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ABSTRACT

Indeterminacy in Dynamic Models: When Diamond Meets Ramsey*

In this paper, we consider an aggregate overlapping generations model with endogenous labour, consumption in both periods of life, homothetic preferences and productive external effects coming from the average capital and labour. We show that under realistic calibrations of the parameters, in particular a large enough share of first period consumption over the wage income, local indeterminacy of equilibria cannot occur with capital externalities alone but that it can occur when there are only, however small, labour externalities. More precisely, under gross substitutability, the existence of multiple equilibria requires a large enough elasticity of capital-labour substitution and a large enough elasticity of the labour supply. We also show that if labour externalities are slightly stronger and the elasticity of labour supply is larger, local indeterminacy occurs in a Cobb-Douglas economy. Finally, we show that, as a consequence of our restriction on first period consumption, a locally indeterminate steady state is generically characterized by an under-accumulation of capital. It follows therefore that while agents live over a finite number of periods, the conditions for the existence of locally indeterminate equilibria are very similar to those obtained within infinite horizon models and that from this point of view, Diamond meets Ramsey.

JEL Classification: C62, E32 and O41
Keywords: capital and labour externalities, endogenous cycles, endogenous labour supply, indeterminacy, overlapping generations and under-accumulation

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1 Introduction

In this paper, we consider an aggregate two periods overlapping generations model with endogenous labor, consumption in both periods of life and productive externalities coming from the average capital and labor. We show that when the share of first period consumption over the wage income is large, local indeterminacy of equilibria does not emerge when there are only capital externalities but that it does when there are only small labor external effects. We prove that sunspot fluctuations are fully compatible with small market imperfections and realistic calibrations for the fundamentals as soon as the elasticity of the labor supply is large enough.

In the recent period, the Ramsey one-sector growth model augmented to include endogenous labor supply and external effects has become a standard framework for the analysis of local indeterminacy and expectations-driven fluctuations based on the existence of sunspot equilibria.[1] While positive capital externalities alone do not provide any mechanism for the occurrence of a continuum of equilibria,[2] Benhabib and Farmer [1] show that strong labor externalities which generate an increasing aggregate labor demand function with respect to wage are necessary for the existence of local indeterminacy in a Cobb-Douglas economy. This conclusion has been widely criticized from the fact that strong increasing returns and a positively sloped aggregate labor demand function cannot be supported empirically. More recently, Pintus [18] considers a general formulation for preferences and technology and relaxes the conditions of Benhabib and Farmer. He shows that local indeterminacy arises under small labor externalities (i.e. a decreasing aggregate labor demand function) and no capital externalities provided that the elasticity of capital-labor substitution is sufficiently larger than one, the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply are large enough.

A similar level of generality has not been experienced by the analysis of aggregate overlapping generations models. Indeed, following Reichlin [20], most of the contributions have focussed on a particular formulation without first period consumption. In such a framework, Cazzavillan [4] then shows that local indeterminacy easily occurs for a large set of parameters values

[1]See the recent survey of Benhabib and Farmer [2].

[2]See Kehoe [14], Boldrin and Rustichini [3].

as soon as small capital externalities are considered. However, assuming that the share of first period consumption over the wage income is equal to zero implies to consider calibrations which remain at variance with standard empirical estimates and prevents from using this model to understand aggregate economic behavior. Extended formulations in which agents consume over their whole life-cycle have been recently considered under perfect competition, i.e. without external effects. But then the uniqueness property of the equilibrium is difficult to destroy.[3] It is shown in particular that local indeterminacy becomes less likely when the share of first period consumption is large.[4] Proceeding as in the framework of infinite horizon models, our objective is then to analyze the occurrence of expectations-driven fluctuations within an aggregate overlapping generations model augmented to include capital and labor externalities in production, which is general enough to provide room for local indeterminacy under realistic parameterizations for the fundamentals.

In order to keep a tractable analysis while introducing simultaneously capital and labor externalities together with current consumption, we need to introduce two crucial simplifying assumptions on preferences: we consider indeed a life-cycle utility function which is firstly, separable between consumption and leisure, and secondly, linearly homogeneous with respect to young and old consumptions. This last assumption generates a fundamental simplification by allowing to write the capital accumulation equation as a function of the share of young agents' consumption over the wage income and then to provide a simple characterization of the local stability properties of equilibrium paths. Within this framework, we will prove that while agents live over a finite number of periods, the conditions for the existence of locally indeterminate equilibria are very similar to those obtained within infinite horizon models and that from this point of view, *Diamond meets Ramsey*.

We consider a model with consumption in both periods of life and we assume that the share of first period consumption over the wage income is large enough to be compatible with standard estimates. Our main results show that the existence of multiple equilibria is a quite robust property of the model which does not require any non-conventional restrictions.

[3]See Nourry [16].

[4]See Cazzavillan and Pintus [7], Nourry and Venditti [17].

We first prove that local indeterminacy of equilibria cannot be generated, under realistic calibrations for the parameters, when there are no externalities (as in Cazzavillan and Pintus [7], Nourry [16], Nourry and Venditti [17]) or when capital externalities are significantly larger than labor externalities (as in Cazzavillan and Pintus [8]). We concentrate on the focal case where capital externalities are absent and show that local indeterminacy occurs with small labor externalities if the elasticity of capital-labor substitution and the elasticity of labor supply are large enough (the lower bounds for these elasticities tending to infinity as the labor externalities go to zero). Considering extremely low market imperfections then implies that capital and labor are more substitutable than in the usual Cobb-Douglas specification, but local indeterminacy appears to be compatible with standard calibrations for the structural parameters. We prove however that even with Cobb-Douglas preferences and technology, locally indeterminate equilibria may also occur but require slightly larger labor externalities, still compatible with a negative slope of the aggregate labor demand function, and a larger elasticity of the labor supply. Finally, we show that, as a consequence of our restriction on first period consumption, the locally indeterminate normalized steady state is always characterized by an under-accumulation of capital.[5] To summarize, the simplifying feature introduced by our assumption of homothetic preferences will allow us to show from direct comparisons with Benhabib and Farmer [1] and Pintus [18] that contrary to what is suggested by earlier contributions within particular OLG frameworks, the existence of local indeterminacy in Diamond-type [9] models is based on similar mechanisms as in Ramsey-type [19] models.

This paper is organized as follows: The next section sets up the basic model. In section 3 we prove the existence of a normalized steady state. Section 4 contains the derivation of the characteristic polynomial and presents the geometrical method used for the local dynamic analysis. In section 5 we present our main results on local indeterminacy. Section 6 presents some detailed comparisons with infinite horizon models while section 7 contains some concluding comments. All the proofs are gathered in a final appendix.

[5]Under-accumulation is obtained when the stationary capital stock is less than the golden rule, i.e. the interest rate is positive. Under perfect competition, this property is associated with dynamic efficiency.

2 The model

Consider a perfectly competitive world where economic activity is performed over infinite discrete time in which there are identical non altruistic agents. Each agent lives for two periods: he works during the first, supplying elastically an amount of labor l such that $0 \leq l \leq \ell$, with $\ell > 0$ (possibly infinite) his endowment of labor. He has preferences for his consumptions (c, when he is young, and $\hat{c}$, when he is old), and derives disutility from labor according to the following function

$$u(c, \hat{c}) - v(l/B)$$

with $B > 0$ a scaling parameter.

Assumption 1.
i) $u(c, \hat{c})$ *is* $\mathbf{C}^r$ *over* $\mathbb{R}^2_+$ *for* r *large enough, increasing with respect to each argument* ($u_1(c, \hat{c}) > 0$, $u_2(c, \hat{c}) > 0$)*, concave and homogeneous of degree one over* $\mathbb{R}^2_{++}$*. Moreover, for all* $c, \hat{c} > 0$, $\lim_{\hat{c}/c \to 0} u_1/u_2 = 0$ *and* $\lim_{\hat{c}/c \to +\infty} u_1/u_2 = +\infty$*.*
ii) $v(l/B)$ *is* $\mathbf{C}^r$ *over* $[0, \ell/B]$ *for* r *large enough, increasing* ($v'(l/B) > 0$) *and convex* ($v''(l/B) > 0$) *over* $(0, \ell/B)$*. Moreover,* $\lim_{l \to 0} v'(l/B) = 0$ *and* $\lim_{l \to \ell} v'(l/B) = +\infty$*.*

Homogeneity is introduced to characterize preferences in terms of standard elasticities and in particular to explicitely write the capital accumulation equation as a function of the share of young agents' consumption over the wage income.

Each agent is assumed to have one child so that population is constant and normalized to one. Under perfect foresight, and considering the wage rate w_t and the interest factor R_{t+1} as given, he maximizes his utility function over his life-cycle as follows:

$$\begin{array}{cl} \max\limits_{c_t, \hat{c}_{t+1}, l_t, K_{t+1}} & u(c_t, \hat{c}_{t+1}) - v(l_t/B) \\ s.t. & w_t l_t = c_t + K_{t+1} \\ & R_{t+1} K_{t+1} = \hat{c}_{t+1} \\ & 0 \leq l_t \leq \ell \end{array} \tag{1}$$

Notice from the first budget constraint that all the savings of young agents are allocated to productive capital and we assume total depreciation of capital. Assumption 1 implies the existence and uniqueness of interior solutions for optimal saving, i.e. the amount of capital K_{t+1}, and labor supply l_t. Using the homogeneity of $u(c_t, \hat{c}_{t+1})$, the first order conditions can be written as:

$$\begin{aligned}
\frac{u_1(1, \hat{c}_{t+1}/c_t)}{u_2(1, \hat{c}_{t+1}/c_t)} &\equiv g(\hat{c}_{t+1}/c_t) = R_{t+1} && (2)\\
u_1(1, \hat{c}_{t+1}/c_t) w_t &= v'(l_t/B)/B && (3)\\
c_t + \frac{\hat{c}_{t+1}}{R_{t+1}} &= w_t l_t && (4)\\
K_{t+1} &= w_t l_t - c_t && (5)
\end{aligned}$$

with $g'(\hat{c}/c) > 0$. It follows that:

$$\hat{c}_{t+1}/c_t = g^{-1}(R_{t+1}) \equiv h(R_{t+1}) \tag{6}$$

Moreover combining (2) and (4) we derive:

$$c_t = \frac{u_1(1, \hat{c}_{t+1}/c_t)}{u(1, \hat{c}_{t+1}/c_t)} w_t l_t \equiv \alpha(R_{t+1}) w_t l_t \tag{7}$$

with $\alpha(R) \in (0, 1)$ the propensity to consume of the young, or equivalently the share of first period consumption over the wage income. We also conclude that the first order condition (5) becomes:

$$K_{t+1} = (1 - \alpha(R_{t+1})) w_t l_t \tag{8}$$

For future reference we may compute the elasticity of intertemporal substitution in consumption which is obtained as the inverse of the elasticity of the marginal rate of substitution (2):

$$\gamma(R) = \frac{R}{g'(h(R))h(R)} = -\left(\frac{u_{11}(1,\hat{c}/c)}{u_1(1,\hat{c}/c)} + \frac{u_{22}(1,\hat{c}/c)(\hat{c}/c)}{u_2(1,\hat{c}/c)}\right)^{-1} > 0 \tag{9}$$

and the elasticity of labor supply with respect to the wage rate:

$$\epsilon_l(l/B) = \frac{v'(l/B)}{(l/B)v''(l/B)} > 0 \tag{10}$$

Moreover, the elasticity of the propensity to consume $\alpha(R)$ derives as:

$$\alpha'(R)\frac{R}{\alpha(R)} = (1 - \gamma(R))(1 - \alpha(R)) \tag{11}$$

It follows that the saving function is increasing with the interest factor R if and only if $\gamma(R) > 1$.

Consider now the technological side of the model. The final output is produced using capital K and labor L. Although production takes place under constant returns to scale, we assume that each of the many firms benefits from positive externalities due to the contributions of the average levels of capital and labor, respectively $\bar{K}$ and $\bar{L}$. Capital external effects are usually interpreted as coming from learning by doing while labor externalities are associated with thick market effects. The production function of a representative firm is thus $AF(K,L)e(\bar{K},\bar{L})$, with $F(K,L)$ homogeneous of degree one, $e(\bar{K},\bar{L})$ increasing in each argument and $A>0$ a scaling parameter. Denoting, for any $L \neq 0$, $x = K/L$ the capital stock per labor unit, we may define the production function in intensive form as $Af(x)e(\bar{K},\bar{L})$.[6]

Assumption 2. *$f(x)$ is $\mathbf{C}^r$ over $\mathbb{R}_{++}$ for r large enough, increasing ($f'(x)>0$) and concave ($f''(x)<0$) over $\mathbb{R}_{++}$.*

All firms being identical, the competitive equilibrium conditions imply that $\bar{K} = K$ and $\bar{L} = L$. The interest factor R_t and the wage rate w_t then satisfy:

$$\begin{aligned} R_t &= Af'(x_t)e(K_t,L_t) && (12)\\ w_t &= A[f(x_t) - x_t f'(x_t)]e(K_t,L_t) \equiv Aw(x_t)e(K_t,L_t) && (13)\end{aligned}$$

We may also compute the share of capital in total income:

$$s(x) = \frac{xf'(x)}{f(x)} \in (0,1) \tag{14}$$

the elasticity of capital-labor substitution:

$$\sigma(x) = -\frac{(1-s(x))f'(x)}{xf''(x)} > 0 \tag{15}$$

and the elasticities of $e(K,L)$ with respect to capital and labor:

$$\varepsilon_{e,K}(K,L) = \frac{e_1(K,L)K}{e(K,L)}, \quad \varepsilon_{e,L}(K,L) = \frac{e_2(K,L)L}{e(K,L)} \tag{16}$$

We consider positive externalities:

Assumption 3. *For any given $K, L > 0$, $\varepsilon_{e,K}(K,L) \geq 0$ and $\varepsilon_{e,L}(K,L) \geq 0$.*

Assuming that the number of firms is normalized to one, the equilibrium on the labor market implies $l_t = L_t$. We then derive from (3) and (8) the dynamical system characterizing competitive equilibrium paths:

$$\begin{aligned} K_{t+1} &= \left[1-\alpha\left(Af'(x_{t+1})e(K_{t+1},l_{t+1})\right)\right]Aw(x_t)e(K_t,l_t)l_t \\ v'(l_t/B)/B &= u_1\left(1,h\left(Af'(x_{t+1})e(K_{t+1},l_{t+1})\right)\right)Aw(x_t)e(K_t,l_t)\end{aligned} \tag{17}$$

with $x_t = K_t/l_t$ and K_0 given.

[6] Our formulation is very close to the one considered in Cazzavillan and Pintus [6]. However although these authors define the model with capital and labor externalities, they only study the dynamic properties of equilibrium paths with capital external effects. Notice also that the same type of capital and labor externalities are considered by Cazzavillan, Lloyd-Braga and Pintus [5] in an infinite horizon model with heterogeneous agents (workers and capitalists) and a finance constraint on wage income.

3 Steady state

A steady state is a pair (K^*, l^*) such that:

$$\begin{aligned} K^* &= [1-\alpha\left(Af'(K^*/l^*)e(K^*,l^*)\right)] Aw(K^*/l^*)e(K^*,l^*)l^* \\ v'(l^*/B)/B &= u_1\left(1, h\left(Af'(K^*/l^*)e(K^*,l^*)\right)\right) Aw(K^*/l^*)e(K^*,l^*) \end{aligned} \tag{18}$$

As in the overlapping generations model with exogenous labor supply, the existence of a non-trivial steady state is not guaranteed even with some strengthened Inada conditions.[7] To simplify the analysis, we will follow the procedure introduced in Cazzavillan, Lloyd-Braga and Pintus [5] and use the scaling parameters A and B in order to give conditions for the existence of a normalized steady state $(K^*, l^*) = (1, 1)$.

Proposition 1. *Under Assumptions 1-2, let* $V(B) = v'(1/B)/B$. *Then* $(K^*, l^*) = (1, 1)$ *is a steady state of the dynamical system (17) if and only if* $\lim_{z\to+\infty}(1-\alpha(z))z > f'(1)/w(1)$ *and* $A^* > 0$, $B^* > 0$ *are respectively the unique solutions of*

$$\begin{aligned} 1 &= [1-\alpha\left(A^*f'(1)e(1,1)\right)] A^*w(1)e(1,1) \\ B^* &= V^{-1}\left(u_1\left(1, h\left(A^*f'(1)e(1,1)\right)\right) A^*w(1)e(1,1)\right) \end{aligned}$$

Proof: See Appendix 8.1. □

To illustrate Proposition 1, consider a standard example of CES preferences: $u(c,\hat{c}) = [\delta c^{-\rho} + (1-\delta)\hat{c}^{-\rho}]^{-1/\rho}$ and $v(l/B) = (l/B)^{1+\beta}/(1+\beta)$ with $\delta \in (0,1)$, $\rho > -1$ and $\beta > 0$. We easily get $\hat{c}/c = h(R) = [(1-\delta)R/\delta]^{1/(1+\rho)}$, $V(B) = (1/B)^{1+\beta}$ and equations (18) with $(K^*, l^*) = (1, 1)$ reduce to:

$$\begin{aligned} 1 &= \frac{(1-\delta)\left[\frac{1-\delta}{\delta}f'(1)e(1,1)\right]^{-\frac{\rho}{1+\rho}} A^{1/(1+\rho)}}{\delta + (1-\delta)\left[\frac{1-\delta}{\delta}f'(1)e(1,1)\right]^{-\frac{\rho}{1+\rho}} A^{-\frac{\rho}{1+\rho}}} w(1)e(1,1) \\ B &= \left(\delta\left(\delta + (1-\delta)\left[\frac{1-\delta}{\delta}Af'(1)e(1,1)\right]^{-\frac{\rho}{1+\rho}}\right)^{-\frac{1+\rho}{\rho}} Aw(1)e(1,1)\right)^{-\frac{1}{1+\beta}} \end{aligned}$$

It follows that for any $\rho > -1$, $\lim_{z\to+\infty}(1-\alpha(z))z = +\infty$ and there exists a unique A^* solution of the first equation. The corresponding value for B^* is obtained substituting A^* in the second equation.

In the rest of the paper we will assume that the conditions of Proposition 1 are satisfied in order to guarantee the existence of one normalized steady state (NSS in the sequel).

[7] See Galor and Ryder [11].

Assumption 4. *$\lim_{z\to+\infty}(1-\alpha(z))z > f'(1)/w(1)$, $A = A^*$ and $B = B^*$.*

Before studying the local stability properties of the NSS, we may provide a more precise characterization. Let us first evaluate all the shares and elasticities previously defined at $(1,1)$. From (7), (9), (10), (14), (15) and (16), we consider indeed $\alpha(A^*f'(1)e(1,1)) = \alpha$, $\gamma(A^*f'(1)e(1,1)) = \gamma$, $\epsilon_l(1/B^*) = \epsilon_l$, $s(1) = s$, $\sigma(1) = \sigma$, $\varepsilon_{e,K}(1,1) = \varepsilon_{e,K}$ and $\varepsilon_{e,L}(1,1) = \varepsilon_{e,L}$.

Proposition 2. *Under Assumptions 1-4, the NSS $(1,1)$ is characterized by a positive interest rate, i.e. $R^* \geq 1$, if and only if $\alpha \geq (1-2s)/(1-s) \equiv \alpha_1$.*

Proof: See Appendix 8.2. □

Notice that the bound α_1 will be positive as soon as the share of capital in total income is less than $1/2$. In such a case, which corresponds to a standard configuration, a positive stationary interest rate requires a high enough share of first period consumption over the wage income.

4 Characteristic polynomial and geometrical method

Let us linearize the dynamical system (17) around the NSS $(1,1)$. We get the following Proposition:

Proposition 3. *Under Assumptions 1-4, the characteristic polynomial is*

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T} + \mathcal{D} \tag{19}$$

with

$$\begin{aligned}
\mathcal{D} &= \left(\tfrac{1+\epsilon_l}{\epsilon_l}\right)\tfrac{s+\sigma\varepsilon_{e,K}}{(1-\alpha)(1-s+\sigma\varepsilon_{e,L})} \\
\mathcal{T} &= \tfrac{1}{(1-\alpha)(1-s+\sigma\varepsilon_{e,L})}\Big(\varepsilon_{e,K}(1-\sigma)(1-\alpha\gamma) + \varepsilon_{e,L}(1-\sigma-\alpha\gamma) \\
&+ 1-\sigma-\alpha\gamma(1-s) + \frac{1+\epsilon_l}{\epsilon_l}[\sigma-\alpha(1-\gamma)(1-s-\sigma\varepsilon_{e,K})]\Big)
\end{aligned}$$

Proof: See Appendix 8.3. □

Our aim is to discuss the local indeterminacy properties of equilibria, i.e. the existence of a continuum of equilibrium paths starting from the same intial capital stock and converging to the NSS. Our model consists in one predetermined variable, the capital stock, and one forward variable, the labor supply. Therefore, the NSS is locally indeterminate if and only if the local stable manifold is two-dimensional.

4.1 The Δ-half-line

As in Grandmont, Pintus and de Vilder [13], we study the variations of the trace $\mathcal{T}$ and the determinant $\mathcal{D}$ in the $(\mathcal{T}, \mathcal{D})$ plane as one of the parameters of interest is made to vary continuously in its admissible range. This methodology allows to fully characterize the local stability of $(K^*, l^*) = (1, 1)$, as well as the occurrence of local bifurcations. Let us then start by considering the locus of points $(\mathcal{T}(\epsilon_l), \mathcal{D}(\epsilon_l))$ as the elasticity of labor supply ϵ_l continuously changes in $(0, +\infty)$. From Proposition 3, we may easily see that $\mathcal{T}$ and $\mathcal{D}$ are linear with respect to $(1+\epsilon_l)/\epsilon_l$. The locus is then defined by the following linear relationship $\Delta(\mathcal{T})$:

$$\mathcal{D} = \Delta(\mathcal{T}) = \mathcal{S}\mathcal{T} - \mathcal{S}\tfrac{\varepsilon_{e,K}(1-\sigma)(1-\alpha\gamma)+\varepsilon_{e,L}(1-\sigma-\alpha\gamma)+1-\sigma-\alpha\gamma(1-s)}{(1-\alpha)(1-s+\sigma\varepsilon_{e,L})}$$

where the slope $\mathcal{S}$ of $\Delta(\mathcal{T})$ is

$$\mathcal{S} = \tfrac{s+\sigma\varepsilon_{e,K}}{\sigma-\alpha(1-\gamma)(1-s-\sigma\varepsilon_{e,K})}, \tag{20}$$

Figure 1 provides an illustration of $\Delta(\mathcal{T})$. We also introduce three other relevant lines: line AC ($\mathcal{D} = \mathcal{T} - 1$) along which one characteristic root is equal to 1, line AB ($\mathcal{D} = -\mathcal{T} - 1$) along which one characteristic root is equal to -1 and segment BC ($\mathcal{D} = 1$, $|\mathcal{T}| < 2$) along which the characteristic roots are complex conjugate with modulus equal to 1. These lines divide the space $(\mathcal{T}, \mathcal{D})$ into three different types of regions according to the number of characteristic roots with modulus less than 1. When $(\mathcal{T}, \mathcal{D})$ belongs to the interior of triangle ABC, the NSS is locally indeterminate. If the $\Delta(\mathcal{T})$ line crosses segment AB as ϵ_l goes through $\epsilon_l^F \in (0, +\infty)$ then a flip bifurcation is generically expected to occur and the NSS becomes a saddle-point. If $\Delta(\mathcal{T})$ crosses segment BC in its interior as ϵ_l goes through $\epsilon_l^H \in (0, +\infty)$ then a Hopf bifurcation is generically expected to occur and the NSS becomes locally unstable. Finally, if $\Delta(\mathcal{T})$ crosses segment AC as ϵ_l goes through $\epsilon_l^T \in (0, +\infty)$ then one characteristic root crosses 1 and the NSS becomes a saddle-point. Under Assumption 4, the existence of the NSS is always ensured and thereby a saddle-node bifurcation cannot occur. Therefore depending on the number of steady states, the critical value ϵ_l^T will be associated with an exchange of stability between the NSS and another (resp. two others) steady state through a transcritical (resp. pitchfork) bifurcation. However, pitchfork bifurcations require some non-generic condition.[8] In order to simplify the exposition we then concentrate on the generic case and we associate in the rest of the paper the existence of one eigenvalue equal to 1 to a transcritical bifurcation.

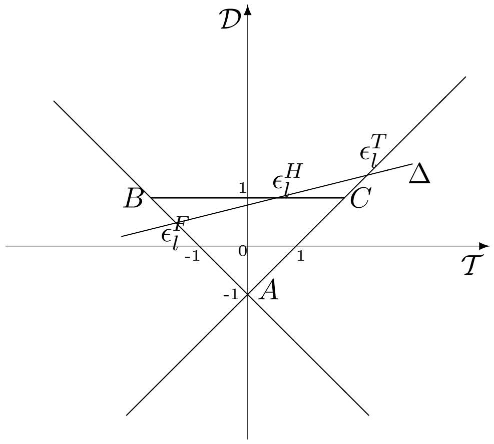

Figure 1: Stability triangle and $\Delta(\mathcal{T})$ line.

As $\epsilon_l \in (0, +\infty)$, only a part of $\Delta(\mathcal{T})$ is relevant. We need therefore to compute the starting and end points of the pair $(\mathcal{T}(\epsilon_l), \mathcal{D}(\epsilon_l))$. Straightforward computations give

$$\begin{aligned} &\lim_{\epsilon_l \to +\infty} \mathcal{D}(\epsilon_l) = \mathcal{D}_1 = \tfrac{s+\sigma\varepsilon_{e,K}}{(1-\alpha)(1-s+\sigma\varepsilon_{e,L})} \\ &\lim_{\epsilon_l \to +\infty} \mathcal{T}(\epsilon_l) = \mathcal{T}_1 = \tfrac{\varepsilon_{e,K}(1-\sigma-\alpha(\gamma-\sigma))+\varepsilon_{e,L}(1-\sigma-\alpha\gamma)+s\alpha+1-\alpha}{(1-\alpha)(1-s+\sigma\varepsilon_{e,L})} \end{aligned} \tag{21}$$

In graphical terms, under Assumption 3, since $\mathcal{D}$ decreases with ϵ_l and $\lim_{\epsilon_l \to 0} \mathcal{D}(\epsilon_l) = +\infty$, the relevant part of $\Delta(\mathcal{T})$ is thus a half-line starting in $(\mathcal{T}_1, \mathcal{D}_1)$ for $\epsilon_l = +\infty$ and pointing upwards (to the left when $\mathcal{S} < 0$, or to the right when $\mathcal{S} > 0$) as ϵ_l decreases to 0.

4.2 A necessary condition for local indeterminacy

From now on $\Delta(\mathcal{T})$ is the half-line described by $(\mathcal{T}(\epsilon_l), \mathcal{D}(\epsilon_l))$ when ϵ_l covers $(0, +\infty)$. Since $\Delta(\mathcal{T})$ is pointing upward, a necessary condition for the occurrence of local indeterminacy, i.e. for $(\mathcal{T}, \mathcal{D})$ to belong to the interior of triangle ABC, is $\mathcal{D}_1 < 1$. Notice then that, for fixed values α, s, $\varepsilon_{e,L}$ and

[8] Some second derivative of the map which defines the dynamical system (17) needs to be equal to zero. As shown in Ruelle [21], this is a non-generic configuration.

$\varepsilon_{e,K}$, $\mathcal{D}_1$ is a monotone function of the elasticity of capital-labor substitution σ with

$$\lim_{\sigma\to 0}\mathcal{D}_1 = \mathcal{D}_1^0 = \frac{s}{(1-\alpha)(1-s)}, \quad \lim_{\sigma\to+\infty}\mathcal{D}_1 = \mathcal{D}_1^\infty = \frac{\varepsilon_{e,K}}{(1-\alpha)\varepsilon_{e,L}} \tag{22}$$

We focus on realistic cases in which the share of first period consumption over the wage income α is significant. Specifically, we consider that

$$\alpha > 1 - \frac{s}{1-s} \equiv \alpha_1 \tag{23}$$

with α_1 the bound exhibited in Proposition 2 to show the positiveness of the interest rate. For realistic calibrations with $s \approx 1/3$, (23) means $\alpha > 0.5$. Notice also that (23) always implies $\mathcal{D}_1^0 > 1$. In that case, a necessary condition for the existence of a locally indeterminate NSS is $\mathcal{D}_1^\infty < 1$ which is equivalent to

$$\frac{\varepsilon_{e,K}}{\varepsilon_{e,L}} < 1 - \alpha < \frac{s}{1-s} \tag{24}$$

Consider first the case without externalities, i.e. $\varepsilon_{e,K} = \varepsilon_{e,L} = 0$. From (21), $\mathcal{D}_1 = \mathcal{D}_1^0 > 1$ and local indeterminacy cannot occur when young agents consume a realistically large portion of their wage income. This confirms the conclusions of Cazzavillan and Pintus [7], Nourry and Venditti [17].

Consider now the general case with $\varepsilon_{e,K} \geq 0$ and (24).[9] This implies that capital externalities are significantly lower than labor externalities.[10] To simplify matters, we consider the limit case with $\varepsilon_{e,K} = 0$ and $\varepsilon_{e,L} > 0$ in which $\mathcal{D}_1^\infty = 0$. In fact this configuration is a proxy for situations in which $\varepsilon_{e,K}/\varepsilon_{e,L} = \beta$, with β a small fixed parameter, where the same phenomena as here are likely to be obtained by continuity.[11]

[9] Cazzavillan [4] considers a formulation with capital externalities only ($\varepsilon_{e,K} > 0$ and $\varepsilon_{e,L} = 0$) but assumes $\alpha = 0$. He gets local indeterminacy with small $\varepsilon_{e,K}$ and $\sigma > s$. Seegmuller [22] exploits a configuration with $\varepsilon_{e,K} \geq 0$ and $\varepsilon_{e,L} > 0$ assuming also $\alpha = 0$, and gets similar conclusions as Cazzavillan [4].

[10] Under (24), if $\varepsilon_{e,K} > 0$ and $\varepsilon_{e,L} = 0$, we easily derive that local indeterminacy is not a possible outcome when $\alpha > \alpha_1$. A similar result has been independently shown by Cazzavillan and Pintus [8] in an OLG model with capital externalities only and totally separable preferences over $(c, \hat{c}, l)$.

[11] Considering an infinite horizon model with heterogeneous agents (workers and capitalists), a finance constraint on wage income and externalities in production, Cazzavillan, Lloyd-Braga and Pintus [5] show that local indeterminacy occurs with large elasticities of capital-labor substitution if the contribution of labor to externalities is sufficiently strong with respect to the contribution of capital. As in our model, this amounts to a small β. Using the same kind of framework but with general market imperfections, Seegmuller [23] extends this conclusion to a formulation in which equilibrium prices are given by general functions of capital and labor that differ from the marginal productivities of inputs.

In order to focus on realistic calibrations, besides the previous restrictions on the share of capital in total income and the share of first period consumption over the wage income, we also assume gross substitutability. To summarize, we introduce the following assumption:

Assumption 5. $\varepsilon_{e,K} = 0$, $s \leq 1/2$, $\alpha > \alpha_1$, $\gamma \geq 1$.

4.3 The Δ_1-segment

Let us now keep fixed α, s, γ at values satisfying Assumption 5. Then, from (20), $\mathcal{S} > 0$ so that $\Delta(\mathcal{T})$ is a half-line pointing upwards to the right as illustrated in Figure 1. Considering small given values for $\varepsilon_{e,L}$, we study how the half-line Δ evolves in the $(\mathcal{T}, \mathcal{D})$ plane as the elasticity of capital-labor substitution σ is made to vary continuously in $(0, +\infty)$. We thus have now to analyze how its starting point $(\mathcal{T}_1(\sigma), \mathcal{D}_1(\sigma))$, given in (21), and its slope $\mathcal{S}$, as given in (20), change with σ. Proceeding in a similar way as for the Δ-half-line, we may define a linear relationship linking the initial points $\mathcal{T}_1$ and $\mathcal{D}_1$ for different values of $\sigma \in [0, +\infty)$:

$$\mathcal{D}_1 = \Delta_1(\mathcal{T}_1) = \mathcal{S}_1\left(\mathcal{T}_1 + \frac{1}{1-\alpha}\right), \text{ where } \mathcal{S}_1 = \frac{s}{1+\varepsilon_{e,L}(1-\alpha\gamma)+(1-\alpha)(1-s)} \quad (25)$$

Since $\sigma \in (0, +\infty)$, only a part of $\Delta_1(\mathcal{T}_1)$ is relevant for our analysis, its extremities being given by the limit values of $(\mathcal{T}_1, \mathcal{D}_1)$ when σ goes from 0 to $+\infty$. When $\sigma = 0$ we get

$$(\mathcal{D}_1^0, \mathcal{T}_1^0) = \left(\frac{s}{(1-\alpha)(1-s)}, \frac{\varepsilon_{e,L}(1-\alpha\gamma)+s\alpha+1-\alpha}{(1-\alpha)(1-s)}\right) \quad (26)$$

and from (21)-(22), we easily derive that, under Assumption 5, $\mathcal{D}_1$ decreases with σ and $\mathcal{D}_1^\infty = 0$, $\mathcal{T}_1^\infty = -1/(1-\alpha) < -2$. Therefore, from now on, $\Delta_1(\mathcal{T}_1)$ is the segment described by $(\mathcal{T}_1(\sigma), \mathcal{D}_1(\sigma))$ when σ covers $[0, +\infty)$.

In graphical terms, the segment $\Delta_1(\mathcal{T}_1)$ is characterized by a "reversed" orientation as σ decreases from $+\infty$ to 0: it starts in $(\mathcal{D}_1^\infty, \mathcal{T}_1^\infty)$ which is located on the negative side of the abscissa axis, points upwards, to the left when $\mathcal{S}_1 < 0$, or to the right when $\mathcal{S}_1 > 0$, and ends in $(\mathcal{T}_1^0, \mathcal{D}_1^0)$ with $\mathcal{D}_1^0 > 1$. Moreover the slope $\mathcal{S}$ of the Δ-half-line decreases with σ and converges to zero while σ increases to $+\infty$. An illustration of $\Delta_1(\mathcal{T}_1)$ with $\mathcal{S}_1 > 0$ is given in Figure 2, where we also plot two examples of $\Delta(\mathcal{T})$ for different values of σ. When $\sigma = +\infty$, the Δ-half-line is merged with the abscissa axis. As σ decreases, the starting point of Δ shifts to the right along Δ_1 while the slope $\mathcal{S}$ increases. The Δ-half-line therefore moves counter-clockwise.

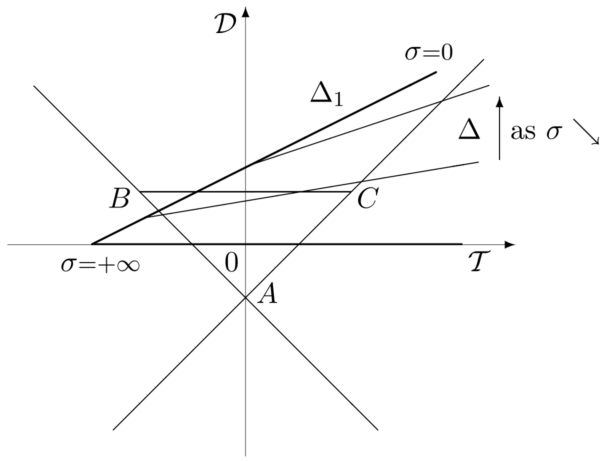

Figure 2: $\Delta_1(\mathcal{T}_1)$ segment

Our main objective is to give conditions for local indeterminacy of equilibria under small labor externalities. With such a restriction, it is easy to see fom (25) that $\mathcal{S}_1 > 0$ so that $\Delta_1(\mathcal{T}_1)$ is pointing upwards to the right as illustrated in Figure 2. A critical issue is to study the intersections of $\Delta_1(\mathcal{T}_1)$ with lines AC, AB and BC. As shown in Lemma 1 below, depending on the share of first period consumption over the wage income and the elasticity of intertemporal substitution in consumption, these intersections may be actually classified in two simple basic cases depending on whether or not $\Delta_1(\mathcal{T}_1)$ crosses the interior of segments $[AB]$ and $[BC]$.

Lemma 1. *Under Assumptions 1-5, for given s, α and γ, there exists $\hat{\varepsilon} > 0$ such that for any $\varepsilon_{e,L} \in (0, \hat{\varepsilon})$ the following results hold:*

i) The slope of the segment $\Delta_1(\mathcal{T}_1)$ satisfies $\mathcal{S}_1 \in (0, 1)$;

ii) There exists $\alpha_2 = (3 - \sqrt{1 + 8s})/(2(1 - s)) \in (\alpha_1, 1)$ such that the segment $\Delta_1(\mathcal{T}_1)$ crosses the interior of $[AB]$ and $[BC]$ iff $\alpha \in (\alpha_1, \alpha_2)$;

iii) The segment $\Delta_1(\mathcal{T}_1)$ does not cross the interior of $[AC]$;

iv) The segment $\Delta_1(\mathcal{T}_1)$ crosses the line AC iff $1 - \alpha\gamma > 0$.

Proof: See Appendix 8.4. □

The basic locations of $\Delta_1(\mathcal{T}_1)$ are summarized in Figures 3 and 4. It is worth noticing that, although the bound $\hat{\varepsilon}$ depends on s, α and γ, the critical values α_1 and α_2 depend only on s. Hence Lemma 1 and Figures 3-4 classify all cases that may occur. In particular, the segment $\Delta_1(\mathcal{T}_1)$ will cross the interior of the triangle ABC if and only if $\alpha \in (\alpha_1, \alpha_2)$.

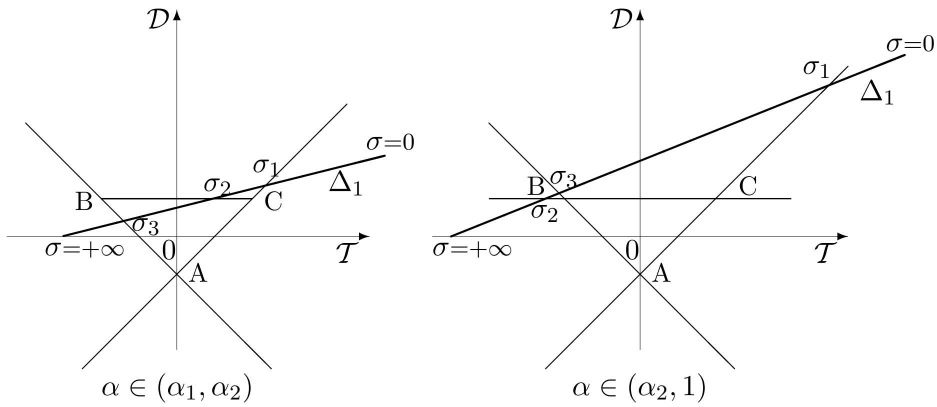

Figure 3: Δ_1-segment with $\varepsilon_{e,L} \in (0, \hat{\varepsilon})$ and $1 - \alpha\gamma > 0$.

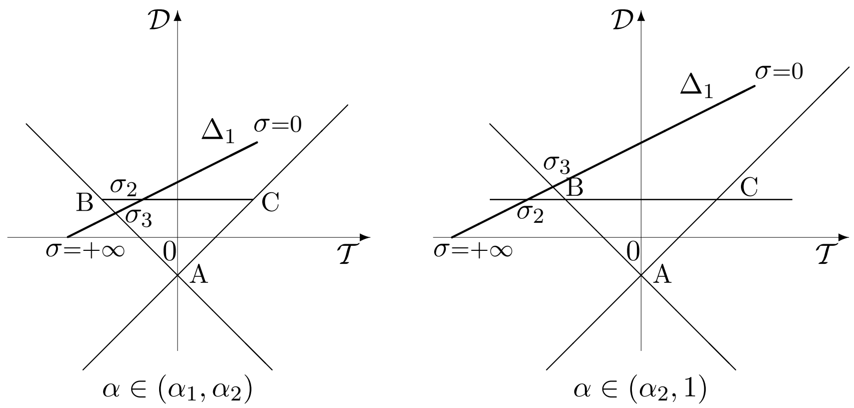

Figure 4: Δ_1-segment with $\varepsilon_{e,L} \in (0, \hat{\varepsilon})$ and $1 - \alpha\gamma < 0$.

The main distinction between Figures 3 and 4, according to whether $1 - \alpha\gamma > 0$ or $1 - \alpha\gamma < 0$, concerns the possible intersection of Δ_1 with line AC. Of course, as clearly shown above, each intersection of Δ_1 with a particular line is associated with a specific value for the elasticity of capital-labor substitution. More precisely, σ_3 corresponds to the intersection with line AB, σ_2 corresponds to the intersection with line BC and σ_1 corresponds to the intersection with line AC, the latter occurring for $\sigma_1 > 0$ only when $1 - \alpha\gamma > 0$.

5 Indeterminacy with first period consumption

Focussing on cases in which the share of first period consumption in wage income has realistic values, we imposed through Assumption 5 the restriction $\alpha \in (\alpha_1, 1)$. Lemma 1 shows however that within this interval, we need to distinguish two cases depending on whether α is lower or greater than α_2. We will also assume in the next two sections that, for a given share of first period consumption α, the elasticity of intertemporal substitution in consumption γ satisfies $1 - \alpha\gamma > 0$. The converse case with $1 - \alpha\gamma < 0$ can be easily deduced afterward from the consideration of Figure 4.

5.1 The case $\alpha \in (\alpha_1, \alpha_2)$ and $\gamma \in (1, 1/\alpha)$

When $\alpha \in (\alpha_1, \alpha_2)$, consider the localization of Δ_1 derived from Lemma 1. We get the following Figure 5 which gives a complete picture of the local stability properties of the NSS. The Δ half-lines corresponding to the critical values of σ are mentioned as, for instance, $\Delta(\sigma_5)$ that is associated with the amount of capital-labor substitution σ_5 for which Δ goes through C, so that for σ above σ_5, Hopf bifurcations are ruled out. Or $\Delta(\sigma_4)$ that is associated with the value σ_4 for which Δ has a slope greater than 1 for σ above σ_4.

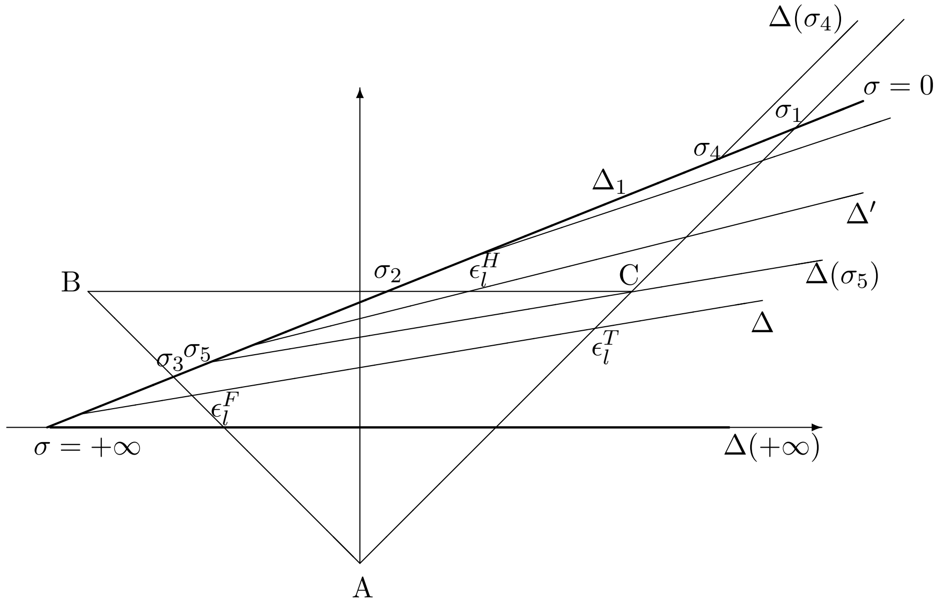

Figure 5: $\alpha \in (\alpha_1, \alpha_2)$ and $1 - \alpha\gamma > 0$.

Consider for instance the half-line Δ corresponding to an elasticity of capital-labor substitution $\sigma > \sigma_3$. The NSS is therefore saddle-point stable for any $\epsilon_l \in (\epsilon_l^F, +\infty)$. A flip bifurcation occurs when ϵ_l crosses ϵ_l^F from above and the NSS becomes locally indeterminate for any $\epsilon_l \in (\epsilon_l^T, \epsilon_l^F)$. Then a transcritical bifurcation occurs when ϵ_l crosses ϵ_l^T from above and the NSS is finally saddle-point stable for any $\epsilon_l < \epsilon_l^T$. If we consider on the contrary the line Δ' corresponding to an elasticity of capital-labor substitution $\sigma \in (\sigma_2, \sigma_5)$, the NSS is locally indeterminate for any $\epsilon_l \in (\epsilon_l^H, +\infty)$. A Hopf bifurcation occurs when ϵ_l crosses ϵ_l^H from above and the NSS becomes locally unstable for any $\epsilon_l \in (\epsilon_l^T, \epsilon_l^H)$. Then a transcritical bifurcation occurs when ϵ_l crosses ϵ_l^T from above and the NSS is finally saddle-point stable for any $\epsilon_l < \epsilon_l^T$.[12]

As this clearly appears on Figure 5, the occurrence of local indeterminacy requires the elasticity of capital-labor substitution to be greater than σ_2 which is a decreasing function of the amount of labor externalities $\varepsilon_{e,L}$. However, as we will show later on, under standard calibrations for the fundamentals, a low σ_2 might remain compatible with small $\varepsilon_{e,L}$.

All the detailed results on local stability of the NSS and bifurcations are now gathered into the following Proposition:

Proposition 4. *Under Assumptions 1-5, let s be fixed as well as $\alpha \in (\alpha_1, \alpha_2)$ and $\gamma \in (1, 1/\alpha)$. There is an $\bar{\varepsilon} > 0$ such that for any $\varepsilon_{e,L} \in (0, \bar{\varepsilon})$ there exist σ_1, σ_2, σ_3, σ_4 and σ_5, with $\sigma_3 > \sigma_5 > \sigma_2 > \sigma_4 > \sigma_1$, such that the following results generically hold:*

i) $\sigma \in (\sigma_3, +\infty)$. The NSS $(1,1)$ is a saddle-point for $\epsilon_l \in (\epsilon_l^F, +\infty)$, undergoes a flip bifurcation at $\epsilon_l = \epsilon_l^F$, becomes locally indeterminate for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^F)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

ii) $\sigma \in (\sigma_5, \sigma_3)$. The NSS $(1,1)$ is locally indeterminate for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation $\epsilon_l = \epsilon_l^T$ and becomes again a saddle-point for $\epsilon_l < \epsilon_l^T$.

iii) $\sigma \in (\sigma_2, \sigma_5)$. The NSS $(1,1)$ is locally indeterminate for $\epsilon_l \in (\epsilon_l^H, +\infty)$, undergoes a Hopf bifurcation at $\epsilon_l = \epsilon_l^H$, becomes a source for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^H)$, undergoes a transcritical bifurcation $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

[12] The expressions of these bifurcation values are given in Appendix 8.5.

iv) $\sigma \in (\sigma_4, \sigma_2)$. *The NSS* $(1,1)$ *is a source for* $\epsilon_l \in (\epsilon_l^T, +\infty)$*, undergoes a transcritical bifurcation at* $\epsilon_l = \epsilon_l^T$ *and becomes a saddle-point for* $\epsilon_l < \epsilon_l^T$*.*

v) $\sigma \in (\sigma_1, \sigma_4)$. *The NSS* $(1,1)$ *is a source for any* $\epsilon_l \in (0, +\infty)$*.*

vi) $\sigma \in (0, \sigma_1)$. *The NSS* $(1,1)$ *is a saddle-point for* $\epsilon_l \in (\epsilon_l^T, +\infty)$*, undergoes a transcritical bifurcation at* $\epsilon_l = \epsilon_l^T$ *and becomes a source for* $\epsilon_l < \epsilon_l^T$*.*

Proof: See Appendix 8.5. □

Proposition 4 shows that with small labor externalities and a share of first period consumption over the wage income compatible with empirical estimates, local indeterminacy of equilibria requires a large enough elasticity of capital-labor substitution ($\sigma > \sigma_2$) and a large enough elasticity of the labor supply ($\epsilon_l \geq \max\{\epsilon_l^H, \epsilon_l^T\}$). As explicitely stated in Proposition 4 and clearly apparent in Appendix 8.5, the critical value σ_2 for the elasticity of capital-labor substitution and the bifurcation values ϵ_l^F, ϵ_l^H, ϵ_l^T for the elasticity of labor supply depend on $\varepsilon_{e,L}$. In particular σ_2 and ϵ_l^T tend to infinity as the labor externality goes to zero. This property simply follows from the fact that in the limit case with $\varepsilon_{e,L} = 0 = \varepsilon_{e,K}$, the steady state is necessarily locally determinate when $\alpha > \alpha_1$. However, as shown later on, although we consider standard calibrations for α and s with small labor externalities, σ_2 remains no too large and local indeterminacy is obtained under realistic values for σ provided that ϵ_l is large enough. Notice also that there is no particular restriction on the elasticity of intertemporal substitution in consumption which is only assumed to be greater than one in order to meet the gross substitutability axiom. Finally it is worth pointing out that for a given elasticity of capital-labor substitution σ, flip and Hopf bifurcations cannot occur consecutively. They require indeed different values for σ.

5.2 The case $\alpha \in (\alpha_2, 1)$ and $\gamma \in (1, 1/\alpha)$

We may now consider the cases with a higher share of first period consumption over the wage income. The important point is the fact that the critical value α_2 only depends on the share of capital in total income s. For a given value of s, we have analyzed in the previous section the case $\alpha < \alpha_2$. We now increase the value of α by considering $\alpha > \alpha_2$. As shown in Lemma 1, the main consequence of this increase is that the segment $\Delta_1(\mathcal{T}_1)$ does not cross anymore the interior of $[AB]$ and $[BC]$. A direct implication of this together with the fact that σ_2 is an increasing function of α (with $\lim_{\alpha\to 1}\sigma_2 = +\infty$) while σ_3 is a decreasing function of α, is that the ranking of the critical values σ_2 and σ_3 is reversed with $\sigma_2 > \sigma_3$. Moreover, we can define a new lower bound for the elasticity of capital-labor substitution, denoted σ_6, which corresponds to the crossing of the Δ half-line with point B and which is greater than σ_2. The intersection of Δ with triangle ABC and thus the occurrence of local indeterminacy then require σ to be greater than σ_6.

The results detailed in the next Proposition are summarized in Figure 6.

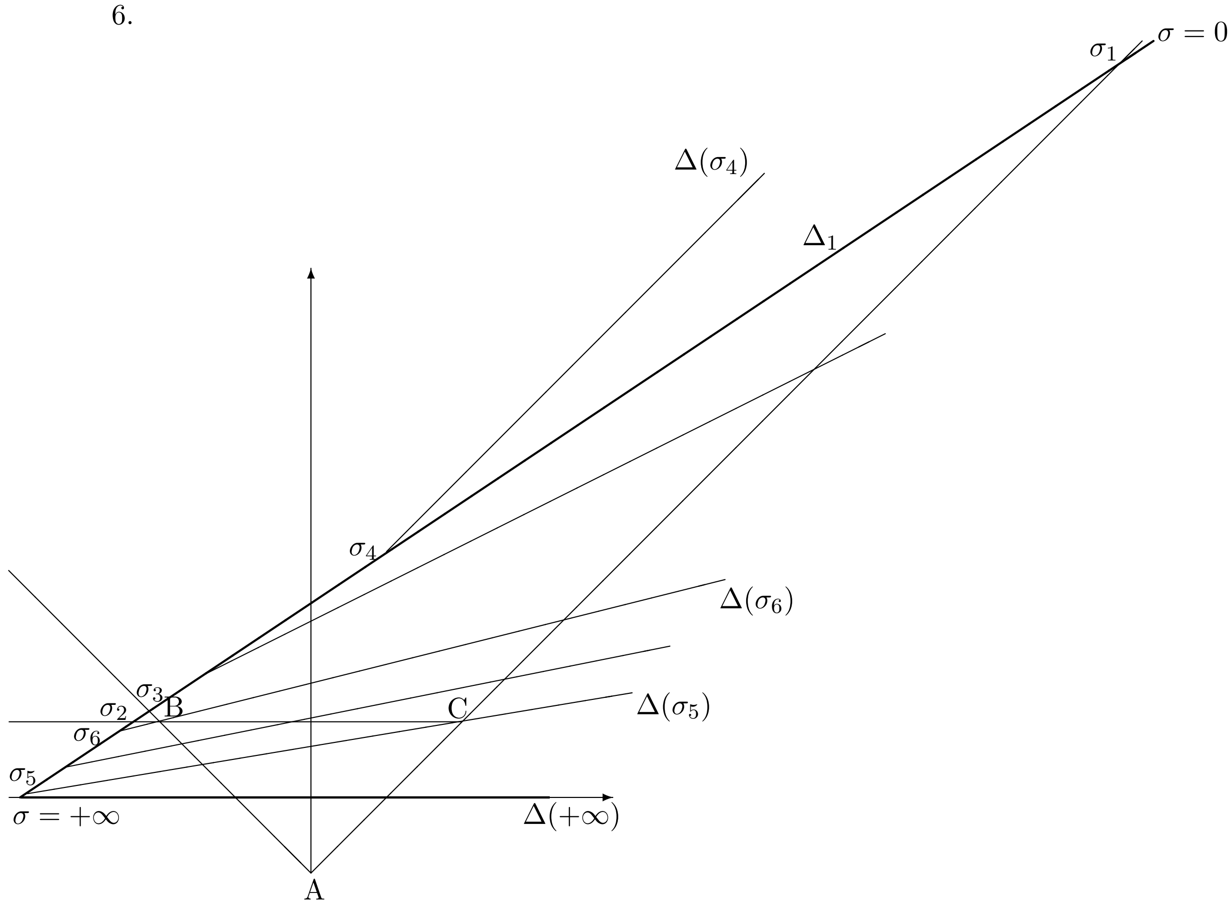

Figure 6: $\alpha \in (\alpha_2, 1)$ and $\gamma \in (1, 1/\alpha)$

Proposition 5. *Under Assumptions 1-5, let s be fixed as well as $\alpha \in (\alpha_2, 1)$ and $\gamma \in (1, 1/\alpha)$. There is an $\bar{\varepsilon} > 0$ such that for any $\varepsilon_{e,L} \in (0, \bar{\varepsilon})$ there* *exist σ_1, σ_2, σ_3, σ_4, σ_5 and σ_6, with $\sigma_5 > \sigma_6 > \sigma_2 > \sigma_3 > \sigma_4 > \sigma_1$, such that the following results generically hold:*

i) $\sigma \in (\sigma_5, +\infty)$. The NSS $(1,1)$ is a saddle-point for $\epsilon_l \in (\epsilon_l^F, +\infty)$, undergoes a flip bifurcation at $\epsilon_l = \epsilon_l^F$, becomes locally indeterminate for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^F)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes again a saddle-point for $\epsilon_l < \epsilon_l^T$.

ii) $\sigma \in (\sigma_6, \sigma_5)$. The NSS $(1,1)$ is a saddle-point for $\epsilon_l \in (\epsilon_l^F, +\infty)$, undergoes a flip bifurcation at $\epsilon_l = \epsilon_l^F$, becomes locally indeterminate for $\epsilon_l \in (\epsilon_l^H, \epsilon_l^F)$, undergoes a Hopf bifurcation at $\epsilon_l = \epsilon_l^H$, becomes a source for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^H)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes again a saddle-point for $\epsilon_l < \epsilon_l^T$.

iii) $\sigma \in (\sigma_3, \sigma_6)$. The NSS $(1,1)$ is a saddle point for $\epsilon_l \in (\epsilon_l^F, +\infty)$, undergoes a flip bifurcation at $\epsilon_l = \epsilon_l^F$, becomes a source for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^F)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes again a saddle-point for $\epsilon_l < \epsilon_l^T$.

iv) $\sigma \in (\sigma_4, \sigma_3)$. The NSS $(1,1)$ is a source for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

v) $\sigma \in (\sigma_1, \sigma_4)$. The NSS $(1,1)$ is a source for any $\epsilon_l \in (0, +\infty)$.

vi) $\sigma \in (0, \sigma_1)$. The NSS $(1,1)$ is a saddle-point for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a source for $\epsilon_l < \epsilon_l^T$.

Proof: See Appendix 8.6. □

When the share of first period consumption over the wage income is larger, the occurrence of local indeterminacy is based on similar conditions as in Proposition 4: a large enough elasticity of capital-labor substitution ($\sigma > \sigma_6 > \sigma_2$) and a large enough elasticity of the labor supply ($\epsilon_l \geq \max\{\epsilon_l^H, \epsilon_l^T\}$), with σ_2 and ϵ_l^T tending to infinity as the labor externality goes to zero. However, from the point of view of calibrations and bifurcations, there exist two significant differences with the conclusions given in the previous section. Firstly, as explicitly stated in Proposition 5 and clearly apparent in Appendix 8.5, the critical values σ_i are not the same as before since they depend on α. As σ_2 is increasing with respect to α, we then find a trade-off between the values of α and σ: local indeterminacy with larger shares of first period consumption over the wage income requires larger elasticities of capital-labor substitution. Secondly, for a given $\sigma \in (\sigma_6, \sigma_5)$, flip and Hopf bifurcations now occur consecutively.

Remark: Up to now, we have assumed that for a given share of first period consumption α, the elasticity of intertemporal substitution in consumption γ satisfies $\gamma < 1/\alpha$. The converse case with $\gamma > 1/\alpha$ can be easily deduced from the consideration of Figure 4. Actually Figures 5 and 6 still apply, except that the segment Δ_1 stops before crossing AC. In other words we have $\sigma_1 < 0$. It follows that case vi) disappears in Propositions 4 and 5 while v) now occurs for $\sigma \in (0, \sigma_4)$. Notice also that for large γ, $\sigma_4 < 0$ and case v) also disappears while iv) occurs for $\sigma \in (0, \sigma_2)$ in Proposition 4 or $\sigma \in (0, \sigma_3)$ in Proposition 5.

5.3 Economic intuitions

In order to provide some economic intuitions, it is convenient to compute the following derivatives: from equation (3) and using (33) in Appendix 8.3, we get:

$$\frac{dl_t}{dw_t}\frac{w_t}{l_t} = \epsilon_l > 0, \quad \frac{dl_t}{dR_{t+1}}\frac{R_{t+1}}{l_t} = (1-\alpha)\epsilon_l > 0 \tag{27}$$

Considering now the general formulation for the interest factor (12) and the definition of the elasticity of capital-labor substitution (15), we obtain:

$$\frac{dR_{t+1}}{dK_{t+1}}\frac{K_{t+1}}{R_{t+1}} = -\frac{1-s}{\sigma} + \varepsilon_{e,K}, \quad \frac{dR_{t+1}}{dL_{t+1}}\frac{L_{t+1}}{R_{t+1}} = \frac{1-s}{\sigma} + \varepsilon_{e,L} > 0 \tag{28}$$

Depending on the size of external effects, the variations of inputs affect differently the interest factor. Capital externalities dampen the negative effect of capital. Notice that when $\varepsilon_{e,K}$ is small, dR/dK remains negative. On the contrary, labor externalities amplify the positive effect of labor. This distinction will be at the core of the mechanism generating indeterminacy.

As shown previously, local indeterminacy is closely related to deterministic fluctuations, based on the existence of flip and Hopf bifurcations, and stochastic fluctuations, based on the existence of sunspot equilibria. Our intuitive argument is therefore based on the existence of a mechanism generating cyclical equilibrium paths.

We first consider the model without external effects, i.e. $\varepsilon_{e,K} = \varepsilon_{e,L} = 0$. We know that local indeterminacy of equilibria is only possible if the elasticity of capital-labor substitution is lower than the share of capital in total income and the first period consumption is a small fraction of wage income.[13] The intuition for these results is the following: Let us start at time t from the steady state and assume an instantaneous increase in the capital stock K_t. This generates two opposite effects: a first *contemporary* effect consists in an increase in the wage rate w_t which implies from equation (27) an increase in the labor supply l_t. Since $K_{t+1} = (1 - \alpha)w_t l_t$, a higher capital stock in the next period is expected. But there exists an *expectation* effect which plays in the opposite direction: a higher K_{t+1} is followed by a decrease in the interest factor which implies, from equation (27), a decrease in the current labor supply l_t. A cyclical path will then be obtained if the *expectation* effect is strong enough to dominate the *contemporary* effect and to generate a decrease in the wage income which would decrease savings at time t and capital at time $t + 1$. It clearly appears that such a scenario will occur if the elasticity of capital-labor substitution and the share of first period consumption over the wage income are sufficiently small. Indeed equation (28) shows that a strong effect on the interest factor is obtained when σ is low while equation (27) shows that a strong effect on the labor supply is obtained when α is low (and ϵ_l large).

Consider now the model with capital externalities only, i.e. $\varepsilon_{e,K} > 0$ and $\varepsilon_{e,L} = 0$. Using a formulation without first period consumption, Cazzavillan [4] shows that local indeterminacy might occur while the elasticity of substitution is higher than the share of capital.[14] Since $\varepsilon_{e,K} > 0$, a higher capital stock K_t generates a stronger increase in wage w_t which then implies a stronger increase in the labor supply l_t. When $\alpha = 0$ (or low enough), the *contemporary* effect consists in a large increase in K_{t+1}. Although the interest factor is less sensitive to capital variations (see equation (28)), and even with a larger elasticity of capital-labor substitution, the decrease in R_{t+1} generates a strong enough decrease in the labor supply l_t. The *expectation* effect may thus be large enough to lower savings and the next period capital stock. On the contrary, when α is large, the "small" expected increase in K_{t+1} does not generate a strong enough reduction in R_{t+1} which is then not sufficient to significantly lower the current labor supply l_t. The cyclical reversal of the capital stock is thus less likely. This explains why,

[13] See Reichlin [20], Lloyd-Braga [15], Cazzavillan and Pintus [7] and Nourry and Venditti [17].

[14] Lloyd-Braga [15] shows that the same result also holds when increasing returns to scale internal to the firm are considered.

no matter the size of capital externalities, local indeterminacy cannot occur when $\varepsilon_{e,L} = 0$ and $\alpha > \alpha_1$.

Consider finally the model with labor externalities only, i.e. $\varepsilon_{e,K} = 0$ and $\varepsilon_{e,L} > 0$. Since $\varepsilon_{e,K} = 0$, a higher capital stock K_t generates increases in wage w_t and thus in the labor supply l_t which are similar to those obtained in the model without external effects. Even if the share α is large, the increase in K_{t+1} implies a decrease in R_{t+1} which is maximal since $\varepsilon_{e,K} = 0$. A decrease in the current labor supply l_t then follows. But due to the presence of labor externalities, the *expectation* effect is amplified by a general equilibrium, or feedback, effect on the interest factor (see dR/dl in (28)) which is stronger than in the previous cases. Indeed, the increase in K_{t+1} following the *contemporary* effect implies through capital-labor substitution in the technology a decrease of the demand of labor L_{t+1} which amplifies the expected decrease of R_{t+1}. As a result, a strong enough decrease in the wage income of period t and thus a cyclical reversal of the capital stock become compatible with a large share of first period consumption.

5.4 A numerical illustration

We may now numerically illustrate our main findings in order to show that local indeterminacy occurs under plausible parameters values, in particular with small labor externalities and a share of first period consumption over the wage income compatible with standard estimates. From that point of view, notice that over the period $1950-2000$ for the U.S., the annual ratio of consumption expenditures over GDP averages at 67%.[15] Total consumption over GDP is given at the steady state by:

$$\frac{c+\hat{c}}{y} = (1-s)\alpha + s$$

Considering a standard value for the share of capital in total income, $s = 1/3$, we easily compute the bounds $\alpha_1 = 1/2$ and $\alpha_2 \approx 0.81385$. Therefore, assuming $\alpha = 0.51$ implies a ratio of consumption expenditures over GDP of 67.3% compatible with the previous estimate. We thus consider the theoretical results given in Section 5.1 with $\alpha \in (\alpha_1, \alpha_2)$.

Recent papers have questioned the empirical relevance of Cobb-Douglas technologies and unitary elasticities of capital-labor substitution. Duffy and

[15] This number has been computed using the Penn World Data set available at http://www.bized.ac.uk/dataserv/penndata/pennhome.htm.

Papageorgiou [10] for instance consider a panel of 82 countries over a 28-year period to estimate a CES production function specification. They find that for the entire sample of countries the assumption of unitary elasticity of substitution is rejected. Moreover, dividing the sample of countries up into several subsamples, they find that capital and labor have an elasticity of substitution significantly greater than unity (i.e. $\sigma \in [1.14, 3.24]$) in the richest group of countries.

The elasticity of intertemporal substitution in consumption is fixed slightly greater than one at $\gamma = 1.1$ in order to guarantee gross substitutability. It follows that $1 - \alpha\gamma > 0$ and the upper bound on the amount of labor externalities considered in Proposition 4 is the following: $\bar{\varepsilon} \approx 1.048\%$.[16] Although we have shown that the lower bound σ_2 tends to infinity as $\varepsilon_{e,L}$ goes to zero, we may easily find some realistic calibrations for s, α and γ such that local indeterminacy arises with very small labor externalities and not too large elasticities of capital-labor substitution. Assuming indeed $\varepsilon_{e,L} = 1\%$, we may derive all the critical values for σ: $\sigma_1 \approx 0.29463$, $\sigma_2 \approx 1.36$, $\sigma_3 \approx 259.68$, $\sigma_4 \approx 0.29933$ and $\sigma_5 \approx 2.376$.

i) Consider first the case $\sigma \in (\sigma_5, \sigma_3) = (2.376, 259.68)$. In order to be compatible with the estimates given by Duffy and Papageorgiou we assume more precisely that $\sigma \in (2.376, 3.24)$. The steady state is thus locally indeterminate for large values of ϵ_l, i.e. $\epsilon_l > \epsilon_l^T \approx 67$.

ii) Consider now the case $\sigma \in (\sigma_2, \sigma_5) = (1.36, 2.376)$. The steady state will be locally indeterminate for any $\epsilon_l > \epsilon_l^H$ with ϵ_l^H a Hopf bifurcation value. Assuming that $\sigma \in (1.37, 2.37)$ we similarly get large values of ϵ_l, i.e. $\epsilon_l^H \in (67.39, 7194.24)$.

While labor externalities are restricted to be small, these numerical illustrations show that local indeterminacy of equilibria and endogenous fluctuations rely on plausible values for the elasticity of capital-labor substitution and the elasticity of intertemporal substitution in consumption as soon as the labor supply is sufficiently elastic. It is also worth mentioning that all our indeterminacy results have been obtained with a decreasing aggregate labor demand function, i.e. $\varepsilon_{e,L} - s/\sigma < 0$.

[16] We have indeed $\bar{\varepsilon} = \hat{\varepsilon}^3_{e,L}$ with $\hat{\varepsilon}^3_{e,L}$ defined in Appendix 8.5.

6 Comparisons with infinite horizon models: Diamond meets Ramsey

We have proved that in an OLG model with elastic labor supply, as soon as the share of first period consumption over the wage income is large enough to be consistent with standard estimates, the consideration of capital externalities alone does not provide any room for the existence of multiple equilibria. On the contrary, we have shown that if the elasticity of the labor supply is large enough, local indeterminacy easily occurs with small externalities on labor, a decreasing aggregate labor demand function, an elasticity of intertemporal substitution in consumption slightly greater than unity and a large enough elasticity of capital-labor substitution. These theoretical results have been confirmed by numerical illustrations in which the consideration of extremely small labor externalities implies some elasticity of capital-labor substitution greater than unity. A short survey of the literature on infinite horizon models allows now to show that the conditions for local indeterminacy in Diamond-type and Ramsey-type models are actually very close qualitatively.

One-sector infinitely-lived agent models with inelastic labor, Romer-type capital externalities and increasing returns at the social level have been considered initally by Kehoe [14] and Boldrin and Rustichini [3]. They show that local indeterminacy requires very strong negative externalities which improve enough the private marginal productivity of capital to destroy concavity of the technology at the social level. Obviously these conditions cannot be met by usual Cobb-Douglas or CES technologies. When standard positive externalities are considered, it is shown on the contrary that the steady state is either saddle-point stable or totally unstable.

Using a Cobb-Douglas aggregate technology, Benhabib and Farmer [1] have shown that local indeterminacy in one-sector models actually requires the consideration of elastic labor supply and aggregate externalities on labor. They assume separable preferences over consumption and leisure with a unitary elasticity of intertemporal substitution in consumption and an infinitely elastic labor supply. In such a framework, their main conclusion is the following: in order to get local indeterminacy of equilibria, externalities and thus the degree of increasing returns to scale must be large enough to imply that the aggregate labor demand curve should be upward-sloping and steeper than the aggregate labor supply curve. This is obviously a non-standard configuration for the labor market.

More recently, Pintus [18], by considering a general separable utility function and a general technology with constant returns to scale at the private level and productive externalities, shows that the conditions of Benhabib and Farmer [1] are not necessary. Local indeterminacy may indeed arise with a standard decreasing equilibrium labor demand function and small externalities on labor provided that the elasticity of capital-labor substitution is significantly greater than one and the elasticity of the labor supply is large enough.[17] Moreover capital externalities alone are not sufficient to generate the occurrence of multiple equilibria. Therefore, the same conditions in both types of models allow to get local indeterminacy. From this point of view *Diamond meets Ramsey*.

On the quantitative side however there exist some differences. Pintus [18] provides the following numerical illustration: considering an infinitely elastic labor supply and some labor externalities of 5%, local indeterminacy arises when the elasticity of intertemporal substitution in consumption is greater than 16.6 and the elasticity of capital-labor substitution belongs to $(2.4, 6.65)$. In Section 5.4, we show that, under similar parameterizations for the share of capital in total income and the elasticity of capital-labor substitution, the existence of sunspot fluctuations is obtained under much smaller external effects (1% instead of 5%), a smaller elasticity of intertemporal substitution in consumption and a smaller elasticity of labor supply.

In order to complete the comparison between Diamond-type and Ramsey-type models, we will first focus on the existence of local indeterminacy when preferences and technology are Cobb-Douglas and we will explore the under- versus over-accumulation properties of the NSS.

6.1 Sunspot fluctuations in Cobb-Douglas economies

Considering an OLG model with consumption in both periods of life and no external effects, Nourry and Venditti [17] show that local indeterminacy cannot occur when the elasticity of capital-labor substitution is equal to one. All our previous results on local indeterminacy have been obtained under

[17] As in our OLG model, the lower bounds on the elasticity of capital-labor substitution and the elasticity of the labor supply tends to infinity as the labor externality goes to zero.

the assumption $\sigma > \sigma_2$, with σ_2 a decreasing function of the amount of labor externalities $\varepsilon_{e,L}$. Numerical simulation based on extremely small values of $\varepsilon_{e,L}$ have then shown that $\sigma_2 > 1$. One question remains therefore open: when preferences and technology are Cobb-Douglas, is it possible to get local indeterminacy in the presence of larger but still empirically plausible labor externalities ? As suggested by the main conclusions of the previous section, we have to introduce a lower bound $\underline{\varepsilon} > 0$ for $\varepsilon_{e,L}$ to get a positive answer. However, we will be able to prove that $\underline{\varepsilon}$ remains rather small under realistic calibrations as soon as the elasticity of the labor supply is very large.

Assumption 6. *$\sigma = \gamma = 1$.*

Using $u(c, \hat{c}) = c^{\alpha}\hat{c}^{1-\alpha}$ and $F(K, L) = K^s L^{1-s}$, the share of first period consumption over the wage income α and the share of capital in total income s are constant. We also assume that $v(l/B) = (l/B)^{1+\beta}/(1+\beta)$ and $e(L) = L^{\phi}$ with $\beta > 0$ and $\phi > 0$. It follows that the elasticity of labor supply $\epsilon_l = 1/\beta$ and the elasticity of labor externalities $\varepsilon_{e,L} = \phi$ are also constant. In such a simplified framework, we easily derive existence and uniqueness of the steady state.

Proposition 6. *Under Assumptions 1, 2 and 6, there exists a unique steady state (K^*, l^*) solution of (18) if and only if $\epsilon_l \neq (1-s)/\varepsilon_{e,L}$. Moreover, $(K^*, l^*) = (1, 1)$ when the scaling parameters satisfy*[18]

$$A = \frac{1}{(1-s)(1-\alpha)} \text{ and } B = \left[\frac{1-\alpha}{\alpha}\left(\frac{\alpha(1-s)}{s}\right)^{1-\alpha}\right]^{\frac{1}{1+\beta}} \tag{29}$$

Proof: See Appendix 8.7. □

From Proposition 3 the determinant and trace simplify as:

$$\mathcal{D} = \left(\frac{1+\epsilon_l}{\epsilon_l}\right)\frac{s}{(1-\alpha)(1-s+\varepsilon_{e,L})}, \quad \mathcal{T} = \left(\frac{1+\epsilon_l}{\epsilon_l}\right)\frac{1}{(1-\alpha)(1-s+\varepsilon_{e,L})} - \frac{\alpha}{1-\alpha}$$

so that the Δ line becomes:

$$\mathcal{D} = \Delta(\mathcal{T}) = s\mathcal{T} + \frac{s\alpha}{1-\alpha}$$

As previously, the starting point $(\mathcal{T}_1, \mathcal{D}_1)$ is obtained when $\epsilon_l = +\infty$:

$$\mathcal{D}_1 = \frac{s}{(1-\alpha)(1-s+\varepsilon_{e,L})}, \quad \mathcal{T}_1 = \frac{1}{(1-\alpha)(1-s+\varepsilon_{e,L})} - \frac{\alpha}{1-\alpha}$$

[18]When A and B satisfy (29) and $\epsilon_l = (1-s)/\varepsilon_{e,L}$, there exists a continuum of steady states with $K^* = l^* > 0$. A similar property has also been exhibited by Cazzavillan [4] in an OLG model with $\varepsilon_{e,K} > 0$, $\varepsilon_{e,L} = 0$, $\alpha = 0$ and a Cobb-Douglas technology.

In order to simplify the analysis and to provide direct comparisons with Benhabib and Farmer [1], we also restrict the value of the share of capital in total income to a standard value:

Assumption 7. $s = 1/3$.

The starting point $(\mathcal{T}_1, \mathcal{D}_1)$ lies within the triangle ABC if and only if $\mathcal{D}_1 < 1$ and $|\mathcal{T}_1| < 1 + \mathcal{D}_1$. While $\mathcal{D}_1 - \mathcal{T}_1 + 1 > 0$ for any $\varepsilon_{e,L} > 0$, we easily get:

$$\begin{array}{lcl} \mathcal{D}_1 < 1 & \Leftrightarrow & \varepsilon_{e,L} > \underline{\varepsilon} \equiv \frac{1}{3(1-\alpha)} - \frac{2}{3} \\ \mathcal{D}_1 + \mathcal{T}_1 + 1 > 0 & \Leftrightarrow & \varepsilon_{e,L} < \bar{\varepsilon} \equiv \frac{4}{3(2\alpha-1)} - \frac{2}{3} \end{array}$$

with $\bar{\varepsilon} > \underline{\varepsilon}$ if and only if $\alpha < \bar{\alpha} \equiv 5/6$. Since $\bar{\alpha} > \alpha_1 = 1/2$, we conclude that under $\alpha \in (1/2, 5/6)$ and $\varepsilon_{e,L} \in (\underline{\varepsilon}, \bar{\varepsilon})$, $\Delta(\mathcal{T})$ is a segment having its starting point within the triangle ABC and its slope lower than 1. We may then summarize all these results in Figure 7:

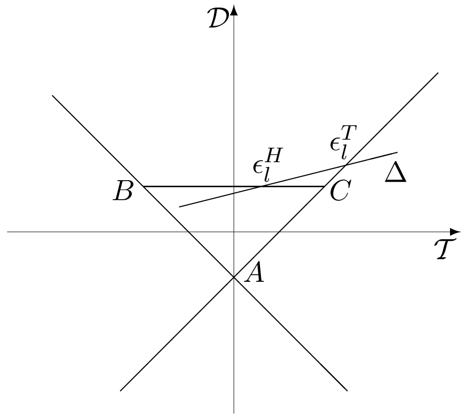

Figure 7: Indeterminacy in a Cobb-Douglas economy

Straightforward computations give the values of the critical bounds ϵ_l^H and ϵ_l^T for the elasticity of labor supply ϵ_l which are respectively associated with the existence of two complex conjugate roots with modulus equal to 1 or one real root equal to 1:

$$\epsilon_l^H = \frac{1}{3(1-\alpha)(\varepsilon_{e,L} - \underline{\varepsilon})}, \quad \epsilon_l^T = \frac{2}{3\varepsilon_{e,L}} \tag{30}$$

While ϵ_l^H is associated with a Hopf bifurcation, ϵ_l^T is exactly equal to the critical value for ϵ_l exhibited in Proposition 6: existence and uniqueness of the steady state are obtained provided $\epsilon_l \neq \epsilon_l^T$. As mentioned in footnote 18, when $\epsilon_l = \epsilon_l^T$ and the scaling parameters A and B satisfy (29), there exist a continuum of steady states with $K^* = l^* > 0$. It follows that ϵ_l^T cannot be a transcritical bifurcation value. To simplify the formulation, and since we are mainly interested in the existence of local indeterminacy for the NSS, we will assume that $\epsilon_l \neq \epsilon_l^T$. We may then summarize all the results in the following Proposition:

Proposition 7. **Local indeterminacy in a Cobb-Douglas economy** *Under Assumptions 1-7, let $\alpha \in (1/2, 5/6)$, $\varepsilon_{e,L} \in (\underline{\varepsilon}, \bar{\varepsilon})$ and consider the critical bounds (30) for the elasticity of labor supply. Assume also that $\epsilon_l \neq \epsilon_l^T$. Then the NSS $(1,1)$ is locally indeterminate for $\epsilon_l \in (\epsilon_l^H, +\infty)$, undergoes a Hopf bifurcation at $\epsilon_l = \epsilon_l^H$, becomes a source for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^H)$, and is a saddle-point for $\epsilon_l < \epsilon_l^T$.*

Therefore, as in Ramsey-type models, local indeterminacy might occur in OLG models with Cobb-Douglas preferences and technology. Assuming a significant first period consumption, this result requires a large enough elasticity of the labor supply ($\epsilon_l > \epsilon_l^H$) and large enough labor externalities ($\varepsilon_{e,L} > \underline{\varepsilon}$). However, the lower bound $\underline{\varepsilon}$ remains not too large when standard calibrations are used for α. Local indeterminacy thus relies on parameters values that appear to be much more empirically plausible than in the contribution of Benhabib and Farmer [1].[19] Considering again that $\alpha = 0.51$, we derive the bounds $\underline{\varepsilon} \approx 0.0136$ and $\bar{\varepsilon} \approx 66$. It follows that with 1.4% of labor externalities, the steady state is locally indeterminate for any $\epsilon_l > \epsilon_l^H \approx 1722.2$. Notice that since $\sigma = 1$, local indeterminacy requires a slightly larger value of $\varepsilon_{e,L}$ than in cases in which $\sigma > 1$, but labor externalities remains extremely small. Of course, the elasticity of the labor supply needs to be very large but remains inferior to the values required within infinite-horizon models. Moreover, even with Cobb-Douglas preferences and technology, local indeterminacy remains compatible with a decreasing aggregate labor demand function, i.e. $\varepsilon_{e,L} - s/\sigma < 0$.

6.2 Under- versus over-accumulation of capital

Finitely-lived agents models, such that the OLG model, and infinitely-lived agents models, such that the Ramsey model, are often distinguished with

[19] In Benhabib and Farmer [1], although the labor supply is assumed to be infinitely elastic, externalities are required to be extremely large (more than 43%).

respect to their efficiency properties. While the long-run equilibrium in a Ramsey economy is given by the modified golden rule which is dynamically efficient, in an OLG economy the steady state may be dynamically inefficient if there is an over-accumulation of capital with respect to the golden rule, i.e. if the stationary interest factor R^* is strictly less than 1.[20]

In presence of productive externalities, such a distinction is not obvious since in both models the equilibrium is affected by these market imperfections. However, the analysis of under- or over-accumulation drastically differs depending on whether the external effects come from capital or labor. In presence of capital externalities the criterium based on the interest factor cannot be used as previously since the definition of the golden rule is directly affected by the externalities. On the contrary, if there are only labor externalities, the golden rule capital stock is defined as in the standard model without market imperfection. In our framework, we may define total stationary consumption as:

$$c + \hat{c} = wl + RK - K = AF(K, l)e(l) - K$$

Maximizing total consumption with respect to the capital stock gives:

$$AF_1(K, l)e(l) = Af'(k)e(l) = R = 1$$

In such a case, we find the same conclusion as in models without market imperfection: in presence of labor externalities only, the steady state of a Ramsey model is always given by the modified golden rule which implies under-accumulation, while the stationay interest factor R^* of an OLG model is not necessarily greater than 1. We may then wonder in which case the NSS is characterized by an under-accumulation of capital. We obviously derive from Proposition 2:

Corollary 1. *Under Assumptions 1-4, the NSS* $(1, 1)$ *is characterized by an under-accumulation of capital if and only if* $\alpha \geq \alpha_1$.

This Corollary shows that under-accumulation of capital is obtained as soon as the share of first period consumption in total income is high enough. This conclusion then implies that all our previous results concerning local indeterminacy of equilibria are associated with under-accumulation of capital as in Ramsey models augmented to include labor externalities. In other

[20]See Galor and Ryder [12] with constant population.

words, it proves that the existence of expectations-driven fluctuations does not require any special or exceptional restrictions but relies on commonly accepted characterizations for the fundamentals.

7 Concluding comments

In this paper we have studied an OLG model with consumption in both periods of life, homothetic preferences and in which the share of first period consumption over the wage income is large enough to be compatible with standard estimates. We have shown that under gross substitutability, local indeterminacy of equilibria cannot occur with capital externalities alone but that it can occur when there are only labor externalities. More precisely, for a small amount of labor external effects, the existence of a continuum of equilibrium paths requires the elasticity of capital-labor substitution and the elasticity of the labor supply to be large enough. Moreover, we have proved that local indeterminacy is compatible with Cobb-Douglas specifications for preferences and technology and a negatively sloped aggregate labor demand function as soon as the elasticity of the labor supply is very large. As a consequence of the restriction on the first period consumption, we have finally shown that a locally indeterminate steady state is characterized by an under-accumulation of capital. We have exhibited a clear mechanism based on labor external effects which explains these results and which allows to get expectations-driven fluctuations under standard parameterizations for the fundamentals.

8 Appendix

8.1 Proof of Proposition 1

Let $V(B) = v'(1/B)/B$. Under Assumption 1, we get $V'(B) < 0$ so that $V(B)$ is invertible. $(K^*, l^*) = (1, 1)$ is a steady state if and only if there exist values for A and B such that:

$$\begin{aligned} 1 &= \left[1 - \alpha\left(Af'(1)e(1,1)\right)\right] Aw(1)e(1,1) \equiv G(A) && (31) \\ B &= V^{-1}\left(u_1\left(1, h\left(Af'(1)e(1,1)\right)\right) Aw(1)e(1,1)\right) && (32) \end{aligned}$$

Consider the elasticity of the propensity to consume $\alpha(R)$ given by (11), and recall that the elasticity of intertemporal substitution in consumption

$\gamma(R)$ may be greater or less than 1. It follows that $\alpha(R)$ may be increasing or decreasing depending on the value of $\gamma(R)$. We easily derive from (31) and (11) the elasticity of $G(A)$ as:

$$\frac{G'(A)A}{G(A)} = 1 - \alpha(R)(1 - \gamma(R))$$

We then conclude that $G'(A)A/G(A)$ is positive and thus $G(A)$ is a monotone increasing function for any $\gamma(R) > 0$. Recall finally that for any $R \geq 0$, $\alpha(R) \in (0, 1)$. It follows that $\lim_{z\to 0}(1 - \alpha(z)) \leq 1$ and thus:

$$\lim_{A\to 0} \left[1 - \alpha\left(Af'(1)e(1,1)\right)\right] A = 0$$

Then there exists a unique $A^* > 0$ solution of (31) if and only if $\lim_{z\to+\infty}(1-\alpha(z))z > f'(1)/w(1)$. B^* is obtained considering A^* into equation (32). $\square$

8.2 Proof of Proposition 2

The stationary interest factor corresponding to the NSS is given by $R^* = A^*f'(1)e(1,1)$. Proposition 1 gives the following value for the scaling parameter A^*:

$$A^* = \frac{1}{(1-\alpha)w(1)e(1,1)}$$

It follows that the stationary interest factor satisfies:

$$R^* = \frac{f'(1)e(1,1)}{(1-\alpha)w(1)e(1,1)} = \left[(1-\alpha)\frac{f(1)-f'(1)}{f'(1)}\right]^{-1} = \frac{s}{(1-\alpha)(1-s)}$$

Therefore $R^* \geq 1$ if and only if $\alpha \geq (1 - 2s)/(1 - s)$. $\square$

8.3 Proof of Proposition 3

Consider the first order condition (2) $u_1(1, \hat{c}/c) = u_2(1, \hat{c}/c)R$. Since from (6) $\hat{c}/c$ is a function of R, the differentiation with respect to R gives:

$$u_{12}(1, \hat{c}/c)\frac{d\hat{c}/c}{dR} = u_{22}(1, \hat{c}/c)R\frac{d\hat{c}/c}{dR} + \frac{u_1(1,\hat{c}/c)}{R}$$

Using the homogeneity of $u(c, \hat{c})$, this equation simplifies to:

$$u_{12}(1, \hat{c}/c)\frac{d\hat{c}/c}{dR}\left(1 + \frac{c}{\hat{c}}R\right) = \frac{u_1(1,\hat{c}/c)}{R}$$

Notice now that from the budget constraints and equation (7) we get $\hat{c}/c = (1 - \alpha(R))R/\alpha(R)$. We finally obtain from all this:

$$\frac{du_1(1,h(R))}{dR}\frac{R}{u_1(1,h(R))} = u_{12}(1, \hat{c}/c)\frac{d\hat{c}/c}{dR}\frac{R}{u_1(1,h(R))} = 1 - \alpha(R) \tag{33}$$

We may therefore differentiate the dynamical system (17) around the steady state $(K^*, l^*) = (1, 1)$. Tedious computations based on (33) give:

$$\begin{pmatrix} dK_{t+1} \\ dl_{t+1} \end{pmatrix} = \begin{pmatrix} 1+\alpha(1-\gamma)\left(\varepsilon_{e,K} - \frac{1-s}{\sigma}\right) & \alpha(1-\gamma)\left(\varepsilon_{e,L} + \frac{1-s}{\sigma}\right) \\ (1-\alpha)\left(\varepsilon_{e,K} - \frac{1-s}{\sigma}\right) & (1-\alpha)\left(\varepsilon_{e,L} + \frac{1-s}{\sigma}\right) \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} \varepsilon_{e,K} + \frac{s}{\sigma} & \varepsilon_{e,L} + \frac{\sigma - s}{\sigma} \\ -\left(\varepsilon_{e,K} + \frac{s}{\sigma}\right) & \frac{1}{\epsilon_l} - \left(\varepsilon_{e,L} - \frac{s}{\sigma}\right) \end{pmatrix} \begin{pmatrix} dK_t \\ dl_t \end{pmatrix}$$

The result follows after straightforward simplifications. □

8.4 Proof of Lemma 1

Let Assumptions 1-5 hold.

1- When $1 - \alpha\gamma > 0$ straightforward computations show that:

i) $\mathcal{S}_1 \in (0, 1)$;

ii) $\Delta_1(2) > 1$ if and only if $\varepsilon_{e,L} < \frac{-\alpha^2(1-s)+\alpha(3-4s)+2(2s-1)}{(1-\alpha)(1-\alpha\gamma)} \equiv \varepsilon^1_{e,L}$. Then Δ_1 does not cross the interior of segment $[AC]$;

iii) there exists $\alpha_2 \equiv \frac{3-\sqrt{1+8s}}{2(1-s)} \in (\alpha_1, 1)$, such that $\Delta_1(-2) < 1$ if and only if $\alpha \in (\alpha_1, \alpha_2)$ and Δ_1 crosses the interior of segments $[AB]$ and $[BC]$;

iv) on the contrary, when $\alpha \in (\alpha_2, 1)$, $\Delta_1(-2) > 1$ if and only if $\varepsilon_{e,L} < \frac{-\alpha^2(1-s)+3\alpha-2}{(1-\alpha)(1-\alpha\gamma)} \equiv \varepsilon^2_{e,L}$. Then Δ_1 does not cross the interior of segment $[AB]$;

v) When $\sigma = 0$ we derive from (26) $\mathcal{D}^0_1 - \mathcal{T}^0_1 + 1 = -\varepsilon_{e,L}\frac{1-\alpha\gamma}{(1-\alpha)(1-s)} < 0$ so that Δ_1 crosses the line AC.

2- When $1 - \alpha\gamma < 0$ we similarly get:

i) $\mathcal{S}_1 > 0$ if and only if $\varepsilon_{e,L} < \frac{1+(1-\alpha)(1-s)}{\alpha\gamma-1} \equiv \varepsilon^3_{e,L}$, and $\mathcal{S}_1 < 1$ if and only if $\varepsilon_{e,L} < \frac{(1-s)(2-\alpha)}{\alpha\gamma-1} \equiv \varepsilon^4_{e,L}$. By construction we have $\varepsilon^4_{e,L} < \varepsilon^3_{e,L}$.

ii) $\Delta_1(2) > 1$;

iii) when $\alpha \in (\alpha_1, \alpha_2)$, $\Delta_1(-2) < 1$ if and only if $\varepsilon_{e,L} < \varepsilon^2_{e,L}$,

iv) when $\alpha \in (\alpha_2, 1)$, $\Delta_1(-2) > 1$.

v) When $\sigma = 0$ then get $\mathcal{D}^0_1 - \mathcal{T}^0_1 + 1 > 0$ and Δ_1 cannot cross AC.

The result is obtained by choosing $\hat{\varepsilon}$ as follows:

- if $1 - \alpha\gamma > 0$, $\hat{\varepsilon} = \min\{\varepsilon^1_{e,L}, \varepsilon^2_{e,L}\}$;
- if $1 - \alpha\gamma < 0$, $\hat{\varepsilon} = \min\{\varepsilon^4_{e,L}, \varepsilon^2_{e,L}\}$.

□

8.5 Proof of Proposition 4

Before proving Proposition 4 we have to examine the intersections of $\Delta_1(\mathcal{T}_1)$ with the lines AC, AB and BC, and to study the values of $\mathcal{D}$ when $\mathcal{T} = \pm 2$.

Lemma 2. *Under Assumptions 1-5, let $\varepsilon_{e,L} \in (0, \hat{\varepsilon})$. Then:*

i) when $1 - \alpha\gamma \geq 0$, there exists $\sigma_1 \geq 0$ such that $\mathcal{D}_1 \geq \mathcal{T}_1 - 1$ if and only if $\sigma \geq \sigma_1$,

ii) when $1 - \alpha\gamma < 0$, $\mathcal{D}_1 > \mathcal{T}_1 - 1$,

iii) there exists $\sigma_2 > 0$ such that $\mathcal{D}_1 \leq 1$ if and only if $\sigma \geq \sigma_2$,

iv) there exists $\sigma_3 \geq 0$ such that $\mathcal{D}_1 \leq -\mathcal{T}_1 - 1$ if and only if $\sigma \geq \sigma_3$,

v) there exists $\sigma_4 \geq 0$ such that $\mathcal{S} \leq 1$ if $\sigma \geq \sigma_4$,

vi) there exists $\sigma_5 > \sigma_2$ such that $\Delta(2) = 1$,

vii) if there exists $\sigma_6 > 0$ such that $\Delta(-2) = 1$, then $\sigma_6 > \sigma_2$.

Proof: Under Assumptions 1-5, we derive from direct computations:
i) and ii): $\mathcal{D}_1 \geq \mathcal{T}_1 - 1$ if and only if $\sigma \geq \frac{1-\alpha\gamma}{2-\alpha} \equiv \sigma_1$. It follows that $\sigma_1 \geq 0$ if and only if $1 - \alpha\gamma \geq 0$.
iii) $\mathcal{D}_1 \leq 1$ if and only if $\sigma \geq \frac{(1-s)(\alpha-\alpha_1)}{(1-\alpha)\varepsilon_{e,L}} \equiv \sigma_2$.
iv) $\mathcal{D}_1 \leq -\mathcal{T}_1 - 1$ if and only if $\sigma \geq \frac{2(1-\alpha(1-s))}{\alpha\varepsilon_{e,L}} + \frac{1-\alpha\gamma}{\alpha} \equiv \sigma_3$.
v) $\mathcal{S} \leq 1$ if and only if $\sigma \geq (1-s)(\alpha - \alpha_1 + 1 - \alpha\gamma) \equiv \sigma_4$. Notice that depending on the value of $1-\alpha\gamma$ this bound may be negative, in which case there is no restriction on the value of σ.
vi) Assumption 5 implies $\mathcal{S} > 0$. By continuity of the family of half-lines Δ as σ decreases from $+\infty$ to 0, there exists σ_5 such that $\Delta(2) = 1$. We then get from simple geometrical considerations $\sigma_5 > \sigma_2$.
vii) similarly we may also find σ_6 as the value of σ such that $\Delta(-2) = 1$, and if $\sigma_6 > 0$ geometrical considerations show that it must satisfy $\sigma_6 > \sigma_2$.
$\square$

A last step consists in ranking all the critical bounds σ_i, $i = 1, \ldots, 6$.

Lemma 3. *Under Assumptions 1-5, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon_{e,L} < \bar{\varepsilon}$ the following results hold:*

a) when $\alpha \in (\alpha_1, \alpha_2)$, $\sigma_3 > \sigma_5 > \sigma_2 > \sigma_4 > \sigma_1$,

b) when $\alpha \in (\alpha_2, 1)$, $\sigma_5 > \sigma_6 > \sigma_2 > \sigma_3 > \sigma_4 > \sigma_1$.

Proof: Let Assumptions 1-5 hold. Consider the bounds σ_i introduced in Lemma 2. Direct computations prove the following facts:

i) Geometrical arguments show that $\sigma_3 > \sigma_2 > \sigma_1$ when $\alpha \in (\alpha_1, \alpha_2)$ and $\sigma_2 > \sigma_3 > \sigma_1$ when $\alpha \in (\alpha_2, 1)$. Moreover for any $\alpha \in (\alpha_1, 1)$, $\sigma_4 > \sigma_1$.
ii) We easily derive from point iv) in the proof of Lemma 2 that when $1 - \alpha\gamma \geq \alpha$, $\sigma_3 > 0$ for any $\varepsilon_{e,L} > 0$. On the contrary, when $1 - \alpha\gamma < \alpha$, there exists $\hat{\varepsilon}^1_{e,L} \equiv \frac{2(1-\alpha(1-s))}{\alpha\gamma-1} > 0$ such that $\sigma_3 > 0$ if $\varepsilon_{e,L} < \hat{\varepsilon}^1_{e,L}$.
iii) Consider point v) in the proof of Lemma 2 and the expression of α_1. Straightforward computations give $\sigma_4 = s + \alpha(1-\gamma)(1-s)$. Assumption 5 then implies $\sigma_4 < s \leq 1/2$.
iv) We know from Lemma 2 that $\sigma_5 > \sigma_2$. When $\alpha \in (\alpha_2, 1)$, we then derive from point i) above that $\sigma_5 > \sigma_2 > \sigma_3$. On the contrary, when $\alpha \in (\alpha_1, \alpha_2)$, we have to compare σ_5 and σ_3. We derive from iv) and vi) in the proof of Lemma 2 that $\lim_{\varepsilon_{e,L} \to 0} \sigma_3 = +\infty$ while the solution σ_5 of equation $\Delta(2) = 1$ remains finite when $\varepsilon_{e,L}$ tends to 0. It follows that there exists $\hat{\varepsilon}^2_{e,L} > 0$ such that $\sigma_3 > \sigma_5$ if $\varepsilon_{e,L} < \hat{\varepsilon}^2_{e,L}$.
v) Recall that $\sigma_2 > 0$. Moreover $\mathcal{S} \leq \mathcal{S}_1$ if and only if $\sigma \geq 1 + (1 - \alpha\gamma)(1 - s + \varepsilon_{e,L}) \equiv \sigma_7$. Since $\lim_{\varepsilon_{e,L} \to 0} \sigma_2 = +\infty$ while $\lim_{\varepsilon_{e,L} \to 0} \sigma_7 = 1 + (1-\alpha\gamma)(1-s)$, there exists $\hat{\varepsilon}^3_{e,L} > 0$ such that $\sigma_2 > \sigma_7$ if $\varepsilon_{e,L} < \hat{\varepsilon}^3_{e,L}$. Then under Assumption 5, if $\sigma = \sigma_2$ we get $\mathcal{S} \in (0, \mathcal{S}_1)$. Therefore the bound $\sigma_6 > 0$ cannot exist if $\alpha \in (\alpha_1, \alpha_2)$ since in this case $\Delta_1(-2) < 1$, while if $\alpha \in (\alpha_2, 1)$, obvious geometrical considerations imply $\sigma_6 < \sigma_5$. Recall finally from Lemma 2 that $\sigma_6 > \sigma_2$.
Considering the bound $\hat{\varepsilon}$ introduced in Lemma 1, the rest of the proof is obtained by choosing the value of $\tilde{\varepsilon}$ as follows:
a) When $\alpha \in (\alpha_1, \alpha_2)$, we know that $\sigma_3 > \sigma_2 > 0$ and the final argument is based on the bound given in iv) and v): $\tilde{\varepsilon} = min\{\hat{\varepsilon}, \hat{\varepsilon}^2_{e,L}, \hat{\varepsilon}^3_{e,L}\}$.
b) When $\alpha \in (\alpha_2, 1)$, we know that $\sigma_2 > \sigma_3$ and the final argument is now based on the bounds given in ii), iv) and v):

- if $1 - \alpha\gamma > 0$, $\bar{\varepsilon} = min\{\hat{\varepsilon}, \hat{\varepsilon}^2_{e,L}, \hat{\varepsilon}^3_{e,L}\}$,
- if $1 - \alpha\gamma < 0$, $\bar{\varepsilon} = min\{\hat{\varepsilon}, \hat{\varepsilon}^1_{e,L}, \hat{\varepsilon}^2_{e,L}, \hat{\varepsilon}^3_{e,L}\}$. □

We may now prove Proposition 4. All the local stability results are derived from Lemmas 1-3 and Figure 5, considering the bound $\bar{\varepsilon}$ introduced in Lemma 3. For a given value of σ, it remains now to compute the bifurcation values of the elasticity of labor supply ϵ_l. The flip bifurcation value ϵ_l^F is such that $\Delta(\mathcal{T})$ crosses [AB], i.e. is a solution of $\mathcal{D} + \mathcal{T} + 1 = 0$. The Hopf bifurcation value ϵ_l^H is such that $\Delta(\mathcal{T})$ crosses [BC], i.e. is a solution of $\mathcal{D} = 1$.

The transcritical bifurcation value ϵ_l^T is such that $\Delta(\mathcal{T})$ crosses [AC], i.e. is a solution of $\mathcal{D} - \mathcal{T} + 1 = 0$. Considering Proposition 3, straightforward computations give the expressions of these bifurcation values:

$$\epsilon_l^F = \frac{2s+\sigma-\sigma_4}{\varepsilon_{e,L}\alpha(\sigma-\sigma_3)}, \quad \epsilon_l^H = \frac{s}{\varepsilon_{e,L}(1-\alpha)(\sigma-\sigma_2)}, \quad \epsilon_l^T = \frac{\sigma-\sigma_4}{\varepsilon_{e,L}(2-\alpha)(\sigma-\sigma_1)} \tag{34}$$

with $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ defined in the proof of Lemma 2. □

8.6 Proof of Proposition 5

All the local stability results are derived from Lemmas 1-3 and Figure 6, considering the bound $\bar{\varepsilon}$ introduced in Lemma 3. As in Proposition 4, the bifurcation values of ϵ_l are given by (34). □

8.7 Proof of Proposition 6

Since the interest factor is given by $R = Asx^{s-1}l^{\phi}$ with $x = K/l$, we easily get from the capital accumulation equation (8) that $R^* = s/(1-s)(1-\alpha)$ is the unique stationary solution. It follows that

$$x^* = \left[A(1-s)(1-\alpha)l^{\phi}\right]^{\frac{1}{1-s}} \tag{35}$$

Consider now the second equation of system (18). Since $\hat{c}/c = (1-\alpha)R/\alpha$ and $w = A(1-s)x^s l^{\phi}$, we get when $R = R^*$ and $x = x^*$ that l^* is the solution of the following equation:

$$l^{\frac{\beta(1-s)-\phi}{1-s}} = B^{1+\beta}\alpha\left(\frac{s}{\alpha(1-s)}\right)^{1-\alpha}\left[A(1-s)\right]^{\frac{1}{1-s}}(1-\alpha)^{\frac{s}{1-s}} \tag{36}$$

Then there exists a unique solution l^* if and only if $1/\beta \neq (1-s)/\phi$. Moreover, a NSS such that $(K^*, l^*) = (1, 1)$ can be defined if A is derived from (35) with $x = l = 1$, namely

$$A = \frac{1}{(1-s)(1-\alpha)} \tag{37}$$

and B is derived from (36) with $l = 1$ and A given by (37), namely

$$B = \left[\frac{1-\alpha}{\alpha}\left(\frac{\alpha(1-s)}{s}\right)^{1-\alpha}\right]^{\frac{1}{1+\beta}}$$

□

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