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ABSTRACT

Risk Management with Benchmarking*

Portfolio theory must address the fact that, in reality, portfolio managers are evaluated relative to a benchmark, and therefore adopt risk management practices to account for the benchmark performance. We capture this risk management consideration by allowing a prespecified shortfall from a target benchmark-linked return, consistent with growing interest in such practice. In a dynamic setting, we demonstrate how a risk averse portfolio manager optimally under- or overperforms a target benchmark under different economic conditions, depending on his attitude towards risk and choice of the benchmark. The analysis therefore illustrates how investors can achieve their desired performance profile for funds under management through an appropriate combined choice of the benchmark and money manager. We consider a variety of extensions, and also highlight the ability of our setting to shed some light on documented return patterns across segments of the money management industry.

JEL Classification: D81, G11 and G23

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1. Introduction

Relative performance evaluation is widespread in many sectors of the financial industry.¹ Such practice leads professional portfolio managers to monitor the amount by which their portfolio return deviates from a benchmark, the so-called tracking error, and to adopt investment strategies that explicitly take relative performance, or benchmarking, into account. A primary concern of benchmarked managers is the possibility that their portfolio underperforms its benchmark. Such downside risk may be completely hedged (or insured against) by specifying a minimum benchmark-linked return via portfolio insurance (Basak (1995), Grossman and Zhou (1996)) for a riskless money market benchmark, or via minimum performance constraints (Teplá (2001)) for a stochastic benchmark. However, a serious shortcoming of this (strict) downside hedging with respect to a benchmark is that it may be prove very costly to fully insure the downside, limiting the upside potential of the strategy. Moreover, the minimum required return must be lower than that from buying-and-holding the benchmark; overperforming a tradable benchmark as a goal is ruled out because it is infeasible (by no arbitrage). In this paper, we consider a more flexible risk management framework, where the manager is able to target overperforming (beating) the benchmark return by a minimum amount, or underperforming by not more than a maximum amount. These targets are feasible since not delivering a target return is allowed with a prespecified shortfall probability. Such a “tracking error constraint” with a potential shortfall is intuitively appealing since managers, or those who evaluate their performance, may tolerate various forms of shortfall in order to meet other goals (like beating the stock market in some states). As a result, the use of such a downside risk measure is indeed rapidly spreading in practice, and has also been advocated in the professional literature (RISK (1998, 2000a, 2000b), Jorion (2000, Chapter 17)), beckoning further investigations. While in the academic literature, shortfall-based risk management practices have been argued to have adverse implications (Artzner, Delbaen, Eber, and Heath (1999), Basak and Shapiro (2001), Alexander and Baptista (2004)), in this paper we will demonstrate that in conjunction with benchmarking such practices offer a variety of attractive features.

Our primary objective is to investigate the optimal dynamic behavior of a manager, henceforth the *benchmarker*, striving to meet a tracking error constraint in a standard utility maximizing framework.² Consistent with leading industry practice, the benchmark is taken to be the stock market. We adopt the familiar Black and Scholes (1973) economy for the financial investment opportunities, and assume the benchmarker is guided by constant relative risk aversion preferences. Although our setting includes important risk-management practices as special cases, many of our

¹Relative performance evaluation is used almost universally: see discussions in, e.g., Fung and Hsieh (1997), p. 276; Chan, Karceski, and Lakonishok (1999), p. 956; the *Economist* (September 1, 2001), p. 60-61.

²In particular, our analysis combines standard absolute performance consideration (via the manager’s utility) with relative-performance concerns (embedded as a risk management constraint), a combination relevant in many financial situations (Chow (1995), Kritzman and Rich (1998)).

findings fall outside the predictions of existing work. Throughout the analysis, we compare the benchmarker's optimal behavior to that of the downside hedger and the "normal" manager who faces no benchmark constraints. Our focus is on the implications of benchmarking. An explicit treatment of market imperfections or behavioral underpinnings leading to benchmark-based constraints is beyond the scope of this paper, and we take the tracking error constraint as given. While benchmarking is not necessarily always warranted under delegated portfolio management (Admati and Pfleiderer (1997)), Basak, Pavlova, and Shapiro (2005) provide an analysis of managerial incentives in which benchmarking practices may be beneficial for fund investors. As an example of benchmarking arising in a behavioral setting, Gómez, Priestley, and Zapatero (2004) capture "keeping up with the Joneses"-type behavior by modeling investors who compare their consumption to a benchmark given by peer consumption. Benchmarking could also be argued to be in the spirit of Keynes's (1958) "beauty contest," in which investors are guided not by their individual expectations, but by their expectations of peer expectations.

Risk management with benchmarking, when shortfall is allowed, emerges as rich in implications. Absent benchmarking considerations, a manager's optimal (normal) policy is driven by its sensitivity (given by the manager's risk tolerance) to changing economic conditions (represented by changes in state prices). Under benchmarking, our analysis identifies economies characterized by the sensitivity of the benchmark relative to that of the normal policy, and additionally relative to unity, in which the manager exhibits distinct patterns of economic behavior in choosing his optimal horizon wealth and trading strategies. In economies in which the benchmark reacts *less* to changes in economic conditions than the normal policy, the benchmark beats the normal policy in economic downturns (bad states), but underperforms in upturns (good states). Consequently, a downside hedger maintains the normal-type policy in good states and matches the allowed underperformance level in bad states. When shortfall is allowed, the benchmarker additionally optimally chooses in which states to fall short of the target return. Here, he identifies the states with the highest cost of matching his target versus following the normal-type policy, so that the benefit from reverting to the normal-type policy is highest. When benchmark sensitivity is at or below unity shortfall occurs in bad states, whereas for benchmark sensitivity above unity it occurs in intermediate states. In the former case, losses in bad states are higher than for the normal policy; in the latter case as well as with downside hedging, they are lower. The practical usefulness of our framework is underscored by the fact that losses under benchmarking can be *further* reduced relative to those under downside hedging. This is due to the fact that while the downside hedger and the benchmarker both match their target returns in bad states, for the downside hedger this necessarily entails underperforming the stock market, whereas the benchmarker can target overperformance. As the latter behavior does account for risk aversion, it may be appealing to some investors, as well as merit regulatory consideration.

In economies in which the benchmark reacts *more* to changes in economic conditions than the normal policy, the benchmark beats the normal policy in good states. This leads the downside hedger to match a minimum target return in good states, while adopting the normal policy in bad states. This is in sharp contrast to the findings of related work on portfolio insurance and Value-at-Risk, in which good states are not insured. Shortfall, when allowed, occurs in good or intermediate states. In the latter case as well as with downside hedging, the gains in good states are higher than for the normal policy. The novelty of our analysis is to demonstrate that, since the benchmarker can target overperformance while the downside hedger cannot, the benchmarker's gains can be chosen to be even higher in good states. Finally, when the benchmark and normal policy have *equal* sensitivities, the manager matches the target return in all states except the shortfall ones (either good or bad). Studying the benchmarker's optimal investment dynamics, we uncover further properties. For instance, in certain economies, small changes in economic conditions can at times result in significant portfolio changes, possibly shifting between large leveraged and short positions. This is consistent with the puzzling, yet observed phenomenon, where seemingly small arrivals of news regarding fundamentals may at times cause considerable portfolio rebalancing, yet hardly no reaction at others. Overall, our results suggest that, under benchmarking, investors can achieve any of a range of different performance behaviors either by entrusting their funds to a money manager with an appropriate risk appetite or through a suitable choice of benchmark. The latter is facilitated by our analysis remaining valid for hybrid benchmarks of money and stock markets (Section 4.2).

The usefulness of our analysis is further highlighted by its ability to rationally generate optimal investment behavior that may shed some light on several intriguing, observed return patterns of mutual funds and hedge funds. For example, there is growing empirical evidence (Moskowitz (2000), Kosowski (2002)) suggesting that equity mutual funds, on average, overperform the market in recessions but underperform in non-recessions, and that (up until the mid 1990s) in recessions the absolute performance of funds is better than in non-recessions. Whereas we illustrate that this behavior cannot be reconciled with risk management practices such as Value-at-Risk, portfolio insurance, or downside hedging, it can indeed arise in economies with benchmarking. Moreover, evidence on trend-following hedge funds indicates a straddle-like performance across states of the world, where the funds, on average, perform equally well in bad and good states, underperforming the market in the latter (Fung and Hsieh (2001)). Our results can also shed some light on the economic environments that may support this hedge-fund evidence.

Closely related to our analysis of tracking error are the works of Roll (1992), Brennan (1993), Gómez and Zapatero (2003), and Jorion (2003), within a static mean-variance framework. Roll (1992) studies the portfolio problem of minimizing the tracking error variance for a given expected tracking error, referred to as the TEV criterion. Accordingly, he derives the TEV frontier in the

mean-variance tracking error space, and demonstrates that TEV efficient portfolios are not total return mean-variance efficient.³ Jorion (2003) further describes TEV-constrained portfolios by an ellipse in the total return mean-variance space. Both consider how imposing additional constraints can move the optimal portfolio closer to the total mean-variance frontier. Brennan (1993) and Gómez and Zapatero (2003) study the equilibrium implications of this type of benchmarking and derive a two-beta CAPM, where a new risk factor arises due to benchmarking. Gómez and Zapatero also provide strong empirical support for their model. Bajeux-Besnainou, Portrait, and Tergny (2003) extend Roll's analysis to a dynamic continuous-time setting, and additionally incorporate portfolio insurance and downside hedging constraints. Also in continuous-time, Teplá (2001) solves the utility maximization problem under downside hedging with respect to a stochastic benchmark, which is not necessarily the stock market. Finally, Browne (1999) studies a number of objective functions involving a stochastic benchmark, including minimizing the expected time to reach the benchmark, and maximizing the probability of beating the benchmark without underperforming it by a given amount.

Section 2 describes the economy with benchmarking practices, and presents new results for downside hedging. Section 3 solves the benchmarker's optimization problem. Section 4 discusses alternative formulations and extensions. Section 5 concludes the paper. Proofs are in the appendix.

2. The Economic Setting and Benchmarking Practices

2.1 The Economy

We consider a continuous-time, finite-horizon, $[0, T]$ economy. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined a 1-dimensional Brownian motion w . All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by w . All stated (in)equalities involving random variables hold P -almost surely.

Financial investment opportunities are given by an instantaneously riskless money market account and a risky stock, as in the Black and Scholes (1973) economy. The money market provides a constant interest rate r . The stock price, S , follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dw_t,$$

where the stock instantaneous mean return, μ , and standard deviation, σ , are constant. Dynamic market completeness (under no-arbitrage) implies the existence of a unique state price density process, ξ , given by

$$d\xi_t = -r\xi_t dt - \kappa\xi_t dw_t,$$

³Foster and Stutzer (2002) provide a more general approach for ranking relative fund performance based on overperformance probability, but do not study optimal portfolio choice.

where $\kappa \equiv (\mu - r)/\sigma$ is the constant market price of risk in the economy. As is well known (e.g., Karatzas and Shreve (1998)), the state price density serves as the driving economic state variable in an agent's dynamic investment problem.⁴ The quantity $\xi_T(\omega)$ is interpreted as the Arrow-Debreu price per unit probability P of one unit of wealth in state $\omega \in \Omega$ at time T .

An agent in this economy is endowed at time zero with an initial wealth of W_0 . The agent chooses an investment policy, θ , where θ_t denotes the fraction of wealth invested in the stock at time t . The agent's wealth process W then follows

$$dW_t = [r + \theta_t(\mu - r)] W_t dt + \theta_t \sigma W_t dw_t .$$

We assume that the agent has constant relative risk aversion (CRRA) preferences, $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and acts to maximize expected utility of horizon wealth, W_T .

With no further restrictions or considerations, this *normal* manager chooses the optimal horizon wealth to be (Merton (1971), Cox and Huang (1989))

$$W_T^N = I(y^N \xi_T) = \frac{1}{(y^N \xi_T)^{1/\gamma}} ,$$

where $I(\cdot)$ denotes the inverse of $u'(\cdot)$, and $y^N > 0$ solves $E[\xi_T I(y^N \xi_T)] = W_0$. As demonstrated in Sections 2.3–4, an important feature of this horizon wealth is its elasticity with respect to the economic state variable ξ_T , which is a constant given by

$$\frac{\partial W_T^N}{\partial \xi_T} \frac{\xi_T}{W_T^N} = -\frac{1}{\gamma} .$$

Henceforth, we refer to the quantity $1/\gamma$ as the *sensitivity of the normal* policy to economic conditions.

2.2 Benchmarking the Stock Market

Given the importance and prevalence of benchmarking in practice, our objective is to model an agent who manages the relative performance, or tracking error, of his portfolio along with other objectives. Specifically, consistent with industry-wide practices and the academic literature, we define the *tracking error* of an agent's horizon wealth relative to a benchmark X as:

$$R_T^W - R_T^X = \frac{1}{T} \ln \frac{W_T}{W_0} - \frac{1}{T} \ln \frac{X_T}{X_0} ,$$

where R_T denotes the continuously compounded return over the horizon $[0, T]$. The benchmark X represents the level of a portfolio, or an index, or any economic indicator. To embed benchmarking

⁴In this Black and Scholes-type setting, the focus is on systematic risk. The extension to multiple sources of uncertainty with multiple stocks is discussed in Section 4.1, where systematic risk is not the only consideration. However, the main insights of our Black and Scholes economy remain.

within a risk management framework, we assume that the manager abides by the following “tracking error constraint:”⁵

$$P(R_T^W - R_T^X \geq \varepsilon) \geq 1 - \alpha . \quad (1)$$

The constraint (1) states that the manager maintains his tracking error to be above some pre-specified level ε with confidence $1 - \alpha$. The case of $\varepsilon > 0$ corresponds to a manager aiming to overperform (beat) the return on the benchmark by at least ε , and $\varepsilon < 0$ to a manager aiming to not underperform the benchmark return by more than $|\varepsilon|$. The realizations of the manager’s return, R_T^W , below the *target return*, $R_T^X + \varepsilon$, are those of an unacceptable shortfall, and we refer to α as the *shortfall probability*. That is, the manager permits the performance of his portfolio to deteriorate below the target return ($R_T^W < R_T^X + \varepsilon$) with probability α .

In this paper, we focus on the most common, natural choice of benchmark: the stock market (Section 4.2 extends the analysis to hybrid benchmarks). In this case, $R_T^X = R_T^S = \frac{1}{T} \ln \frac{S_T}{S_0}$, and the constraint (1) leads the manager to strive to maintain his horizon wealth above a level given by:

$$X_T = W_0 e^{(R_T^S + \varepsilon)T} = e^{\varepsilon T} \frac{W_0}{S_0} S_T . \quad (2)$$

This is the wealth generated by investing the initial endowment at the target return, or equivalently a tracking error adjusted initial wealth, $X_0 = e^{\varepsilon T} W_0$, in the stock market. We refer to X_T as the *horizon benchmark level*, and note that terminal wealth W_T may fall short of the benchmark level with probability α (because comparing W_T to X_T is equivalent to comparing R_T^W to the target return $R_T^X + \varepsilon$). Although R_T^X is independent of ε (by definition), we incorporate ε into the definition of the benchmark level X_T in (2) to highlight that the manager’s wealth is determined by targeting stock market performance R_T^S , adjusted for the required overperformance ($\varepsilon > 0$), or allowed underperformance ($\varepsilon < 0$). Since in our economic setting S_T is decreasing in ξ_T , so is X_T ; as economic conditions deteriorate at the horizon, so does the benchmark level. We note that the stock market level being decreasing in the state price density is consistent with all related equilibria studied in the literature (e.g., normal pure-exchange economy of Lucas (1978)). An important quantity identified in the subsequent analysis is the elasticity of the horizon benchmark level with respect to the economic state variable ξ_T , which is a constant given by

$$\frac{\partial X_T}{\partial \xi_T} \frac{\xi_T}{X_T} = -\frac{\sigma}{\kappa} .$$

We refer to the quantity σ/κ as the *sensitivity of the benchmark* to economic conditions, and assume $\sigma/\kappa > 0$ without loss of generality (see Section 3.1).

Our reduced form tracking error constraint (1), has the convenient property that it nests other cases of interest investigated in the literature. When $\alpha = 1$, it nests the normal manager, who is

⁵While our definition of tracking error as a difference in returns is consistent with that adopted in the academic literature, in some practitioners’ literature tracking error refers to the standard deviation of this difference.

not concerned with benchmarking. For the case of the benchmark being the riskless money market account, $R_T^X = r$, the formulation reduces to Value-at-Risk based risk management (Basak and Shapiro (2001)). When $\alpha = 0$ the constraint is a “hard constraint,” nesting the case of portfolio insurance (Basak (1995), Grossman and Zhou (1996)) for the money market benchmark, and the case of minimum performance constraint (Teplá (2001)) for the stock market benchmark. Although the optimization problem accounting for the hard constraint (Section 2.3) represents time-consistent planning (e.g., Johnsen and Donaldson (1985)), for the general tracking error constraint it may not. A plausible mechanism for a commitment strategy may stem from the fact that a manager’s performance in practice is evaluated ex post, i.e., backtested on a repeated basis with implicit penalties, such as outflows of funds, imposed when falling short of a target return. However, such an analysis is beyond the scope of the current paper.

2.3 Downside Hedging

In this Section, we derive new results for the downside hedger who has to maintain his wealth above a minimum benchmarked-linked return at all times (tracking error constraint with $\alpha = 0$). A shortcoming of this approach, which prohibits shortfall, is that the downside hedger cannot target outperforming the stock market, $\varepsilon > 0$, as the problem is only feasible for $\varepsilon < 0$. The case of matching the benchmark return, $\varepsilon = 0$, leads to the trivial policy of investing all wealth in the benchmark. The downside hedger solves the following optimization problem:

$$\begin{aligned} \max_{W_T} E[u(W_T)] \quad & \text{subject to } E[\xi_T W_T] \leq W_0, \\ \text{and } R_T^W - R_T^X & \geq \varepsilon. \end{aligned} \quad (3)$$

The downside hedger’s optimal behavior is reported in Proposition 1 and depicted in Figure 1. Although aspects of downside hedging have previously been studied in the literature, the results below are new since we are able to exploit the dependence of the stock market benchmark on economic conditions (level of ξ_T). These results also establish a valuable comparison for the benchmarker’s behavior in Section 3.

Proposition 1. *The optimal horizon wealth of a downside hedger, for $\varepsilon < 0$, is given by,*

(a) *for economies with $\sigma/\kappa < 1/\gamma$:*

$$W_T^H = \begin{cases} I(y^H \xi_T) & \text{if } \xi_T < \underline{\xi} \\ X_T & \text{if } \underline{\xi} \leq \xi_T, \end{cases}$$

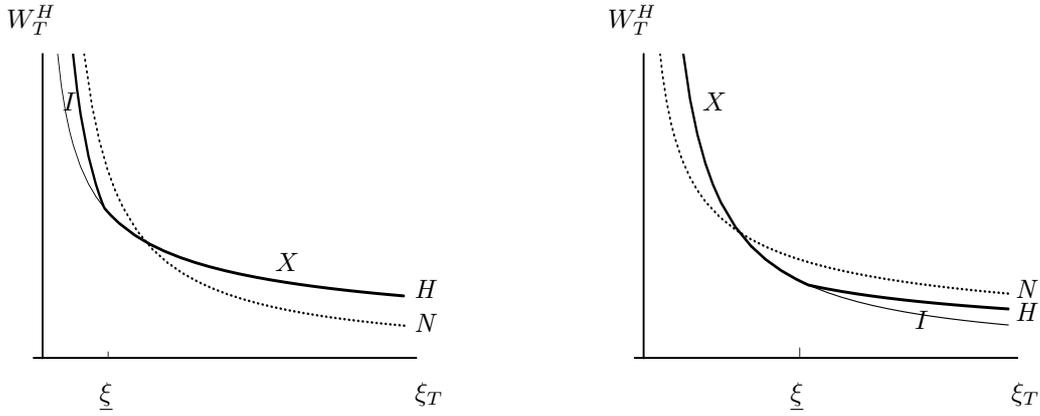
(b) *for economies with $\sigma/\kappa > 1/\gamma$:*

$$W_T^H = \begin{cases} X_T & \text{if } \xi_T < \underline{\xi} \\ I(y^H \xi_T) & \text{if } \underline{\xi} \leq \xi_T, \end{cases}$$

(c) *for economies with $\sigma/\kappa = 1/\gamma$: $W_T^H = I(y^H \xi_T)$,*

where in all economies $y^H > 0$ solves $E[\xi_T W_T^H] = W_0$, $\underline{\xi} = (y^H A^\gamma)^{1/(\gamma\sigma/\kappa-1)}$, and $A = W_0 \exp[(\varepsilon + (\mu - \sigma^2/2) - (r + \kappa^2/2)\sigma/\kappa)T]$. When $\varepsilon = 0$, then $W_T^H = X_T$, and when $\varepsilon > 0$, downside hedging is not feasible.

In economies where the benchmark is less sensitive to economic conditions than the normal policy, $\sigma/\kappa < 1/\gamma$ (Proposition 1(a), Figure 1(a)), the downside hedger's optimal behavior is similar to that of a portfolio insurer. When the benchmark reacts less to changes in economic conditions than the normal policy, the benchmark performs worse in good states ($\xi_T < \underline{\xi}$) and better in bad states ($\underline{\xi} \leq \xi_T$), as compared to the normal policy. Consequently, to meet the tracking error constraint (1), the benchmark level X_T (equation (2) with $\varepsilon < 0$) is matched in bad states, while a normal-type policy $I(y^H \xi_T)$ is adopted in good states.⁶ As with standard portfolio insurance, gains are lower in good states, and losses are lower in bad states compared to the normal policy.



(a) For economies with benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$.

(b) For economies with benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$.

Figure 1: Optimal horizon wealth, W_T^H , of a downside hedger (Proposition 1, bold plot, H), and of the normal manager, W_T^N (dotted plot, N). The thin solid plot denotes the benchmark level, X_T .

In economies where the benchmark reacts more to changes in economic conditions than a normal policy, $\sigma/\kappa > 1/\gamma$ (Proposition 1(b), Figure 1(b)), the benchmark performs better in good states ($\xi_T < \underline{\xi}$) and worse in bad states ($\underline{\xi} \leq \xi_T$), as compared to the normal policy. Consequently, it is now the good states that are insured, in contrast to the findings of related work on portfolio insurance and Value-at-Risk, where it is always the bad states that are insured. Gains are therefore

⁶Throughout, we use the term *normal* policy to refer to $I(y^N \xi_T)$, the optimal horizon wealth of the normal manager as defined in Section 2.1. A *normal-type* policy has the general form $I(y \xi_T)$, and therefore the same sensitivity to economic conditions as the normal policy, but differs in that $y \neq y^N$ so that, for a given value of ξ_T , the normal and normal-type policies will have different levels of wealth. For expositional convenience, the function I appears in our figures without the $y \xi_T$ argument.

higher in good states, and losses higher in bad states compared to the normal policy. When the stock market sensitivity equals normal sensitivity (Proposition 1(c)), the benchmark and normal policies respond similarly to economic fluctuations, the normal policy delivers the stock-market return in all states, and hence $W_T^H = W_T^N$.

3. Optimization under Risk Management with Benchmarking

In this Section, we solve the optimization problem of the benchmarker, who is required to maintain his tracking error relative to the stock market return to be above some prespecified level ε with a given confidence $1 - \alpha$, over an investment horizon $[0, T]$.

3.1 Manager's Optimization with Benchmarking

The dynamic optimization problem of the benchmarker can be restated as the following static variational problem using the martingale representation approach (Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987)):

$$\begin{aligned} & \max_{W_T} E[u(W_T)] \\ \text{subject to} & \quad E[\xi_T W_T] \leq W_0, \\ & \quad P(R_T^W - R_T^X \geq \varepsilon) \geq 1 - \alpha. \end{aligned} \tag{4}$$

One of the analytical subtleties here stems from the fact that the tracking error constraint complicates the problem not only by introducing nonconcavity into the maximization (as with benchmarking the money market), but also by linking the nature of the nonconcavity to the state-dependent characteristics of the benchmark. Proposition 2 presents the optimal solution⁷ and identifies six types of economies, as depicted in Figure 2, each characterized by the sensitivities of the normal policy and the benchmark to changes in the state of the economy. We note that none of the six economies may be ruled out on empirical grounds, as the managerial profile, γ , need not coincide with that of a representative agent.⁸

⁷The solution is in closed form, up to the constant Lagrange multiplier y^B . A feasibility condition for a solution is $\min_{\mathcal{A}} E[\xi_T X_T 1_{\{\xi_T \in \mathcal{A}\}}] \leq W_0$, where $\mathcal{A} \equiv \mathcal{R}_+ \setminus [a, b]$, and $0 \leq a < b \leq \infty$ are such that, $P(a \leq \xi_T < b) = \alpha$.

⁸To assess the plausibility of each economy, consider a risk premium of 6%, in line with the Mehra and Prescott (1985) estimate, with an accepted value for market volatility of 18%, which translates into benchmark sensitivity $\sigma/\kappa = 0.54$. Then, economy (a1) arises for $\gamma < 1.85$, (b1) for $\gamma > 1.85$, and (c1) for $\gamma = 1.85$. On the other hand, recent studies such as those by Pastor and Stambaugh (2001) and Fama and French (2002) suggest a lower value for the risk premium. For a risk premium of 3%, which is within the 2.55%–4.32% range estimated by Fama and French (2002), and using the above volatility value, corresponding to benchmark sensitivity $\sigma/\kappa = 1.08$, economy (a2) arises for $\gamma < 0.93$, (b2) for $\gamma > 0.93$, and (c2) for $\gamma = 0.93$. We also note that although economies (c1) and (c2) appear as knife-edge cases in the parameter space, these two economies are of interest as under both cases in the absence of benchmarking restrictions, the manager would be fully invested in the stock market benchmark.

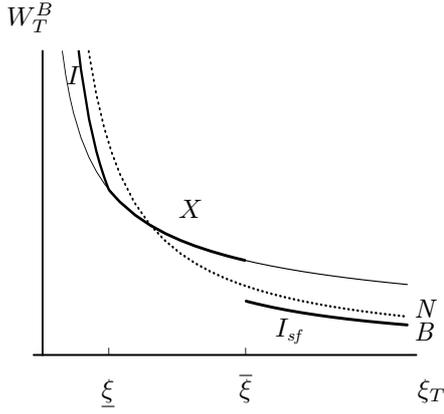
Proposition 2. *The optimal horizon wealth of a benchmarker, W_T^B , is reported in Table I, where in all economies $y^B > 0$ solves $E[\xi_T W_T^B] = W_0$, $\bar{\xi}$ and ξ^* denote downward and upward discontinuities in W_T^B , respectively (Figure 2), $\underline{\xi} = (y^B A^\gamma)^{1/(\gamma\sigma/\kappa - 1)}$, A is as in Proposition 1, and $g(\xi) = (\gamma(y^B \xi)^{(\gamma-1)/\gamma} - (A/\xi^{\sigma/\kappa})^{1-\gamma}) / (1-\gamma) + y^B A \xi^{1-\sigma/\kappa}$. For economies with $\sigma/\kappa = 1/\gamma$: if $\varepsilon < 0$, $W_T^B = I(y^B \xi_T)$; if $\varepsilon = 0$, $W_T^B = X_T$. In the remaining economies, when the tracking error constraint (1) is not binding, then $W_T^B = I(y^B \xi_T)$, $y^B = y^N$.*

Table I: Optimal Horizon Wealth of a Benchmarker, W_T^B .

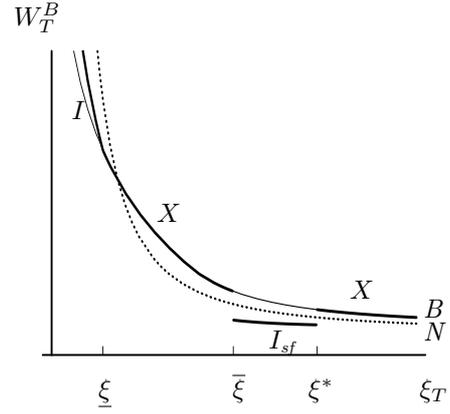
	$\sigma/\kappa < 1$	$\sigma/\kappa \geq 1$
	Economy (a1)	Economy (a2)
$\sigma/\kappa < 1/\gamma$	$W_T^B = \begin{cases} I(y^B \xi_T) & \text{if } \xi_T < \underline{\xi} \\ X_T & \text{if } \underline{\xi} \leq \xi_T < \bar{\xi} \\ I(y^B \xi_T) & \text{if } \bar{\xi} \leq \xi_T, \end{cases}$ <p>where $\bar{\xi}$ satisfies $P(\bar{\xi} \leq \xi_T) = \alpha$</p>	$W_T^B = \begin{cases} I(y^B \xi_T) & \text{if } \xi_T < \underline{\xi} \\ X_T & \text{if } \underline{\xi} \leq \xi_T < \bar{\xi} \\ I(y^B \xi_T) & \text{if } \bar{\xi} \leq \xi_T < \xi^* \\ X_T & \text{if } \xi^* \leq \xi_T, \end{cases}$ <p>where $\bar{\xi}, \xi$ satisfy $P(\bar{\xi} \leq \xi_T < \xi^*) = \alpha$, $g(\bar{\xi}) = g(\xi^*)$</p>
	Economy (b1)	Economy (b2)
$\sigma/\kappa > 1/\gamma$	$W_T^B = \begin{cases} X_T & \text{if } \xi_T < \bar{\xi} \\ I(y^B \xi_T) & \text{if } \bar{\xi} \leq \xi_T < \xi^* \\ X_T & \text{if } \xi^* \leq \xi_T < \underline{\xi} \\ I(y^B \xi_T) & \text{if } \underline{\xi} \leq \xi_T, \end{cases}$ <p>where $\bar{\xi}, \xi$ satisfy $P(\bar{\xi} \leq \xi_T < \xi^*) = \alpha$, $g(\bar{\xi}) = g(\xi^*)$</p>	$W_T^B = \begin{cases} I(y^B \xi_T) & \text{if } \xi_T < \xi^* \\ X_T & \text{if } \xi^* \leq \xi_T < \underline{\xi} \\ I(y^B \xi_T) & \text{if } \underline{\xi} \leq \xi_T, \end{cases}$ <p>where ξ^* satisfies $P(\xi_T < \xi^*) = \alpha$</p>
	Economy (c1)	Economy (c2)
$\sigma/\kappa = 1/\gamma$ for $\varepsilon > 0$	$W_T^B = \begin{cases} X_T & \text{if } \xi_T < \bar{\xi} \\ I(y^B \xi_T) & \text{if } \bar{\xi} \leq \xi_T, \end{cases}$ <p>where $\bar{\xi}$ satisfies $P(\bar{\xi} \leq \xi_T) = \alpha$</p>	$W_T^B = \begin{cases} I(y^B \xi_T) & \text{if } \xi_T < \xi^* \\ X_T & \text{if } \xi^* \leq \xi_T, \end{cases}$ <p>where ξ^* satisfies $P(\xi_T < \xi^*) = \alpha$</p>

In economies where the benchmark is less sensitive than the normal policy, $\sigma/\kappa < 1/\gamma$, downside hedging (when feasible) leads to matching the benchmark level in bad states (Proposition 1(a)). When shortfall is allowed, the key difference is that the benchmarker can choose in which “ α -fraction” of the states to fall short of the benchmark level and revert to a normal-type policy, $I(y^B \xi_T)$. Here, he identifies the states with the highest cost of matching the target versus following the normal-type policy, so that the benefit from reverting to the normal-type policy is highest.⁹ Proposition 2 reveals that the choice depends on whether the benchmark sensitivity is

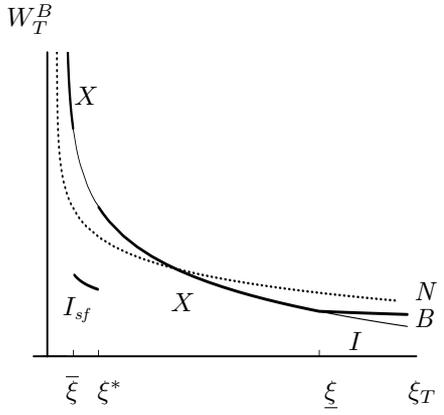
⁹As discussed in the Appendix, this benefit, captured by $g(\xi)$ in Proposition 2, is driven by the trade-off between the *state-contingent relative cost*, $\xi_T X_T - \xi_T I(y^B \xi_T)$, and the *state-contingent relative utility*, $u(X_T) - u(I(y^B \xi_T))$, of matching the benchmark versus following the normal-type policy. Where this benefit is largest (high, low or intermediate values of ξ_T) depends on whether the normal policy sensitivity, $1/\gamma$, and the benchmark sensitivity, σ/κ , are above or below unity as well as on which of the two is larger. However, in terms of distinct economic behaviors, the absolute value of $1/\gamma$ does not play a role.



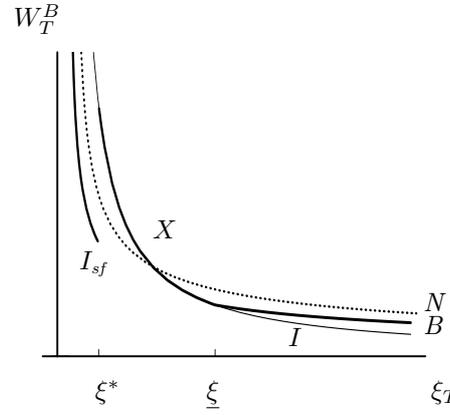
(a1) Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa < 1$.



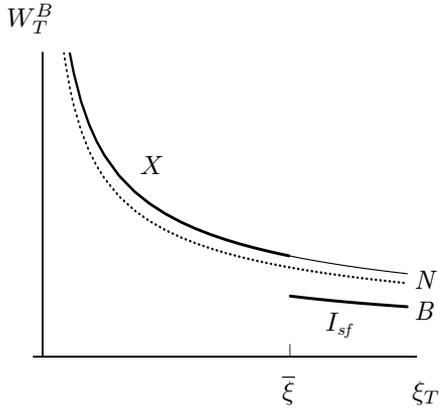
(a2) Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa \geq 1$.



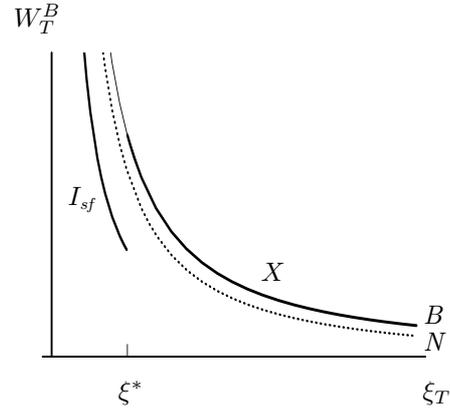
(b1) Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa < 1$.



(b2) Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa \geq 1$.



(c1) Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa < 1$.



(c2) Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa \geq 1$.

Figure 2: Optimal horizon wealth, W_T^B , of the benchmarker (Proposition 2, bold plot, B), and the normal manager, W_T^N (dotted plot, N). The thin solid plot denotes the benchmark level, X_T . Both I and I_{sf} represent the normal-type policy, with the latter denoting this policy across the shortfall (sf) states.¹⁰

below or above unity. For benchmark sensitivity below unity (economy (a1)), $\sigma/\kappa < 1$, the cost is highest in bad states, leading the benchmarker to revert to the normal policy in those states, and causing the single downward discontinuity at $\bar{\xi}$ (Figure 2(a1)).¹¹ On the other hand, for benchmark sensitivity above unity (economy (a2)), $\sigma/\kappa > 1$, it is now the “intermediate-bad” states in which the benchmark is least affordable, leading the benchmarker to revert to the normal policy in those states, causing the two discontinuities at $\bar{\xi}$ and ξ^* (Figure 2(a2)). An important economic implication of this analysis is that, in bad states, whereas in economy (a1) the benchmarker falls short of the target return and generates larger losses than the normal policy, in economy (a2) the benchmarker matches the target return and performs better than both the normal policy and possibly the stock market (for $\varepsilon > 0$). In these bad states, the benchmarker in economy (a2) can also perform better than the downside hedger, who also matches the target return in these states but necessarily underperforms the stock market (as $\varepsilon < 0$). This markedly different economic behaviour suggests that an appropriate combined choice of benchmark and manager as in economy (a2) could be of value to investors and may also merit regulatory consideration.

In economies where the benchmark reacts more to changes in economic conditions than the normal policy, $\sigma/\kappa > 1/\gamma$, downside hedging (when feasible) leads to matching the stock market benchmark in good states (Proposition 1(b)). The benchmarker, who is allowed a shortfall, reverts to the normal-type policy in good states when benchmark sensitivity exceeds unity (economy (b2)), and in “intermediate-good” states when benchmark sensitivity is below unity (economy (b1)), resulting in the discontinuities shown in Figures 2(b2) and 2(b1). In these economies, the main difference in the performance of the strategies now arises in the good states. In these states, the benchmarker in economy (b2) falls short of the target return and underperforms the normal policy, whereas the benchmarker in economy (b1) once again outperforms both the normal policy and the benchmark for $\varepsilon > 0$ (and thereby also the downside hedger who has $\varepsilon < 0$).

Finally, when the benchmark and normal policy have *equal* sensitivities, the benchmarker matches the target return in all states except the shortfall ones (either good or bad). Consequently,

¹⁰In Figures 2–4, the parameter values are as follows. In all economies, $W_0 = 1$, $r = 0.05$, $T = 1$. In (a1) and (b2), $\alpha = 0.01$, $\varepsilon = -0.025$, in all other economies $\alpha = 0.05$, $\varepsilon = 0.03$. In (a1), $\sigma/\kappa=0.5$, $1/\gamma=1$, $\sigma=0.13$, then $y^B=1.15$, $\xi=0.73$, $\bar{\xi}=2.23$. In (a2), $\sigma/\kappa=1.91$, $1/\gamma=3.23$, $\sigma=0.33$, then $y^B=1.55$, $\xi=0.35$, $\bar{\xi}=1.24$, $\xi^*=1.82$. In (b1), $\sigma/\kappa=0.83$, $1/\gamma=0.17$, $\sigma=0.45$, then $y^B=29.62$, $\bar{\xi}=0.08$, $\xi^*=0.34$, $\xi=2.67$. In (b2), $\sigma/\kappa=1$, $1/\gamma=0.5$, $\sigma=0.3$, then $y^B=1.35$, $\xi^*=0.26$, $\xi=1.28$. In (c1), $\sigma/\kappa=0.5$, $1/\gamma=0.5$, $\sigma=0.2$, then $y^B=2.38$, $\bar{\xi}=1.70$. In (c2), $\sigma/\kappa=1.25$, $1/\gamma=1.25$, $\sigma=0.3$, then $y^B=1.80$, $\xi^*=0.62$.

¹¹The Value-at-Risk manager (Basak and Shapiro (2001)) acts similarly to the benchmarker in economy (a1). This is the special case of a money market benchmark with zero sensitivity, where the benchmark is least affordable in bad states compared to any normal policy, as the latter is adversely affected in bad states for any (risk averse) preferences. As argued in the literature, this case inherits some unattractive features, such as higher losses in bad states than the normal policy. However, from our analysis (Proposition 2, Figure 2), such adverse effects are not robust to changes in the economic environment for a general benchmarking practice with benchmark sensitivity σ/κ . Hence, the money market benchmark case is somewhat restrictive as it is limited to only the type of behavior in economy (a1).

It is also evident that the case of negative sensitivity ($\sigma/\kappa < 0$) is captured by the solution in Proposition 2(a1). The only difference is that in Figure 2(a1), X_T in the intermediate region will be depicted as increasing in ξ_T (and similarly, in the bad states in Figure 1(a)).

unlike in other economies, the benchmarker never exceeds the target return, although matching this return does involve overperforming the stock market and hence results in either lower losses than the normal policy in bad states, as in economy (a2), or higher gains in good states, as in economy (b1).¹² In all economies, the wealth of the benchmarker in states in which he follows a normal-type policy (whether shortfall or not) is lower than that of the normal manager. This is because the benchmarker gives up some wealth in these states in order to be able to match the target return in others. However, where the states are not shortfall ones, the normal-type policy beats the target return.

The usefulness of our model is further highlighted by its ability to generate optimal investment behavior, which may shed some light on recently documented return patterns in money management. For equity mutual funds, Moskowitz (2000) and Kosowski (2002) present evidence suggesting that in recessions the funds, on average, overperform the market, while underperforming in non-recessions. Although such return patterns could hypothetically be attributed to cash balances held by these funds, the evidence of Kosowski (2002) is against this, showing that their cash holdings tend to fall significantly in recessions. Additionally, in absolute terms, funds perform better in recessions than otherwise, excluding late 1990 boom years. To see how such patterns may arise, note that our economies (b2) and (c2) agree with the relative performance feature when $\varepsilon > 0$ (and non-recessions identified by $\xi_T < \xi^*$). With a sufficiently high ε , and a plausible distribution of ξ_T , the absolute performance feature may also arise with the average performance over states in the $[\xi^*, \infty)$ region being higher than in good states $[0, \xi^*)$ in (b2) and (c2). This behavior, however, could not have been generated by other leading risk management approaches. The Value-at-Risk approach, portfolio insurance, or downside hedging with a more sensitive benchmark (Figure 1(b)) prescribe results for bad states that are inconsistent with respect to absolute fund performance. Downside hedging with a less sensitive benchmark (Figure 1(a)) cannot reconcile either the relative or absolute return fund patterns. Moreover, for trend-following hedge funds, Fung and Hsieh (2001) document a straddle-like return pattern, where the funds underperform the market in good states and overperform in bad, while in absolute terms their returns are similar in good and bad states. This can be generated in our setting by an appropriate adjustment of parameters, as for example economies (b2) and (c2) readily deliver such behavior relative to the market, and for an appropriate $\varepsilon > 0$ can support the absolute performance as well.

3.2 Investment Policies with Benchmarking

The benchmarker's optimal horizon wealth (Proposition 2) can be expressed as the wealth generated by a normal policy plus an option to exchange this wealth for the horizon benchmark level, plus a short binary option position with exercise range corresponding to the shortfall region, and payoff

¹²When the normal policy and benchmark have the same sensitivity, the normal policy matches the stock market return in all states, and hence for $\varepsilon \leq 0$ the tracking error constraint (1) never binds.

given by the shortfall amount. While these options are written on the normal policy, given that the normal policy corresponds to a managed position in the benchmark, the optimal horizon wealth can therefore be generated through a static strategy involving options on the benchmark (stock market index). Proposition 3 presents explicit expressions for the benchmarker's optimal wealth and portfolio strategies before the planning horizon, and also reports new results for the special case of downside hedging. The expression for optimal pre-horizon wealth in Proposition 3(i) can be understood as the pre-horizon value of the option package in each economy, explaining the appearance of the Black and Scholes (1973)-type terms, as well as the non-monotonous patterns (due to the binary option) in Figures 3–5.

Proposition 3.

(i) *The time- t optimal wealth of the benchmarker is given by:*

$$\begin{aligned}
W_t^B = & \left[1_{\{a1,b1,b2,c1\}} + \mathcal{N}(d(\gamma, \underline{\xi}))1_{\{a1,a2\}} - \mathcal{N}(d(\gamma, \bar{\xi}))1_{\{a1,a2,b1,c1\}} \right. \\
& \left. + \mathcal{N}(d(\gamma, \xi^*))1_{\{a2,b1,b2,c2\}} - \mathcal{N}(d(\gamma, \underline{\xi}))1_{\{b1,b2\}} \right] Z(\gamma)(y^B \xi_t)^{-1/\gamma} \\
& + \left[1_{\{a2,c2\}} - \mathcal{N}(d(\kappa/\sigma, \underline{\xi}))1_{\{a1,a2\}} + \mathcal{N}(d(\kappa/\sigma, \bar{\xi}))1_{\{a1,a2,b1,c1\}} \right. \\
& \left. - \mathcal{N}(d(\kappa/\sigma, \xi^*))1_{\{a2,b1,b2,c2\}} + \mathcal{N}(d(\kappa/\sigma, \underline{\xi}))1_{\{b1,b2\}} \right] Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa}, \quad (5)
\end{aligned}$$

where the arguments of the indicator function $1_{\{\cdot\}}$ refer to the economies identified in Proposition 2, $\mathcal{N}(\cdot)$ is the standard-normal cumulative distribution function, y^B is as in Proposition 2, and

$$Z(v) \equiv e^{\frac{1-v}{v} \left(r + \frac{\kappa^2}{2v} \right) (T-t)}, \quad d(v, x) \equiv \frac{\ln \frac{x}{\xi(t)} + \left(r + \frac{2-v}{2v} \kappa^2 \right) (T-t)}{\kappa \sqrt{T-t}}.$$

(ii) *The fraction of wealth invested in stocks is:*

$$\theta_t^B = q_t^B \theta^N,$$

where $\theta^N = \kappa/(\gamma\sigma)$ is the optimal fraction of wealth invested in the stock under the normal policy, and q_t^B , the exposure relative to the normal policy, is given by

$$\begin{aligned}
q_t^B = & 1 + \left[1_{\{a2\}} - \mathcal{N}(d(\kappa/\sigma, \underline{\xi}))1_{\{a1,a2\}} + \mathcal{N}(d(\kappa/\sigma, \bar{\xi}))1_{\{a1,a2,b1\}} \right. \\
& \left. - \mathcal{N}(d(\kappa/\sigma, \xi^*))1_{\{a2,b1,b2\}} + \mathcal{N}(d(\kappa/\sigma, \underline{\xi}))1_{\{b1,b2\}} \right] (\gamma\sigma/\kappa - 1) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} / W_t^B \\
& + \left\{ \left[\varphi(d(\gamma, \underline{\xi}))1_{\{a1,a2\}} - \varphi(d(\gamma, \bar{\xi}))1_{\{a1,a2,b1,c1\}} + \varphi(d(\gamma, \xi^*))1_{\{a2,b1,b2,c2\}} \right. \right. \\
& \left. \left. - \varphi(d(\gamma, \underline{\xi}))1_{\{b1,b2\}} \right] Z(\gamma)(y^B \xi_t)^{-1/\gamma} \right. \\
& \left. - \left[\varphi(d(\kappa/\sigma, \underline{\xi}))1_{\{a1,a2\}} - \varphi(d(\kappa/\sigma, \bar{\xi}))1_{\{a1,a2,b1,c1\}} + \varphi(d(\kappa/\sigma, \xi^*))1_{\{a2,b1,b2,c2\}} \right. \right. \\
& \left. \left. - \varphi(d(\kappa/\sigma, \underline{\xi}))1_{\{b1,b2\}} \right] Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} \right\} \gamma / (W_t^B \kappa \sqrt{T-t}), \quad (6)
\end{aligned}$$

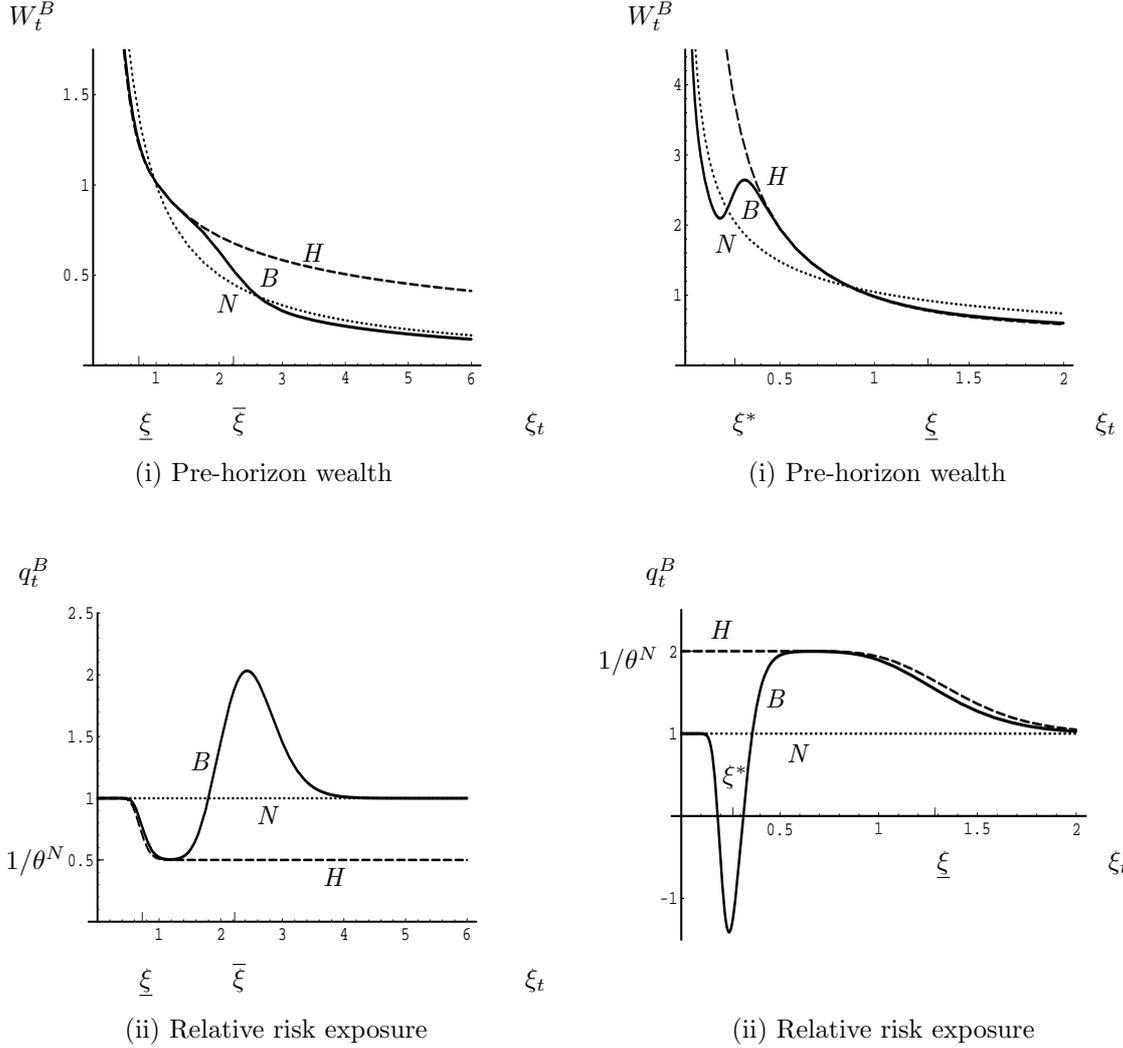
where $\varphi(\cdot)$ is the standard-normal probability density function.

(iii) When $\varepsilon < 0$, $\sigma/\kappa \neq 1/\gamma$, and the optimal policies for downside hedging are given by (5) and (6), for $\alpha = 0$, so that in (a1) $\bar{\xi} = \infty$, in (b2) $\xi^* = 0$, and in (a2) and (b1) $\bar{\xi} = \xi^*$. When $\bar{\xi} = \infty$, $\sigma/\kappa = 1/\gamma$, (5) and (6) coincide with the normal policy. When $\varepsilon = 0$, X_t is the optimal policy, with relative risk exposure of $1/\theta^N$.

Figure 3 presents the results for economies (a1) and (b2) when the benchmarker's goal is merely to limit underperformance ($\varepsilon < 0$), allowing us to simultaneously study the policies of the downside hedger.¹³ In economy (a1), both the benchmarker and the downside hedger match the horizon benchmark level in intermediate states, with the former choosing to fall short in bad states. Thus, both managers' pre-horizon wealth behaves similarly to that of a normal manager in good states, tending to the benchmark in intermediate states. In bad states, the downside hedger's pre-horizon wealth continues to track the less sensitive benchmark, while the benchmarker reverts back to normal behavior. Similarly, the risk exposure for both managers resembles the normal policy in good states, and as ξ_t increases, decreases towards $1/\theta^N < 1$, the relative risk exposure required to replicate the benchmark. In bad states, the downside hedger remains invested in the benchmark. The benchmarker, however, increases his exposure back up to, then above, and finally back down to the normal policy as ξ_t increases. In states near $\bar{\xi}$, there is a fair chance that the benchmarker matches the benchmark, but only if he takes a large stock position and the economy does not experience a downturn ($\xi_T < \bar{\xi}$).

In economy (b2), the benchmarker falls short of the horizon benchmark level in good states. Thus, in the region of ξ^* , the benchmarker reduces, rather than increases, his stock market exposure, possibly even taking a short position, to allow him to increase his wealth and match the benchmark if economic conditions deteriorate ($\xi_T > \xi^*$). A noteworthy feature of economy (b2) is that, due to the upward discontinuity at ξ^* of the horizon policy, over a region of the state space, the benchmarker's pre-horizon wealth increases, rather than decreases for deteriorating economic conditions. This is in contrast to standard results where optimal wealth suffers as economic conditions deteriorate. A byproduct of this behavior is that the same wealth level may be observed under three different economic scenarios (e.g., consider the $W_t^B = 2.5$ level obtained for three different values of ξ_t in Figure 3 for economy (b2)), suggesting caution in attempting to infer the state of the economy by observing portfolio wealth alone. This increasing wealth feature is also present in economies (a2), (b1) around ξ^* , and (c2) as well (for brevity not depicted in the figures).

¹³In Figure 3, the downside hedger's pre-horizon wealth appears to be always greater than or equal to that of the benchmarker. Values of ξ_t for which the benchmarker's pre-horizon wealth (clearly) exceeds that of the downside hedger fall outside the plotted range (which is chosen to highlight the behavior of pre-horizon wealth around the discontinuity points).



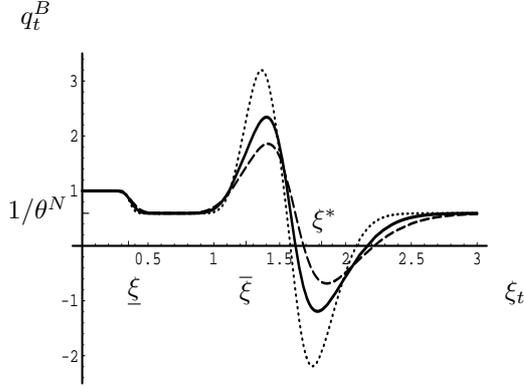
Economy (a1): Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa < 1$.

Economy (b2): Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa \geq 1$.

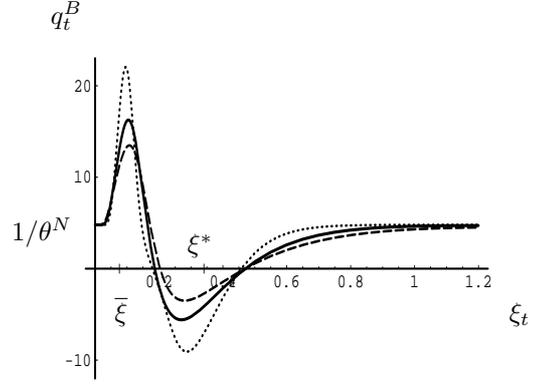
Figure 3: The time- t (i) wealth and (ii) exposure to risky assets relative to the normal policy (Proposition 3), for the benchmarker (solid plots, B), the downside hedger (dashed plots, H), and the normal manager (dotted plots, N), in economies (a1) and (b2). Here, $t = 0.8$, and all remaining parameter values are as in Figure 2 for (a1) and (b2).

The remaining economies can be analyzed analogously. Optimal risk exposure tends to 1 whenever the benchmarker or downside hedger acts like a normal manager, and to $1/\theta^N$ whenever he tracks the benchmark. Downward discontinuities in optimal horizon wealth lead to increased pre-horizon risk exposures at those values of ξ_t (as in economy (a1)), while upward discontinuities lead to reduced, possibly negative, risk exposures (as in economy (b2)). This leads to interesting portfolio behavior in economies (a2) and (b1), which feature both upward and downward discontinuities in

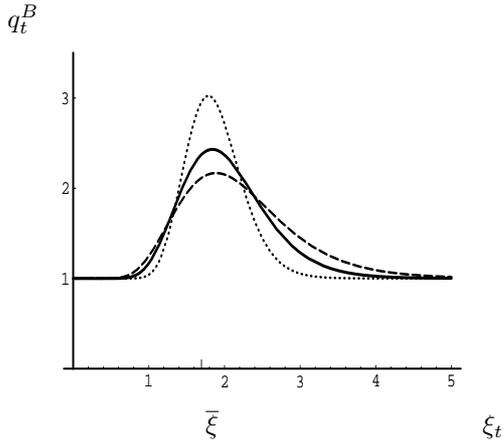
optimal horizon wealth. As Figure 4 illustrates, in these economies, considerable shifts in portfolio composition can occur, possibly from leveraged to short positions, and vice versa, upon relatively minor changes in economic conditions (as ξ_t changes).¹⁴ Hence, if shortfall-based risk management



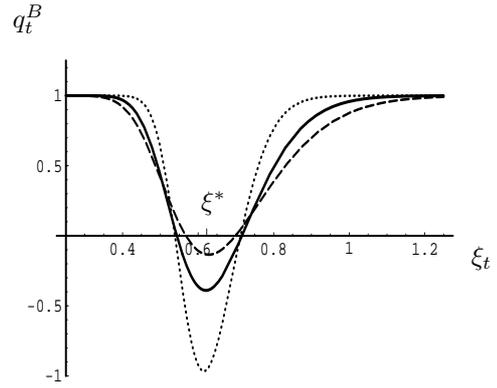
Economy (a2): Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa \geq 1$.



Economy (b1): Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa < 1$.



Economy (c1): Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa < 1$.



Economy (c2): Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa \geq 1$.

Figure 4: The time- t exposure to risky assets relative to the normal policy, for the benchmarker (Proposition 3) for economies (a2), (b1), (c1) and (c2). The solid, dashed, and dotted plots represent $t = 0.5$, $t = 0.25$, and $t = 0.75$, respectively. All remaining parameter values are as in Figure 2 for (a2), (b1), (c1) and (c2).

¹⁴In our initial analysis with one risky investment opportunity, such investment behavior is obviously permissible for hedge funds. Equity mutual fund managers are likely to face borrowing and shortsale constraints, which in this initial analysis we ignore for simplicity. However, our main insights do not rely on the presence of short positions per se, but more generally on the manager's desire to take on "bearish" positions. Indeed, in reality with many available investment opportunities, in line with our extension in Section 4.1, such bearish positions may be implemented by mutual funds through exposure to securities that are of a contrarian nature relative to the broad market.

is indeed explicitly or implicitly being followed by institutional investors, our results suggest a potential explanation to the puzzling, but yet observed phenomena, where seemingly small arrivals of news regarding fundamentals may at times carry no considerable reaction from market participants, but at other times cause significant portfolio rebalancing. Clearly, in economies (a2) and (b1) (as in other economies), the nature of the risk management practice (α , ε) as well as the actual state of the economy (ξ_t) determines how pronounced the impact of external news is.

The sensitivities of the risk exposures to various parameteres are illustrated in Figures 4–5, and are typical across all economies. From Figure 4, the shorter the time horizon, the more the benchmarker deviates from the normal policy in the region for which chances of shortfall are highest, amplifying portfolio swings in that region with possible implications for financial stability. From Figure 5, the benchmarker deviates further from the normal policy as α decreases and as ε increases, in each case reflecting the greater influence of the tracking error constraint. The effect is most pronounced in the region of maximum exposure around $\bar{\xi}$, as around $\underline{\xi}$, the risk exposure is bounded below by $1/\theta^N$ ($= 0.5$ for the figure parameters). The maximum exposure for decreasing α , or increasing ε , occurs at higher values of ξ_t (since the shortfall region is shrinking, or is fixed).

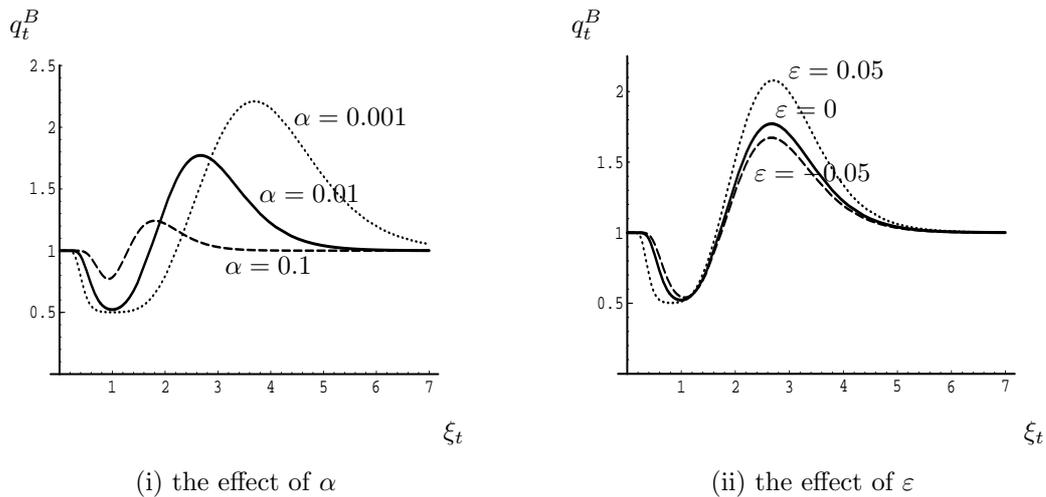


Figure 5: The benchmarker’s relative risk exposure for varying levels of (i) $\alpha \in \{0.001, 0.01, 0.1\}$, and (ii) $\varepsilon \in \{-0.05, 0, 0.05\}$ in economy (a1). The solid plots represent the following parameter values: $\sigma/\kappa = 0.5$, $1/\gamma = 1$, $\alpha = 0.01$, $\varepsilon = 0$, $r = 0.05$, $\sigma = 0.2$, and $W_0 = 1$. Then $y^B = 1.28$, $\underline{\xi} = 0.55$, and $\bar{\xi} = 2.23$.

4. Alternative Formulations and Extensions

4.1 Multiple Sources of Uncertainty with Multiple Stocks

When there are multiple sources of uncertainty, our results regarding benchmarking the stock market remain the same provided stock market fluctuations are driven by “systematic” uncertainty,

as captured by the state price density process. If, instead, one is interested in benchmarking some sector of the market that is also affected by “idiosyncratic” uncertainty, our insights are still applicable. Consider, for example, an economy with uncertainty generated by two Brownian motions (w_1, w_2) , and financial investment opportunities given by the money market account, and two risky stocks (S, Q) , each with a price following a geometric Brownian motion. Suppose that the manager has logarithmic preferences ($\gamma = 1$), and benchmarks the performance of the first stock ($R_T^X = R_T^S$), with allowed shortfall probability α . Suppose that an exactly matched performance is desired ($\varepsilon = 0$) so that $X_T = S_T$ (normalizing $W_0 = S_0$). The benchmarker’s optimal policy is then:

$$W_T^B = \begin{cases} I(y^B \xi_T) = 1/(y^B \xi_T) & \text{if } \xi_T < 1/(y^B S_T) & \text{(I)} \\ S_T & \text{if } 1/(y^B S_T) \leq \xi_T < c/S_T & \text{(II)} \\ I(y^B \xi_T) = 1/(y^B \xi_T) & \text{if } c/S_T \leq \xi_T, & \text{(III)} \end{cases}$$

where c satisfies $P(\xi_T S_T \geq c) = \alpha$, and y^B the budget constraint. The optimal policy, exhibits three distinct patterns of behavior over three regions of the (S_T, ξ_T) state space, where region (III) is the shortfall region (in which $W_T^B < S_T$). However, it is the correlation between S_T and ξ_T that determines the location of each region within the (S_T, ξ_T) plane.

When the benchmark, S_T , represents a dominant sector within the economy, it is driven mainly by ξ_T . The manager’s problem is then effectively one-dimensional, and the solution resembles one of the three-region policies in Proposition 2(a1)(b2) (depicted in Figure 2(a1)(b2)), depending on the underlying parameters ($\gamma = 1$ implies either economy (a1) or (b2)). As the correlation between S_T and ξ_T weakens, S_T can take many values upon a given realization of ξ_T . Whether the optimal policy follows the benchmark or normal behavior is determined by considering the relative sensitivities and state-contingent relative costs of the two types of behavior, as in Section 3. If S_T has low sensitivity with respect to ξ_T , it will tend to not rise significantly in good states (low ξ_T) and to not decrease significantly in bad states (high ξ_T). Then, it is the condition for region (I) that will hold in good states (as $\xi_T S_T$ is low), and the condition for region (III) that will hold in bad states (as $\xi_T S_T$ is high). Consequently, the shortfall region in the (S_T, ξ_T) plane will be where ξ_T is high and S_T not too low. The opposite holds when S_T is highly sensitive with respect to ξ_T .

4.2 Hybrid Benchmarks and Other Extensions

While the analysis so far has focused on stock market benchmarks, in practice there appears to be considerable interest in composite benchmarks that combine money market and stock market returns. In the simplest version of such a benchmark, the benchmark return is a weighted average of the returns over the period on the money market account and the stock market: $R_T^X = \beta r + (1 - \beta)R_T^S$, where $0 \leq \beta \leq 1$. The horizon level for this hybrid benchmark is given by

$$X_T = W_0 e^{(\beta r + \varepsilon)T} \left(\frac{S_T}{S_0} \right)^{1-\beta} .$$

Clearly, for $\beta = 1$ and $\beta = 0$, the money market and stock market benchmarks obtain. Moreover, our analysis goes through using this hybrid level, and using the corresponding benchmark sensitivity $(1 - \beta)\sigma/\kappa$ (instead of σ/κ).¹⁵ The applicability of our analysis for the hybrid benchmark offers important flexibility in the benchmark choice. By choosing the appropriate benchmark (via choice of β) one can lead a manager, with a given risk profile, to follow a particularly desirable policy out of those presented in Proposition 2/Figure 2. Note that all the benchmarks discussed in this paper are tradeable (investable), as is commonly the case in the mutual fund industry. A logical extension would be to consider non-tradeable benchmarks, such as exist for various hedge fund styles.

4.3 Benchmarking with Limited Expected Relative Losses

We have so far considered the most basic shortfall approach, captured by the quantile-based tracking error constraint (1), which focuses on the shortfall probability α of not meeting the target return $R_T^X + \varepsilon$. An alternative approach is to limit both the probability and magnitude of the shortfall, and the simplest way to achieve that is to adopt an expectations-based constraint that limits the losses relative to the horizon benchmark level in (2):

$$E \left[\xi_T \left(e^{(R_T^X + \varepsilon)T} - e^{R_T^W T} \right) 1_{\{R_T^W - R_T^X < \varepsilon\}} \right] \leq \beta . \quad (7)$$

Such an approach is of further interest as it has been argued to remedy some of the shortcomings of Value-at-Risk with the money market benchmark (Artzner, Delbaen, Eber and Heath (1999), Basak and Shapiro (2001)). The optimal policy of a manager benchmarking the stock market ($R_T^X = R_T^S$) subject to limited expected relative losses (LERL) as in (7)¹⁶ is reported in Proposition 4 and depicted in Figure 6.

Proposition 4. *The optimal horizon wealth of a manager benchmarking the stock market subject to limited expected relative losses is given by,*

(a) *for economies with $\sigma/\kappa < 1/\gamma$:*

$$W_T^L = \begin{cases} I(z_1 \xi_T) & \text{if } \xi_T < \underline{\xi} \\ X_T & \text{if } \underline{\xi} \leq \xi_T < \bar{\xi} \\ I((z_1 - z_2)\xi_T) & \text{if } \bar{\xi} \leq \xi_T , \end{cases}$$

¹⁵These results extend straightforwardly to the case of a dynamically managed hybrid benchmark which at all times maintains weights δ and $(1 - \delta)$ in the money market and stock market, respectively, for which the benchmark sensitivity is $(1 - \delta)\sigma/\kappa$.

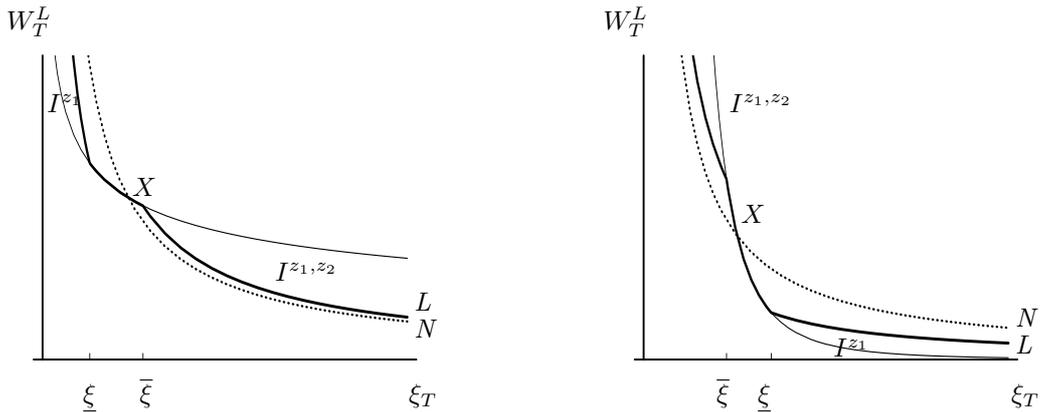
¹⁶The constraint in (7) is cast in units of wealth, because as we saw in Section 3, terminal wealth is in fact the relevant choice variable. Another expectations-based constraint to consider is $E \left[\left((R_T^X + \varepsilon) - R_T^W \right) 1_{\{R_T^W - R_T^X < \varepsilon\}} \right] \leq \eta$. Although this constraint is less tractable to analyze, our insights can be extended to this case as well.

(b) for economies with $\sigma/\kappa > 1/\gamma$:

$$W_T^L = \begin{cases} I((z_1 - z_2)\xi_T) & \text{if } \xi_T < \bar{\xi} \\ X_T & \text{if } \bar{\xi} \leq \xi_T < \underline{\xi} \\ I(z_1\xi_T) & \text{if } \underline{\xi} \leq \xi_T, \end{cases}$$

(c) for economies with $\sigma/\kappa = 1/\gamma$: $W_T^L = I((z_1 - z_2)\xi_T)$ coincides with W_T^N , where in all economies $z_1 > z_2 \geq 0$ solve $E[\xi_T W_T^L] = W_0$ with (7) holding with equality, $\underline{\xi} = (z_1^\gamma)^{1/(\gamma\sigma/\kappa-1)}$, $\bar{\xi} = ((z_1 - z_2)^\gamma)^{1/(\gamma\sigma/\kappa-1)}$, and A is as in Proposition 1.

In economies where the stock market is less sensitive than the normal policy (Figure 6(a)), the LERL approach guarantees lower losses in bad states than the normal policy, similarly to benchmarking the money market with zero sensitivity. However, since it is most cost effective to fall short of the benchmark in bad states, risk management with LERL is less desirable for those interested in beating the stock market in bad states. It is the quantile shortfall approach that can allow market overperformance in bad states (Proposition 2(a2)(c2)).



(a) For economies with benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$.

(b) For economies with benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$.

Figure 6: Optimal horizon wealth, W_T^L , of a manager (Proposition 4) benchmarking the stock market with LERL (bold plot, L), and of the normal manager, W_T^N (dotted plot, N). The thin solid plot denotes the benchmark level, X_T .

Moreover, in economies where the stock market is more sensitive than the normal policy (Figure 6(b)), the LERL approach leads to larger losses in bad states, similarly to the quantile-based approach. However, the quantile shortfall approach offers the additional flexibility of beating the stock market in good states. Therefore, contrary to the case of benchmarking the money market, the expectations based risk measure is not unambiguously more desirable than the quantile measure on a gain/loss basis.

5. Conclusion

In this paper, we focus on an important feature of the money management industry – relative performance evaluation – which leads to risk management practices that account for benchmarking. A rigorous understanding of this practice is in its infancy in the academic literature, not least because of the analytical difficulty of the problem. We approach the issue by combining a tracking error constraint with utility maximizing behavior. This, in turn, provides a rich set of theoretical results, as well as guidance for investors on how to select managers/benchmarks in order to achieve a desired investment performance profile. It would be of interest to explore further the cross-sectional implications of our analysis for the money management industry, such as identifying combinations of benchmarks and risk attitudes that may explain a particular mutual fund or hedge fund style.

A natural extension of our investigation would be to incorporate investment restrictions (Cvitanović and Karatzas (1992), Detemple and Murthy (1997)) that a money manager may be faced with, though this appears not so tractable given the current state of the field. Another direction for future research would be to extend our analysis to a richer setting, where the manager has superior information or ability, or expends costly effort. Finally, there is room to study the implications of benchmarking in other institutional settings, such as that of a pension fund manager who is interested in limiting a shortfall relative to future liabilities that are affected by uncertain retirement patterns.

Appendix: Proofs

Proof of Proposition 1. See proof of Proposition 2 for $\alpha = 0$. ■

Proof of Proposition 2. When the constraint is binding, the optimality of the solutions in Table I for each of the economies (a1)-(c2) is most conveniently proved case by case, for the associated ranges of the benchmark and normal sensitivities. The logic of the proof in each economy is to adapt the convex-duality approach (see Karatzas and Shreve (1998)) to a non-concave problem, and to show sufficiency for optimality of the stated solution. Lemmas 1 and 2 below deal with the state dependency of the problem introduced by the stochastic benchmark. Since economy (a2) is a case with an optimal policy of four distinct regions and two discontinuities across the state space, it represents, to the best of our knowledge, a somewhat different case compared to the existing literature, and hence we first focus on the proof in this economy. We then show how the proof proceeds for the other economies in a similar manner. To save notation, we suppress below the superscript B on the Lagrange multiplier y .

Lemma 1. For $1 < \sigma/\kappa < 1/\gamma$, and $\bar{\xi}, \xi^*$ satisfying $g(\bar{\xi}) = g(\xi^*)$ and $\underline{\xi} < \bar{\xi} < \xi^*$, we have $g(\xi) < g(\bar{\xi})$ for $\underline{\xi} < \xi < \bar{\xi}$ or $\xi^* < \xi$, and $g(\xi) > g(\bar{\xi})$ for $\bar{\xi} < \xi < \xi^*$.

Proof: Note that $g(\underline{\xi}) = 0$, and since $\gamma - 1 < 0$ and $1 - \sigma/\kappa < 0$, we obtain $\lim_{\xi \rightarrow \infty} g(\xi) = 0$. $\frac{\partial g(\xi)}{\partial \xi} = \xi^{-\sigma/\kappa} f(\xi)$, where

$$f(\xi) = -y^{(\gamma-1)/\gamma} \xi^{(\sigma/\kappa-1/\gamma)} + (\sigma/\kappa) A^{1-\gamma} \xi^{\gamma(\sigma/\kappa-1/\gamma)} + yA(1 - \sigma/\kappa).$$

We also have $f(\underline{\xi}) = 0$, and because $\sigma/\kappa - 1/\gamma < 0$, we obtain $\lim_{\xi \rightarrow \infty} f(\xi) = yA(1 - \sigma/\kappa) < 0$. It is thus left to show $\exists \tilde{\xi} > \underline{\xi}$ such that $f(\xi) > 0$ for $\underline{\xi} < \xi < \tilde{\xi}$, and $f(\xi) < 0$ for $\tilde{\xi} < \xi$, as this will establish the desired properties of g over $(\underline{\xi}, \infty)$. To that end, it is immediate to verify that $\frac{\partial f(\xi)}{\partial \xi} > 0$ if, and only if, $\xi < \check{\xi}$, where $\check{\xi} = ((1/\gamma)/(\sigma/\kappa))^{1/((1-\gamma)(1/\gamma-\sigma/\kappa))} \underline{\xi} > \underline{\xi}$, with the latter inequality holding because $\check{\xi}$ multiplies $\underline{\xi}$ by a constant greater than unity raised to a positive power. The continuity of f , while decreasing over $(\check{\xi}, \infty)$ towards its negative limit for $\xi \rightarrow \infty$, guarantees the uniqueness of $\tilde{\xi}$, as required. ■

Lemma 2. For $1 < \sigma/\kappa < 1/\gamma$, let $W(\xi) = (y\xi)^{-1/\gamma} 1_{\{\xi < \underline{\xi}, \text{ or } \bar{\xi} \leq \xi < \xi^*\}} + A\xi^{-\sigma/\kappa} 1_{\{\xi \leq \xi < \bar{\xi}, \text{ or } \xi^* \leq \xi\}}$, $x = g(\bar{\xi})$, and $h(W, \xi) = u(W) - y\xi W + x 1_{\{W \geq A\xi^{-\sigma/\kappa}\}}$. Then, $\forall \xi \geq 0$, $W(\xi) = \arg \max_W h(W, \xi)$.

Proof: For a given $\xi \geq 0$, $h(W, \xi)$ is not concave in W . However, its local maxima are attained at $I(y\xi) = (y\xi)^{-1/\gamma}$ or at $A\xi^{-\sigma/\kappa}$. To find the global maximizer, we compare the value of h at these two candidate points. When $\xi < \underline{\xi}$, then $(y\xi)^{-1/\gamma} > A\xi^{-\sigma/\kappa}$, and hence $h((y\xi)^{-1/\gamma}, \xi) > h(A\xi^{-\sigma/\kappa}, \xi)$, so that $(y\xi)^{-1/\gamma}$ is the global maximizer. When $\underline{\xi} \leq \xi$, then $(y\xi)^{-1/\gamma} < A\xi^{-\sigma/\kappa}$, and

from the definitions of $g(\cdot)$, $h(\cdot)$ and x we get

$$h((y\xi)^{-1/\gamma}, \xi) - h(A\xi^{-\sigma/\kappa}, \xi) = g(\xi) - x .$$

From Lemma 1, $g(\xi) < x$ for $\underline{\xi} \leq \xi < \bar{\xi}$ or $\xi^* \leq \xi$, and the global maximizer in these regions is $A\xi^{-\sigma/\kappa}$. On the other hand, for $\bar{\xi} \leq \xi < \xi^*$, $g(\xi) > x$, and $(y\xi)^{-1/\gamma}$ is the global maximizer. ■

The benchmark horizon level in (2) satisfies

$$X_T = W_0 e^{\varepsilon T} S_T / S_0 = W_0 e^{(\varepsilon + \mu - \sigma^2/2)T + \sigma w_T} = A \xi_T^{-\sigma/\kappa} , \quad (\text{A1})$$

where the second and third equalities follow from the terminal values of S_T and ξ_T , respectively, as implied by their geometric Brownian motion dynamics. Next, let W_T^B be as in Table I (a2), and let W_T be any candidate optimal solution for economy (a2), satisfying the tracking error constraint and the static budget constraint in (4). We then have

$$\begin{aligned} & E[u(W_T^B)] - E[u(W_T)] \\ &= E[u(W_T^B)] - yW_0 + x(1 - \alpha) - (E[u(W_T)] - yW_0 + x(1 - \alpha)) \\ &\geq E[u(W_T^B)] - E[y\xi_T W_T^B] + E[x1_{\{W_T^B \geq X_T\}}] - (E[u(W_T)] - E[y\xi_T W_T] + E[x1_{\{W_T \geq X_T\}}]) \geq 0, \end{aligned}$$

where the first inequality follows from the budget constraint and the tracking error constraint holding with equality for W_T^B , and holding with equality or inequality for W_T . The second inequality follows from Lemma 2, after substituting (A1) in the expression for W_T^B in Table I (a2), with $y = y^B$. This establishes the optimality of W_T^B in Table I (a2) for economy (a2).

From Lemma 1, it is evident that in economy (a2) there are unique values of $\bar{\xi}$ and ξ^* satisfying $P(\bar{\xi} \leq \xi_T < \xi^*) = \alpha$. For any other values $\bar{\xi}_o$ and ξ_o^* with $g(\bar{\xi}_o) = g(\xi_o^*)$, given the established properties of g in economy (a2), we have either $\underline{\xi} < \bar{\xi} < \bar{\xi}_o < \xi_o^* < \xi^*$, or $\underline{\xi} < \bar{\xi}_o < \bar{\xi} < \xi^* < \xi_o^*$, and consequently $P(\bar{\xi}_o \leq \xi_T < \xi_o^*) \neq \alpha$. Lemma 1 further implies that as $\alpha \rightarrow 0$ in economy (a2), we have $\bar{\xi} \rightarrow \xi^*$, and we obtain the solution in Proposition 1(a).

For the remaining economies, the proof follows similar steps, where Lemma 1 is modified to establish that in economy (a1) $g(\xi) > g(\bar{\xi})$ for $\underline{\xi} < \bar{\xi} < \xi$; (b1) $g(\xi) > g(\bar{\xi})$ for $\bar{\xi} < \xi < \xi^* < \underline{\xi}$; (b2) $g(\xi) > g(\xi^*)$ for $\xi < \xi^* < \underline{\xi}$; (c1) $g(\xi) > g(\bar{\xi})$ for $\bar{\xi} < \xi$; (c2) $g(\xi) > g(\xi^*)$ for $\xi < \xi^*$. Lemma 2 then proceeds to verify for a given economy which one of the two candidate solutions is the global maximizer within each region of the state space.

Finally, we note that the function $g(\xi_T) = y[\xi_T X_T - \xi_T I(y\xi_T)] - [u(X_T) - u(I(y\xi_T))]$ captures the benefit of reverting to the normal-type policy versus matching the benchmark, as given by the trade-off between the *state-contingent relative cost*, $\xi_T X_T - \xi_T I(y\xi_T)$, and the *state-contingent relative utility*, $u(X_T) - u(I(y\xi_T))$, of matching the benchmark versus following the normal policy). Hence, shortfall occurs in the α -probability states in which $g(\xi_T)$ is largest. For economies (a2) and (b1) this occurs for intermediate values of ξ_T , in (a1) and (c1) for high ξ_T , and in (b2)

and (c2) for low ξ_T . For the special, zero measure, case of $\sigma/\kappa = 1/\gamma = 1$, $g(\xi)$ is constant, and so if $\varepsilon < 0$, $W_T^B = I(y^B \xi_T)$; if $\varepsilon = 0$, $W_T^B = X_T$; and if $\varepsilon > 0$, $W_T^B = \{I(y^B \xi_T) \text{ or } X_T\}$, with $P(I(y^B \xi_T) < X_T) = \alpha$. In the latter case, either candidate solution can be used, with the state contingent relative costs being constant, $A - y^{B-1/\gamma} > 0$, where the inequality is because $A = e^{\varepsilon T} y^{N-1/\gamma}$, $\varepsilon > 0$, and also $y^B > y^N$ for the static budget constraint to hold with equality. ■

Proof of Proposition 3. (i) Using the dynamics of the state price density process and agent's wealth, Itô's Lemma implies that $\xi_t W_t^B$ is a martingale:

$$W_t^B = E \left[\xi_T W_T^B \middle| \mathcal{F}_t \right] / \xi_t . \quad (\text{A2})$$

When r and κ are constant, conditional on \mathcal{F}_t , $\ln \xi_T$ is normally distributed with variance $\kappa^2(T-t)$ and mean $\ln \xi_t - (r + \frac{\kappa^2}{2})(T-t)$. For each economy, substituting the expression for W_T^B in Proposition 2 into (A2), and evaluating the expectation over the relevant regions of ξ_T yields (5).

(ii) For each economy, applying Itô's Lemma to (5), results in an expression for σ_t^B , the diffusion term of W_t^B . The expression for θ_t^B follows from the fact that, from the agent's wealth process, σ_t^B must equal $\sigma_t \theta_t^B W_t^B$. Normalizing θ_t^B by the well-known quantity θ^N yields q_t^B .

(iii) For completeness, we present here the solution for the downside hedger, obtained in economy (a1) for $\bar{\xi} = \infty$ and (a2) for $\bar{\xi} = \xi^*$, when $\varepsilon < 0$:

$$\begin{aligned} W_t^H &= \mathcal{N}(d(\gamma, \underline{\xi})) Z(\gamma) (y^H \xi_t)^{-1/\gamma} + \left[1 - \mathcal{N}(d(\kappa/\sigma, \underline{\xi})) \right] Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} , \\ q_t^H &= 1 + \left[1 - \mathcal{N}(d(\kappa/\sigma, \underline{\xi})) \right] (\gamma\sigma/\kappa - 1) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} / W_t^H \\ &\quad + \left(\varphi(d(\gamma, \underline{\xi})) Z(\gamma) (y^H \xi_t)^{-1/\gamma} - \varphi(d(\kappa/\sigma, \underline{\xi})) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} \right) \gamma / (W_t^H \kappa \sqrt{T-t}) , \end{aligned}$$

and, in economy (b1) for $\bar{\xi} = \xi^*$, and in economy (b2) for $\xi^* = 0$, when $\varepsilon < 0$:

$$\begin{aligned} W_t^H &= \left[1 - \mathcal{N}(d(\gamma, \underline{\xi})) \right] Z(\gamma) (y^H \xi_t)^{-1/\gamma} + \mathcal{N}(d(\kappa/\sigma, \underline{\xi})) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} , \\ q_t^H &= 1 + \mathcal{N}(d(\kappa/\sigma, \underline{\xi})) (\gamma\sigma/\kappa - 1) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} / W_t^H \\ &\quad + \left(-\varphi(d(\gamma, \underline{\xi})) Z(\gamma) (y^H \xi_t)^{-1/\gamma} + \varphi(d(\kappa/\sigma, \underline{\xi})) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} \right) \gamma / (W_t^H \kappa \sqrt{T-t}) . \end{aligned}$$

In all cases, y^H is as in Proposition 1. ■

Proof of Proposition 4. The proof is analogous to the proof of Proposition 2, with the appropriate counterparts of Lemmas 1 and 2, and is therefore omitted. ■

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