DISCUSSION PAPER SERIES

No. 5132

CONTESTS WITH TIES

Chen Cohen and Aner Sela

INDUSTRIAL ORGANIZATION

Centre for Economic Policy Research

www.cepr.org

Available online at: **www.cepr.org/pubs/dps/DP5132.asp**

ISSN 0265-8003

CONTESTS WITH TIES

Chen Cohen, Ben-Gurion University
Aner Sela, Ben Gurion University of the Negev and CEPR

Discussion Paper No. 5132
July 2005

Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR, UK
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: www.cepr.org

CEPR Discussion Paper No. 5132

July 2005

ABSTRACT

Contests with Ties

We study all-pay contests in which there is a positive probability of a tied outcome. We analyse both one-stage contests and multi-stage contests with tie-breaks. We demonstrate that in symmetric two-player contests, the designer does not have an incentive to award a prize in a case of a tie. Consequently, in symmetric multi-stage two-player contests, the designer should allow an unlimited number of tie-breaks until a winner is decided.

JEL Classification: D44 and D72
Keywords: all-pay auctions and contests

Chen Cohen
Economics Department
Ben-Gurion University
Beer-Sheva 84105
ISRAEL
Email: levih@bgu.ac.il

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=100651

Aner Sela
Department of Economics
Ben-Gurion University of the Negev
Beer-Sheva 84105
ISRAEL
Tel: (972 8) 647 2309
Fax: (972 8) 647 2941
Email: anersela@bgumail.bgu.ac.il

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=156699

Submitted 27 June 2005

1 Introduction

In winner-take-all contests, all contestants including those who do not win the prize, incur costs as a result of their efforts, but only the winner receives the prize. Some winner-take-all contests are decided by objective valuations such as tests of skill and ability but others are resolved on the basis of considerably more subjective evaluations. If the performances of the best contestants are the same or almost the same but not distinguishable, the contests end without either side winning. Such a situation is referred to as a tie. The most common situations where ties are encountered are sport competitions.

In 1982 the English soccer league changed one of the basic rules in the European soccer leagues: it decided to award three points for a win rather than two points, while continue to award one point for a tie. This policy was adopted in France in 1995, in Germany and Spain in 1996 and afterwards in the rest of Europe. This raises the question we address in the present paper: what is the optimal prize in the case of a tie in relative to the prize for winning? Should the policy be as in chess where both players earn half a point in case of a tie and the winner earns one point, or alternatively as in soccer where three points are awarded for a win and one for a tie? Furthermore, should prizes be awarded in the case of a tie at all? In some contests ties are not broken and the prize is shared equally, while in others, tie-breaks are hold until the tie is broken and one of the players wins the entire prize. In basketball, for example, a tied score after the regulation period is broken by a five-minute extension and if the game is still tied after that, another five minutes is used to break the tie and so on until one of the team wins.

Ties occur not only in sport contests. In political elections with more than two candidates, if no candidate succeeds achieving a majority, a runoff is often hold between the two candidates receiving the highest numbers of votes. In elections for a new Pope, voting is repeated indefinitely until the necessary majority is reached. In contrast, the Nobel prize is a contest that allows "ties", often awarding the prize to several people who share it equally.[1]

We investigate the optimal policy in contests with ties by initially applying the model of all-pay contests in which the contestant with the highest effort wins the prize. To allow a positive probability of a tie we assume in contrast to most of the literature on all-pay contests that the sets of possible strategies are finite.[2] The designer of the contest must therefore determine not only the size of the prize in a case that a single player exerts the highest effort, but also the size of the prizes in the case of a tie when several players exert the highest effort. In this case the designer actually determines the probability of winning for each of the tied winners where the

[1]For example, in 1994 the Nobel prize in economics was awarded to J. Harsany, J. Nash and R Selten for their pioneering analysis of equilibria in the theory of noncooperative games.

[2]Studies on all-pay auction models with complete information include, among others, Hillman and Riley (1989), Baye et al. (1993), Clark and Riis (1998) and Che and Gale (1998)). Studies on all-pay auction models with incomplete information about the prize's value to different contestants include, among others, Amman and Leininger (1996) Krishna and Morgan (1997) and Gavious et al. (2003).

sum of these probabilities may be less than one.[3]

In Section 2 we analyze the equilibrium strategies in all-pay contests (auctions) with complete information where the sets of strategies are finite. Some of the equilibrium strategies are similar to those of the standard all-pay contest where the sets of strategies are not finite (Hillman and Riley (1989) and Baye et al. (1993, 1996)), but there is a meaningful distinction between both models. For example, in contrast to the standard model, in our asymmetric two-player contest the probability of the stronger player to win the contest is not necessarily higher than the probability of the weaker player to win the contest, and the expected payoff of the weaker player is not necessarily zero. Moreover, in the case of a contest with more than two players there is always an equilibrium in which only two players are active (exert some effort) and the rest of the players are passive (do not exert any effort). However, in contrast to the standard model, being passive may be profitable.

In Section 3 we investigate the optimal value of the prize for the contest designer in the case of a tie, or equivalently, the optimal probability of winning for each player who exerts the highest effort. We show that in symmetric two-player contests, if the sum of the winners' probabilities to win the contest in a tie is less than one, independent of the value of these probabilities of winning, there is a unique symmetric equilibrium.[4]

[3]Other works on all-pay auctions in which the designer determines the optimal reward structure include, among others, Barut and Kovenock (1998), Moldovanu and Sela (2001, 2004).

[4]The designer's expected payoff is equal to the average of his expected payoffs if the sum of the players' probabilities is equal to 1.

Consequently, offering a prize in a case of a tie does not affect the contestants' efforts, and if it causes disutility for the contest designer, he should not award such a prize. On the other hand, if the prize does not cause any disutility for the contest designer, he could award prizes in the case of a tie but the sum of these prizes should be less than the prize in a case of a win (i.e., the sum of the players' probabilities of winning is smaller than one). This suggests that the decision of the English soccer league to award three points for a win rather than two points was induced an improvement that increased the total effort exerted by the players.[5]

In Section 4 we study multistage all-pay contests with tie-breaks. In the case of a tie, the players continue to compete until one of the players exerts the highest effort and wins the contest. We assume that the players' valuations decrease with stages. We show that there is a unique sub-game perfect equilibrium in a symmetric two-player contest where the players' strategies do not depend on their valuations in the next stages (that is, they do not depend on the discount factor) although there is a positive probability at each stage that the contest will proceed to the next stage. Consequently, the number of stages do not affect the players' strategies in each stage and a contest designer seeking to maximize total effort should not limit the number of stages (tie-breaks) and allow the contest to continue until a winner is decided.

[5]The introduction of the award of three points for a win in the English league made a dramatic jump in the average number of goals by away teams (see Dobson and Goddard (2001)).

2 One-Stage Contests with one or more winners

There are n players competing for a single prize in a one-stage contest. The value of winning in the contest for player i is v_i. Valuations are common knowledge. We model the match between the players as an all-pay auction: each player exerts an effort $x \in \{0, 1, 2, 3, ...\}$, all players bear the cost of their efforts and the player with the highest effort wins. In the case of a tie in which $h \leq n$ players exert the highest effort, we assume initially that each of the tied players wins with probability $1/h$. In a two-player contest, if players exert efforts of $x_i, x_j \in \{0, 1, 2, 3, ...\}$ then the payoff for player i is given by

$$u_i(x_i, x_j) = \begin{cases} -x_i & \text{if } \ x_i < x_j \\ \frac{v_i}{2} - x_i & \text{if } \ x_i = x_j \\ v_i - x_i & \text{if } \ x_i > x_j \end{cases}$$

and can be constructed analogously for player j.

Proposition 1 *Consider two symmetric players with the same valuation v who compete in an all-pay contest for a unique prize.*

1. Let $v = 2k + 1,\ k \in \{1, 2, 3, ...\}$. Then, there is a symmetric mixed strategy equilibrium where

- *Each player chooses every effort $x \in \{\ 0, 1, ..., v - 1\}$ with the same probability of $p_x = \frac{1}{v}$.*
- *The expected payoff of each player is 0.5.*

2. Let $v = 2k$ *,* $k \in \{1, 2, 3, ...\}$*. Then, there is a symmetric mixed strategy equilibrium where*

- *Each player chooses every effort level* $x \in \{0, 2, 4, ...\}$ *with the same probability of* $p_x = \frac{2R}{v}$*.*
- *Each player chooses every effort level* $x \in \{1, 3, ...\}$ *with the same probability of* $p_x = \frac{2(1-R)}{v}$*.*
- *The expected payoff of every player is* $R, 0 \leq R \leq 1$*.*

According to Proposition 1 in our model of all-pay contests in which the set of possible efforts is finite, there is an equilibrium where the players' efforts are uniformly distributed. This holds for the standard model of all-pay contests where the set of possible efforts is not finite (see Hillman and Riley (1989) and Baye et al. (1993,1996)). However, while in the standard model the probability of a tie (the players exert the same effort) is zero, in our model there is a positive probability for a tie in the contest. Moreover, in contrast to the symmetric standard model of all-pay contests, in our symmetric model the expected payoff of both players is not zero. These results hold independently of the size of the money unit (which is in our model equal to 1). However when the money unit approaches zero our results are consistent with the standard all-pay contest.

The generalization of our model of all-pay contests for the case of $n > 2$ players is simple since there is always an equilibrium in which only two players are active (exert some effort) and the rest of the players are passive (exert no effort). However, in contrast to the standard model, in our model being passive may be profitable.

Proposition 2 *Consider $n > 2$ symmetric players with the same valuation v who compete in an all-pay contest for a unique prize.*

1. If $v = 2k + 1$, $k \in \{1, 2, 3, ...\}$ there is an equilibrium in which $n - 2$ players do not exert any effort. The other two players use the symmetric mixed strategy equilibrium where:

- *The probability of every effort level $x \in \{2, 4, ..., v - 1\}$ is $p_x = \frac{4}{nv}$.*
- *The probability of every effort level $x \in \{1, 3, ..., v - 2\}$ is $p_x = \frac{2n-4}{nv}$.*
- *The probability of $x = 0$ is $p_0 = 1 - \left[\frac{v-1}{2}\frac{4}{nv} + \frac{v-1}{2}\frac{2n-4}{nv}\right]$.*
- *The expected payoff of each of the active players is $\frac{1}{n}$ and the expected payoff of each of the passive players is $\frac{1}{vn}$.*

2. If $v = 2k$, $k \in \{1, 2, 3, ...\}$ there is an equilibrium in which $n-2$ players do not exert any effort. The other two players use the symmetric mixed strategy equilibrium where:

- *The probability of every effort level $x \in \{2, 4, ...\}$ is zero.*
- *The probability of every effort level $x \in \{1, 3, ...\}$ is $p_x = \frac{2}{v}$.*
- *The expected payoff of each player is zero.*

According to Proposition 2, when players have positive payoffs, the expected payoffs of the active players do not depend on the players' valuation v, while the expected payoffs of the passive players may decrease in the players' valuation v. The seller's payoff, however, increases both in v and in n.

By Propositions 1 and 2, independent of the number of players, there is a positive probability that symmetric contests will not be decided. However, if the players are asymmetric, then there are contests in which ties are not possible.

Proposition 3 *Consider two asymmetric players, $v_1 > v_2$, who compete in an all-pay contest for a unique prize.*

1. Let $v_2 = 2k$, $k \in \{1, 2, 3, ...\}$. Then there is a mixed strategy equilibrium where

- *Player 1 chooses every effort $x_1 \in \{1, 3, 5, ..., v_2 - 1\}$ with the same probability of $p_{x_1} = \frac{2}{v_2}$.*
- *Player 2 chooses the effort $x_2 = 0$ with probability $q_0 = \frac{v_1 - v_2 + 2}{v_1}$ and every effort $x_2 \in \{2, 4, ..., v_2 - 2\}$ with the same probability of $q_{x_2} = \frac{2}{v_1}$.*
- *The expected payoff of player 1 is $v_1 - v_2 + 1$ and the expected payoff of player 2 is* 0.

2. Let $v_2 = 2k + 1$, $k \in \{1, 2, 3, ...\}$. Then there is a mixed strategy equilibrium where

- *Player 1 chooses the effort $x = v_2$ with probability of $p_{v_2} = \frac{1}{v_2}$ and every effort $x_1 \in \{1, 3, 5, ..., v_2 - 2\}$ with the same probability of $p_{x_1} = \frac{2}{v_2}$.*

- *Player 2 chooses the effort* $x_2 = 0$ *with probability of* $q_0 = \frac{v_1 - v_2 + 1}{v_1}$ *and every effort* $x_2 \in \{2, 4, ..., v_2 - 1\}$ *with the same probability of* $q_{x_2} = \frac{2}{v_1}$.

- *The expected payoff of player 1 is* $v_1 - v_2$ *and the expected payoff of player 2 is* 0.

In contradiction to Proposition 3, the following example shows that ties are possible even in asymmetric contests. The example also shows that in a two-player contest the probability of the strong player to win the contest is not necessarily higher than the probability of the weak player to win. Moreover, the expected payoff of the weak player is not necessarily zero.

Example 4 *Consider an all-pay contest with two asymmetric players where* $v_1 = 4$ *and* $v_2 = 3$. *Then there is a mixed strategy equilibrium where*

- *Player* 2 *chooses the efforts of* $x_2 = 0$ *and* $x_2 = 2$ *with the same probability of* $\frac{1}{2}$.

- *Player* 1 *chooses the efforts of* $x_1 = 0, x_1 = 1$ *and* $x_1 = 2$ *with the same probability of* $\frac{1}{3}$.

- *The probability of winning is the same for both players although the expected payoff of player* 1 *is equal to* 1 *and is larger than the expected payoff of player* 2 *which is equal to* 0.5.

3 One-Stage Contests with or without Winners

There are n players competing for a single prize in a one-stage all-pay auction. Each player exerts an effort $x \in \{0, 1, 2, 3, ...\}$, all players bear the cost of their efforts and the player with the highest effort wins. Now we assume that in the case of a tie, the sum of the winners' probabilities of winning is less than 1. Consider a two-player contest in which the probability of winning of each player is $\frac{1}{m}, m \in \{3, ...\}$ and players exert efforts of $x_i, x_j \in \{0, 1, 2, 3, ...\}$.The payoff for player i is

$$u_i(x_i, x_j) = \begin{cases} -x_i & \text{if } x_i < x_j \\ \frac{v_i}{m} - x_i & \text{if } x_i = x_j \\ v_i - x_i & \text{if } x_i > x_j \end{cases}$$

and can be constructed analogously for player j.

Theorem 5 *Consider two players with the same valuation v who compete in a one-stage all-pay contest for a unique prize. If the probability of winning of each player in the case of a tie is smaller than one half, then, independent of the players' probabilities of winning, there is a unique symmetric equilibrium in which each player chooses every effort $x \in \{ 0, 1, ..., v-1\}$ with the same probability of $p_x = \frac{1}{v}$.*

Proof. Denote by p_x the probability that each player chooses an effort level of x. Then, the expected payoff of a player that exerts an effort of x is:

$$v(p_0 + p_1 + ... + p_{x-1} + \frac{p_x}{m}) - x \quad \text{for all} \quad x = 0, 1, ..., v-1.$$

This system of linear equations can be written by the following matrix form:

$$A * y = b \tag{1}$$

where

$$A = \begin{bmatrix} \frac{v}{m} & 0 & 0 & 0 & 0 & . & . & -1 \\ v & \frac{v}{m} & 0 & 0 & 0 & . & . & -1 \\ v & v & \frac{v}{m} & 0 & 0 & . & . & -1 \\ v & v & v & \frac{v}{m} & 0 & . & . & -1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & 1 & . & . & 0 \end{bmatrix}_{v\text{x}v} \quad y = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ . \\ . \\ p_{v-1} \\ k \end{bmatrix}_{v\text{x}1} \quad b = \begin{bmatrix} 0 \\ 1 \\ 2 \\ . \\ . \\ v_{-1} \\ 1 \end{bmatrix}_{v\text{x}1}$$

Formally, matrix A is given by:

- $a_{x,x} = \frac{v}{m}$ for all $x < v$ and $a_{x,x} = 0$ for $x = v$.

- $a_{x,y} = v$ for all $v > x > y$ and $a_{x,y} = 0$ for all $x < y < v$.

- $a_{v,y} = 1$ for all $y < v$, $a_{x,v} = -1$ for all $x < v$ and $a_{v,v} = 0$.

The vector y is given by:

- $y_x = p_x$ for all $x < v$ and $y_v = k$ where k is each player's expected payoff.

The vector b is given by:

- $y_x = x - 1$ for all $x < v$ and $y_v = 1$.

It can be verified that independent of the value of m, the system of linear equations (1) has the solution $p_x = \frac{1}{v}$ for all $x \in \{0, 1, 2, ..., v-1\}$ and $k = \frac{1}{m}$. Below we prove the uniqueness of this solution. ■

Lemma 6 *In every solution $y = (p_0, p_1, , ..., p_{v-1}, k)$ of the equation system (1), $p_x \neq 0$ for every $0 \leq x \leq v-1$.*

Proof. See the Appendix.

In the following we show that the solution of (1) that satisfies the condition of Lemma 6 is unique. If $v = 1$ the equation system (1) has the reduced form

$$\begin{bmatrix} \frac{v}{m} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this case there is unique solution since $\det(A_1) = 1$. If $v = 2$ the equation system (1) has the reduced form

$$\begin{bmatrix} \frac{v}{m} & 0 & -1 \\ v & \frac{v}{m} & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

In this case there is a unique solution since $\det(A_2) = 2\frac{v}{m} - v < 0$.

For every $v \geq 1$ we have

$$\det(A_v) = (-1)^{v+1} \sum_{s=0}^{v-1} b_s$$

where

$$b_s = (-1)^s \left[(\frac{v}{m})^s (v - \frac{v}{m})^{v-1-s} \right]$$

Note that $b_s > 0$ for $s = 0, 2, 4, ...$ and $b_s < 0$ for $s = 1, 3, 5, ...$

Since $(v - \frac{v}{m}) > \frac{v}{m}$ we obtain that $|b_s| > |b_{s+1}|$ for all $s \geq 0$. Therefore for every $v \geq 1$, $\det(A_v) \neq 0$. ■

By Theorem (5) the probability of each player to win the contest in the case of a tie does not affect his strategy. Thus, if we assume that the prize causes a disutility for the contest designer, we can conclude that in a contest with two symmetric players, the contest designer should not award any prize in the case of a tie.

The following result shows that in asymmetric contests ties are not possible.

Proposition 7 *Consider an all-pay contest with two asymmetric players where* $v_1 > v_2$*, and where the probability of each player to win the contest in the case of a tie is smaller than one half. Then, there is no equilibrium with a tie. Moreover, the expected payoff of player* 2 *is zero and the expected payoff of player* 1 *is either* $v_1 - v_2$ *or* $v_1 - v_2 + 1$.

Proof. See the Appendix. ■

Note that by Example 4 we show that in decided contests the expected payoff of both players might be positive even if they are asymmetric. Proposition 7 shows that when the sum of the players' probabilities of winning is less than 1, only the stronger player has a positive expected payoff as well as in the standard asymmetric model where the sets of efforts are not finite.

4 Multi-Stage Contests with a Single Winner

There are two players competing for a variable prize in a multi-stage contest. The value of winning in the contest for player i at stage $t \geq 0$ is $v_{i,t} = \delta^t v_i, 0 < \delta < 1$. Valuations in every stage are common knowledge. We model the match between the players in every stage as an all-pay auction: each player at stage t exerts an effort $x_t \in \{0, 1, 2, 3, ...\}$ and the player with the highest effort wins and the game is over. However, in the case of a tie where players exert the same effort in stage t, they compete again in the next stage until one of the players wins the contest (one player exerts a higher effort than his opponent). That is, if players exert efforts of $x_{i,t}, x_{j,t}$ $\in \{0, 1, 2, 3, ...\}$ in stage t then the payoff for player i in this stage is given by

$$u_i(x_{i,t}, x_{j,t}) = \begin{cases} \delta^t v_i - x_i \text{ if } x_{i,t} > x_{j,t} \\ E_{i,t} - x_{i,t} \text{ if } x_i = x_j \\ -x_{i,t} \text{ if } x_{i,t} < x_{j,t} \end{cases}$$

where $E_{i,t}$ is the expected payoff for player i later than stage t (and can be constructed analogously for player j).

Proposition 8 *Consider two symmetric players with the same valuation who compete in a multi-stage all-pay contest for a variable prize v_t. Then, there is a unique symmetric sub-game perfect equilibrium in which at stage t each player chooses every effort $x_t \in \{0, 1, ..., v_t - 1\}$ with the same probability of $\frac{1}{v_t}$.*

Proof. By the assumption of symmetric equilibrium strategies and $0 < \delta < 1$ we obtain that $E_t < \frac{v_t}{2}$. Since the expected payoff of each player is smaller than half of his valuation in every stage $t \geq 0$, the result is obtained by Theorem 5. ■

Note that there is a positive probability that the contest will take place in any finite stage $t \geq 0$. Nevertheless, the equilibrium strategy in each stage depends only on the prize that is awarded in that stage, i.e., v_t , but the equilibrium strategy does not depend on the prizes that will be awarded in the next stages.

By Proposition 7 we obtain that in the case of asymmetric contests with tie-breaks the prize in the case of a tie is not relevant since players exert different levels of effort such that the contest will be decided already in the first stage.

Proposition 9 *Consider two asymmetric players, $v_i \neq v_j$, that compete in a multi-stage all-pay contest for a variable prize. Then, there is a sub-game perfect equilibrium where the contest is decided in the first stage.*

The conclusion from Propositions 8 and 9 is that the contest designer who maximizes the total effort should not limit the number of tie breaks since the length of the contest does not have any effect on the players' strategies at each stage of the contest.

5 Appendix

5.1 Proof of Lemma 6

We prove Lemma 6 by the following four lemmas:

Lemma 6.1: The equation system (1) does not have a solution $y = (p_0, p_1, , ..., p_{v-1}, k)$ where $p_x \neq 0$, $v - 1 > x > 0$ and $p_{x+1} = p_{x-1} = 0$.

Proof: Since the effort of x weakly dominates the effort of $x - 1$ we have

$$-1 + p_x \frac{v}{m} \geq 0. \tag{2}$$

Likewise, since effort of x weakly dominants effort of $x + 1$ we have

$$1 + p_x \frac{v}{m} \geq p_x v. \tag{3}$$

Since $m > 2$, we obtain from (3) that

$$p_x \frac{v}{m} < 1 \tag{4}$$

But the inequalities in (2) and (4) contradict each other.

Lemma 6.2: The equation system (1) does not have a solution $y = (p_0, p_1, , ..., p_{v-1}, k)$ where $p_{x+i} \neq 0, v - 1 > x > 0, i = 0, ..., j, j < v - x - 1$ and $p_{x+j+1} = p_{x-1} = 0$.

Proof: Since the effort of x weakly dominates the effort of $x - 1$ we have

$$-1 + p_x \frac{v}{m} \geq 0 \tag{5}$$

Now, since the player is indifferent to the efforts of x and $x + 1$ we have

$$1 + p_x \frac{v}{m} = p_x v + p_{x+1} \frac{v}{m} \tag{6}$$

Since $m > 2$, by (6) we have $\frac{v}{m}(p_x + p_{x+1}) \leq 1$ and therefore

$$p_x \frac{v}{m} \leq 1 \tag{7}$$

But, the inequalities in (5) and (7) contradict each other.

Lemma 6.3: The equation system (1) does not have a solution $y = (p_0, p_1, , ..., p_{v-1}, k)$ where $p_i \neq 0$ for all $0 \leq i \leq x < v - 1$ and $p_{x+1} = 0$.

Proof: The expected payoff from an effort of 0 is $p_0 \frac{v}{m}$. Since the players are indifferent to the efforts of 0 and 1 we have

$$p_0 \frac{v}{m} + 1 = p_0 v + p_1 \frac{v}{m}. \tag{8}$$

By equation (8) since $m > 2$ we obtain that $\frac{v}{m}(p_0 + p_1) \leq 1$, and therefore $p_0 \frac{v}{m} < 1$. That is, the expected payoff of each player is smaller than 1. But this is a contradiction, since every player can choose an effort of $v - 1$ which gives him a payoff of 1 for sure.

Lemma 6.4: The equation system (1) does not have a solution $y = (p_0, p_1, , ..., p_{v-1}, k)$ where $p_{v-i} \neq 0$, for all $i = 1, ..., v - x$, $x > 0$.

Proof: Since the effort of x weakly dominates the effort of $x - 1$ we have

$$p_x \frac{v}{m} \geq 1. \tag{9}$$

Since the players are indifferent to the efforts of x and $x + 1$ we have

$$p_x \frac{v}{m} + 1 = p_x v + p_{x+1} \frac{v}{m} \tag{10}$$

Since $m > 2$, equation (10) implies that

$$p_x \frac{v}{m} + p_{x+1} \frac{v}{m} < 1. \tag{11}$$

Since $p_{x+1} > 0$, by (9)+(11) we obtain the contradiction $1 \leq p_x \frac{v}{m} < 1$.

Combining the above four lemmas yields that in every solution $y = (p_0, p_1, , ..., p_{v-1}, k)$ of the equation system (1), $p_x \neq 0$ for every $0 \leq x \leq v - 1$.

5.2 Proof of Proposition 7

We wish to show that in an all-pay contest with two asymmetric players where $v_1 > v_2$, there is no equilibrium with a tie. First we show that the smallest levels of effort of both players in equilibrium are not identical and afterwards we will show that any levels of effort of both players in equilibrium are not identical neither.

Let p_x^i be the probability that player i chooses the effort of x. Suppose that for $i = 1, 2$, $p_x^i \neq 0$, $x \geq 0$, and $p_y^i = 0$ for all $y < x$. Player 1, who exerts the effort of x, has the expected payoff of

$$v_1 \frac{p_x^2}{m} > 1 \tag{12}$$

Since x weakly dominates $x + 1$ for player 1 we have

$$v_1 \frac{p_x^2}{m} + 1 \geq v_1 (p_x^2 + \frac{p_{x+1}^2}{m}) \tag{13}$$

Since $p_{x+1}^2 \geq 0$ and $m > 2$, equation (13) implies that

$$v_1 \frac{p_x^2}{m} (m - 1) < 1 \tag{14}$$

But equation (14) contradicts equation (12). Thus, the smallest levels of effort of both players are not identical, and therefore the smallest effort of player 2 is necessarily $x = 0$ and his expected payoff is zero. The highest effort of player 1 is necessarily larger or equal to $v_2 - 1$ and his expected payoff is larger or equal to the difference of the players' valuations.

We now show that both players do not exert identical efforts in equilibrium, that is, ties are not possible at all. Suppose now that there is a tie, and the tie with the lowest effort is on $x > 0$. Then if $p^2_{x-1} = 0$, we have the same argument as the one above. Otherwise, if $p^2_{x-1} \neq 0$ ($p^1_{x-1} = 0$) since for player 2, x weakly dominates $x-1$ we have

$$v_2 \frac{p^1_x}{m} \geq 1. \tag{15}$$

Similarly, since for player 2, x weakly dominates $x+1$ we have

$$v_2 \frac{p^1_x}{m} + 1 \geq v_2 (p^1_x + \frac{p^1_{x+1}}{m}). \tag{16}$$

Thus, since $m > 2$, equation (16) implies that

$$v_2 \frac{p^1_x}{m}(m-1) < 1 \tag{17}$$

Combining inequalities (15) and (17) yields a contradiction.

6 References

1. Amman, E., and Leininger,W. (1996), "Asymmetric All-pay Auctions with Incomplete Information: The Two-Player Case," *Games and Economic Behavior*, 14, 1-18

2. Baye, M., Kovenock, D., and de Vries, C. (1993), "Rigging the Lobbying Process," *American Economic Review*, 83, 289-294.

3. Baye, M., Kovenock, D., and de Vries, C. (1996), "The All-Pay Auction with Complete Information," *Economic Theory*, 8, 291-305.

4. Barut, Y., and Kovenock, D. (1998), "The Symmetric Multiple Prize All-Pay Auction with Complete Information," *European Journal of Political Economy*, 14, 627-644.

5. Che, Y-K, and Gale, I. (1998), "Caps on Political Lobbying," *American Economic Review,* 88 , 643-651.

6. Clark, D., and Riis, C. (1998), "Competition over More than One Prize," *American Economic Review*, 88, 276-289.

7. Dobson, S., and Goddard, J. (2001), The Economics of Football, Cambridge University Press.

8. Gavious, A., Moldovanu, B., and Sela, A. (2003), "Bid Costs and Endogenous Bid Caps," *Rand Journal of Economics,* 33(4), 709-722.

9. Hilman, A., and Riley, J. G. (1989), "Politically Contestable Rents and Transfers," *Economics and Politics*, 17-39.

10. Krishna, V., and Morgan, J. (1997), "An Analysis of the War of Attrition and the All-Pay Auction," *Journal of Economic Theory*, 343-362.

11. Moldovanu, B., and Sela, A. (2001), "The Optimal Allocation of Prizes in Contests," *American Economic Review*, 91, 542-558.

12. Moldovanu, B., and Sela, A. (2004), "Contest Architecture," *Journal of Economic Theory*, forthcoming.