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**Gerard J van den Berg,** Vrije Universiteit Amsterdam and CEPR

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Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR, UK
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: www.cepr.org

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# ABSTRACT

## On the Uniqueness of Optimal Prices Set by Monopolistic Sellers*

This paper considers price determination by monopolistic sellers who know the distribution of valuations among the potential buyers. We derive a novel condition under which the optimal price set by the monopolist is unique. In many settings, this condition is easy to interpret, and it is valid for a very wide range of distributions of valuations. The results carry over to the optimal minimum price in independent private value auctions. In addition, they can be fruitfully applied in the analysis of quantity discount price policies.

JEL Classification: D42, D44, L12 and L42
Keywords: auction, hazard rate, local maxima, minimum price, monopoly, quantity discount, regularity and reservation price

Gerard J van den Berg
Department of Economics
Free University
De Boelelaan 1105
NL-1081 HV Amsterdam
THE NETHERLANDS

Tel: (31 20) 444 6132
Fax: (31 20) 444 6020
Email: gjvdberg@xs4all.nl

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# 1 Introduction

On August 20, 2000, Henri Theil, one of the most eminent post-war econometricians in the world, passed away at the age of 75. Theil left his mark in the area for example by having developed the two-stage least squares estimation method (Theil, 1953) and by his influential econometrics handbook (Theil, 1971). Perhaps less well known is that Theil's first research project was economic-theoretical, albeit with a strong empirical motivation. In his Master Thesis (Theil, 1948), Henri Theil analyzes price setting behavior by an entrepreneur in a product market characterized by monopolistic competition.[1] He derives the optimal price given a parametric functional form assumption on the demand curve faced by the entrepreneur. The generic form of the first order condition of this maximization problem plays an important role in many models in the field of industrial organization. Obviously, if there is only one good to sell and the entrepreneur only knows the distribution of the valuations of the potential buyers then the demand curve can be replaced by one minus the distribution function of valuations. In this paper we prove the uniqueness of the solution of such maximization problems, under conditions that are very general and easy to interpret. We establish a link to the econometric analysis of such models.

To shape thoughts, we briefly consider the most basic model to which the analysis can be applied. This model deals with an individual who wants to sell a single item and who will meet only one potential buyer.[2] The seller himself values the item by an amount of $x$ monetary units, and he will set a price $p$ before meeting a potential buyer. Each potential buyer has his own valuation $v$ for this item. The function $F(v)$ is the distribution function of $v$ across the population of potential buyers. Thus, the probability that the item will not be sold is $F(p)$. The problem is to choose the price that maximizes the expected profit of the seller. For a given price, the objective function is $p(1-F(p))+xF(p)$. Under regularity conditions to be discussed below, the optimal price satisfies the first-order condition

$$p = x + \frac{1 - F(p)}{f(p)} \tag{1}$$

in which $f(.)$ denotes the derivative of $F(.)$ We use $\overline{F}(.)$ as short-hand notation for $1 - F(.)$.

[1] See Heertje (2002) for an English summary and a critical examination of this thesis.

[2] It is straightforward to generalize this setup by allowing the seller to meet more than one potential buyer (see e.g. Wolfstetter, 1996).

The above framework is a model for a monopoly market or a market with monopolistic competition. However, the first-order condition (1) also describes optimal strategies in other models. Notably, consider a standard IPV (Independent Private Value) auction of some item, with risk-neutral and symmetric bidders and with payments that only depend on bids. If the item is not bought by any bidder then it will be destroyed. The seller may publicly announce a minimum price or reservation price. Under certain conditions, equation (1) with $x = 0$ describes the optimal value of this minimum price (see e.g. Laffont and Maskin, 1980, McAfee and McMillan, 1987, Matthews, 1995, and Wolfstetter, 1996). Note that this optimal minimum price does not depend on the actual number of bidders.

In all cases, it is useful to know whether the optimal price is unique for all possible $x$. A market is more transparent if equilibrium is unique, and this reduces the uncertainty and computational costs of the market participants. Also, theoretical comparative statics analyses of a model are facilitated if it can be ruled out that the equilibrium solution jumps in response to a marginal change in a value of a structural determinant. To see the econometric relevance, notice that uniqueness of the optimal price is closely related to the absence of non-global local maxima of the expected profit as a function of the price. In the absence of these features, iterative estimation procedures may lead to incorrect inference.[3] In addition, standard asymptotic theory may not be applicable.

It is obviously useful to have a sufficient condition for uniqueness that does not depend on the value of $x$. Intuitively, it follows from equation (1) that the assumption that $v - \overline{F}(v)/f(v)$ increases everywhere can be a sufficient condition for uniqueness. In the literature, two approaches have been taken. At the one extreme, one simply assumes that $v - \overline{F}(v)/f(v)$ strictly increases in $v$ everywhere (see e.g. Bulow and Roberts, 1989, Matthews, 1995, Wolfstetter, 1996, Eso, 2002). At the other extreme, one adopts a sufficient condition for the latter, namely that $f(v)/\overline{F}(v)$ increases in $v$ everywhere (see e.g. Wolfstetter, 1996, and Eso, 2002). This condition on $F$ is called the increasing hazard rate property or increasing failure rate (IFR) property.

Both approaches have disadvantages. To start with, consider the first assumption (that $v - \overline{F}(v)/f(v)$ increases in $v$ everywhere). One may give an economic interpretation to it by noting that $v - \overline{F}(v)/f(v)$ is the marginal revenue when offering the item for sale to the buyer with valuation $v$ at a take-it-or-leave-it price equal to $p = v$ (Bulow and Roberts, 1989). The assumption then states that

[3] Theil had a strong interest in numerical optimization procedures in complex models; see e.g. Theil and Van de Panne (1960).

this "marginal revenue" increases in $v$ everywhere. In practice it may however be difficult to assess whether $F$ would have such a property. It is not related to well-known and well-understood characteristics of distribution functions, so that it is difficult to get an intuitive feeling of its meaning from that point of view.

Now consider the second assumption (that $F$ has the IFR property). The hazard rate is a well-understood characterization of a distribution. For example, if $F$ has the IFR property then the probability that the valuation lies in $[v, v+dv)$, with $dv$ small, conditional on the valuation being at least as large as $v$, increases as $v$ increases. It is also well known which parametric families of distributions satisfy the IFR property. However, there are numerous (members of) well-known families of distributions that do not satisfy the IFR property. These include all members of the families of log-normal distributions, Pareto distributions[4], Singh-Maddala distributions, log-logistic distributions, generalized beta distributions of the second kind (GB2), log-uniform distributions, t distributions truncated from below at zero, and F distributions, and subsets of the families of gamma distributions and Weibull distributions (see e.g. McDonald, 1984, Majumder and Chakravarty, 1990, and Van den Berg, 1994, for descriptions of these families).

Note that, in particular, the families that are typically used to model the distributions of income, wages, and other indicators of well-being across the population, and that fit the corresponding data well, do not have the IFR property. Indeed, there is much empirical evidence that confirms that income distributions have a failure rate that decreases after a certain point (see for example Singh and Maddala, 1976). Singh and Maddala (1976) also provide theoretical reasons for income-related distributions to violate IFR. The basic idea is that the ability to make more money may well increase with one's income, which would imply that the expected residual income increases with the point at which the income distribution is truncated from below. IFR predicts the opposite. Returning to our model, if the willingness to pay for the item by potential buyers is a fixed fraction of their income, and if the income distribution does not have the IFR property, then the distribution $F$ does not have this property either.

In this paper we introduce a new sufficient condition for uniqueness, called the increasing proportionate failure rate (IPFR) property. This property was first defined by Singh and Maddala (1976) as a desirable property of distributions for income-related variables, and it described in great detail in Section 2. It is uniformly weaker than IFR, and includes all families of distributions mentioned above that do not satisfy IFR. It is non-nested with respect to the assumption that $v - \overline{F}(v)/f(v)$ increases in $v$ everywhere, but compared to the latter as-

[4]Contrary to the statement in Tirole (1988), page 156, that these satisfy IFR.

sumption the former is easier to relate to well-understood properties of (income) distributions. For example, it is implied by the assumption that the income-share elasticity of a distribution (Esteban, 1986) decreases in its argument.

Recall that Theil (1948) made a parametric functional form assumption on the demand curve faced by the entrepreneur. In particular, he assumed a constant elasticity demand curve. In the context of the above model, this means that $F$ has a Pareto distribution. This assumption does not satisfy IFR, but we show that it nevertheless leads to a unique optimal price.

In this paper we also demonstrate the usefulness of our condition for the analysis of nonlinear price policies. In that literature, IFR is usually given as a condition for quantity discounting (see the overviews in Tirole, 1988, Wilson, 1993, and Armstrong, 1996). We show that this condition can be weakened at no cost.

The outline of the paper is as follows. Section 2 deals with existence and uniqueness of the optimal price. At the end of Section 2 we discuss the econometric implications. Section 3 deals with the wider use of the sufficient condition for uniqueness. Section 4 concludes.

# 2 Sufficient conditions for a unique price

## 2.1 Existence

For convenience we adopt the terminology of the monopoly seller model of the previous section. Recall that for $x = 0$ we obtain the relevant equation in IPV auction models. The following weak assumptions ensure that attention is restricted to economically meaningful cases and guarantee the existence of the optimal price.

**Assumption 1 : Distribution of valuations.** *$F(v)$ is a continuous non-decreasing function of $v$ on $[0, \infty)$, with $F(0) = 0$ and $\lim_{v\to\infty} F(v) = 1$. There is an interval $(\alpha, \beta)$ such that $0 < F(v) < 1$ on this interval while $F(v) = 0$ or $F(v) = 1$ outside it, with $0 \leq \alpha < \beta \leq \infty$. $F(v)$ has a continuous positive derivative $f$ on $(\alpha, \beta)$, and $\lim_{v\downarrow\alpha} f(v)$ exists. Finally, $\mathrm{E}_F(v) < \infty$.*

**Assumption 2 : Outside option.** *$0 \leq x < \beta$.*

We restrict attention to distributions with connected support for expositional reasons. This is also why we impose that $\lim_{v\downarrow\alpha} f(v)$ exists. Neither restriction is essential. In the sequel we write $f(\alpha) := \lim_{v\downarrow\alpha} f(v)$.

The assumption that $x < \beta$ prevents that the seller decides not to consider trade at all. In the literature, the assumption that $F$ has a finite mean is often not mentioned. However, as we shall demonstrate, without it the existence of the optimal price is not guaranteed.

**Proposition 1 : Existence.** *Under Assumptions 1 and 2 there exists an optimal price $p$ satisfying $\alpha \le p < \beta$ and $p > x$.*

*Proof.* For a given set price, profits equal $\pi(p) := \overline{F}(p)(p-x)+x$. A price $p < x$ is not optimal because then $\pi(p) < \pi(x)$ (note that $x < \beta$ so that $\overline{F}(x) > 0$). Define $\widetilde{\alpha} = \max\{\alpha, x\}$. Suppose first that $\beta < \infty$. Prices $p \ge \beta$ and $p = x$ are not optimal because then $\pi(p) = x < \pi(\frac{1}{2}\widetilde{\alpha} + \frac{1}{2}\beta)$. If $\beta = \infty$ then prices $p \ge \beta$ and $p = x$ are not optimal because then $\pi(p) = x < \pi(\widetilde{\alpha}+1)$. A price $p < \alpha$ is not optimal because then $\pi(p) < \pi(\alpha)$. So the optimal price, if it exists, satisfies the inequalities in the proposition.

The value of $\pi(p)$ at $\widetilde{\alpha}$ equals $\widetilde{\alpha}$, which is a non-negative finite number. If $\beta < \infty$ then $\pi(\beta) = x$. If $\beta = \infty$ then $\lim_{p\uparrow\beta} \pi(p) = x + \lim_{p\to\infty} p\overline{F}(p)$. The latter limit is zero if $F$ has a finite mean (see e.g. Feller, 1971). We conclude that in general $\lim_{p\uparrow\beta} \pi(p) = x$, which is also a non-negative finite number. The function $\pi(p)$ is continuous on $[\widetilde{\alpha}, \beta)$. By Brouwer's fixed point theorem, this means that it has a maximum value in this interval. □

For an example of non-existence, consider the case $\overline{F}(v) = v/\alpha$ on $v \in (\alpha, \infty)$. Then $\mathrm{E}_F(v) = \infty$; $\pi(p)$ increases everywhere, and the optimal price tends to infinity.

## 2.2 Uniqueness

Let $\theta(v) := f(v)/\overline{F}(v)$ denote the hazard rate or failure rate of $F$. It is defined on $(\alpha, \beta)$, but by Assumption 1 we can extend this to $[\alpha, \beta)$. We adopt the following assumption,

**Assumption 3 : Increasing proportionate failure rate (IPFR).** $v \cdot \theta(v)$ *is strictly increasing in* $v$ *on* $(\alpha, \beta)$.

We first present the main result. Then we reformulate the assumption in a number of ways, and we discuss its usefulness and generality. We then also give results for when $v\theta(v)$ is constant.

**Proposition 2 : Uniqueness.** *Under Assumptions 1–3, the optimal price exists and is unique.*

*Proof.* From Proposition 1, optimal prices exist. They can equal $\alpha$ and/or values in $(\alpha, \beta)$. In the latter case the values follow from the first order condition $(p-x)\theta(p) = 1$. In both cases, from Proposition 1, optimal prices strictly exceed $x$. We therefore restrict attention to candidate optimal prices in the interval $[\widetilde{\alpha}, \beta)$, with again $\widetilde{\alpha} := \max\{\alpha, x\}$.

We define the function $h(v) := v\theta(v)$ on $[\alpha, \beta)$. Note that $h$ and the profit function $\pi$ in the proof of Proposition 1 are related by way of

$$\frac{d\pi(p)}{dp} = \overline{F}(p)\left(1 - \frac{p-x}{p}h(p)\right) \tag{2}$$

on $(\alpha, \beta)$.

Assumption 3 states that $h$ is strictly increasing on $(\alpha, \beta)$. This implies that $(p-x)h(p)/p$ is strictly increasing on $(\widetilde{\alpha}, \beta)$. If $(\widetilde{\alpha} - x)h(\widetilde{\alpha})/\widetilde{\alpha} > 1$ (implying that $\widetilde{\alpha} = \alpha > x$) then certainly $(p-x)h(p)/p > 1$ for all $p > \alpha$, and this keeps $d\pi(p)/dp$ in equation (2) negative for all these larger $p$, so then the unique optimal price equals $\alpha$. If $(\widetilde{\alpha} - x)h(\widetilde{\alpha})/\widetilde{\alpha} < 1$ then, by the same line of reasoning, there is a unique interior optimal price. Note that it is not possible that $(p-x)h(p)/p$ stays below 1 as $p$ becomes large, because this would violate Proposition 1. If $(\widetilde{\alpha} - x)h(\widetilde{\alpha})/\widetilde{\alpha} = 1$ (implying that $\widetilde{\alpha} = \alpha > x$) then $\lim_{p\downarrow\alpha} d\pi(p)/dp = 0$ while $d\pi(p)/dp < 0$ for all $p > \alpha$, so then again the unique optimal price equals $\alpha$. □

To describe the conditions under which Assumption 3 holds true, it is useful to reformulate it. To this aim we define the concept of log concavity. A function $g(y)$ is called log concave if there is an interval $I$ such that $\log g(y)$ is concave on $I$ and $g(y)$ is positive on $I$ but vanishes exterior to $I$. This is equivalent to saying that $g(y)$ is a Pólya frequency function of order 2 (see Karlin, 1968). Strict log concavity can be defined analogously. Assumption 3 holds if and only if $\overline{F}(e^y)$ is a strictly log concave function of $y$. Alternatively, let $y := \log v$ denote the log valuations, with distribution function $F_y(y)$. Then Assumption 3 holds if and only if $\overline{F}_y(y)$ is a strictly log concave function of $y$, and this holds if and only if the distribution of $y$ satisfies IFR. IPFR can not be characterized in terms of moment restrictions (Van den Berg, 1994).

We now provide some sufficient conditions for IPFR. First, IFR implies IPFR. Secondly, IFR and IPFR are implied by the condition that $f(v)$ is strictly log concave. Thirdly, IPFR is implied by the condition that $f(e^y)$ is strictly log concave (Van den Berg, 1994). Fourthly, the latter condition holds if and only if the density $f_y(y)$ of $y := \log v$ is strictly log concave, and if and only if $vf'(v)/f(v)$ (if existent) is decreasing on $(\alpha, \beta)$. Because $1 + vf'(v)/f(v)$ is equal to the income-share elasticity of a distribution (Esteban, 1986), this condition is equivalent to stating that the income-share elasticity of $F$ is decreasing in its argument.[5] We also note that IPFR implies that the conditional mean $\mathrm{E}_F(v|v > v_0)$ as a function of $v_0$ has an elasticity smaller than 1.

Now let us examine for which parametric families of distributions Assumption 3 is satisfied. First, as IPFR encompasses IFR, the former is satisfied by all members of families with IFR, like the exponential and beta families, the families of normal, logistic, and extreme value distributions that are truncated from below at zero, and the family of uniform distributions for which the lower point of support is non-negative. (Recall that we are only interested in distributions satisfying Assumption 1, which implies that the mean is finite and that $\Pr(v \leq 0) = 0$. Secondly, as noted in Section 1, IPFR is also satisfied by all members of the families of log-normal, Pareto, Singh-Maddala, log-logistic, GB2, F, gamma, Weibull, and log-uniform distributions, and the family of t distributions truncated from below at zero (see Van den Berg, 1994, for descriptions of these families and derivations).[6]

It is easily seen that truncation from below or above does not affect the IPFR property. Van den Berg (1994) shows that any distribution not satisfying IPFR can be made to satisfy it by truncating it from above at a sufficiently small value.

For our purposes it is particularly useful to point out that IPFR is not affected by scaling the valuation variable. As mentioned in Section 1, the distribution families that are typically used to model the distributions of income, wages, and other indicators of well-being across the population, and that fit the corresponding data well, all have the IPFR property. If the willingness to pay for the item by potential buyers is a fixed fraction of their income, then the distribution $F$ automatically inherits the IPFR property.

If $F$ is a left-translation of a distribution with the IPFR property then $F$ also has this property. Right-translation of a distribution with the IPFR property may result in a distribution not satisfying it. But if $F$ is the result of a right-translation with an amount $\delta \leq x$ then the original maximization problem can be redefined by subtracting $\delta$ from $v$ and $x$ such that we obtain a problem satisfying Assumptions 1–3, and uniqueness follows.[7]

[5]The results concerning the distribution of $\log v$ enable additional characterizations of Assumption 3 and sufficient conditions for it, by using characterizations (like those in Dharmadhikari and Joag-dev, 1988) of distributions with IFR or with a log concave density.

[6]Both Assumption 3 and the assumption that $v - \overline{F}(v)/f(v)$ increases in $v$ everywhere are violated if the density $f(v)$ displays a sufficiently high peak, or if $v$ is a discrete random variable.

[7]The restriction $\delta \leq x$ is not necessary; it follows here from the economic requirement in Assumption 2 that the translated outside option value $x - \delta$ satisfies $x - \delta \geq 0$.

We now turn to the econometric relevance of the IPFR property. The proof of Proposition 2 shows that with IPFR the profit function $\pi(p)$ does not have local non-global maxima.[8] This implies that standard iterative numerical optimization procedures, when applied to $\pi(p)$, always converge to the optimal price. This property is useful in the estimation of models in which the optimal price plays a role. Suppose that one aims to estimate some (possibly non-linear) model in which the optimal price plays a role, and in which the unknown parameters include $x$ and/or parameters of $F$. Typically, estimation involves the iterative maximization of an objective function. To calculate the value of the objective function, the optimal price has to be calculated numerically as the value that maximizes $\pi(p)$. If the latter has local non-global maxima then the numerical procedure may converge to an incorrect value, resulting in incorrect inference.

Also, if local non-global maxima in $\pi(p)$ are possible then at certain parameter values there may be multiple optimal prices, and a small change in a parameter value may entail a jump of the optimal price from one maximum to another. More in general, if the optimal price is not continuous in the model parameters then the conditions for application of standard asymptotic theory may be violated. However, from the proof of Proposition 2 it follows that the optimal price is a continuous function of the cost parameter $x$.

Assumption 3 is strict in the sense that it rules out that $v\theta(v)$ is constant on an interval. From the proof of Proposition 2 it follows immediately that this restriction can be removed provided that $x > 0$. This substantiates the claim in Section 1 that Theil (1948)'s model leads to a unique price. Now let us consider the case with $x = 0$ and constant $v\theta(v)$. The latter corresponds to the family of Pareto distributions for $F$, with $\overline{F}(v) = (\alpha/v)^\eta$ with $\alpha > 0$ and $\eta > 1$ (the latter ensures that $\mathrm{E}_F(v) < \infty$). In this case the objective function equals $\alpha^\eta/p^{\eta-1}$ on $(\alpha, \infty)$, so the unique optimal price equals $\alpha$.[9]

We now discuss the relation between IPFR and the assumption that $v - \overline{F}(v)/f(v)$ increases in $v$ everywhere. We show that the two conditions on $F$ are

[8] Basically, the proof establishes strict pseudoconcavity of the profit function. Note that the proof only considers $p \geq \max\{\alpha, x\}$. However, first of all, for $p < x$ there holds that $\pi(p) < x = \pi(x)$, so these values of $p$ can be discarded immediately. Secondly, for $p \leq \alpha$ there holds that $\pi(p) = p$, which strictly increases in $p$.

[9] If $\overline{F}$ is a general demand function rather than a distribution function then obviously it is possible that $\overline{F}(0) = \infty$, and this can give rise to non-existence of the optimal price (see Wolfstetter, 1999). Note that under the demand function interpretation, IPFR means that the price elasticity of demand (in absolute value) increases in the price.

not nested.[10] By writing

$$v - \overline{F}(v)/f(v) = v \cdot \left(1 - \frac{1}{v\theta(v)}\right)$$

it is clear that the statement that IPFR implies that $v - \overline{F}(v)/f(v)$ increases in $v$ everywhere is true if $v\theta(v) > 1$ everywhere. However, if $v\theta(v) < 1$ then $v\theta(v)$ may increase in $v$ even though $v - \overline{F}(v)/f(v)$ does not increase in $v$. For example, take a Weibull distribution for $F$ with $\theta(v) = \frac{1}{2}v^{-1/2}$ with support $(0, \infty)$. This satisfies Assumption 1 except that $\lim_{v \downarrow \alpha} f(v) = \infty$, but the latter is irrelevant here, because we are only going to consider values of $v$ sufficiently far away from 0,[11] and the optimal price is bounded away from 0 anyway. The distribution $F$ satisfies IPFR. However, $v - \overline{F}(v)/f(v)$ decreases in $v$ for all $v < 1$. The unique optimal price equals $x + 2 + 2\sqrt{x+1}$ and this is larger than or equal to 4 for all $x \geq 0$.[12]

Locally, in an optimum, $v - \overline{F}(v)/f(v)$ always increases in $v$ (with $v \equiv p$), as this captures the local second order condition of the optimization problem (McAfee and McMillan, 1987).[13] Therefore, around an optimum, the condition that $v - \overline{F}(v)/f(v)$ increases in $v$ is weaker than the condition that $v\theta(v)$ increases in $v$. But this is not informative on uniqueness of the optimum, because if $v - \overline{F}(v)/f(v)$ decreases in $v$ for some other $v$ then there may be multiple optimal prices.[14]

[10]This contradicts the statement in Armstrong (1996), page 61, that IPFR implies the other condition.

[11]Alternatively, we may take $F$ such that $\theta(v) = \frac{1}{2}v^{-1/2}$ on $(\varepsilon, \infty)$ and $\theta(v) = \frac{1}{2}\varepsilon^{-1/2}$ on $(0, \varepsilon)$ for sufficiently small $\varepsilon$, and consider values $v$ satisfying $v > \varepsilon$.

[12]Laffont and Maskin (1980) present an alternative sufficient condition for uniqueness in the special case where $x = 0$ and $F$ has support on $[0, 1]$, but it does not carry over to the general case.

[13]As noted in Section 1, $v - \overline{F}(v)/f(v)$ may be interpreted as a marginal revenue. Let $\overline{F}$ be a general demand function rather than a distribution function, so that $q := \overline{F}(p)$ is the demand at price $p$. Then the assumption that $v - \overline{F}(v)/f(v)$ increases in $v$ everywhere means that the firm's revenue function $p(q) \cdot q$ is concave in $q$.

[14]It should be noted that in the analysis of auctions there are often other reasons to adopt the assumption that $v - \overline{F}(v)/f(v)$ increases in $v$ everywhere. For example, in an English auction it ensures that the winning bidder's expected payment is increasing in his valuation (see e.g. McAfee and McMillan, 1987) so that the seller does not use a randomized strategy. In general the assumption is often made in the explicit derivation of optimality from the seller's point of view of standard auctions, among the large set of possible auctions. See e.g. Matthews (1995) for a lucid account.

# 3 Some further applications

In the theory of nonlinear pricing with heterogeneous consumers, the latter may purchase any nonnegative quantity of a certain good (see the overviews in Tirole, 1988, and Wilson, 1993; Salanié, 1997, calls this the Standard Model of Adverse Selection). An issue of interest is whether in equilibrium the seller applies quantity discounts. Let a consumer's utility $U$ as a function of the quantity $q$ of the good be heterogeneous across consumers by way of a multiplicative term $v$, so that one can write $U(q, v) = vU_0(q)$. Also, take unit production costs $c$ to be constant. An assumption commonly made in this literature is that the distribution of $v$ satisfies IFR (see e.g. Tirole, 1988, Wilson, 1993, and Armstrong, 1996). IFR ensures that $v - \overline{F}(v)/f(v)$ increases everywhere, and this ensures that in equilibrium consumers with higher $v$ will have higher $q$. IFR subsequently also ensures that the optimal payment schedule is such that quantity discounts are applied. (In that case the optimal schedule can also be implemented by a menu of two-part tariffs. Tirole, 1988, demonstrates that these results can be translated in terms of the choice of optimal Ramsey prices.) However, for quantity discounts to be optimal, the IFR condition can be replaced at no cost by the weaker IPFR condition. To see this, let $T(q)$ denote the equilibrium price of the bundle $q$ and let $p(q) := T'(q)$ denote the corresponding price of an extra unit. The equilibrium price-cost margin satisfies $(p - c)/p = 1/(v\theta(v))$, with $p = p(q(v))$. Clearly, given that $q$ increases in $v$, IPFR is equivalent to $p$ being a decreasing function of $q$. This in turn implies that the average price $T(q)/q$ decreases in the amount $q$ purchased (see Tirole, 1988, for detailed derivations, and Wilson, 1993, for extensions).

McAfee and McMillan (1987) discuss the optimal minimum price in standard auctions when bidders collude by forming a cartel. Under certain assumptions, it is optimal for the seller to behave as if it faces only one bidder whose distribution of valuations is the distribution of the maximum valuation out of a random sample from the actual distribution of valuations across single bidders, the sample size $n$ being equal to the number of cartel members. From the previous section it then follows that the optimal minimum price is unique if the distribution of the maximum valuation satisfies Assumption 3. This distribution has the distribution function $(F(v))^n$. It is not difficult to show that if $F$ satisfies Assumption 3 then $F^n$ also satisfies Assumption 3, so that Assumption 3 is in fact weaker for $F^n$ than for $F$. Our results can thus be straightforwardly applied to this model.

## 4 Conclusion

We have developed a novel condition for uniqueness of the optimal price set by a monopolist and the optimal minimum price set in auctions. It is easy to interpret and it is valid for many possible distributions of valuations, including virtually all distributions derived from income distributions. The results can be fruitfully applied to the analysis of models of price formation in many markets with imperfect competition.

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