}}_{\theta_0^c}\| \leq \left\|\frac{1}{\sqrt{T}}\mathbf{F}\right\|\|\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0 + I_r)^{-1} - I_r\| + \left\|\frac{1}{\sqrt{nT}}\mathbf{E}\right\|\|\sqrt{n}\Lambda_0\Psi_{0,d}^{-1}(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0 + I_r)^{-1}\|$$

Assumptions A implies:

$$\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0 + I_r)^{-1} - I_r = (\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0 + I_r)^{-1} = O\left(\frac{1}{n}\right) \text{ as } n \to \infty$$

Further, we have: $\|\Lambda_0\Psi_{0,d}^{-1}(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0+I_r)^{-1}\| \leq \|\Lambda_0'\Psi_{0,d}^{-1/2}\|\|\Psi_{0,d}^{-1/2}\|\|(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0+I_r)^{-1}\|$.

As assumptions A also imply: $\|\Psi_{0,d}^{-1/2}\| \leq \frac{1}{\sqrt{\lambda_{min}(\Psi_0)}} = O(1)$ as $n \to \infty$

and: $\|\Lambda_0'\Psi_{0,d}^{-1/2}\| = \|\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\|^{1/2} \leq \frac{1}{\lambda_{min}(\Psi_0)}\|\Lambda_0'\Lambda_0\|^{1/2} = O(\sqrt{n})$ as $n \to \infty$,

result 2 then follows from the previous result of this lemma and the fact that by assumption B we have $\left\|\frac{1}{\sqrt{T}}\mathbf{F}\right\| = O_p(1)$.

Result 3 is a direct consequence of Assumption B (i) and the Markov inequality. In fact:

$$\frac{1}{nT}\mathrm{Tr}\left(\mathbf{E}\Psi_{0,d}^{-1}\mathbf{E}'\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\frac{1}{T}\sum_{t=1}^{T}e_{it}^2}{\psi_{0,ii}}\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{\psi_{0,ii}}{\hat{\psi}_{0,ii}} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

To obtain result 4, notice that:

$$\frac{1}{nT}\mathrm{Tr}\hat{\mathbf{F}}_{\theta_0^c}'\hat{\mathbf{F}}_{\theta_0^c} \leq \frac{r}{nT}\left\|\hat{\mathbf{F}}_{\theta_0^c}\right\|^2 = \frac{r}{nT}\left\|\mathbf{F}+\hat{\mathbf{F}}_{\theta_0^c}-\mathbf{F}\right\|^2 \leq \frac{2r}{n}\left(\|\frac{1}{\sqrt{T}}\mathbf{F}\|^2 + \frac{1}{T}\|\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}\|^2\right)$$

As $\|\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}\|^2 \leq \mathrm{Tr}\left(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}\right)'\left(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}\right)$, the desired rate follows from Assumption B (iii) and result 2.

Concerning result 5, notice that, by assumptions A:

$\log\left|I_r + \Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\right| = \log(n) + \log\left|\frac{I_r}{n} + \frac{\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0}{}\right|$, with:

$\log\left|\frac{I_r}{n} + \frac{\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0}{n}\right| \simeq \log\left|\frac{\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0}{n}\right| \leq r\log\frac{\lambda_{max}\left(\frac{\Lambda_0'\Lambda_0}{n}\right)}{\lambda_{min}(\Psi_0)} = O(1)$ as $n \to \infty$. Q.E.D.

Corollary Under the same assumptions of Lemma 1, we have:

$$\frac{1}{nT}\mathcal{L}(\mathbf{X};\theta_0^c) = -\frac{1}{2n}\log(2\pi) - \frac{1}{2}\log|\Psi_{0,d}| - \frac{1}{2} + O_p\left(\frac{\log(n)}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right), \text{ as } n,T \to \infty$$

Proof

The only term for which the asymptotic behavior is not a direct consequence of Lemma 1 is the following:

$\frac{1}{nT}\mathrm{Tr}\left((\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c})\Lambda_0'+\mathbf{E}\right)\Psi_{0,d}^{-1}\left((\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c})\Lambda_0'+\mathbf{E}\right)'$

$= \frac{1}{nT}\mathrm{Tr}\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c})'(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}) - 2\frac{1}{nT}\mathrm{Tr}\Lambda_0'\Psi_{0,d}^{-1}\mathbf{E}'(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}) + \frac{1}{nT}\mathrm{Tr}\Psi_{0,d}^{-1}\mathbf{E}'\mathbf{E}$

Let us analyze the three terms in the summation separately.

The asymptotic behavior of the third term in the summation is a direct consequence on Lemma 1 (3).

The asymptotic behavior of the first term follows from Assumption A and Lemma 1 (2):

$$\frac{1}{nT}\mathrm{Tr}\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c})'(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}) \leq \frac{1}{nT}\lambda_{max}\left(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\right)\mathrm{Tr}(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c})'(\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c})$$

We know (see the proof of lemma 1) that $\frac{1}{n}\lambda_{max}\left(\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\right) = \frac{1}{n}\|\Lambda_0'\Psi_{0,d}^{-1}\Lambda_0\| = O(1)$ so that the result directly follows from lemma 1 (2).

For the second term:

$$\begin{aligned}\tfrac{1}{nT}\mathrm{Tr}\Lambda_0'\Psi_{0,d}^{-1}\mathbf{E}'(\mathbf{F}-\hat{\mathbf{F}}_\theta) &\leq r\left\|\tfrac{\mathbf{E}'\mathbf{E}}{nT}\right\|^{1/2}\left\|\tfrac{\Lambda_0'\Lambda_0}{n}\right\|\tfrac{1}{\left(\lambda_{min}\Psi_{0,d}\right)^2}\tfrac{1}{\sqrt{T}}\|\mathbf{F}-\hat{\mathbf{F}}_{\theta_0^c}\| \\ &= O_p\left(\tfrac{1}{n}\right)+O_p\left(\tfrac{1}{\sqrt{T}}\right)\end{aligned}$$

where the last equality follows for Lemma 1 (1-2) and Assumptions A and B.

This drastic simplification is due to the fact that under the simple approximating model the expected common factor converge to the true ones (Lemma 1 (i)). The expected values of the common factors, $\hat{\mathbf{F}}_{\theta_0^c}$, are essentially the coefficients of an OLS regression of the observation, $\mathbf{X}$, on the factor loadings, Λ_0. If data are gaussian and the restrictions in θ_0^c are satisfied, then such estimates of the common factors are the most efficient. However, the estimates are still consistent under the weaker assumptions A (i) and A (ii). This result also tells us that a large cross-section solves the common factors indeterminacy we have with a finite cross-section dimension.

Consider now the likelihood evaluated at its maximum where $\hat{\theta} := \left\{\hat{A}(L); \hat{H}; \hat{\Lambda}; \hat{\Psi}_d\right\}$ are the Maximum Likelihood estimates of the parameters, with $\hat{\theta} \in \Theta^c$. We will denote by $\hat{\mathbf{F}}_{\hat{\theta}}$ the corresponding estimates of the common factors.

The likelihood at its maximum takes the form (see equation(6.2)):

$$\begin{aligned}\mathcal{L}(\mathbf{X};\hat{\theta}) = \; & -\tfrac{nT}{2}\log(2\pi) - \tfrac{T}{2}\log|\hat{\Psi}_d| - \tfrac{1}{2}\mathrm{Tr}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')\hat{\Psi}_d^{-1}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')' \\ & -\tfrac{1}{2}\mathrm{vec}(\hat{\mathbf{F}}_{\hat{\theta}}')'\mathbf{\Phi}_{\hat{\theta}}^{-1}\mathrm{vec}(\hat{\mathbf{F}}_{\hat{\theta}}') - \tfrac{1}{2}\log|\mathbf{\Omega}_{\hat{\theta}}|\end{aligned}$$

Assumption C below insures that the constraints on the size of the idiosyncratic variance that is imposed in the maximization is not binding, that is $\theta_0^c \in \Theta^c$. Consequently, $\mathcal{L}(\mathbf{X};\hat{\theta}) \geq \mathcal{L}(\mathbf{X};\theta_0)$. Using the Corollary, this implies:

$$\begin{aligned} 0 \geq \tfrac{2}{nT}\left(\mathcal{L}(\mathbf{X};\theta_0^c) - \mathcal{L}(\mathbf{X};\hat{\theta})\right) &= \tfrac{1}{n}\log|\hat{\Psi}_d| + \tfrac{1}{nT}\mathrm{Tr}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')\hat{\Psi}_d^{-1}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')' \\ &+ \tfrac{1}{nT}\mathrm{vec}(\hat{\mathbf{F}}'_{\hat{\theta}})'\hat{\mathbf{\Phi}}_{\hat{\theta}}^{-1}\mathrm{vec}(\hat{\mathbf{F}}'_{\hat{\theta}}) + \tfrac{1}{nT}\log|\hat{\mathbf{\Omega}}_{\hat{\theta}}| \\ &- \tfrac{1}{n}\log|\Psi_{0,d}| - 1 + O_p\left(\tfrac{1}{\sqrt{T}}\right) + O_p\left(\tfrac{\log(n)}{n}\right) \end{aligned}$$

Lemma 2 Under assumptions A, B, and C, we have:

$$\begin{aligned} \tfrac{1}{nT}\mathrm{Tr}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')\hat{\Psi}_d^{-1}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')' \geq\ & \tfrac{1}{nT}\mathrm{Tr}(\Lambda_0'\hat{\Psi}_d^{-1}\Lambda_0)'(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H}) \\ & -2\sqrt{\tfrac{1}{T}\mathrm{Tr}((\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H}))}\sqrt{O_p\left(\tfrac{1}{\sqrt{T}}\right)+O_p\left(\tfrac{1}{n}\right)} \\ & + \tfrac{1}{n}\textstyle\sum_{i=1}^n \tfrac{\psi_{0,ii}}{\hat{\psi}_{ii}} + O_p\left(\tfrac{1}{\sqrt{T}}\right) + O_p\left(\tfrac{1}{n}\right) \end{aligned}$$

where $\hat{H} = \left(\hat{\mathbf{F}}'_{\hat{\theta}}\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}\hat{\mathbf{F}}'_{\hat{\theta}}\mathbf{F}$ is the coefficient of the OLS projection of $\mathbf{F}$ on $\hat{\mathbf{F}}_{\hat{\theta}}$

Proof Consider the coefficients of the OLS projection of $\mathbf{X}$ on $\hat{\mathbf{F}}_{\hat{\theta}}$:

$$\hat{\hat{\Lambda}} = \mathbf{X}'\hat{\mathbf{F}}_{\hat{\theta}}\left(\hat{\mathbf{F}}'_{\hat{\theta}}\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}$$

Least squares properties imply that:

$$\tfrac{1}{nT}\mathrm{Tr}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')\hat{\Psi}_d^{-1}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\Lambda}')' \geq \tfrac{1}{nT}\mathrm{Tr}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\hat{\Lambda}}')\hat{\Psi}_d^{-1}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\hat{\Lambda}}')'$$

Notice that:

$$\begin{aligned} (\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\hat{\Lambda}}') &= \left(\mathbf{F}\Lambda_0' + \mathbf{E} - \hat{\mathbf{F}}_{\hat{\theta}}\left(\hat{\mathbf{F}}'_{\hat{\theta}}\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}\hat{\mathbf{F}}'_{\hat{\theta}}\mathbf{F}\mathbf{\Lambda_0'} - \hat{\mathbf{F}}_{\hat{\theta}}\left(\hat{\mathbf{F}}'_{\hat{\theta}}\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}\hat{\mathbf{F}}'_{\hat{\theta}}\mathbf{E}\right) \\ &= (\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})\Lambda_0' + (I_T - P_{\hat{F}_{\hat{\theta}}})\mathbf{E} \end{aligned}$$

where $\hat{H} = \left(\hat{\mathbf{F}}'_{\hat{\theta}}\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}\hat{\mathbf{F}}'_{\hat{\theta}}\mathbf{F}$ is the coefficient of the OLS projection of $\mathbf{F}$ on $\hat{\mathbf{F}}_{\hat{\theta}}$ and $P_{\hat{F}_{\hat{\theta}}} = \hat{\mathbf{F}}_{\hat{\theta}}\left(\hat{\mathbf{F}}'_{\hat{\theta}}\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}\hat{\mathbf{F}}'_{\hat{\theta}}$ is the projection matrix associated with $\hat{\mathbf{F}}_{\hat{\theta}}$.

Consequently:

$$\begin{aligned}\tfrac{1}{nT}\mathrm{Tr}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\hat{\Lambda}}')\hat{\Psi}_d^{-1}(\mathbf{X}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{\hat{\Lambda}}')' &= \tfrac{1}{nT}\mathrm{Tr}(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})\Lambda_0'\hat{\Psi}_d^{-1}\Lambda_0(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})' \\ &+\tfrac{1}{nT}\mathrm{Tr}(I_T-P_{\hat{F}_{\hat{\theta}}})\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'(I_T-P_{\hat{F}_{\hat{\theta}}}) \\ &+2\tfrac{1}{nT}\mathrm{Tr}(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})\Lambda_0'\hat{\Psi}_d^{-1}\mathbf{E}'(I_T-P_{\hat{F}_{\hat{\theta}}})\end{aligned}$$

We have:

$$\begin{aligned}\tfrac{1}{nT}\mathrm{Tr}(I_T-P_{\hat{F}_{\hat{\theta}}})\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'(I_T-P_{\hat{F}_{\hat{\theta}}}) &= \tfrac{1}{nT}\mathrm{Tr}\left(\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'(I_T-P_{\hat{F}_{\hat{\theta}}})\right) \\ &= \tfrac{1}{nT}\mathrm{Tr}\left(\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'\right)-\tfrac{1}{nT}\mathrm{Tr}\left(\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'P_{\hat{F}_{\hat{\theta}}}\right)\end{aligned}$$

By assumption B (ii):

$$\frac{1}{nT}\mathrm{Tr}\left(\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\frac{1}{T}\sum_{t=1}^{T}e_{it}^2}{\hat{\psi}_{ii}}\right)=\frac{1}{n}\sum_{i=1}^{n}\frac{\psi_{0,ii}}{\hat{\psi}_{ii}}+O_p\left(\frac{1}{\sqrt{T}}\right)$$

Furthermore:

$$\frac{1}{nT}\mathrm{Tr}\left(\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'P_{\hat{F}_{\hat{\theta}}}\right)=\frac{1}{nT}\mathrm{Tr}\left(\hat{\mathbf{F}}_{\hat{\theta}}'\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'\hat{\mathbf{F}}_{\hat{\theta}}\left(\hat{\mathbf{F}}_{\hat{\theta}}'\hat{\mathbf{F}}_{\hat{\theta}}\right)^{-1}\right)\leq r\frac{1}{nT}\lambda_{max}\left(\mathbf{E}\hat{\Psi}_d^{-1}\mathbf{E}'\right)=O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{n}\right)$$

Finally,

$$\begin{aligned}\tfrac{1}{nT}\left|\mathrm{Tr}(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})\Lambda_0'\hat{\Psi}_d^{-1}\mathbf{E}'(I_T-P_{\hat{F}_{\hat{\theta}}})\right| &\leq \sqrt{\tfrac{1}{T}\mathrm{Tr}(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})}\sqrt{\tfrac{1}{n^2T}\mathrm{Tr}(\Lambda_0'\hat{\Psi}_d^{-1}\mathbf{E}'\mathbf{E}\hat{\Psi}_d^{-1}\Lambda_0)} \\ &= \sqrt{\tfrac{1}{T}\mathrm{Tr}(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F}-\hat{\mathbf{F}}_{\hat{\theta}}\hat{H})}\sqrt{O_p\left(\tfrac{1}{\sqrt{T}}\right)+O_p\left(\tfrac{1}{n}\right)}\end{aligned}$$

The desired result follows. Q.E.D.

To prepare the proof of Proposition 1, notice first that $\mathrm{vec}(\hat{\mathbf{F}}_{\hat{\theta}}')'\mathbf{\Phi}_{\hat{\theta}}^{-1}\mathrm{vec}(\hat{\mathbf{F}}_{\hat{\theta}}')\geq 0$.

Moreover, it can be shown that: $\log|\mathbf{\Omega}_{\hat{\theta}}|>0$.

Indeed, if we denote $\mathbf{\Sigma}_\theta=\mathrm{E}_\theta\left[(\mathrm{vec}\mathbf{X}')(\mathrm{vec}\mathbf{X}')'\right]$, we have:

$$\mathbf{\Sigma}_\theta=(I_T\otimes\Lambda)\,\mathbf{\Phi}_\theta\,(I_T\otimes\Lambda)'+(I_T\otimes\Psi_d)$$

It can the be checked that

$$\Sigma_\theta^{-1} = \left(I_T \otimes \Psi_d^{-1}\right) - \left(I_T \otimes \Psi_d^{-1}\Lambda\right)\left(\Phi_\theta^{-1} + I_T \otimes \Lambda'\Psi_d^{-1}\Lambda\right)^{-1}\left(I_T \otimes \Lambda'\Psi_d^{-1}\right)$$

and that $\Omega_\theta = I_{rT} + \left(I_T \otimes \Lambda'\Psi_d^{-1}\Lambda\right)\Phi_\theta$.

It then follows that $\Omega_\theta > I_{rT}$, so that $\log|\mathbf{\Omega}_{\hat{\theta}}| > 0$. This property holds for all $A(L)$ and H satisfying R1.

Finally:

$$\frac{1}{n}\log|\hat{\Psi}_d| + \frac{1}{n}\sum_{i=1}^{n}\frac{\psi_{0,ii}}{\hat{\psi}_{ii}} - \frac{1}{n}\log|\hat{\Psi}_{0d}| - 1 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\psi_{0i}}{\hat{\psi}_i} - \log\left(\frac{\psi_{0i}}{\hat{\psi}_i}\right) - 1\right) \geq 0$$

Using the fact that $\frac{n}{\log(n)} = O(n)$, we then obtain:

$$\begin{aligned}
0 &\geq \tfrac{2}{nT}\left(\mathcal{L}(\mathbf{X};\theta_0^c) - \mathcal{L}(\mathbf{X};\hat{\theta})\right) \\
&\geq \tfrac{1}{nT}\mathrm{Tr}(\Lambda_0'\hat{\Psi}_d^{-1}\Lambda_0)(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H}) \\
&\quad -2\sqrt{\tfrac{1}{T}\mathrm{Tr}(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})}O_p\left(\sqrt{\tfrac{1}{\Delta_{nT}}}\right) + O_p\left(\tfrac{1}{\Delta_{nT}}\right)
\end{aligned}$$

where

$$\Delta_{nT} = \min\left\{\sqrt{T}, \frac{n}{\log(n)}\right\}$$

We can now prove our main result.

Proof of Proposition 1

$$\begin{aligned}
0 &\geq \tfrac{1}{nT}\mathrm{Tr}(\Lambda_0'\hat{\Psi}_d^{-1}\Lambda_0)(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H}) \\
&\quad -2\sqrt{\tfrac{1}{T}\mathrm{Tr}(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})}O_p\left(\sqrt{\tfrac{1}{\Delta_{nT}}}\right) + O_p\left(\tfrac{1}{\Delta_{nT}}\right) \\
&\geq \lambda_{min}\left(\frac{\Lambda_0'\hat{\Psi}_d^{-1}\Lambda_0}{n}\right)\tfrac{1}{T}\mathrm{Tr}(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H}) \\
&\quad -2O_p\left(\sqrt{\tfrac{1}{\Delta_{nT}}}\right)\sqrt{\tfrac{1}{T}\mathrm{Tr}(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})} + O_p\left(\tfrac{1}{\Delta_{nT}}\right) \\
&= \lambda_{min}\left(\frac{\Lambda_0'\hat{\Psi}_d^{-1}\Lambda_0}{n}\right)V_{nT} - 2\sqrt{V_{nT}}O_p\left(\sqrt{\tfrac{1}{\Delta_{nT}}}\right) + O_p\left(\tfrac{1}{\Delta_{nT}}\right)
\end{aligned}$$

where $V_{nT} = \frac{1}{T}\mathrm{Tr}(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})'(\mathbf{F} - \hat{\mathbf{F}}_{\hat{\theta}}\hat{H})$.

Since $\liminf_{n,T\to\infty} \lambda_{min}\left(\frac{\Lambda_0'\hat{\Psi}_d^{-1}\Lambda_0}{n}\right) > 0$, we have:

$$V_{nT} - \sqrt{V_{nT}}O_p\left(\sqrt{\frac{1}{\Delta_{nT}}}\right) + O_p\left(\frac{1}{\Delta_{nT}}\right) \leq 0 \tag{6.6}$$

This implies that: $V_{nT} = O_p\left(\frac{1}{\Delta_{nT}}\right)$

In order to proves this, it is actually sufficient to notice that for any T and n we have a second order polynomial: $y^2 + by + c$ with $y := \sqrt{V_{nT}}$, $b = O_p\left(\sqrt{\frac{1}{\Delta_{nT}}}\right)$, $c = O_p\left(\frac{1}{\Delta_{nT}}\right)$ which is supposed to take a negative value in y.

This is possible only if the following conditions are satisfied:

a) the discriminant is positive, i.e. $c < \frac{1}{4}b^2$ (which is possible since $b^2 = O_p\left(\frac{1}{\Delta_{nT}}\right)$)

b) y is between the two roots of the polynomial, i.e.

$$\frac{1}{2}\left(b - \sqrt{b^2 - 4c}\right) \leq y \leq \frac{1}{2}\left(b + \sqrt{b^2 + 4c}\right)$$

This implies $y = O_p\left(\sqrt{\frac{1}{\Delta_{nT}}}\right)$ and hence $V_{nT} := y^2 = O_p\left(\frac{1}{\Delta_{nT}}\right)$.

The fact that Proposition 1 holds for any $A(L)$ and H is easily proved by noticing that:
a) $A(L), H$ only enter in $\text{vec}(\hat{\mathbf{F}}_{\hat{\theta}}')'\mathbf{\Phi}_{\hat{\theta}}^{-1}\text{vec}(\hat{\mathbf{F}}_{\hat{\theta}}')$ and $\log\left(I_{rT} + \mathbf{\Phi}_{\hat{\theta}}\mathbf{\Gamma}_{\hat{\theta}}\right)$ and the proof only requires these quantities to be positive.
b) imposing restrictions on $A(L)$ and H in the approximating model, we define a parameter space $\tilde{\Theta}^c \subseteq \Theta^c$ for which we still have $\theta_0^c \in \Theta_c$ and hence $\mathcal{L}(\mathbf{X}; \hat{\theta}) \geq \mathcal{L}(\mathbf{X}; \theta_0)$.

Q.E.D.

Proof of Remark 1

If the maximization is run for a number of common factors $\tilde{r} > r$ the new model will encompass the previous one and hence $\mathcal{L}(\mathbf{X}; \hat{\theta}) \geq \mathcal{L}(\mathbf{X}; \theta_0)$. This is all we need for Proposition 1 to hold.

Proof of Remark 2

This case does not follow immediately from the proof of Proposition 1. In fact, under the approximating model of the principal components we have a restricted parameter space, say Θ_{pc}^c, that does not necessarily contains θ_0^c defined above for which the idiosyncratic component is left unrestricted. However, if we replace in the proof of Proposition 1 θ_0^c with

$$\theta_0^{pc} := \left\{A(L) = I_r; H = I_r, \Lambda = \Lambda_0; \Psi_d = \sigma_0^2 I_n\right\}$$

where $\sigma_0^2 = \frac{1}{n}\mathrm{Tr}\Psi_0$, the result will follow along the same lines since we would have $\theta_0^{pc} \in \Theta_{pc}^c$ and hence $\mathcal{L}(\mathbf{X};\hat{\theta}) \geq \mathcal{L}(\mathbf{X};\theta_0^{pc})$. In addition it is possible to show that $\mathbf{F}_{\theta_0^{pc}}$ have the same asymptotic properties of $\mathbf{F}_{\theta_0^c}$. A detailed proof is available under request.