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## DISCRETIONARY POLICY, MULTIPLE EQUILIBRIA, AND MONETARY INSTRUMENTS

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# DISCRETIONARY POLICY, MULTIPLE EQUILIBRIA, AND MONETARY INSTRUMENTS

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# ABSTRACT

## Discretionary Policy, Multiple Equilibria, and Monetary Instruments*

This paper examines monetary policy implementation in a sticky price model. The central bank's plan under discretionary optimization is entirely forward-looking and exhibits multiple equilibrium solutions if transactions frictions are not negligibly small. The central bank can then implement stable history dependent equilibrium sequences that are consistent with its plan by inertial interest rate adjustments or by money transfers. These equilibria can be associated with lower welfare losses than a forward-looking solution implemented by interest rate adjustments. The welfare gain from a history dependent implementation tends to rise with the strength of transactions frictions and the degree of price flexibility. It is further shown that the central bank's plan can uniquely be implemented in a history dependent way by money transfers, whereas inertial interest rate adjustments cannot avoid equilibrium multiplicity.

JEL Classification: E32, E51 and E52
Keywords: equilibrium indeterminacy, history dependence, Monetary policy implementation, money growth policy and optimal discretionary policy

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## 1 Introduction

Does it matter how a particular plan of a central bank is implemented? In general, discretionary policy leads to suboptimal outcomes, while an optimal commitment policy, which implements a superior allocation, is not time consistent (see Kydland and Prescott, 1977).[3] Discretionary policies are further known to allow for the possibility of rational expectations equilibrium multiplicity (see Albanesi et al., 2003, and King and Wolman, 2004).[4] Both, inefficiency and indeterminacy are due to the characteristic feature of discretionary policymaking not to account for private sector expectations about policy actions, and to the lack of history dependence when the private sector is forward-looking (see Woodford, 2003a, 2003b). In this paper, we show that a plan of a central bank acting under discretion can be implemented in a history dependent way, even when the private sector is entirely forward-looking. The conduct of monetary policy depends on past conditions when the central bank applies either the interest rate in an inertial way or the money growth rate as its instrument. The induced history dependence is able to raise household welfare compared to the case where the central bank implements its plan by purely forward-looking interest rate adjustments. In order to avoid equilibrium indeterminacy the central bank should control the growth rate of nominal money balances.

Previous studies on the monetary instrument choice have mainly focused on the stabilization and welfare implications of particular rules for different instruments, such as Poole (1970), Sargent and Wallace (1975), Carlstrom and Fuerst (1995), Collard and Dellas (2004), or Gavin et al. (2004). In contrast to these studies, we examine different reaction functions for monetary policy instruments under a particular plan of an optimizing central bank. Throughout the paper, we restrict our attention to the realistic case where the central bank cannot commit itself to a once-and-for-all-policy, and acts in a discretionary way. We consider a framework with conflicting macroeconomic distortions, implying that the central bank faces a trade-off, since it cannot eliminate more than one friction with a single instrument. The central bank's optimal plan under discretion can then exhibit multiple equilibrium solutions. This has also been shown in several recent studies on monetary discretion in New Keynesian models, where distortions induced by price rigidities are accompanied by distortions due to monopolistic competition (see King and Wolman, 2004, and Siu 2005) or transactions frictions (see Albanesi et al., 2003, Brueckner and Schabert, 2005, and Kurozumi, 2005). Once a particular plan is consistent with more than one allocation and equilibrium price system, the operational procedure of monetary policy and the instrument choice can matter.

The novel idea of this paper is that different instruments, which are designed to im-

[3] Exceptions of the latter principle are examined in Alvarez et al. (2004). In particular, they show that commitment policies can be time consistent when the Friedman rule is optimal.

[4] The existence of multiple equilibria under discretionary monetary policy is further examined in Brueckner and Schabert (2005), Kurozumi (2005), and Siu (2005).

plement such a (not uniquely determined) plan of the central bank, can lead to different macroeconomic outcomes and therefore to different results regarding equilibrium determinacy and social welfare. Specifically, we consider three means of monetary policy implementation: *i.*) a forward-looking and *ii.*) an inertial reaction function for the risk-free nominal interest rate, as well as *iii.*) a forward-looking reaction function for money transfers (which are equivalent to open market asset purchases/sales). If the plan is implemented by *i.*), the conduct of monetary policy by a central bank that acts in a discretionary way is entirely forward-looking. However, when the monetary instrument is set in a history dependent way, such as under an inertial interest rate reaction function, monetary policy becomes history dependent. The same result holds for the case *iii.*), where the central bank adjusts the money stock via lump-sum transfers in a forward-looking way in order to implement its plan. The reason is that a central bank operation that is meant to adjust the supply of money has to take into account the preexisting stock of outstanding money.[5] Put differently, the beginning-of-period stock of money contains non-negligible information for money transfers required to obtain a particular end-of-period stock of money. Hence, when money transfers, i.e., the money growth rate, serves as the instrument monetary policy becomes history dependent, even if the central bank does – in contrast to *ii.*) – not consider past conditions as indicators for adjustments of its instruments.[6]

We apply a standard New Keynesian framework with transactions frictions (modeled by money-in-the-utility-function) and with cost-push shocks. The central bank's plan under discretionary optimization is shown to allow for equilibrium multiplicity. Since interest rate changes are associated with non-negligible welfare costs, the central bank abstains from choosing a plan that is associated with strong (active) adjustments of the interest rate, which would lead to equilibrium uniqueness (since the Taylor-principle applies in our model). The likelihood of equilibrium multiplicity under discretionary policy thereby increases with the severity of transactions frictions.[7] However, the central bank can avoid equilibrium indeterminacy by designing a reaction function for the prevailing instrument in an appropriate way. For example, there exist interest rate reaction functions of type *i.*) that uniquely implement an entirely forward-looking solution to the central bank's plan. There further exists a money supply reaction function that uniquely implements a history dependent and stable solution to the plan. In contrast, it is shown that a reaction function of type *ii.*) cannot implement a history dependent solution of the plan in a stable and unique way. Thus, under an inertial interest rate reaction function the problem of

[5] Related studies on the equilibrium behavior of sticky price models have also shown that real money serves as a relevant endogenous state variable when the central bank controls the money growth rate (see e.g., Evans and Honkapohja, 2003, and Schabert, 2005).

[6] A legal restriction that demands monetary policy to be conducted via money transfers (or, equivalently, via open market asset purchases/sales) can thus serve as a commitment device to a history dependent policy implementation.

[7] In the limiting case, where the distortion due to transactions frictions is negligible, the central bank's plan under discretionary exhibits a unique solution (see also Jensen, 2002).

equilibrium indeterminacy cannot be avoided, implying that monetary policy allows for non-fundamental equilibria and thus endogenous fluctuations.

Given that the reaction functions *i.*) − *iii.*) lead to different equilibria, which are all consistent with the central bank's plan, we compare the welfare implications of different means of monetary policy implementation.[8] As stressed by Woodford (2003a), history dependence can be beneficial for social welfare when the private sector behavior is forward-looking. Based on this principle, Walsh (2003) and Woodford (2003b) have shown that social welfare can be raised under a discretionary monetary policy by introducing lagged endogenous variables in the central banker's objective, inducing the plan under discretionary optimization to be history dependent. Corresponding to these results, we find that equilibria under a history dependent implementation of monetary policy, i.e., under *ii.*) and *iii.*), can be associated with higher social welfare compared to the unique equilibrium under an entirely forward-looking interest rate setting, even if they are all consistent with the same plan. Social welfare is thus raised by a history dependent central bank behavior that is induced by monetary policy implementation, rather than by a particular plan based on preferences of a central banker which deviate from social welfare.

In general, history dependence affects the expectations about future realizations of macroeconomic aggregates and therefore their conditional variances.[9] Here, the relevance of a lagged variable for the evolution of macroeconomic aggregates under a reaction function *ii.*) or *iii.*) can lower welfare-reducing macroeconomic fluctuations, as forecast error variances tend to decrease with the introduction of relevant state variables. However, an extension of the state space increases the support of the stochastic variables, which might raise the variances of macroeconomic aggregates. We find that this effect on the variances of macroeconomic aggregates is less important when the autocorrelation of the common state variable, i.e., the cost-push shock, is high. The extent to which social welfare is raised under a history dependent implementation of the plan further depends on the particular economic structure. Specifically, we find that the welfare gain from a history dependent implementation increases with the severity of the distortion due to transactions frictions and with the degree of price flexibility. The main conclusion from the welfare analysis is therefore that a central bank should implement its plan by forward-looking interest rate adjustments only if the aggregate shock is not very persistent or transactions frictions are negligible. Otherwise, it should implement the plan in a history dependent way.

To summarize, this paper contributes to the analysis of discretionary monetary policy

[8] We disregard the possibility that the central bank controls both instruments simultaneously, which is for example considered in Adao et al. (2003). They demonstrate that the optimal allocation under sticky prices can welfare dominate the optimal allocation under flexible prices, if the central bank sets the nominal interest rate equal to a sufficiently small value and, simultaneously, controls money supply.

[9] If a central bank takes into account the impact of monetary policy on expectations of a forward-looking private sector, its plan would exhibit history dependence. See, for example, Woodford (2003a) for a so-called optimal commitment policy under a timeless perspective.

in two novel ways. Given an environment where monetary discretion allows for equilibrium multiplicity, we, firstly, show that a potentially welfare-enhancing history dependence can be introduced by monetary policy implementation, in particular, by an inertial interest rate policy or a money growth policy. Secondly, the problem of equilibrium multiplicity under monetary discretion can be eliminated if the central bank implements its plan by state contingent money injections, but not by an inertial interest rate policy.[10] Finally, the analysis shows that the application of the minimum state variable criterion (e.g., for the solution to the equilibrium under the central bank's plan) would select an equilibrium allocation which can be welfare dominated by alternative history dependent allocations.

The remainder is organized as follows. In section 2 we describe the model. Section 3 presents the equilibrium behavior under different reaction functions. In section 4, we examine the central bank's plan under discretionary optimization and its implementation. In section 5, we compare social welfare under different equilibrium solutions. Section 6 concludes.

## 2 The model

In this section we describe the macroeconomic framework, which closely relates to the model in Woodford (2003a, section 4.3). There are three sectors, the household sector, the production sector, and the public sector. Cost-push shocks, which stem from exogenous changes in the elasticity of substitution of individual labor services, are the only source of uncertainty. There are no information asymmetries between the three sectors. Nominal (real) variables are denoted by upper-case (lower-case) letters.

There is a continuum of infinitely lived households indexed with $j \in [0,1]$. Households have identical asset endowments and identical preferences. Household $j$ maximizes the expected sum of a discounted stream of instantaneous utilities $U$ :

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_{jt}, l_{jt}, M_{jt}/P_t\right), \qquad (1)$$

where $E_0$ is the expectation operator conditional on the time 0 information set, and $\beta \in (0,1)$ is the subjective discount factor. The instantaneous utility $U$ is assumed to be increasing in consumption $c$ and real balances $M/P$, decreasing in working time $l$, strictly concave, twice continuously differentiable, and to satisfy the usual Inada conditions. Instantaneous utility $U$ is further assumed to be separable in the utility from private

[10] This result corresponds to the property of nominal (in)determinacy under money growth (interest rate) policy, which has for example been examined by Sargent and Wallace (1974). While a money growth policy facilitates nominal determinacy under perfectly flexible prices, it causes beginning-of-period real balances to be relevant for equilibrium determination when prices are not perfectly flexible. The predetermined value of real money then serves as an equilibrium selection criterion, which rules out solutions with extraneous states that would allow for endogenous fluctuations.

consumption and from real balances, and in the disutility of working time, $U(c_{jt}, l_{jt}) = u(c_{jt}) - v(l_{jt}) + \nu(M_{jt}/P_t)$.

At the beginning of period $t$ household $j$ is endowed with holdings of money $M_{jt-1}$ and a portfolio of state contingent claims on other households yielding a (random) payment $Z_{jt}$. Before the goods market opens, households enter the asset market, where they can adjust their portfolio and receive government transfers. Let $q_{t,t+1}$ denote the period $t$ price of one unit of currency in a particular state of period $t+1$ normalized by the probability of occurrence of that state, conditional on the information available in period $t$. Then, the price of a random payoff $Z_{jt+1}$ in period $t+1$ is given by $E_t[q_{t,t+1}Z_{jt+1}]$. Households further receive wage payments and dividends $D_{it}$ from monopolistically competitive firms indexed with $i \in [0,1]$. The budget constraint of household $j$ can be written as

$$M_{jt} \leq M_{jt-1} + Z_{jt} - E_t[q_{t,t+1}Z_{jt+1}] + P_t w_{jt} l_{jt} + P_t \tau_t - P_t c_{jt} + \int_0^1 D_{j,it} di, \tag{2}$$

where $P_t$ denotes the aggregate price level and $w_{jt}$ the (individual) real wage rate. Lump-sum money transfers $P_t\tau_t$, which households receive in the asset market, serve as a central bank instrument. As will be demonstrated below, money supply can equivalently be specified by assuming that money is injected via open market operations, instead of via lump-sum transfers. We further assume that households have to fulfill a no-Ponzi game condition, $\lim_{s\to\infty} E_t q_{t,t+s}(M_{jt+s} + Z_{jt+1+s}) \geq 0$.

We assume that households monopolistically supply differentiated labor services $l_j$, which are transformed into aggregate labor input $l_t$ where $l_t^{1-1/\eta_t} = \int_0^1 l_{jt}^{1-1/\eta_t} dj$. The elasticity of substitution between differentiated labor services $\eta_t > 1$ is allowed to vary exogenously over time. Cost minimization then leads to the following labor demand $l_{jt} = (w_{jt}/w_t)^{-\eta_t} l_t$, with $w_t^{1-\eta_t} = \int_0^1 w_{jt}^{1-\eta_t} dj$, where $w_t$ denotes aggregate real wage rate. Maximizing the objective (1), subject to the budget constraint (2), the labor demand condition, and the no-Ponzi-game condition, for given initial values $Z_{j0}$ and $M_{j,-1}$ leads to the following first order conditions:

$$\begin{aligned} u_c(c_{jt}) &= \lambda_{jt}, \quad v_l(l_{jt}) = \xi_t^{-1} w_{jt} \lambda_{jt}, \\ \lambda_{jt} - \nu_m(M_{jt}/P_t) &= \beta E_t \frac{\lambda_{jt+1}}{\pi_{t+1}}, \quad q_{t,t+1} = \frac{\beta}{\pi_{t+1}} \frac{\lambda_{jt+1}}{\lambda_{jt}}, \end{aligned} \tag{3}$$

where $\pi$ denotes the inflation rate ($\pi_t = P_t/P_{t-1}$), $\lambda$ the shadow price of wealth and $\xi$ the wage mark-up where $\xi_t = \eta_t/(\eta_t - 1)$. The stochastic properties of $\xi_t$ will be discussed below. Furthermore, the budget constraint (2) holds with equality and the transversality condition, $\lim_{s\to\infty} \beta^{t+s} E_t[\lambda_{jt+s}(M_{jt+s} + Z_{jt+1+s})/P_{t+s}] = 0$, must be satisfied. The one-period nominal interest rate on a risk-free portfolio, which serves as an alternative central bank instrument, is defined as follows

$$R_t = [E_t q_{t,t+1}]^{-1}. \tag{4}$$

Using (4), money demand can be written as $\nu_m(M_{jt}/P_t) = u_c(c_{jt})\,(R_t - 1)\,/R_t$. The final consumption good is an aggregate of differentiated goods produced by monopolistically competitive firms indexed with $i \in [0, 1]$. The CES aggregator of differentiated goods is defined as $y_t^{\frac{\epsilon-1}{\epsilon}} = \int_0^1 y_{it}^{\frac{\epsilon-1}{\epsilon}} di$, with $\epsilon > 1$, where $y_t$ is the number of units of the final good, $y_{it}$ the amount produced by firm $i$, and $\epsilon$ the constant elasticity of substitution between these differentiated goods. Let $P_{it}$ and $P_t$ denote the price of good $i$ set by firm $i$ and the price index for the final good. The demand for each differentiated good is $y_{it} = (P_{it}/P_t)^{-\epsilon} y_t$, with $P_t^{1-\epsilon} = \int_0^1 P_{it}^{1-\epsilon} di$. A firm $i$ produces good $y_i$ employing a technology which is linear in labor: $y_{it} = l_{it}$, where $l_t = \int_0^1 l_{it} di$. Hence, labor demand satisfies: $mc_{it} = w_t$, where $mc_{it} = mc_t$ denotes real marginal costs.

We consider a nominal rigidity in form of staggered price setting as developed by Calvo (1983). Each period firms may reset their prices with the probability $1 - \phi$ independently of the time elapsed since the last price setting. The fraction $\phi \in (0, 1)$ of firms is assumed to adopt the previous period's prices according $P_{it} = P_{it-1}$. In each period a measure $1-\phi$ of randomly selected firms sets new prices $\widetilde{P}_{it}$ in order to maximize the expected sum of discounted future dividends ($D_{it} = (P_{it} - P_t mc_t)\, y_{it}$): $\max_{\widetilde{P}_{it}} E_t \sum_{s=0}^{\infty} \phi^s q_{t,t+s}(\widetilde{P}_{it} y_{it+s} - P_{t+s} mc_{t+s} y_{it+s})$, s.t. $y_{it+s} = \widetilde{P}_{it}{}^{-\epsilon} P_{t+s}^{\epsilon} y_{t+s}$. The first order condition is given by

$$\widetilde{P}_{it} = \frac{\epsilon}{\epsilon - 1} \frac{E_t \sum_{s=0}^{\infty} \phi^s \left[ q_{t,t+s} y_{t+s} P_{t+s}^{\epsilon+1} mc_{t+s} \right]}{E_t \sum_{s=0}^{\infty} \phi^s \left[ q_{t,t+s} y_{t+s} P_{t+s}^{\epsilon} \right]}. \tag{5}$$

Aggregate output is $y_t = (P_t^*/P_t)^{\epsilon} l_t$, where $(P_t^*)^{-\epsilon} = \int_0^1 P_{it}^{-\epsilon} di$ and thus $(P_t^*)^{-\epsilon} = \phi \left(P_{t-1}^*\right)^{-\epsilon} + (1-\phi)\widetilde{P}_t^{-\epsilon}$.

The central bank is assumed to trade with households in the asset markets. There, it adjusts the outstanding stock of money via lump-sum transfers $P_t \tau_t$. Its budget constraint is given by $P_t \tau_t = M_t - M_{t-1} = (\mu_t - 1)\, M_{t-1}$, where $\mu_t$ denotes the gross money growth rate, $\mu_t = M_t/M_{t-1}$. It should be noted that we can (equivalently) model money supply by assuming that money and government bonds $B$ are exclusively traded in open market operations.

For example, open market operations can be specified by "holding fiscal policy constant in the face of a government asset exchange" (see Sargent and Smith, 1987): $(\mu_t - 1)\, M_{t-1} = -B_t + R_{t-1}^b B_{t-1}$ and $P_t \tau_t = 0$, like in Alvarez et al. (2002).[11] A monetary expansion is then brought about by an open market bond purchase, which implies $\mu_t > 1$. A corresponding initial value for total government liabilities would be equal to zero, $B_{-1} + M_{-1} = 0$, which is consistent with government solvency, $\lim_{s\to\infty}(B_{t+s} + M_{t+s})\Pi_{v=1}^{s}(1/R_{t+v}) = 0$ and $M_{-1} > 0$.[12] This alternative specification is then equivalent to the former specification.

[11] The households' budget constraint would then be given by $B_{jt} + M_{jt} \leq R_{t-1}^b B_{jt-1} + M_{jt-1} + Z_{jt} - E_t[q_{t,t+1} Z_{jt+1}] + P_t w_{jt} l_{jt} - P_t c_{jt} + \int_0^1 D_{j,it} di$, and the first order condition on bond holdings by $u_c(c_{jt}) = \beta R_t^b E_t[u_c(c_{jt+1})/\pi_{t+1}]$, implying $R_t^b = R_t$.

[12] Alternatively, open market operations can be specified without restricting transfers (i.e., lump-sum

Finally, the central bank is assumed to set either the risk-free nominal interest rate $R_t$ or the money growth rate $\mu_t = m_t\pi_t/m_{t-1}$, where $m$ denotes real balances $m_t = M_t/P_t$. The equilibrium for $R_t > 1$ is defined as follows.

**Definition 1** *A rational expectations equilibrium for $R_t > 1$ is a set of sequences $\{y_t,\, l_t,\, P_t^*,\, P_t,\, \widetilde{P}_t,\, mc_t,\, w_t,\, m_t,\, R_t\}_{t=0}^{\infty}$ satisfying the firms' first order conditions $mc_t = w_t$, (5) with $\widetilde{P}_{it} = \widetilde{P}_t$, and $P_t^{1-\epsilon} = \phi\left(P_{t-1}\right)^{1-\epsilon} + (1-\phi)\,\widetilde{P}_t^{1-\epsilon}$, the households' first order conditions $u_c(y_t)w_t = v_l(l_t)\xi_t$, $u_c(y_t)/P_t = \beta R_t E_t\left[u_c(y_{t+1})/P_{t+1}\right]$, $\nu_m(m_t)/u_c(y_t) = (R_t - 1)/R_t$, the aggregate resource constraint $y_t = (P_t^*/P_t)^{\epsilon} l_t$, where $(P_t^*)^{-\epsilon} = \phi\left(P_{t-1}^*\right)^{-\epsilon} + (1-\phi)\widetilde{P}_t^{-\epsilon}$, and the transversality condition, given a monetary policy, a sequence $\{\varepsilon_t\}_{t=0}^{\infty}$, and initial values $P_{-1} > 0$, $P_{-1}^* > 0$, and $m_{-1}P_{-1} = M_{-1} > 0$.*

The stochastic process for the wage mark-up $\xi_t$ is assumed to satisfy $\widehat{\xi}_t = \rho\widehat{\xi}_{t-1} + \varepsilon_t$, where $\widehat{\xi}_t = \log\xi_t - \overline{\xi}$ and $\rho \in [0,1)$. The innovations are assumed to be normally distributed with mean zero and a constant variance, $\varepsilon_t \sim N(0, var_e)$.

## 3 Equilibrium behavior under different reaction functions

In this section we present the log-linearized version of the model described in the previous section and summarize the main equilibrium properties of the model under different reaction functions for the monetary instrument. The equilibrium conditions given in definition 1 are log-linearized at the deterministic steady state.[13] Given that our analysis focuses on the stabilization properties of monetary policy, we abstract from long-run effects of different monetary policy regimes and assume that they are consistent with the same steady state. We further assume that the steady state is characterized by a constant price level, such that the steady state values (marked with bars) for the inflation rate, the money growth rate, and the interest rate are given by $\overline{\pi} = \overline{\mu} = 1$ and $\overline{R} = 1/\beta > 1$. A steady state is then characterized by uniquely determined values for output $u_c(\overline{y})/v_l(\overline{y}) = \Omega$, where $\Omega = \overline{\xi}\frac{\epsilon}{\epsilon-1} > 1$, and for real balances $\nu_m(\overline{m}) = u_c(\overline{y})\,(1-\beta)$. Throughout the paper, $\widehat{x}_t$ denotes the percent deviation of a generic variable $x_t$ from its steady state value $\overline{x} : \widehat{x} = \log(x_t) - \log(\overline{x})$. An equilibrium of the log-linear model is defined as follows:

A rational expectations equilibrium of the log-linear approximation to the model at the steady state is a set of sequences $\{\widehat{\pi}_t,\, \widehat{m}_t,\, \widehat{y}_t,\, \widehat{R}_t\}_{t=0}^{\infty}$ satisfying

$$\sigma\widehat{y}_t = \sigma E_t\widehat{y}_{t+1} - \widehat{R}_t + E_t\widehat{\pi}_{t+1}, \tag{6}$$

$$\widehat{\pi}_t = \omega\widehat{y}_t + \beta E_t\widehat{\pi}_{t+1} + \chi\widehat{\xi}_t, \tag{7}$$

$$\widehat{m}_t = (\sigma/\sigma_m)\widehat{y}_t - \left[\sigma_m\left(\overline{R} - 1\right)\right]^{-1}\widehat{R}_t, \tag{8}$$

taxes) to be equal to zero (see Alvarez et al. 2002), such that government solvency can be guaranteed at any off-equilibrium price level sequence, even if $B_{-1} + M_{-1} > 0$.

[13] Throughout the paper, we implicitly assume that the bounds on the mark-up fluctuations are sufficiently tight, such that the central bank can always ensure the nominal interest rate to be larger than one, $R_t > 1$.

where $\omega = \chi(\vartheta+\sigma)$, $\sigma = -u_{cc}(\overline{c})\overline{c}/u_c(\overline{c}) > 0$, $\vartheta = v_{ll}(\overline{l})\overline{l}/v_l(\overline{l}) > 0$, $\sigma_m = -\overline{m}\nu_{mm}(\overline{m})/\nu_m(\overline{m}) > 0$, and $\chi = (1-\phi)(1-\beta\phi)/\phi > 0$, the transversality condition, for a monetary policy, a sequence $\{\widehat{\xi}_t\}_{t=0}^{\infty}$, and given initial values for nominal balances $M_{-1}$ and the price level $P_{-1}$.[14]

In what follows, we consider three types of monetary policy regimes which are characterized by state contingent adjustments of the prevailing central bank instrument. The first monetary policy regime is characterized by a forward-looking reaction function for the risk-free nominal interest rate

$$\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t + \rho_\xi \widehat{\xi}_t. \tag{9}$$

The reaction function (9) allows for an explicit feedback from the exogenous state (the cost-push shock). This is the main difference to widely applied interest rate feedback rules, which are specified without a feedback from private sector shocks, $\rho_\xi = 0$.

For the second regime we consider a history dependent reaction function for the risk-free nominal interest rate, which allows for interest rates smoothing

$$\widehat{R}_t = \rho_R \widehat{R}_{t-1} + \rho_\pi^s \widehat{\pi}_t + \rho_y^s \widehat{y}_t + \rho_\xi^s \widehat{\xi}_t, \qquad \rho_R \in (0,1), \tag{10}$$

where $\rho_\pi^s/(1-\rho_R)$ and $\rho_y^s/(1-\rho_R)$ measure the long-run feedback from inflation and output. In contrast to (9), the inertial reaction function (10) features a feedback from past conditions, $\widehat{R}_{t-1}$. By adjusting the current interest rate contingent to changes in a lagged variable, central bank behavior becomes history dependent under (10).

Under the third regime, the central bank applies a reaction function for the money growth rate:

$$\widehat{\mu}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t + \mu_\xi \widehat{\xi}_t. \tag{11}$$

Since $\widehat{\mu}_t = \widehat{m}_t + \widehat{\pi}_t - \widehat{m}_{t-1}$, the reaction function (11) introduces beginning-of-period real balances, $\widehat{m}_{t-1}$, as a backward-looking element. Monetary policy might therefore be history dependent, even though past conditions are not considered as policy indicators (in contrast to 10).

The following lemma summarizes main properties of the fundamental solutions for the equilibrium sequences for the endogenous variables $\mathbf{x}_t' = [\widehat{\pi}_t\ \widehat{m}_t\ \widehat{y}_t\ \widehat{R}_t]$ for monetary policy satisfying (9), (10), or (11). It should be noted that the fundamental solution, i.e., the minimum state variable solution, is identical with the uniquely determined and stable solution in our framework.[15] The derivation of the conditions in part one and two of the lemma can be found in Woodford (2001, 2003a). The proof of the third part relates to

[14]Note that $\widehat{y}_t$ can be interpreted as a measure for the output gap (measured by output deviations from an efficient value), since any deviation of current output from its steady state value is induced by a distortionary shock.

[15]See McCallum (2004) for a comprehensive discussion of the relation between determinate solutions and the minimum state variable solution in rational expectations models.

the analysis in Schabert (2005) and is provided in appendix 7.1.

**Lemma 1** *Consider the fundamental solution for the equilibrium sequences* $\{\mathbf{x}_t\}_{t=0}^{\infty}$ *satisfying (6)-(8) and either (9), (10), or (11).*

1. *Under an interest rate policy (9) it takes the form* $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{\xi}_t)$*, and is the unique stable equilibrium solution if and only if* $\rho_\pi + [(1-\beta)/\omega]\rho_y > 1$.

2. *Under an inertial interest rate policy (10) it takes the form* $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{R}_{t-1}, \widehat{\xi}_t)$*, and is the unique stable equilibrium solution if and only if* $\rho_\pi^s + [(1-\beta)/\omega]\rho_y^s > 1 - \rho_R$*. Then, the single stable eigenvalue* $\delta_R = \partial\widehat{R}_t/\partial\widehat{R}_{t-1}$ *satisfies* $\delta_R \in (0,1)$.

3. *Under a money growth policy (11) it takes the form* $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{m}_{t-1}, \widehat{\xi}_t)$*, and is the unique stable equilibrium solution if and only if if and only if i.)* $\mu_\pi + \frac{1-\beta}{\omega}\mu_y < 1$ *and* $\mu_\pi + \mu_y\frac{1+\beta}{\omega} < 1 + 2\frac{\omega+\sigma(R+1)(\beta+1)}{(R-1)\omega\sigma_m}$ *leading to non-oscillatory equilibrium sequences,* $\delta_m \in (0,1)$ *where* $\delta_m = \partial\widehat{m}_t/\partial\widehat{m}_{t-1}$*, or ii.)* $\mu_\pi + \frac{1-\beta}{\omega}\mu_y > 1$ *and* $\mu_\pi + \mu_y\frac{1+\beta}{\omega} > 1 + 2\frac{\omega+\sigma(R+1)(\beta+1)}{(R-1)\omega\sigma_m}$ *leading to oscillatory equilibrium sequences,* $\delta_m \in (-1,0)$.

The properties summarized in the first two parts of lemma 1 are well established and are, therefore, not discussed in further detail. Subsequently, we will refer to the notion of an active (passive) interest rate policy, which is defined as an interest rate setting satisfying $\rho_\pi + [(1-\beta)/\omega]\rho_y > 1$ $(< 1)$ or $(\rho_\pi^s + [(1-\beta)/\omega]\rho_y^s)/(1-\rho_R) > 1$ $(< 1)$. According to lemma 1 part 3, the central bank is able to ensure the existence of a unique and stable (non-oscillatory) equilibrium if the response of money supply to a rise in output or inflation is sufficiently small, in the sense that $\mu_\pi + \frac{1-\beta}{\omega}\mu_y < 1$ and $\mu_\pi + \mu_y\frac{1+\beta}{\omega} < 1 + 2\frac{\omega+\sigma(R+1)(\beta+1)}{(R-1)\omega\sigma_m}$ are satisfied. If, however, money supply satisfies $\mu_\pi + [(1-\beta)/\omega]\mu_y > 1$, then any rise in inflation (or output) causes the central bank to raise the stock of nominal balances, which tends to increase the price level. Given that prices are not fully flexible, a rise in nominal balances is accompanied by a rise in real balances, which tends to lower the nominal interest rate by (8) and to raise aggregate demand by (6). Hence, monetary policy stimulates real activity and further increases inflation, such that self-fulfilling expectations or explosive equilibrium sequences are possible. The likelihood for explosiveness thereby increases with the price rigidity, i.e., with the fraction of firms that do not set prices in an optimal way.

It should be noted that condition $\mu_\pi + \frac{1-\beta}{\omega}\mu_y < 1$ ensures the existence of exactly one stable eigenvalue, which lies between zero and one, and therefore the existence and the uniqueness of stable and non-oscillatory equilibrium sequences. The condition $\mu_\pi + \mu_y\frac{1+\beta}{\omega} > 1 + 2\frac{\omega+\sigma(R+1)(\beta+1)}{(R-1)\omega\sigma_m}$, which is hardly binding for reasonable parameter values, further guarantees that there is no additional negative and stable eigenvalue, which would allow for an alternative stable solution characterized by equilibrium sequences that oscillate around the steady state.

## 4 Monetary policy under discretion

In this section we firstly characterize the central bank's plan under discretionary optimization and discuss the existence of multiple solutions to the plan. In the second part, we establish the existence of reactions functions of the type (9), (10), or (11) that implement the central bank's plan and examine their ability to solve the indeterminacy problem. Throughout the subsequent analysis, we repeatedly apply some standard parameter values for $\sigma$, $\vartheta$, $\beta$, $\phi$, and $\epsilon$ for demonstrative purposes. They are given in table 1. We set (the inverse of) the intertemporal substitution elasticities equal to two, $\sigma = \vartheta = \sigma_m = 2$, implying the income elasticity of money demand to equal one (see 8). We further set $\beta = 0.99$, $\epsilon = 6$, and $\phi = 0.8$; the latter being consistent with empirical evidence provided by Gali and Gertler (1999).[16] As an alternative, we consider a lower value for the fraction of non-optimizing price setters ($\phi = 0.5$), which might be more consistent with recent evidence from disaggregate US data (see Bils and Klenow, 2004). Finally, for the steady state velocity $\nu = \overline{y}/\overline{m}$ we use the value 2 for the benchmark case and, alternatively, the value 0.44, which is taken from Christiano et al. (2005).

**Table 1** Benchmark parameter values

| $\sigma$ | $\sigma_m$ | $\vartheta$ | $\beta$ | $\phi$ | $\epsilon$ | $\overline{\pi}$ | $\nu$ |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 0.99 | 0.8 | 6 | 1 | 2 |

### 4.1 Discretionary policy and equilibrium multiplicity

We now examine the plan of a central bank that aims to maximize social welfare. We realistically assume that the central bank does not have access to a technology which enables a commitment to a once-and-for-all policy. Thus, we assume that it aims to maximize social welfare in a discretionary way. We follow Woodford (2003a) and apply a linear-quadratic approximation of household welfare at the undistorted steady state. Since we want to abstract from long-run distortions due to monopolistic competition we assume that an unspecified (lump-sum financed) subsidy ensures $\Omega = 1$. We further assume that the long-run distortion due to transactions frictions is negligible.[17] Applying a second-order Taylor-expansion of household welfare and of the private sector equilibrium conditions at the undistorted steady state, leads to the following objective, as shown by

[16] Given these parameter values, the composite coefficients in (7) equal $\chi = 0.052$, and $\omega = 0.208$.

[17] This can be rationalized by an (unspecified) constant interest rate on money holdings $R^m$, which is set by the central bank in a way that minimizes welfare costs of money holdings in the steady state, $\overline{R}^m \to 1/\beta$ (see Woodford, 2003a). The steady state velocity would then relate to a long-run satiation level of money holdings, which is characterized by $\overline{m} = \overline{y}/\nu$.

Woodford (2003a, section 6.4.1)[18]

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U_t \approx \max \left[ \overline{U} - \Upsilon E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} \left( \hat{\pi}_t^2 + \alpha \widehat{y}_t^2 + \varphi \widehat{R}_t^2 \right) \right], \tag{12}$$

$$\text{where} \quad \alpha = \frac{\omega}{\epsilon}, \quad \text{and} \quad \varphi = \frac{1}{\sigma_m \nu} \frac{1}{\overline{R} - 1} \frac{1}{\sigma + \vartheta} \frac{\omega}{\epsilon},$$

and $\Upsilon > 0$. Applying the parameter values in table 1, the weights in the loss function are $\alpha = 0.0347$ and $\varphi = 0.215$. The loss function weight $\varphi$ on the interest rate variance, which provides a measure for the severity of the distortion induced by transactions frictions, is thus 6.2-times larger than the weight on output fluctuations for our benchmark parametrization.[19] The ratio of the weights $\varphi/\alpha$, which will be crucial for the subsequent analysis, is similar to Woodford's (2003b) value (4.9) and much smaller than Walsh's (2005) value (25.7). Evidently, the weight on the inflation variance is still larger than the other weights, indicating that the predominant distortion is induced by the price rigidity. In the subsequent section we will apply alternative values for the velocity $\nu$ and for the interest elasticity of money (induced by changes in $\sigma_m$), which alter the welfare costs of interest rate changes, and we change the fraction of non-optimizing firms $\phi$.

The central bank's problem under discretion can be summarized by a simple linear-quadratic set-up, where (12) serves as the policy objective and the linear equilibrium conditions (6)-(8) as constraints. Taking expectations as given, leads to the following first order conditions for all periods $t \geq 0 : \widehat{\pi}_t + \phi_{2t} = 0$, $\alpha \widehat{y}_t - \chi \sigma \phi_{2t} + \sigma \phi_{1t} = 0$, and $\varphi \widehat{R}_t + \phi_{1t} = 0$, where $\phi_{1t}$ and $\phi_{2t}$ denote the multiplier on the constraints (6) and (7), respectively. We can then define a central bank's plan as follows.

**Definition 2** *A central bank's plan is a set of sequences* $\{\widehat{\pi}_t, \widehat{m}_t, \widehat{y}_t, \widehat{R}_t\}_{t=0}^{\infty}$ *satisfying*

$$\sigma \varphi \widehat{R}_t = \alpha \widehat{y}_t + \omega \widehat{\pi}_t, \ \forall t \geq 0, \tag{13}$$

*(6)-(8), and the transversality condition, given* $\{\widehat{\xi}_t\}_{t=0}^{\infty}$ *and an initial value* $\widehat{m}_{-1}$.

According to the first part of lemma 1, a monetary policy satisfying (9) is associated with a unique equilibrium solution if and only if $\rho_\pi + \frac{1-\beta}{\omega}\rho_y > 1$. The central bank's first order condition (13), which is also called "targeting rule", implies the relation between the interest rate, inflation, and output to satisfy $\rho_\pi = \frac{\omega}{\sigma\varphi}$ and $\rho_y = \frac{\alpha}{\sigma\varphi}$ (as well as $\rho_\xi = 0$). The central bank's plan therefore exhibits a unique solution only if the weight $\varphi$ is sufficiently small. Otherwise, discretionary policy is associated with equilibrium multiplicity, which has also been shown by Brueckner and Schabert (2005) and Kurozumi (2005) for an isomorphic model and by Albanesi et al. (2003), King and Wolman (2004), and Siu (2005)

[18] This approximation of household welfare is for example also applied in Woodford (2003b), Brueckner and Schabert (2005), Kurozumi (2005), or Walsh (2005) for isomorphic models.

[19] The coefficient $\varphi$ would be equal to zero in a "cashless" version of this model (see Woodford, 2003a).

for models with different price setting schemes. The condition for the existence of multiple equilibria is summarized in the following lemma.

**Lemma 2** *The central bank's plan exhibits a unique stable solution if and only if $\varphi < \varphi^*$, where $\varphi^* = \frac{\omega + (1-\beta)/\epsilon}{\sigma}$. This solution takes the form $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{\xi}_t)$. If $\varphi > \varphi^*$, there further exist stable autoregressive solutions to the plan.*

According to lemma 2, the central bank's plan under discretion is associated with a unique solution if the distortion induced by transactions frictions are sufficiently small such that $\varphi < \varphi^*$.[20] Applying the parameter values in table 1, leads to a threshold equal to $\varphi^* = 0.105$. Hence, the benchmark value for the interest rate weight ($\varphi = 0.215$) clearly exceeds this threshold, indicating that there exist multiple solutions to the plan. When transactions frictions are non-negligible, the central bank is not willing to strongly stabilize inflation and the output gap, since the associated interest rate adjustments lead to welfare losses. If $\varphi > \varphi^*$, interest rates are adjusted in a passive way, $\rho_\pi + \frac{1-\beta}{\omega}\rho_y < 1$, which allows for multiple equilibria (see lemma 1).[21] Then, there exists a stable solution without any endogenous state variable, $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{\xi}_t)$, as well as stable autoregressive equilibrium solutions that feature a lagged endogenous state variable, $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{y}_{t-1}, \widehat{\xi}_t)$, $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{\pi}_{t-1}, \widehat{\xi}_t)$, $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{R}_{t-1}, \widehat{\xi}_t)$, or $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{m}_{t-1}, \widehat{\xi}_t)$. Further, there exist non-fundamental solutions, featuring an extraneous state variable, that allow for expectations to become self-fulfilling, i.e., for sunspot equilibria. In the subsequent analysis we will not apply the latter type of solutions.

## 4.2 Implementing the plan under discretionary optimization

In this section we take a closer look at the implementation of the central bank's plan given in definition 2. We examine if and how a central bank can implement its plan with reaction functions of the form (9), (10), and (11). In particular, we want to assess if the plan can be implemented in a stable and unique way, such that explosive equilibrium sequences and endogenous fluctuations are avoided. Evidently, a forward-looking interest rate reaction function (9) can uniquely implement the plan if $\varphi < \varphi^*$, since the central bank's first order condition (13) can be interpreted as a specific case with $\rho_\xi = 0$. If $\varphi > \varphi^*$, the central bank can design forward-looking reaction functions with $\rho_\xi \neq 0$ which uniquely implement its plan. The following proposition summarizes this result.

**Proposition 1** *Suppose that the central bank controls the interest rate in a forward-looking way. Then, there exist infinitely many reaction functions (9) which uniquely implement the central bank's plan. They are in general characterized by $\rho_\xi \neq 0$.*

[20] This corresponds to the result in Albanesi et al. (2003), who show that multiple equilibria can arise under discretion in a (non-linearized) sticky price model where transactions frictions are induced by a cash-in-advance constraint.

[21] For the benchmark parameter values, the targeting rule can be written as $\widehat{R}_t = 0.485 \cdot \widehat{\pi}_t + 0.08 \cdot \widehat{y}_t$.

**Proof.** The fundamental equilibrium solution under (9) is characterized by $\widehat{\pi}_t = \eta_1 \widehat{\xi}_t$ and $\widehat{y}_t = \eta_2 \widehat{\xi}_t$, and therefore $\widehat{\pi}_t = \frac{\eta_1}{\eta_2} \widehat{y}_t$. Lemma 1 part 1 then implies that for any $\zeta_\pi$ and $\zeta_y$ satisfying $(\zeta_\pi + \frac{\omega}{\sigma\varphi}) + [(1-\beta)/\omega](\zeta_y + \frac{\alpha}{\sigma\varphi}) > 1$ there exists an interest rate reaction function $\widehat{R}_t = (\zeta_\pi + \frac{\omega}{\sigma\varphi})\widehat{\pi}_t + (\zeta_y + \frac{\alpha}{\sigma\varphi})\widehat{y}_t - (\zeta_\pi \eta_1 + \zeta_y \eta_2)\widehat{\xi}_t$ that uniquely implements the fundamental solution. ■

Hence, regardless whether its plan exhibits a unique solution ($\varphi < \varphi^*$) or multiple solutions ($\varphi > \varphi^*$), the central bank can always uniquely implement the fundamental solution by choosing a particular forward-looking reaction function of the type (9). To be more precise, it can design a forward-looking reaction function, which is characterized by a feedback from inflation and output which is strong enough to rule out multiple solutions (by satisfying $\rho_\pi + [(1-\beta)/\omega]\rho_y > 1$). At the same time, an appropriate feedback from the exogenous state variable ensures that the implemented equilibrium is consistent with the fundamental solution to the plan.

When the central bank applies an inertial interest rate reaction function (10) this picture changes. If transactions frictions are very small such that $\varphi < \varphi^*$, there is a unique stable solution to the central bank's plan of the form $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{\xi}_t)$. According to lemma 1 part 2, the central bank can therefore not apply an inertial interest rate reaction function to implement its plan (in a stable way). If transactions frictions are sufficiently large, $\varphi > \varphi^*$, the central bank can in principle implement a stable set of sequences of the form $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{R}_{t-1}, \widehat{\xi}_t)$ which are consistent with its plan. An analysis of the feedback coefficients in (10) however shows that such an equilibrium is not uniquely determined.

**Proposition 2** *Suppose that the central bank controls the interest rate according to an inertial reaction function (10). If $\varphi < \varphi^*$, it cannot implement its plan in a stable way. If $\varphi > \varphi^*$, it cannot implement its plan in a unique way.*

**Proof.** Consider an inertial reaction function $\widehat{R}_t = \rho_R^* \widehat{R}_{t-1} + \rho_\pi^* \widehat{\pi}_t + \rho_y^* \widehat{y}_t + \rho_\xi^* \widehat{\xi}_t$ with $\rho_R^* \in (0,1)$, which implements a set of equilibrium sequences $\{\widehat{\mathbf{x}}_t^*\}_{t=0}^\infty$ consistent with the plan under discretionary optimization. According to lemma 1 part 2, the fundamental equilibrium solution then takes the form $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{R}_{t-1}, \widehat{\xi}_t)$. According to lemma 2, the sequences $\{\widehat{\mathbf{x}}_t^*\}_{t=0}^\infty$ are unstable if $\varphi < \varphi^*$ and stable if $\varphi > \varphi^*$ (see also appendix 7.4). Now suppose that $\varphi > \varphi^*$ (A1) and that $\rho_\pi^* + [(1-\beta)/\omega]\rho_y^* > 1 - \rho_R^*$ (A2) are satisfied. Then, the set of sequences $\{\widehat{\mathbf{x}}_t^*\}_{t=0}^\infty$ would be uniquely determined and stable, and the interest rate solution would read $\widehat{R}_t = \delta_R \widehat{R}_{t-1} + \delta_{Re} \widehat{\xi}_t$ with $\delta_R \in (0,1)$. Combining the latter with the reaction function would lead to the equilibrium relation $\widehat{R}_t = \frac{\rho_\pi^*}{1-\rho_R^*/\delta_R}\widehat{\pi}_t + \frac{\rho_y^*}{1-\rho_R^*/\delta_R}\widehat{y}_t + \frac{\rho_\xi^* - \rho_R \delta_{Re}/\delta_R}{1-\rho_R^*/\delta_R}\widehat{\xi}_t$. Given that the solution satisfies $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{R}_{t-1}, \widehat{\xi}_t)$, it follows immediately that for any given $\rho_R^* \neq \delta_R$ there exist exactly one set of coefficients $\{\rho_\pi^*, \rho_y^*, \rho_\xi^*\}$ that is consistent with the central bank's first order condition (13). These coefficients have to satisfy $\rho_\pi^*/(1-\rho_R^*/\delta_R) = \omega/(\sigma\varphi) > 0$, $\rho_y^*/(1-\rho_R^*/\delta_R) = \alpha/(\sigma\varphi) > 0$, and $\rho_\xi^* = \rho_R^* \delta_{Re}/\delta_R$. If $\rho_R^* > \delta_R$, the coefficients $\rho_\pi^*$ and $\rho_y^*$ have to be negative, which contradicts assumption (A2). If $\rho_R^* < \delta_R$, assumption (A2) implies that the coefficients satisfy $\frac{\rho_\pi^*}{1-\rho_R^*/\delta_R} + \frac{1-\beta}{\omega}\frac{\rho_y^*}{1-\rho_R^*/\delta_R} > 1$. This contradicts assumption (A1), which implies $\frac{\omega}{\sigma\varphi} + \frac{1-\beta}{\omega}\frac{\alpha}{\sigma\varphi} < 1$. Hence, if $\varphi > \varphi^*$ the central bank's plan cannot uniquely be implemented. ■

Proposition 2 indicates that the central bank's plan cannot be implemented by an inertial interest rate reaction function in a stable *and* unique way. If transactions frictions are sufficiently large such that $\varphi > \varphi^*$, the central bank's first order condition (13) requires passive (short-run) interest rate adjustments. In order to implement equilibrium sequences that are consistent with this behavior, an inertial reaction function (10) has to exhibit feedback coefficients that imply interest rates to be passively adjusted in the long-run, $\rho_\pi^s + [(1-\beta)/\omega]\rho_y^s < 1 - \rho_R$, which allows for further solutions that exhibit two endogenous state variables (see lemma 1 part 2). Thus, when the central bank applies an inertial interest rate reaction function to implement its plan in a history dependent way, it cannot avoid equilibrium multiplicity and therefore allows for endogenous fluctuations.

Now consider the case where the central bank uses lump-sum money transfers as its instrument and controls the money growth rate in a state contingent way (11). The minimum state variable solution for a rational expectations equilibrium then takes the form $\widehat{m}_t = \delta_m \widehat{m}_{t-1} + \delta_{me}\widehat{\xi}_t$, $\widehat{\pi}_t = \delta_{\pi m}\widehat{m}_{t-1} + \delta_{\pi e}\widehat{\xi}_t$, $\widehat{y}_t = \delta_{ym}\widehat{m}_{t-1} + \delta_{ye}\widehat{\xi}_t$, and $\widehat{R}_t = \delta_{Rm}\widehat{m}_{t-1} + \delta_{Re}\widehat{\xi}_t$ (see lemma 1 part 1). We want to assess whether there exists a money growth reaction function of the form (11) that can implement the central bank's plan.[22]

**Lemma 3** *Suppose that the central bank uses money transfers as its instrument. Then, there exists a money growth reaction function (11) that implements equilibrium sequences that are consistent with the plan. It satisfies*

$$\mu_\pi = \kappa_1(\mu_\pi, \mu_y), \quad \mu_y = \kappa_2(\mu_\pi, \mu_y), \quad \mu_\xi = \kappa_3(\mu_\pi, \mu_y, \mu_\xi), \tag{14}$$

*where* $\kappa_1 = 1 - \frac{1}{\sigma_m}(1 - \frac{\omega}{\sigma\varphi}\frac{\overline{R}-\delta_m}{\overline{R}-1}\frac{1}{\delta_m})$, $\kappa_2 = \frac{\alpha}{\sigma\varphi}\frac{\overline{R}-\delta_m}{\overline{R}-1}\frac{1}{\delta_m\sigma_m}$, $\kappa_3 = \frac{-1}{\rho\sigma_m}[(\overline{R}-1)^{-1}(\varkappa\delta_{Re} + \delta_{Rm}\delta_{me}) + \sigma_m\mu_y(\delta_{me}\delta_{ym} + \delta_{ye}\varkappa) + (\delta_{em}\delta_{\pi m} + \delta_{\pi e}\varkappa)((\mu_\pi - 1)\sigma_m - 1)]$, *and* $\varkappa = (\rho - \delta_m)$.

**Proof.** See appendix 7.2.

According to lemma 3, the central bank's plan can in principle be implemented by a money growth reaction function (11). It remains to analyze whether a money growth reaction function can implement the plan in a stable and unique way. The following proposition refers to the particular reaction function characterized in lemma 3.

**Proposition 3** *Suppose that the central bank implements its plan with a money growth reaction function satisfying (11) and (14). If $\varphi > \varphi^*$, the equilibrium sequences are stable, non-oscillatory, and uniquely determined. If $\varphi < \varphi^*$, they are unstable.*

[22] This analysis relates to Schabert (2005), where the implementation of interest rate targets via money supply adjustments is examined for different specifications of aggregate supply and for money demand.

**Proof.** See appendix 7.3.

A money growth reaction function of the type (11) can thus implement the central bank's plan in a stable, non-oscillatory, and unique way, if transactions frictions are sufficiently large ($\varphi > \varphi^*$). Otherwise ($\varphi < \varphi^*$), a money growth reaction function cannot implement the plan in a stable way, which corresponds to the case of inertial interest rate adjustments (see proposition 2). In contrast to the latter case, a central bank can avoid equilibrium multiplicity by applying the money growth rate as its instrument. Thus, a money growth policy ensures a unique determination of the plan while corresponding (passive) interest rate policies allow for multiple equilibria in our sticky price model. This result corresponds to the well-known property of money growth policy to facilitate nominal determinacy when prices are perfectly flexible (see Sargent and Wallace, 1975). While the predetermined value of beginning-of-period real balances serves as equilibrium selection criterion under a money growth reaction function (11), the mere introduction of the lagged interest rate as a policy indicator is not sufficient for this purpose.

## 5 Monetary instruments and social welfare

In this section we examine social welfare when the central bank applies different instruments in order to implement its plan under discretionary optimization. To compare the welfare implications of different monetary policy regimes, we focus on the case $\varphi > \varphi^*$ (see lemma 2). We restrict our attention to stable fundamental (minimum state variable) solutions which are characterized in lemma 1. To be more precise, in the case where the central bank sets the interest rate in a forward-looking way (9) or controls the money growth rate according to (11), we assume that it applies a particular reaction function for the prevailing instrument that ensures its plan to be implemented in a unique and stable way.[23] As shown in proposition 2, this is not possible for the case where the central bank applies an inertial interest rate reaction function (10).

The fundamental solution under an inertial interest rate policy reads $\widehat{\pi}_t = \eta_1\widehat{\xi}_t$, $\widehat{y}_t = \eta_2\widehat{\xi}_t$, $\widehat{R}_t = \eta_3\widehat{\xi}_t$ and $\widehat{m}_t = \eta_4\widehat{\xi}_t$ (see part 1 of lemma 1). If the central bank applies an inertial interest rate reaction function or a money growth reaction function, the equilibrium sequences become history dependent. Under an inertial interest rate reaction function, the solution takes the form $\widehat{R}_t = \rho_1\widehat{R}_{t-1} + \rho_2\widehat{\xi}_t$ , $\widehat{\pi}_t = \rho_3\widehat{R}_{t-1} + \rho_4\widehat{\xi}_t$, and $\widehat{y}_t = \rho_5\widehat{R}_{t-1} + \rho_6\widehat{\xi}_t$ (see lemma 1 part 2). Under a money growth reaction function it takes the form $\widehat{m}_t = \delta_1\widehat{m}_{t-1} + \delta_2\widehat{\xi}_t$, $\widehat{\pi}_t = \delta_3\widehat{m}_{t-1} + \delta_4\widehat{\xi}_t$, $\widehat{y}_t = \delta_5\widehat{m}_{t-1} + \delta_6\widehat{\xi}_t$, and $\widehat{R}_t = \delta_7\widehat{m}_{t-1} + \delta_8\widehat{\xi}_t$ (see lemma 1 part 3). The solution coefficients are derived in appendix 7.4.

Before we turn to the welfare comparison, which will be based on the unconditional variances of the endogenous variables, we briefly want to assess the difference between the

[23] The existence of such reaction functions have been established in proposition 1 for an interest rate regime and in proposition 3 for a money growth regime for $\varphi > \varphi^*$.

*conditional* variances of a forward-looking solution and of a history dependent solution to the central bank's plan. In particular, we compare the conditional variance of inflation which is implemented by a forward-looking interest rate reaction function, $var^{IR}\left(\widehat{\pi}_t^2\right) = \eta_1^2 var\left(\widehat{\xi}_t\right)$, to its counterpart under a money growth reaction function, $var^{MG}\left(\widehat{\pi}_t^2\right) = \delta_3^2 \widehat{m}_{t-1}^2 + \delta_4^2 var(\widehat{\xi}_t)$. Since, the solution coefficients (given in appendix 7.4) are in general too complex to compare these variances, we apply the simplifying parameter values $\sigma = 1$, $\sigma_m = 1$, $\vartheta = 0$, and $v = 1$. We then obtain tractable expressions for the limiting case where the discount factor converges to one $\beta \to 1$.[24] The ratio of the variances for the limiting case is then

$$\frac{\lim_{\beta\to 1} var^{MG}\left(\widehat{\pi}_t^2\right)}{\lim_{\beta\to 1} var^{IR}\left(\widehat{\pi}_t^2\right)} = \left(\frac{\delta_1}{\epsilon}\frac{\Delta}{(1-\rho)\chi}\right)^2 \frac{\widehat{m}_{t-1}^2}{var(\widehat{\xi}_t)} + \left[\frac{\Delta}{\Delta + \delta_1(\omega + 1 - \delta_1 + 1 - \rho)}\right]^2, \quad (15)$$

where $\Delta = \omega\rho - (1-\rho)^2$. Suppose that the autocorrelation of cost-push shocks is sufficiently large such that $\rho/(1-\rho)^2 > 1/\omega$. Then, $\Delta > 0$ and the term in the square brackets in (15) is smaller than one, given that the solution under the money growth reaction function is stable and non-oscillatory $\delta_1 \in (0,1)$. Thus, for any given value $\widehat{m}_{t-1}$, the inflation variance under a money growth policy can be smaller than under a forward-looking interest rate policy, $\frac{\lim_{\beta\to 1} var^{MG}(\widehat{\pi}_t^2)}{\lim_{\beta\to 1} var^{IR}(\widehat{\pi}_t^2)} < 1$, if the variance of the cost-push shock $var(\widehat{\xi}_t)$ is sufficiently large. If, however, the autocorrelation is small $\rho/(1-\rho)^2 < 1/\omega$, the inflation variance for a history dependent solution is always larger then for a purely forward-looking solution. Under a history dependent solution, the responses to a shock can be spread out over time and might not die out after the shock disappears. This effect tends to raise the variance, in particular, when shocks are not very persistent. If the autocorrelation of the common exogenous state is large, the macroeconomic responses to shocks can persist even if there is no endogenous state variable. If the variance of the exogenous state $var(\widehat{\xi}_t)$ is further large enough, then a history dependent solution can be associated with a smaller variance, as shock responses are smoothed. This principle also applies for the unconditional variances, which will be demonstrated in the subsequent welfare analysis.

For the welfare analysis we apply the second order approximation to household welfare (12). Since policy implementation is – by assumption – ensured to be steady state invariant, we use the welfare measure $E_0 \sum_{t=0}^{\infty} \beta^t L_t$, where $L_t = var(\widehat{\pi}_t) + \alpha var(\widehat{y}_t) + \varphi var(\widehat{R}_t)$ and $var(\widehat{x}_t)$ denotes the variance of a generic variable $\widehat{x}_t$. Let $var_x$ denote its *unconditional* variance, i.e., the variance conditional upon the state in period $t = 0$. We assume that the state in the initial period is identical with the steady state, such that

[24] For the limiting case $\beta \to 1$, the variances are given by $\lim_{\beta\to 1} var^{MG}(\widehat{\pi}_t^2) = (-\frac{\delta_1}{\epsilon})^2 \widehat{m}_{t-1}^2 + (\frac{-(1-\rho)\chi}{\omega\rho-(1-\rho)^2+\delta_1(\omega+1-\delta_1+1-\rho)})^2 var(\widehat{\xi}_t)$ and $\lim_{\beta\to 1} var^{IR}(\widehat{\pi}_t^2) = (\frac{-(1-\rho)\chi}{\omega\rho-(1-\rho)^2})^2 var(\widehat{\xi}_t)$.

$E_0 \sum_{t=0}^{\infty} \beta^t L_t = \sum_{t=0}^{\infty} \beta^t L$ where

$$L = var_\pi + \alpha var_y + \varphi var_R. \tag{16}$$

Since the discount factor is constant, $L$ provides a measure for the welfare ranking of allocations implemented by different policy regimes. Given the solution coefficients under the reaction functions (9)-(11), which are derived in appendix 7.4, we compute values for the variances.

The unconditional variances for the fundamental solution under a forward-looking interest rate reaction function are $var_\pi = \eta_1^2 var_\xi$, $var_y = \eta_2^2 var_\xi$, and $var_R = \eta_3^2 var_\xi$, where $var_\xi = (1-\rho^2)^{-1} var_e$ denotes the variance of the cost-push shock. The unconditional variances for the fundamental solution under an inertial interest rate reaction function are $var_\pi = \left(\rho_3^2 \rho_2^2 (1-\rho_1^2)^{-1} + \rho_4^2\right) var_\xi$, $var_y = \left(\rho_5^2 \rho_2^2 (1-\rho_1^2)^{-1} + \rho_6^2\right) var_\xi$, and $var_R = \rho_2^2 (1-\rho_1^2)^{-1} var_\xi$. The unconditional variances for the fundamental solution under a money growth policy are given by $var_\pi = \left(\delta_3^2 \delta_2^2 (1-\delta_1^2)^{-1} + \delta_4^2\right) var_\xi$, $var_y = \left(\delta_5^2 \delta_2^2 (1-\delta_1^2)^{-1} + \delta_6^2\right) var_\xi$, and $var_R = \left(\delta_7^2 \delta_2^2 (1-\delta_1^2)^{-1} + \delta_8^2\right) var_\xi$.

**Table 2** Welfare losses $L/var_\xi$ for alternative instruments

| $\rho$ | i.) Forward-looking interest rate policy | ii.) Backward-looking interest rate policy# | iii.) Forward-looking money growth policy |
|---|---|---|---|
| 0.95 | 0.016 | 0.0020 | 0.0019 |
| 0.9 | 0.079 | 0.0021 | 0.0021 |
| 0.8 | 0.57 | 0.0030 | 0.0032 |
| 0.7 | 0.049 | 0.0047 | 0.0051 |
| 0.6 | 0.020 | 0.0075 | 0.0082 |
| 0.5 | 0.011 | 0.012 | 0.013 |
| 0.4 | 0.0074 | 0.021 | 0.023 |

Note: The eigenvalue under ii.) and iii.) equals 0.83 and # indicates indeterminacy.

Table 2 presents relative welfare losses $L/var_\xi$ of the three equilibrium solutions for the parameter values in table 1. (The associated unconditional variances can be found in table A1 in appendix 7.5.) The results are reported for various values for the autocorrelation of cost-push shocks $\rho$. It should be noted that we present *relative* variances $var_x/var_\xi$, in order to abstract from changes in variances of endogenous variables that are solely due to changes in $var_\xi$ induced by different degrees of autocorrelation, $\rho$. The (relative) loss $L/var_\xi$ under the unique solution for a forward-looking interest rate policy *i.*) changes with $\rho$ in a non-monotonic way. For high values ($\rho > 0.8$) the relative loss decreases with $\rho$ since the variance of the cost-push shock $var_\xi$ rises more strongly with higher values for $\rho$ than the unconditional variances of endogenous variables. For $\rho < 0.8$, this effect is reversed. In contrast, under the history dependent solutions implemented by an inertial interest rate policy or a money growth policy, the relative loss monotonically decreases with $\rho$. These solutions exhibit a backward-looking element that is independent of the shock persistence, namely, an endogenous state variable with a non-zero eigenvalue (which equals 0.83 for the benchmark parameter values). As a consequence, the variances of endogenous variables are much less affected by $\rho$ than $var_\xi$. It should be noted that empirical evidence suggests the autocorrelation coefficient $\rho$ to be high. For example, Ireland's (2004) estimation of a similar model leads to $\rho = 0.95$.

Overall, the welfare losses for both history dependent solutions are closely related, though the loss under *ii.*) is almost always slightly smaller than under *iii.*). For $\rho = 0.95$, the relative loss (0.016) for a purely forward-looking solution induced by a forward-looking interest rate policy is clearly larger than those under an inertial interest rate policy (0.0020) and a money growth policy (0.0019). As demonstrated for the conditional inflation variances (see 15), forecast error variances can be reduced by the inclusion of a relevant lagged endogenous variable in the information set. Yet, unconditional variances can increase with the eigenvalues of endogenous variables, which enlarge the support of their distributions. Depending on whether the former or the latter effect dominates, a history dependent solution can, therefore, lead to higher or lower welfare losses. The latter effect becomes less relevant if the common (exogenous) state already exhibits a high eigenvalue $\rho$. For our benchmark parameterization, $\rho > 0.5$ is sufficient for this. Then, social welfare is higher when the central bank implements its plan in a history dependent way, i.e., by *ii.*) or *iii.*). If the autocorrelation coefficient $\rho$ is small (here $\rho \leq 0.5$), the welfare-reducing impact of the endogenous state on the variance of macroeconomic variables prevails, such that social welfare is higher under a forward-looking interest rate policy.

Table 3 further presents corresponding results for the case where the macroeconomic distortion due to transactions frictions is smaller. This is induced by setting $\sigma$ and $\sigma_m$ equal to 4 such that the interest elasticity of money demand is half as large, while the income elasticity still equals one. As a consequence, the loss function weight on output fluctuations rises to $\alpha = 0.07$, whereas the weight on interest rate fluctuations falls to $\varphi = 0.107$ (while $\varphi^*$ equals 0.104). Given that the distortion due to transactions frictions is less costly, a welfare gain from a history dependent implementation of the time consistent plan requires a higher value for $\rho$ than before. A forward-looking interest rate policy then leads to lower welfare losses if $\rho \leq 0.8$. We further examined the case where the steady state velocity $\nu$ is lowered to a value of 0.44, which is taken from Christiano et al. (2005). This evidently emphasizes the role of money and therefore the welfare costs of interest rate changes measured by $\varphi$ (see 12), while it leaves the private sector equilibrium conditions unaffected. The weight $\varphi$ then almost equals the weight on the inflation variance $\varphi = 0.98$, while $\alpha$ equals 0.035. A a consequence, the threshold for $\rho$ falls to 0.4 (see table A2 in appendix 7.5).[25]

**Table 3** Welfare losses $L/var_\xi$ for a lower interest rate elasticity ($\sigma = \sigma_m = 4$)

| $\rho$ | i.) Forward-looking interest rate policy | ii.) Backward-looking interest rate policy# | iii.) Forward-looking money growth policy |
|---|---|---|---|
| 0.95 | 0.44 | 0.017 | 0.017 |
| 0.9 | 0.12 | 0.029 | 0.030 |
| 0.8 | 0.037 | 0.057 | 0.058 |
| 0.7 | 0.020 | 0.092 | 0.094 |
| 0.6 | 0.012 | 0.14 | 0.15 |
| 0.5 | 0.0084 | 0.22 | 0.23 |
| 0.4 | 0.0061 | 0.37 | 0.38 |

Note: The eigenvalue under ii.) and iii.) equals 0.98 and # indicates indeterminacy.

To get an intuition for the effect of transactions frictions on the welfare ranking, suppose that the autocorrelation $\rho$ equals zero, such that $E_t\widehat{y}_{t+1} = E_t\widehat{\pi}_{t+1} = 0$ under an interest rate policy. Further consider a cost-push shock that tends to raise inflation. Then, a reduction in output (and inflation) requires a strong interest rate adjustment since the aggregate demand condition $\sigma(\widehat{y}_t - E_t\widehat{y}_{t+1}) = -(\widehat{R}_t - E_t\widehat{\pi}_{t+1})$ reduces to $\widehat{y}_t = -\sigma^{-1}\widehat{R}_t$. Under a history dependent solution, a reduction in current output implies $\widehat{y}_t < E_t\widehat{y}_{t+1} < 0 \Rightarrow |\widehat{y}_t - E_t\widehat{y}_{t+1}| < |\widehat{y}_t|$ in a stable and non-oscillatory equilibrium. As a consequence, smaller interest rate changes are required as long as monetary policy stabilizes expected inflation by applying small or negative values for $\mu_\pi$ and for $\mu_y$. Thus, the change in expectation formation can reduce the interest rate variance and, according to the central bank's first order condition (13), also the variances of the other endogenous variables. As a consequence, welfare losses can be reduced under a history dependent solution, while

[25] It should be noted that the relative losses under the history dependent solutions are then also not strictly decreasing in $\rho$, which is (partially) due to the lower eigenvalue 0.75.

the welfare gain increases with the interest elasticity of money demand and decreases with the velocity. Further, when prices are more flexible, future inflation is expected to return faster to its steady state value, which also tends to reduce the required increase in the nominal interest rate and, thus, welfare losses. When, for example, the fraction of non-optimizing price setters is set to a smaller value ($\phi = 0.5$), which might be more in line with evidence from disaggregate data (see Bils and Klenow, 2004), there is a welfare gain of a history dependent implementation even for an autocorrelation of $\rho = 0.4$ (see table 4).

**Table 4** Welfare losses $L/var_\xi$ under more flexible prices ($\phi = 0.5$)

| $\rho$ | i.) Forward-looking Interest rate policy | ii.) Backward-looking Interest rate policy# | iii.) Interest rate policy Money growth policy |
|---|---|---|---|
| 0.95 | 0.041 | 0.022 | 0.022 |
| 0.9 | 0.077 | 0.025 | 0.026 |
| 0.8 | 0.36 | 0.037 | 0.039 |
| 0.7 | 3.66 | 0.063 | 0.065 |
| 0.6 | 19.21 | 0.11 | 0.11 |
| 0.5 | 1.68 | 0.20 | 0.21 |
| 0.4 | 0.69 | 0.38 | 0.39 |

Note: The eigenvalue under ii.) and iii.) equals 0.63 and # indicates indeterminacy.

## 6 Conclusion

When money is held to reduce transactions costs, a central bank should abstain from strong adjustments of nominal interest rates. The latter might however be necessary for the stabilization of prices in an environment where price movements are associated with welfare costs. If the central bank acts in a discretionary way such a trade-off can lead to an optimal policy plan which fails to uniquely pin down an allocation and equilibrium price system. Or, put in terms of New Keynesian macroeconomics, the central bank's plan under discretionary optimization can imply interest rate adjustments that violate the Taylor-principle. Once a policy plan is consistent with multiple equilibria, different central bank operating procedures can be associated with different macroeconomic outcomes.

In this paper we apply a standard New Keynesian model and compare social welfare of different equilibrium solutions to the central bank's plan under different means of monetary policy implementation. The central bank either sets the nominal interest rate in a forward-looking way or in an inertial way, or adjusts the money stock via lump-sum transfers. Since the central bank acts under discretion, it does not account for its impact on private sector expectations such that its plan does not exhibit any backward-looking element. However, monetary policy can be history dependent if the central bank implements its plan by inertial interest rate adjustments or by money transfers. By providing a link to past conditions (i.e., to lagged interest rates or to the preexisting money stock), monetary policy alters the way private sector expectations are built and can thereby affect macroeconomic fluctuations. As responses to aggregate shocks are smoothed out, a history dependent monetary policy implementation can reduce welfare losses compared to an entirely forward-looking conduct of monetary policy. In particular, this welfare gain increases with the persistence of cost-push shocks, with the interest rate elasticity of money demand, and with the degree of price flexibility. However, the central bank can only avoid a history dependent equilibrium to exhibit real indeterminacy if it implements its plan by money supply adjustments. The predetermined stock of money then becomes a relevant state variable and serves as an equilibrium selection criterion. Correspondingly, the interest rate should be applied as the monetary policy instrument if transactions frictions are negligibly small compared to distortions induced by the price rigidity.

The results in this paper can further be interpreted in an alternative way. Studies on optimal monetary policy usually apply stylized models where the issue of policy implementation is not explicitly considered. While some real world central banks might be able to change interest rate targets by mere announcements, many central banks (including the US Federal Reserve) in principle implement operating targets by quantity adjustments in open market operations. A reduction of monetary policy to forward-looking interest rate adjustments can therefore overemphasize problems that originate in the lack of history dependence. These problems might in fact be less severe if one considers the underlying money supply behavior, which is in general not independent from past conditions, i.e., from the preexisting stock of money.

# 7 Appendix

## 7.1 Proof of lemma 1

To establish the claim made in the third part of the lemma, we eliminate the interest rate with the money demand condition (8). The model under a money growth policy can then be summarized by (6), the reaction function (11), and $R\sigma\widehat{y}_t = \sigma E_t\widehat{y}_{t+1} + (R-1)\,\sigma_m\widehat{m}_t + E_t\widehat{\pi}_{t+1}$, where we used $R = \overline{R}$ for convenience. The model can further be written as $(\widehat{m}_t \;\; E_t\widehat{\pi}_{t+1} \;\; E_t\widehat{x}_{t+1})' = A\,(\widehat{m}_{t-1} \;\; \widehat{\pi}_t \;\; \widehat{x}_t)' + (-\chi \;\; 0 \;\; 0)\,\widehat{\xi}_t$, where

$$A = \begin{pmatrix} 1 & \mu_\pi - 1 & \mu_y \\ 0 & \frac{1}{\beta} & -\frac{1}{\beta}\omega \\ \frac{\sigma_m(1-R)}{\sigma} & -\frac{1}{\sigma\beta} + \frac{\sigma_m(\mu_\pi-1)(1-R)}{\sigma} & R + \frac{\omega}{\sigma\beta} + \frac{\sigma_m\mu_y(1-R)}{\sigma} \end{pmatrix}.$$

The characteristic polynomial of $A$ is given by

$$Q(X) = X^3 + X^2\frac{(R-1)\,\beta\sigma_m\mu_y - \omega - [1+(R+1)\,\beta]\,\sigma}{\beta\sigma} \tag{17}$$
$$+X\frac{\omega + [1+(1+\beta)\,R]\sigma + \left(\omega - \mu_y - \omega\mu_\pi\right)(R-1)\sigma_m}{\beta\sigma} - \frac{R}{\beta}.$$

Given that the model exhibits one predetermined, $\widehat{m}_{t-1}$, and two jump variables, $\widehat{\pi}_t$ and $\widehat{c}_t$, stability and uniqueness of equilibrium sequences require exactly one stable eigenvalue. To derive the conditions therefore, we use that the value of $Q(X)$ at $X = 0 : Q(0) = -R\beta^{-1} < -1$. Thus, $\det(A) = -Q(0) > 1$ implying that there are either two or zero negative eigenvalues, and that there is at least one unstable eigenvalue. The existence of a stable root lying between zero and one, thus, requires $Q(1) > 0$. Examining $Q(X)$ at $X = 1$, which is given by

$$Q(1) = \frac{(R-1)\,\sigma_m}{\beta\sigma}\left(\omega\,(1-\mu_\pi) - (1-\beta)\,\mu_y\right),$$

reveals that the value $Q(1)$ depends on the elasticities $\mu_\pi$ and $\mu_y$ :

$$\mu_\pi + \frac{1-\beta}{\omega}\mu_y < 1.$$

While this ensures $X_1 \in (0,1)$ and thus the existence of a solution with stable and non-oscillatory equilibrium sequences, uniqueness additionally requires the remaining roots, $X_2$ and $X_3$, to lie outside the unit circle. For this, we assess $Q(X)$ at $X = -1$, which is given by

$$Q(-1) = \frac{1}{\beta\sigma}\left\{\sigma_m\,(R-1)\left[\mu_\pi\omega + \mu_y\,(1+\beta)\right] - [2+(R-1)\sigma_m]\,\omega - 2\sigma\,(R+1)\,(\beta+1)\right\}.$$

As $\det(\mathbf{A}) > 1$, two further stable roots (either complex or real) cannot exist, since they would necessarily lead to a determinant with an absolute value that is smaller than one.

Thus for $Q(-1) < 0$, there exists exactly one stable eigenvalue. This is ensured by

$$\mu_\pi + \mu_y \frac{1+\beta}{\omega} < 1 + 2\frac{\omega + \sigma\left(R+1\right)\left(\beta+1\right)}{\left(R-1\right)\omega\sigma_m}.$$

Hence, the equilibrium sequences are stable and uniquely determined if and only if $i.)$ $\mu_\pi + \frac{1-\beta}{\omega}\mu_y < 1$ and $\mu_\pi + \mu_y \frac{1+\beta}{\omega} < 1 + 2\frac{\omega+\sigma(R+1)(\beta+1)}{(R-1)\omega\sigma_m}$ leading to non-oscillatory equilibrium sequences, $X_1 \in (0,1)$, or $ii.)$ $\mu_\pi + \frac{1-\beta}{\omega}\mu_y > 1$ and $\mu_\pi + \mu_y \frac{1+\beta}{\omega} > 1 + 2\frac{\omega+\sigma(R+1)(\beta+1)}{(R-1)\omega\sigma_m}$ leading to oscillatory equilibrium sequences, $X_1 \in (-1,0)$. ■

## 7.2 Proof of lemma 3

To characterize the equilibrium behavior of nominal interest rates under a state contingent money growth policy $\widehat{\mu}_t = \widehat{m}_t + \widehat{\pi}_t - \widehat{m}_{t-1} = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t + \mu_\xi \widehat{\xi}_t$, we use the equilibrium condition $\overline{R}\sigma\widehat{y}_t = \sigma\widehat{y}_{t+1} + \left(\overline{R}-1\right)\sigma_m \widehat{m}_t + \widehat{\pi}_{t+1}$ and money demand $\sigma_m \widehat{m}_t + \frac{1}{\overline{R}-1}\widehat{R}_t = \sigma\widehat{y}_t$, to get $\sigma_m E_t \widehat{\mu}_{t+1} = \left(\sigma_m - 1\right) E_t \widehat{\pi}_{t+1} - \frac{1}{\overline{R}-1} E_t \widehat{R}_{t+1} + \frac{\overline{R}}{\overline{R}-1}\widehat{R}_t$, which together with the money growth reaction function (11) leads to,

$$\frac{\overline{R}}{\overline{R}-1}\widehat{R}_t - \frac{1}{\overline{R}-1}E_t\widehat{R}_{t+1} = \left(\sigma_m\mu_\pi - \left(\sigma_m - 1\right)\right) E_t\widehat{\pi}_{t+1} + \mu_y \sigma_m E_t \widehat{y}_{t+1} + \mu_\xi \sigma_m \rho \widehat{\xi}_t.$$

Now use that the fundamental solution under a money growth policy implies $E_t\widehat{R}_{t+1} = \delta_m \widehat{R}_t + \left(\left(\rho - \delta_m\right)\delta_{Re} + \delta_{Rm}\delta_{me}\right)\widehat{\xi}_t$. Thus, the current nominal interest rate is characterized by the following equilibrium relation

$$\begin{aligned}\widehat{R}_t = & \frac{\overline{R}-1}{\overline{R}-\delta_m}\left[\left(\sigma_m\mu_\pi - \left(\sigma_m-1\right)\right)E_t\widehat{\pi}_{t+1} + \mu_y\sigma_m E_t\widehat{y}_{t+1}\right] \\ & + \frac{\overline{R}-1}{\overline{R}-\delta_m}\left(\mu_\xi\sigma_m\rho + \frac{1}{\overline{R}-1}\left(\left(\rho-\delta_m\right)\delta_{Re} + \delta_{Rm}\delta_{me}\right)\right)\widehat{\xi}_t.\end{aligned}$$

Further using that $E_t\widehat{\pi}_{t+1} = \delta_m\widehat{\pi}_t + \left(\left(\rho-\delta_m\right)\delta_{\pi e} + \delta_{\pi m}\delta_{me}\right)\widehat{\xi}_t$, and $E_t\widehat{y}_{t+1} = \delta_m\widehat{y}_t + \left(\left(\rho-\delta_m\right)\delta_{ye} + \delta_{ym}\delta_{me}\right)\widehat{\xi}_t$, we can rewrite this expression as

$$\begin{aligned}\widehat{R}_t = & \left[\sigma_m\left(\mu_\pi - 1\right) + 1\right]\frac{\overline{R}-1}{\overline{R}-\delta_m}\delta_m\widehat{\pi}_t + \mu_y\sigma_m\frac{\overline{R}-1}{\overline{R}-\delta_m}\delta_m\widehat{y}_t \\ & + \frac{\overline{R}-1}{\overline{R}-\delta_m}\left\{\begin{array}{c}\mu_\xi\sigma_m\rho + \frac{(\rho-\delta_m)\delta_{Re}+\delta_{Rm}\delta_{me}}{\overline{R}-1} \\ +\left(\sigma_m\mu_\pi - \left(\sigma_m-1\right)\right)\left(\left(\rho-\delta_m\right)\delta_{\pi e} + \delta_{\pi m}\delta_{me}\right) + \mu_y\sigma_m\left(\left(\rho-\delta_m\right)\delta_{ye} + \delta_{ym}\delta_{me}\right)\end{array}\right\}\widehat{\xi}_t.\end{aligned}$$

We further know that there exists a unique value for $\mu_\xi^*$, such that the term in the curly brackets equals zero if $\mu_\xi = \mu_\xi^*$, since all solution coefficients in the curly brackets are either independent of $\mu_\xi$, such as $\delta_m$, $\delta_{Rm}$, $\delta_{\pi m}$, and $\delta_{\pi m}$, or are linear in $\mu_\xi^*$, such as $\delta_{me}$,

$\delta_{Re}$, $\delta_{ye}$, and $\delta_{\pi e}$. The value of $\mu_\xi^*$ is given by

$$\mu_\xi^* = \frac{-1}{\rho\sigma_m}\Big[\left(\overline{R}-1\right)^{-1}\left(\left(\rho-\delta_m\right)\delta_{Re}+\delta_{Rm}\delta_{me}\right)+\sigma_m\mu_y\left(\delta_{em}\delta_{my}+\delta_{ey}\left(\rho-\delta_m\right)\right) \qquad (18)$$
$$+\left(\delta_{em}\delta_{\pi m}+\delta_{e\pi}\left(\rho-\delta_m\right)\right)\left(\left(\mu_\pi-1\right)\sigma_m-1\right)\Big].$$

Then, we end up with an expression which takes the form of the first order condition (13). This imposes the following restrictions on the partial derivatives $\partial\widehat{R}_t/\partial\widehat{\pi}_t$ and $\partial\widehat{R}_t/\partial\widehat{y}_t$ :

$$\partial\widehat{R}_t/\partial\widehat{\pi}_t = \left[\sigma_m\left(\mu_\pi-1\right)+1\right]\Gamma\delta_m \quad \text{and} \quad \partial\widehat{R}_t/\partial\widehat{y}_t = \mu_y\sigma_m\Gamma\delta_m, \quad \text{where } \Gamma = \frac{\overline{R}-1}{\overline{R}-\delta_m},$$

which are satisfied by equilibrium sequences implemented by a money growth reaction function. Hence, a money growth reaction function satisfying (18),

$$\mu_\pi = 1-\frac{1}{\sigma_m}\left(1-\frac{\omega}{\sigma\varphi}\frac{\overline{R}-\delta_m}{\overline{R}-1}\frac{1}{\delta_m}\right), \quad \text{and} \quad \mu_y = \frac{\alpha}{\sigma\varphi}\frac{\overline{R}-\delta_m}{\overline{R}-1}\frac{1}{\delta_m\sigma_m},$$

implements a set of equilibrium sequences which are consistent with the plan. ■

## 7.3 Proof of proposition 3

From lemma 1 and 2 it follows immediately that a history dependent solution under a money growth reaction function (11) has to be unstable if $\varphi < \varphi^*$. It remains to examine the stability and uniqueness properties of this solution when $\varphi > \varphi^*$. For this we consider the characteristic polynomial of the model (6)-(8) and (11), which has been derived in the proof of lemma (1), $Q(X) = X^3 + X^2[(R-1)\beta\sigma_m\mu_y - \omega - \sigma\beta(1+R) - \sigma](\beta\sigma)^{-1} + X[\omega + \sigma(1+R(1+\beta)) + (R-1)\sigma_m(\omega-\mu_y-\omega\mu_\pi)](\beta\sigma)^{-1} - R/\beta$ (see 17). The roots $X$ of this polynomial are functions of the reaction function parameter $\mu_\pi$ and $\mu_y$. The values for the latter have to satisfy $\mu_\pi = 1+\frac{1}{\sigma_m}\left(\frac{\omega}{\sigma\varphi}\frac{R-X}{R-1}\frac{1}{X}-1\right)$ and $\mu_y = \frac{\alpha}{\sigma\varphi}\frac{R-X}{R-1}\frac{1}{X\sigma_m}$ (see lemma 3), in order to implement the central bank's plan, and they are functions of the particular eigenvalue. Eliminating the reaction function parameter with these conditions, we end up with the following cubic equation for the eigenvalues $X$:

$$Q(X) = 0 \Leftrightarrow 0 = X^3 - X^2\frac{\alpha\beta+\sigma\omega\varphi+\sigma^2\varphi+\sigma^2\beta\varphi+R\sigma^2\beta\varphi}{\varphi\sigma^2\beta}$$
$$+X\frac{\alpha+R\alpha\beta+R\sigma\omega\varphi+\omega^2+\sigma^2\varphi+R\sigma^2\varphi+R\sigma^2\beta\varphi}{\varphi\sigma^2\beta} - \frac{\alpha+\omega^2+\sigma^2\varphi}{\varphi\sigma^2\beta}R.$$

It can immediately be seen that $Q(0) = -\frac{R}{\sigma^2\beta\varphi}\left(\alpha+\omega^2+\sigma^2\varphi\right) < -1$, implying that the product of the eigenvalues exceeds one. Hence, there is at least one unstable eigenvalue and either no or two negative eigenvalues. Assessing the value of $Q(X)$ at $X=1$, $Q(1) = \frac{R-1}{\varphi\sigma^2\beta}\left(\sigma\omega\varphi-\alpha(1-\beta)-\omega^2\right)$, reveals that $\varphi > \varphi^* = \frac{\alpha}{\sigma}\frac{1-\beta}{\omega}+\frac{\omega}{\sigma} \Leftrightarrow Q(1) > 0$. Thus, there exists one stable eigenvalue $X_1$ if and only if $\varphi > \frac{\alpha}{\sigma}\frac{1-\beta}{\omega}+\frac{\omega}{\sigma}$. It satisfies $X_1 \in (0,1)$. We further use the second derivative of $Q(X)$ at $X = 1$, which is strictly negative

$$Q''(1) = -\frac{2}{\varphi\sigma^2\beta}[\alpha\beta + \sigma\varphi(\omega + \sigma(1-\beta) + (R-1)\sigma\beta)] < 0.$$

If the roots $X_2$ and $X_3$ are real, this evidently ensures the existence of exactly one stable eigenvalue. When the roots $X_2$ and $X_3$ are complex, they can be written as $X_2, X_3 = h \pm vi$. Using $X_1 + X_2 + X_3 = b$ for the cubic polynomial $Q(X) = X^3 + bX^2 + cX + d$, we further know that $h$ then satisfies $h = (b - X_1)/2$. Given that $Q''(1) = 6 + 2b < 0 \Leftrightarrow b < -3$, we know that $h > 1$ and that $X_2$ and $X_3$ are unstable, if $\varphi > \varphi^* \Leftrightarrow X_1 \in (0,1)$. Hence, under a money growth reaction function (14) there are two unstable and one stable (positive) eigenvalue if $\varphi > \varphi^*$. ■

### 7.4 Appendix to the solutions of the central bank's plan

**Forward-looking interest rate policy** If the central bank applies an interest rate reaction function of the form (9), the fundamental solution satisfies $\widehat{\mathbf{x}}_t = \mathbf{x}(\widehat{\xi}_t)$. Under the central bank's plan the equilibrium can be summarized by a two dimensional system in inflation and output satisfying (6) and $(\sigma - \frac{\alpha}{\sigma\varphi})\widehat{y}_t = \sigma E_t \widehat{y}_{t+1} + \frac{\omega}{\sigma\varphi}\widehat{\pi}_t + E_t\widehat{\pi}_{t+1}$. The generic solution for the coefficients derived above, thus reduce to $\widehat{\pi}_t = \eta_1\widehat{\xi}_t$, $\widehat{y}_t = \eta_2\widehat{\xi}_t$, and $\widehat{R}_t = \eta_3\widehat{\xi}_t$. These coefficients are given by $\eta_1 = (\alpha + \sigma^2\varphi - \sigma^2\rho\varphi)\chi F$, $\eta_2 = (\sigma\rho\varphi - \omega)\chi F$, and $\eta_3 = (\sigma\omega + \alpha\rho - \sigma\omega\rho)\chi F$, where $F = [\alpha - \alpha\beta\rho - \sigma\omega\rho\varphi + \omega^2 + \sigma^2\varphi(1 - \rho - \beta\rho + \beta\rho^2)]^{-1}$.

**Inertial interest rate policy** As for the previous regime, it is sufficient for our purpose to solve the equilibrium under an inertial interest rate reaction function (10) for the sequences of inflation, output, and the interest rate. In order to be consistent with the plan these sequences have to satisfy (6), (7), and (13). Eliminating output with the latter, $\widehat{y}_t = \frac{\sigma\varphi}{\alpha}\widehat{R}_t - \frac{\omega}{\alpha}\widehat{\pi}_t$, leads to the following set of equilibrium conditions for inflation and the interest rate

$$(1 + \sigma^2\varphi/\alpha)\widehat{R}_t - \sigma\frac{\omega}{\alpha}\widehat{\pi}_t = E_t\frac{\sigma^2\varphi}{\alpha}\widehat{R}_{t+1} + (1 - \sigma\omega/\alpha)E_t\widehat{\pi}_{t+1},$$
$$(1 + \omega^2/\alpha)\widehat{\pi}_t = \omega\frac{\sigma\varphi}{\alpha}\widehat{R}_t + \beta E_t\widehat{\pi}_{t+1} + \chi\widehat{\xi}_t.$$

The generic form of the minimum state variable solution for inflation and the interest rate under an inertial interest rate reaction function is given by $\widehat{R}_t = \rho_1\widehat{R}_{t-1} + \rho_2\widehat{\xi}_t$ and $\widehat{\pi}_t = \rho_3\widehat{R}_{t-1} + \rho_4\widehat{\xi}_t$. Applying these solutions, leads to the following set of conditions for the undetermined coefficients

$$0 = \sigma\omega\rho_3 - \alpha\rho_1 + \alpha\rho_1\rho_3 - \sigma\omega\rho_1\rho_3 - \sigma^2\varphi\rho_1 + \sigma^2\varphi\rho_1^2,$$
$$0 = \sigma\omega\varphi\rho_1 - \alpha\rho_3 + \alpha\beta\rho_1\rho_3 - \omega^2\rho_3,$$
$$0 = \sigma\omega\rho_4 - \alpha\rho_2 + \alpha\rho\rho_4 - \sigma\omega\rho\rho_4 + \alpha\rho_2\rho_3 - \sigma\omega\rho_2\rho_3 - \sigma^2\varphi\rho_2 + \sigma^2\rho\varphi\rho_2 + \sigma^2\varphi\rho_1\rho_2,$$
$$0 = \alpha\chi - \alpha\rho_4 + \alpha\beta\rho\rho_4 + \sigma\omega\varphi\rho_2 + \alpha\beta\rho_2\rho_3 - \omega^2\rho_4.$$

Combining the first two conditions, gives $\rho_3 = -\frac{\sigma\omega\varphi\rho_1}{-\alpha+\alpha\beta\rho_1-\omega^2}$ and the following condition

$$\left[\rho_1^2 - \rho_1\frac{\alpha\beta+\sigma\omega\varphi+\sigma^2\varphi+\sigma^2\beta\varphi}{\sigma^2\beta\varphi} + \frac{\alpha+\omega^2+\sigma^2\varphi}{\sigma^2\beta\varphi}\right]\rho_1 = 0.$$

One solution is evidently given by $\rho_1 = 0$, which leads to the previous forward-looking solution. To assess the existence of another solution, let $G(\rho_1)$ denote the quadratic polynomial in the square brackets. Since $G(0) = \beta^{-1} + (\alpha+\omega^2)/(\varphi\beta\sigma^2) > 1$ and $G(1) = [\alpha(1-\beta) + (\omega-\sigma\varphi)\omega]/(\varphi\beta\sigma^2)$, we can conclude that there exists exactly one stable and strictly positive root if and only if $\varphi > \varphi^*$ (see also proposition 2). The remaining conditions for the undetermined coefficients imply

$$\rho_2 = \alpha\frac{\sigma\omega+\alpha\rho-\sigma\omega\rho}{\alpha\beta\rho-\alpha-\omega^2}\frac{\chi}{\Xi}\left[\left(1-\frac{\sigma\omega+\alpha\rho-\sigma\omega\rho}{\alpha\beta\rho-\alpha-\omega^2}\left(\sigma\omega\varphi+\alpha\sigma\beta\omega\varphi\frac{\rho_1}{\omega^2+\alpha(1-\beta\rho_1)}\right)/\Xi\right)\right]^{-1},$$

$$\rho_4 = -\alpha\frac{\chi}{\alpha\beta\rho-\alpha-\omega^2}\left[\left(1-\frac{\sigma\omega+\alpha\rho-\sigma\omega\rho}{\alpha\beta\rho-\alpha-\omega^2}\left(\sigma\omega\varphi+\alpha\sigma\beta\omega\varphi\frac{\rho_1}{\omega^2+\alpha(1-\beta\rho_1)}\right)/\Xi\right)\right]^{-1},$$

$$\text{where } \Xi = \sigma^2\rho\varphi - \sigma^2\varphi - \alpha + \sigma^2\varphi\rho_1 + \sigma\omega\varphi\rho_1\frac{\alpha-\sigma\omega}{\omega^2+\alpha(1-\beta\rho_1)}.$$

The coefficients for the output solution $\widehat{y}_t = \rho_5\widehat{R}_{t-1}+\rho_6\widehat{\xi}_t$ can easily be derived by applying $\widehat{y}_t = \frac{\sigma\varphi}{\alpha}\widehat{R}_t - \frac{\omega}{\alpha}\widehat{\pi}_t$. They have to satisfy

$$\rho_5 = \frac{\sigma\varphi}{\alpha}\rho_1 - \frac{\omega}{\alpha}\rho_3, \quad \text{and} \quad \rho_6 = \frac{\sigma\varphi}{\alpha}\rho_2 - \frac{\omega}{\alpha}\rho_4,$$

which completes the minimum state variable solution under an inertial interest rate reaction function.

**Money growth policy** In order to derive the solution under a money growth reaction function that is consistent with the central bank's plan, we use the central bank's first order condition, $\sigma\varphi\widehat{R}_t = \alpha\widehat{y}_t+\omega\widehat{\pi}_t$, and money demand, $\widehat{R}_t = \sigma(R-1)\widehat{y}_t - \sigma_m(R-1)\widehat{m}_t$, to summarize the equilibrium by (6), and

$$R\sigma\widehat{y}_t = \sigma E_t\widehat{y}_{t+1} + (R-1)\sigma_m\widehat{m}_t + E_t\widehat{\pi}_{t+1},$$
$$\left(\varphi\sigma^2(R-1)-\alpha\right)\widehat{y}_t = \varphi\sigma\sigma_m(R-1)\widehat{m}_t + \omega\widehat{\pi}_t.$$

The fundamental solution under a money growth reaction function takes the form

$$\widehat{m}_t = \delta_1\widehat{m}_{t-1} + \delta_2\widehat{\xi}_t, \quad \widehat{\pi}_t = \delta_3\widehat{m}_{t-1} + \delta_4\widehat{\xi}_t, \quad \text{and} \quad \widehat{y}_t = \delta_5\widehat{m}_{t-1} + \delta_6\widehat{\xi}_t.$$

The set of equilibrium conditions in inflation, output, and real balances can be reduced, by eliminating output with $\widehat{y}_t = \psi_1\widehat{m}_t + \psi_2\widehat{\pi}_t$, where $\psi_1 = \frac{\varphi\sigma\sigma_m(R-1)}{(\varphi\sigma^2(R-1)-\alpha)}$ and $\psi_2 = \frac{\omega}{(\varphi\sigma^2(R-1)-\alpha)}$. Hence, the equilibrium can be summarized by the following two dimensional system in $\widehat{m}_t$ and $\widehat{\pi}_t$

$$(1-\omega\psi_2)\,\widehat{\pi}_t = \omega\psi_1\widehat{m}_t + \beta\widehat{\pi}_{t+1} + \chi\widehat{\xi}_t,$$
$$R\sigma\psi_2\widehat{\pi}_t = \sigma\psi_1\widehat{m}_{t+1} + (1+\sigma\psi_2)\,\widehat{\pi}_{t+1} + ((R-1)\,\sigma_m - R\sigma\psi_1)\,\widehat{m}_t.$$

Applying the solutions $\widehat{m}_t = \delta_1\widehat{m}_{t-1} + \delta_2\widehat{\xi}_t$ and $\widehat{\pi}_t = \delta_3\widehat{m}_{t-1} + \delta_4\widehat{\xi}_t$, we end up with the following set of conditions for the undetermined coefficients

$$\begin{aligned}
0 &= \beta\delta_1\delta_3 + \omega\delta_1\psi_1 - \delta_3\,(1-\omega\psi_2)\,,\\
0 &= \sigma\delta_1^2\psi_1 - R\sigma\delta_3\psi_2 + \delta_1\delta_3\,(\sigma\psi_2+1) + \delta_1\,(\sigma_m\,(R-1) - R\sigma\psi_1)\,,\\
0 &= \chi + \omega\delta_2\psi_1 + \beta\,(\rho\delta_4 + \delta_2\delta_3) - \delta_4\,(1-\omega\psi_2)\,,\\
0 &= \sigma\psi_1\,(\rho\delta_2 + \delta_1\delta_2) - R\sigma\delta_4\psi_2 + (\rho\delta_4 + \delta_2\delta_3)\,(\sigma\psi_2+1) + \delta_2\,(\sigma_m\,(R-1) - R\sigma\psi_1)\,.
\end{aligned}$$

The first two conditions can be combined to give $\delta_3 = \frac{\omega\delta_1\psi_1}{(1-\omega\psi_2)-\beta\delta_1}$ and the following condition, where $\delta_1$ is the eigenvalue of real balances:

$$\delta_1\left[\delta_1^2 - \delta_1\frac{\psi_1\,(\sigma+\omega+R\sigma\beta) + \beta\sigma_m\,(1-R)}{\sigma\beta\psi_1} - \frac{\sigma_m\,(R-1)\,(1-\omega\psi_2) - R\sigma\psi_1}{\sigma\beta\psi_1}\right] = 0.$$

Evidently, there is one solution characterized by a zero eigenvalue $\delta_1 = 0$. Let $K(\delta_1)$ denote the quadratic polynomial in the square brackets. As $K(\delta_1)$ is strictly positive at $\delta_1 = 0$, $K(0) = \varphi^{-1}\beta^{-1}\sigma^{-2}(\alpha+\omega^2+\sigma^2\varphi) > 1$, and satisfies $K(1) = \varphi^{-1}\beta^{-1}\sigma^{-2}(\alpha-\alpha\beta-\sigma\omega\varphi+\omega^2)$, we can conclude that there exists exactly one stable and non-zero root of $K(\delta_1)$, if and only if $\varphi > \varphi^*$. Thus, when this condition is satisfied, the solution with $\delta_1 > 0$ is stable and uniquely determined. Combining the remaining two equations, we end up with the following conditions for the coefficients $\delta_2$ and $\delta_4$ :

$$\delta_2 = -\frac{\chi\,(\rho\,(\sigma\psi_2+1) - R\sigma\psi_2)}{\psi_3}, \qquad \delta_4 = \frac{\chi + \delta_2\beta\delta_3 + \delta_2\omega\psi_1}{1-\beta\rho-\omega\psi_2},$$
$$\text{where } \psi_3 = \begin{bmatrix} (\sigma\psi_1\,(\rho+\delta_1-R) + \sigma_m\,(R-1) + \delta_3\,(\sigma\psi_2+1))\,(1-\omega\psi_2-\beta\rho) \\ +(\rho\,(\sigma\psi_2+1) - R\sigma\psi_2)\,(\beta\delta_3+\omega\psi_1) \end{bmatrix}.$$

In order to solve for output we apply $\widehat{m}_t = \delta_1\widehat{m}_{t-1} + \delta_2\widehat{\xi}_t$ and $\widehat{\pi}_t = \delta_3\widehat{m}_{t-1} + \delta_4\widehat{\xi}_t$, leading to $\widehat{y}_t = \delta_5\widehat{m}_{t-1} + \delta_6\widehat{\xi}_t$, where

$$\delta_5 = \delta_3\psi_2 + \delta_1\psi_1, \quad \text{and} \quad \delta_6 = \delta_4\psi_2 + \delta_2\psi_1.$$

Finally, we solve for the interest rate using $\widehat{R}_t = \sigma\,(R-1)\,\widehat{y}_t - \sigma_m\,(R-1)\,\widehat{m}_t$ to give the solution for the nominal interest rate $\widehat{R}_t = \delta_7\widehat{m}_{t-1} + \delta_8\widehat{\xi}_t$, where

$$\delta_7 = (R-1)\,(\sigma\delta_5 - \sigma_m\delta_1)\,, \quad \text{and} \quad \delta_8 = (R-1)\,(\sigma\delta_6 - \sigma_m\delta_2)\,.$$

This completes the solution under a money growth reaction function (11) and (14).

### 7.5 Monetary instruments and social welfare (further results)

**Table A1** Variances for alternative instruments

| $\rho$ | i.) Forward-looking Interest rate policy | | | ii.) Forward-looking Interest rate policy$^{\#}$ | | | iii.) Backward-looking Money growth policy | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\frac{var_\pi}{var_\xi}$ | $\frac{var_y}{var_\xi}$ | $\frac{var_R}{var_\xi}$ | $\frac{var_\pi}{var_\xi}$ | $\frac{var_y}{var_\xi}$ | $\frac{var_R}{var_\xi}$ | $\frac{var_\pi}{var_\xi}$ | $\frac{var_y}{var_\xi}$ | $\frac{var_R}{var_\xi}$ |
| 0.95 | 0.012 | 0.080 | 0.0057 | $7.6/10^4$ | 0.033 | $2.8/10^4$ | $6.1/10^4$ | 0.035 | $3.3/10^4$ |
| 0.9 | 0.069 | 0.15 | 0.025 | 0.0010 | 0.028 | $4.5/10^4$ | $9.9/10^4$ | 0.029 | $5.1/10^4$ |
| 0.8 | 0.53 | 0.23 | 0.15 | 0.0021 | 0.022 | $8.6/10^4$ | 0.0022 | 0.023 | $9.6/10^4$ |
| 0.7 | 0.046 | 0.0046 | 0.012 | 0.0037 | 0.020 | 0.0014 | 0.0041 | 0.021 | 0.0016 |
| 0.6 | 0.019 | $3.2/10^4$ | 0.0046 | 0.0063 | 0.020 | 0.0023 | 0.0069 | 0.021 | 0.0025 |
| 0.5 | 0.011 | $0.21/10^5$ | 0.0025 | 0.011 | 0.022 | 0.0037 | 0.012 | 0.023 | 0.0040 |
| 0.4 | 0.0071 | $3.1/10^5$ | 0.0016 | 0.019 | 0.028 | 0.0063 | 0.021 | 0.030 | 0.0068 |

Note: The eigenvalue under ii.) and iii.) equals 0.83 and # indicates indeterminacy.

**Table A2** Welfare losses $L/var_\xi$ for a smaller velocity ($\nu = 0.44$)

| $\rho$ | i.) Forward-looking Interest rate policy | ii.) Backward-looking Interest rate policy$^{\#}$ | iii.) Forward-looking Money growth policy |
|---|---|---|---|
| 0.95 | 0.0037 | 0.0018 | 0.0021 |
| 0.9 | 0.0096 | 0.0017 | 0.0021 |
| 0.8 | 0.16 | 0.0018 | 0.0028 |
| 0.7 | 0.25 | 0.0025 | 0.0043 |
| 0.6 | 0.035 | 0.0039 | 0.0067 |
| 0.5 | 0.015 | 0.0064 | 0.011 |
| 0.4 | 0.0089 | 0.011 | 0.019 |

Note: The eigenvalue under ii.) and iii.) equals 0.75 and # indicates indeterminacy.

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