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ABSTRACT

Optimal Procurement When Both Price and Quality Matter*

A buyer seeks to procure a good characterized by its price and its quality from suppliers who have private information about their cost structure (fixed cost + marginal cost of providing quality). We solve for the optimal buying mechanism, i.e. the procedure that maximizes the buyer's expected utility, and discuss its properties. Many of the properties of the optimal buying mechanism when information is one-dimensional (Laffont and Tirole, 1987; Che, 1993) no longer hold when we introduce private information about the fixed costs. We compare the performance of the optimal scheme to that of buying procedures used in practice, namely a quasilinear scoring auction and negotiation. Specifically, we characterize an upper bound to what a quasilinear scoring auction and negotiation can achieve, and compare the performance of these procedures numerically. Quasilinear scoring auctions are able to extract a good proportion of the surplus from being strategic. Negotiation does less well. In fact, our results suggest that negotiation does worse than holding a simple scoring auction where the buyer reveals his preference.

JEL Classification: C78, D44, D82, L22 and L24
Keywords: bargaining, differentiated product, multi-attribute auction, multidimensional screening, negotiation, optimal auction, procurement and scoring auction

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1 Introduction

Procurement rarely involves considerations solely based on price. Instead, concerns about the quality of the good or service provided are often important to the final decision. In this paper, we consider how a buyer who cares about quality should structure his purchasing process in order to maximize his expected utility when suppliers compete for a single procurement contract.

The two distinguishing features of our model are that suppliers' private information about their cost structure is multidimensional and that quality is endogenously determined as part of the procurement process. US State Highway Authorities' procurement for highway repair jobs illustrates these aspects of the contracting environment.[1] For high density traffic areas, these agencies care about the cost of the job and the time in which the job will be completed. A contractor may be able to speed up the job by hiring extra labor, using some equipment more intensively, or shifting some resources from other jobs. Hence, suppliers' quality (here, the time they need to complete the job) is not fixed but is endogenous, with increased quality incurring a higher cost. Moreover, the marginal cost of quality is likely to vary across potential contractors in a way that is not observable to their competitors. Therefore, it represents one dimension of private information. However, there are other sources of unobserved cost heterogeneity. These include the contractors' material costs, existing contractual obligations and its organizational structure, which combine to determine the fixed cost of undertaking a job at any quality level. Thus, private information is likely to be better captured by a multidimensional parameter.

We examine the optimal procurement mechanism in a model where each potential supplier has private information about two components of her cost structure: her fixed cost and her marginal cost of providing quality. Costs on each dimension can be high or low, and we allow for any pattern of correlation between a supplier's fixed cost and her marginal cost. Across bidders, costs are independently distributed: a supplier's cost realization is independent of her competitors' cost realizations. The buyer's objective is to maximize his expected utility subject to the suppliers' participation and incentive compatibility constraints.

The first contribution of this paper is to characterize the optimal buying mechanism in this multidimensional private information environment (Theorem 1). Laffont and Tirole (1987) and Che (1993) characterize the optimal buying mechanism when private information is one-dimensional (the marginal cost of providing quality). Under some regularity conditions, Che (1993) shows that the optimal buying scheme distorts the quality provided by the suppliers downwards relative to their first best levels. By acting as if he does not care much about quality, the buyer reduces the

[1] See for instance Arizona Department of Transport (2002) and Herbsman et al. (1995).

differentiation between suppliers and thus increases the level of competition among them. The optimal level of distortion is independent of the number of suppliers, a property known as the "separation between screening and selection" (Laffont and Tirole, 1987). In addition, except for the presence of a reserve price, the contract is always allocated efficiently. Finally, Che shows that a scoring auction with a scoring rule that is linear in price implements the optimal scheme.[2]

The optimal scheme when private information is multidimensional differs significantly from the optimal scheme when private information is one-dimensional. First, the optimal scheme depends on the number of bidders, i.e. the separation property no longer holds. The reason is that the ordering of types, which determines the optimal quality distortion, is endogenous in multidimensional environments. The number of suppliers, through its effect on the probabilities of winning, affects which incentive compatibility constraints bind most and therefore the ordering of types.

Second, the optimal scheme involves two sources of inefficiencies: a productive inefficiency because quality levels differ from their first best levels, and an allocative inefficiency because the probabilities of winning differ from the first best probabilities of winning. This is in contrast with the optimal scheme in one–dimensional environments, which only involves a productive inefficiency. As in the one-dimensional scheme, the qualities of the suppliers with the high marginal cost for quality are usually distorted downwards. In addition, the optimal scheme sometimes allocates the contract to a high marginal cost supplier over a low marginal cost supplier where efficiency would require the opposite. The optimal scheme in one-dimensional environments is sometimes summarized as involving a "bias against quality." This shortcut is misleading in multidimensional environments because the optimal scheme here sometimes requires a high marginal cost supplier to win more often than what is dictated by efficiency.

Third, a quasilinear scoring auction cannot in general implement the optimal scheme in multidimensional environments. The reason is that the optimal scheme generically requires that the high marginal cost suppliers produce different levels of quality, depending on their fixed costs. In contrast, a quasilinear scoring auction induces the same level of quality for suppliers with the same marginal costs.

The optimal buying mechanism is complex. It depends finely on the exact parameters of the problem (cost structure and distribution of costs) and is not easily amenable to implementation by a simple-looking auction format. This suggests that, for practical purposes, simpler buying procedures that perform well in a variety of environments might be more useful.

[2]A scoring auction is an auction where bidders submit bids on several dimensions (price and quality); the scoring rule reduces these bids into one dimension: the score; and the bidder with the highest score wins. A scoring auction is said to be quasilinear when it uses a scoring rule linear in price (or an increasing monotonic transformation thereof).

The second contribution of this paper is to evaluate the relative performance of such simpler procedures against the benchmark of the optimal mechanism. We consider three procedures: the buyer-optimal efficient auction, a quasilinear scoring auction and negotiation. The buyer-optimal efficient auction is the efficient auction that maximizes the buyer's expected utility. It can be implemented by a scoring auction where the scoring rule corresponds to the true preference of the buyer. A quasilinear scoring auction is a scoring auction where the scoring rule is linear in price. We characterize the allocations that can be implemented by a quasilinear scoring auction (Theorem 2) and use this result to generate an upper bound to what quasilinear scoring auctions can achieve. Finally, negotiation is favored by many professional buyers when quality matters. The term itself refers to various more or less structured procedures. We take the idea that suppliers do not compete directly with one another as the key distinguishing feature of negotiation and characterize an upper bound to what any negotiation procedure can achieve in our environment (Theorem 3).

By definition, all three procedures underperform against the optimal buying mechanism. Our numerical results suggest the following about the relative performance of these simpler mechanisms. First, quasilinear scoring auctions capture a significant proportion of the difference between the buyer's expected utility from the optimal mechanism and that from the buyer-optimal efficient mechanism, which represents the surplus available from being a strategic buyer. Second, negotiation performs poorly. In fact, we find that the buyer is better off holding an efficient auction than negotiating. This is a significant result given the emphasis on negotiation in practice, and the simplicity of the efficient auction alternative. Third, as the number of suppliers increases, the buyer's expected utility in all three procedures converge (as expected) to that of the optimal buying mechanism. However, this happens at different speeds. Most interestingly, the optimal quasilinear auction converges quickly to the optimal mechanism and faster than the other mechanisms considered.

The third contribution to this paper is to the multidimensional screening literature. Rochet and Stole (2003) present a recent survey of the contracting applications of multidimensional screening. Auction applications include the optimal multi-unit auction problems studied by Armstrong (2000), Avery and Henderschott (2000), Manelli and Vincent (2004) and Malakhov and Vohra (2004), and the optimal auction with externalities studied by Jehiel et al. (1999). Unlike contracting environments, our problem involves a resource constraint given that the contract can only be allocated to one supplier. Unlike multi-unit auction environments, quality in our problem introduces some non-linearity. Hence, none of the existing characterization results applies to our problem and the method we use to solve for the solution is somewhat different from the methods used in these papers (even if the underlying principle is the same). A straightforward variant of our model is one of single-agent single-principal contracting where stochastic contracts have been conjectured to improve on non stochastic contracts. We argue in section 6 that our results imply that such contracts cannot improve on non stochastic contracts, absent competition.

This paper is related to the literature on procurement. This literature is organized around several themes, including the question of how to take factors other than price into account in the procurement process (Laffont and Tirole, 1987, Che, 1993, Branco, 1997, Ganuza and Pechlivanos, 2000, Rezende, 2003, de Frutos and Pechlivanos, 2004), the impact of the potential non-contractibility of quality (Manelli and Vincent, 1995, Morand and Thomas, 2002), and the impact of moral hazard and renegotiation (Bajari and Tadelis, 2001, Bajari, McMillan and Tadelis, 2004).

Our paper fits squarely into the first group and we abstract from the other issues. Our contribution to this literature is twofold. First, we extend prior analyses of optimal procurement to the richer environment where private information is multidimensional. Our analysis shows that the assumption of one-dimensional private information is not a trivial one: many properties of the optimal procurement process no longer holds when we move to multidimensional settings. Second, we evaluate existing buying procedures against the benchmark of the optimal scheme. Other papers compare the performance of different procedures: Dasgupta and Spulber (1989), Che (1993) and Chen-Ritzo et al. (2003) compare the scoring auction, which turns out to be optimal in their setting, with price-only auctions, Asker and Cantillon (2004) compare the scoring auction with price-only auctions and menu auctions, Manelli and Vincent (1995) and Bulow and Klemperer (1996) compare (two different models of) negotiation with auctions. Except for Asker and Cantillon (2004), all these papers are restricted to one-dimensional private information.

2 Model

2.1 Environment

We consider a buyer who wants to buy an indivisible good for which there are N potential suppliers. The good is characterized by its price, p, and its quality, q.

Preferences. The buyer values the good (p, q) at $v(q) - p$, where $v_q > 0$ (we assume that $v_q(0) = \infty$ and $\lim_{q\to\infty} v_q(q) = 0$ to ensure an interior solution) and $v_{qq} < 0$. Supplier i's profit from selling good (p, q) is given by $p - \theta_1^i - \theta_2^i q$, where $\theta_1^i \in \{l, h\}$ and $\theta_2^i \in \{L, H\}$ ($l < h$ and $0 < L < H$.)[3] Given the binomial support of θ_1 and θ_2, there are four supplier types: (h, H), (l, H), (h, L), (l, L),

[3] The linearity of costs in quality ensures that the buyer's optimization problem is concave - see Lemma 1 and the accompanying discussion.

which we denote for brevity hH, lH, hL and lL. We will sometimes use $(\theta_{1k}, \theta_{2k})$ to denote supplier type k. For example, $(\theta_{1lH}, \theta_{2lH}) = (l, H)$. Note that the buyer and the suppliers are risk neutral.

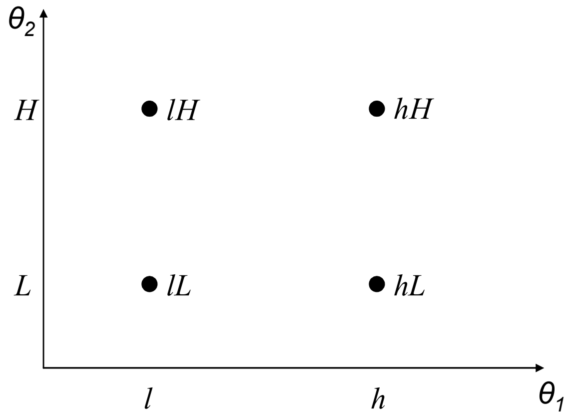


Figure 1: Cost structure

Social welfare. Let $W_k(q) = v(q) - \theta_{1k} - \theta_{2k}q$, the social welfare associated with giving the contract to type k with quality q. Define $W_k^{FB} = \max_q W_k(q)$. Given the single crossing condition, $q_{lH}^{FB} = q_{hH}^{FB} < q_{hL}^{FB} = q_{lL}^{FB}$ (to save on notation we will use the short-hand notation $\overline{q}$ and $\underline{q}$ to describe the first best levels of qualities, $\overline{q} < \underline{q}$). An expression that will play a role in the analysis is $W_{lH}(q) - W_{hL}(q)$. Its derivative is negative, $\frac{d}{dq}[W_{lH}(q) - W_{hL}(q)] < 0$.

Our assumptions thus far yield an incomplete ordering of types in terms of the first best levels of welfare they generate. To simplify the analysis, we restrict attention to the case where $W_{lH}^{FB} < W_{hL}^{FB}$. Under this specification, having a low marginal cost for delivering a higher quality product is more important than having a low fixed cost, at least in a first best scenario. This case includes, as a limit, the case where firms only differ in their marginal cost parameter, which has been studied by Laffont and Tirole (1987), Che (1993) and Branco (1997). The natural ordering of types is thus $lL \succ hL \succ lH \succ hH$.

Information. Preferences are common knowledge among suppliers and the buyer, with the exception of suppliers' types, $(\theta_{1i}, \theta_{2i})$, $i = 1, ..., N$, which are privately observed by each supplier. Types are independently and symmetrically distributed across suppliers, in the sense that the probability of supplier i being of some type is independent of other suppliers' types, but the ex-ante distribution of types is the same for all bidders. Thus we can write the probability of each type in the population is given by $\alpha_k > 0$, $k \in \{hH, lH, hL, lL\}$. Notice that we do not put any restriction on the α_k's except for the fact that they need to sum to one. Any pattern of correlation among a supplier's fixed cost and her marginal cost is allowed.

2.2 An optimal buying mechanism

The buyer's problem is to find a mechanism that maximizes his expected utility from the procurement process. For simplicity, we assume that the buyer buys with probability one (that is, we assume non exclusion).[4] A direct revelation mechanism in this setting is a mapping from the announcements of all suppliers, $\{\theta_{1i}, \theta_{2i}\}_{i=1}^{N}$, into probabilities of getting the contract, the level of quality to deliver and a money transfer.

Given that the buyer's preference over quality levels is strictly concave, there is no loss of generality in restricting attention to quality levels that are only a function of suppliers' types. Let q_k denote the quality level to be delivered by a type k supplier. This, together with suppliers' risk neutrality, implies that suppliers' payoffs and thus behavior only depends on their *expected* probabilities of winning and their *expected* payment. Let x_k be the probability of winning the contract conditional on being type k and let m_k the expected payment she receives. Finally, let U_k denote type k's equilibrium expected utility. We have: $U_k = m_k - x_k(\theta_{1k} + \theta_{2k}q_k)$.

With these simplifications and notation, the buyer's expected utility from the mechanism is given by

$$F(x_k, q_k, U_k) = N \sum_{k\in\{hH,lH,hL,lL\}} \alpha_k(x_k W_k(q_k) - U_k) \tag{1}$$

The buyer seeks to maximize this expression over contracts (x_k, q_k, U_k), subject to suppliers' incentive compatibility (IC) constraints:

$$U_k \geq U_j + x_j(\theta_{1j} - \theta_{1k}) + x_j q_j(\theta_{2j} - \theta_{2k}) \qquad \text{for all } k, j \in \{hH, lH, hL, lL\}, \tag{2}$$

individual rationality (IR) constraints:

$$U_k \geq 0 \qquad \text{for all } k \in \{hH, lH, hL, lL\}, \tag{3}$$

and subject to the feasibility constraint that the probability of awarding the contract to a subset of the types is always less than or equal to the probability of such types in the population:

$$N \sum_{k\in K} \alpha_k x_k \leq 1 - (1 - \sum_{k\in K} \alpha_k)^N \text{ for all subsets } K \text{ of } \{hH, lH, hL, lL\} \tag{4}$$

Finally, non exclusion imposes that

$$N \sum_{k\in\{hH,lH,hL,lL\}} \alpha_k x_k = 1 \tag{5}$$

[4] This is not particularly restrictive. It is easy to find parameter values such that all virtual welfares in the solution are positive making exclusion non-optimal.

Border (1991) guarantees that the feasibility constraint is both necessary and sufficient for the expected probabilities x_k to be derived from a real allocation mechanism. This ensures that the solution to the maximization problem of (1) subject to (2), (3), (4) and (5) is implementable.

3 A heuristic approach

Formally, our design problem belongs to the class of multidimensional screening problems. The difficulty in such problems is that the "ordering" of types, or, more precisely, the direction in which the IC constraints are binding is endogenous. In this section, we describe our approach and explore the economic trade-offs underlying the optimal buying procedure.

Our approach is to start with the Buyer-Optimal Efficient Mechanism. The Buyer-Optimal Efficient Mechanism is the mechanism that implements the efficient allocation in the way most favorable to the buyer. Efficiency requires that qualities are set such that $q_{lL} = q_{hL} = \underline{q}$ and $q_{hL} = q_{hH} = \overline{q}$, and that the probabilities of winning corresponds to the allocation that gives priority to lL over hL over lH over hH. That is,

$$
\begin{aligned}
x_{lL}^{FB} &= \frac{1-(1-\alpha_{lL})^N}{N\alpha_{lL}} \\
x_{hL}^{FB} &= \frac{(1-\alpha_{lL})^N-(1-\alpha_{lL}-\alpha_{hL})^N}{N\alpha_{hL}} \\
x_{lH}^{FB} &= \frac{(1-\alpha_{hL}-\alpha_{lL})^N-(1-\alpha_{hL}-\alpha_{lL}-\alpha_{lH})^N}{N\alpha_{lH}} \\
x_{hH}^{FB} &= \frac{\alpha_{hH}^{N-1}}{N}
\end{aligned}
$$

Efficiency does not pin down payments to suppliers. The Buyer-Optimal Efficient Mechanism sets payments to maximize the buyer's expected utility while satisfying all incentive compatibility constraints.

From this starting point we progressively increase the buyer's expected utility by adjusting the conditional probabilities of winning (the x's) and the quantities (the q's) until there is no further scope for improvement. At this point we will have reached the global maximum, a result established formally in Section 4 where we prove that the buyer's optimization problem is concave and that the first order conditions are both necessary and sufficient for a global maximum. In many cases, this process will lead to new IC constraints binding. These changes in the pattern of binding IC constraints give rise to many of the interesting economic implications of the model.

We first illustrate our approach for the previously studied case where private information is one-dimensional. The exercise also serves to highlight the similarities and differences between the one-dimensional problem and its multidimensional counterpart.

3.1 One dimensional private information

Suppose first that $\alpha_{hH} = \alpha_{hL} = 0$ so that private information is essentially one-dimensional. All suppliers have low fixed costs. With probability α_{lL}, they have a low marginal cost of providing quality ($\theta_2 = L$); with probability α_{lH}, they have a high cost of providing quality ($\theta_2 = H$). Consider the Buyer-Optimal Efficient Mechanism. To ensure efficiency, qualities must be set equal to $q_{lH} = \overline{q}$ and $q_{lL} = \underline{q}$. The contract is awarded to a type lL supplier whenever there is one among the suppliers, otherwise to a type lH supplier. Thus, $x^{FB}_{lH} = \frac{\alpha^{N-1}_{lH}}{N}$ and $x^{FB}_{lL} = \frac{1-\alpha^{N}_{lH}}{N\alpha_{lL}}$. To ensure incentive compatibility while maximizing the buyer's expected utility, payments are set such that $U_{lH} = 0$ and $U_{lL} = x^{FB}_{lH}\overline{q}(H - L)$ (a type lH supplier has no incentive to pretend she is of type lL since $x^{FB}_{lH} < x^{FB}_{lL}$ and $\overline{q} < \underline{q}$). Under this scheme, the buyer's expected utility is given by:

$$\alpha_{lH}x^{FB}_{lH}W_{lH}(\overline{q}) + \alpha_{lL}[x^{FB}_{lL}W_{lL}(\underline{q}) - U_{lL}]$$

or, to highlight the virtual welfare associated with each type, that is, the welfare each type generates, corrected by the incentive cost due to its presence:

$$\alpha_{lH}x^{FB}_{lH}[W_{lH}(\overline{q}) - \frac{\alpha_{lL}}{\alpha_{lH}}\overline{q}(H - L)] + \alpha_{lL}x^{FB}_{lL}W_{lL}(\underline{q}) \tag{6}$$

($W_{lH}(\overline{q}) - \frac{\alpha_{lL}}{\alpha_{lH}}\overline{q}(H - L)$ is supplier lH's virtual welfare). Now consider reducing the quality delivered by lH. Clearly, this would have no first order effect on $W_{lH}(q_{lH})$ but this would decrease the information rent of supplier lL. In the buyer-optimal scheme, of course, q_{lH} should be decreased further until

$$q^{*}_{lH} = \arg\max\{W_{lH}(q_{lH}) - \frac{\alpha_{lL}}{\alpha_{lH}}q_{lH}(H - L)\} \tag{7}$$

This reduces supplier lL's rent since we can now set $U_{lL} = x^{FB}_{lH}q^{*}_{lH}(H - L)$ and still satisfy all incentive compatibility constraints. In (7), the optimal distortion is independent of the probabilities of winning (and thus is also independent of the number of suppliers), a property that Laffont and Tirole (1987) have coined the "dichotomy property" between selection and incentives.

Can the buyer use the probabilities of winning as a second instrument to extract more rents from the suppliers? Consider again (6). Given the award rule and the feasibility constraint (4), x_{lL} is set as high as possible so there is no way to increase the buyer's utility by changing x_{lL}. However, supplier lL's rent is increasing in x_{lH}. If $W_{lH}(q^{*}_{lH}) - \frac{\alpha_{lL}}{\alpha_{lH}}q^{*}_{lH}(H - L)$, the virtual welfare generated by type lH, is negative, the buyer can increase his utility by excluding types lH (and setting $x_{lH} = 0$).

To summarize: In the one-dimensional model, the buyer distorts the qualities of the "high cost" suppliers downward in order to reduce the informational rents of the "low cost" suppliers. That is, there is a systematic bias against quality. Winning probabilities do not play any role, except for possibly excluding some of the "high cost" types from the market.[5]

So far, we have described the direct revelation mechanism that maximizes the buyer's expected utility. An open question is whether a simple mechanism exists that implements it. Che (1993) showed that a quasilinear scoring auction can play that role. In a quasilinear scoring auction, the buyer announces a scoring rule that is linear in price, $\widetilde{v}(q) - p$; suppliers submit price-quality offers, and the offer generating the highest score wins. Obligations can be determined, for example, by the second score, i.e. the winner must deliver a score equal to the score of the second best offer. The environment that Che considers has continuous types so that the equivalent of (7) is

$$q^*(\theta) = \arg\max\{v(q) - \theta q - \frac{F(\theta)}{f(\theta)}q\} \tag{8}$$

where f and F denote the density and cumulative distribution of θ respectively. Che proposes to set $\widetilde{v}(q) = v(q) - \Delta(q)$ where the distortion term is given by

$$\Delta(q) = \int_0^q \frac{F\left(q^{*-1}(s)\right)}{f\left(q^{*-1}(s)\right)} ds \text{ for } q \in \left[q^*(H), q^*(L)\right] \tag{9}$$

Suppliers in scoring auctions always choose $(p^*(\theta), q^*(\theta))$ to maximize the score their offer generates, given their profit target. This means they set $q^*(\theta) = \arg\max\{v(q) - \Delta(q) - \theta q\}$. It is easy to check that this yields the same solution as problem (8) when $\Delta(q)$ is given by (9). Moreover, in the dominant strategy equilibrium of the second score auction (where the winner must deliver a score equal to the second highest score), suppliers submit bids generating scores equal to $\max\{v(q) - \Delta(q) - \theta q\}$.

It is not hard to arrive at the discrete analog to Che's scoring auction. The optimal scoring rule needs to satisfy three conditions: (1) Given the scoring rule, lH maximizes her score by choosing q^*_{lH} and lL maximizes her score by choosing $\underline{q}$, (2) the scoring rule must be such that the equilibrium score generated by lL is larger than that generated by lH, (3) the payment must be such that lH gets no rent and lL has an expected payoff equal to $x^{FB}_{lH}(H - L)q^*_{lH}$. Conditions (1) and (2) were already present in the continuous types environment. The following scoring rule satisfies them: $\widetilde{v}(q) = v(q) - \frac{\alpha_{lL}}{\alpha_{lH}}(H - L)q + 1_{\{q \geq \underline{q}\}}\varepsilon$ where $\varepsilon > 0$ is chosen such that condition (1) is satisfied (this is always possible given their differential marginal costs). Condition (3) is specific to the discrete types environment. Indeed, when private information is discrete, allocations (in our case, the

[5] With the usual monotone hazard rate assumption this is true, even in a model with more than two types. The monotone hazard rate assumption ensures that, under the optimal choice of qualities, the ranking of the virtual welfares corresponds to the natural ranking of types. Therefore, so does the optimal choice of probabilities of winning.

probability of winning and the level of quality) no longer pin down expected payoffs.[6] This implies that a second score format may not be sufficient to extract as much rent as possible. Additional type-specific payments may be necessary. One possibility is to require down-payments for offers whose quality exceed a certain level.

3.2 Two dimensional private information

We now replicate this reasoning when private information is two-dimensional. In the Buyer-Optimal Efficient Mechanism (BOEM), qualities must be such that $q_{lH} = q_{hH} = \overline{q}$ and $q_{hL} = q_{lL} = \underline{q}$. The contract is awarded giving preference to the type that generates the highest social welfare, i.e. $x_k = x_k^{FB}$.

Let $U_{k,j}$ be the expected utility of a type k pretending she is of type j. Also, let $\Delta\theta_1 = h - l$ and $\Delta\theta_2 = H - L$. To satisfy incentive compatibility, while minimizing suppliers' rents, suppliers' expected utilities must be set such that $U_k = \max_{j \neq k} U_{k,j}$. Let $U_{hH} = 0$ (we can check ex post that this will satisfy supplier hH's incentive compatibility constraints).

Given the parameters of the model, we first argue that $U_{hL} = \max\{U_{hL,hH}, U_{hL,lH}\}$, i.e., that imitating lL is dominated for supplier hL. Indeed, $U_{hL,lL} = U_{lL} - x_{lL}\Delta\theta_1$. If $U_{lL} = U_{lL,hL}$, we get $U_{hL,lL} = U_{hL} + (x_{hL} - x_{lL})\Delta\theta_1 < U_{hL}$. If instead, $U_{lL} = U_{lL,lH}$, we have $U_{hL,lL} = U_{lH} + x_{lH}\Delta\theta_2\overline{q} - x_{lL}\Delta\theta_1 < U_{lH} + x_{lH}\Delta\theta_2\overline{q} - x_{lH}\Delta\theta_1 = U_{hL,lH} \leq U_{hL}$ (we can rule out $U_{lL} = U_{lL,hH}$ since it is dominated for supplier lL).

Second, we argue that $U_{lH} = U_{lH,hH}$. The first alternative for supplier lH is that he imitates hL. Her expected payoff in this case is $U_{lH,hL} = U_{hL} + x_{hL}\Delta\theta_1 - x_{hL}\underline{q}\Delta\theta_2$ where $U_{hL} = \max\{U_{hL,hH}, U_{hL,lH}\}$. It can be shown that $U_{lH,hL} < U_{lH,hH}$.[7] The second alternative is $U_{lH,lL} = U_{lL} - x_{lL}\Delta\theta_2\underline{q}$. When $U_{lL} = U_{lL,lH}$, $U_{lH,lL} = U_{lH} + x_{lH}\Delta\theta_2\overline{q} - x_{lL}\Delta\theta_2\underline{q} < U_{lH}$. When $U_{lL} = U_{lL,hL}$, $U_{lH,lL} = U_{hL} + x_{hL}\Delta\theta_1 - x_{lL}\Delta\theta_2\underline{q} < U_{lH,hL} = U_{hL} + x_{hL}\Delta\theta_1 - x_{hL}\Delta\theta_2\underline{q}$.

Finally, we show that $U_{lL} = U_{lL,hL}$. When $U_{hL} = U_{hL,lH}$, $U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}\overline{q}\Delta\theta_2 + x_{hH}\Delta\theta_1 > U_{lL,lH} = x_{lH}\overline{q}\Delta\theta_2 + x_{hH}\Delta\theta_1$ since $x_{hL} > x_{lH}$. When $U_{hL} = U_{hL,hH}$, the claim follows from the fact that $x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_2\overline{q} > x_{lH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] > x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ (the last inequality being implied by $U_{hL,hH} > U_{hL,lH}$).

[6] See Fudenberg and Tirole (1991), p. 253, for an illustration of this general issue in working with discrete type spaces.

[7] Consider first $U_{hL} = U_{hL,hH}$. The requirement of $U_{lH,hL} < U_{lH,hH}$ is equivalent to $x_{hL}[W_{lH}(\underline{q}) - W_{hL}(\underline{q})] - x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] < 0$. The first term is negative since, by assumption, $W_{lH}(\overline{q}) - W_{hL}(\underline{q}) < 0$. The second term may be positive or negative, but even when it is negative, $x_{hL}[W_{lH}(\underline{q}) - W_{hL}(\underline{q})] < x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ since $\frac{d}{dq}[W_{lH} - W_{hL}] < 0$ and $x_{hH} < x_{hL}$. When $U_{hL} = U_{hL,lH}$, $U_{lH,hL} < U_{lH,hH}$ is equivalent to $x_{hL}[W_{lH}(\underline{q}) - W_{hL}(\underline{q})] - x_{lH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] < 0$. This holds by exactly the same reasoning.

This leads us to:

$$\begin{aligned} U_{lH} &= U_{lH,hH} = x_{hH}^{FB}\Delta\theta_1 \\ U_{hH} &= 0 \\ U_{hL} &= \max\{U_{hL,hH}, U_{hL,lH}\} = \max\{x_{hH}^{FB}\overline{q}\Delta\theta_2, -x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] + x_{hH}^{FB}\Delta\theta_1\} \\ U_{lL} &= U_{lL,hL} = x_{hL}^{FB}\Delta\theta_1 + U_{hL} \end{aligned}$$

In practice, this generates two cases depending on the sign of $W_{lH}(\overline{q}) - W_{hL}(\overline{q})$. When $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) > 0$, $U_{hL,hH} > U_{hL,lH}$. The binding constraints are represented in Figure 2(a). When $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) < 0$, $U_{hL,lH} > U_{hL,hH}$. This case is represented in Figure 2(b).

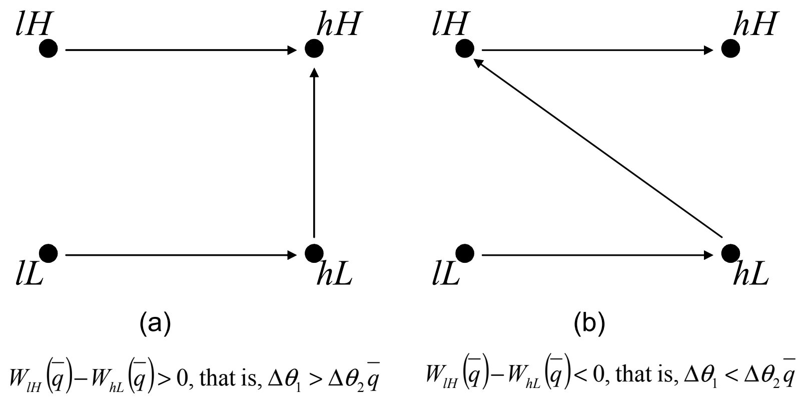


Figure 2: Binding constraints in the Buyer-Optimal Efficient Mechanism (BOEM).

Consider the first case. The buyer's expected utility, $\sum_k \alpha_k[x_k^{FB}W_k(q_k) - U_k]$, is given by:

$$\alpha_{lH}x_{lH}^{FB}W_{lH}(q_{lH}) - \alpha_{lH}x_{hH}^{FB}\Delta\theta_1 + \alpha_{hH}x_{hH}^{FB}W_{hH}(q_{hH}) + \alpha_{hL}x_{hL}^{FB}W_{hL}(q_{hL}) - \alpha_{hL}x_{hH}^{FB}q_{hH}\Delta\theta_2$$
$$+\alpha_{lL}x_{lL}^{FB}W_{lL}(q_{lL}) - \alpha_{lL}x_{hL}^{FB}\Delta\theta_1 - \alpha_{lL}x_{hH}^{FB}q_{hH}\Delta\theta_2$$

(where all qualities are initially equal to the first best qualities) or, to highlight the virtual welfare generated by each supplier:

$$\alpha_{lH}x_{lH}^{FB}W_{lH}(q_{lH}) + \alpha_{hH}x_{hH}^{FB}[W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2] \tag{10}$$
$$+\alpha_{hL}x_{hL}^{FB}[W_{hL}(q_{hL}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] + \alpha_{lL}x_{lL}^{FB}W_{lL}(q_{lL})$$

The rents of suppliers lL and hL depend positively on q_{hH} and the buyer can increase his expected utility by decreasing q_{hH}, ideally until

$$q_{hH}^2 = \arg\max\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2\} \tag{11}$$

Suppose further that, despite this, the binding IC constraints for hL and lL remain the same, that is, $x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] > x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)]$ (this ensures that both $U_{hL,hH} > U_{hL,lH}$ and $U_{lL,hL} > U_{lL,lH}$).[8] Now reconsider (10). There is no further scope for improvement by distorting the qualities. Furthermore, the virtual welfare of lL is clearly the largest of all so that it is optimal to set $x_{lL} = x_{lL}^{FB}$. In contrast, the relative ranking of the virtual welfare of lH and hL is unclear. If $W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 > W_{lH}(\overline{q})$, the virtual welfare generated by supplier hL remains larger than that of lH so the optimal allocation is the first best allocation.

Suppose instead that the virtual welfare associated with lH is larger than that associated with hL, itself larger than the virtual welfare associated with hH. Then, the buyer would rather increase the probability of giving the contract to supplier lH over hL. As we increase x_{lH} and decrease x_{hL} concurrently (keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL} = \frac{1-\alpha_{hH}^N}{N}$ so that the contract is always allocated), the buyer's expected utility increases. This process does not affect the qualities nor, consequently, the virtual welfares. Thus it should continue until a new IC constraint binds or we reach an upper bound to x_{lH}. Increasing x_{lH} (and decreasing x_{hL}) only affects supplier lL's rent, so the only IC constraint to worry about is that $U_{lL} \geq U_{lL,lH}$.

Practically, define $x_{lH}^{\max}$ as the expected probability with which a type lH wins when she has priority over all the other types but lL.[9] Similarly, define $x_{hL}^{\min}$ as hL's expected probability of winning when lH and lL have priority over hL.[10] Thus $x_{hL}^{\min} < x_{lH}^{\max}$. Suppose that at $x_{lH}^{\max}$ and $x_{hL}^{\min}$, $U_{lL,hL} = x_{hL}^{\min}\Delta\theta_1 + x_{hH}q_{hH}^2\Delta\theta_2$ is still larger than $U_{lL,lH} = x_{lH}^{\max}\overline{q}\Delta\theta_2 + x_{hH}^{FB}\Delta\theta_1$. Then no other constraint binds in the process of increasing x_{lH} and decreasing x_{hL}. The qualities and probabilities are then all optimized given the binding constraints. If the first order conditions are sufficient, which we will show in Lemma 1, we have reached the solution: $x_{lL} = x_{lL}^{FB} > x_{lH} = x_{lH}^{\max} > x_{hL} = x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$ and $q_k = q_k^{FB}$, except for $q_{hH} = q_{hH}^2$.

Note that the lowest virtual welfare in both cases is supplier hH's. It is given by $W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2$. As long as this expression is positive, it is optimal to set $x_{hH} = x_{hH}^{FB}$ even when we do not impose non exclusion. This will be the case for $W_{hH}(q_{hH})$ high enough. This is the sense in which imposing non exclusion is not restrictive (cf. footnote 4 above).

The preceding describes Solutions 1.1.a and 1.1.b in Theorem 1 below. They illustrate the differences with the one-dimensional model. First, unlike the solution to the one-dimensional model, the probabilities of winning play a screening role beyond simple exclusion. This is intuitive. In the

[8] Recall that $W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2) > W_{lH}(\overline{q}) - W_{hL}(\overline{q})$ since $W_{lH}(q) - W_{hL}(q)$ is decreasing in q.

[9] Formally, $x_{lH}^{\max}$ is defined implicitly by $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH}^{\max}) = 1 - (\alpha_{hL} + \alpha_{hH})^N$ (cf. the feasibility constraint in (4)).

[10] Formally, $x_{hL}^{\min}$ is implicitly defined by $N(\alpha_{lH}x_{lH}^{\max} + \alpha_{hL}x_{hL}^{\min} + \alpha_{lL}x_{lL}^{FB}) = 1 - \alpha_{hH}^N$.

one-dimensional model, we had two instruments but only one dimension to screen over. Here, we have two dimensions to screen over, θ_1 and θ_2. The consequence is that we now have two kinds of inefficiencies due to screening: A productive inefficiency because some qualities are distorted relative to the first best level of provision, and an allocative inefficiency because winning probabilities differ from the first best winning probabilities.

The second difference with the one-dimensional model is that which constraint binds may change as we adjust the qualities. This is now a familiar property of all multidimensional screening models.

The third and final difference between the one-dimensional model and the two-dimensional model is the loss of the dichotomy property between incentives and selection. Indeed, despite the fact that the optimal choice of quality appears to be independent of the probabilities of winning in (11), the fact that the binding IC constraints are configured in this particular way at $q_{hH} = q^2_{hH}$ depends on the probabilities of winning. An implication is that, unlike in the one-dimensional model, the optimal buying mechanism will depend on the number of suppliers.[11]

4 Characterization of the Optimal Mechanism

We now turn to the formal characterization of the optimal buying procedure. Our approach relies on the following result:

Lemma 1: *The first order conditions of the maximization problem (1) subject to (2), (3), (4) and (5) are necessary and sufficient for a global maximum.*

The proof of Lemma 1 is in Appendix B. Lemma 1 is not trivial because, in its stated version, the maximization problem of the buyer is neither concave, nor even quasi-concave. However, we show that it can be turned into a concave programming problem by a change of variables. The remaining subtlety is to show that the first order conditions of the new problem are equivalent to those of the original problem. This allows us to continue working with the q's and the x's which are more easily interpretable.

We next turn our attention to simplifying the problem in other ways. Manipulation of the feasibility constraints gives Lemma 2:

[11] A tempting interpretation of the dichotomy property in the one-dimensional model is that adding an instrument via the competition among suppliers is unnecessary because we already had one instrument and one dimension. This interpretation is misleading. In fact, the qualitative properties of the solution would change in a model with two types of qualities à la Armstrong and Rochet (1999), when we add a third instrument (competition) to the two existing ones (levels of each type of quality). The reason is the one stated in the main text. The presence of the probabilities of winning affects the configuration of the binding constraints.

Lemma 2: *Consider the feasibility constraints*

$$N\sum_{k\in K}\alpha_k x_k \leq 1-(1-\sum_{k\in K}\alpha_k)^N \text{ for all subsets } K \text{ of } \{hH, lH, hL, lL\} \tag{12}$$

and define an n-type constraint as a feasibility constraint with the relevant subset K having n elements. The following statements hold:

i. At most one one-type constraint binds, at most one two-type constraint binds and at most one three-type constraint binds.

ii. These binding constraints are nested, in the sense that the type in the binding one-type constraint must belong to the binding two-type constraint, and so on.

The proof of Lemma 2 is in Appendix B. The intuition is as follows. Suppose that, at the solution, the contract is allocated according to the following order of priority: $lL \succ lH \succ hL \succ hH$, i.e. give the contract to a type lL if there is one, otherwise to a type lH if there is one, and so on. This means that the ex-ante probability that a lL type gets the contract is the probability that there is at least one type lL suppliers among the N suppliers, i.e. $N\alpha_{lL}x_{lL} = 1-(1-\alpha_{lL})^N$. Thus the one-type constraint binds for lL. It cannot bind for any other types because a binding constraint for another type would imply that *that* type has priority over all other types in the allocation, a contradiction. Next, $lL \succ lH \succ hL \succ hH$ also means that the contract is allocated to a type lL or lH whenever there is one among the N suppliers. This means that the ex-ante probability of a type lL or lH winning, $N(\alpha_{lL}x_{lL}+\alpha_{lH}x_{lH})$, is the probability that there is at least one of these types among the suppliers, $1-(1-\alpha_{lL}-\alpha_{lH})^N$. Thus the two-type constraint binds for $\{lL, lH\}$, showing that the binding constraints are indeed nested. Statement (i) of Lemma 2 suggests that it could be the case that, say, no one-type constraint binds. This will be the case, for instance if the order of priority is $lL \sim lH \succ hL \succ hH$, that is, lL and lH have priority over all the other types, but if there are a lL type and a lH type, the buyer allocates the contract among them randomly. In this case, no one-type constraint binds. Finally, note that the suppliers' expected probabilities are weakly aligned with their order of priority in the sense that, if $k \succ j$, then $x_k > x_j$ but if $k \sim j$, then $x_k \gtreqless x_j$.

Lastly, standard manipulation of the incentive compatibility constraints and the individual rationality constraints allows us to order the probabilities of winning in a limited way.

Lemma 3: *At any solution, $x_{lH} \geq x_{hH}$, $x_{lL} \geq x_{hL}$ and $U_{hH} = 0$*

The full characterization of the solution to the buyer's design problem is provided in the next Theorem. Figures 3 and 4 provide a graphic representation the binding incentive compatibility constraints in each solution.

Theorem 1: Characterization of the optimal buying mechanism

Let $q_{hH}^2 = \arg\max\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q_{hH}\Delta\theta_2\}$ *and let* $q_{lH}^2 = \arg\max\{W_{lH}(q_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q_{lH}\Delta\theta_2\}$.

Part I: When $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) > 0$, *the probabilities of winning and quality levels in the optimal buying mechanism are as given in Table 1. Details on the conditions for the specific solutions are given, along with the proof, in Appendix C.*

Part II: When $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) < 0$, *the probabilities of winning and quality levels in the optimal buying mechanism are as given in Table 2. Details on the conditions for the specific solutions are given, along with the proof, in Appendix C.*

Table 1: Probabilities of winning and quality levels when $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) > 0$

Solution	Probabilities of Winning	q_{lL}	q_{hL}	q_{lH}	q_{hH}
Condition: $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] \leq x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$					
1.1.a	$x_k = x_k^{FB}$	$\underline{q}$	$\underline{q}$	$\overline{q}$	q_{hH}^2
1.1.b	$x_{lL} = x_{lL}^{FB} > x_{lH} = x_{lH}^{\max} > x_{hL} = x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$\overline{q}$	q_{hH}^2
1.1.c	$x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ \overline{q})$	$(q_{hH}^2, \overline{q})$
1.1.d	$x_{lL} = x_{lL}^{FB} > x_{lH} = x_{lH}^{\max} > x_{hL}^{\min} \geq x_{hL} > x_{hH} \geq x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ \overline{q})$	$(q_{hH}^2, \overline{q})$
1.1.e	$x_{lL} = x_{lL}^{FB} > x_{lH} > x_{hL} = x_{hH} > x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ q_{hH})$	$(q_{hH}^2, \overline{q})$
Condition: $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] > x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$					
1.2.a*	$x_k = x_k^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ \overline{q})$	$(q_{hH}^2, \overline{q})$
1.2.b*	$x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} > x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ \overline{q})$	$(q_{hH}^2, \overline{q})$
1.2.c*	$x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} = x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ \overline{q})$	$(q_{hH}^2, \overline{q})$
Other relevant solutions are 1.1.b, 1.1.c, 1.1.d and 1.1.e					

Additional conditions for individual solutions can be found in the Appendix.

* Under the condition that $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) > 0$ we can tighten the bound on q_{hH} so that $q_{hH} \in (q_{hH}^2, q_{lH})$

Table 2: Probabilities of winning and quality levels when $W_{lH}\left(\overline{q}\right) - W_{hL}\left(\overline{q}\right) < 0$

Solution	Probabilities of Winning (x's)	q_{lL}	q_{hL}	q_{lH}	q_{hH}
Condition: $x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] \geq x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)]$					
2.1.a	$x_k = x_k^{FB}$	$\underline{q}$	$\underline{q}$	q_{lH}^2	$\overline{q}$
2.1.b	$x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} = x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	q_{lH}^2	$\overline{q}$
2.1.c	$x_{lL} = x_{lL}^{FB} > x_{hL} = x_{hL}^{FB} > x_{lH}^{FB} > x_{lH} > x_{hH} > x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ q_{hH})$	$(q_{hH}^2,\ \overline{q})$
2.1.d	$x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{lH} = x_{hL} > x_{lH}^{FB} > x_{hH} > x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ q_{hH})$	$(q_{hH}^2,\ \overline{q})$
2.1.e	$x_{lL} = x_{lL}^{FB} > x_{lH} > x_{hL} = x_{hH} > x_{hH}^{FB}$	$\underline{q}$	$\underline{q}$	$(q_{lH}^2,\ q_{hH})$	$(q_{hH}^2,\ \overline{q})$
Condition: $x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] < x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)]$					
The relevant solutions are 1.2.a, 1.2.b, 1.2.c, 2.1.c, 2.1.d and 2.1.e					

Additional conditions for the individual solutions are found in the Appendix

Tables 1 and 2 present the main features of the solution. The second column describes the x's and the last four columns describe the values of the qualities at the solution (an interval means that the optimal level of quality lies in this interval). For instance, Solution 1.2.b has $x_{lL} = x_{lL}^{FB}$ which is greater than $x_{hL}(< x_{hL}^{FB})$. This is in turn greater than $x_{lH}\left(> x_{lH}^{FB}\right)$ and $x_{hH} = x_{hH}^{FB}$. Both q_{lL} and q_{hL} are at the first best levels and $q_{hH} \in (q_{hH}^2, \overline{q})$ and $q_{lH} \in (q_{lH}^2, \overline{q})$. Both are distorted below the first best level. The conditions that define each solution depend on the resulting binding constraints and virtual welfares. The binding constraints are a function of the probabilities of winning and the qualities, and thus ultimately of all the parameters of the model. The virtual welfares are a function of $\Delta\theta_1$, $\Delta\theta_2$, $v(.)$ and the α_k's. The only way in which they are affected by the number of suppliers, N, is through the value of the Lagrange multipliers when several IC constraints are binding out of one type. The value of the objective function and the value of the control variables at the solution are continuous in the parameters of the model.

The following patterns emerge from the tables: First, the solution describing the optimal scheme depends on the number of suppliers as well as the usual parameters of the environment (distributions of types and cost structure). The reason is that the number of bidders affect the probabilities of winning and thus which incentive compatibility constraints bind. This can be seen already in the condition that separates the solutions 1.1. from 1.2. in Table 1. As N increases, both x_{hH}^{FB} and x_{lH}^{FB} decrease but x_{hH}^{FB} decreases even more than x_{lH}^{FB}. As a result, we are more likely to be in one of the solutions 1.1.

Second, there is some downward distortion in the quality provided by the high cost marginal suppliers. The quality provided by the low marginal cost suppliers is never distorted. It is also apparent that the kind of distortion that can be generated by a scoring rule linear in price (i.e. such that $q_{lL} = q_{hL}$ and $q_{lH} = q_{hH}$) is generically not part of the solution, unlike in the one-dimensional model. The reason is that the optimal qualities depend on the configuration of binding constraints. Equal quality levels for types with the same marginal cost require a very specific combination of binding constraints and distributions of types. This is never an issue in one-dimensional models since there is only one type for any given marginal cost.

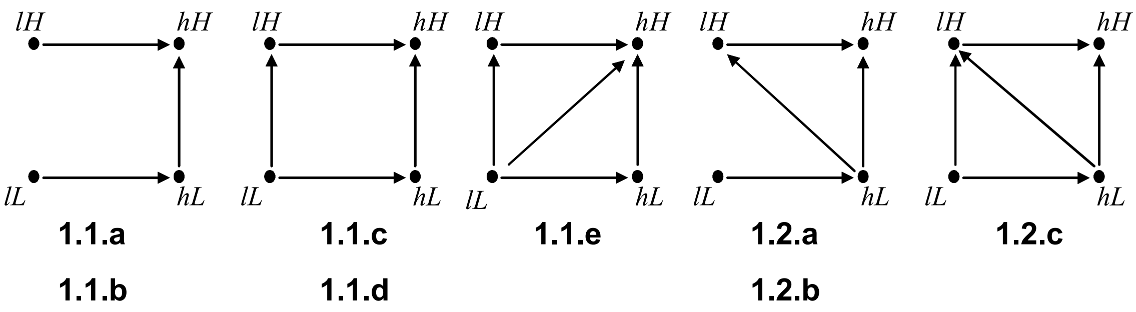


Figure 3: Binding IC constraints at the solution when $\Delta\theta_1 - \Delta\theta_2\overline{q} < 0$

Third, probabilities of winning are also often distorted. Specifically, the probabilities of winning of the high marginal cost suppliers are sometimes distorted upwards, whereas the probability of winning of low marginal cost supplier hL is sometimes distorted downwards. The allocation of supplier lL is never distorted.

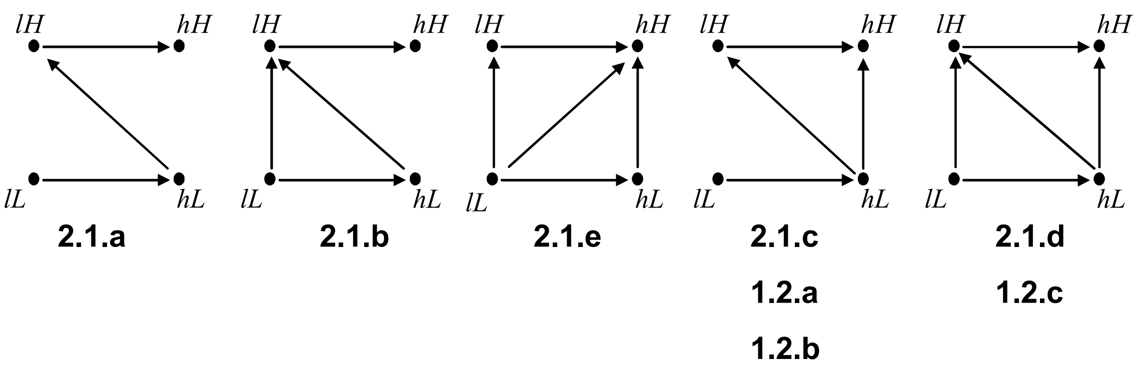


Figure 4: Binding IC constraints at the solution when $\Delta\theta_1 - \Delta\theta_2\overline{q} < 0$

Putting these two aspects together - productive and allocative distortions - we find no systematic "bias against quality" in the two-dimensional model, unlike in the one-dimensional model (Laffont and Tirole, 1987 and Che, 1993). While the economic conclusions differ, the underlying economic motivation is the same: reducing the suppliers' rents. The qualities of the high marginal cost types are distorted downwards to reduce the low marginal cost supplier's benefit from imitating them. As illustrated in Figures 3 and 4, all binding constraints between suppliers with different marginal costs are from the low marginal cost supplier to the high marginal cost supplier so this "trick" is effective. This is also the case in the one-dimensional model where the distortion of high-cost types' quality lowers the informational rents of the low cost types. Similarly, the reason why supplier hL's probability of winning is sometimes below her first best level is to reduce supplier lL's rent when the incentive compatibility constraint $\mathrm{IC}_{lL,hL}$ is binding. In each case, the optimal level of distortion balances a trade-off between the costs in terms of lost social welfare and the benefits in terms of reduced rents.

4.1 Comparative Statics

To illustrate the properties of the solution, we carry out several comparative statics exercises. To investigate the pattern of incentive compatibility across the type space, we computed the solution for a parametrized version of the model. Figure 5 shows the results of this simulation exercise for $N = 2$ in an environment where $\alpha_{lH} = \alpha_{hL}$, $\alpha_{lL} = \alpha_{hH}$, $v(q) = 3q^{\frac{1}{2}}$, $\Delta\theta_2 = 1$, and $l = L = 1$. The graph shows which solution applies as a function of $\Delta\theta_1$ and α_{lH}. The space is bounded on the right by the value of $\Delta\theta_1$ that equates $W_{lH}^{FB} = W_{hL}^{FB}$. It is bounded from above by $\alpha_{lH} = 0.5$ (i.e. $\alpha_{hH} = 0$) which corresponds to perfect negative correlation in types. When the number of suppliers is increased to seven $(N = 7)$, the partition of the type space is depicted in Figure 6. Figures 5 and 6 illustrate the comparative statics with respect to N, $\Delta\theta_1$ and α_{lH}. To draw out the intuitions and trade-offs involved in the determination of the optimal mechanism we discuss the impacts of changing N and $\Delta\theta_1$.[12]

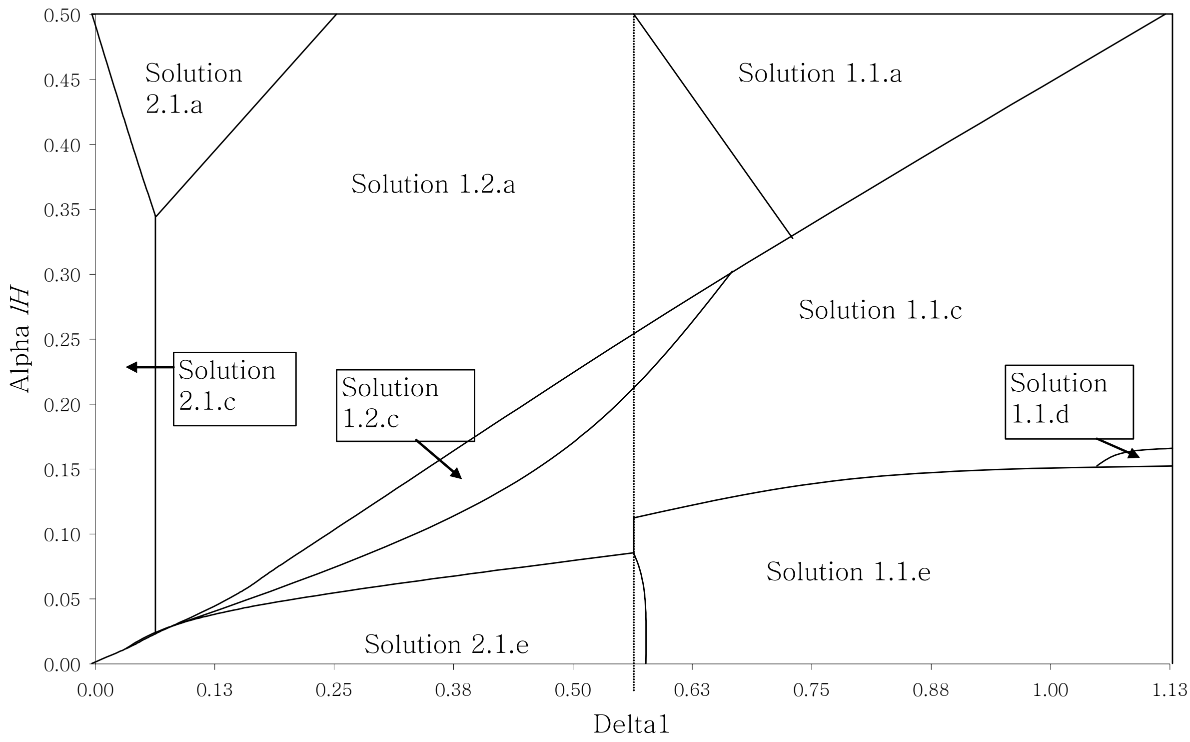
4.1.1 Comparative statics with respect to N

As is clear from Figures 5 and 6, the number of suppliers affects the position of the solutions in the environment.[13]

The only boundary that is completely unaffected by the change in the number of suppliers is the boundary marking the division between Parts I and II of Theorem 1 (the dashed vertical line at Delta 1 = 0.56). The condition that defines it, $\Delta\theta_1 - \Delta\theta_2\overline{q} = 0$, corresponds to the point at which

[12] The Matlab code for these simulations is available at pages.stern.nyu.edu/~jasker/ and www.people.hbs.edu/ecantillon.

[13] Moreover, within a solution, the exact values for q's are often a function of N as well.

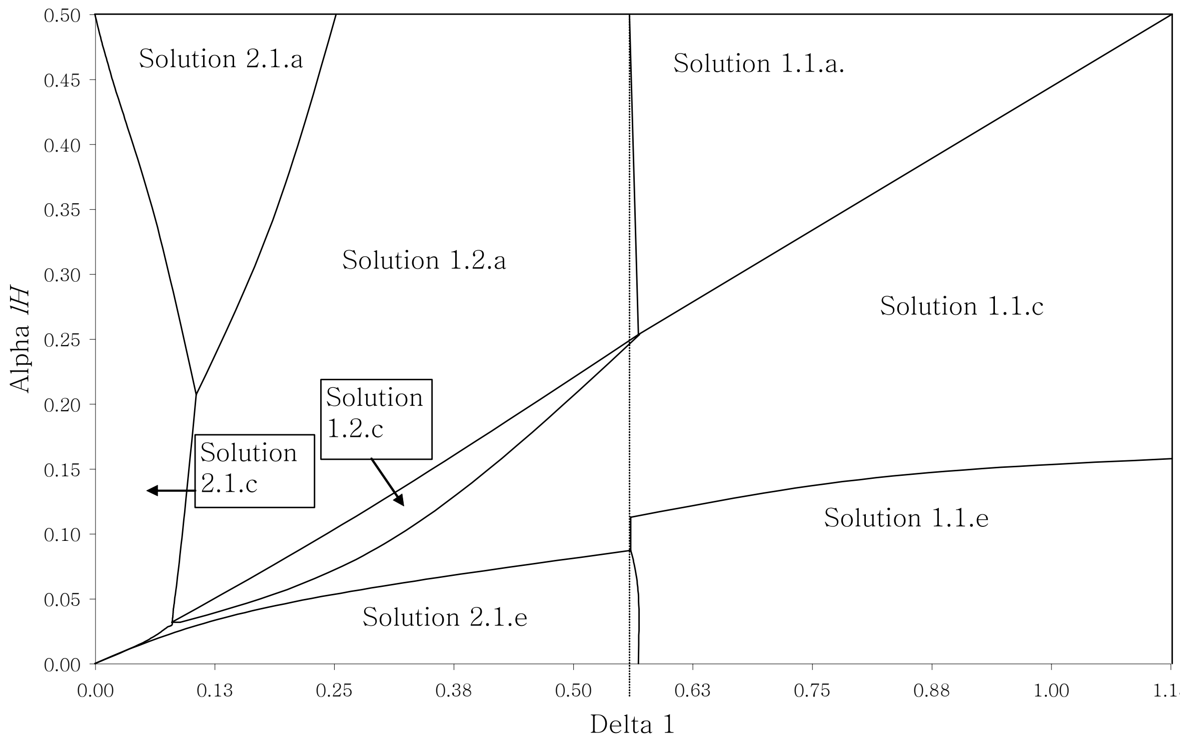


: Figure 5: Partition of the Type Space by Solution. $N = 2$. Other parameters are: $\alpha_{lH} = \alpha_{hL}$, $\alpha_{lL} = \alpha_{hH}$, $v(q) = 3q^{\frac{1}{2}}$, $\Delta\theta_2 = 1$, $l = L = 1$. The dashed line intersecting the x-axis at 0.56 is the division between Part 1 and Part 2 of the proof of Theorem 1. Solution 2.1.d occurs in the region of (0.07,0.02) but is too small to be able to be shown on the diagram.

the binding IC constraint for type hL in the Buyer Optimal Efficient Mechanism (BOEM) changes from binding toward lH to binding toward hH ($IC_{hL,lH}$ ceases to bind and $IC_{hL,hH}$ binds instead). This condition is independent of the number of suppliers.

Three boundaries are highly sensitive to the number of suppliers. These are the boundary extending between Solutions 2.1.a and 1.2.a, the boundary extending between Solutions 1.2.a and 1.1.a, and the boundary between 1.2.c and 1.1.c. Consider the boundary between 1.2.a and 1.1.a. This marks the points at which $IC_{hL,hH}$ ceases to bind and $IC_{hL,lH}$ binds when the q's are optimized relative to the BOEM. Given that $U_{hL,hH} = x_{hH}^{FB}\Delta\theta_2 q_{hH}^2$ and $U_{hL,lH} = x_{hH}^{FB}\Delta\theta_1 - x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$, the condition for $U_{hL,lH} \geq U_{hL,hH}$ can be rewritten as,

$$\frac{x_{lH}^{FB}}{x_{hH}^{FB}} \geq \frac{[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)]}{[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]}$$



: Figure 6: Partition of the Type Space by Solution. $N = 7$. Other parameters are: $\alpha_{lH} = \alpha_{hL}$, $\alpha_{lL} = \alpha_{hH}$, $v(q) = 3q^{\frac{1}{2}}$, $\Delta\theta_2 = 1$, $l = L = 1$. The dashed line intersecting the x-axis at 0.56 is the division between Part 1 and Part 2 of the proof of Theorem 1.

It is a direct function of x's and therefore of the number of suppliers. Since the q's are fixed (and independent of the number of bidders), the only thing determining the position of the boundary in the type space is the relative magnitude of x_{hH}^{FB} and x_{lH}^{FB}. The left side of this expression increases with N (x_{hH}^{FB} approaches zero faster than x_{lH}^{FB}) while the right side is invariant to N. Since the ratio on the right hand side is increasing in $\Delta\theta_1$, the boundary shifts to the left in Figure 6.[14] The same intuition applies for the boundary between 1.2.c and 1.1.c which is defined by $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] = x_{lH}^{\max}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$.

Several boundaries are only somewhat affected by the increase in the number of suppliers. The boundary between Solution 1.1.a and 1.1.c is invariant to the number of suppliers between 0.76 and 1.125, but the rest of the boundary does change, although only very subtly. The invariant part

[14]For the boundary extending between 1.2.a and 2.1.a, switch all subscripts on the x's and q's in the preceding discussion (that is, if they read lH make them hH, and vice versa).

marks the points at which the virtual surplus of lH equals that of hL after the quantities have been optimized relative to the BOEM (the x's are as in the BOEM). The part that does vary reflects the same equality, but is sensitive to changes in the number of suppliers since optimizing quantities from those in the BOEM leads to $IC_{hL,lH}$ and $IC_{hL,hH}$ both binding. This leads to a need to set quantities so that $x_{lH}^{FB}\left[W_{lH}\left(q_{lH}\right)-W_{hL}\left(q_{lH}\right)\right]=x_{hH}^{FB}\left[W_{lH}\left(q_{hH}\right)-W_{hL}\left(q_{hH}\right)\right]$. The sensitivity is via the x's, which are functions of the number of suppliers, but this is mitigated by the ability of the q's to adjust. The other boundaries that are only somewhat affected are those of 2.1.c, 2.1.e and 1.1.e. The economic intuition is the same as the boundary already discussed.

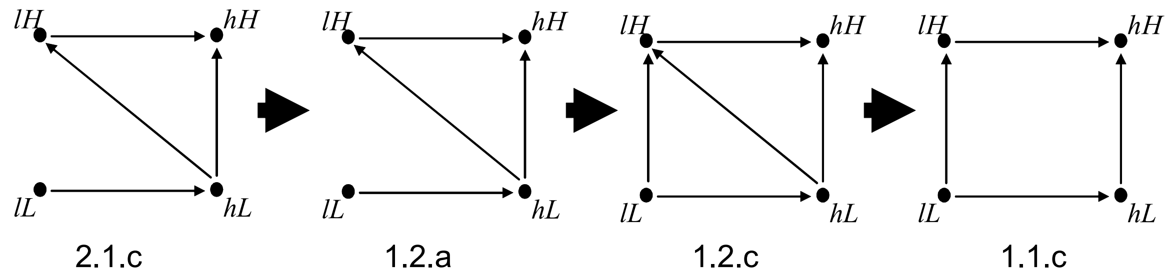
4.1.2 Comparative statics with respect to $\Delta\theta_1$

Having discussed the way the boundaries work, we now discuss how changing other parameters affects the shape of the optimal mechanism. To illustrate, in Figure 5, taking $\alpha_{lH} = 0.25$ (implying all types have the same probability), we can see how varying $\Delta\theta_1$ affects the mechanism. Recall that when $\Delta\theta_1$ is small, the pattern of binding IC constraints in the BOEM is the one in Figure 2(b). At the solution (Solution 2.1.c), hL is indifferent between both lH and hH (two IC constraints binding). This is due to the fact that, when the q's are optimized relative to the BOEM, the virtual welfare of lH is less than that from hH. This induces an incentive to decrease x_{lH} and increase x_{hH}. Doing so makes hH more attractive to hL than lH, although were lH not bearing some of the burden of reducing hL and lL's informational rents, hL would prefer to imitate lH (by assumption). Thus q_{hH} and q_{lH} are distorted downward from the first best, as both are used to reduce the informational rents of lL and hL. The probabilities of winning for lH and hH are also distorted. In the limit, as $\Delta\theta_1 \to 0$, this case approaches that considered by Che (1993), and indeed we can check that, at the solution, q_{lH} converges to q_{hH} and x_{hH} converges to x_{lH}, exactly as in Che (1993).

As $\Delta\theta_1$ increases we move to Solution 1.2.a. Here we are in a similar position to 2.1.c, with $IC_{hL,lH}$ and $IC_{hL,hH}$ binding, only here, with higher fixed costs, there is no incentive to distort the allocation: $x_k = x_k^{FB}$.

As $\Delta\theta_1$ rises further we move from Solution 1.2.a to Solution 1.2.c. At this point, $\Delta\theta_1$ is so high that in the BOEM the virtual welfare of lH is greater than that of hL under the pattern of binding IC constraints in solution 1.2.a. As a result, the buyer will start to increase x_{lH} and decrease x_{hL}. When $x_{lH} = x_{hL}$ the lL type becomes indifferent between imitating hL and lH and so $\text{IC}_{lL,lH}$ starts to bind.

The last transition is from Solution 1.2.c to Solution 1.1.c. As before, $\Delta\theta_1$ has risen to the point



: Figure 7: The changing IC constraints, by solution, as $\Delta\theta_1$ increases when $\alpha_{lH} = 0.25$

that the virtual welfare from lH, under the BOEM, is higher than that of hL. However, $\Delta\theta_1$ is large enough that hL no longer has an incentive to imitate lH (hL's fixed cost is too high relative to lH's). Instead, $\text{IC}_{hL,lH}$ ceases to bind. As x_{lH} increases lL will prefer to switch to lH over hL. This results in $\text{IC}_{lL,lH}$ binding, in addition to $\text{IC}_{lL,hL}$. However, lH and hL continue to share some of the burden of extracting informational rents from lL, resulting in distortions on q_{lH}, q_{hH}, x_{lH} and x_{hL}.

4.1.3 Comparative statics with respect to α_{lH}

We now consider the effect of the correlation between suppliers' fixed costs and their marginal costs. For this exercise, we fix $\Delta\theta_1 = 0.8$ and increase α_{lH} progressively. Note that, in the BOEM, the binding constraints are as in Figure 2(a).

When α_{lH} is small we find ourselves in Solution 1.1.e. Here lL is indifferent between the three other types. Because the hL type is very uncommon, his virtual welfare, $W_{hL}(q) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$ is very small, and actually smaller than that of lH and hH. This induces the buyer to decrease x_{hL} at the benefit of lH and hH. When $\text{IC}_{lL,lH}$ starts binding, the fact that lH is also very infrequent in the population leads to large distortion in her level of quality and thus her virtual welfare. The high level of distortion in the x's and q's eventually make imitating hH attractive for the lL type. In the limit, as $\alpha_{lH} \to 0$, the environment converges to a one-dimensional environment.

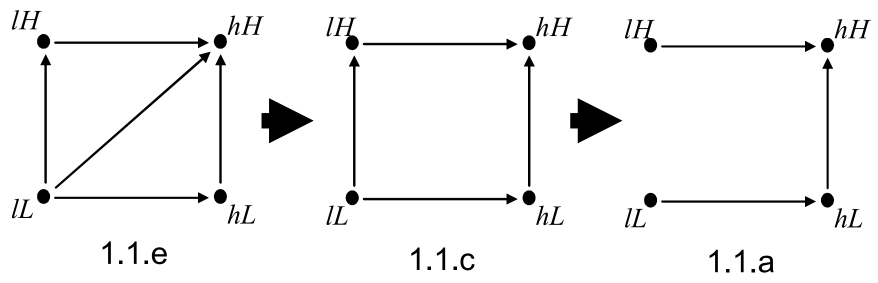


Figure 8: The changing IC constraints, by solution, as α_{lH} increases when $\Delta\theta_1 = 0.8$

As the correlation between the fixed and marginal costs decreases (α_{lH} increases), we move from Solution 1.1.e to 1.1.c. The probability of a lH or hL type in the population is now higher. As a result, the cost of distorting their allocation is higher (this is seen in a higher virtual welfare). The resulting distortion, both in the x's and the q's, is lower and IL never finds it attractive to imitate the hH type.

As the correlation between the fixed and marginal costs becomes strongly negative we find ourselves in Solution 1.1.a. Now the probability of a hL type in the population is sufficiently high that the cost of using hL to extract information rents from lL is high relative to the expected value of the hL type itself. Put differently, the virtual welfare associated with hL is large, and larger than that of hL and hH. As a result, there is no incentive to distort the probabilities of winning away from the first best levels, and thus the pattern of IC constraints is as in the BOEM configuration.

5 Comparison with commonly used procedures

The optimal scheme has two disadvantages. First, it depends finely on the parameters of the environment, a feature that does not make it robust in the sense of Wilson (1987). Second, it is complex and does not seem to be implementable using a transparent procedure.[15] Yet, transparency is desirable and even often a requirement, as in public procurement. This suggests that, for practical purposes, second best solutions that are simple and robust performers in a variety of settings are likely to be more useful. Commonly used procedures are obvious candidates. They include quasilinear scoring auctions, price-only auctions with minimum quality standards, menu auctions

[15] Some cases from Theorem 1 can be implemented by the following variant of a scoring auction. There is a default contract (p_{hH}, q_{hH}), a scoring rule $\widetilde{v}(q) - p$ and a threshold $\underline{S}$. The rules are as follow. Bidders submit bids (p, q) which are evaluated according to the scoring rule. If at least one bid generates a higher score than $\underline{S}$, the winner is the supplier with the highest score. Otherwise, the default contract is allocated randomly among the suppliers. In practice, the default contract will be allocated only to supplier hH, and the scoring rule will induce lH to choose a level of quality $q \neq q_{hH}$, as required. For example, Solution 1.1.a can be implemented by such a scoring auction. However, not all cases in Tables 1 and 2 are amenable to such scoring auction. This will be clear after Theorem 2.

where suppliers can submit several price-quality offers, and negotiation. Asker and Cantillon (2004) have shown that quasilinear scoring auctions yield a higher expected utility to the buyer than a price-only auction with minimum standards, and that they dominate menu auctions when a second price or an ascending format is used. Hence, our contenders for second best procedures are a quasilinear scoring auction or negotiation.

We proceed by first investigating the quasilinear scoring auction. In a quasilinear auction, the buyer announces a scoring rule that is linear in price, $S(q) = \widetilde{v}(q) - p$ (with $\widetilde{v}'(q) \geq 0$ and $\widetilde{v}''(q) \leq 0$), suppliers submit price-quality bids (p, q), and the winner is the supplier whose bid generates the highest score according to the scoring rule. The winner's resulting obligation depends on the auction format. For example, in a first score scoring auction, the winner must deliver a quality level at a price that matches the score of his bid. In a second score scoring auction, the winner must deliver a quality level at a price that matches the second highest score submitted. Quasilinear scoring auctions are used by the Italian procurement agency, Consip, the US State Highways Authorities, some independent system operators for the procurement of electricity reserve supply and they are supported by several procurement softwares.[16]

Quasilinear scoring auctions put some structure on suppliers' bidding behavior. First, given a scoring rule $\widetilde{v}(q) - p$, suppliers choose their bids to maximize the score they generate given their profit target π, i.e. they solve $\max_{(p,q)}\{\widetilde{v}(q) - p\}$ subject to $p - \theta_{1i} - \theta_{2i}q = \pi$. Substituting for p inside the maximizer yields

$$\max_{q}\{\widetilde{v}(q) - \theta_{1i} - \theta_{2i}q - \pi\} \tag{13}$$

A property of the solution is that it is independent of π, the profit target, and of θ_{1i}, the fixed cost. Second, a standard incentive compatibility argument establishes that the ordering of suppliers' winning probabilities must correspond to their ability to generate a higher score (intuitively, define $\max_q\{\widetilde{v}(q) - \theta_{1i} - \theta_{2i}q\}$ as the supplier's type). Thus, a scoring rule will implement a particular allocation as part of a scoring auction if two conditions hold:

1. [incentive constraint] Given the scoring rule, suppliers maximize (13) by choosing the level of quality assigned by the allocation.

2. [ranking constraint] The ranking of $\max_q\{\widetilde{v}(q) - \theta_{1i} - \theta_{2i}q\}$ and, thus, the ranking of the scores is consistent with the assigned probabilities of winning.

[16]See Bushnell and Oren, 1994, and Chao and Wilson, 2001 for electricity reserve auctions and the websites of eBreviate, PurchasePro, Clarus, IBM/DigitalUnion, Oracle Sourcing, Verticalnet and Perfect for procurement softwares.

The next Theorem characterizes the set of allocations that can be implemented by a quasilinear scoring auction.

Theorem 2: *The solution to the original problem can be implemented as a quasilinear scoring auction if and only if (1)* $q_{lH} = q_{hH}$, $q_{hL} = q_{lL}$ *with* $q_{lH}, q_{hH} \leq q_{hL}, q_{lL}$, *(2)* $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} = \alpha_{lH}x_{lH}^{FB} + \alpha_{hL}x_{hL}^{FB}$, $x_{hH} = x_{hH}^{FB}$ *and* $x_{lL} = x_{lL}^{FB}$, *(3)* $\Delta\theta_1 - \Delta\theta_2 q_{hL} \leq 0$ *when* $x_{hL} > x_{hL}^{\min}$ *and (4)* $\Delta\theta_1 - \Delta\theta_2 q_{lH} \geq 0$ *whenever the optimal solution is such that* $x_{lH} > x_{lH}^{FB}$.

Theorem 2 clarifies the constraints that a quasilinear scoring auction places on the possible allocations. Its proof can be found in Appendix B. The first condition says that two suppliers with the same marginal cost of quality must be providing the same level of quality. Moreover, suppliers with a lower marginal cost of quality must deliver weakly higher levels of quality at equilibrium. These two properties follow from the structure of (13). The second condition says that, at equilibrium, type lL must win over any other type, and that type hH must lose against any other type. The reason is that type lL generates the highest value for $\max_q\{\widetilde{v}(q) - \theta_{1i} - \theta_{2i}q\}$ for any scoring rule and that type hH generates the lowest such value. The third and fourth conditions follow from the combination of the incentive constraint and the ranking constraint. Finally, to prove the sufficiency part of the claim, we construct a quasilinear scoring rule that implements the allocation under conditions (1) through (4).

An immediate consequence of Theorem 2 is that the Buyer-Optimal Efficient Mechanism (BOEM) can be implemented by a quasilinear scoring auction. Such a scoring auction has a scoring rule that corresponds to the buyer's preferences and uses a second score format (with down-payments to account for the discrete nature of the type space).[17] It is both robust, in the sense that it does not depend on the parameters of the environment, and transparent.

Theorem 2 also clarifies why even a variant of a quasilinear scoring auction with a default contract, as described in footnote 15, is unable to implement the optimal buying mechanism. For example, solution 1.1.d. in Table 1 requires $x_{hH} > x_{hH}^{FB}$, which is clearly ruled out by Theorem 2.

We proceed by computing the buyer's expected utility in the optimal quasilinear scoring auction. This will give us an upper bound on the performance of quasilinear scoring auctions in this environment. Theorem 2 suggests the ways in which the original problem must be adjusted in order to solve for the optimal quasilinear auction. It contains two difficulties. First, conditions (1) and (2) destroy the concavity of the optimization problem. Indeed, while they are linear in the control variables when expressed in the x, q space, they are not when expressed in the z_1, z_2 space (cf. the

[17] Recall from our discussion in Section 3.1. that these down payments are only needed because of the discrete nature of the type space.

proof of Lemma 1). Moreover, substituting (1) and (2) into the objective function destroys the concavity, and quasiconcavity of the objective function. Second, constraints (3) and (4) imply that the constraint set is no longer closed. This could, in principle, affect the existence of a solution.

We address these issues in two steps. In a first step, we integrate conditions (1) and (2) into the problem and solve for the first order conditions using the same progressive approach as before. We stop there if the solution satisfies constraints (3) and (4). In principle, we only have the guarantee that this is a local maximum so we should interpret our numerical results for the optimal quasilinear scoring auction as being about the lower bound to the optimal quasilinear scoring auction. None of the conclusions are affected by this caveat. If the outcome of the first step does not satisfy (3) or (4), we incorporate (3) and (4) explicitly and solve for the new optimum.[18]

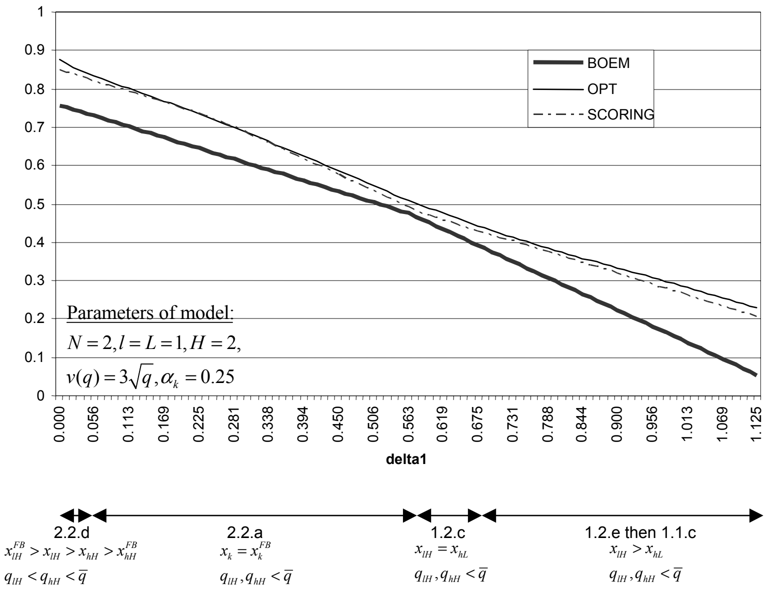


Figure 9: Buyer's expected utility from the different mechanisms, as a function of $\Delta\theta_1$.

Figure 9 illustrates how the optimal quasilinear scoring auction performs relative to the optimal scheme and to the BOEM in an environment similar to the one studied in Figure 5. As $\Delta\theta_1$ increases, fixed costs become relatively more important as a source of adverse selection and the maximum level of welfare decreases since suppliers' costs increase. Figure 9 shows that the expected utility from all three mechanisms is also decreasing in $\Delta\theta_1$. The kink in the BOEM curve corresponds to the value of $\Delta\theta_1$ for which $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) = 0$, when the binding constraints in the efficient mechanism switch from $\mathrm{IC}_{lL,hL}$, $\mathrm{IC}_{hL,lH}$ and $\mathrm{IC}_{lH,hH}$ to $\mathrm{IC}_{lL,hL}$, $\mathrm{IC}_{hL,hH}$ and $\mathrm{IC}_{lH,hH}$: this increases the weight of $\Delta\theta_1$ in the buyer's expected surplus.[19] The BOEM curve is correspondingly steeper.

[18] A separate appendix deriving the solution for the optimal scoring auction presented in the next Figures is available at pages.stern.nyu.edu/∼jasker/ and www.people.hbs.edu/ecantillon (also reproduced as appendix D below).

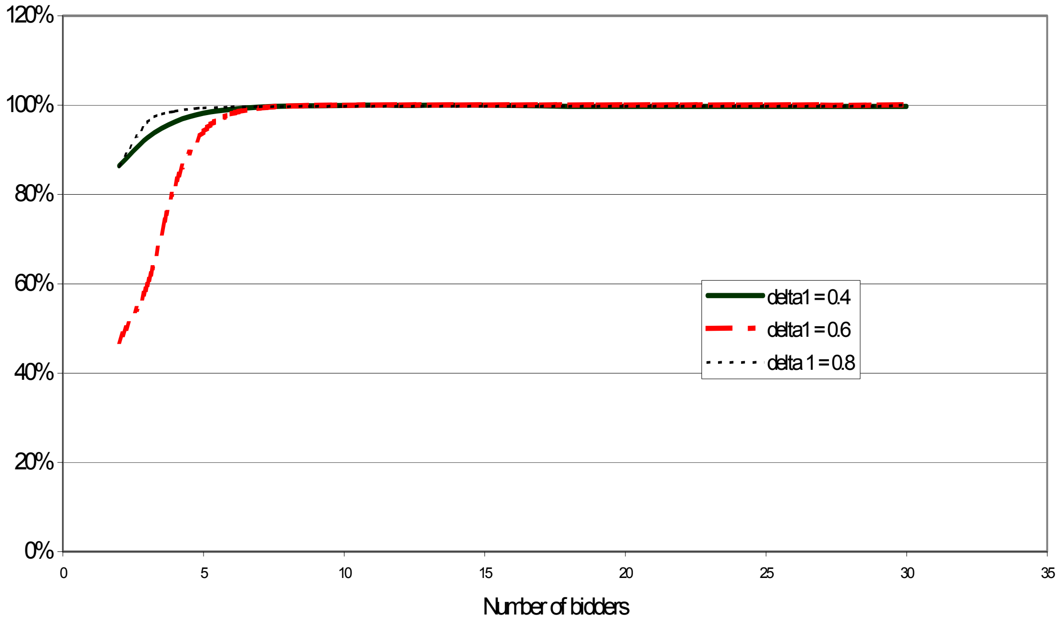
[19] Recall, the expressions for the buyer's expected utility (terms depending on $\Delta\theta_1$ underlined):

The difference in expected utility between the optimal buying mechanism and the BOEM represents the potential gain from being a strategic buyer. Indeed, the buyer can always obtain the expected utility from the BOEM simply by revealing his true preferences and holding a second score scoring auction. The optimal quasilinear scoring auction captures a large part of the potential surplus, for most values of $\Delta\theta_1$. For values of $\Delta\theta_1$ around 0.25, the optimal quasilinear scoring rule does basically as well as the optimal mechanism. This area corresponds to Solution 2.2.a in the solution, which can be implemented by a quasilinear scoring rule for some values of the parameters.

As $\Delta\theta_1$ tends to 0, the source of adverse selection reduces to one dimension, the marginal cost. In this case, Che (1993) has shown that a quasilinear scoring rule implements the optimal mechanism. The reason why the expected utility from the optimal quasilinear scoring rule does not converge to the expected utility of the optimal mechanism in our graph is that there is some discontinuity in the optimal quasilinear scoring auction at $\Delta\theta_1 = 0$. As long as $\Delta\theta_1 > 0$, the nature of the scoring rule imposes that lH generates a strictly higher score at equilibrium than hH. Thus $x_{lH} \geq x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$ (Theorem 2). This leaves some informational rent to lH and increases the rents of hL and lL relative to the case where $x_{lH} = x_{hH}$. When $\Delta\theta_1 = 0$, suppliers lH and hH are essentially the same. The optimal scoring rule will thus set $x_{lH} = x_{hH}$ and leave no rent to supplier lH.

The next figure plots the percentage of the surplus, that is, the difference between the buyer's expected utility in the optimal scheme and in the BOEM, that the optimal quasilinear scoring rule captures as N increases. The exercise is done for three values of $\Delta\theta_1$, $\Delta\theta_1 = 0.4$, 0.6 and 0.8 (the rest of the parameters is as in Figure 9). The optimal quasilinear scoring auction captures most of the surplus as soon as the number of suppliers is larger than five. Two effects are at play here. As N increases, the probability that any of these three mechanisms (the BOEM, the optimal mechanism and the optimal quasilinear scoring rule) allocates the contract to a type lL increases. This reduces the difference among them. In particular, the surplus available from being a strategic buyer, i.e. the difference between the optimal mechanism and the BOEM decreases. At the same time, like the optimal mechanism (but unlike the BOEM), the optimal quasilinear scoring rule can change the type of the supplier who is second in the priority order (from type hL to type lH). As N increases, the types with higher priority matter more. This provides some intuition for why the *share* of the surplus that the optimal quasilinear scoring rule captures increases.

1. When $\Delta\theta_1 < \Delta\theta_2\overline{q}$: $\alpha_{lH}x_{lH}^{FB}W_{lH}(q_{lH}) - \underline{\alpha_{lH}x_{hH}^{FB}\Delta\theta_1} + \alpha_{hH}x_{hH}^{FB}W_{hH}(q_{hH}) + \alpha_{hL}x_{hL}^{FB}W_{hL}(q_{hL}) + \underline{\alpha_{hL}(x_{lH}^{FB} - x_{hH}^{FB})\Delta\theta_1} - \alpha_{hL}x_{lH}^{FB}q_{lH}\Delta\theta_2 + \alpha_{lL}x_{lL}W_{lL}(q_{lL}) - \underline{\alpha_{lL}x_{hL}^{FB}\Delta\theta_1} + \underline{\alpha_{lL}(x_{lH}^{FB} - x_{hH}^{FB})\Delta\theta_1} - \alpha_{lL}x_{lH}^{FB}q_{lH}\Delta\theta_2$
2. When $\Delta\theta_1 > \Delta\theta_2\overline{q}$: $\alpha_{lH}x_{lH}^{FB}W_{lH}(q_{lH}) - \underline{\alpha_{lH}x_{hH}^{FB}\Delta\theta_1} + \alpha_{hH}x_{hH}^{FB}W_{hH}(q_{hH}) + \alpha_{hL}x_{hL}^{FB}W_{hL}(q_{hL}) - \alpha_{hL}q_{hH}\Delta\theta_2 + \alpha_{lL}x_{lL}^{FB}W_{lL}(q_{lL}) - \underline{\alpha_{lL}x_{hL}^{FB}\Delta\theta_1} - \alpha_{lL}q_{hH}\Delta\theta_2$ when $\Delta\theta_1 > \Delta\theta_2\overline{q}$.



: Figure 10: Performance of the optimal quasilinear scoring rule (% of (OPT - BOEM) captured).

We now turn to negotiation. There are many ways to model negotiation in our environment. The main advantage that buyers see in using negotiation over auctions is that it allows them to incorporate all dimensions of the product in the purchasing process.[20] Hence, we adopt a model of negotiation where, indeed, all dimensions of the product are discussed. The exact boundary between auction mechanisms and negotiation is somewhat arbitrary. We take the use of direct competition among bidders, including allowing bidders to make counter-offers, as the distinguishing feature of an auction. Hence, we adopt a negotiation procedure in the spirit of Manelli and Vincent (1995):[21]

Negotiation consists of a sequence of take-it-or-leave-it offers. Each offer is a menu of optimal screening contracts: it maximizes the buyer's expected utility subject to the suppliers' incen-

[20] See e.g. the purchaser survey in the August 2004 issue of the magazine "Purchasing."

[21] Like Manelli and Vincent, the negotiation process takes the form of a sequence of take-it-or-leave it offers. However, unlike them, we allow the buyer to make these offers contingent on the delivered level of quality since quality is verifiable in our model.

For a review of early models of bargaining under asymmetric information, see the survey by Kennan and Wilson (1993) and Fudenberg and Tirole (1991). These early models tended to consider bargaining in a static environments where interaction ceases after the contract was signed, as is the case here. More recent research includes explorations of repeated bargaining environments (e.g. Kennan, 2001) or environments, like ours, where offers are multidimensional (e.g. Wang, 1998). Our model is distinct in having a known end period and being a multilateral negotiation.

tive compatibility constraints. The level of payment and the level of qualities are adjusted to take into account that the contract can be offered to another supplier in case negotiation breaks down. Each supplier only receives one offer.

This definition requires several comments. First, to make it comparable to the other mechanisms we need to impose that at least one buyer's offer is always accepted. This means that the contract offered to the last supplier (meaning all previous suppliers have rejected the offer from the buyer) must satisfy that supplier's individual rationality constraint. Second, the mechanism provides an upper bound to what negotiation can actually achieve. Indeed, it gives all bargaining power to the buyer who can make take-it-or-leave-it offers[22] to each supplier sequentially, and it does not restrict the number of partners the buyer negotiates with.

The optimal bargaining procedure solves a simple dynamic programming problem. Since each supplier is only made an offer once, she will accept any contract that satisfies her individual rationality constraint. In addition, non exclusion requires that the contracts offered in the last round satisfy the individual rationality constraint of all suppliers. Let V_N denote the buyer's expected utility when he reaches round N of the negotiation. We construct the optimal bargaining procedure iteratively. At round $N-1$, the buyer's offer trades off extracting more surplus from supplier $N-1$ with the risk of not meeting her individual rationality constraint and being forced to make an offer to supplier N (and getting V_N as a result). The next Lemma shows that, in any round, we can without loss of generality consider a menu of two contracts, one targeted at the low marginal cost suppliers, and the other targeted at the high marginal cost suppliers. The intuition is that contracts of the form (p, q) are unable to screen over suppliers' fixed cost, θ_1 (the same property shows up in Rochet and Stole, 2002).

Lemma 4: *In any round, the buyer offers at most two contracts.*

Proof: Towards a contradiction, suppose the buyer offers three contracts and, without loss of generality, suppose that one, (p_{lL}, q_{lL}), is targeted at lL and another, (p_{hL}, q_{hL}) is targeted at hL. Incentive compatibility requires:

$$\begin{aligned} p_{lL} - l - Lq_{lL} &\geq p_{hL} - l - Lq_{hL} \\ p_{hL} - h - Lq_{hL} &\geq p_{lL} - h - Lq_{lL} \end{aligned}$$

Thus $p_{lL} - Lq_{lL} = p_{hL} - Lq_{hL}$. Incentive compatibility also requires that neither lH or hH be tempted by the (p_{lL}, q_{lL}) and (p_{hL}, q_{hL}) contracts. This requires $p - Hq \geq \max\{p_{lL} - Hq_{lL},$

[22] Wang (1998) shows that the optimal bargaining mechanism in a single-agent single-principal model with multi-dimensional private information and quality like here takes the form of a take-it-or-leave-it offer.

$p_{hL} - Hq_{hL}\}$ for the contract targeted at lH and possibly hH, if any.[23] This constraint defines the contracts on the isoprofit locus for suppliers hL and lL, $\{(p,q) : p_{lL} - Lq_{lL} = p_{hL} - Lq_{hL}\}$, that are incentive compatible. Since the buyer has strictly convex preferences, there is a unique contract on this locus that maximizes his utility. QED

Let (p_1^n, q_1^n) and (p_2^n, q_2^n) denote the menu of contracts offered in round n, with the convention that $q_1^n \leq q_2^n$. In round N, the buyer's optimization problem is given by

$$\max_{(p_1,q_1),(p_2,q_2)}(\alpha_{lH} + \alpha_{hH})(v(q_1) - p_1) + (\alpha_{hL} + \alpha_{lL})(v(q_2) - p_2)$$

subject to suppliers' IR and IC constraints. Following standard arguments, supplier hH's IR constraint is binding, i.e. $p_1 = h + Hq_1$, and only the downward IC constraint is binding, i.e. $p_2 - Lq_2 = p_1 - Lq_1$. Substituting for p_1 and p_2 in the objective function yields:

$$\max_{(p_1,q_1),(p_2,q_2)}(\alpha_{lH} + \alpha_{hH})(v(q_1) - h - Hq_1) + (\alpha_{hL} + \alpha_{lL})(v(q_2) - h - Lq_2 - \Delta\theta_2 q_1)$$

This establishes Lemma 5:

Lemma 5: *The optimal take-it-or-leave it offer in round N is given by* (p_1^N, q_1^N), where $p_1^N = h + Hq_1$ and $q_1^N = \arg\max\{v(q) - Hq - \frac{(\alpha_{hL}+\alpha_{lL})}{(\alpha_{lH}+\alpha_{hH})}q\Delta\theta_2\}$, *and* $(p_2^N, q_2^N) = (p_1^N + L(\underline{q} - q_1), \underline{q})$.

We are now ready to characterize the offers made in earlier periods and the equilibrium behavior of suppliers.

Theorem 3: *(1) At any round n, suppliers of types lH and hH accept the contract (p_1^n, q_1^n) if it satisfies their IR constraint, and likewise, suppliers of types lL and hL accept contract (p_2^n, q_2^n) if it satisfies their IR constraint.*
(2) At any round $n < N$, the buyer offers one of the four following pairs of contracts, depending on which one yields the largest continuation value (conditional on the condition in the third column being satisfied):

K_n	Offers (p_1, q_1) and (p_2, q_2)	V_n
lL	$(l + L\underline{q}, \underline{q})$	$\alpha_{lL} W_{lL}^{FB} + (1 - \alpha_{lL}) V_{n+1}$
$\{lL, lH\}$	$(l + Hq_1, q_1)$	$\alpha_{lL}(W_{lL}^{FB} - \Delta\theta_2 q_1) +$
	$(l + L\underline{q} + \Delta\theta_2 q_1, \underline{q})$	$\alpha_{lH} W_{lH}(q_1) + (\alpha_{hL} + \alpha_{hH}) V_{n+1}$
	$q_1 = \arg\max\{v(q) - Hq - \frac{\alpha_{lL}}{\alpha_{lH}}\Delta\theta_2 q\}$	Condition: $\Delta\theta_1 - \Delta\theta_2 q_1 \geq 0$
$\{lL, hL\}$	$(h + L\underline{q}, \underline{q})$	$(\alpha_{lL} + \alpha_{hL}) W_{hL}^{FB} + (\alpha_{lH} + \alpha_{hH}) V_{n+1}$
$\{lL, hL, lH\}$	$(l + Hq_1, q_1)$	$(\alpha_{lL} + \alpha_{hL})(W_{lL}^{FB} - \Delta\theta_2 q_1)$
	$(l + L\underline{q} + \Delta\theta_2 q_1, \underline{q})$	$\alpha_{lH} W_{lH}(q_1) + \alpha_{hH} V_{n+1}$
	$q_1 = \arg\max\{v(q) - Hq - \frac{(\alpha_{lL}+\alpha_{hL})}{\alpha_{lH}}\Delta\theta_2 q\}$	Condition: $\Delta\theta_1 - \Delta\theta_2 q_1 \leq 0$

[23] Indeed, the optimal mechanism may exclude hH and lH.

(The first column in the table indicates the set of supplier types who will accept the buyer's offer in round n, and the third column indicates the buyer's continuation value, V_n.).

At round N, the buyer offers the menu of contracts defined in Lemma 5.

Proof: Since suppliers are only made an offer once, they accept this offer whenever it contains a contract that satisfies their IR constraint. By design, the contracts will be such that (p_1^n, q_1^n) (resp. (p_2^n, q_2^n)) is the contract chosen by the high (resp. low) marginal cost suppliers.

Let K_n be the set of supplier types who accept round n offer. Given suppliers' cost structure, $K_n \in \{lL, \{lL, lH\}, \{lL, hL\}, \{lL, lH, hL\}, \{lL, lH, hL, hH\}\}$. We first argue that $K_n \neq \{lL, lH, hL, hH\}$, for $n < N$, i.e. some exclusion is optimal. By offering the single contract $(l + L\underline{q}, \underline{q})$ that excludes all suppliers but lL, and the round N contract forever after, the buyer guarantees himself an expected utility of $\alpha_{lL} W_{lL}^{FB} + (\alpha_{lH} + \alpha_{hH} + \alpha_{hL}) V_N > V_N$, his continuation value at round n if $K_n = \{lL, lH, hL, hH\}$. We now examine the optimal offers for the other three inclusion sets.

$\underline{K_n = \{lL, lH\}}$: lH's IR constraint is binding, $p_1 = l + Hq_1$ and lL's IC constraint is binding, $p_2 = p_1 - Lq_1 + Lq_2$. The optimal qualities solve $\alpha_{lH}(v(q_1) - l - Hq_1) + \alpha_{lL}(v(q_2) - l - Lq_2 - \Delta\theta_2 q_1)$, thus $q_1 = \arg\max\{v(q) - Hq - \frac{\alpha_{lL}}{\alpha_{lH}} \Delta\theta_2 q\}$. For this solution to be feasible we need in addition that hL is indeed excluded, i.e. that $p_1 - h - Lq_1 \leq 0$, i.e. $\Delta\theta_1 - \Delta\theta_2 q_1 \geq 0$.

$\underline{K_n = \{hL, lL\}}$: Only one contract is offered in this case: $(h + L\underline{q}, \underline{q})$. For this solution to be feasible, lH's IR constraint must be violated, i.e. $\Delta\theta_1 - \Delta\theta_2 \underline{q} \leq 0$, which is automatically satisfied in our model.

$\underline{K_n = \{lH, hL, lL\}}$: As before the binding IC constraint is from hL and lL to lH (this will hold as long as $q_1 \leq q_2$), hence $p_2 - Lq_2 = p_1 - Lq_1$. We need to distinguish between two scenarios depending on which, from lH or hL's IR constraints, is binding. Case 1: lH's IR constraint is binding at the optimum, i.e., $p_1 = l + Hq_1$. Substituting into the buyer's utility function, the resulting qualities solve $\alpha_{lH}(v(q_1) - l - Hq_1) + (\alpha_{hL} + \alpha_{lL})(v(q_2) - l - Lq_2 - \Delta\theta_2 q_1)$. The last condition to check is that hL's IR constraint is indeed satisfied i.e. $\Delta\theta_1 - \Delta\theta_2 q_1 \leq 0$. Case 2: hL's IR constraint is binding at the optimum, i.e. $p_2 = h + Lq_2$. Hence the resulting qualities must solve $\alpha_{lH}(v(q_1) - h - Lq_1) + (\alpha_{hL} + \alpha_{lL})(v(q_2) - h - Lq_2)$, that is, $q_1 = q_2 = \underline{q}$: the buyer offers a single contract. lH's IR constraint is satisfied if $\Delta\theta_1 - \Delta\theta_2 \underline{q} \geq 0$. (This is ruled out by assumption). QED.

Negotiation puts some structure on the outcome. As the negotiation progresses (and the buyer's offers are rejected), the buyer's continuation value decreases. He has less suppliers in front of him and his bargaining power decreases accordingly. Similarly, the offers that the buyer makes early in the negotiation are "tougher" in the sense that they satisfy the IR constraints of fewer supplier types than the offers that he makes later in the game. These properties are formalized in the following Theorem:

Theorem 4: *Let V_n, the continuation value of the buyer when he reaches the n^{th} supplier in the sequential negotiation and let K_n be the set of suppliers who accept the buyer's offer in round n. Then $V_n > V_{n+1}$ and $|K_n| \leq |K_{n+1}|$ for all $n < N$.*

Proof: The maximum surplus the buyer can extract from a supplier is $v(\underline{q}) - l - L\underline{q}$, the maximum surplus generated by supplier lL. Since $\alpha_{lL} < 1$, $V_n < v(\underline{q}) - l - L\underline{q}$ for all n. At the same time, at any $n < N$, an available strategy for the buyer is to offer the contract $(l + L\underline{q}, \underline{q})$ that is only accepted by supplier lL. The resulting payoff is $V_n = \alpha_{lL}(v(\underline{q}) - l - L\underline{q}) + (1 - \alpha_{lL})V_{n+1} > V_{n+1}$. Let x be the part of the buyer's continuation value that is acquired in round n when $K_n = \{lL\}$, i.e. $V_n = x + (1 - \alpha_{lL})V_{n+1}$. Similarly let y be the part of the buyer's continuation value when $K_n = \{lL, lH\}$. In round n, the buyer prefers $K_n = \{lL\}$ to $K_n = \{lL, lH\}$ is $x + \alpha_{lH}V_{n+1} > y$. Since V_n is decreasing, $K_n = \{lL\}$ preferred to $K_n = \{lL, lH\}$ in round n, implies that $K_{n'} = \{lL\}$ preferred to $K_{n'} = \{lL, lH\}$ for all $n' < n$. A similar argument establishes that if $K_n = \{lL, hL\}$ is preferred to $K_n = \{lL, lH, hL\}$, it is also preferred for $n' < n$. The same argument also applies when we replace $K_n = \{lL, lH\}$ by $K_n = \{lL, hL\}$. The claim follows. QED.

A consequence of Theorem 4 is that the buyer's expected value from negotiation increases with N.

We now discuss the relative performance of negotiation. Since everything in our model is observable and verifiable, negotiation must do less well than the optimal procurement mechanism. The comparison with the quasilinear scoring auction is more ambiguous. On the one hand, for any given round, the outcome in the negotiation is restricted to two contracts, (p_1, q_1) and (p_2, q_2), whereas a quasilinear scoring auction introduces more flexibility in terms of realized prices. On the other hand, the quasilinear scoring auction restricts the qualities to two levels, whereas the multi-stage format of the negotiation allows for multiple levels of qualities (across periods).

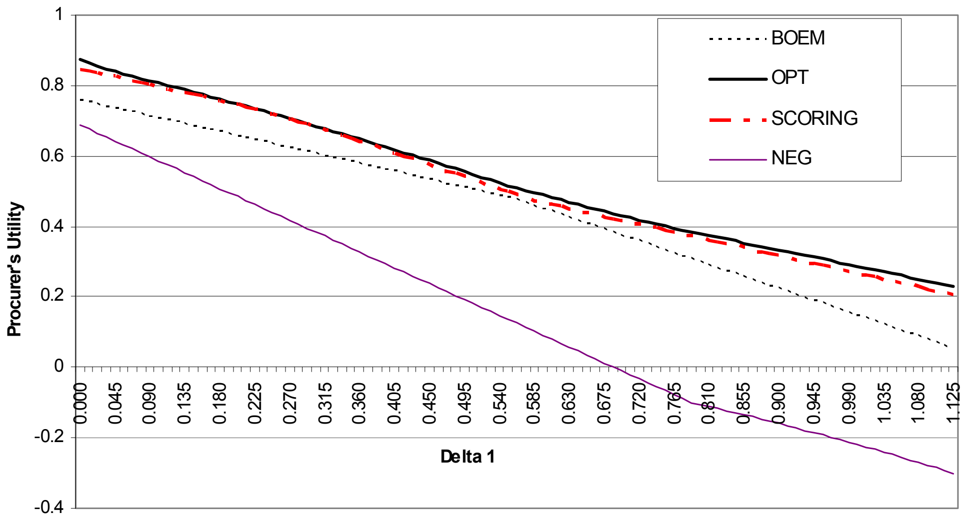


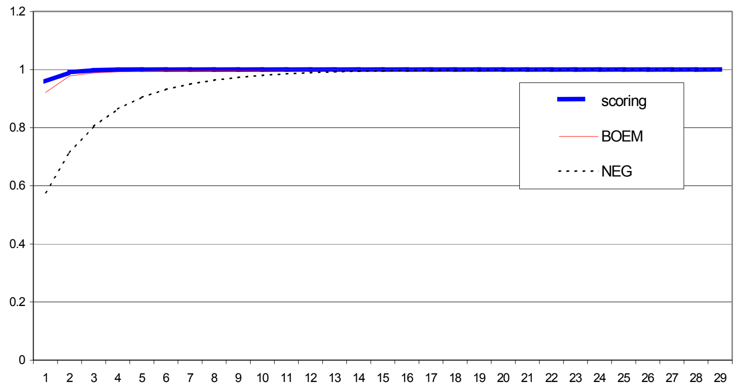
Figure 11: Mechanism performance vs Negotiation (parameters as for Figure 7)

Figure 11 compares the outcome under negotiation when $N = 2$, for the same environment as in Figure 9. The main result from Figure 11 is that negotiation does very badly, even compared to the BOEM. This is despite our choice of a negotiation procedure that is as favorable as possible to negotiation (in other words, the performance of negotiation in Figure 11 is actually an upper bound to what any reasonable model of negotiation can achieve). Direct competition among suppliers is a powerful tool to extract surplus for the buyer.[24] This result is all the more striking given the extensive use of negotiation by practitioners in environments where quality matters.

Finally, we note that, as the number of potential suppliers increases, the buyer's expected utility in all mechanisms converges to that in the optimal mechanism. This is because the probability of allocating the object to a type lL converges to one, and all mechanisms generate the same quality levels for this type. A remaining issue is when mechanism choice matters. The next picture provides some insight into this question. The expected utility from the optimal quasilinear scoring auction and the BOEM converge very fast to that of the optimal mechanism. Convergence for negotiation is slower.

We summarize the insights from this section. First, a well-designed quasilinear scoring auction captures a significant portion of the surplus from being a strategic buyer. It has the advantage of being transparent, but the disadvantage of requiring a lot of information about the environment

[24] This result is in the spirit of Bulow and Klemperer (1996), though both the model of negotiation and the thought experiment are different.



: Figure 12: Relative performance as N varies, normalized by the utility in the optimal mechanism ($l = L = 1$, $h = 2$, $H = 1.6$, $v(q) = 3\sqrt{q}$)

in order to fine-tune the scoring rule. Second, negotiation, which is favored by practitioners on grounds of "simplicity", does very poorly. It does even worse than holding a straightforward scoring auction when the buyer reveals his preference. Thus, if simplicity is a requirement of the buying procedure, our results suggest that buyers should be using scoring auctions where they reveal their true preferences. The BOEM has the added advantages that it is transparent and robust to the environment, unlike negotiation.

6 Relationship to the multidimensional screening literature

Our model is one of multidimensional screening (see Rochet and Stole, 2003 for a recent survey). In this section, we discuss how it relates to other models studied in that literature. This discussion sheds further light on some properties of the solution and suggests some extensions.

Consider again the buyer's optimization problem in the optimal buying mechanism:

$$\max_{x_k,q_k,U_k} F(x_k,q_k,U_k) = \max_{x_k,q_k,U_k} N \sum_{k\in\{hH,lH,hL,lL\}} \alpha_k(x_k W_k(q_k) - U_k) \quad (14)$$

subject to

$$U_k \geq U_j + x_j(\theta_{1j} - \theta_{1k}) + x_j q_j(\theta_{2j} - \theta_{2k}) \quad \forall\, k,j \in \{hH,lH,hL,lL\} \quad (15)$$

$$U_k \geq 0 \quad \forall\, k \in \{hH,lH,hL,lL\} \quad (16)$$

$$N\sum_{k\in K} \alpha_k x_k \leq 1-(1-\sum_{k\in K}\alpha_k)^N \quad \forall\, K \subset \{hH,lH,hL,lL\} \quad (17)$$

$$N \sum_{k\in\{hH,lH,hL,lL\}} \alpha_k x_k = 1 \quad (18)$$

Suppose we fixed the q_k's. Then, the problem of $\max_{x_k,U_k} F(x_k,q_k,U_k)$ subject to (15), (16), (17) and (18) is a linear programming problem. It corresponds to a procurement auction with type-specific predetermined levels of quality. The optimal multi-unit auction problems studied in Armstrong (2000), Avery and Henderschott (2000), Malakhov and Vohra (2004), and Manelli and Vincent (2004) are also linear programming problems. A second kind of auction environment that fits this category is the single object auction with externalities studied by Jehiel at al. (1999).[25] Candidates for a solution in a linear programming problem are extreme points. The standard solution technique is to characterize the parameter space over which these extreme points are indeed solutions.[26]

Our auction problem is not a linear programming problem but instead a concave programming problem. This is reflected in the solution: both the value of the objective function and the value of the control variables are continuous in the parameters of the model.

Suppose we now fixed the x_k's. Then, the problem of $\max_{q_k,U_k} F(x_k,q_k,U_k)$ subject to (15) and (16) is a concave programming problem (since W_k is concave). It is a two-dimensional private information problem with a single instrument as in Armstrong (1999), or Laffont, Maskin and Rochet (1987) and Rochet and Stole (2002) for continuous types analogues. Dana (1993), Armstrong and Rochet (1999) and Walckiers (2004) study models with multiple instruments. Their models also belong to this category of concave programming problems. The standard solution technique used

[25] In Jehiel et al., bidders have multidimensional private information about their utility under every possible allocation of the object among them and their opponents. The authors show that in a class of anonymous mechanisms where bidders submit one dimensional bids, there is a sufficient statistic for bidders' private information so that their model effectively reduces to one with one instrument and one dimension of private information.

[26] In a continuous type environment, Manelli and Vincent (2004) argue that, generically, all extreme points of this problem are solutions of the optimal multi-unit auction problem in the sense that, for each extreme point, there exists a non degenerate set of distributions such that this extreme point is a solution.

in these papers is to posit a set of binding constraints and characterize the parameter space over which the first order conditions are satisfied given these binding constraints.

Our problem belongs to this category (even if, in its original form, it is neither concave nor even quasi-concave - see Lemma 1). However, unlike these problems, we have many more constraints: on top of the standard four individual rationality and 12 incentive compatibility constraints that these problems have in their discrete form, we have 15 feasibility constraints. Moreover, because it is intrinsically non separable, our problem lacks the symmetry of the models in Dana (1993), Armstrong and Rochet (1999) and Walckiers (2004). The consequences are twofold. First, the number of "Solutions", i.e., configurations of binding constraints at the optimum is larger. This is seen in Tables 1 and 2 (Armstrong and Rochet (1999) have at most six solutions to consider). Second, it is harder to reduce *a priori* the number of constraints that are likely to bind. By seeking incremental improvements from the buyer-optimal efficient mechanism, our constructive approach to the characterization of the solution guarantees that we cover the entire parameter space.

Finally, we note a potential extension / application of our model to stochastic optimal contracts. Consider the following situations. An agency regulates a monopoly with costs $\theta_1 + \theta_2 q$ where q can be interpreted as the contractible quantity or quality produced (when θ_1 is known, this corresponds to the problem studied in Baron and Myerson, 1982). A monopolist is facing consumers with utility $\theta_1 + \theta_2 q$, where q is the quality or quantity of the product the monopolist produces (when θ_1 is known this corresponds to the problem studied in Mussa and Rosen, 1978). Rochet and Stole (2002) study the optimal contract in these cases when the principal is limited to one instrument, q. They rule out stochastic contracts where the agent is, in addition, given a probability of being allowed to produce or getting the good, x. In fact they note (p. 283): "The restrictions to non-random price schedules comes possibly at a slight cost: in general, a firm may benefit by asking its potential customers to announce their private information (θ_1, θ_2), followed by a lottery which determines whether or not the customer consumes the product, and if so, the quality of the product. By randomizing over the probability of consumption, the firm may be able to sort over θ_1 as well as θ_2."

The optimal stochastic contract problem in these situations corresponds to the problem studied here, without the feasibility constraint and the non exclusion constraint (the only constraint on the x's is that $x_k \in [0, 1]$). The problem can again be shown to be concave after the appropriate change of variables and the same heuristic solution technique used here applies. Given a set of binding IC constraints and resulting quality levels, the optimal probabilities solve a linear program with constraints $x_k \in [0, 1]$. The solution is to set $x_k = 1$ whenever the virtual welfare associated with type k is positive, and zero otherwise. In other words, at least in the discrete version of Rochet and Stole's model, there is no role for stochastic contracts and no way to screen over θ_1 beyond exclusion which was already available with only q as an instrument.

7 Concluding remarks

In this paper, we have derived the optimal procurement mechanism when both quality and price matter. We found that the recommendations based on models in which suppliers have one-dimensional private information do not carry over to the richer environment where suppliers can have a fixed and variable cost for providing quality. In this richer environment, we find that: (1) the optimal scheme depends on the number of suppliers, (2) the optimal solution involves two sources of inefficiencies: a productive inefficiency because quality levels differ from their first best levels and an allocative inefficiency because the probabilities of winning differ from the first best level of probabilities of winning, and (3) suppliers with the same marginal cost of quality generically supply different levels of quality in the optimal scheme.

Though these results have been derived in the simple two-by-two case, the lessons are likely to generalize to more general discrete types environments as well as to continuous types environments. Indeed, what drives the first result is the endogeneity of the direction in which the incentive compatibility constraints bind. This endogeneity is present in more general multidimensional environments. What drives the second result is the combination of having two instruments and the endogeneity of the relative ranking of the virtual welfares. Again, this effect will be present in general environments. Finally, we have argued that an outcome compatible with a quasilinear scoring auction requires a delicate combination of specific hazard rates together with a specific configuration of binding constraints. This will be even more true in general environments.

The optimal buying mechanism depends finely on the details of the environment and, as a result, may not be very useful for practical application. We have characterized an upper bound to what quasilinear scoring auctions and negotiation can achieve in our environment. Both procedures are commonly used and the existing literature indicates that they are likely contenders for the title of "second best solution." We have compared negotiation and quasilinear auctions with the optimal buying mechanism and found that quasilinear scoring auctions are strong performers, while negotiation does very badly. In fact, taken literally, our results suggest a buyer is better off using a scoring auction where he reveals his true preference than negotiation. Moreover, this procedure is both simple and transparent.

We conclude by suggesting some areas for further research. First, we have assumed that quality is endogenous (i.e. it can be varied by suppliers) and contractible. It is easy to think of examples where this is implausible. For example, a supplier might own a technology that allows her to produce specific levels of quality, or the quality of the product might be observable but impossible to verify so that it cannot be contracted upon. Exogenous qualities reduce the problem to one of a standard asymmetric auction, implying that the optimal mechanism may be biased in favor of one supplier (Naegelen, 2002 and Rezende, 2003). Exogenous qualities can easily be incorporated in our model by increasing the cost of changing qualities and introducing some asymmetry in the initial level of qualities (otherwise, the problem would not be very interesting). Non contractible quality is a more serious issue because it raises the possibility that, whatever mechanism is chosen, adverse selection may be present. Non contractible quality also introduces some role to reputation in repeated procurement situations. Even more realistic, the product to be purchased may involve some contractible dimensions and some others which are not. Contracting in this case may lead to underprovision of quality in other dimensions, a result familiar from Holmstrom and Milgrom (1991). All these extensions seem interesting and worthwhile. However, we note that, while these extensions may change the optimal mechanism, they are unlikely to provide support for negotiation over scoring auctions.

Another important assumption in our analysis is that the buyer knows his preference over the different price - quality pairs. This might be a strong assumption in some procurement situations, especially for complex products. In this case, negotiation may help the buyer become informed about his choice set. This may provide an edge to negotiation over auctions.

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Appendix A

For future reference, this appendix reproduces the optimization problem of the buyer with $U_{hH}=0$ (Lemma 3) and with the subset of the IC constraints that happen to bind at the optimum.

$$\max_{\{x_k,q_k,U_k\}} \alpha_{lH}\left[x_{lH}W_{lH}(q_{lH})-U_{lH}\right]+\alpha_{hH}x_{hH}W_{hH}(q_{hH})+\alpha_{hL}\left[x_{hL}W_{hL}(q_{hL})-U_{hL}\right]+\alpha_{lL}\left[x_{lL}W_{lL}(q_{lL})-U_{lL}\right]$$

subject to:

$$\begin{aligned}
U_{lH} &\geq x_{hH}\Delta\theta_1 && \text{(IC 1)}\\
U_{hL} &\geq U_{lH}-x_{lH}\left[W_{lH}\left(q_{lH}\right)-W_{hL}\left(q_{lH}\right)\right] && \text{(IC 2)}\\
U_{hL} &\geq x_{hH}q_{hH}\Delta\theta_2 && \text{(IC 3)}\\
U_{lL} &\geq U_{lH}+x_{lH}q_{lH}\Delta\theta_2 && \text{(IC 4)}\\
U_{lL} &\geq U_{hL}+x_{hL}\Delta\theta_1 && \text{(IC 5)}\\
U_{lL} &\geq U_{hH}+x_{hH}\Delta\theta_1+x_{hH}q_{hH}\Delta\theta_2 && \text{(IC 6)}\\
N\sum_{k\in K}\alpha_k x_k &\leq 1-(1-\sum_{k\in K}\alpha_k)^N \text{ for all subsets } K \text{ of } \{lH,hH,hL,lL\} && \text{(feasibility)}
\end{aligned}$$

(We omit the non exclusion constraint). The associated Lagrangian is given by:

$$\begin{aligned}
&\alpha_{lH}\left[x_{lH}W_{lH}(q_{lH})-U_{lH}\right]+\alpha_{hH}x_{hH}W_{hH}(q_{hH})+\alpha_{hL}\left[x_{hL}W_{hL}(q_{hL})-U_{hL}\right]+\alpha_{lL}\left[x_{lL}W_{lL}(q_{lL})-U_{lL}\right] \quad (20)\\
&+\lambda_1\left[U_{lH}-x_{hH}\Delta\theta_1\right]+\lambda_2\left[U_{hL}-U_{lH}+x_{lH}\left(W_{lH}\left(q_{lH}\right)-W_{hL}\left(q_{lH}\right)\right)\right]\\
&+\lambda_3\left[U_{hL}-x_{hH}q_{hH}\Delta\theta_2\right]+\lambda_4\left[U_{lL}-U_{lH}-x_{lH}q_{lH}\Delta\theta_2\right]+\lambda_5\left[U_{lL}-U_{hL}-x_{hL}\Delta\theta_1\right]\\
&\lambda_6\left[U_{lL}-x_{hH}\Delta\theta_1-x_{hH}q_{hH}\Delta\theta_2\right]-\sum\gamma_K\left[N\sum_{k\in K}\alpha_k x_k-1+(1-\sum_{k\in K}\alpha_k)^N\right]
\end{aligned}$$

(where λ_i is the Lagrangian multiplier associated with IC constraint i, and γ_K is the multiplier associated with feasibility constraint K). Figure 13 provides a graphical representation of these IC constraints together with their associated multipliers. A dotted line means that a constraint may bind at the optimum. A full line means it always binds.

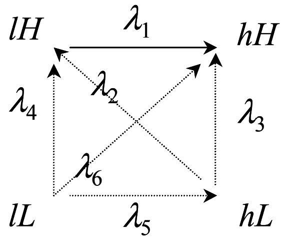

Figure 13: Potentially binding constraints at the solution

The Kuhn-Tucker conditions of this program are standard. For future reference, we only reproduce those with respect to U_k :

$$\lambda_1 - \lambda_2 - \lambda_4 = \alpha_{lH} \quad (21)$$

$$\lambda_2 + \lambda_3 - \lambda_5 = \alpha_{hL} \quad (22)$$

$$\lambda_4 + \lambda_5 + \lambda_6 = \alpha_{lL} \quad (23)$$

Appendix B

Lemma 1: *The first order conditions of the maximization problem (1) subject to (2), (3), (4) and (5) are necessary and sufficient for a global maximum.*

Proof of Lemma 1: Consider the following change of variables: $z_{1k} = x_k$, $z_{2k} = x_k q_k$. Let $\widetilde{F}(z_{1k}, z_{2k}, U_k) = N \sum_{k=lH,hH,hL,lL} \alpha_k (z_{1k} W_k(\frac{z_{2k}}{z_{1k}}) - U_k)$. The problem becomes:

$$\max_{z_{1k},z_{2k},U_k} \widetilde{F}(z_{1k}, z_{2k}, U_k) \qquad \text{s.t.}$$

$$U_k \geq U_j + z_{1j}(\theta_{1j} - \theta_{1k}) + z_{2j}(\theta_{2j} - \theta_{2k}) \qquad \text{for all } k, j \in \{hH, lH, hL, lL\}$$

$$U_k \geq 0 \qquad \text{for all } k \in \{hH, lH, hL, lL\}$$

$$N \sum_{k \in K} \alpha_k z_{1k} \leq 1 - (1 - \sum_{k \in K} \alpha_k)^N \text{ for all subsets } K \text{ of } \{hH, lH, hL, lL\}$$

$$N \sum_{k \in \{hH,lH,hL,lL\}} \alpha_k z_{1k} = 1$$

The constraints are linear in the control variables so the constraint qualification holds and the objective function is concave.[27] The first order conditions are thus necessary and sufficient for a global maximum. To prove that the first order conditions of the original problem are also necessary and sufficient, we need to check that the first order conditions of the two problems are equivalent. To see this, let $G(x_k, q_k, U_k)$ gather all constraint terms of the Lagrangian of the original problem, and $\widetilde{G}(z_{1k}, z_{2k}, U_k)$ gather the constraint terms of the Lagrangian of the transformed problem. We must show that (x_k^*, q_k^*, U_k^*) solves the first order conditions of $\max_{x_k,q_k,U_k} F(x_k, q_k, U_k) + G(x_k, q_k, U_k)$ if and only if $(x_k^*, x_k^* q_k^*, U_k^*)$ solves the first order conditions of $\max_{z_{1k},z_{2k},U_k} \widetilde{F}(z_{1k}, z_{2k}, U_k) + \widetilde{G}(z_{1k}, z_{2k}, U_k)$. The first order conditions with respect to U_k are identical. The first order condition with respect to q_k, $F_{q_k}(x_k^*, q_k^*, U_k^*) + G_{q_k}(x_k^*, q_k^*, U_k^*) = 0$, takes the form

$$N \alpha_k x_k^* W_k'(q_k^*) - \sum \lambda_l x_k^* (\theta_{2k} - \theta_{2l}) = 0$$

[27] The hessian is block diagonal with each block given by $\begin{bmatrix} \alpha_k \frac{z_{2k}^2}{z_{1k}^3} W'' & -\alpha_k \frac{z_{2k}}{z_{1k}^2} W'' & 0 \\ -\alpha_k \frac{z_{2k}}{z_{1k}^2} W'' & \alpha_k \frac{W''}{z_{1k}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(where λ_l are the Lagrangian multipliers of the constraints). This is equivalent to the first order conditions of the transformed problem with respect to z_{2k},

$$N\alpha_k W_k'(\frac{z_{2k}}{z_{1k}}) - \sum \lambda_l(\theta_{2k} - \theta_{2l}) = 0 \tag{24}$$

as long as $x_k^* > 0$ for all k, a consequence of the non exclusion condition (5). Finally, the first order condition with respect to x_k, $F_{x_k}(x_k^*, q_k^*, U_k^*) + G_{x_k}(x_k^*, q_k^*, U_k^*) = 0$ takes the form:

$$N\alpha_k W_k(q_k^*) - \sum \lambda_l[(\theta_{1k} - \theta_{1l}) + q_k^*(\theta_{2j} - \theta_{2l})] - N \sum_{K \text{ st } k\in K} \gamma_K \alpha_k = 0 \tag{25}$$

The first order condition of the transformed problem takes the form:

$$N\alpha_k W_k(\frac{z_{2k}}{z_{1k}}) - N\alpha_k \frac{z_{2k}}{z_{1k}} W_k'(\frac{z_{2k}}{z_{1k}}) - \sum \lambda_l(\theta_{1j} - \theta_{1l}) - N \sum_{K \text{ st } k\in K} \gamma_K \alpha_k = 0$$

This is equivalent to (25) as soon as (24) holds. QED

Lemma 2: *Consider the feasibility constraints*

$$N \sum_{k\in K} \alpha_k x_k \le 1 - (1 - \sum_{k\in K} \alpha_k)^N \text{ for all subsets } K \text{ of } \{hH, lH, hL, lL\}$$

and define an n-type constraint as a feasibility constraint with the relevant subset K having n elements. The following statements hold:

i. At most one one-type constraint binds, at most one two-type constraint binds and at most one three-type constraint binds.

ii. These binding constraints are nested, in the sense that the type in the binding one-type constraint must belong to the binding two-type constraint, and so on.

Proof of Lemma 2: The claim relies on the fact that the function $f(t) = t^N$ for $N \ge 2$ is strictly convex. There are two generic cases to rule out: two constraints binding with no type in common, and two non nested constraints binding with some type in common.

Case 1: No overlap. Suppose, towards a contradiction, that the constraint for lH and the constraint for $\{hH, hL\}$ bind. Then, from (12), $N(\alpha_{lH}x_{lH} + \alpha_{hH}x_{hH} + \alpha_{hL}x_{hL}) = 2-(1-\alpha_{lH})^N-(1-\alpha_{hH}-\alpha_{hL})^N > 1-(1-\alpha_{lH}-\alpha_{hH}-\alpha_{hL})^N$ since $1+(1-\alpha_{lH}-\alpha_{hH}-\alpha_{hL}) = (1-\alpha_{lH})+(1-\alpha_{hH}-\alpha_{hL})$ and $(1-\alpha_{lH})$ and $(1-\alpha_{hH}-\alpha_{hL})$ lie in $(1-\alpha_{lH}-\alpha_{hH}-\alpha_{hL}, 1)$. That is, (12) is violated for $\{lH, hH, hL\}$. All cases with no overlap are proved in this way.

Case 2: Some overlap. Suppose, towards a contradiction that the constraint for $\{lH, hH\}$, and that for $\{hH, hL\}$ is binding. Since (12) holds for hH, this means that

$$\begin{aligned} N(\alpha_{lH}x_{lH} + \alpha_{hH}x_{hH} + \alpha_{hL}x_{hL}) &\ge 1-(1-\alpha_{lH}-\alpha_{hH})^N - (1-\alpha_{hH}-\alpha_{hL})^N + (1-\alpha_{hH})^N \\ &> 1-(1-\alpha_{lH}-\alpha_{hH}-\alpha_{hL})^N \text{ by convexity} \end{aligned}$$

This contradicts (12) for $\{lH, hH, hL\}$. All cases with some overlap are proved in this way.
This proves that binding constraints are nested and that they cannot be more than one constraint of a type to bind. QED.

Theorem 2: *The solution to the original problem can be implemented as a quasilinear scoring auction if and only if (1)* $q_{lH} = q_{hH}$, $q_{hL} = q_{lL}$ *with* $q_{lH}, q_{hH} \leq q_{hL}, q_{lL}$, *(2)* $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} = \alpha_{lH}x_{lH}^{FB} + \alpha_{hL}x_{hL}^{FB}$, $x_{hH} = x_{hH}^{FB}$ *and* $x_{lL} = x_{lL}^{FB}$, *(3)* $\Delta\theta_1 - \Delta\theta_2 q_{hL} \leq 0$ *when* $x_{hL} > x_{hL}^{\min}$ *and (4)* $\Delta\theta_1 - \Delta\theta_2 q_{lH} \geq 0$ *whenever the optimal solution is such that* $x_{lH} > x_{lH}^{FB}$.

Proof of Theorem 2: Let $S_k(q) = \widetilde{v}(q) - \theta_{1k} - \theta_{2k}q$. We first prove the necessary conditions. Recall from the discussion in the main text that, in a scoring auction, suppliers select their offers to maximize the score they generate, given their profit target, $\{\widetilde{v}(q) - \theta_{1i} - \theta_{2i}q - \pi\}$. The solution only depends on suppliers' marginal cost, which establishes condition (1) given that $\theta_{2lH} = \theta_{2hH} > \theta_{2hL} = \theta_{2lL}$. Condition (2) follows from the fact that lL can always generate a strictly higher score than either lH and hL for all choices of the scoring rule $\widetilde{v}(.)$. Similarly, both lH and hL can always generate a strictly higher score than hH so they must win against a hH type.
When $x_{hL} > x_{hL}^{\min}$, $S_{hL}(q_{hL}) \geq S_{lH}(q_{lH})$, else lH should have priority over hL in the allocation. This implies that

$$\begin{aligned} \widetilde{v}(q_{hL}) - h - Lq_{hL} &\geq \widetilde{v}(q_{lH}) - l - Hq_{lH}, \text{ that is,} \\ \Delta\theta_1 - \Delta\theta_2 q_{hL} &\leq \widetilde{v}(q_{hL}) - \widetilde{v}(q_{lH}) - \overline{\theta}_2(q_{hL} - q_{lH}) \end{aligned}$$

In addition, incentive compatibility requires that lH generates a higher score by choosing q_{lH} than q_{hL}, i.e.

$$\widetilde{v}(q_{hL}) - \widetilde{v}(q_{lH}) - H(q_{hL} - q_{lH}) \leq 0$$

Combining both inequalities yields condition (3). Similarly, when $x_{lH} > x_{lH}^{FB}$, $S_{lH}(q_{lH}) \geq S_{hL}(q_{hL})$, else hL should have priority in the allocation. This implies $\Delta\theta_1 - \Delta\theta_2 q_{lH} + L(q_{hL} - q_{lH}) + \widetilde{v}(q_{lH}) - \widetilde{v}(q_{hL}) \geq 0$. In addition, hL must be generating a higher score by choosing q_{hL} over q_{lH}, i.e. $L(q_{hL} - q_{lH}) + \widetilde{v}(q_{lH}) - \widetilde{v}(q_{hL}) \leq 0$. Combining both inequalities yields condition (4).
To prove sufficiency, we construct a scoring rule that implements the intended allocation in a second score auction (in a second score auction, it is a dominant strategy to submit bids generating scores $S_k(q_k) = \max_q\{\widetilde{v}(q) - \theta_{1k} - \theta_{2k}q\}$). Consider

$$\widetilde{v}(q) = v(q)1_{\{q \leq q_{lH}\}} + v(q_{lH})1_{\{q > q_{lH}\}} + \epsilon 1_{\{q \geq q_{hL}\}}$$

For this scoring auction to implement the outcome, two conditions must be satisfied. First, suppliers must be choosing the assigned qualities when they maximize their scores. Second, the ranking of the scores must (weakly) correspond to the assigned ranking of types in the allocation.

Given the shape of this scoring rule, the two relevant choices are q_{lH} and q_{hL}. lH prefers q_{lH} to q_{hL} if and only if $v(q_{lH})-l-Hq_{lH} \geq v(q_{lH})+\varepsilon-l-Hq_{hL}$ i.e. $\varepsilon \leq H(q_{hL}-q_{lH})$ (hH's preferences yield the same condition). hL prefers q_{hL} to q_{lH} if and only if $v(q_{lH})+\varepsilon-h-Lq_{hL} \geq v(q_{lH})-h-Lq_{lH}$, i.e. $\varepsilon \geq L(q_{hL}-q_{lH})$ (lL's preferences yield the same condition). Hence, suppliers choose their assigned qualities if ε satisfies the following inequalities:

$$L(q_{hL}-q_{lH}) \leq \varepsilon \leq H(q_{hL}-q_{lH}), \tag{26}$$

which is possible by condition (1). Next, hL generates a higher score if and only if $S_{hL}(q_{hL}) = v(q_{lH})+\varepsilon-h-Lq_{hL} \geq S_{lH}(q_{lH}) = v(q_{lH})-l-Hq_{lH}$ i.e.

$$\varepsilon \geq \Delta\theta_1 - Hq_{lH} + Lq_{hL} = \Delta\theta_1 - \Delta\theta_2 q_{hL} + H(q_{hL}-q_{lH}) \tag{27}$$

lH generates a higher score otherwise. Inequalities (26) and (27) are always compatible if $\Delta\theta_1 - \Delta\theta_2 q_{hL} \leq 0$ holds. When the solution is such that $x_{lH} > x_{lH}^{FB}$, we need $S_{lH}(q_{lH}) \geq S_{hL}(q_{hL})$:

$$\varepsilon \leq \Delta\theta_1 - Hq_{lH} + Lq_{hL} = \Delta\theta_1 - \Delta\theta_2 q_{lH} + L(q_{hL}-q_{lH})$$

instead. It is compatible with (26) if $\Delta\theta_1 - \Delta\theta_2 q_{lH} \geq 0$. QED.

Appendix C: Characterization of the Optimal Buying Mechanism

Preliminaries

We first define the notation that we will be using for some of the x_k variables when they take specific values. When x_{lH} takes its maximum value conditional on lL keeping priority in the contract allocation, we will denote it $x_{lH}^{\max}$. Formally, $x_{lH}^{\max}$ is defined by the equation

$$N\left(\alpha_{lH}x_{lH}^{\max} + \alpha_{lL}x_{lL}^{FB}\right) = 1-(\alpha_{hL}+\alpha_{hH})^N$$

By Border (1991), this implies the following allocation: When there is a type lL, give the contract to lL, if not, give priority to a type lH if there is one. Conversely, $x_{hL}^{\min}$ corresponds to the expected probability of winning for hL when lH and lL have priority over hL (but hL maintains priority over hH). Formally,

$$N\left(\alpha_{lH}x_{lH}^{\max} + \alpha_{hL}x_{hL}^{\min} + \alpha_{lL}x_{lL}^{FB}\right) = 1-\alpha_{hH}^N$$

Finally, $\overline{x}$ is defined such that $x_{lH} = x_{hL}$ and they have priority over hH in the allocation, that is

$$N\left((\alpha_{lH}+\alpha_{hL})\overline{x} + \alpha_{lL}x_{lL}^{FB}\right) = 1-\alpha_{hH}^N$$

The proof of Theorem 1 uses the following result repeatedly:

Lemma 6: *Suppose $U_{lH} = x_{hH}\Delta\theta_1$. (1) Suppose further that $U_{hL,lH} \geq U_{hL,hH}$. Then, $x_{hL} > x_{lH}$ if and only if $U_{lL,hL} > U_{lL,lH}$. (2) Suppose now that $U_{hL,lH} \leq U_{hL,hH}$. Then $U_{lL,hL} \geq U_{lL,lH}$ when $x_{hL} \geq x_{lH}$.*

Proof: The result follows directly from a comparison of $U_{lL,lH}$ and $U_{lL,hL}$ (when $U_{hL,lH} \geq U_{hL,hH}$) :

$$U_{lL,lH} = x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1 \quad U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1$$

When $U_{hL,hH} \geq U_{hL,lH}$, $U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2$. Since $U_{hL,hH} \geq U_{hL,lH}$ is equivalent to $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] \leq x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$, the condition implies $U_{lL,hL} \geq U_{lL,lH}$ when $x_{hL} > x_{lH}$. QED.

Lemma 7: *Suppose that $IC_{hL,hH}$ is satisfied. Then $x_{hL} \geq x_{hH} \Longrightarrow IC_{lL,hH}$ is satisfied.*

Proof: $\text{IC}_{hL,hH}$ satisfied means that $U_{lL,hL} \stackrel{\text{defn}}{=} U_{hL} + x_{hL}\Delta\theta_1 \geq U_{hH} + x_{hH}\Delta\theta_2 q_{hH} + x_{hL}\Delta\theta_1$. On this other hand, $U_{lL,hH} = U_{hH} + x_{hH}\Delta\theta_2 q_{hH} + x_{hH}\Delta\theta_1$. Clearly, $U_{lL,hH} \leq U_{lL,hL}$ as long as $x_{hL} \geq x_{hH}$. QED

We are now ready to prove Theorem 1. The proof proceeds by progressively partioning the space of parameters into sets of parameters for which the solution shares the same binding IC and feasibility constraints. The logic of the proof is pretty simple, even if the mechanics can be involved. For this reason an exhaustive exposition of the proof of part I, scenario 1 is presented. The arguments in the rest of the proof are presented more briefly where they mirror those in part I, scenario 1.

Proof of part I of Theorem 1: $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) > 0$ i.e. $\Delta\theta_1 > \overline{q}\Delta\theta_2$

The binding constraints in the buyer-optimal efficient mechanism are $\text{IC}_{lH,hH}$, $\text{IC}_{hL,hH}$ and $\text{IC}_{lL,hL}$. The buyer's resulting expected utility is given by

$$\begin{aligned} &\alpha_{lH}x_{lH}W_{lH}(q_{lH}) + \alpha_{hH}x_{hH}[W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2] \\ &+\alpha_{hL}x_{hL}[W_{hL}(q_{hL}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] + \alpha_{lL}x_{lL}W_{lL}(q_{lL}) \end{aligned} \tag{28}$$

(where, again, we have highlighted the virtual welfares associated with each type). Keeping the probabilities fixed at $x_k = x_k^{FB}$, optimizing the q's in (28) requires that only q_{hH} be adjusted away from the efficient level and set equal to

$$q_{hH}^2 = \arg\max\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2\} \tag{29}$$

This reduces the informational rents of hL and lL. From Lemma 6(2), we know that $U_{lL,hL} \geq U_{lL,lH}$ as long as $U_{hL,hH} \geq U_{hL,lH}$. Hence, we need to consider only two scenarios:

Scenario 1: At q_{hH}^2, $U_{hL,hH} \geq U_{hL,lH}$, that is,

$$x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] \leq x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] \quad (30)$$

In this case, all IC constraints remain satisfied as we decrease q_{hH} to q_{hH}^2.
We now consider the optimization of the probabilities of winning. From (28) and the model assumptions, the virtual welfare associated with lL is the largest. Moreover, the virtual welfare associated with lH is larger than that associated with hH. Thus, we need to consider three cases depending on the relative ranking of the virtual welfare of hL with respect to the virtual welfares of hH and lH.

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{lH}(\overline{q}) > W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q_{hH}^2 \Delta\theta_2$

   [Solution 1.1.a]

   The optimal probabilities of winning are $x_k = x_k^{FB}$ since the ranking of the virtual welfares corresponds to the ranking of the first best welfares. All IC constraints are satisfied given the arguments above. The x's and q's are optimized given the binding constraints; $q_{lH} = \overline{q}$, $q_{hH} = q_{hH}^2$ and $q_{hL} = q_{lL} = \underline{q}$.

2. $VW_{lH} > VW_{hL} \geq VW_{hH} : W_{lH}(\overline{q}) > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q_{hH}^2 \Delta\theta_2$

   In this case, type lH generates a higher level of virtual welfare than type hL. Thus, the buyer would rather give the contract to supplier lH than to supplier hL, i.e. he would like to change the order of priority in the allocation. Increasing x_{lH} while decreasing x_{hL} concurrently (keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB}$ constant) does not initially affect any of the virtual welfares and it increases the buyer's expected utility. This process continues until either a new IC constraint binds or we have reach $x_{lH} = x_{lH}^{\max}$.

   We now argue that the only potentially new binding constraint is $\text{IC}_{lL,lH}$. To see this consider the following:

   (a) <u>hL's IC constraints:</u> Given that $U_{hL,lH} = U_{lH} - x_{lH}[\Delta\theta_1 - \Delta\theta_2\overline{q}]$ and that U_{lH} is not affected by the process, the incentives for hL to imitate lH have actually decreased. $\text{IC}_{hL,lL}$ remains satisfied as well since $\text{IC}_{lL,hL}$ is binding and $x_{lL} > x_{hL}$.

   (b) <u>lH's IC constraints:</u> Because $U_{lH,hL} = U_{hL} + x_{hL}(\Delta\theta_1 - \Delta\theta_2 q_{hH}^2)$ and $U_{lH,lL} = U_{hL} + x_{hL}\Delta\theta_1 - x_{lL}\Delta\theta_2\underline{q}$, the incentives for lH to imitate hL and lL have decreased ($U_{hL} = x_{hH}\Delta\theta_1 q_{hH}^2$ is not affected by the process).

   (c) <u>hH's IC constraints:</u> hH continues to have no incentive to imitate hH, hL or lL given that $\text{IC}_{lH,hH}$ and $\text{IC}_{hL,hH}$ are binding, and $U_{hH,lL}$ is not affected by the process.

(d) $\underline{lL\text{'s IC constraint:}}$ By Lemma 7, $\text{IC}_{lL,hH}$ is not affected by the process. By Lemma 6(2), $\text{IC}_{lL,lH}$ remains satisfied as long as $x_{lH} \leq x_{hL}$, but it could start binding afterwards.

Thus, we continue to increase x_{lH} at the cost of x_{hL} until either $x_{lH} = x_{lH}^{\max}$ or $\text{IC}_{lL,lH}$ starts binding, whichever comes first.

(a) $x_{lH} = x_{lH}^{\max}$ first. **[Solution 1.1.b]**

This means that $U_{lL,hL} \geq U_{lL,lH}$ even when x_{lH} reaches its maximum. This corresponds to the solution because there are no more opportunities to increase the buyer's expected utility: the q's are optimized given the binding IC constraints, the x's are optimized given the virtual welfare and the feasibility constraints. The solution is thus: $q_{lH} = \overline{q}$, $q_{hH} = q_{hH}^2$, $q_{hL} = q_{lL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB} > x_{lH} = x_{lH}^{\max} > x_{hL} = x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$. By the argument just above, all IC constraints are satisfied.

(b) $\text{IC}_{lL,lH}$ starts binding. **[Solution 1.1.c]**

At that point, $U_{lL,lH} = U_{lL,hL}$, that is, $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] = x_{hL}\Delta\theta_1 - x_{lH}\overline{q}\Delta\theta_2$ (note that by Lemma 6(2), this happens at $x_{lH} > x_{hL}$).

We now argue that we should be looking for a solution where both $\text{IC}_{lL,hL}$ and $\text{IC}_{lL,lH}$ are binding. Indeed, if only $\text{IC}_{lL,lH}$ binds, the virtual welfare associated with hL is W_{hL}^{FB} which is greater than the virtual welfare associated with lH. Thus the buyer would want to set x_{hL} back to x_{hL}^{FB}, but this would bring us back to the starting point.

Thus the buyer further increases his expected utility by increasing x_{lH} and decreasing x_{hL} while keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB}$ constant *and* $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}\overline{q}\Delta\theta_2$. This requires that we adjust q_{hH} and q_{lH}.

Such change corresponds to a change in the value of the Lagrangian multiplier on the $\text{IC}_{lL,lH}$ constraint. Using the expressions in (20) to (23), we can rewrite the expressions for lH and hH's virtual welfares as follows:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}q_{lH}\Delta\theta_2\} \tag{31}$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH} + \lambda_4}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL} + \alpha_{lL} - \lambda_4}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{32}$$

where λ_4 is the Lagrangian multiplier on the $\text{IC}_{lL,lH}$ constraint.

Thus, practically, we increase x_{lH} and decrease x_{hL} concurrently to keep $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB}$ constant. This implies a new value for q_{hH} and q_{lH} to ensure that $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2$. These correspond to a new value for λ_4 through (32). Specifically, λ_4 increases.

This process increases the virtual welfare associated with hL, $W_{hL}(\overline{q}) - \frac{\alpha_{lL}-\lambda_4}{\alpha_{hL}}\Delta\theta_1$, and decreases the virtual welfare associated with lH and hH (see (31) and (32)).

It continues until we have either reached the upper bound to x_{lH}, $x_{lH}^{\max}$, or the virtual welfares associated with lH and hL become equal:

$$\max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4^*}{\alpha_{lH}}q_{lH}\Delta\theta_2\} = W_{hL}(\underline{q}) - \frac{\alpha_{lL}-\lambda_4^*}{\alpha_{hL}}\Delta\theta_1$$

whichever comes first. Thus $\lambda_4 \in (0, \lambda_4^*) \subset (0, \alpha_{lL})$ as required by (23).

This defines the solution: $x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (31) and (32), q_{lH}, $q_{hH} < \overline{q}$. The x's are optimized given the virtual welfares and the feasibility constraints. The q's are optimized given the binding constraints.

All IC constraints remain satisfied. The arguments for this are the same as those we made above, except for $\text{IC}_{hL,lH}$, which follows because $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] \overset{U_{lL,hL}=U_{lL,lH}}{=} x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2 < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ when $x_{lH} > x_{hL}$.

3. $VW_{lH} \geq VW_{hH} > VW_{hL}$: $W_{lH}(\overline{q}) > W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}^2\Delta\theta_2 > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$.

   In this case, the ideal ordering of types in the allocation is $lL \succ lH \succ hH \succ hL$. The buyer increases his expected utility by decreasing x_{hL}, first to the benefit of x_{lH} (that is, keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB}$ constant), and then to the benefit of x_{hH} (that is, keeping $N(\alpha_{lH}x_{lH}^{\max} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB} + \alpha_{hH}x_{hH}) = 1$).

   This process initially does not affect any of the virtual welfares until a new IC constraint binds. By the same arguments as in point 2 above, we can establish that the first binding constraint is $\text{IC}_{lL,lH}$. When it binds $x_{hH}[\Delta\theta_1 - \Delta\theta_2 q_{hH}^2] = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_2\overline{q}$. At this point, $x_{lH} > x_{hL} > x_{hH}$ (the first inequality comes from Lemma 6(2)).

   Once this happens, any further improvement requires that we keep $U_{lL,hL} = U_{lL,lH}$ (otherwise, if $U_{lL,hL} < U_{lL,lH}$, $\text{IC}_{lL,hL}$ ceases to bind, the virtual welfare associated with hL bounces back to W_{hL}^{FB} and thus we get back to the starting point). We are thus in a similar situation as in point 2 above. Any further change in the x's requires some changes in the q's and thus in the value of the multiplier on the IC constraints. Using the expressions in (20) to (23), the resulting virtual welfares associated with lH, hH and hL are:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \quad (33)$$

$$VW_{hH} = \max_{hH}\{W_{hH}(q_{hH}) - \frac{(\alpha_{lH}+\lambda_4)}{\alpha_{hH}}\Delta\theta_1 - \frac{(\alpha_{hL}+\alpha_{lL}-\lambda_4)}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \quad (34)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\alpha_{lL}-\lambda_4}{\alpha_{hL}}\Delta\theta_1 \quad (35)$$

where $\lambda_4 \in (0, \alpha_{lL})$ is such that $U_{lL,hL} = U_{lL,lH}$ i.e. $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2$ for the current value of x_{hL} (x_{lH} and x_{hH} are well-defined once x_{hL} is defined given that lH has priority hH). Practically, a decrease in x_{hL} is associated with an increase in q_{hH}, a decrease in q_{lH} and an increase in λ_4. This decreases VW_{lH} and VW_{hH} and increases VW_{hL}.

The difference relative to Solution 1.1.c is what ends this process. Here, the process ends when either a new IC constraint binds or the relative ranking of virtual welfare changes.[28] The only new IC constraint that can bind is $\text{IC}_{lL,hH}$. This happens at $x_{hL} = x_{hH}$. Thus we need to distinguish the following cases depending on which event happens first:

(a) We have reached $VW_{lH} \geq VW_{hH} = VW_{hL}$ and $x_{lH} = x_{lH}^{\max}$. Then this is the solution. The buyer is indifferent between hH and hL. The qualities are given by the value of λ_4 that solves for $VW_{hH} = VW_{hL}$ in (34) and (35), $q_{lL} = q_{hL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB}$, $x_{lH} = x_{lH}^{\max} > x_{hL}^{\min} \geq x_{hL} > x_{hH} \geq x_{hH}^{FB}$. **[Solution 1.1.d]**

(b) We have reached $VW_{lH} \geq VW_{hH} = VW_{hL}$ at $x_{lH} < x_{lH}^{\max}$. Then the buyer can further increase his expected utility by decreasing x_{hL} and increasing x_{lH} keeping $U_{lL,lH} = U_{lL,hL}$. This further decreases VW_{lH} and VW_{hH} and increases VW_{hL}. The process stops when either $VW_{lH} = VW_{hL}$ or $x_{lH} = x_{lH}^{\max}$, whichever comes earlier. At the solution the q's are defined from (34) and (35) for the value of λ_4 at which the process stops, $q_{lL} = q_{hL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB}$, $x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min}$ and $x_{hH} = x_{hH}^{FB}$. This corresponds to **Solution 1.1.c.** above.

(c) We have reached $VW_{lH} = VW_{hH} > VW_{hL}$. (note that this implies that $q_{lH} < q_{hL}$ given (33) and (34)). The buyer further increases his expected utility by decreasing x_{hL} and adjusting x_{lH} and x_{hH} in a way that preserves $VW_{lH} = VW_{hH}$ and $U_{lL,lH} = U_{lL,hL}$.[29]

[28] No feasibility constraint binds in the process. Indeed, the only potential feasibility constraint would involve x_{hH} hitting its maximum but this never occurs before $x_{hH} = x_{hL}$.

[29] The feasibility constraints on the x's are $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hH}x_{hH}) \leq 1 - \alpha_{hL}^N$ and $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$

Thus λ_4 is fixed and the virtual welfares are not affected. This process continues until $x_{hL} = x_{hH}$ $(< x_{lH})$ at which point $U_{lL,hH}$ starts binding. At this stage we have:

$$\begin{aligned} U_{lL,lH} &= x_{hH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 = U_{lL,hH} = x_{hH}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \\ &= U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \end{aligned}$$

Using the expressions in (20) to (23), the virtual welfares are given by

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \quad (36)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH} - \frac{\alpha_{lH}+\lambda_4}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_4}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \quad (37)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\alpha_{lL}-\lambda_4-\lambda_6}{\alpha_{hL}}\Delta\theta_1 \quad (38)$$

where λ_4 and λ_6 are the multipliers on the $\text{IC}_{lL,lH}$ and $\text{IC}_{lL,hH}$ constraint respectively. There exists a value for λ_4 and λ_6 such that $VW_{lH} = VW_{hH} = VW_{hL}$ and $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$ and $N(\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB} + \alpha_{hH}x_{hH}) = 1$. Indeed, we have five equations and five unknowns: λ_4, λ_6, x_{lH}, x_{hL} and x_{hH} (from (38) and the fact $VW_{lH} = VW_{hL}$, we know that $\alpha_{lL} - \lambda_4 - \lambda_6 > 0$, thus $\alpha_{hL} + \alpha_{lL} - \lambda_4$ in (37) is ensured to be positive which is required by the non negative constraint on the multipliers).

These values for λ_4 and λ_6 correspond to the solution. At the solution, $x_{lH} > x_{hH} = x_{hL}$ (implied by $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$), $q_{lH} < q_{hL} < \overline{q}$ and $q_{hL} = q_{hH} = \underline{q}$. The buyer is indifferent among lH, hH and hL and the x's are thus optimized. The q's are optimized given the binding constraints and the value of the multipliers. No new constraint binds in the process. The argument for this is identical as the one in point 2, except for $\text{IC}_{hL,lH}$, which follows because $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] \overset{U_{lL,hL}=U_{lL,lH}}{=} x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2 < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ when $x_{lH} > x_{hL}$. **[Solution 1.1.e]**

(d) We have reached $x_{hH} = x_{hL}$. At this point, $\text{IC}_{lL,hH}$ starts binding. The rest of the argument is as in point (c) above: There exists a value for λ_4 and λ_6 such that $VW_{lH} = VW_{hH} = VW_{hL}$ and $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$ and $N(\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB} + \alpha_{hH}x_{hH}) = 1$. The solution is thus Solution 1.1.e.

Scenario 2: At q_{hH}^2, $U_{hL,hH} < U_{hL,lH}$, that is, $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] > x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$.

In this case, $\text{IC}_{hL,lH}$ becomes binding as we decrease q_{hH}. To reduce hL and lL's rents further, one now needs to decrease q_{lH} at the same time as q_{hH} in such a way that $U_{hL,hH} = U_{hL,lH}$, i.e.,

$x_{hH}^{FB}[W_{lH}(q_{hH})-W_{hL}(q_{hH})]=x_{lH}^{FB}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]$. (Note that this implies that $q_{lH}>q_{hH}$.) Formally, using (20) to (23) in Appendix A, we let q_{lH} and q_{hH} solve:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]\} \quad (39)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH})-\frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1-\frac{(\alpha_{hL}+\alpha_{lL}-\lambda_2^*)}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \quad (40)$$

for the value of $\lambda_2^* \in (0, \alpha_{hL}+\alpha_{lL})$ such that $x_{hH}^{FB}[W_{lH}(q_{hH})-W_{hL}(q_{hH})] = x_{lH}^{FB}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]$ (λ_2 is the multiplier on $\text{IC}_{hL,lH}$). Such value for λ_2 always exists. When $\lambda_2^*=0$, $q_{lH}=\overline{q}$ and $q_{hH}=q_{hH}^2$ so that $x_{hH}^{FB}[W_{lH}(q_{hH}^2)-W_{hL}(q_{hH}^2)] > x_{lH}^{FB}[W_{lH}(\overline{q})-W_{hL}(\overline{q})]$ from the definition of scenario 2. When $\lambda_2^*=\alpha_{hL}+\alpha_{lL}$, $q_{lH}<q_{hH}=\overline{q}$ and $x_{hH}^{FB}[W_{lH}(\overline{q})-W_{hL}(\overline{q})] < x_{lH}^{FB}[W_{lH}(q_{lH}^2)-W_{hL}(q_{lH}^2)]$).

Relative to the BOEM, only the rents of hL and lL have decreased. The IC constraint of hL is taken care of by construction, and $U_{lL,hL}\geq U_{lL,lH}$ from Lemma 6(1). Hence, all IC constraints remain satisfied.

We now optimize over the x's. Notice that $VW_{lH}=\max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]\} > W_{lH}(\overline{q}) > VW_{hH}$. Hence, we need to consider three cases depending on the relative ranking of the virtual welfare associated with hL.

1. $VW_{hL} \geq VW_{lH} > VW_{hH}$: $W_{hL}(\underline{q})-\frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq \max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]\}$ **[Solution 1.2.a]**

   The optimal probabilities are thus $x_k=x_k^{FB}$. The values of q_{lH} and q_{hH} are defined in (39) and (40) and $\overline{q}>q_{lH}>q_{hH}>q_{hH}^2$, $q_{hL}=q_{lL}=\underline{q}$.

2. $VW_{lH} > VW_{hL} \geq VW_{hH}$: $\max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]\} > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q_{hH}^2)-\frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1-\frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}^2\Delta\theta_2$ (note that the condition is on VW_{hH} evaluated at $\lambda_2=0$).

   At the current value of λ_2, the buyer prefers to give the contract to lH over hL. As we progressively increase x_{lH} at the expense of x_{hL}, while keeping $x_{hH}^{FB}[W_{lH}(q_{hH})-W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{hL})-W_{hL}(q_{hL})]$, we decrease λ_2 (i.e. increase q_{lH} and decrease q_{hH} - from (39) and (40)). This decreases VW_{lH} and increases VW_{hH}.

   This process continues until the relative ordering of virtual welfares changes or the binding IC constraints change (at least of one these two events happen before we reach the feasibility constraint $x_{lH}=x_{lH}^{\max}$). Specifically, the two IC constraints we need to worry about are $\text{IC}_{hL,lH}$ which stops binding when $\lambda_2=0$, and $\text{IC}_{lL,lH}$ which starts binding when $x_{lH}=x_{hL}$. This yields three cases depending on which event happens first:

(a) $VW_{lH} = VW_{hL}$ first (note that given the assumption of this case, $VW_{hL} \geq VW_{hH}$ always): We have then reached the solution. At the solution, the probabilities of winning are: $x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} > x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$ where x_{lH} and x_{hL} are defined implicitly by $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ for the values of q_{hH} and q_{lH} that solve (39) and (40) at the current value of λ_2 ($q_{hH} < q_{lH}$). The x's are optimized given the virtual welfares. The q's are optimized given the binding constraints and the value of λ_2. **[Solution 1.2.b]**

(b) $\lambda_2 = 0$ first. $\text{IC}_{hL,lH}$ ceases to bind and $q_{hH} = q_{hH}^2$ and $q_{lH} = \overline{q}$. As x_{lH} further increases and x_{hL} decreases, the buyer increases his expected utility. None of the virtual welfares are affected in the process, and thus this continues until we either reach $x_{lH} = x_{lH}^{\max}$ or $\text{IC}_{lL,lH}$ starts binding (this happens when $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] = x_{hL}\Delta\theta_1 - x_{hL}\Delta\theta_2\overline{q}$.).

In the first case, we are as in **Solution 1.1.b:** $x_{lL} = x_{lL}^{FB} > x_{lH} = x_{lH}^{\max} > x_{hL} = x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{hH} = q_{hH}^2$ and $q_{lH} = \overline{q}$. The x's are optimized given that, by assumption, $VW_{hL} \geq VW_{hH}$.

In the second case, we are as in **Solution 1.1c.** Thus, $x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (31) and (32), q_{lH}, $q_{hH} < \overline{q}$.

(c) $x_{lH} = x_{hL}$ first. At this point, $\text{IC}_{lL,lH}$ starts binding. Based on the expressions from (20), reworked using the equalities (21) to (23), the associated virtual welfares are given by:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_3)}{\alpha_{lH}}\Delta\theta_2 q_{lH} + \frac{\alpha_{hL} + \lambda_5 - \lambda_3}{\alpha_{lH}}\Delta\theta_1\} \quad (41)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\lambda_3}{\alpha_{hH}}\Delta\theta_2 q_{hH} - \frac{\alpha_{lH} + \alpha_{hL} + \alpha_{lL} - \lambda_3}{\alpha_{hH}}\Delta\theta_1\} \quad (42)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\lambda_5}{\alpha_{hL}}\Delta\theta_1 \quad (43)$$

There exist values for λ_3 and λ_5 such that (1) $\overline{x}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ and (2) $VW_{lH} = VW_{hL}$. To see this, note that the progressive adjustment of x_{lH} until $x_{lH} = x_{hL}$ implies that there exists a value for λ_3 that satisfies condition (1). Once λ_3 is fixed, there is a value of λ_5 that ensures condition (2). Indeed for any feasible λ_3, when $\lambda_5 = 0$, the virtual welfare of hL is greater. When $\lambda_5 = \alpha_{lL}$ and $\lambda_2 = \alpha_{hL} + \alpha_{lL} - \lambda_3$, this follows from the fact that we have assume that $VW_{lH} > VW_{hL}$ when $\text{IC}_{lL,lH}$ becomes binding.

Note that $\lambda_2 = \alpha_{hL} - \lambda_3 + \lambda_5$. If the implied λ_2 is positive, this is the solution: $x_{lL} =$

$x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} = \overline{x} = x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$ and the q's solving (41) through (43) above for the values of λ_3 and λ_5 that satisfy conditions (1) and (2) (in particular, $q_{lH} > q_{hH}$). The x's are optimized given the virtual welfares: the buyer is indifferent between lH and hL and $VW_{lH} > VW_{hH}$ follows from the comparison between (41) and (42) when $q_{lH} > q_{hH}$. The q's are optimized given the binding constraints and the value of the multipliers. [**Solution 1.2.c**]

If the implied λ_2 is strictly negative, then $\text{IC}_{hL,lH}$ ceases to bind at some point. We are then in the same situation as in **Solution 1.1.c**. At the solution, $x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (31) and (32), $q_{lH}, q_{hH} < \overline{q}$.

3. $VW_{lH} > VW_{hH} > VW_{hL} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 < W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q_{hH}^2 \Delta\theta_2$ (note that the condition is on VW_{hH} evaluated at $\lambda_2 = 0$).

In this case, we ideally want to decrease x_{hL}, first to the benefit of x_{lH} (then, possibly to the benefit of x_{hH}). Doing this while keeping $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$, requires that we decrease λ_2 (cf. (39) and (40)). This decreases VW_{lH} and increases VW_{hH}, but given the condition on this case, the ordering of virtual welfares is not affected. Thus, this process continues until, either we reach $\lambda_2 = 0$ (and thus $\text{IC}_{hL,lH}$ ceases to bind) or $x_{lH} = x_{hL}$ (and thus $\text{IC}_{lL,lH}$ starts binding).

(a) We reach $x_{lH} = x_{hL}$ when $\lambda_2 > 0$: This implies that $IC_{lL,lH}$ becomes binding in the process. Optimizing from now on with constraints $IC_{lH,hH}$, $IC_{lL,lH}$, $IC_{lL,hL}$, $IC_{hL,lH}$ and $IC_{hL,hH}$ binding requires that we keep $x_{lH} = x_{hL}$. The virtual welfares are given by (41), (42) and (43). Like in part 1, scenario 2, case 2c, we proceed by first looking for values of λ_3, λ_5 and q's such that (1) $\overline{x}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$, i.e. $U_{lL,hL} = U_{lL,lH}$ and $U_{hL,hH} = U_{hL,hH}$ and (2) $VW_{lH} = VW_{hL}$.

If the implied λ_2 is positive, then this is the solution (solution 1.2.c) because condition (1) implies that $q_{lH} > q_{hH}$, which in turn ensures that $VW_{lH} = VW_{hL} > VW_{hH}$. The x's are optimized, and so are the q's.

If the implied λ_2 is negative, then we are as in part I, scenario 1, case 3: the binding constraints are $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{lH,hH}$ and $\text{IC}_{hL,hH}$. This leads to solutions 1.1.c, 1.1.d or 1.1.e.

(b) We reach $\lambda_2 = 0$ when $x_{lH} \leq x_{hL}$. We can continue to increase x_{lH} at the expense of x_{hL}, and afterwards if necessary increase x_{hH} at the expense of x_{hL} until $IC_{lL,lH}$ starts binding. ($IC_{hL,lH}$ no longer binds because increasing x_{lH} beyond x_{hL} means that

$x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$). The case then reduces to part 1, scenario 1, case 3, implying one of solutions 1.1.c, 1.1.d or 1.1.e apply.

Proof of part II of Theorem 1: $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) < 0$ i.e. $\Delta\theta_1 < \overline{q}\Delta\theta_2$

The binding constraints in the buyer-optimal efficient mechanism are $\mathrm{IC}_{lH,hH}$, $\mathrm{IC}_{hL,lH}$ and $\mathrm{IC}_{lL,hL}$. The buyer's resulting expected utility is given by

$$\alpha_{lH}x_{lH}[W_{lH}(q_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}\Delta\theta_2] + \alpha_{hH}x_{hH}[W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1]$$
$$+\alpha_{hL}x_{hL}[W_{hL}(q_{hL}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] + \alpha_{lL}x_{lL}W_{lL}(q_{lL}) \tag{44}$$

Keeping the probabilities fixed at $x_k = x_k^{FB}$, optimizing the q's requires that q_{lH} be set equal to

$$q_{lH}^2 = \arg\max\{W_{lH}(q_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}\Delta\theta_2\} \tag{45}$$

This reduces the informational rents of hL and lL. By Lemma 6(1), we know that $U_{lL,hL} > U_{lL,lH}$ as long as $U_{hL,lH} \geq U_{hL,hH}$. Hence, we need to consider only two scenarios, depending on whether $IC_{hL,hH}$ binds at q_{LH}^2:

Scenario 1: At q_{lH}^2, $U_{hL,lH} \geq U_{hL,hH}$, i.e., $x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] \leq x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ In this case, all IC constraints remain satisfied as we decrease q_{lH} to q_{lH}^2. Note that $W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2) \equiv \Delta\theta_1 - \Delta\theta_2 q_{lH}^2 < 0$. We now consider the optimization of the probabilities of winning. From (44), the virtual welfare associated with lL is the largest. This leaves four cases depending on the relative ranking of lH, hH and hL:

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH} : [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] \geq [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2] \geq [W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1]$ **[Solution 2.1.a]**

   The optimal probabilities of winning are $x_k = x_k^{FB}$ since the ranking of the virtual welfares corresponds to the ranking of the first best welfares. All IC constraints are satisfied. The x's and q's are optimized given the binding constraints.

2. $VW_{lH} > VW_{hH} \geq VW_{hL} : [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2] > W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 \geq [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1]$; or

   $VW_{lH} > VW_{hL} \geq VW_{hH} : [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2] > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] \geq W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1$

   The buyer would like to increase x_{lH} at the expense of x_{hL}. Doing this does not affect the supplier hL's IC constraint: $U_{hL,lH} \geq U_{hL,hH}$ corresponds to $x_{lH}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] \leq x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ and $W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2) < 0$. Moreover, as long as $x_{hL} > x_{lH}$, the change in x_{lH} does not affect lL's IC constraint either (Lemma 6(1)). Thus, changing x_{lH} does not initially affect the virtual welfares.

When we reach $x_{lH} = x_{hL} = \overline{x}$, $\text{IC}_{lL,lH}$ starts binding since $U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}q_{lH}^2\Delta\theta_2 + x_{hH}\Delta\theta_1$ and $U_{lL,lH} = x_{lH}q_{lH}^2\Delta\theta_2 + x_{hH}\Delta\theta_1$. Define $\lambda_5^* \in (0, \alpha_{lL})$, the value of λ_5 that equalizes the virtual welfares associated with lH and hL:

$$W_{lH}(q_{lH}^2) + \frac{(\alpha_{hL} + \lambda_5^*)}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL} + \alpha_{lL}}{\alpha_{lH}}\Delta\theta_2 q_{lH}^2 = W_{hL}(\underline{q}) - \frac{\lambda_5^*}{\alpha_{hL}}\Delta\theta_1 \qquad (46)$$

(from (20) to (23)). Such a value for λ_5 exists. When $\lambda_5 = 0$, the virtual welfare associated with hL is larger. When $\lambda_5 = \alpha_{lL}$, the virtual welfare of lH is bigger by assumption. Note that this process does not affect the virtual welfare associated with hH, which remains unchanged.

(a) **[Solution 2.1.b]** If at λ_5^*, $VW_{lH} = VW_{hL} > VW_{hH}$, then the solution is $q_{lH} = q_{lH}^2$, $q_{hH} = \overline{q}$ and $q_{hL} = q_{lL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB}, x_{hH} = x_{hH}^{FB}$, and $x_{lH} = x_{hL} = \overline{x}$. All IC constraints are satisfied. The q's and the x's are optimized given the binding constraints (in particular, the buyer is indifferent between lH and hL, but strictly prefer these to hH).

(b) If at λ_5^*, $VW_{lH} = VW_{hL} < VW_{hH}$, the buyer prefers hH to lH or hL. He increases his expected utility by raising x_{hH} while keeping $U_{lL,lH} = U_{lL,hL}$, that is, $x_{lH} = x_{hL}$, and $\lambda_5 = \lambda_5^*$. This process does not initially affect any of the virtual welfares until $\text{IC}_{hL,hH}$ starts binding (this happens at $x_{hL} = x_{lH} > x_{hH}$ given that $q_{lH} = q_{lH}^2 < q_{hH} = \overline{q}$ when $U_{hL,hH} \leq U_{hL,lH}$).

From then on, $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. The expressions for the resulting virtual welfares are given by:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_3)}{\alpha_{lH}}\Delta\theta_2 q_{lH} + \frac{\alpha_{hL} + \lambda_5 - \lambda_3}{\alpha_{lH}}\Delta\theta_1\} \qquad (47)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\lambda_3}{\alpha_{hH}}\Delta\theta_2 q_{hH} - \frac{\alpha_{lH} + \alpha_{hL} + \alpha_{lL} - \lambda_3}{\alpha_{hH}}\Delta\theta_1\} \qquad (48)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\lambda_5}{\alpha_{hL}}\Delta\theta_1 \qquad (49)$$

The buyer increases his expected utility by continuing to increase x_{hH} at the cost of x_{hL} and x_{lH}, while satisfying: (1) $U_{lL,lH} = U_{lL,hL}$ (thus $x_{lH} = x_{hL}$), (2) $U_{hL,hH} = U_{hL,lH}$, that is $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$, and (3) $VW_{lH} = VW_{hL}$. This requires an increase in λ_3 and a decrease in λ_5, i.e. a rise in q_{lH} and a decrease in q_{hH} (nonetheless, $q_{lH}^2 < q_{lH} < q_{hH}$ remains as long as $VW_{lH} \leq VW_{hH}$ as is apparent from (47) and (48)).[30]

This process stops when either $VW_{hH} = VW_{lH} = VW_{hL}$ or we hit a non negativity constraint for the multiplier $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$.

i. **[Solution 2.1.d]** Suppose $VW_{hH} = VW_{lH} = VW_{hL}$ at a point where $\lambda_2 \geq 0$. Then we have reached the solution. The q's are defined from (47) and (48) for the values of λ_3 and λ_5 that equalize the virtual welfares (note that this implies that $q_{lH} < q_{hH}$, so that, in turn, $U_{hL,hH} = U_{hL,lH}$ implies $x_{lH} > x_{hH}$ as required for incentive compatibility). The x's are such that $x_{lL} = x_{lL}^{FB}$, and $x_{lH}^{FB} > x_{lH} = x_{hL} > x_{hH} > x_{hH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.[31] All IC constraints are satisfied. The q's are optimized given the binding constraints. The x's are optimized given the resulting virtual welfares (the buyer is indifferent among lH, hL and hH).

ii. **[Solution 2.1.e]** Suppose λ_2 reaches zero at a point where $VW_{hH} > VW_{lH} = VW_{hL}$.

Let λ_5^{**}, the value of λ_5 at this point. We also have $q_{lH}^2 < q_{lH} < q_{hH}$ and $x_{lH} = x_{hL} > x_{hH}$ at this point. The buyer further increases his utility by increasing x_{hH} at the cost of x_{lH} and x_{hL}, while keeping $U_{lL,lH} = U_{lL,hL}$ and $VW_{lH} = VW_{hL}$ (i.e. $\lambda_5 = \lambda_5^{**}$ and the q's are fixed at $q_{lH} < q_{hH}$).[32] This process at first does not affect the virtual welfares (since λ_5 is fixed, we keep having $VW_{hH} > VW_{lH} = VW_{hL}$), until IC$_{lL,hH}$ starts binding.[33] At this stage we have:

$$\begin{aligned} U_{lL,lH} &= x_{hH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 = U_{lL,hH} = x_{hH}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \\ &= U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \end{aligned}$$

thus $x_{hL} = x_{hH} < x_{lH}$. To keep increasing the buyer's welfare while satisfying all three constraints out of lL requires that we keep $x_{hL} = x_{hH}$. Thus we increase both x_{hL} and x_{hH} at the expense of x_{lH} (this will indeed increase the buyer's utility since $VW_{hH} > VW_{lH} = VW_{hL}$), and adjust the q's as needed, that is, we

[30] Formally, we have four equations (the three constraints mentioned in the text, plus the feasibility constraint $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$) and five unknowns: x_{hH}, x_{hL}, x_{lH} and λ_3 and λ_5 (the q's are determined on the basis of the λ's by (47) and (48)). Thus any value for x_{hH} pins down the other variables.

[31] No other feasibility constraint for the probabilities of winning binds, except for the one-type constraint for x_{lL}.

[32] The exact way in which x_{lH} and x_{hL} are decreased is determined by $U_{lL,lH} = U_{lL,hL}$, i.e. $x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1 = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2$ and the feasibility constraint $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

[33] This is the only constraint that can bind in the process. No new constraint can bind out of lH since $U_{lH} = x_{hH}\Delta\theta_1$ increases and alternatives decrease. No new constraint can bind out of hL because $\Delta\theta_1 - \Delta\theta_2 q_{hH} < 0$ given that $q_{hH} > q_{lH} > q_{lH}^2$ and $VW_{hH} \geq VW_{lH}$.

increase q_{lH} and decrease q_{hH}. We do this until $VW_{lH} = VW_{hL} = VW_{hH}$. We have then reached the solution. At the solution, $q_{lH} < q_{hH}$ and $x_{lL} = x_{lL}^{FB}$, and $x_{lH}^{FB} > x_{lH} > x_{hL} = x_{hH} > x_{hH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

3. $VW_{hL} > VW_{hH} > VW_{lH} : [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] > [W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1] > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2]$; or
$VW_{hH} > VW_{hL} > VW_{lH} : W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2]$

   In this case, the buyer would like to increase x_{hH} at the expense of x_{lH}. As we increase x_{hH} and decrease x_{lH}, we reach a point where $x_{lH}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] = x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$, that is, $\text{IC}_{hL,hH}$ starts binding.

   A candidate solution is defined by the value of $\lambda_2 \in (0,\ \alpha_{hL} + \alpha_{lL})$ that equates VW_{lH} and VW_{hH} :

$$\max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{50}$$

   (from (20) to (23)). Such value for λ_2 exists since the virtual welfare of lH is larger than that of hH at $\lambda_2 = 0$, and smaller at $\lambda_2 = \alpha_{hL} + \alpha_{lL}$ by assumption. By inspection of (50), this happens at $\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}} < \frac{\lambda_2^*}{\alpha_{lH}}$ that is, the resulting q's are such that $q_{lH}^2 < q_{lH} < q_{hH}$. Finally, we require that $U_{hL,lH} = U_{hL,hH}$, that is, $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ which implies that $x_{lH} > x_{hH}$ as required by incentive compatibility.

   This process only affected VW_{hL} and VW_{hH}. If $VW_{hL} > VW_{hH} = VW_{lH}$ at this point, then this is indeed the solution. The other variables are set such that $x_{lL} = x_{lL}^{FB}$, $x_{hL} = x_{hL}^{FB}$, and $q_{hL} = q_{lL} = \overline{q}$. The q's are optimized given the values of the multipliers and the binding constraints. The x's are optimized given the resulting virtual welfares. All IC constraints are satisfied ($\text{IC}_{lL,lH}$ satisfied given Lemma 6(1)). **[Solution 2.1.c]**

   If $VW_{hL} < VW_{hH} = VW_{lH}$, the buyer can further increase his expected utility by increasing x_{hH} and x_{lH} at the cost of x_{hL}. He does so while keeping $\lambda_2 = \lambda_2^*$ so that $VW_{hH} = VW_{lH}$. The exact way in which x_{hH} and x_{lH} are increased is pinned down by $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. This process does not affect the virtual welfare, until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (47) - (49). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines: we look for a solution where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$, or $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{lL,hH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$. [**Solution 2.1.d or 2.1.e**]

4. $VW_{hH} > VW_{lH} > VW_{hL}$: $W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 > [W_{lH}(q^2_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q^2_{lH}\Delta\theta_2] > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1]$

   Given the ordering of virtual welfares, the buyer is first tempted to increase x_{hH} at the expense of x_{hL}.[34] Two things can happen in the process: (1) $\text{IC}_{lL,lH}$ starts binding (this happens at $x^{FB}_{lH} = x_{hL}$ because $U_{lL,lH} = x_{lH}\Delta\theta_2 q^2_{lH} + x_{hH}\Delta\theta_1$ and $U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}\Delta\theta_2 q^2_{lH} + x_{hH}\Delta\theta_1$), (2) $\text{IC}_{hL,hH}$ starts binding (this happens at a point where $x_{hH} < x^{FB}_{lH}$ since $x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] = x^{FB}_{lH}[W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH})]$ at that point, and $W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH}) < 0$ from the definition of scenario 1). We examine each case in turn.

   (a) $IC_{lL,lH}$ binds first ($x^{FB}_{lH} = x_{hL}$)

   Let λ^*_5, the value of λ_5 that equalizes VW_{lH} and VW_{hL}. This was defined in (46). We now have $VW_{hH} > VW_{lH} = VW_{hL}$. Thus the buyer can increase his welfare by increasing x_{hH}. The rest of the solution is as described in 2(b) above. [**Solution 2.1.d or Solution 2.1.e**].

   (b) $\text{IC}_{hL,hH}$ binds first:

   This happens at $x_{hL} > x^{FB}_{lH} > x_{hH}$ (the first inequality comes from the fact that $\text{IC}_{hL,hH}$ binds first; the second inequality comes from the fact that $q_{lH} < q_{hH} = \overline{q}$ at the point where $\text{IC}_{hL,hH}$ starts binding). Increasing further x_{hH} at the expense of x_{hL}, while keeping $x^{FB}_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ requires that we decrease q_{hH} and increase q_{lH}. This corresponds to a rise in λ_3, a decrease in VW_{hH} and an increase in VW_{lH}. This process stops when either $VW_{lH} = VW_{hH}$ or $x_{lH} = x_{hL}$ whichever comes first (note at this stage $x_{lH} = x_{hL} > x_{hH}$ and $IC_{lL,lH}$ starts binding). If $VW_{lH} = VW_{hH}$ first, we can continue to increase the buyer's utility by decreasing x_{hL}, this time to the benefit of both lH and hH while keeping $VW_{lH} = VW_{hH}$ and $U_{hL,hH} = U_{hL,lH}$ (note that this implies $q_{lH} < q_{hH}$ and $x_{hL} > x_{hH}$). This process continues until $x_{hL} = x_{lH}$ at which point $IC_{lL,lH}$ starts binding.

   Thus, in both events, we reach a point where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. From then on, the virtual welfares are those defined in (47) -

[34] That is, keeping the equality $N(\alpha_{lL}x^{FB}_{lL} + \alpha_{lH}x^{FB}_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

(49). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines: we look for a solution where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$, or $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{lL,hH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$. **[Solution 2.1.d or 2.1.e]**

Scenario 2: At q_{lH}^2, $U_{hL,hH} > U_{hL,lH}$ that is, $x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] > x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ In this case, $\text{IC}_{hL,hH}$ becomes binding as we decrease q_{lH} towards q_{lH}^2. To decrease the rents of hL and lL, we now need to decrease q_{lH} and q_{hH}, holding $U_{hL,hH} = U_{hL,lH}$. The optimal q's are defined by:

$$\begin{aligned} q_{lH}^* &= \arg\max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \\ q_{hH}^* &= \arg\max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH} + \lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL} + \alpha_{lL} - \lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \end{aligned}$$

where $\lambda_2^* \in (0, \alpha_{hL} + \alpha_{lL})$ is chosen such that $x_{lH}^{FB}[W_{lH}(q_{lH}^*) - W_{hL}(q_{lH}^*)] = x_{hH}^{FB}[W_{lH}(q_{hH}^*) - W_{hL}(q_{hH}^*)]$. Note that the sign of $W_{lH}(q_{lH}^*) - W_{hL}(q_{lH}^*) = \Delta\theta_1 - \Delta\theta_2 q_{lH}^*$ is not pinned down *a priori* so that q_{lH} and q_{hH} cannot be ranked. No other new constraint binds in the process (Lemma 6(1)).

We now consider the optimization of the probabilities of winning. We need to consider five cases:

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH}$: $W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* \geq W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^*$.

   The optimal probabilities of winning are $x_k = x_k^{FB}$. This corresponds to **Solution 1.2.a** except that q_{lH} and q_{hH} cannot be ranked *a priori*.

2. $VW_{lH} > VW_{hL} \geq VW_{hH}$: $W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2}{\alpha_{hH}}\Delta\theta_2 q_{hH}^*$

   $VW_{lH} > VW_{hH} > VW_{hL}$: $W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* > W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$

   The buyer would like to increase x_{lH} at the expense of x_{hL}. Doing this while keeping $U_{hL,hH} = U_{hL,lH}$ requires that we adjust the q's and thus λ_2. Specifically, if $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* > 0$, we need to decrease λ_2, otherwise, we need to increase it. In both cases, VW_{lH} goes down and VW_{hH} goes up. This process continues until either a new IC constraint binds or the relative ranking of the virtual welfare changes. Since $x_{lH} > x_{lH}^{FB} > x_{hH}$, the only IC constraint to worry about is $\text{IC}_{lL,lH}$. This gives us three cases to consider depending on which event happens first:

(a) $VW_{lH} = VW_{hL} \geq VW_{hH}$: We have reached the solution: $x_{lL} = x_{lL}^{FB}$, $x_{hH} = x_{hH}^{FB}$ and $x_{hL}^{FB} > x_{hL} > x_{lH} > x_{lH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL}) = 1 - \alpha_{hH}^{N}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} determined by the value of λ_2 that equates $VW_{hH} = VW_{lH}$. This corresponds to **Solution 1.2.b**.

(b) $VW_{lH} = VW_{hH} > VW_{hL}$: Note that this means that $q_{lH} < q_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH} < 0$ since $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. The buyer continues to increase his expected utility by decreasing x_{hL}, this time, to the benefit of both x_{lH} and x_{hH}, doing so while keeping $VW_{lH} = VW_{hH}$ and $U_{hL,lL} = U_{hL,lH}$. Thus λ_2 is fixed and so are q_{lH} and q_{hH}. Therefore $x_{lH} > x_{hH}$. This process continues until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. From then on, the virtual welfares are those defined in (47) - (49). (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in scenario 1, 2(b) above ($VW_{hH} > VW_{lH} = VW_{hL}$), and the proof thus proceeds along the same lines. [**Solution 2.1.d or 2.1.e**]

(c) $x_{hL} = x_{lH}$, i.e. $\text{IC}_{lL,lH}$ starts binding. From then on, $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. The virtual welfares are those defined in (47) - (49). (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. If $VW_{lH} = VW_{hL} > VW_{hH}$, we have reached the solution: $x_{lL} = x_{lL}^{FB}$, $x_{hL}^{FB} > x_{lH} = x_{hL} = \overline{x} > x_{lH}^{FB}$, $x_{hH} = x_{hH}^{FB}$, q_{lH}, $q_{hH} < \overline{q}$ and $q_{lL} = q_{hL} = \overline{q}$. All IC constraints are satisfied and the q's and x's are optimal given the resulting virtual welfares. [**Solution 1.2.c**]

If $VW_{lH} = VW_{hL} < VW_{hH}$, we can conclude that $q_{lH} < q_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH} < 0$ since $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. We are thus in the same situation as in scenario 1, 2(b) above. [**Solution 2.1.d or 2.1.e**]

3. $VW_{hL} > VW_{hH} > VW_{lH}$: $W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 > W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* > W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^*$.

(Note that this implies $q_{lH}^* < q_{hH}^*$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* < 0$ given that $x_{lH}^{FB}[W_{lH}(q_{lH}^*) - W_{hL}(q_{lH}^*)] = x_{hH}^{FB}[W_{lH}(q_{hH}^*) - W_{hL}(q_{hH}^*)]$). The buyer wants to increase x_{hH} at the expense of x_{lH}. This requires adjusting λ_2 to maintain the equality $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) -$

$W_{hL}(q_{hH})]$. Specifically, λ_2 decreases, q_{lH} increases and q_{hH} decreases, until $VW_{hH} = VW_{lH}$. This occurs at $x_{lH} > x_{hH}$. Indeed, at $x_{lH} = x_{hH}$, $q_{lH} = q_{hH}$ thus $\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2}{\alpha_{hH}}\Delta\theta_2 q_{hH} = \frac{\lambda_2}{\alpha_{lH}}\Delta\theta_2 q_{lH}$ implying that $VW_{hH} < VW_{lH}$. The solution is thus $x_{lL} = x_{lL}^{FB}, x_{hL} = x_{hL}^{FB}$ and $x_{lH}^{FB} > x_{lH} > x_{hH} > x_{hH}^{FB}$ and $q_{lH} < q_{lH} < q_{hH} < \overline{q}$. This corresponds to **solution 2.1.c**

4. $VW_{hH} \geq VW_{lH} \geq VW_{hL}$: $W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* \geq W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* \geq W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$

   Note that this implies that $q_{lH}^* < q_{hH}^*$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* < 0$. Define $\lambda_2^{**} \in (0, \lambda_2^*)$ such that

$$\max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\lambda_2^{**}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^{**}}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} = \max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^{**}}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^{**}}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \quad (51)$$

   This implies $q_{lH}^* < q_{lH} < q_{hH} < q_{hH}^*$ and $VW_{lH} = VW_{hH} > VW_{hL}$.

   >From there, the buyer can increase his expected utility by increasing x_{hH} and x_{lH} at the cost of x_{hL}. He does so while keeping $\lambda_2 = \lambda_2^{**}$ so that $VW_{hH} = VW_{lH}$. The exact way in which x_{hH} and x_{lH} are increased is pinned down by $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. This process does not affect the virtual welfare, until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (47) - (49) (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above. **[Solution 2.1.d or 2.1.e]**

5. $VW_{hH} > VW_{hL} > VW_{lH}$: $W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 > W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^*$

   We are again in a situation where $q_{lH}^* < q_{hH}^*$ and $\Delta\theta_1 - \Delta\theta_2 q_{hH}^* < 0$. The buyer would like to increase x_{hH} at the expense of x_{lH}. Doing so while keeping $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ requires an adjustment in λ_2, leading to VW_{lH} decreasing and VW_{hH} increasing. This process continues until we reach λ_2^{**} which corresponds to $VW_{lH} = VW_{hH}$ (as defined in (51)). Since $\Delta\theta_1 - \Delta\theta_2 q_{hH}^* < 0$, the corresponding qualities and x's are such that $q_{lH}^* < q_{lH} < q_{hH} < q_{hH}^*$ and $x_{hH} < x_{lH}$.

   We now need to distinguish two cases depending whether $VW_{hL} > VW_{lH} = VW_{hH}$ or $VW_{lH} = VW_{hH} > VW_{hL}$.

   (a) $VW_{hL} > VW_{lH} = VW_{hH}$: Then we have reached the solution: $x_{lL} = x_{lL}^{FB}$, $x_{hL} = x_{hL}^{FB}$ and $x_{lH}^{FB} > x_{lH} > x_{hH} > x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and $q_{lH}^* < q_{lH} < q_{hH} < q_{hH}^*$ as defined by

(51). This corresponds to **Solution 2.1.c.**

(b) $VW_{lH} = VW_{hH} > VW_{hL}$: the buyer further increases his expected utility by increases x_{lH} and x_{hH} at the expense of x_{hL} while keeping $VW_{hH} = VW_{lH}$ (that is keeping λ_2 and the q's fixed) and $U_{hL,lH} = U_{hL,hH}$ (thus $x_{hH} < x_{lH}$). This process does not affect the virtual welfares until $x_{hL} = x_{lH}$ and $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (47) - (49) (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines. [**Solution 2.1.d or 2.1.e**]