DISCUSSION PAPER SERIES

No. 5583

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INDUSTRIAL ORGANIZATION

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Available online at: www.cepr.org/pubs/dps/DP5583.asp

ISSN 0265-8003

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Discussion Paper No. 5583
March 2006

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CEPR Discussion Paper No. 5583

March 2006

ABSTRACT

Strategic Complementarities in Multi-Stage Games*

We provide sufficient conditions in finite-horizon multi-stage games for the value function of each player, associated to extremal Markov perfect equilibria, to display strategic complementarities and for the contemporaneous equilibrium to be increasing in the state variables.

JEL Classification: C73 and L13
Keywords: adjustment costs, learning curve, Markov game, supermodularity and two-stage game

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* The author is grateful to Federico Echenique and Rabah Amir for helpful conversations on the topic, and the DGI of the Ministerio de Educación y Ciencia in Spain for financial support.

Submitted 17 February 2006

1 Introduction

In many dynamic models, we are interested in whether the actions and state variables have strategic complementarity or substitutability properties. The reason for this is that many qualitative features of the dynamics will be driven by these properties. This applies, for example, both to models of investment or pricing with adjustment costs and to models where state variables are research and development (R&D) or advertising stocks.

In this paper we explore the issue in the context of Markov-perfect equilibria of finite-horizon discrete-time games. Markov strategies depend only on (state) variables that condense the direct effect of the past on the current payoff. A Markov-perfect equilibrium (MPE) is a subgame-perfect equilibrium in Markov strategies.

A particular question of interest is whether and how static complementarities translate into dynamic complementarities. A payoff function displays strategic complementarity when the incremental benefit of any action of a player is increasing in the other actions of the player as well as in the actions of rivals. Then we say that actions are strategic complements. We can think of dynamic strategic complementarity in at least two ways. First, we can think of "contemporaneous" strategic complementarity when the value function at a MPE displays strategic complementarity for each player. We can think of "intertemporal" strategic complementarity when the policy function at a MPE is monotone increasing in the state variables. For example, when each player controls a set of state variables, we can say that there is intertemporal strategic complementarity when a player raising the state variables under its control today will, in turn, increase the state variables of the rivals tomorrow. We could define the strategic substitutability properties of equilibria similarly.

Consider a two-stage game to illustrate what we are after. Players invest in some variable at the first stage, say advertising, and then compete in the second stage, say in prices. Suppose, furthermore, that strategies at the second stage are strategic complements. The questions we are trying to answer are as follows.

- When are advertising strategies strategic complements when price competition with the same property is anticipated? In this case, when rivals increase advertising expenditure, we also want to. That is, under what conditions are the strategies in the reduced-form first-stage game, obtained by folding back second-stage payoffs at a subgame-perfect equilibrium, also strategic complements?
- When does an increase in advertising expenditure by any firm at the first stage induce higher prices at the second?

The exercise is also of interest because the dynamic strategic complementarity or substitutability properties have a bearing on whether an initial dominance of a firm is reinforced or fades away (see, for example, Athey and Schmutzler (2001)).

Echenique (2004) has argued forcefully that static complementarities assumptions are not sufficient to guarantee extensive-form complementarities. Indeed, it is easy to construct two-stage games examples where at the second stage the complementarities properties are fulfilled (at least in weak form such as in Milgrom and Shanon (1994)), second-stage equilibria are monotone in first-stage actions, but the induced first-period game does not have complementarities. In this paper we provide a positive result that requires the full force of supermodularity of payoffs in addition to some properties of the second-stage equilibrium (or, more generally for Markov games, of the contemporaneous equilibrium given state variables).

Consider a MPE of a n-player finite-horizon game. We assume that the reader is familiar with the basic lattice-theoretic tools and results (see Appendix 4.1 for a summary of the method and results). The main result is that if:

- the current payoff of each player has the complementarity property in any pair of variables (increasing differences) and positive spillovers;
- the law of motion is increasing and supermodular in actions and state variables;
- the contemporaneous equilibrium is supermodular in the state variables; and
- payoffs, the law of motion and the equilibrium fulfill a convexity property;

then the contemporaneous equilibrium is increasing in the state variables and the value function is supermodular.

These conditions are stringent and some are on nonprimitives (i.e. on the equilibrium). However, they are easy to check. For example, we know that a linear-quadratic finite-horizon game will have a linear MPE (Kydland, 1975). The equilibrium may be quite cumbersome to compute, or may not have a closed-form solution, but we know already that it will fulfil trivially the supermodularity and convexity properties (the same applies to the law of motion). From the payoff properties, we can then derive the desired results without any need to get into computations. It is worth noting that the linear-quadratic model is the workhorse model for Markov games.

This paper is organized as follows. In Section 2 we present the model and results. In Section 3 we present examples of two-stage games, dynamic games with learning or network effects, and dynamic games with adjustment costs. In Appendix A we summarize some of the lattice-theoretic methodology for the reader and provide some proofs.

2 Model and results

Consider a n-player finite-horizon discrete time, $t = 1, \dots, T$, Markov game. A Markov strategy depends only on state variables, denoted by y and lying in a compact rectangle of Euclidean space, that condense the direct effect of the past on the current payoff. Let the payoff of player i in period t be $\pi_i^t(x^t, y^t)$, where x_j^t is the vector of actions of player j, lying in a compact rectangle of Euclidean space $A_j^t(y^t)$, $x^t \in A^t(y^t) \equiv \prod_{i=1}^n A_i^t(y^t)$ is the current action profile vector and y^t is a vector of state variables evolving according to $y^t = f^t(x^{t-1}, y^{t-1})$. We allow, therefore, for state variables to condition the set of feasible actions. As $A(y)$ is a compact rectangle we can write $A(y) = [\underline{a}(y), \overline{a}(y)]$. Endow the Euclidean space with the usual component-wise order. We say that $A(y)$ is ascending in y if both $\underline{a}(\cdot)$ and $\overline{a}(\cdot)$ are increasing in y.[1]

We say that $\pi_i(x, y)$ displays convex nonnegative (nonpositive) spillovers in x_{-i} if π_i is increasing (decreasing) and convex in x_{jh} for any $j \neq i$ and action h of player j. (In the differentiable case, $\partial\pi_i/\partial x_{jh} \geq (\leq) 0$ and $\partial^2\pi_i/(\partial x_{jh})^2 \geq 0$ for $j \neq i$ and action h of player j.)

We drop the time superscripts unless there is risk of confusion. As an illustration, consider the class of Markov games in which $\pi_i(x, y)$ is the current payoff for player i, with y the action profile in the previous period (state variables) and $x \in A(y)$ the current action profile. This simple class of games encompasses two-stage games and games of simultaneous moves with adjustment costs or of alternating moves.

[1] By "increasing" or "decreasing" we always mean weakly.

Players maximize the discounted sum of profits (without loss of generality in our finite-horizon game we let the discount factor equal one: $\delta = 1$). A MPE is a subgame-perfect equilibrium in Markov strategies. That is, a MPE is a set of strategies optimal for any firm and for any state of system, given the strategies of rivals.

2.1 The result for a general Markov game

An extremal equilibrium is either the largest or the smallest element in the equilibrium set. The following proposition provides conditions under which an extremal MPE of the dynamic game is monotone increasing in the state variables and has an associated value function which is supermodular. See the Appendix for the definitions and basics of supermodular games.

Proposition 1 *Suppose that for each i and any period, $\pi_i(x, y)$ is continuous, supermodular in (x, y) (i.e. displays increasing differences in any pair of variables) and has convex nonnegative (nonpositive) spillovers in x_{jh}, $j \neq i$ and all h, and is increasing (decreasing) and convex in each y_k. Suppose also that $A_i(y)$ is ascending in y. Suppose that $f_k(x, y)$ is continuous, supermodular (submodular) in (x, y) and increasing and convex (concave) in each x_{ih} and in each y_k. Consider an extremal MPE where at any period, for given states variables y, $x^*_{ih}(y)$ is continuous, supermodular (submodular) in y and convex (concave) in each y_k for any player i and action h. Then $x^*_{ih}(y)$ is increasing in y, and the value function $V_i(y)$ associated to the extremal MPE is continuous, supermodular in y, and increasing (decreasing) and convex in each y_k for any i.*

Proof. Consider an extremal equilibrium $x^*(y)$ of the game defined for player i by the payoffs $\pi_i(x, y)$ and strategy set $A_i(y)$ for a given y. This corresponds to the game in the last period. As $\pi_i(x, y)$ is continuous in x (or just in x_i) the game is supermodular and extremal equilibria exist (see, for example, Vives (1990a)). In the last period there is no continuation value and we have that

$$V_i(y) \equiv \pi_i(x^*(y), y) = \max_{x_i \in A_i(y)} \phi_i(x_i, y),$$

where $\phi_i(x_i, y) \equiv \pi_i(x_i, x^*_{-i}(y), y)$. Note that $x^*_{jh}(y)$ increases in y for any h because π_i is supermodular in x_i, has increasing differences in (x_i, y), and $A_i(y)$ is ascending in y (see Appendix 4.1).

We show that under the assumptions $V_i(y)$ is (1) supermodular in y and (2) increasing (decreasing) and convex in each y_k.

(1) It follows that $\phi_i(x_i, y)$ has increasing differences in (x_i, y) because: (i) π_i has increasing differences in $(x_i, (x_{-i}, y))$; (ii) $x^*_j(y)$ increases in y for $j \neq i$. Furthermore, we have that $\phi_i(x_i, y)$ is supermodular in y for any x_i. This follows because each $x^*_{jh}(\cdot)$ is increasing and supermodular (submodular) in y, $\pi_i(x_i, x_{-i}, y)$ is supermodular in (x_{-i}, y) (i.e. displays increasing differences in any pair of components of the vector (x_{-i}, y)) for any x_i, and is increasing (decreasing) and convex in x_{jh}, $j \neq i$, for all h, y, and all x_k, $k \neq j$ (see Lemma 7 in Appendix 4.1; a differentiable regular version is given in Appendix 4.2 for the benefit of the reader). We conclude that $V_i(y)$ is supermodular in y as supermodularity is preserved under the maximization operation. Furthermore, the function $V_i(y)$ is continuous as $\pi_i(\cdot)$ is continuous and $x^*(y)$ is also continuous by assumption.

(2) $V_i(y)$ is increasing (decreasing) in y_k because π_i is increasing (decreasing) in x_{jh}, $j \neq i$, and we know that $x^*_{jh}(y)$ is increasing in y, and π_i is increasing (decreasing) in y_k. $V_i(y)$ is convex in each y_k because π_i has increasing differences in all pairs of variables, π_i is increasing (decreasing) and convex in x_{jh}, $j \neq i$, and increasing (decreasing) and convex in y_k. Also, $x^*_{jh}(y)$ is increasing and convex (concave)

in each y_k. (Appendix 4.2 contains a differentiable regular version of the result for the benefit of the reader.)

Consider now a generic period before the last and, for given states variables y, a continuation extremal MPE with continuation value function $W_i(z)$ where $z = f(x, y)$, and let the current extremal equilibrium of the continuation game be $x^*(y)$. Suppose that $W_i(z)$ is continuous, supermodular in z and increasing (decreasing) and convex in each z_k. Player i solves

$$\max_{x_i \in A_i(y)} \{\pi_i(x_i, x^*_{-i}(y), y) + W_i(f(x_i, x^*_{-i}(y), y))\}.$$

Now, given that $W_i(z)$ is supermodular and increasing (decreasing) and convex in each z_k, $W_i(f(x, y))$ will be supermodular in (x, y) because for each k, $f_k(x, y)$ is increasing and supermodular (submodular) in (x, y) (see Lemma 7 in Appendix A.1). Furthermore, $W_i(f(x, y))$ will be increasing (decreasing) and convex in each x_{jh} and in each y_k, given that $W_i(z)$ is supermodular, increasing (decreasing), and convex in each z_k and $f_k(x, y)$ is increasing in (x, y), because for any k, $f_k(x, y)$ is convex (concave) in each x_{jh} and in each y_k.

In consequence, and under the assumptions, $\psi_i(x, y) \equiv \pi_i(x, y) + W_i(f(x, y))$ is continuous, supermodular in (x, y), and has convex nonnegative (nonpositive) spillovers in x_{jh}, $j \neq i$ and all h (because this holds for both $\pi_i(\cdot)$ and $W_i(f(\cdot))$), and therefore, as in (1), extremal equilibria will exist and $x^*_{jh}(y)$ increases in y for any h (because ψ_i is supermodular in x_i, has increasing differences in (x_i, y), and $A_i(y)$ is ascending in y). The value function

$$V_i(y) \equiv \pi_i(x^*(y), y) + W_i(f(x^*(y), y))$$

will be supermodular in y provided that the extremal equilibrium $x^*(y)$ is supermodular (submodular) in y. Furthermore, $V_i(y)$ will be increasing (decreasing) and convex in each y_k, provided that $x^*_{jh}(y)$ is convex (concave) in each y_k because the property holds for both $\pi_i(x^*(y), y)$ and $W_i(f(x^*(y), y))$ (for $\pi_i(x^*(y), y)$ exactly as in (1), and for $W_i(f(x^*(y), y))$ as $W_i(f(x, y))$ is increasing (decreasing) and convex in each x_{jh} and in each y_k, and $x^*_{jh}(y)$ is increasing in y for any j and h).

We have the desired result by backwards induction. ■

Remark. The proposition does not assert the existence of an extremal MPE. (However, given our assumptions, the last stage of the game is a supermodular static game and, therefore, extremal equilibria exist.) Existence results of MPE using lattice-theoretic tools can be found in Curtat (1996), Sleet (2001), and Amir (2005).

The sufficient conditions required for the preservation of supermodularity of the value functions are quite stringent and require knowledge of the supermodularity and convexity properties of the contemporaneous equilibrium $x^*_i(y)$. However, the conditions on the equilibrium are easily checked. A case where they are easily fulfilled is when the game has a linear MPE. This is the case in the general linear-quadratic formulation of Kydland (1975). Indeed, when Markov equilibria are linear they are supermodular (and submodular) and the convexity property is fulfilled trivially. We give an example in Section 3.3.

The result cannot be extended to the case where each payoff function $\pi_i(x, y)$ fulfils the ordinal complementarity conditions or single-crossing property in any pair of variables (Milgrom and Shanon, 1994). Indeed, it is easy to construct examples where each payoff fulfills the single-crossing property for all pairs of variables while the property is not preserved in the reduced-form first-period payoffs (Echenique, 2004). Supermodularity/increasing differences cannot be weakened to the ordinal single-crossing property. This happens even though the simultaneous move ("open loop") game would be an ordinal game of strategic complementarities and even though the second-period equilibrium is monotone in first-period choices.

A special, albeit common, case has the current payoff $\pi_i(x, y_i)$ of player i depending on a (scalar) state variable y_i, that affects only the payoff of player i, and that depends only on accumulated past actions of the player and $x_i \in A_i(y_i)$ where $A_i(y_i)$ is an interval ascending in y_i. In this case $z_i = f_i(x_i, y_i) = x_i + y_i$ and the conditions on f are automatically fulfilled. This is an instance of the class of simple Markov games to which we now turn.

2.2 Simple Markov games

We define a simple Markov game as that in which each player has its "own" vector of state variables (with dimension equal to the vector of actions to simplify notation) and the law of motion is linear and depends only on "own" variables, that is, each $f_i(x, y)$ is linear in (x_i, y_i). This allows, among others, for the case where the state variables y are the action profile in the previous period (i.e. $f_i(x_i, y_i) = x_i$ or $f(x, y) = x$) and the case where the state vector of player i is just the accumulated past actions of the player $f_i(x_i, y_i) = x_i + y_i$.

Corollary 2 (Simple Markov game) *Consider the class of simple Markov games. Suppose that for each i and any period, $\pi_i(x, y)$ is continuous, supermodular in (x, y), and has convex nonnegative (nonpositive) spillovers in x_{jh} and in y_{jh}, $j \neq i$ and all h. Suppose also that $A_i(y)$ is ascending in y. Consider an extremal MPE where at any period, for given state variables y, for any player i, and action h, $x^*_{ih}(y)$ is continuous, supermodular (submodular) in y, and convex (concave) in each y_{jk} for any player j and action k. Then $x^*_{ih}(y)$ is increasing in y, and for any i the value function $V_i(y)$ associated to the extremal MPE is continuous, supermodular in y, and increasing (decreasing) and convex in each y_{jk} for any player $j \neq i$ and action k.*

Proof. Note first that if $f(x, y)$ is linear then, trivially, $f_{ih}(x, y)$ is supermodular (submodular) in (x, y) and increasing and convex (concave) in each x_{ih} and in each y_{ik}. From the proof of Proposition 1, given that $f_i(x, y)$ depends only (and linearly) on (x_i, y_i), it is clear that to preserve supermodularity in the value function it is only required that $\pi_i(x, y)$ has convex nonnegative (nonpositive) spillovers in the state variables of rivals: y_{jh}, $j \neq i$ and all h. This in turn implies that the value function is increasing (decreasing) and convex in each of the state variables of rivals, but the property need not hold for own state variables. ■

Let us extend the result to the case of a strategic substitutes duopoly in simple Markov games.

Corollary 3 (Strategic substitutes duopoly) *Consider a simple Markov duopoly in which for all i, $\pi_i(x, y)$ is continuous, supermodular in $(x_i, -x_j, y_i, -y_j)$ (i.e. has increasing differences in any pair of variables in $(x_i, -x_j, y_i, -y_j)$), $j \neq i$, and convex nonpositive (nonnegative) spillovers in x_{jh} and in y_{jh}, $j \neq i$ and all h. Suppose that $A_i(y)$ is ascending in $(y_i, -y_j)$. Consider an extremal MPE where at any period, for given states variables y, for any player i and action h, $x^*_{ih}(y)$ is continuous, submodular (supermodular) in $(y_i, -y_j)$, and concave (convex) in each y_{jk}, $j \neq i$ and all k. Then for any i, $x^*_{ih}(y)$ is increasing in y_i and decreasing in y_j, and the value function $V_i(y)$ associated to the extremal MPE is continuous, supermodular in $(y_i, -y_j)$ and decreasing (increasing) and convex in each y_{jk}, $j \neq i$ and all k.*

Proof. The proof follows from Corollary 3 by changing the sign of the action space of one player. For completeness, a step-by-step argument is given in Appendix A.3 as well as a check in the differentiable regular case. ■

For two-stage games let, with some abuse of notation, $\pi_i(x, y)$ denote the *total* payoff of player i, where y denotes the profile of actions in the first stage and $x \in A(y)$ the profile in the second stage. Then, if we are only interested in the supermodularity or submodularity of the first-stage value function, we can obviously do with less assumptions.

Corollary 4 (Two-stage game) *Consider a two-stage game and suppose that for each i, $\pi_i(x, y)$ is continuous, supermodular in (x, y) (i.e. displays increasing differences in any pair of variables), and has convex nonnegative (nonpositive) spillovers in x_{jh}, $j \neq i$ and all h. Suppose also that $A_i(y)$ is ascending in y. For a given first-stage action profile y, consider an extremal equilibrium of the second-stage $x^*(y)$ and assume that for any player i and action h, $x^*_{ih}(y)$ is supermodular (submodular) in y for any h. Let $V_i(y) \equiv \pi_i(x^*(y), y)$. Then $x^*_{ih}(y)$ is increasing in y for any h and $V_i(y)$ is supermodular in y.*

Proof. This follows as in the proof of Proposition 1(1). ■

Corollary 5 (Two-stage strategic substitutes duopoly) *In the duopoly case suppose that for all i, $\pi_i(x, y)$ is continuous, supermodular in $(x_i, -x_j, y_i, -y_j)$, $j \neq i$, and convex nonpositive (nonnegative) spillovers in x_{jh}, $j \neq i$ and all h. Suppose that $A_i(y)$ is ascending in $(y_i, -y_j)$. For a given first-stage action profile y, consider an extremal equilibrium of the second stage $x^*(y)$ and assume that for any player i and action h, $x^*_{ih}(y)$ is submodular (supermodular) in $(y_i, -y_j)$ for any h. Let $V_i(y) \equiv \pi_i(x^*(y), y)$. Then $x^*_{ih}(y)$ is increasing in $(y_i, -y_j)$ for any h and $V_i(y)$ is supermodular in $(y_i, -y_j)$.*

3 Examples

We consider first two-stage games in Section 3.1 and then finite-horizon games in Sections 3.2 and 3.3.

3.1 Investment followed by market competition

3.1.1 Advertising and price competition

Consider a n-firm differentiated product Bertrand oligopoly where firm i produces a variety and chooses advertising effort y_i at the first stage, which has a positive influence on its (smooth) demand $D_i(x; y_i)$ where x is the price vector. Suppose also that goods are gross substitutes, $\partial D_i/\partial x_j \geq 0$ for $j \neq i$, and that demand is downward sloping, $\partial D_i/\partial x_i < 0$, and $\partial D_i/\partial y_i > 0$. Let

$$\pi_i = (x_i - c_i)D_i(x; y_i) - G_i(y_i)$$

where c_i is the constant marginal cost of firm i and G_i is the (smooth) cost of advertising, with $G'_i > 0$. Suppose also that there are natural finite upper bounds for x_i and y_i. Profits π_i are strictly supermodular in (x_i, y_i) if

$$\frac{\partial^2 \pi_i}{\partial x_i \partial y_i} = (x_i - c_i)\frac{\partial^2 D_i}{\partial x_i \partial y_i} + \frac{\partial D_i}{\partial y_i} > 0.$$

A sufficient condition for the condition to hold is that $\partial^2 D_i/\partial x_i \partial y_i \geq 0$ for $x_i - c_i \geq 0$. This amounts to requiring that advertising increases the customers willingness to pay. Furthermore, π_i has increasing differences in $((x_i, y_i), (x_{-i}, y_{-i}))$ if $\partial^2 D_i/\partial x_i \partial x_j \geq 0$ for $j \neq i$ given that $\partial D_i/\partial x_i \partial y_j = 0$, $j \neq i$.

Under the assumptions made, $\pi_i(x, y)$ has increasing differences in any pair of arguments and $\partial \pi_i/\partial x_j = (x_i - c_i)\partial D_i/\partial x_j \geq 0$ for $x_i - c_i \geq 0$. Then the first-stage value function at extremal equilibria $x^*(y)$

is supermodular in y provided that $\partial^2 D_i/(\partial x_j)^2 \geq 0$ (implying that $\partial^2 \pi_i/(\partial x_j)^2 \geq 0$) and $x_i^*(y)$ is supermodular in y for any i.

The assumptions are fulfilled in the classical linear differentiated product Bertrand competition model with constant marginal costs when either advertising or investment in product quality raises the demand intercept of the firm exerting the effort (Vives, 1985). In this case, there is a unique price equilibrium at the second stage that is linear in the first-stage advertising efforts (and, therefore, supermodular). This result is also obtained when advertising increases the willingness to pay for the product of the firm by lowering the absolute value of the slope of demand (Vives, 1990b). Then the second-stage equilibrium is nonlinear in y, but still supermodular in y. In Section 3.1.3 we check those properties of linear demand models.

3.1.2 Cournot duopoly

Consider a Cournot duopoly in which outputs are substitutes $\partial P_i/\partial x_j \leq 0$ and also strategic substitutes $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} \leq 0$, $j \neq i$, and y_i is the cost-reduction effort by firm i. Let

$$\pi_i = P_i(x_1, x_2)x_i - C_i(x_i, y_i)$$

with $\frac{\partial^2 C_i}{\partial x_i \partial y_i} \leq 0$.[2] We have that $\frac{\partial^2 \pi_i}{\partial x_i \partial y_i} \geq 0$, $\frac{\partial^2 \pi_i}{\partial x_i \partial y_j} = \frac{\partial^2 \pi_i}{\partial y_i \partial y_j} = 0$, and $\partial \pi_i/\partial x_j = x_i \; \partial P_i/\partial x_j \leq 0$, $j \neq i$. According to Corollary 5, cost-reduction investments are strategic substitutes at the first stage provided that $\partial^2 P_i/(\partial x_j)^2 \geq 0$ and $x_i^*(y)$ is submodular in $(y_i, -y_j)$ for any i.

With linear demand and costs there is a unique equilibrium at the second stage. In Section 3.1.3 we check such an example. A version of this case was studied by Amir and Wooders (2000). The authors assumed that the reduced-form payoff function at the first stage is strictly submodular and, as an example, they provided a linear Cournot duopoly. This follows directly from our approach because of the linearity of equilibria.

Capacity constraints. The case where a capacity investment y_i is made at the first stage that determines a capacity constraint $x_i \leq y_i$ at the market stage can also be accommodated. Under the assumptions above, suppose that $C_i(x_i, y_i) = f_i(y_i)$ with f_i increasing for $x_i \leq y_i$ and infinite otherwise. In this case, the action set at the second stage $A_i(y) = [0, y_i]$ is ascending in y_i and the assumptions for a strategic substitutes duopoly are fulfilled. The conclusion is that capacity investments at the first stage will be strategic substitutes.

3.1.3 The case with second-stage linear equilibria

Consider the following duopoly with second-stage quadratic payoffs (a modification of the model in Vives (1990b)) for firm i:

$$\pi_i(x, y) = (\alpha_i(y_i) - (\beta + \omega_i(y_i))x_i - \beta x_j)x_i - G_i(y_i)$$

for $j \neq i$, $i = 1, 2$, where $\alpha_i(y_i) > 0$, $\omega_i(y_i) \in (\underline{\omega}, \infty)$ with $\underline{\omega} \geq 0$, $\beta + \omega_i(y_i) > 0$, and $G_i : (\underline{\omega}, \infty) \to R_{++}$ is smooth increasing and strictly convex with $G_i' > 0$ where y_i is the first-period effort choice. This effort can potentially influence a level variable α_i (the demand intercept or unit cost) or a slope variable ω_i.

We have that $\partial \pi_i/\partial x_j = -\beta x_i$ and $\partial^2 \pi_i/(\partial x_j)^2 = 0$ for $j \neq i$. Furthermore, $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} = -\beta$ and $\frac{\partial^2 \pi_i}{\partial x_i \partial y_j} = \frac{\partial^2 \pi_i}{\partial x_j \partial y_j} = \frac{\partial^2 \pi_i}{\partial y_i \partial y_j} = 0$.

[2] A sufficient condition to have strategic substitutability (decreasing best responses) is that $\partial P_i/\partial x_j < 0$ and $\log P_i$ is submodular in (x_i, x_{-i}) (see Vives (1999, p. 151)).

This model encompasses the following cases.

- Bertrand competition with differentiated products and constant marginal production costs (set to zero for simplicity). Demand for product i is given by $\alpha_i(y_i) - (\beta + \omega_i(y_i))x_i - \beta x_j$ where x_i is the price set by firm i, $\beta < 0$ and $\underline{\omega} = 2|\beta|$. We have then that $\beta + \omega_i > |\beta|$ and the own effect dominates the cross effect in demand.

- Cournot competition with differentiated products and constant marginal production costs (set to zero for simplicity). The inverse demand for product i is given by $\alpha_i(y_i) - (\beta + \omega_i(y_i))x_i - \beta x_j$ where x_i is the quantity set by firm i and $\beta > 0$. Then we have that $\beta + \omega_i > \beta$ and the own effect dominates the cross effect in inverse demand. First-stage effort can also be interpreted to reduce the constant marginal production cost and, therefore, increase α_i.

We consider the following two cases.

(1) Investment only has a positive effect on the demand intercept $\alpha_i(y_i)$ with $\alpha_i^{'}(y_i) > 0$ ($\omega_i(y_i) = \omega_i$ for all y_i). Then $\frac{\partial^2 \pi_i}{\partial x_i \partial y_i} = \alpha_i^{'} > 0$.

(2) Investment only has a positive effect on the slope $\omega_i(y_i)$ with $\omega_i^{'} < 0$ ($\alpha_i(y_i) = \alpha_i$ for all y_i). A lower ω_i will mean a better competitive position for player i. Then $\frac{\partial^2 \pi_i}{\partial x_i \partial y_i} = -2x_i\omega_i^{'} > 0$. In the Bertrand case, effort increases the willingness to pay of consumers while in the Cournot case it expands the market for the firm.[3]

For a given y at the second stage, there is a unique linear equilibrium

$$x_i^* = \frac{2(\beta + \omega_j)\alpha_i - \beta\alpha_j}{4(\beta + \omega_1)(\beta + \omega_2) - \beta^2}$$

for $j \neq i$, $i = 1, 2$ provided that $2(\beta + \omega_j)\alpha_i - \beta\alpha_j > 0$, which we assume throughout (see Singh and Vives (1984)).

In case (1), the equilibrium x_i is linear in y and therefore both supermodular and submodular in (y_i, y_j). The assumptions of Corollary 5 are fulfilled in the Bertrand case $\beta < 0$ (then $\partial\pi_i/\partial x_j > 0$ and $\partial^2\pi_i/\partial x_i \partial x_j > 0$) and, therefore, first-stage strategies are strategic complements. Note that x_i^* is increasing in (y_i, y_j). The assumptions of Corollary 6 are fulfilled in the Cournot case $\beta > 0$ (then $\partial\pi_i/\partial x_j < 0$ and $\partial^2\pi_i/\partial x_i \partial x_j < 0$) and, therefore, first-stage strategies are strategic substitutes. Note that x_i^* is increasing in $(y_i, -y_j)$.

In case (2), it can be checked that

$$\frac{\partial x_i^*}{\partial \omega_i} < 0,$$

$$\operatorname{sign} \frac{\partial x_i^*}{\partial \omega_j} = \operatorname{sign} \beta$$

and

$$\operatorname{sign} \frac{\partial^2 x_i^*}{\partial \omega_i \partial \omega_j} = \operatorname{sign} -\beta.$$

As $\omega_i^{'} < 0$, $i = 1, 2$ we have that

$$\frac{\partial x_i^*}{\partial y_i} > 0,$$

[3] In this case the model also admits the interpretation of Cournot competition with homogenous product with increasing (linear) marginal production costs. This interpretation is apparent if we allow the market payoff to be rewritten as $(\alpha(y_i) - \beta(x_i + x_j))x_i - \omega_i(y_i)x_i^2$. Then investment lowers the slope of marginal costs.

$$\text{sign}\,\frac{\partial x_i^*}{\partial y_j} = \text{sign} -\beta$$

and

$$\text{sign}\,\frac{\partial^2 x_i^*}{\partial y_i \partial y_j} = \text{sign}\left\{\frac{\partial^2 x_i}{\partial \omega_i \partial \omega_j}\omega_i^{'}\omega_j^{'}\right\} = \text{sign} -\beta.$$

Again, the assumptions of Corollary 5 are fulfilled in the Bertrand case $\beta < 0$. Note that x_i^* is increasing in (y_i, y_j). In the Cournot case $\beta > 0$ and, according to Corollary 6, it would be required that x_i^* be submodular in $(y_i, -y_j)$ or, equivalently, supermodular in (y_i, y_j), when in fact $\frac{\partial^2 x_i^*}{\partial y_i \partial y_j} < 0$. Despite this, the result holds: x_i^* is increasing in $(y_i, -y_j)$ and the value function is submodular in (y_i, y_j). (This should come as no surprise as the conditions stated for the results are sufficient but not necessary.) Indeed, it can be directly checked that

$$\text{sign}\,\frac{\partial^2 V_i}{\partial y_i \partial y_j} = \text{sign} -\beta.$$

3.2 Dynamic competition with learning or network effects

Consider T-period competition with learning curve or network externalities in a differentiated Cournot duopoly. Actions are current rates of output. The state variables for period t are the inherited accumulated production of each firm, so we are in the context of our simple Markov game. We have

$$\pi_i(x, y) = (P_i(x_1, x_2) - (c - g_i(y)))x_i,$$

where x is the vector of output levels of the firms, c is the constant base marginal cost of production, $g_i(\cdot)$ is the (smooth) learning or network effects curve, increasing in own accumulated sales y_i (own accumulated sales lower costs or expand demand) and increasing (decreasing) in the rival accumulated sales y_j with positive (negative) spillovers. If $g_i(\cdot)$ depends only on y_i then there are no spillovers.[4] We have that $\partial\pi_i/\partial x_j = x_i\ \partial P_i/\partial x_j \leq 0$, $j \neq i$ (substitute goods), $\partial^2\pi_i/\partial x_i \partial y_i = \partial g_i/\partial y_i \geq 0$, and $\partial^2\pi_i/\partial x_i \partial y_j = \partial g_i/\partial y_j \leq 0$, $j \neq i$, and strategic output substitutability ($\partial^2\pi_i/\partial x_i \partial x_j \leq 0$). Furthermore, $\partial\pi_i/\partial y_j = x_i\ \partial g_i/\partial y_j \leq 0$ and $\partial^2\pi_i/\partial x_i \partial y_j = \partial g_i/\partial y_j \leq 0$, $j \neq i$, with nonpositive spillovers. Then the assumptions of Corollary 3 are fulfilled if $\partial^2 P_i/(\partial x_j)^2 \geq 0$, $\partial^2 g_i/(\partial y_j)^2 \geq 0$, and the extremal MPE $x_i^*(y)$ is continuous and submodular in $(y_i, -y_j)$ for any i. This is the case for the duopoly with a linear demand system and with linear g (with some parametric restrictions to insure interior solutions).[5] When the game is linear-quadratic, a linear MPE exists. The result is then that strategic substitutability in regard to accumulated learning or networks stocks is preserved dynamically and the output rate of firm i is decreasing in the accumulated production of rival firm j.

3.3 Dynamic competition with adjustment costs

Competition with adjustment costs provides another illustration. The current payoff to player i is

$$\pi_i(x, y) = u_i(x) - F_i(x, y)$$

where $u_i(x)$ is the current profit in the period and $F_i(x, y)$ is the adjustment cost in going from past actions (y) to current actions (x). This is another instance, therefore, of our simple Markov game. Assume that $F_i(x, x) = 0$, $i = 1, 2$; that is, when actions are not changed, there is no adjustment cost.

[4] See Katz and Shapiro (1986) for a network externalities model.

[5] Fudenberg and Tirole (1983) analyzed the game for n-firm and two periods with linear demand for a homogenous product and linear learning curve. A T-period duopoly version was analyzed by Dasgupta and Stiglitz (1988).

Consider a linear-quadratic T-period competition model where each player has a one-dimensional action and the adjustment cost falls on the action of each player. That is,

$$F_i(x,y) = \frac{\lambda_i}{2}(x_i - y_i)^2,$$

where x_i is the current action and y_i the past action of firm i with $\lambda_i > 0$. Let, as in Section 3.1.3,

$$u_i(x) = (\alpha_i - (\beta + \omega_i)x_i - \beta x_j)x_i.$$

The payoff for player i at any stage is therefore

$$\pi_i(x,y) = (\alpha_i - (\beta + \omega_i)x_i - \beta x_j)x_i - \frac{\lambda_i}{2}(x_i - y_i)^2,$$

where x is the current action profile and y the past action profile. The case $\beta < 0$ corresponds to price competition with menu costs, commonly used in macroeconomics, and the case $\beta > 0$ corresponds to quantity competition with production adjustment costs. This finite-horizon linear-quadratic game has a linear MPE $\{x_i^t(y)\}_{t=1}^T$, $i = 1, 2$ (Kydland, 1975). Note that π_i is indpendent of y_j, $\partial^2\pi_i/\partial x_i \partial y_i = \lambda_i > 0$, $\partial^2\pi_i/\partial x_i \partial y_j = \partial^2\pi_i/\partial y_i \partial y_j = 0$, $\operatorname{sign} \partial\pi_i/\partial x_j = \operatorname{sign} \partial^2\pi_i/\partial x_i \partial x_j = \operatorname{sign} -\beta$, $\partial\pi_i/\partial x_j = -\beta x_i$, and $\partial^2\pi_i/(\partial x_j)^2 = 0$ for $j \neq i$. If $\beta < 0$, then the assumptions of Corollary 2 are fulfilled and, therefore, the value function at any stage $V_i(y)$ will be supermodular in y and the equilibrium x_i will be increasing in y_i and y_j. That is, there will be both contemporaneous strategic complementarity, because $V_i(\cdot)$ is supermodular, and intertemporal strategic complementarity, because the price charged today by firm i will be increasing in yesterday's price of firm j. If $\beta > 0$, then the assumptions of Corollary 3 are fulfilled and, therefore, the value function at any stage $V_i(y)$ will be submodular in (y_i, y_j) and the equilibrium x_i will be increasing in y_i and decreasing in y_j. That is, there will be both contemporaneous strategic substitutability, because $V_i(\cdot)$ is submodular, and intertemporal strategic substitutability as the quantity set today by firm i will be increasing in yesterday's quantity of firm j. In both cases $V_i(\cdot)$ will be convex in y_j.

It is worth noting that no computations are needed to obtain the results, as we know the existence of a linear MPE (which is associated, obviously, to a quadratic value function for each player). The case $\beta < 0$ (Corollary 2) also applies to a n-firm oligopoly.

In the first case, the static strategic complementarity turns into dynamic strategic complementarity, whereas in the second case, static strategic substitutability turns into dynamic strategic substitutability. The linear MPE of the dynamic switching cost model of Beggs and Klemperer (1992) has a similar strategic flavor to the model with $\beta < 0$. In this case, the state variables are the loyal customer bases of every firm.

When the adjustment cost does not fall in the control variable of the player, then things are not so simple. Suppose that production is costly to adjust and firms compete in prices ($\beta < 0$), then

$$F_i(x,y) = \frac{\lambda_i}{2}((\alpha_i - (\beta + \omega_i)x_i - \beta x_j) - (\alpha_i - (\beta + \omega_i)y_i - \beta y_j))^2.$$

We have that $\partial^2\pi_i/\partial x_i \partial y_i = \lambda_i(\beta + \omega_i)^2 > 0$, but $\partial^2\pi_i/\partial x_i \partial y_j = \lambda_i(\beta + \omega_i)\beta < 0$ and neither of Corollaries 2 or 3 apply. In this case, we may have intertemporal strategic complementarity or strategic substitutability.

For example, consider a two-period model with $\lambda_1 > 0$ and $\lambda_2 = 0$. At the last period, firm 2 will price according to its static Bertrand best-reply function as neither firm can manipulate the costs of firm

2. However, an increase in the price of firm 1 in the first period induces a decrease in its output and, therefore, an increase in its marginal cost in the second period. This moves the best response function of firm 1 outwards in the second period. The outcome is higher prices for both firms giving a strategic incentive for firm 1 to raise its price in the first period. The described incentives will be the same whenever λ_2 is close to zero, in which case the second period's best reply of firm 2 will also be affected by changing prices in the first period. This, however, does not happen in the symmetric case. In a continuous-time infinite-horizon differential game *symmetric* version of this model, Jun and Vives (2004) showed that at the linear and stable MPE the value function displays strategic complementarities but today's equilibrium price for firm i is decreasing in yesterday's price of firm j. The reason for this, much as in the learning curve model with price competition, is that a firm wants to make the rival small today in order to induce it to price softly tomorrow. Indeed, a smaller rival will face a stiff cost of increasing its output. A cut in price today will therefore bring a price increase by the rival tomorrow.

4 Appendix

4.1 Summary of lattice-theoretic definitions and results

For the convenience of the reader, we include a few definitions and results of lattice methods. More complete treatments can be found in Vives (1990a; 1999, Ch. 2) and Topkis (1998).

A binary relation $\geq$ on a nonempty set X is a *partial order* if $\geq$ is reflexive, transitive, and antisymmetric. An upper bound on a subset $A \subset X$ is $z \in X$ such that $z \geq x$ for all $x \in A$. A greatest element of A is an element of A that is also an upper bound on A. Lower bounds and least elements are defined analogously. The greatest and least elements of A, when they exist, are denoted by $\max A$ and $\min A$, respectively. A supremum (respectively, infimum) of A is a least upper bound (respectively, greatest lower bound) and is denoted by $\sup A$ (respectively, $\inf A$).

A *lattice* is a partially ordered set $(X, \geq)$ in which any two elements have a supremum and an infimum. A lattice $(X, \geq)$ is *complete* if every nonempty subset has a supremum and an infimum. A subset L of the lattice X is a *sublattice* of X if the supremum and infimum of any two elements of L also belong to L. A compact rectangle in Euclidean space is a complete lattice (and a sublattice of the space).

Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. A function $f\colon X \to T$ is *increasing* if, for x, y in $\in X$, $x \geq y$ implies that $f(x) \geq f(y)$.

A function $g\colon X \to \mathbb{R}$ on a lattice X is *supermodular* if for all x, y in X, $g(\inf(x, y)) + g(\sup(x, y)) \geq g(x) + g(y)$. It is *strictly supermodular* if the inequality is strict for all pairs x, y in X that cannot be compared with respect to $\geq$ (i.e. neither $x \geq y$ nor $y \geq x$ holds). A function f is (*strictly*) *submodular* if $-f$ is (strictly) supermodular.

Let X be a lattice and T a partially ordered set. The function $g\colon X \times T \to R$ has (*strictly*) *increasing differences* in (x, t) if $g(x', t) - g(x, t)$ is (strictly) increasing in t for $x' > x$ or, equivalently, if $g(x, t') - g(x, t)$ is (strictly) increasing in x for $t' > t$. Decreasing differences are defined analogously. If X is a convex subset of $\mathbb{R}^n$ and if $g\colon X \to R$ is twice-continuously differentiable, then g has increasing differences in (x_i, x_j) if and only if $\partial^2 g(x)/\partial x_i \partial x_j \geq 0$ for all x and $i \neq j$.

Supermodularity is, in general, a stronger property than increasing differences: if T is also a lattice and if g is (strictly) supermodular on $X \times T$, then g has (strictly) increasing differences in (x, t). However, the two concepts coincide on the product of linearly ordered sets: if X is such a lattice, then a function $g\colon X \to \mathbb{R}$ is supermodular if and only if it has increasing differences in any pair of variables. This is the case in our paper since the spaces considered are compact rectangles in Euclidean space.

Lemma 6 (Topkis (1998, Lemma 2.6.4)) *If X is a lattice, $f_i(x)$ is increasing and supermodular (submodular) on X for $i = 1, \ldots, k$, Z_i is a convex subset of the reals containing the range of $f_i(x)$ on X for $i = 1, \ldots, k$, and $g(z_1, \ldots, z_k, x)$ is supermodular in $(z_1, \ldots, z_k, x)$ on $(\Pi_{i=1}^k Z_i) \times X$ and is increasing (decreasing) and convex in z_i on Z_i, for $i = 1, \ldots, k$ and for all z_j, $j \neq i$, and all x in X, then $g(f_1(x), \ldots, f_k(x), x)$ is supermodular on X.*

We say that a lattice $X(t)$ is ascending in t, with t belonging to a partially ordered set, if $t \geq t'$ implies that for $x' \in X(t')$ and $x \in X(t)$, we have $\sup(x', x) \in X(t)$ and $\inf(x', x) \in X(t')$. If $X(t)$ is a compact rectangle we can write $X(t) = [\underline{x}(t), \overline{x}(t)]$. Then $X(t)$ is ascending in t if both $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are increasing in t.

Supermodular game. The game $(A_i, \pi_i; i \in N)$ is *supermodular* if, for all i:

- A_i is a compact lattice in Euclidean space; and
- $\pi_i(a_i, a_{-i})$ is continuous, supermodular in a_i, and has increasing differences in (a_i, a_{-i}).

Lemma 7 *Consider a supermodular game in which payoffs and strategy sets are parameterized by t in a partially ordered set T : $A_i(t)$ and $\pi_i(a_i, a_{-i}; t)$. Then there exist extremal (pure-strategy) equilibria $\overline{a}(t)$ and $\underline{a}(t)$ and, if for each i, $A_i(t)$ is ascending in t and π_i has increasing differences in (a_i, t), then $\overline{a}(t)$ and $\underline{a}(t)$ are increasing in t.*

Proof. See, for example, Topkis (1998, Theorem 4.2.2) or Vives (1999, Section 2.2). ■

4.2 Differentiable and regular case

As an illustration let us show in a duopoly case, under the assumptions of Proposition 1 but in the differentiable case with payoffs twice-continuously differentiable in all arguments and the equilibrium $x_j^*(y)$ regular (interior, stable, and differentiable in y, assume that the constraint set $A_i(y)$ does not bind), that in the last stage

(1) $\phi_i(x_i, y) \equiv \pi_i(x_i, x_j^*(y), y)$ is supermodular in y for any x_i; and

(2) $V_i(y) \equiv \pi_i(x^*(y), y)$ is increasing (decreasing) and convex in each y_k.

(1) Proof that $\phi_i(x_i, y) \equiv \pi_i(x_i, x_j^*(y), y)$ is supermodular in y for any x_i. We have that for $j \neq i$

$$\frac{\partial \phi_i}{\partial y_{jk}} = \sum_h \underset{+(-)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{+}{\frac{\partial x_{jh}^*}{\partial y_k}} + \underset{+(-)}{\frac{\partial \pi_i}{\partial y_k}} \geq (\leq) 0$$

and for $j \neq i$ and $k \neq m$

$$\frac{\partial^2 \phi_i}{\partial y_k \partial y_m} = \sum_h \left[\underset{+(-)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{+(-)}{\frac{\partial^2 x_{jh}^*}{\partial y_k \partial y_m}} + \underset{+}{\frac{\partial x_{jh}^*}{\partial y_k}} \left(\sum_{p \neq h} \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}}} \underset{+}{\frac{\partial x_{jp}^*}{\partial y_m}} + \underset{+}{\frac{\partial^2 \pi_i}{(\partial x_{jh})^2}} \underset{+}{\frac{\partial x_{jh}^*}{\partial y_m}} + \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_m}} \right) \right]$$

$$+\sum_h \underset{+}{\frac{\partial^2 \pi_i}{\partial y_k \partial x_{jh}}} \underset{+}{\frac{\partial x^*_{jh}}{\partial y_m}} + \underset{+}{\frac{\partial^2 \pi_i}{\partial y_k \partial y_m}} \geq 0,$$

The inequalities follow directly from our assumptions: $\pi_i(x, y)$

- is supermodular in x_j ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}} \geq 0, p \neq h$) and in y ($\frac{\partial^2 \pi_i}{\partial y_k \partial y_m} \geq 0, k \neq m$),
- has increasing differences in (y_k, y_m) ($\frac{\partial^2 \pi_i}{\partial y_k \partial y_m} \geq 0$, $j \neq i$) and in (x_i, y) ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_m} \geq 0$), and
- has convex nonnegative (nonpositive) spillovers in x_{jh} ($\frac{\partial \pi_i}{\partial x_{jh}} \geq (\leq) 0$ and $\frac{\partial^2 \pi_i}{(\partial x_{jh})^2} \geq 0$) and in y_k and all h;
- $x^*_{jh}(y)$ is supermodular (submodular) in y ($\frac{\partial^2 x^*_{jh}}{\partial y_k \partial y_m} \geq (\leq) 0$);

and from the fact that $\frac{\partial x^*_{jh}}{\partial y_k} \geq 0$ (this holds at regular and stable equilibria given the assumptions).

(2) Proof that $V_i(y) \equiv \pi_i(x^*(y), y)$ is increasing (decreasing) and convex in each y_k. We have that for $j \neq i$

$$\frac{\partial V_i}{\partial y_k} = \sum_h \underset{+(-)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{+}{\frac{\partial x^*_{jh}}{\partial y_k}} + \underset{+(-)}{\frac{\partial \pi_i}{\partial y_k}} \geq (\leq) 0$$

(note that $\frac{\partial \pi_i}{\partial x_{ih}} = 0$ for all h) and

$$\frac{\partial^2 V_i}{(\partial y_k)^2} =$$

$$\sum_h \left[\underset{+(-)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{+(-)}{\frac{\partial^2 x^*_{jh}}{(\partial y_k)^2}} + \underset{+}{\frac{\partial x^*_{jh}}{\partial y_k}} \left(\sum_m \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{im}}} \underset{+}{\frac{\partial x^*_{im}}{\partial y_k}} + \sum_{p \neq h} \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}}} \underset{+}{\frac{\partial x^*_{jp}}{\partial y_k}} + \underset{+}{\frac{\partial^2 \pi_i}{(\partial x_{jh})^2}} \underset{+}{\frac{\partial x^*_{jh}}{\partial y_k}} + \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_k}} \right) \right]$$

$$+ \sum_m \underset{+}{\frac{\partial^2 \pi_i}{\partial y_k \partial x_{im}}} \underset{+}{\frac{\partial x^*_{im}}{\partial y_k}} + \sum_p \underset{+}{\frac{\partial^2 \pi_i}{\partial y_k \partial x_{jp}}} \underset{+}{\frac{\partial x^*_{jp}}{\partial y_k}} + \underset{+}{\frac{\partial^2 \pi_i}{(\partial y_k)^2}} \geq 0.$$

The inequalities follow directly from our assumptions: $\pi_i(x, y)$

- is supermodular in x ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}} \geq 0$, $p \neq h$; $\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{im}}$, $j \neq i$) and in (x_i, y) ($\frac{\partial^2 \pi_i}{\partial y_k \partial x_{im}} \geq 0$), (x_j, y), $j \neq i$ ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_m} \geq 0$), and
- has convex nonnegative (nonpositive) spillovers in x_{jh} ($\frac{\partial \pi_i}{\partial x_{jh}} \leq (\geq) 0$ and $\frac{\partial^2 \pi_i}{(\partial x_{jh})^2} \geq 0$) and in y_k ($\frac{\partial \pi_i}{\partial y_k} \geq 0$ and $\frac{\partial^2 \pi_i}{(\partial y_k)^2} \geq 0$), $j \neq i$ and all h;
- $x^*_{jh}(y)$ is convex (concave) in each y_k and all k ($\frac{\partial^2 x^*_{jh}}{(\partial y_k)^2} \geq (\leq) 0$);

and from the fact that $\frac{\partial x^*_{jh}}{\partial y_k} \geq 0$ for all j (this holds at regular and stable equilibria given the assumptions).

4.3 Strategic substitutes duopoly

Proof of Corollary 3. Consider an extremal equilibrium $x^*(y)$ of the game defined for player i by the payoffs $\pi_i(x, y)$ and strategy set $A_i(y)$ for a given y. If $\pi_i(x, y)$ is continuous in x, then the game is supermodular (considering $(x_i, -x_j)$) and extremal equilibria exist (see, for example, Vives (1999)). In the last period there is no continuation value and we have that

$$V_i(y) \equiv \pi_i(x^*(y), y) = \max_{x_i \in A_i(y)} \phi_i(x_i, y),$$

where $\phi_i(x_i, y) \equiv \pi_i(x_i, x_j^*(y), y)$. Note that $x_j^*(y)$ increases in y_j and decreases in y_i because π_j is supermodular in x_j, has increasing differences in $(x_j, (-y_i, y_j))$, and $A_j(y)$ is ascending in y_j and descending in y_i.

Then it follows that $\phi_i(x_i, y) \equiv \pi_i(x_i, x_j^*(y), y)$ has increasing differences in $(x_i, (y_i, -y_j))$ because: (i) π_i is supermodular in x_i and has increasing differences in $(x_i, -x_j)$ and $(x_i, (y_i, -y_j))$; (ii) $x_j^*(y)$ increases in y_j and decreases in y_i for $j \neq i$.

Furthermore, we have that $\phi_i(x_i, y)$ is supermodular in y_i and in $(y_i, -y_j)$ for any x_i. This follows because each $-x_{jh}^*(\cdot)$ is supermodular (submodular) and increasing in $(y_i, -y_j)$; $\pi_i(x_i, x_j, y)$ is supermodular in $(-x_j, (y_i, -y_j))$ for any x_i, and increasing (decreasing) and convex in $-x_{jh}$, $j \neq i$, for all h, y (see Topkis (1998, Lemma 2.6.4) and a differentiable version in (1) below). We conclude that $V_i(y)$ is supermodular in y_i and in $(y_i, -y_j)$ as supermodularity/increasing differences are preserved under the maximization operation.

$V_i(y)$ is decreasing (increasing) in y_j because π_i is decreasing (increasing) in x_{jh} and in y_{jh}, $j \neq i$, and we know that $x_{jh}^*(y)$ is decreasing in y_i and increasing in y_j. Also, $V_i(y)$ is convex in each y_{jh} because π_i has increasing differences in all pairs of variables, π_i is decreasing (increasing) and convex in x_{jh} and in y_{jh}, $j \neq i$, and $x_{ih}^*(y)$ is increasing in y_i and decreasing in y_j and convex in each y_{jk}, $j \neq i$ and all k.

Consider now a generic period before the last and, for given states variables y, a continuation extremal MPE with continuation value function $W_i(x)$, and let the current extremal equilibrium of the continuation game be $x^*(y)$. Suppose that $W_i(x)$ is supermodular in x_i and in $(x_i, -x_j)$ and decreasing (increasing) and convex in each y_{jh}. Player i solves

$$\max_{x_i \in A_i(y)} \{\pi_i(x_i, x_{-i}^*(y), y) + W_i(x_i, x_{-i}^*(y))\}.$$

Now, given that $W_i(x)$ is supermodular in x_i and in $(x_i, -x_j)$ and decreasing (increasing) and convex in each x_{jh}, and under the assumptions, $\psi_i(x, y) \equiv \pi_i(x, y) + W_i(x)$ is supermodular in x_i and has increasing differences in $(x_i, -x_j), (y_i, -y_j)$ and in $(x_i, (y_i, -y_j))$, and is decreasing (increasing) and convex in each x_{jh}, $j \neq i$ and all h, and therefore, as in (1), the value function

$$V_i(y) \equiv \pi_i(x^*(y), y) + W_i(x^*(y))$$

will be supermodular in y_i and in $(y_i, -y_j)$ provided that the extremal equilibrium $x_{ih}^*(y)$ is submodular (supermodular) in y_i and supermodular (submodular) in (y_i, y_j), and it will be decreasing (increasing) and convex in each y_{jh}.

We have the desired result by backwards induction. ■

For the benefit of the reader, let us now show in a duopoly case, under the assumptions of Corollary 4 but in the differentiable case with payoffs twice-continuously differentiable in all arguments and the equilibrium $x_j^*(y)$ regular (interior, stable and differentiable in y, assume that the constraint set $A_i(y)$ does not bind), that

(1) $\phi_i(x_i, y) \equiv \pi_i(x_i, x_j^*(y), y)$ is supermodular in y_i and in $(y_i, -y_j)$ for any x_i; and

(1) $V_i(y) \equiv \pi_i(x^*(y), y)$ is decreasing (increasing) and convex in each y_{jh}, $j \neq i$.

(1) Proof that $\phi_i(x_i, y) \equiv \pi_i(x_i, x_j^*(y), y)$ is supermodular in $(y_i, -y_j)$ for any x_i. We have that for $j \neq i$

$$\frac{\partial \phi_i}{\partial y_{jk}} = \sum_h \underset{-(+)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{+}{\frac{\partial x_{jh}^*}{\partial y_{jk}}} + \underset{-(+)}{\frac{\partial \pi_i}{\partial y_{jk}}} \leq (\geq) 0$$

and for $j \neq i$ and $k \neq m$

$$\frac{\partial^2 \phi_i}{\partial y_{jk} \partial y_{jm}} = \sum_h \left[\underset{-(+)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{-(+)}{\frac{\partial^2 x_{jh}^*}{\partial y_{jk} \partial y_{jm}}} + \underset{+}{\frac{\partial x_{jh}^*}{\partial y_{jk}}} \left(\sum_{p \neq h} \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}}} \underset{+}{\frac{\partial x_{jp}^*}{\partial y_{jm}}} + \underset{+}{\frac{\partial^2 \pi_i}{(\partial x_{jh})^2}} \underset{+}{\frac{\partial x_{jh}^*}{\partial y_{jm}}} + \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_{jm}}} \right) \right]$$

$$+ \sum_h \underset{+}{\frac{\partial^2 \pi_i}{\partial y_{jk} \partial x_{jh}}} \underset{+}{\frac{\partial x_{jh}^*}{\partial y_{jm}}} + \underset{+}{\frac{\partial^2 \pi_i}{\partial y_{jk} \partial y_{jm}}} \geq 0,$$

and for $j \neq i$

$$\frac{\partial^2 \phi_i}{\partial y_{jk} \partial y_{im}} = \sum_h \left[\underset{-(+)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{+(-)}{\frac{\partial^2 x_{jh}^*}{\partial y_{jk} \partial y_{im}}} + \underset{+}{\frac{\partial x_{jh}^*}{\partial y_{jk}}} \left(\sum_{p \neq h} \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}}} \underset{-}{\frac{\partial x_{jp}^*}{\partial y_{im}}} + \underset{+}{\frac{\partial^2 \pi_i}{(\partial x_{jh})^2}} \underset{-}{\frac{\partial x_{jh}^*}{\partial y_{im}}} + \underset{-}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_{im}}} \right) \right]$$

$$+ \sum_h \underset{+}{\frac{\partial^2 \pi_i}{\partial y_{jk} \partial x_{jh}}} \underset{-}{\frac{\partial x_{jh}^*}{\partial y_{im}}} + \underset{-}{\frac{\partial^2 \pi_i}{\partial y_{jk} \partial y_{im}}} \leq 0.$$

The inequalities follow directly from the smooth version of our increasing differences assumptions: $\pi_i(x, y)$

- is supermodular in x_j ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}} \geq 0$, $p \neq h$) and in y_j ($\frac{\partial^2 \pi_i}{\partial y_{jk} \partial y_{jm}} \geq 0, k \neq m$),
- has increasing differences in $(x_i, -x_j), (y_i, -y_j)$ ($\frac{\partial^2 \pi_i}{\partial y_{jk} \partial y_{im}} \leq 0, j \neq i$) and $(x_i, (y_i, -y_j)), (x_j, (-y_i, y_j))$, $j \neq i$ ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_{jm}} \geq 0$, $\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_{im}} \leq 0$), and
- has convex nonpositive (nonnegative) spillovers in x_{jh} ($\frac{\partial \pi_i}{\partial x_{jh}} \leq (\geq) 0$ and $\frac{\partial^2 \pi_i}{(\partial x_{jh})^2} \geq 0$) and in y_{jh}, $j \neq i$ and all h;
- $x_{jh}^*(y)$ is submodular (supermodular) in y_j ($\frac{\partial^2 x_{jh}^*}{\partial y_{jk} \partial y_{jm}} \leq (\geq) 0$) and supermodular (submodular) in (y_i, y_j) ($\frac{\partial^2 x_{jh}^*}{\partial y_{jk} \partial y_{im}} \geq (\leq) 0$, $j \neq i$);

and from the fact that $\frac{\partial x_{jh}^*}{\partial y_{jk}} \geq 0$ and $\frac{\partial x_{jh}^*}{\partial y_{im}} \leq 0$, $j \neq i$ (this would hold at regular and stable equilibria given the assumptions).

(2) Proof that $V_i(y) \equiv \pi_i(x^*(y), y)$ is decreasing (increasing) and convex in each y_{jk}, $j \neq i$. We have that for $j \neq i$

$$\frac{\partial V_i}{\partial y_{jk}} = \sum_h \underset{-(+)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{+}{\frac{\partial x_{jh}^*}{\partial y_{jk}}} + \underset{-(+)}{\frac{\partial \pi_i}{\partial y_{jk}}} \leq (\geq) 0$$

(note that $\frac{\partial \pi_i}{\partial x_{ih}} = 0$ for all h) and

$$\frac{\partial^2 V_i}{(\partial y_{jk})^2} =$$

$$\sum_h \left[\underset{-(+)}{\frac{\partial \pi_i}{\partial x_{jh}}} \underset{-(+)}{\frac{\partial^2 x^*_{jh}}{(\partial y_{jk})^2}} + \underset{+}{\frac{\partial x^*_{jh}}{\partial y_{jk}}} \left(\sum_m \underset{-}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{im}}} \underset{-}{\frac{\partial x^*_{im}}{\partial y_{jk}}} + \sum_{p \neq h} \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}}} \underset{+}{\frac{\partial x^*_{jp}}{\partial y_{jk}}} + \underset{+}{\frac{\partial^2 \pi_i}{(\partial x_{jh})^2}} \underset{+}{\frac{\partial x^*_{jh}}{\partial y_{jk}}} + \underset{+}{\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_{jk}}} \right) \right]$$

$$+ \sum_m \underset{-}{\frac{\partial^2 \pi_i}{\partial y_{jk} \partial x_{im}}} \underset{-}{\frac{\partial x^*_{im}}{\partial y_{jk}}} + \sum_p \underset{+}{\frac{\partial^2 \pi_i}{\partial y_{jk} \partial x_{jp}}} \underset{+}{\frac{\partial x^*_{jp}}{\partial y_{jk}}} + \underset{+}{\frac{\partial^2 \pi_i}{(\partial y_{jk})^2}} \geq 0.$$

The inequalities follow directly from the smooth version of our assumptions: $\pi_i(x, y)$

- is supermodular in x_j ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{jp}} \geq 0, p \neq h$),
- has increasing differences in $(x_i, -x_j)$ ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial x_{im}} \leq 0$, $j \neq i$) and in $(x_i, (y_i, -y_j))$ ($\frac{\partial^2 \pi_i}{\partial y_{jk} \partial x_{im}} \leq 0$), $(x_j, (-y_i, y_j))$, $j \neq i$ ($\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_{jm}} \geq 0$, $\frac{\partial^2 \pi_i}{\partial x_{jh} \partial y_{im}} \leq 0$), and
- has convex nonpositive (nonnegative) spillovers in x_{jh} ($\frac{\partial \pi_i}{\partial x_{jh}} \leq (\geq) 0$ and $\frac{\partial^2 \pi_i}{(\partial x_{jh})^2} \geq 0$) and in y_{jh} ($\frac{\partial \pi_i}{\partial y_{jk}} \leq 0$ and $\frac{\partial^2 \pi_i}{(\partial y_{jk})^2} \geq 0$), $j \neq i$ and all h;
- $x^*_{jh}(y)$ is concave (convex) in each y_{jk}, $j \neq i$ and all k ($\frac{\partial^2 x^*_{jh}}{(\partial y_{jk})^2} \leq (\geq) 0$);

and from the fact that $\frac{\partial x^*_{jh}}{\partial y_{jk}} \geq 0$ and $\frac{\partial x^*_{im}}{\partial y_{jk}} \leq 0$, $j \neq i$ (this holds at regular and stable equilibria given the assumptions).

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