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Marco Battaglini

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Marco Battaglini, Princeton University and CEPR

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Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR, UK
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: www.cepr.org

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ABSTRACT

Optimality and Renegotiation in Dynamic Contracting*

We characterize the optimal renegotiation-proof contract in a dynamic Principal-Agent model in which the type of the agent may change stochastically over time. Contrary to the case with constant types, the ex ante optimal contract may be renegotiation-proof even if types are highly correlated. The marginal benefit of having some pooling of types in the first period is not monotonic in their persistence level, but the equilibrium level of pooling is non-decreasing in persistence; and, for any level persistence, it is always optimal to partially screen the types by offering a menu of choices to the agent. Despite the non-linearity of the problem, the optimal equilibrium allocation is unique.

JEL Classification: D42 and L51
Keywords: contract theory, dynamic contracts, regulation and renegotiation

Marco Battaglini
Department of Economics
Princeton University
Fisher Hall
Princeton, NJ 08544
USA
Tel: (1 617) 452 2511
Fax: (1 617) 258 6419
Email: mbattagl@princeton.edu

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I Introduction

It is well known that in Principal-Agent models the optimal contract generally prescribes allocative inefficiencies. These distortions increase the principal's utility because they make it easier to screen the agent's types. For this reason, a contract that is ex-ante optimal may not be time-consistent: ex post, after some history, both the principal and the agent may agree on renegotiating these inefficiencies. In seminal papers, Hart and Tirole [1988], Dewatripont [1989], and Laffont and Tirole [1990] have characterized contracts that are robust to this time-inconsistency problem: they have found that renegotiation-proofness always weakens the principal's ability to screen the agent's types and implies lower surplus. All of these works assume that the type of the agent is constant throughout time.

In this paper we characterize the optimal renegotiation-proof contract in a Principal-Agent model in which types may change stochastically across periods. This generalization has a direct interest from an applied perspective since it is natural to assume that the type of an agent, though persistent, may change stochastically. But it also highlights some features of dynamic contracting that could not be seen under the assumption of constant types. Even if types are highly correlated, we show that there is not necessarily a conflict between renegotiation-proofness and optimality, as identified in the previous literature with constant types. Moreover, even when the contract that maximizes welfare ex ante is not renegotiation-proof, the optimal renegotiation-proof contract critically depends on the persistence of the realizations of types over time.

Figure 1 summarizes the main features of the optimal contract as a function of the level of correlation (measured by the probability α that a type is persistent). The optimal contract is characterized by two thresholds on correlation α_1 and α_2 with $\alpha_2 > \alpha_1$, which identify three possible cases. *When $\alpha \leq \alpha_1$ (area I) there is no conflict between optimality and renegotiation-proofness.* This result is perhaps surprising because, as we said, the previous literature showed that this is never true with constant types: however α_1 may be very high, and therefore the results with constant types cannot necessarily be interpreted as describing situations with highly persistent types. As seen in Figure 1, this threshold is a function of the likelihood ratio Γ_0 between the prior probability of the high type, v, and that of the low type (in a two types model). As the fraction of low types is increased, α_1 converges to one; but for realistic parameters' values, α_1 is very high: when, for example, there is a 20% prior probability of high types, then the contract is renegotiation-proof if the types are persistent even more than 80% of the time.

When $\alpha > \alpha_1$, the ex ante optimal contract is no longer time-consistent. Two cases are possible. In $\alpha \in (\alpha_1, \alpha_2]$, the optimal contract is still fully separating in the first

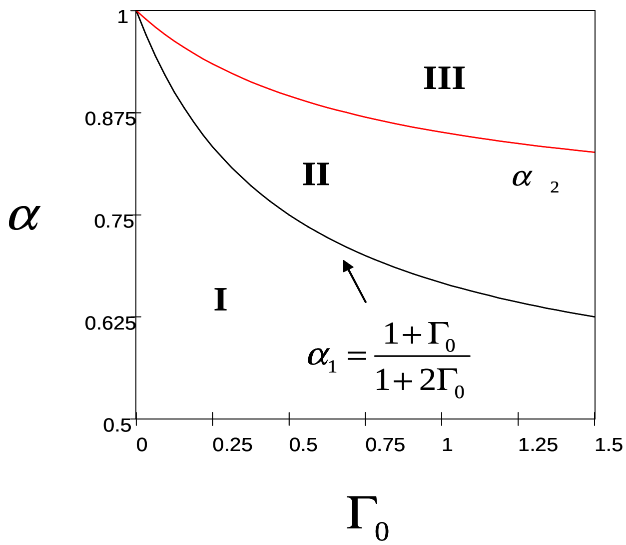

Figure 1: The optimal contract as a function of types' persistence (α) and Γ_0, the likelihood ratio between the prior probability of an efficient and inefficient firm.

period. In this region, the ex ante optimal contract would prescribe an allocation that, in the second period, is more distorted than what would be ex post optimal (given that the types were revealed in the previous period). These distortions would not be credible when $\alpha > \alpha_1$; even in this case, however, *the principal would still find it optimal to have full separation in the first stage, and to partially reduce the distortions at* $t = 2$. This feature of the optimal renegotiation-proof contract is also novel with respect to previous results. When types are constant, in fact, except when the discount factor is small enough, the optimal contract is never fully separating in the first period. On the contrary, the set $(\alpha_1, \alpha_2]$ (with full separation despite the renegotiation constraint being binding) is always non-empty, for any discount factor δ, and for any size of the payoffs in the second period.

If $\alpha > \alpha_2$ the principal finds it optimal to have a pooling equilibrium in which types do not fully separate in the first period. The marginal benefit of having pooling of types is not monotonic in α: however, we prove that the equilibrium level of pooling is monotonically increasing in persistence. This result, therefore, implies that even if types are imperfectly correlated *full pooling is never optimal*: the principal always prefers to offer a menu of choices in the first period. Clearly this result could not be observed in a model with constant types.

To prove these findings we generalize Laffont and Tirole's [1990] extended revelation principle with renegotiation-proofness to the case with variable types, which does not follow directly from previous arguments and it has a clear independent interest.[1] We also establish *uniqueness of the second best equilibrium* allocation. The principal's objective function is non-linear in the degree of pooling: despite this non-linearity, the equilibrium that maximizes welfare is unique in our model for any level of correlation of types.[2]

To understand why the ex ante optimal contract can be renegotiation-proof even with high correlation, and the other results mentioned above, consider the extreme example in which types are stochastic, but serially uncorrelated. Here, it is well known that the principal finds it optimal to offer a contract that is fully efficient in the second period (see Roberts [1982], Baron and Besanko [1984]). Indeed, with uncorrelated types the agent has no informational advantage in the first period with respect to the continuation of the game: the principal therefore can profitably make a take-it-or-leave-it offer for an efficient contract in the second period and extract all the surplus. Efficiency, however, implies that the contract is also Pareto optimal and therefore renegotiation-proof: the principal will never be able to offer a different contract that maintains the agent's utility and increases social welfare as well.

When we modify this benchmark case introducing some degree of correlation of types over time, the principal does not find it optimal anymore to offer a contract that is efficient in the second period, even if correlation is very small. Renegotiation-proofness, however, is guaranteed in this case by the residual uncertainty on realizations of types across periods. Even if in the first period types are perfectly screened, there is still uncertainty on the type in the second period. This implies that at t=*2* the conditionally optimal contract for the principal *still prescribes an allocative inefficiency* in order to screen this residual uncertainty away. We show that the ex ante optimal contract is optimal when the inefficiency in the ex post optimal contract is larger than in the ex ante optimal contract. When this is the case, it is credible that the inefficiency in the second period will be at least as large as promised ex ante, because the principal would be willing to impose an even higher inefficiency in the second period, if she could. On the other hand, the fact that the principal has to guarantee the same utility promised in the ex ante contract assures that the inefficiency will not be larger than promised.

As correlation increases, the ex ante optimal contract is no longer renegotiation-proof,

[1] In a significant contribution, Bester and Stausz [2001] have extended Laffont and Tirole's findings showing that with constant types there is no loss in using a direct mechanism even with more than two types (as in Laffont and Tirole's original work). Laffont and Tirole's result, however, is stronger than a simple "revelation principle," since it not only proves that there is no loss in using a direct mechanism, but also it specifies which types are pooling together, and which are separating in equilibrium. We will extend to the case with variable types this stronger version of the result.

[2] As noted by Laffont and Tirole [1990] for the case with constant types, the non-linearity of the Principal's problem may imply multiple equilibria. In their paper, however, they do not present formal on the number of equilibria.

but it still depends on the level of correlation because this determines the discrepancy between the ex ante optimal level of the allocative inefficiency and the ex post optimal level. Contrary to the case with constant types, we show that the optimal contract is not conditionally optimal for the principal in the second period. The contract, moreover, may continue to be fully separating in the first period even if the renegotiation proofness constraint is binding; and when only partial separation is optimal, the degree of pooling is monotonically increasing in persistence.

This paper connects two distinct lines of research. On the one hand we have the literature, mentioned above, that has studied renegotiation of contracts in dynamic settings when the agent's type is constant across periods. Dewatripont [1989], Hart and Tirole [1988] and Laffont and Tirole [1990] have introduced the notion of "renegotiation-proofness"[3] and characterized the optimal renegotiation-proof contract under different assumptions of the economic environment.[4] Our paper departs from this literature by assuming that types may change stochastically from period to period. For this generalization, we adopt the standard Principal-Agent framework introduced in Laffont and Tirole [1990], and we extend it to the case with variable types: this allows us to have a clear benchmark case to evaluate the impact of types variability across periods and compare it to the case with constant types.

The other literature to which our paper relates is the one on dynamic adverse selection with stochastic types, in which renegotiation is not a concern. In early contributions, Roberts [1982] and Baron and Besanko [1984] have studied the optimal contract with variable types and commitment. Laffont and Tirole [1996] have studied a two-period model of regulation in which the agent's types are variable, but the principal can commit. Dynamic models of pricing in which consumer's types change stochastically are considered, among others, by Battaglini [2003], Biehl [2001], Courty and Li [2000], Hendel and Lizzeri [1999], Hendel, Lizzeri and Siniscalchi (2004), Kennan [2001], Rustichini and Wolinsky [1995]. These papers, however, do not characterize the optimal renegotiation-proof contract because either they focus on the case with full commitment, or on the case in which long-term contracts are ruled out and only spot contracts are possible.

[3]When a contract must be renegotiation-proof, the principal and the agent cannot commit to forgo a Pareto optimal deviation from the initial contract. See below for details and a formal definition. The problem of time-inconsistency in mechanism design was first highlighted by Roberts [1982] and Holmström and Myerson [1983]. Holmström and Myerson [1983] have suggested the alternative concept of a "Durable Equilibrium" that, with respect to the concepts used in the papers mentioned above, has a more cooperative nature.

[4]Dewatripont [1989] characterizes the optimal renegotiation-proof contract in a general framework, but with restrictions on the contract space and assuming pure strategies. Hart and Tirole [1988] consider a model with T periods and two types in which supply could assume two values. Laffont and Tirole [1990] fully characterize the optimal renegotiation-proof contract in a model with two periods and two types with a general supply function.

The connection among these two literatures has previously been partially explored in two papers. Battaglini [2003] has characterized a sufficient condition for renegotiation-proofness of the ex ante optimal contract, but has not fully characterized the optimal renegotiation-proof contract. Blume [1995] studied renegotiation in a model in which a durable good monopolist serves a buyer with time varying valuations. In this model, supply is restricted to take two possible values, and stronger assumptions than in the present paper are made on the transition probabilities.[5] Although this model is insightful in comparing the rental versus the sale of a durable good, it cannot be used to study the relationship between optimality and renegotiation-proofness since optimal supply does not vary continuously with the distribution of types. In our model, moreover, production takes place in every period and the good is not durable.

The paper is organized as follows. Section 2 presents the model. As a benchmark, Section 3 characterizes the ex ante optimal contract with full commitment. Section 4 extends the revelation principle to our setting with variable types and renegotiation. Section 5 characterizes the optimal renegotiation-proof contract with variable types. Section 5 concludes.

II Model

We consider a dynamic Principal-Agent model with two periods. We interpret it as a model of cost regulation, but the setting is general and can be used to describe many other standard Principal-Agent interactions.[6]

In each period a firm (agent) has to realize a public project which generates surplus S and costs $c_t = \theta^t - e_t$. The agent's type is represented by the exogenous cost parameter θ^t. For simplicity, we assume that θ^t may take two values θ_L and θ_H with $\theta_H > \theta_L$ and $\Delta\theta = \theta_H - \theta_L$. The variable e_t is the level of effort that the firm can exert to reduce the project's cost, and may assume values in $\mathbb{R}^+$.

As in Laffont and Tirole [1990], the regulator (principal) observes the realized cost, but not the level of effort or the firm's type. The compensation to the firm, therefore, cannot directly depend on effort and the firm's type, but only on the realized cost. The principal reimburses the cost c_t of realizing the projects and pays an incentive fee s_t. In each period the firm's manager has utility $s_t - \varphi(e_t)$ where s_t is the net compensation for the project (net of the cost) and $\varphi(e_t)$ is a convex cost function. To obtain simple closed

[5] It is assumed that only the low type may change between the first and the second period. The model can be extended only under the condition that the probability that the low type can change is very small.

[6] The same model can for example be used to model a seller-buyer relation a' la Maskin and Riley [1984]; and, indeed, any other textbook Principal-Agent interaction can be seen as particular forms of this model.

forms, we assume $\varphi(e_t) = \frac{\eta}{2}e_t^2$ with $\eta > 0$, but, as it will be evident, this assumption is not essential for the results presented below. In this case, since the socially optimal level of effort is $e^* = \frac{1}{\eta}$, the socially optimal level of cost is state-contingent and equal to $c^* = \theta - e^*$.

The firm's ability changes stochastically over time: the probability that a type persists is $\Pr\left(\theta^{t+1} = \theta_i \,\middle|\, \theta^t = \theta_i\right) = \alpha$ for $i, j = H, L$ and $t \geq 0$. We assume that types are positively correlated ($1/2 \leq \alpha \leq 1$), but we do not make assumptions on the degree of correlation. The initial prior is that a fraction $v \in (0,1)$ of firms have a low cost parameter. It is useful to define the ex ante likelihood ratio $\Gamma_0 = \frac{v}{1-v}$ and the ex post ratios after an efficient type declaration, $\Gamma_L = \frac{\alpha}{1-\alpha}$, and an inefficient type declaration, $\Gamma_H = \frac{1-\alpha}{\alpha}$.

The goal of the regulator is to design a mechanism which maximizes total welfare. We assume that S is large enough so that it is always socially optimal to realize the project. The per-period welfare level is $S - (1+\lambda)(s_t + c_t)$, where $\lambda > 0$ is the distortion due to taxation.[7] The principal's objective function, therefore, is:

$$\sum_{t=1,2} \delta^{t-1} E\left[S - (1+\lambda)(s_t + c_t) + s_t - \varphi(e_t)\right] \tag{1}$$

where the expectation is taken according to the prior and the transition probabilities. The discount factor is δ.

When the principal can commit to a contract, the revelation principle allows us to use a direct mechanism: the problem consists in maximizing (1) in which c_t and s_t are contingent on the realized type, under the incentive compatibility constraints that each type desires to report his type truthfully, and that each type receives a reservation utility $\underline{u}$, which we normalize at zero (see program $\mathcal{P}_I$ in Section 3). The solution of this problem is the *ex ante optimal contract*. However, we require the contract to satisfy an additional constraint, *renegotiation-proofness*.

Definition 1 *A contract is renegotiation-proof if at the beginning of the second period the principal cannot replace it with a new contract which strictly increases social welfare and guarantees at least the same rent to the firm.*

This definition is standard in the literature (see Dewatripont [1988], Hart and Tirole [1988], Laffont and Tirole [1990]). When a contract is not renegotiation-proof, then both the regulator and the regulated firm would agree on renegotiating the initial contract.

[7] The variable λ can be interpreted as the shadow cost of a budget constraint faced by the principal (see Laffont and Tirole [1993] for a more extensive discussion).

When the contract is renegotiation-proof, on the contrary, any attempt to renege the initial contract by the principal would face the opposition of the regulated firm.

Observe that the model is a generalization of the framework introduced by Laffont and Tirole [1990]. Indeed, if we assume $\alpha = 1$, types are perfectly correlated and we are in Laffont and Tirole's world. This feature allows us to directly compare our results with their results with constant types.

III The Benchmark Case With Commitment

As a benchmark, in this section we discuss the optimal regulatory framework when the principal can commit to an ex ante optimal contract.

With commitment, the optimal contract specifies a cost c_{h_t} and a transfer s_{h_t} for any possible history $h_1 \in \{H, L\}$ and $h_2 \in \{HH, LH, LL, HL\}$. As mentioned, the revelation principle guarantees that we can restrict attention to a direct mechanism in which the firm truthfully reports her type at any possible history node. The regulator's problem with commitment is:

$$\min_{\langle \mathbf{c},\mathbf{s}\rangle} \sum_{i=H,L} \Pr(\theta_i) \left\{ \begin{array}{c} (1+\lambda)\left(c_i+\varphi\left(\theta_i-c_i\right)\right)+\lambda\left(s_i-\varphi\left(\theta_i-c_i\right)\right) \\ +\delta \displaystyle\sum_{j=H,L} \Pr\left(\theta_j \left|\theta_i\right.\right) \left[\begin{array}{c} (1+\lambda)\left(c_{ij}+\varphi\left(\theta_j-c_{ij}\right)\right) \\ +\lambda\left(s_{ij}-\varphi\left(\theta_j-c_{ij}\right)\right) \end{array} \right] \end{array} \right\} \quad (\mathcal{P}_I)$$

under the usual constraints:

$$\begin{array}{c} IC_H,\, IC_L,\, IC_i\left(H\right),\, IC_i\left(L\right) \text{ for } i=H,L \\ IR_H,\, IR_L,\, IR_i\left(H\right),\, IR_i\left(L\right) \text{ for } i=H,L \end{array}$$

where IR_H, IR_L, $IR_i\left(H\right)$, $IR_i\left(L\right)$ are the participation constraints which guarantee that after any possible history the firm receives at least her reservation value; and IC_H, IC_L and $IC_i\left(H\right)$, $IC_i\left(L\right)$ for $i = H, L$ are the incentive constraints which guarantee that the regulated agent is willing to report its type truthfully, respectively, in the first and in the second period after a history i.[8]

As it is formally proven in the Appendix, in period 1 and after any history h_2, only the incentive compatibility of the efficient type and the participation constraint of the inefficient type can be binding; and these constraints can be assumed to be binding without loss of generality. Given this, we can write the profit π_{iL} enjoyed by a firm who in the second period is efficient after it has declared to be a type $i = H, L$ in period 1 as:

$$\pi_{iL} = s_{iL} - \varphi(\theta_L - c_{iL}) = \pi_{iH} + \Phi\left(c_{iH}\right) \tag{2}$$

[8] A detailed description of these constraints is found in Section 7.1 in the Appendix.

where $\Phi(c_{iH}) = \varphi(\theta_H - c_{iH}) - \varphi(\theta_L - c_{iH})$, and the second inequality follows from the binding participation constraint for the low type.

Consider now the incentive constraint in the first period. We have:

$$\begin{aligned}
\pi_L &= [s_L - \varphi(\theta_L - c_L)] + \delta[\alpha\pi_{LL} + (1-\alpha)\pi_{LH}] \\
&= [s_H - \varphi(\theta_L - c_H)] + \delta[\alpha\pi_{HL} + (1-\alpha)\pi_{HH}] \\
&= \pi_H + \Phi(c_H) + \delta(2\alpha - 1)[\Phi(c_{HH}) - \pi_{HH}] \\
&= \Phi(c_H) + \delta(2\alpha - 1)\Phi(c_{HH}).
\end{aligned} \tag{3}$$

We can therefore write the objective function as:

$$\min_{c_H, c_{HH}} \left\{ (1+\lambda) \sum_{t=1,2} E\delta^{t-1}\left(c_t + \varphi\left(\theta^t - c_t\right)\right) + \lambda\nu\left[\Phi(c_H) + \delta(2\alpha-1)\Phi(c_{HH})\right] \right\}. \tag{4}$$

The first order condition of (4) easily yields the second best cost-reimbursement mechanism. Solving problem (4), we can see that the ex ante optimal contract with commitment is:

$$e^{\mathcal{C}}_{h_t} = \begin{cases} \frac{1}{\eta} & h_t = L, LL, LH, HL \\ \frac{1-\eta\Gamma_0\frac{\lambda}{1+\lambda}\Delta\theta}{\eta} & h_t = H \\ \frac{1-\eta\left(\frac{2\alpha-1}{\alpha}\right)\Gamma_0\frac{\lambda}{1+\lambda}\Delta\theta}{\eta} & h_t = HH \end{cases} \tag{5}$$

i.e., in the optimal contract effort is efficient after histories L, LL, LH and HL; and lower than efficient after the other histories.

The solution (5) shows the time-inconsistency problem faced by the principal in this stochastic environment.[9] In the optimal regulatory framework, the principal gives up some efficiency in the second period in order to increase efficiency in the first period. If the principal could rewrite the contract in the second period after history HH, for example, she would choose the ex post optimal level $e^P(HH) = \frac{1-\eta\Gamma_H\frac{\lambda}{1+\lambda}\Delta\theta}{\eta}$ and not the (generically different) level $e^{\mathcal{C}}_{HH} = \frac{1-\eta\left(\frac{2\alpha-1}{\alpha}\right)\Gamma_0\frac{\lambda}{1+\lambda}\Delta\theta}{\eta}$ prescribed by (5).

The case with changing types, however, is different from the canonical case with constant types.[10] Even if the history is economically irrelevant, the principal treats the agent differently according to it at $t = 2$. Indeed a firm with a low cost history is treated better than a firm with a high cost history. In the second case, the firm is required to exert less

[9] The optimal contract is fully characterized by (5). Indeed, given (5), the monetary transfers s_t can be found immediately using the binding constraints (2), (3) and the binding individual rationality constraint for the inefficient firm. Because of this, and because it would not add to the results, here and in the following propositions we omit the explicit solution of s_t for simplicity.

[10] Because the focus of the paper is in the optimal renegotiation-proof contract, we do not present here a full narrative discussion of the intuition behind the properties of the optimal contract with commitment, but we highlight only the features that are useful for the discussion of renegotiation-proofness.

effort than the efficient level. Since the equilibrium level of the firm's rent is increasing in the level of effort, a firm with a HH history receives a higher rent in the second period than a firm with a history LH: $e^{\mathcal{C}}_{HH} < e^{\mathcal{C}}_{LH} \Rightarrow \pi_{HH} < \pi_{LH}$.

This feature of the optimal contract is not true when types are constant over time. The benchmark case with constant types follows immediately from (5). If we let $\alpha \to 1$, we have that with constant types the principal offers the optimal static contract in every period:

$$e^{\mathcal{C}}_{h_t} = \begin{cases} \frac{1}{\eta} & h_t = L, LL \\ \frac{1-\eta\Gamma_0\frac{\lambda}{1+\lambda}\Delta\theta}{\eta} & h_t = H, HH \end{cases} . \tag{6}$$

As it can be seen from (6), and also well known in the literature, the optimal contract with constant types, ignores past history and is a simple repetition of the optimal static contract.[11]

Because contracts (5) and (6) are ex post inefficient, they are not credible: both the principal and the agent may agree on renegotiating them. In the following sections, therefore, we will characterize the optimal credible contract that is not renegotiated.

IV The Revelation Principle with Variable Types

To characterize the optimal contract, we first prove a version of the revelation principle that applies when types are variable and renegotiation is possible. This will simplify the space of contracts that we need to study. Then, in the next section, we study the features of the optimal renegotiation-proof contract.

When the contract must be renegotiation-proof, the regulator solves program $\mathcal{P}_I$ with an additional constraint. For any history h_2, the contract must remain conditionally optimal in the second period given the level of utility promised to the firm at $t = 1$:

$$\left\{c^*_{h_2 j}, s^*_{h_2,\, j}\right\}_{j=H,L} \in \arg\min \sum_{j=H,L} \Pr\left(\theta_j \left| h_2\right.\right) \left[\begin{array}{c} (1+\lambda)\left(c_{h_2 j} + \varphi\left(\theta_j - c_{h_2 j}\right)\right) \\ +\lambda\left(s_{h_2 j} - \varphi\left(\theta_j - c_{h_2 j}\right)\right) \end{array} \right] \quad (\mathcal{R})$$
$$\text{s.t. } IC_{h_2}(j) \text{ and } \pi_{h_2,j} \geq \pi^*_{h_2,j} \text{ for } j = H, L$$

where $\Pr\left(\theta_j \left| h_2\right.\right)$ are the posterior probabilities at the beginning of period 2 given the history h_2, and $\pi^*_{h_2,j}$ is the rent promised by the initial contract to a firm of type θ_j after h_2. Observe that there is no loss of generality in assuming that the optimal contract specifies only two options at $t = 2$: indeed, given any posterior, at $t = 2$ the problem of

[11] With constant types histories HL and LH have zero probability. Obviously, however, the optimal contract with constant types must specify the contractual terms after all histories (deviations are possible out of equilibrium). In this sense, the optimal contract can be seen as a repetition of the optimal static contract.

the regulator is a standard static Principal-Agent problem in which the utility promised to the types at $t = 1$ is the reservation level at $t = 2$. We define $\mathcal{P}_{II}$ as the program consisting of $\mathcal{P}_I$ plus $(\mathcal{R})$.

Constraint $(\mathcal{R})$ can be usefully simplified. Let $\widetilde{c}(\mu)$ be the cost level prescribed for the inefficient type in the ex post optimal contract at *t=2* when the posterior at the beginning of the period is μ:[12]

$$\widetilde{c}(\mu) = \theta_H - e^* + \frac{\mu}{1-\mu}\frac{\lambda}{1+\lambda}\Delta\theta.$$

And define $\mu(h_2)$ the posterior probability that the firm is efficient after history h_2. We have:

Lemma 1 *The renegotiation-proofness constraint (R) is satisfied after history h_2 if and only if:*

$$\theta_H - e^* \leq c_{h_2,H} \leq \widetilde{c}(\mu(h_2)) \tag{7}$$

where c_{h_2H} is the cost implemented in the second period after a history $\{h_2, \theta_H\}$.

At this stage we cannot be very specific regarding what the history h_2 can be. Indeed the revelation principle does not hold in this environment; so we cannot assume, without loss of generality, a direct mechanism. This problem was first pointed out in the constant type framework by Laffont and Tirole [1990].

To extend the revelation principle to our environment with variable types, we proceed with two steps. Let's assume that the contract specifies a set A^o of options in period 1. Therefore the possible histories are $H = \{i, iH, iL\}_{i \in A^o}$. In the first step, we rule out double randomizations:

Lemma 2 *Without loss of generality, no pair of contracts i or j can be chosen by both types with positive probability.*

This lemma implies that only three cases are possible in correspondence to the optimal contract. We can have full separation, in which each type selects a different contract; or, if we have a pooling equilibrium, we can have two cases: either one contract is chosen only by the efficient type, and the other contract is chosen by both types with positive probability; or one contract is chosen only by the inefficient type, and the other contract is chosen by both types with positive probability.

The second step rules out the case in which the inefficient type is "pooling alone":

[12] This is simply the cost that would be prescribed for the inefficient type in an optimal static contract if the prior is μ.

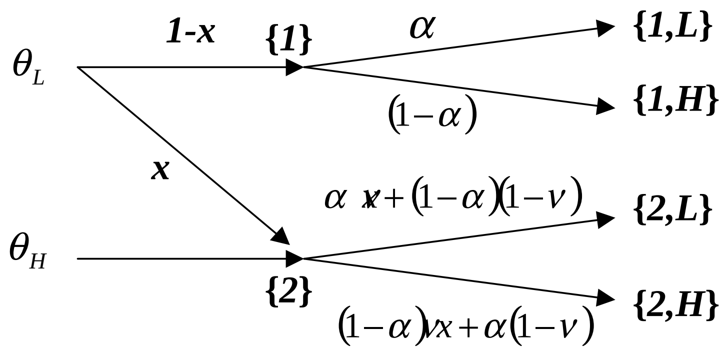

Figure 2: The transition probabilities in equilibrium: x is the probability that the efficient type θ_L pools with the inefficient type θ_H.

Lemma 3 *Without loss of generality, in the equilibrium that maximizes the principal's welfare there is no couple of contracts i and j such that with positive probability one is chosen by both types and the other only by the inefficient type.*

From Lemmata 1 and 2 we know that, without loss of generality, at the optimum we can only have two cases. In the first case, at $t = 1$ the regulator offers some menus that are chosen only by the efficient firm, and other menus that are chosen only by the inefficient firm: in this case the optimal contract is fully separating and the contracts chosen only by one type can be merged without loss of generality. In the second case, the regulator at $t = 1$ offers some menus that are chosen only by the efficient firm, and only one menu that is chosen by both types. In this case too there would be no loss if we merge all the contracts chosen only by the efficient type. We therefore conclude:

Proposition 1 *Without loss of generality, the principal can offer the agent a choice between two contracts, one chosen by the efficient type, and the other chosen by the inefficient type and possibly by the efficient type.*

Figure 2 describes the transition probabilities in the optimal contract. Only the efficient firm may randomize in equilibrium. Contract $\{1\}$ is chosen by the efficient firm only. With probability $x \in [0, 1]$, however, this firm may choose the "inefficient" contract (history $\{2\}$) chosen also by the other type. This choice of the efficient firm affects the transition probabilities in the second period after $\{2\}$, so that, after this history, the probability of an efficient firm is $\alpha vx + (1 - \alpha)(1 - v)$. To simplify the comparison with the case with commitment we call contract $\{1\}$ (respectively, $\{2\}$) the $\{L\}$ (respectively, $\{H\}$) contract.

V The Optimal Renegotiation-Proof Contract

Using Proposition 1 we can now characterize the optimal contract. In this section we show that, as anticipated by Figure 1 in the Introduction, we may have three relevant cases, depending on the persistence of the types. In the first case, the optimal renegotiation-proof contract may coincide with the ex-ante optimal contract with commitment (Area I in Figure 1). In the second case, it may not be ex ante optimal, but still be fully separating (Area II). Finally, in the third case, $x \in (0, 1)$ and we have pooling of types in the first period (Area III) as represented in Figure 1.

These three cases are studied in order in the next three subsections.

V.1 When is the Renegotiation-Proofness constraint irrelevant?

In this section we characterize the threshold on persistence α_1 that guarantees that there is no conflict between ex ante optimality and renegotiation-proofness.

Both with constant and with variable types, the principal finds it optimal to require the inefficient firm to exert less than efficient effort. As it can be seen from (6), when types are constant, the distortions are actually equal in the first and second period, both are $\Gamma_0 \frac{\lambda}{1+\lambda} \Delta\theta$. On the contrary, when there is commitment but types are stochastic, the distortion in the second period is

$$\left(\frac{2\alpha - 1}{\alpha}\right) \Gamma_0 \frac{\lambda}{1+\lambda} \Delta\theta$$

which is strictly smaller than the distortion in the first period if the types are positively correlated ($\alpha > 1/2$). In the limit case when types are uncorrelated ($\alpha \to \frac{1}{2}$), the distortion in the second period is actually zero. Because of this, the principal is promising a contract that, in the second period, is the more efficient the less the firm's types are correlated.

Assume that α is such that the principal at $t = 1$ finds it optimal to promise a contract for $t = 2$ that is actually more efficient than what would be the ex post optimal contract. This occurs when when persistence is below a threshold, $\alpha < \alpha_1 = \frac{1+\Gamma_0}{1+2\Gamma_0}$. In this case, if at $t = 2$ the principal tries to renegotiate the original contract with a contract that has a higher distortion (i.e. that requires an even lower level of effort and a higher cost level), the new contact would generate less welfare than the original: so it would be impossible for both the firm and the principal to be (weakly) better off.

So if the principal wants to renegotiate, he needs to offer a new contract with a distortion lower than what he promised in the initial contract. This however can not be optimal for the principal. Since types change stochastically over time, even in an equilibrium that is fully separating in the first period, some residual uncertainty remains in the second period regarding the firm's type. Because of this, the principal still finds it optimal to use an allocative inefficiency in the second period to minimize the agent's rent and maximize welfare. The principal's payoff is strictly concave in the distortion: if the ex post optimal allocative inefficiency is higher than the optimal ex ante level, then any inefficiency lower than what is ex ante optimal would reduce the principal's payoff. This implies that if the principal renegotiates the contract with a contract with a lower distortion, she would have to pay such a high rent to the efficient firm that her payoff would necessarily go down.

Therefore we have:

Proposition 2 *There is no conflict between renegotiation-proofness and ex ante optimality if*

$$\alpha \leq \frac{1+\Gamma_0}{1+2\Gamma_0} \equiv \alpha_1. \tag{8}$$

Therefore in this case the optimal renegotiation-proof contract $e^L_{h_t}$ *is given by (5).*

As proven in the previous literature, the ex ante optimal contract is never renegotiation-proof when types are constant. This can be easily explained in the light of the previous discussion. When types are constant, the ex post optimal distortion is zero, since after the first period there is no residual uncertainty left (in a fully separating equilibrium): therefore it is always smaller than the ex ante optimal distortion. This implies that ex post the principal always finds it optimal to reduce the distortion; and the agent would agree with the change.

On the contrary, with variable types renegotiation-proofness is not a problem even when types are highly persistent. The threshold α_1 is always larger than $\frac{1}{2}$ (even as $\Gamma_0 \to \infty$), and from Figure 1 we can see that there is no conflict between optimality and renegotiation-proofness even in cases in which correlation is very high. Observe, moreover, that condition (8) is independent of the discount factor or of any other relevant parameter (like, for example, the particular function representing the cost of effort).

Proposition 2, therefore, suggests that the problem of time-inconsistency of contracts may be less important that what is currently believed, at least in many environments. It is natural to interpret the assumption that types are constant as an approximation to the assumption that types are highly correlated. However Proposition 2 shows that this interpretation is not always allowed, since even if types are very persistent, the ex ante optimal cost-reimbursement scheme may be time-consistent. This observation may be particularly relevant in cases in which the Principal-Agent relationship depends of the evolution of new technologies, or new markets in which there is substantial uncertainty regarding the nature of the agent's competitive advantage.

This result also has implications on the type of policy instruments that a regulator should use. It is obvious that a regulatory framework that is state contingent is superior to a rule determined ex ante. If the regulator fears a time-inconsistency problem, however, she might be tempted to choose a fixed rule that, perhaps, is more credible in the long term. Examples of this "tying hands" policies are forced spin-offs, or other interventions on the organizational structure of the regulated firm which are expensive to reverse. Proposition 2, however, shows that ex ante optimal, state contingent rules may not only be superior in terms of welfare, but also credible and immune from renegotiation.

It is finally interesting to note that, as proven by Hart and Tirole (1988) and Laffont and Tirole (1990), except when δ is small, the optimal renegotiation-proof contract is in mixed strategies when types are constant. This is necessary to generate the correct posterior beliefs in the second period that guarantee that there is no Pareto superior contract. These strategies may require considerable strategic sophistication. Except when the interaction is repeated many times, it may seem unrealistic that the play of the game follows the exact equilibrium prediction. This may be a problem in situations in which the Principal-Agent relationship is desultory. When types change over time and (8) is satisfied, on the contrary, the equilibrium is in pure strategies and only requires the firm to report its type.

V.2 When is the Contract Separating?

When $\alpha > \alpha_1$ the ex ante optimal contract is not renegotiation-proof. However, as we prove in this section, the optimal renegotiation-proof contract remains separating, and in pure strategies, until α remains below a second threshold $\alpha_2 > \alpha_1$.

Lemma 1 guarantees that only the efficient firm may randomize between the two contracts in the first period. This however does not necessarily imply that the efficient firm is randomizing (and therefore IC_L is binding); or that we can ignore the incentive constraint of the inefficient firm. To complete the characterization of the optimal renegotiation-proof contract let's define the auxiliary problem $\mathcal{P}_{III}$:

$$\min_{\langle x,\mathbf{c},\mathbf{s}\rangle} \left\{ \begin{array}{c} v\,(1-x)\,C(\theta_L, c_L, s_L) + vxC(\theta_L, c_H, s_H) + (1-v)\,C(\theta_H, c_H, s_H) \\ +\delta \left[\begin{array}{c} \sum\limits_{j=H,L} \Pr\left(\theta_j \,|\theta_L\,\right) \left[v\,(1-x)\,C(\theta_j, c_{Lj}, s_{Lj}) + vxC(\theta_j, c_{Hj}, s_{Hj})\right] \\ \cdot\,(1-v) \sum\limits_{j=H,L} \Pr\left(\theta_j \,|\theta_H\,\right) C(\theta_j, c_{Hj}, s_{Hj}) \end{array} \right] \end{array} \right\} \quad (\mathcal{P}_{III})$$

under the constraint that:

$$\begin{array}{l} IC_L,\, IC_i\,(L) \text{ and } IR_H\ IR_i\,(H) \ \ i = L, H \text{ are binding} \\ \text{and (7) is satisfied.} \end{array}$$

where $C(\theta, c, s) = (1 + \lambda)\,(c + \varphi\,(\theta - c)) + \lambda\,(s - \varphi\,(\theta - c))$ is the net (per period) welfare cost of the project if the type is θ, the transfer is s, and the realized cost of the project is c; $\Pr\left(\theta_j\,|\theta\right)$ is the probability of type θ_j in period two, given type θ in $t = 1$; and x is the probability that the efficient type mimics the inefficient type (by Lemma 1 this is the only possible case).

We say that a cost schedule c_{h_t} is renegotiation-proof optimal if there is a transfer schedule s_{h_t} such that $\{c_{h_t}, s_{h_t}\}$ solves P_{II}. The following lemma characterizes the optimal renegotiation-proof cost schedule.

Lemma 4 *A cost schedule c_{h_t} is renegotiation-proof optimal if and only if it solves $\mathcal{P}_{III}$.*

A few remarks are necessary to interpret this result. The usual approach to solve Principal-Agent models is to solve a "relaxed problem" in which some constraints are ignored, find a solution $\{c_{h_t}, s_{h_t}\}$, and then verify that the solution of this problem solves the general problem too. This is not what we are doing with Lemma 4. Indeed if we only considered the solution of the "usual" relaxed program that ignores the incentive constraints of the inefficient firm and the participation constraints of the efficient firm without *imposing* that the remaining constraints are binding, we would certainly have solutions that do not solve the general program. While in a static setting, the incentive constraint is necessarily binding, and this is sufficient to guarantee that its solution also solves the general program, in a dynamic setting this is not generally true. We can have some solutions in which the constraint is not binding after some history, and that would violate some incentive compatibility constraint of the inefficient firm in $\mathcal{P}_{II}$. Similarly we can have solutions $\{c_{h_t}, s_{h_t}\}$ of $\mathcal{P}_{II}$ in which the incentive compatibility of the high type in the second period is not binding. Lemma 4, however, proves that we can use $\mathcal{P}_{III}$ as an *auxiliary* program to characterize the set of optimal solutions $\mathbf{c} = \{c_{h_t}\}_{\forall h_t}$. While in a static model the relaxed problem is essentially identical to the full problem because at the optimum the omitted constraints are superfluous, program $\mathcal{P}_{III}$ is not identical to the full program, it is only a "tool" to characterize the cost schedule.

We can now find the optimal cost schedule c_{h_t} $\forall h_t$, by solving $\mathcal{P}_{III}$. It is easy to verify that c_L, c_{LH}, c_{iL} for $i = 1, 2$ are set at the efficient level in $\mathcal{P}_{III}$: $c_L = c_{iL} = \theta_L - e^*$ for $i = L, H$ and $c_{LH} = \theta_H - e^*$. If we define

$$\mu\,(\alpha, x) = \frac{vx\alpha + (1 - v)\,(1 - \alpha)}{vx + (1 - v)}$$

as the posterior probability that the firm is efficient after a history H, using some simple algebra we can rewrite $\mathcal{P}_{III}$ and characterize the optimal cost schedule solving the program:

$$\begin{array}{cc} \min_{\{c_{HH},c_H,x\}} & A(x,c_H)+\delta B(x,c_{HH}) \\ s.t. & \theta_H - e^* \leq c_{HH} \leq \widetilde{c}(\mu(\alpha,x)) \end{array} \tag{9}$$

where

$$A(x,c_H) = (1+\lambda)\left\{ \begin{array}{c} v(1-x)(\varphi(e^*)+\theta_L - e^*) \\ +vx(\varphi(\theta_L - c_H)+c_H)+(1-v)(\varphi(\theta_H - c_H)+c_H) \end{array} \right\} + \lambda v[\Phi(c_H)]$$

is the total cost of the project in the first period; and

$$\begin{array}{rcl} B(x,c_{HH}) & = & (1+\lambda)\left[\begin{array}{c} [v\alpha+(1-\alpha)(1-v)](\varphi(e^*)+\theta_L - e^*) \\ +v(1-\alpha)(1-x)(\varphi(e^*)+\theta_H - e^*) \\ +[\alpha(1-v)+xv(1-\alpha)](\varphi(\theta_H - c_{HH})+c_{HH}) \end{array} \right] \\ & & +\lambda v(2\alpha-1)\Phi(c_{HH}) \end{array}$$

is the total cost of the project for the principal in the second period.

The renegotiation constraint must now be binding at the optimum:

Lemma 5 *If $\alpha > \alpha_1$, then $c_{HH} = \widetilde{c}(\mu(\alpha,x))$ in (9).*

This implies that the problem can be written as the minimization of

$$A(x,c_H)+\delta B(x,\widetilde{c}(\mu(\alpha,x))) \tag{10}$$

in which the control variables are x and c_H. This expression makes clear the trade-off in the determination of x, and therefore in the benefit of pooling in the equilibrium contract. An increase in x has a *direct* and an *indirect* effect on welfare. The direct effect always reduces welfare in both periods. In the first period, the higher x is, the less efficient discrimination of types is. In the second period an efficient firm that chooses the "L" path receives an efficient contract even if her type changes; on the contrary, after "H," a firm that is efficient in $t=1$ but inefficient in $t=2$ receives a distorted contract. The higher x is, therefore, the greater the expected distortion, and the lower the expected welfare in the second period too.

On the other hand, an increase in x, by increasing $\widetilde{c}(\mu(\alpha,x))$, also has an indirect, but positive effect on welfare. An increase in x directly increases the probability that a firm is efficient after history $\{H\}$ because more firms that were efficient in the first period choose this path; and, since types are correlated over time, this increases the fraction of efficient firms in the second period too. The increase in the posterior, finally, induces a higher cost level c_{HH}, since by Lemma 5 $c_{HH} = \widetilde{c}(\mu(\alpha, x))$: this is beneficial because it reduces the gap between the ex ante optimal cost level after history $\{HH\}$ and the optimal level of the cost that can be credibly enforced after $\{HH\}$.[13]

It is precisely because the marginal benefit of x is "mediated" by the marginal effect of $\widetilde{c}(\mu(\alpha, x))$ on $B(x, \widetilde{c}(\mu(\alpha, x)))$ that makes full separation of types in $t = 1$ optimal when α is not too much larger than α_1. As proven in Proposition 2, with $\alpha \leq \alpha_1$, the ex ante optimal cost level for the second period is already more efficient than the ex post optimal level: therefore having pooling in the first period would unequivocally reduce welfare. Similarly, when $\alpha = \alpha_1$ the marginal benefit of increasing the cost level in the second period through an increase in x is zero; and when $\alpha > \alpha_1$, but $(\alpha - \alpha_1)$ is not too large, the (indirect) marginal benefit of x is not sufficient to compensate for its (direct) marginal cost. We have:

Proposition 3 *There exists a $\alpha_2 \in (\alpha_1, 1)$ such that if $\alpha \leq \alpha_2$ then the optimal renegotiation-proof contract is fully separating in the first period, and if $\alpha \in (\alpha_1, \alpha_2]$ then:*

$$e^L_{h_t} = \begin{cases} \frac{1}{\eta} & h_t = L, LL, HL, LH \\ \frac{1 - \eta\Gamma_0 \frac{\lambda}{1+\lambda}\Delta\theta}{\eta} & h_t = H \\ \frac{1 - \eta\Gamma_H \frac{\lambda}{1+\lambda}\Delta\theta}{\eta} & h_t = HH \end{cases} . \tag{11}$$

In the optimal renegotiation-proof contract, therefore, effort is efficient after a history L; and it is ex post optimal after history H, given the equilibrium beliefs.

It is interesting to discuss how the optimal contract (11) differs from the ex ante optimal contract (and therefore the optimal renegotiation-proof contract when $\alpha \leq \alpha_1$) and the ex post optimal contract. As in the contract described by (5), also in contract (11) the regulator finds it optimal, independently of the type realization in the second period, to induce an efficient level of effort e^* after a firm declares to be efficient in the first period. This fact, therefore, extends the *Generalized No-Distortion at the Top* principle, shown in models with commitment (cf. Battaglini [2003]), to cases when there is a binding renegotiation-proofness constraint. This is a feature that distinguishes the optimal contract with stochastic and variable types from the case with constant types.

With constant types, the optimal renegotiation-proof contract is always conditionally optimal: given the equilibrium strategies and the implied posterior beliefs, the contract is optimal for the principal in the second period. Even if $\alpha_1 < \alpha < \alpha_2$, when the

[13] With full separation, when $\alpha > \alpha_1$ the ex ante optimal level $c^{\mathcal{C}}_{HH}$ is lower than the optimal level that can be credibly enforced ex post at *t=2*. Pooling at t=1 reduces this gap by increasing the ex post optimal cost level.

renegotiation-proofness constraint is actually binding, however, the optimal renegotiation-proof contract is not ex post optimal. Indeed, after a history $\{L\}$ it would be ex post optimal to introduce some distortion (since there is still uncertainty about the type's realization); however the optimal renegotiation-proof contract prescribes no inefficiency. This shows that while in the second period the renegotiation-proofness constraint is not really binding for the regulator with constant types (she indeed takes a conditionally optimal action), the constraint really binds with changing types.

After a history $\{HL\}$, (5) and (11) are efficient and prescribe a level of effort that coincides with the ex post optimal level. The optimal renegotiation-proof contract after history $\{HH\}$ with $\alpha > \alpha_1$ (i.e., c^L_{HH}), however, differs from the ex ante optimal contract $c^{\mathcal{C}}_{HH}$:

$$c^{\mathcal{C}}_{HH} - c^L_{HH} = \left[\left(\frac{2\alpha - 1}{\alpha}\right)\Gamma_0 - \Gamma_H\right]\frac{\lambda}{1+\lambda}\Delta\theta > 0. \tag{12}$$

Interestingly the optimal renegotiation-proof contract with $\alpha > \alpha_1$ is ex post optimal for the regulator after history H, as in the case with constant types; this cost level, however is strictly lower (more efficient) than what would be ex ante optimal. When $\alpha \leq \alpha_1$, on the contrary, the cost level is ex ante optimal but strictly larger (more inefficient) than the ex post optimal level.

V.3 When Do We Have a Pooling Equilibrium?

In this section we study the conditions for the existence of pooling equilibria and their properties. Propositions 2 and 3 guarantee that no pooling equilibrium exists in $\left(\frac{1}{2}, \alpha_2\right)$. However, they do not prove that a pooling equilibrium exists at all: indeed α_2 could be equal to one. Moreover, the objective function is very non-linear in x and α, so multiplicity of equilibria may be a problem.

V.3.1 Existence and Uniqueness

The key to the existence of an equilibrium with pooling is the relationship between the payoffs in the first and second periods. When the payoffs in the first period are substantially more important than the payoffs in the second period (for example if δ is very small), then the problem is essentially static and, not surprisingly, full separation is optimal: in this case $\alpha_2 = 1$. Intuition would perhaps suggest that when the payoffs of the second period are large compared to the first period, then it always becomes optimal to pool.[14]

[14] The relationship between the payoffs of the first period and the second can be parametrized by δ. In this sense this variable not only reflects the discount factor, but, more in general, the relationship between the payoffs in the two periods. This is the interpretation given in Laffont and Tirole [1990], who, in fact, assume δ to be in $(0, \infty)$.

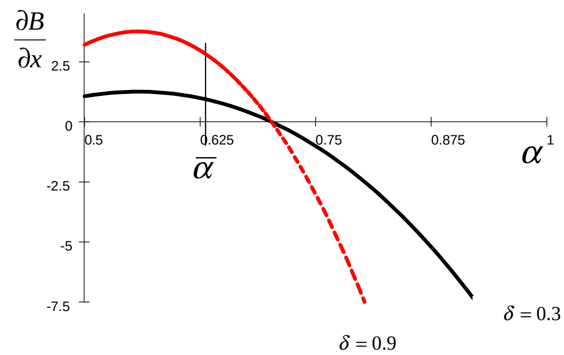

Figure 3: The marginal effect of x on the cost of the project $B\left(x, \widetilde{c}\left(\mu\left(\alpha, x\right)\right)\right)$ in the second period. See footnote 19 for details.

This is always true with constant types since the cost of pooling is concentrated in the first period: therefore an increase in δ increases only the marginal benefit of pooling.[15]

However, this fails to be true when $\alpha < 1$. In this case, the cost of pooling spreads to the second period too: so as we increase the relevance of the payoff in the second period (increasing, for example, δ) we not only increase the benefits of pooling, but also part of its costs. As we discussed before, if an efficient firm declares itself to be inefficient, it would receive an inefficient cost assignment in the second period if its type changes; on the contrary, the cost assignment would always be efficient if it truthfully reports its type at $t = 1$.[16] This fact reduces surplus generated in the second period.

Consider Figure 3.[17] Even if the renegotiation-proofness constraint is binding ($\alpha > \alpha_1$), an increase in the payoffs at $t = 2$ may reduce the incentives to pool at $t = 1$. If we increase the discount factor from $\delta = .3$ to $\delta = .9$ when, for example, $\alpha = \overline{\alpha} \in (\alpha_1, 1)$, then we increase the (positive) marginal effect of an increase of x on social costs $B\left(x, \widetilde{c}\left(\mu\left(\alpha, x\right)\right)\right)$, and this reduces the marginal effect on surplus of x.

This cost of pooling in the second period, however, is proportional to the probability that the inefficient firm changes type, $1 - \alpha$: therefore as $\alpha \to 1$, this cost in the second period fades away. It follows that while it is *not true* that for any $\alpha < 1$ there is a δ large enough that guarantees pooling, it is true that the *lower bound* on the correlation threshold α_2 that guarantees a pooling equilibrium for some large enough δ is certainly

[15]In this case x affects welfare in the second period only indirectly through a change in $c(\mu(\alpha, x))$, therefore we do not have the direct effect described above and an increase in x always (weakly) increases welfare at $t = 2$.

[16]This would not be true if we had a double randomization at $t = 1$. However we can rule this out by Lemma 2.

[17]This figure represents the marginal effect of an increase in x on $B\left(x, \widetilde{c}\left(\mu\left(\alpha, x\right)\right)\right)$ as a function of α, evaluated at $x = 0.3$, $\lambda = .8$, $\upsilon = .8$, $\eta = .5$.

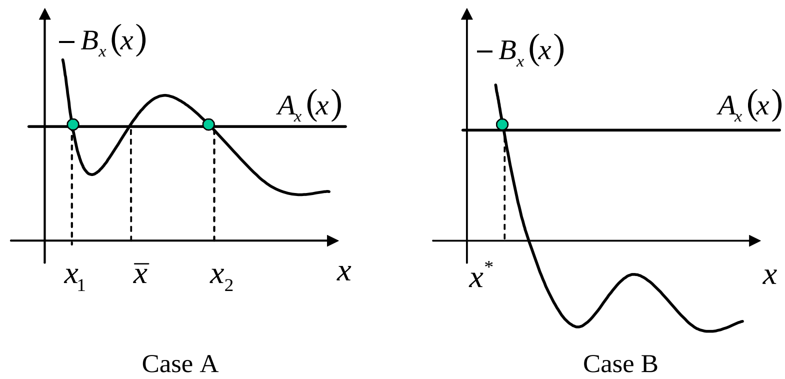

Figure 4: Uniqueness of the solution: only case B is posssible.

lower than one.

Proposition 4 *There exists a $\overline{\delta}$ such that $\alpha_2 < 1$ if $\delta > \overline{\delta}$, and $\alpha_2 = 1$ otherwise. Therefore if $\delta > \overline{\delta}$, there exists at least one pooling equilibrium with α lower than one.*

When $\delta > \overline{\delta}$, therefore there is always a choice of parameters for which the equilibrium cannot be fully separating: in these cases a marginal increase in x starting from $x = 0$ surely increases welfare. However, the indirect "beneficial" effect of x at $t = 2$ is not monotonically increasing in x: on the one hand, an increase in x reduces the marginal effect $\frac{\partial}{\partial \widetilde{c}(\mu(\alpha,x))} B\left(x, \widetilde{c}\left(\mu\left(\alpha, x\right)\right)\right)$; on the other hand, it increases $\frac{\partial}{\partial x}\widetilde{c}\left(\mu\left(\alpha, x\right)\right)$. If case A in Figure 4 were possible, then we might have multiple equilibria (x_1, x_2). The particular structure of the problem, however, guarantees a unique equilibrium. Proposition 5 proves that only case B in Figure 4 is possible.

Proposition 5 *For $\alpha > \alpha_2$ there is a unique optimal level of pooling $x^*(\alpha, \delta) \in (0, 1)$, and the optimal contract is characterized by:*

$$e_{h_t}^L = \begin{cases} \frac{1}{\eta} & h_t = L, LL, HL, LH \\ \frac{1}{\eta} - \frac{v}{vx^*(\alpha,\delta)+(1-v)}\left(\frac{\lambda}{1+\lambda} - x^*(\alpha,\delta)\right)\Delta\theta & h_t = H \\ \frac{1}{\eta} - \frac{vx^*(\alpha,\delta)\alpha+(1-v)(1-\alpha)}{vx^*(\alpha,\delta)(1-\alpha)+\alpha(1-v)}\frac{\lambda}{1+\lambda}\Delta\theta & h_t = HH \end{cases}.$$

Clearly, the equilibrium level of pooling, as measured by $x^*(\alpha, \delta)$, depends on all the parameters of the model. It is however possible to bound its possible range and characterize its sensitivity to the persistence of types.

V.3.2 How much pooling?

To understand how the "degree" of pooling is affected by a change in α, we need to study the impact of α on the first order condition

$$\frac{\partial}{\partial x}\left[A\left(x^*(\alpha,\delta),c_H(\alpha,\delta)\right)+\delta B\left(x^*(\alpha,\delta),\widetilde{c}\left(\mu\left(\alpha,x^*(\alpha,\delta)\right)\right)\right)\right]=0.$$

The first term $\frac{\partial}{\partial x\partial\alpha}A\left(x,c_H\right)$ depends on α only indirectly, through $c_H(\alpha,\delta)$, and $x^*(\alpha,\delta)$: but because of the Envelope Theorem, this is irrelevant for the impact of α on the first order condition. The key variable, therefore, is the impact of correlation on $\frac{\partial}{\partial x}B\left(x,\mu\left(\alpha,x\right),\alpha\right)$:[18]

$$\begin{aligned}\frac{\partial}{\partial x}B\left(x,\mu\left(\alpha,x\right),\alpha\right) &= \frac{\eta\Delta\theta^2\lambda^2}{1+\lambda}\left[(1-\alpha)-(1-x)\,\alpha v\right] \qquad (13)\\ &\cdot\frac{v\,(1-v)\,(2\alpha-1)}{\left[vx\,(1-\alpha)+\alpha\,(1-v)\right]^2} \qquad (14)\\ &+\frac{\eta v\,(1+\lambda)\,(1-\alpha)}{2}\left(\frac{vx\alpha+(1-v)\,(1-\alpha)}{vx\,(1-\alpha)+\alpha\,(1-v)}\frac{\lambda}{1+\lambda}\Delta\theta\right)^2.\end{aligned}$$

As it can be seen from (13) and, for example graphically from Figure 3, this expression is generally not monotonic in α.[19] Exploiting the equilibrium conditions, however, it is possible prove the monotonicity of the solution.

Proposition 6 *There is a unique optimal pooling level $x^*(\alpha,\delta)$ that is non-decreasing in α and strictly increasing for $\alpha>\alpha_2$.*

Laffont and Tirole [1990] proved that when types are constant full-pooling is impossible. Proposition 6, therefore, has an immediate implication.[20]

Proposition 7 *With imperfect persistence pooling is never larger than with constant types, so full pooling is never optimal for any level of correlation α.*

[18] In the expression below we make explicit that α also affects $B\left(x,\mu\left(\alpha,x\right)\right)$ directly by writing $B\left(x,\mu\left(\alpha,x\right),\alpha\right)$.

[19] The intuition why this may occur is as follows. As we increase the probability x that an efficient firm chooses the "inefficient" contract at t=1, we increase the fraction of efficient firms at t=1 that receives an inefficient contract at t=2 because their type have changed: this reduces expected surplus. Note that while an expected distortion for the low type may increase surplus (because it helps to screen the types), an expected distortion for the high type unambiguously reduces welfare. This negative effect of x may be increasing in α: as correlation increases, the contract becomes more inefficient for the low type in the second period. Even if this negative effect is small in absolute terms, when the initial population of efficient firms is large (large v), and the initial level of α is small (so the probability of changing type is high) the negative *marginal* impact of α may be substantial.

[20] The fact that full pooling is not optimal at $\alpha=1$ can also be directly seen from (13).

This result extends Laffont and Tirole's [1990] finding that full pooling is suboptimal with constant types to the general case when types can have any degree of correlation over time, despite the fact that the marginal benefit of pooling is non-monotonic in α. This is therefore a robust finding with practical implications. It is not optimal for the principal to impose a fixed rule at *t=1*; instead, it is always optimal to offer a menu of options which separate the types in a significant way.

V.4 Discussion

Proposition 6 completes the characterization of the optimal renegotiation-proof contract as anticipated in Figure 1 in the Introduction. Since $x^*(\alpha, \delta)$ is monotonic non-decreasing in α, then α_2 is a cutoff threshold: we have a pooling equilibrium if and only if $\alpha > \alpha_2$. Together with Propositions 2-5, we now have a complete picture of the optimal contract. It is useful to summarize the results. When $\alpha \leq \alpha_1$ the optimal renegotiation-proof contract coincides with the ex ante optimal contract. In $\alpha \in (\alpha_1, \alpha_2]$, the optimal renegotiation-proof contract is fully separating in the first period, and conditionally optimal in the second period. Finally for $\alpha > \alpha_2$ we have a pooling equilibrium in which the efficient firm pools with the inefficient firm with probability $x^*(\alpha, \dot{\delta})$. In any of these cases, the optimal contract is unique.

We conclude here with a discussion of the model and its robustness, focusing on the questions that are open for future research: in particular, the time horizon, the types space and the contractual environment.

In our previous analysis we considered the simplest form of dynamics, a model with two periods. Since many environments span a longer time horizon, it would be interesting to extend the basic model to more than two periods. Battaglini [2003] characterizes the ex ante optimal contract with commitment in a model with multiple finite or infinite periods: a full characterization of the optimal renegotiation-proof contract in a infinite horizon model, however, is still a question open for future research. This extension presents no conceptual complications, except that the analysis becomes more tedious; such analysis, however, would allow to study the asymptotic properties of the model as $t \to \infty$. We conjecture that the optimal renegotiation-proof contract would converge to an efficient contract, but we leave the analysis of this issue for future work.

As in Hart and Tirole [1988], Laffont and Tirole [1990], Blume [1998], we assumed that the state variable θ^t is binary. Only two papers considered the more complicated case with many (but constant across periods) types. Dewatripont [1989] considers the case with n finite types, but he has to assume that the equilibrium is in pure strategies and to restrict the set of possible contracts. Laffont and Tirole [1988] consider a model of dynamic contracting with a continuum of types but do not present a complete characterization of the contract (the full characterization is presented in Laffont and Tirole [1990], where two types are assumed): their result is that full separation is not optimal in the first period. When types may change stochastically the analysis is even more complicated. In this case the type of an agent is given not only by θ^t, but also by the shape of the conditional distribution of future realizations. In particular, when there are n types the distribution is a n-1 dimensional object, and we are in a multidimensional screening problem. As it is well known, multidimensionality complicates the analysis of screening problems; our framework allows us to separate the dynamics of the Principal-Agent interaction, which is the focus of the paper, from the conceptually distinct problem of the multidimensionality of types, allowing, therefore, a sharper analysis of the dynamics.

Regarding the contractual environment, in this work we have focused the analysis on long-term contracts with renegotiation. Different assumptions are also possible. One case (plausible in many environments) is that the principal can fully commit to a contract, and renegotiation is not an issue. We have presented this case in Section 3 as a benchmark. The other extreme case is when the principal cannot make any commitment at all, and only spot contracts can be offered. This is certainly an interesting case in some environments, but unrealistic in most long-term principal agent interactions. Indeed, although contract can be renegotiated, it is rarely the case that a non anonymous Principal-Agent interaction is independent from past performance. It is therefore important to understand the limits imposed by renegotiation-proofness on long-term agreements.

VI Conclusion

Since Ramsey's [1927] seminal paper on optimal policy in a dynamic environment, the literature has struggled with a time-inconsistency issue. Indeed the ex ante optimal policy is rarely time-consistent because it can often be substituted in subsequent periods by a Pareto superior policy. In the context of Principal-Agent problems, time-inconsistency of the ex ante optimal contract always reduces welfare when the agent's type is constant over time (as typically postulated in the existing literature) because it makes it more difficult to provide credible incentives that induce the agent to reveal his type.

In this paper we have characterized the optimal renegotiation-proof contract in a Principal-Agent model in which the type of the agent can vary across periods. This case is important both from a strictly theoretical point of view, as well as from a more applied perspective since an agent's types are variable and stochastic in many real world environments. In this case, even when types are very persistent, the Ramsey equilibrium may be renegotiation-proof. This implies that results drawn from models with constant types cannot always be generalized to the case with imperfectly correlated types, even if persistent. Regardless of the specific environment in which the Principal-Agent relationship is staged, a model that postulates constant types would always suggest to a policy maker to guarantee renegotiation-proofness in subsequent periods with some commitment device. Commitment devices may be socially costly and inefficient. A model that recognizes that types are only imperfectly correlated, however, may make clear that these types of "tying ones own hands" are not necessary in some important situations.

Even when the Ramsey equilibrium is not renegotiation-proof, the persistence of types is important in the characterization of the optimal renegotiation-proof contract. We have fully characterized the properties of the contract in this case. Contrary to the case with constant types, even if payoffs in the second period are very high relative to the first period, a fully separating equilibrium may continue to be optimal when the renegotiation-proofness constraint is not satisfied. When the level of correlation is higher than a threshold, however, the principal may not find it optimal any longer to have full separation in the first period. For any degree of correlation, however, full pooling is never optimal. This implies that, regardless of the level of persistence of types, the principal will always find it optimal to offer in the first period a menu of choices to the agent.

VII Appendix

VII.1 Characterization of the Optimal Contract with Commitment

In this Section we complete the argument presented in Section 3 for the characterization of the optimal contract with commitment.

The set of constraints of $\mathcal{P}_I$ is described by the following inequalities. The incentive compatibility constraints are:

$$\begin{aligned} & [s_H - \varphi(\theta_H - c_H)] + \delta\left[\alpha(s_{HH} - \varphi(\theta_H - c_{HH})) + (1-\alpha)\left(s_{HL} - \varphi(\theta_L - c_{HL})\right]\right. \\ \geq \; & [s_L - \varphi(\theta_H - c_L)] + \delta\left[\alpha(s_{LH} - \varphi(\theta_H - c_{LH})) + (1-\alpha)\left(s_{LL} - \varphi(\theta_L - c_{LL})\right]\right. \qquad (IC_H) \end{aligned}$$

$$\begin{aligned} & [s_L - \varphi(\theta_L - c_L)] + \delta\left[(1-\alpha)\left(s_{LH} - \varphi(\theta_H - c_{LH})\right) + \alpha(s_{LL} - \varphi(\theta_L - c_{LL})\right] \\ \geq \; & [s_H - \varphi(\theta_L - c_H)] + \delta\left[\alpha(s_{HL} - \varphi(\theta_L - c_{HL})) + (1-\alpha)\left(s_{HH} - \varphi(\theta_H - c_{HH})\right]\right. \qquad (IC_L) \end{aligned}$$

$$s_{iH} - \varphi(\theta_H - c_{iH}) \geq s_{iL} - \varphi(\theta_H - c_{iL}) \qquad (IC_i(H) \;\; i = H, L)$$

$$s_{iL} - \varphi(\theta_L - c_{iL}) \geq s_{iH} - \varphi(\theta_L - c_{iH}) \qquad (IC_i(L) \;\; i = H, L)$$

and the participation constraints are:

$$[s_H - \varphi(\theta_H - c_H)] + \delta[\alpha(s_{HH} - \varphi(\theta_H - c_{HH})) + (1-\alpha)\left(s_{HL} - \varphi(\theta_L - c_{HL})\right] \geq 0 \quad (IR_H)$$

$$[s_L - \varphi(\theta_L - c_L)] + \delta[(1-\alpha)\left(s_{LH} - \varphi(\theta_H - c_{LH})\right) + \alpha(s_{LL} - \varphi(\theta_L - c_{LL})] \geq 0 \qquad (IR_L)$$

$$s_{iH} - \varphi(\theta_H - c_{iH}) \geq 0 \qquad (IR_i\,(H))$$

$$s_{iL} - \varphi(\theta_L - c_{iL}) \geq 0. \qquad (IR_i\,(L))$$

To complete the argument in Section 3, we only need to show that in the relaxed problem, in which we ignore the incentive constraint of the inefficient type and the participation constraints of the efficient type, the incentive constraint of the efficient type and the participation constraints of the inefficient type can be assumed to be binding without loss of generality; and that the solution of this relaxed problem is a solution of the full problem as well. The strict concavity of the problem guarantees that cost function c_{h_t} found with this procedure must be the unique solution of $\mathcal{P}_I$.

It can be easily verified that the incentive compatibility constraints of the efficient type and the participation constraint of the inefficient type are binding in the first period. It is also simple to see that the participation constraint of the inefficient type must be binding in the second period. Assume that this were not true after a history h and $\pi_{hH} = \kappa_1 > 0$, then we can uniformly reduce s_{hH} and s_{hL} by κ_1; and contextually increase s_h by $\delta\kappa_1$.

The modified contract continues to be a solution of $\mathcal{P}_I$ because it still satisfies the efficient type's incentive constraints in both periods; and it satisfies the participation constraints of the low type with equality.

Consider now the incentive compatibility constraint of the efficient type after a history $h_1 = \{H\}$ and assume, by contradiction, that: $s_{HL} - \varphi(\theta_L - c_{HL}) = \pi_{HL} + \Phi(c_{HH}) + \kappa_2$, with $\kappa_2 > 0$. Modify the contract in the following way. First reduce s_{HL} by κ_2; contextually increase s_H by $\delta(1-\alpha)\kappa_2$. After the change, welfare is constant and the incentive compatibility and participation constraints are satisfied in period 2. In period 1 the participation constraint of the low type is also satisfied; consider the incentive constraint. If the efficient firm mimics the inefficient, the payoff world be:

$$\pi_L - \delta(2\alpha - 1)\kappa_2 < \pi_L.$$

So this constraint remains satisfied too. We can now reduce s_L by $\delta(2\alpha-1)\kappa_2$: the resulting contract satisfies all the constraints with equality and yields higher welfare. So the initial contract was not optimal, a contradiction. We conclude that $IC_H(L)$ is binding. The case for $IC_L(L)$ is analogous.

We now prove that the solution of the relaxed problem is also a solution of the full problem. To see that the second period incentive compatibility constraints of the low type are binding after some history h note that by using the binding $IC_h(L)$ and $IR_h(H)$ constraints we have:

$$\begin{aligned}
s_{hL} - s_{hH} &= \varphi(\theta_{hL} - c_{hL}) - \varphi(\theta_{hL} - c_{hH}) \\
&= \varphi(\theta_{hH} - c_{hL}) - \varphi(\theta_{hH} - c_{hH}) + [\Phi(c_{hH}) - \Phi(c_{hL})] \\
&\Rightarrow s_{hH} - \varphi(\theta_{hH} - c_{hH}) \geq s_{hL} - \varphi(\theta_{hH} - c_{hL})
\end{aligned}$$

where the last passage follows from the fact that by (5) $c_{hH} \geq c_{hL}$ for any h and $\Phi(c)$ is decreasing in c. Similarly, for period one we have:

$$\begin{aligned}
s_L - s_H &= \varphi(\theta_H - c_L) - \varphi(\theta_H - c_H) + \delta[(1-\alpha)(\pi_{HL} - \pi_{LL})] \\
&\quad + [\Phi(c_H) - \Phi(c_L)] + (2\alpha - 1)(\pi_{HL} - \pi_{LL}) \\
&\Rightarrow s_L - \varphi(\theta_L - c_L) + \delta(1-\alpha)\pi_{LH} \geq s_H - \varphi(\theta_L - c_H) \\
&\quad + \delta(1-\alpha)\pi_{HH}
\end{aligned}$$

where the last expression follows by the fact that $\Phi(c_H) - \Phi(c_L) \leq 0$ (since $c_H \geq c_L$ and $\Phi(c)$ is decreasing) and $\pi_{LL} \geq \pi_{HL}$ (since $c_{LL} \leq c_{HL}$). The fact that the participation constraint of the efficient type is satisfied follows from (2) and (3). $\blacksquare$

VII.2 Proof of Lemma 1

Consider the renegotiation constraint (R) after a history "h_2." Total output (i.e. the value of the project net of its cost) is non-increasing with respect to c_{h_2H} in the interval $[\theta_H - e^*, \infty)$ and the principal objective function (i.e. welfare, output net of the cost λ of leaving a rent to the firm) are non-decreasing with respect to c_{h_2H} in the interval $(-\infty, \widetilde{c}(\mu(h_2)]$ (respectively, strictly decreasing and strictly increasing in the interior of the interval). Therefore, when $c_{h_2H} \in [\theta_H - e^*, \widetilde{c}(\mu(h_2))]$, renegotiating the contract in favor of a cost $c > c_{h_2H}$ would imply lower output, making it impossible to obtain a Pareto improvement. On the contrary, replacing the cost prescribed in the original contract with a new cost $c < c_{h_2H}$ would imply lower welfare. We therefore conclude that when $c_{h_2H} \in [\theta_H - e^*, \widetilde{c}(\mu(h_2))]$, c_{h_2H} cannot be profitably renegotiated. Similarly, we can see that the condition is necessary: when c_{h_2H} does not belong to $[\theta_H - e^*, \widetilde{c}(\mu(\alpha, x))]$, the sign of the derivative of output and welfare is the same, so there is a marginal change in c which would improve output and welfare at the same time. ■

VII.3 Proofs of Lemmata 2 and 3

Without loss of generality, the optimal contract specifies only two options at $t = 2$.[21] It can be verified that given a posterior and a level of utility promised by the contract ex ante, the optimal renegotiation-proof contract in the second period can assume only three possible forms:[22] it can be conditionally optimal given the posterior; it can be a sell-out contract, in which the cost is chosen efficiently in the second period; or it can be "rent constrained." In a rent constrained contract the incentive compatibility constraint is binding; but, in order to guarantee to the efficient type the utility promised at $t = 1$, the low type is assigned a lower cost level than what it would be ex post optimal.[23] Except for the sell-out contract, the incentive compatibility constraint of the efficient firm is always binding at $t = 2$. As usual in static Principal-Agent problems, in all these cases the incentive compatibility constraint of the inefficient types is not binding at $t = 2$. It is also easy to prove that the participation constraint of the inefficient type is binding at $t = 1$. We now prove Lemma 2.

Lemma 2 *There is no loss of welfare in assuming that no pair of contracts i or j can be chosen by both types with positive probability.*

[21] Given any posterior, at $t = 2$ the problem of the regulator is a standard static principal-agent problem in which the utility promised to the types at $t = 1$ is the reservation level at $t = 2$.

[22] See previous footnote.

[23] As usual, when the incentive constraint of the efficient firm is binding, her rent is decreasing in the cost chosen by the inefficient firm.

Proof. Assume that it is not true and that, without loss of generality, contracts 1 and 2 are chosen by both types with positive probability: conditionally on choosing contract 1 or 2, the efficient firm chooses contract 2 with probability $x \in (0,1)$, and the inefficient types chooses contract 1 with probability $y \in (0,1)$. Without loss of generality we can label the histories so that the cost in the first period after contract 1 is chosen (history $\{\mathit{1}\}$) is smaller than after 2 is chosen (history $\{\mathit{2}\}$): $c_1 \leq c_2$. We call the contract after history $\{\mathit{1}\}$ (respectively, $\{\mathit{2}\}$) "the efficient (respectively, inefficient) contract." For future reference it is also useful to write the objective function with this double randomization as a function of x and y. We will keep the contracts $j \neq 1,2$ and the probability that they are chosen constant throughout the proof so total cost can be written as

$$(1-p_H)\,K_H + (1-p_L)\,K_L + C(x,y)$$

where $p_i > 0\ \forall i = H, L$ is the probability that type i chooses contracts 1 or 2, K_i is the expected cost associated with type i choosing contracts $j \neq 1,2$, and $C\,(x,y)$ is the expected welfare cost associated with types H and L choosing contracts 1 or 2:

$$C(x,y) = \left\{ \begin{array}{c} p_H v\,(1-x)\,C(\theta_L, c_1, s_1) + p_H v x C(\theta_L, c_2, s_2) \\ +p_L\,(1-v)\,(1-y)\,C(\theta_H, c_2, s_2) + p_L\,(1-v)\,y C(\theta_H, c_1, s_1) \\ +\delta \left[\begin{array}{c} p_H v \displaystyle\sum_{j=H,L} \Pr\left(\theta_j \left| \theta_L \right.\right) \left[(1-x)\,C(\theta_j, c_{1j}, s_{1j}) + x C(\theta_j, c_{2j}, s_{2j})\right] + \\ p_L\,(1-v) \displaystyle\sum_{j=H,L} \Pr\left(\theta_j \left| \theta_H \right.\right) \left[(1-y)\,C(\theta_j, c_{2j}, s_{2j}) + y C(\theta_j, c_{1j}, s_{1j})\right] \end{array} \right] \end{array} \right\} \tag{15}$$

where $C(\theta, c, s) = (1+\lambda)\,(c + \varphi\,(\theta - c)) + \lambda\,(s - \varphi\,(\theta - c))$. We proceed in three steps.

Step 1. We first show that if x, y are interior in $(0,1)$, the incentive compatibility of the efficient firm must be binding in the second period after any history $i = 1,2$. To see this assume, for example, that it is not true after history $\{\mathit{1}\}$. In this case we have a sell-out contract after $\{\mathit{1}\}$, which, since it is efficient, is renegotiation-proof independently of x, y. So the optimal levels of x, y are determined by maximizing welfare under the constraint that the contract is renegotiation-proof after only history $\{\mathit{2}\}$. Assume the posterior after this history is μ_2. We can have two cases. If $c_{2H} \in [\theta_H - e^*, \widetilde{c}\,(\mu_2))$, then, by Lemma 1, a marginal change in x, y would not affect the renegotiation constraint. Since the objective function is linear in x, y, at least one of them must be a corner solution 0 or 1, a contradiction. If $c_{2H} = \widetilde{c}\,(\mu)$, then, since $\widetilde{c}\,(\mu)$ is increasing is μ, the renegotiation constraint is satisfied after a change in x, y if and only if posterior kept larger or equal to μ_2. The solution x, y, therefore, must minimize (15) under the constraint that

$$\alpha v p_L x + (1-\alpha)\,(1-v)\,p_H y \geq \mu_2\left[v p_L x + (1-v)\,p_H y\right]. \tag{16}$$

But this is still a linear problem with linear constraints, so implies a corner solution 0 or 1, a contradiction.

Step 2. We now show that if x, y are interior, then conditional on a firm choosing contract *{1}* at $t = 1$, the probability that the firm is efficient (γ_1) is higher than the probability conditional on choosing contract *{2}* (γ_2). For this purpose, it is useful to re-write the firm's profits in terms of the cost levels. If y is interior, the inefficient type must be indifferent between the two contracts; therefore the profit if she chooses contract 2, $\pi_H(2)$, must be equal to the profits if she chooses contract 1, $\pi_H(1)$:

$$\begin{aligned}
\pi_H(2) &= \pi_H(1) = s_1 - \varphi(\theta_H - c_1) + \delta \begin{bmatrix} \alpha(s_{1H} - \varphi(\theta_H - c_{1H})) \\ +(1-\alpha)(s_{1L} - \varphi(\theta_L - c_{1L})) \end{bmatrix} \\
&= \pi_L(1) - [\Phi(c_1) + \delta(2\alpha - 1)\Phi(c_{1H})]
\end{aligned}$$

where the last equality follows from the fact that by Step 1, the incentive compatibility of the efficient type is binding at $t = 2$ and therefore $s_{1L} - \varphi(\theta_L - c_{1L}) = \Phi(c_{1H})$; and since the participation constraint the inefficient type is binding at $t = 2$, $s_{1H} - \varphi(\theta_H - c_{1H}) = 0$. But since the participation constraint of the inefficient firm is binding at $t = 1$, we have $\pi_H(2) = 0$ and:

$$\pi_L(1) = [\Phi(c_1) + \delta(2\alpha - 1)\Phi(c_{1H})].$$

A similar argument yields:

$$\pi_L(2) = [\Phi(c_2) + \delta(2\alpha - 1)\Phi(c_{2H})].$$

So by the indifference condition of the efficient firm, x interior implies:

$$[\Phi(c_1) + \delta(2\alpha - 1)\Phi(c_{1H})] = [\Phi(c_2) + \delta(2\alpha - 1)\Phi(c_{2H})]. \tag{17}$$

The indifference condition of the inefficient firm is automatically guaranteed by an appropriate choice of the monetary transfers: since it is independent of the choice of the costs, we can ignore it. We now consider two cases.

Case 2.1. Assume that $c_{2H} < \widetilde{c}(\mu_2)$ and $c_{1H} < \widetilde{c}(\mu_1)$.[24] It is easy to verify that in the optimal solution it cannot be that $c_{iH} < \theta_H - e^* \ \forall i \in 1, 2$: by Lemma 1, we conclude that the renegotiation-proofness constraint is not binding after histories 1 and 2. The optimal contract after these histories, therefore, is a solution of the minimization of $C(x, y)$ without the renegotiation-proofness constraint. However this problem is linear in x and y, so they cannot be interior as assumed, a contradiction.

Case 2.2. We can therefore restrict the analysis to only two cases: $c_{2H} = \widetilde{c}(\mu_2)$ and $c_{1H} \leq \widetilde{c}(\mu_1)$ (Case 2.2.1) or $c_{2H} < \widetilde{c}(\mu_2)$ and $c_{1H} = \widetilde{c}(\mu_1)$ (Case 2.2.2).

[24] As proven in Proposition 2, this case occurs if $\alpha < \alpha_1$. See Section 5.1 for details.

Case 2.2.1. Note that:

$$\gamma_i = \frac{\mu_i - (1-\alpha)}{2\alpha - 1} \quad \forall i = 1, 2$$

where μ_i is the posterior probability at $t = 2$ that the firm is efficient after history $i = 1, 2$. Since $\alpha \geq \frac{1}{2}$, $\gamma_1 \geq \gamma_2$ if and only if $\mu_1 \geq \mu_2$. If $x \in (0, 1)$, the indifference condition for the efficient firm implies:

$$[\Phi(c_{1H}) - \Phi(c_{2H})] = -\frac{\Phi(c_1) - \Phi(c_2)}{\delta(2\alpha - 1)} \leq 0 \tag{18}$$

so $c_{1H} \geq c_{2H}$ since $\Phi(\cdot)$ is decreasing. But then, since $\widetilde{c}(\cdot)$ is increasing, we have that $\widetilde{c}(\mu_2) = c_{2H} \leq c_{1H} \leq \widetilde{c}(\mu_1)$ implies $\mu_1 \geq \mu_2$, and $\gamma_1 \geq \gamma_2$.

Case 2.2.2. Let us define $c_{ND}(\gamma)$ the optimal cost that the principal would choose if she knew that the probability that a firms is efficient is γ and she could not discriminate between the two types (ND stands for no-discrimination). Assume that $c_1 < c_{ND}(\gamma_1)$. Consider a marginal increase in c_1 and a reduction in c_{1H} such that the incentives for the efficient firm to mix are preserved (condition (18)). This change does not violate the renegotiation constraint. Since the principal's objective function is concave in c_1, the change in c_1 increase welfare. However since c_{1H} is conditionally optimal, its change has only a second order effect. It follows that the total change increases welfare, a contradiction. It must be that $c_1 \geq c_{ND}(\gamma_1)$. Assume now that $c_2 \geq c_{ND}(\gamma_2)$. Consider a marginal increase in c_{2H} and a reduction in c_2 so that the indifference condition for the inefficient firm is preserved. This change does not violate the renegotiation constraint since $c_{2H} < \widetilde{c}(\mu_2)$. By concavity, both changes weakly increase welfare and at least one strictly increases it, a contradiction: therefore $c_2 < c_{ND}(\gamma_2)$. We conclude that $c_{ND}(\gamma_1) \leq c_1 \leq c_2 < c_{ND}(\gamma_2)$, which implies $\gamma_1 \geq \gamma_2$.

Step 3. It can be verified that $c_2 \geq c_{ND}(\gamma_2)$ if and only if $c_1 \leq c_{ND}(\gamma_1)$. Indeed if this were not the case, it would be possible to simultaneously increase (or decrease) c_1 and c_2 keeping all incentives constraints satisfied in the first period. This would imply that after any history the cost would be nearer to the first best and therefore welfare (which is concave) would be higher in the first period and nothing would change in the second period, a contradiction. Therefore we can focus on two possible cases need to be considered: $c_{ND}(\gamma_1) \leq c_1 \leq c_2 \leq c_{ND}(\gamma_2)$ (Case 3.1) and $c_1 < c_{ND}(\gamma_1) \leq c_{ND}(\gamma_2) < c_2$ (Case 3.2). We claim that if both x and y are interior, we can reduce at least one of them without loss of generality. Compared to the case with constant types, the analysis is complicated by the fact that the sign of the marginal effect of x on the cost function is ambiguous. With constant types a marginal effect of x affects only the first period: now it affects both. Given that $c_1 \leq c_2$ implies that $c_{1H} \geq c_{2H}$ (and therefore if contract 1 is more efficient in the first period than it must be less efficient in the second) shifting types from branch $\{2\}$ to branch $\{1\}$ may actually reduce welfare.

Case 3.1. Differentiating (15) we have ("$a \propto b$" stands for "a it is proportional to b"):

$$\frac{\partial C(x,y)}{\partial y} \propto \left\{ \begin{array}{c} [C(\theta_H, c_1, s_1) - C(\theta_H, c_2, s_2)] + \\ \delta \sum_{j=H,L} \Pr(\theta_j \,|\theta_H)\, [C(\theta_j, c_{1j}, s_{1j}) - C(\theta_j, c_{2j}, s_{2j})] \end{array} \right\}.$$

Since the expected discounted profit of the inefficient firm is the same in the two contracts:

$$\frac{\partial C(x,y)}{\partial y} \propto \left\{ \begin{array}{c} [c_1 + \varphi(\theta_H - c_1) - c_2 - \varphi(\theta_H - c_2)] + \\ \delta \sum_{j=H,L} \Pr(\theta_j \,|\theta_H)\, [c_{1j} + \varphi(\theta_j - c_{1j}) - c_{2j} - \varphi(\theta_j - c_{2j})] \end{array} \right\}. \tag{19}$$

We have that:

$$[c_1 + \varphi(\theta_H - c_1) - c_2 - \varphi(\theta_H - c_2)] \geq 0, \tag{20}$$

since $c_1 \leq c_2 \leq c_{ND}(\gamma_2) \leq \theta_H - e^*$. Similarly since $c_{1H} \geq c_{2H} \geq \theta_H - e^*$ (see Case 2.2.1) and $c_{1L} = c_{2L} = \theta_L - e^*$, we have that the second term of (19) is also non-negative. Note moreover that by (18) if $c_1 = c_2$ then $c_{1H} = c_{2H} \geq \theta_H - e^*$: since $c_{1L} = c_{2L} = \theta_L - e^*$, then the two contracts are identical and can be merged without loss. So we can assume without loss of generality that $c_1 < c_2$, and (20) is strict. Therefore:

$$\frac{\partial C(x,y)}{\partial y} > 0.$$

We can now consider two cases.

Case 3.1.1. Assume that

$$\frac{\partial C(x,y)}{\partial x} \leq 0$$

then consider a marginal increase in x and a contextual marginal decrease in y such that the posterior probability after history $\{1\}$ is unchanged. The changes in x, y must reduce total welfare cost. If $c_{2H} \in [\theta_H - e^*, \widetilde{c}(\mu))$ or if the posterior after history $\{2\}$ is increased, then, by Lemma 1, a marginal change in x, y would not affect the renegotiation constraint: so the contract would remain renegotiation-proof. If $c_{2H} = \widetilde{c}(\mu)$ and the posterior decreases after the change, then we marginally reduce c_{2H} so that the contract remains renegotiation-proof after history $\{2\}$ too. In order to preserve the indifference condition at $t = 1$ between the two contracts for the inefficient firm, we can marginally increase c_2. Since c_{2H} is conditionally optimal, its change would have only a second order effect on welfare. Since, on the contrary $c_2 \leq c_{ND}(\gamma_2)$, we would (weakly) increase in welfare at $t = 1$. The total effect, therefore, strictly increases welfare, a contradiction.

Case 3.1.2. Assume now:

$$\frac{\partial C(x,y)}{\partial x} > 0.$$

Consider a marginal decrease in x. By Lemma 1 the contract remains renegotiation-proof after history $\{1\}$. If $c_{2H} \in [\theta_H - e^*, \widetilde{c}(\mu))$, then, by Lemma 1, a marginal change in x, y would not affect the renegotiation constraint: so the contract would remain renegotiation-proof. If $c_{2H} = \widetilde{c}(\mu)$, then we marginally reduce c_{2H} so that the contract remains renegotiation-proof after history $\{2\}$ too. In order to preserve the indifference condition at $t = 1$ between the two contracts for the inefficient firm, we can marginally increase c_2. Since c_{2H} is conditionally optimal, its change would have only a second order effect on welfare. Since, on the contrary $c_2 \leq c_{ND}(\gamma_2) < \theta_L - e^*$, we would not have a first order decrease in welfare at $t = 1$.

Case 3.2. Assume that $c_{1H} = \widetilde{c}(\mu_1)$, where μ_1 is the posterior probability of an efficient firm at $t = 2$ after history $\{1\}$. We can reduce c_{1H} by ε, for $\varepsilon > 0$ small and simultaneously increase c_1 so that the indifference condition between the two contracts is preserved for the efficient firm. The change in c_{1H} has zero marginal effect, since $\widetilde{c}(\mu_1)$ is conditionally optimal; on the contrary the increase in c_1 strictly increases welfare since $c_1 < c_{ND}(\gamma_1)$. So welfare strictly increases, a contradiction: it must be that $c_{1H} < \widetilde{c}(\mu_1)$. The principal's solution for x, y, therefore minimizes (15), which is linear in x, y under the linear constraint (16), which guarantees that the posterior in the second period after history $\{2\}$ is at least as large as μ_2. This program cannot have both x and y interior, so a double randomization cannot be optimal in this case either. ■

We now prove Lemma 3.

Lemma 3 *Without loss of generality, in the equilibrium that maximizes the principal's welfare there is no couple of contracts i and j such that with positive probability one is chosen by both types and the other only by the inefficient type.*

Proof. Assume by contradiction that, conditional on choosing contracts 1 or 2, the efficient type chooses contract 1 with probability $1 - x = 1$ and the other type chooses contract 1 with probability $y \in (0, 1)$ and contract 2 with probability $1 - y$. We consider two cases.

Case 1. Assume first that the efficient firm strictly prefers contract 1 to contract 2.

Step 1.1. Contract 2 must be efficient: i.e. both in period 1 and in period 2 after any possible history it prescribes an efficient cost level. Clearly the contract is efficient if at time 2 the firm's type is L. Assume it is not efficient at time 2 if the firm's type is H. In this case effort must be distorted below the efficient level, otherwise it is easy to verify that an increase in e would unambiguously increase welfare and respect all the constraints. Consider therefore a new contract in which we marginally reduce cost after history $\{2, H\}$: $c^*_{2,H} = c_{2,H} - \varepsilon$. This (weakly) increases the rent that the efficient type receives at $t = 2$ after contract 2. Let us reduce the monetary payment at $t = 1$ if contract 2 is chosen so that the inefficient firm's profit is constant. This new contract is renegotiation-proof, and does not affect incentives at $t = 1$ (since the efficient firm's incentive compatibility was not binding), and it is more efficient than the original: so welfare must be higher, a contradiction. The fact that the contract is efficient at $t = 1$ is proven in a similar way.

Step 1.2. If we marginally reduce y, contract 1 remains renegotiation-proof by Lemma 1 because the posterior increases after history $\{1\}$; and contract 2 is efficient, so renegotiation-proof. The firm's rents are unchanged after all histories and welfare is higher, since an efficient contract is chosen with higher probability. So the original contract could not be optimal, a contradiction.

Case 2. Assume now that the efficient firm does not randomize, but she is also indifferent between the two contracts. Assume that $C(0, y)$ is non-decreasing in y. Then if we marginally reduce y we do not reduce welfare. Contract 1 remains renegotiation-proof since the posterior that the firm is efficient is increasing. Contract 2 is chosen only by inefficient firms, so the posterior that the firm is efficient at $t = 2$ is equal to $1 - \alpha$, independent of y: therefore the change does not affect the renegotiation-proofness constraint either. Since $C(0, y)$ is linear in y, it must be globally non decreasing in y, so we can reduce y to zero without reducing welfare and respecting the renegotiation-proofness constraint. Assume therefore that $C(0, y)$ is strictly decreasing at y. If we marginally increase y, contract 2 remains renegotiation-proof. If $c_{1H} \in [\theta_H - e^*, \widetilde{c}(\mu_1))$, then a marginal change in y would not affect the renegotiation constraint: so the change strictly increases welfare without violating the renegotiation-proofness constraint, a contradiction. If $c_{1H} = \widetilde{c}(\mu_1)$, then we marginally reduce c_{1H} so that contract 1 remains renegotiation-proof too. This change has only a second order effect on welfare, while the decrease in y strictly increases welfare. Therefore the new contract is strictly better after the increase, again a contradiction. ■

VII.4 Proof of Proposition 2

Given Lemma 1 we only need to show that the cost function implied in the ex ante optimal contract is in $[\theta_H - e^*, \widetilde{c}(\mu(\alpha, x))]$ If $h_1 = L$, then in the ex ante optimal contract $c^{\mathcal{C}}_{LH} = \theta_H - e^*$ and the constraint is satisfied. After $h_1 = H$, since $c^{\mathcal{C}}_{HH}$ is always larger than $\theta_H - e^*$, the ex ante contract is renegotiation-proof if $c^{\mathcal{C}}_{HH} \leq \widetilde{c}(\mu(\alpha, x))$, which is true if and only if $\frac{2\alpha-1}{\alpha}\Gamma_0 \leq \Gamma_H$, which can be written as (8). ■

VII.5 Proof of Lemma 4

We first present the "If" part of statement, proving that any solution $c(h_t)$ of P_{III} solves the general problem. Then we present the "only if" part, that any solution $c(h_t)$ of $\mathcal{P}_{II}$ solves P_{III}.

"If" We first prove that the value of program P_{II} in which constraints IC_H, $IC_i\,(H)$ and $IR_i\,(L)$ for $i = H, L$ are ignored is identical to the value of program P_{III}: i.e. if there is a solution $\{c(h_t), s(h_t)\}$ of the first problem, then there exists a solution $\{c(h_t), s'(h_t)\}$ of the second problem which yields the same value to the principal. Then we show that the solution of P_{III} satisfies all the constraints of P_{II}. These two steps imply that the solution of P_{III} solves P_{II}.

It is easy to verify that in the relaxed problem IC_L, IR_H, $IR_H\,(H)$ and $IR_L\,(H)$ are binding following the same steps as in Proposition 1. Therefore we focus on $IC_i\,(L)$ for $i = H, L$. Consider first $IC_H(L)$. Assume that this constraint is not binding. As in Proposition 1, we now show that we can modify the contract respecting all the constraints and increase (at least weakly) welfare. Differently from Proposition 1, however, we need to respect the incentive compatibility constraint in the first period which guarantees that the efficient firm is, if necessary, willing to mix between the two options. Modify the contract as follows. Reduce s_{HL} by $\varepsilon > 0$ and increase s_H by $\delta\,(1-\alpha)\,\varepsilon$. Let us also reduce s_L by $\delta\,(2\alpha-1)\,\varepsilon$. All the other terms of the contract remain constant, moreover the probability x that contract $\{L\}$ is chosen in the first period by the efficient firm is also unchanged. This modification of the initial contract does not affect any incentive or participation constraint in the first or second period. In the first period, moreover, the efficient firm remains indifferent (if it was before the change) between the two contracts and therefore is still willing to mix with probability x. Since $\mu\,(\alpha, x)$ and c_{HH} are unchanged, the renegotiation constraint (7) continues to be satisfied as well. However in the new contract welfare is higher because the present value of expected payments is reduced by $\delta v\,(2\alpha-1)\,\varepsilon > 0$. The case of $IC_L\,(L)$ is analogous, however now the new contract with binding constraints is only weakly superior for the principal.

Given that IC_L and $IC_H\,(L)$ are binding, in $\mathcal{P}_{III}$ the efficient firm receives a rent:

$$\Phi\,(\theta_H - c_H) + (2\alpha - 1)\,\Phi\,(\theta_H - c_{HH})\,.$$

And given that the participation constraints are binding the inefficient firm receives zero. It follows that the optimal solution of $\mathcal{P}_{III}$ is characterized by (9). In the proof of Proposition 1, where we showed that the solution of the relaxed problem with binding constraints solves all the constraints of $\mathcal{P}_I$, we only used the fact that constraints of this problem are binding and that $c_L \le c_H$ and $c_{LL} \le c_{HL}$, both of these two properties are still satisfied in (9). Therefore all the steps in Proposition 1 can be replicated to prove that the solution of the relaxed problem with binding constraints solves all the constraints of $\mathcal{P}_{II}$.

"Only if" By the "If" part, the value of program $\mathcal{P}_{II}$ must be identical to the value of $\mathcal{P}_{III}$, and the value of $\mathcal{P}_{III}$ is identical to the value of $\mathcal{P}_{II}$ in which only the constraints IC_L, $IC_i(L)$ and $IR_i(H)$ for $i = H, L$ and (7) are considered. This implies that any solution $\{c(h_t), s(h_t)\}$ of $\mathcal{P}_{II}$ is also a solution of this relaxed problem. But then we know that there must be a solution $\{c(h_t), s'(h_t)\}$ which solves $\mathcal{P}_{III}$.

VII.6 Proof of Lemma 5

Assume that $c_{HH} < \widetilde{c}(\mu(\alpha, x))$. Then, since $x \in [0,1]$, program (9) can be represented by the Kuhn-Tucker Lagrangian:

$$\mathcal{L} = A(x, c_H) + \delta B(x, c_{HH}) + \xi[c_{HH} - (\theta_H - e^*)]$$

where $\xi \leq 0$ is the Kuhn-Tucker multiplier of the constraint $c_{HH} \geq (\theta_H - e^*)$ and $x \in [0,1]$. The optimal solution must satisfy the first order condition for c_{HH}: $\delta \frac{\partial}{\partial c_{HH}} B(x, c_{HH}) + \xi \geq 0$, which implies

$$\frac{\partial}{\partial c_{HH}} B(x, c_{HH}) \geq 0. \tag{21}$$

We must also satisfy the Kuhn-Tucker condition for x: $x \cdot \frac{\partial}{\partial x}[A(x, c_H) + \delta B(x, c_{HH})] = 0$, which implies that $x = 0$ since

$$\begin{aligned} \tfrac{\partial}{\partial x}[A(x, c_H) + \delta B(x, c_{HH})] &= v[(\varphi(\theta_L - c_H) + c_H) - (\varphi(e^*) + \theta_L - e^*)] \\ &+\delta v(1-\alpha)[(\varphi(\theta_H - c_{HH}) + c_{HH}) - (\varphi(e^*) + \theta_H - e^*)] > 0. \end{aligned}$$

As an immediate consequence, we have that $\frac{\mu(\alpha,x)}{1-\mu(\alpha,x)} = \frac{(1-\alpha)}{\alpha}$. The threshold α_1 is defined by $\frac{(1-\alpha_1)}{\alpha_1} = \Gamma_0\left(\frac{2\alpha_1 - 1}{\alpha_1}\right)$; therefore $\alpha > \alpha_1$ implies $\frac{\mu(\alpha,x)}{1-\mu(\alpha,x)} < \Gamma_0\left(\frac{2\alpha-1}{\alpha}\right)$, and

$$\begin{aligned} \widetilde{c}(\mu(\alpha, x)) &= e^* - \frac{\lambda}{1+\lambda}\frac{\mu(\alpha, x)}{1 - \mu(\alpha, x)}\Delta\theta \\ &> e^* - \Gamma_0\left(\frac{2\alpha - 1}{\alpha}\right)\Delta\theta = c_{HH}^C \end{aligned}$$

that is: $\widetilde{c}(\mu(\alpha, x))$ is larger than the ex ante optimal solution of the unconstrained problem c_{HH}^C. Since $B(x, c_{HH})$ is strictly convex in c_{HH}, this implies that $\frac{\partial}{\partial c_{HH}} B(x, c_{HH}) < 0$: but this contradicts (21). We therefore conclude that $c_{HH} = \widetilde{c}(\mu(\alpha, x))$. ■

VII.7 Proof of Proposition 3

Consider the derivative of $A(x, c_H) + \delta B(x, \widetilde{c}(\mu(\alpha, x)))$ evaluated at $\alpha = \alpha_1$. Regarding the first term, we have $\frac{\partial}{\partial x} A(x, c_H) = v[(\varphi(\theta_L - c_H) + c_H) - (\varphi(e^*) + \theta_L - e^*)] > 0$, and for the second term:

$$
\begin{aligned}
& \frac{\partial}{\partial x} B(x, \widetilde{c}(\mu(\alpha, x))) && (22)\\
= & \left[\frac{\partial}{\partial \widetilde{c}(\mu(\alpha, x))} B(x, \widetilde{c}(\mu(\alpha, x))) \frac{\partial}{\partial x}\widetilde{c}(\mu(\alpha, x)) + \frac{\partial}{\partial x} B(x, \widetilde{c}(\mu(\alpha, x)))\right] && (23)\\
= & (1+\lambda)\left\{\begin{array}{c} [\alpha(1-v) + xv(1-\alpha)] \cdot \\ \left[\begin{array}{c} (1-\varphi'(\theta_H - \widetilde{c}(\mu(\alpha, x)))) \\ +\frac{\lambda}{1+\lambda}\Gamma_0\left(\frac{2\alpha-1}{\alpha(1-v)+xv(1-\alpha)}\right)\Phi'(\widetilde{c}(\mu(\alpha, x))) \end{array}\right] \cdot \frac{\partial}{\partial x}\widetilde{c}(\mu(\alpha, x)) \\ +v(1-\alpha)[(\varphi(\theta_H - \widetilde{c}(\mu(\alpha, x))) + \widetilde{c}(\mu(\alpha, x))) - (\varphi(e^*) + \theta_H - e^*)] \end{array}\right\}
\end{aligned}
$$

where $\varphi'(\cdot)$ and $\Phi'(\cdot)$ are the first order derivatives. Lets now evaluate this expression at $\alpha = \alpha_1$. By definition of α_1, $\frac{(1-\alpha_1)}{\alpha_1} = \Gamma_0\left(\frac{2\alpha_1 - 1}{\alpha_1}\right)$; therefore:

$$
\begin{aligned}
& \frac{\partial}{\partial x} B(x, \widetilde{c}(\mu(\alpha_1, x)))\\
= & (1+\lambda)\left.\left\{\begin{array}{c} [\alpha(1-v) + xv(1-\alpha)] \cdot \\ \left[\begin{array}{c} 1-\varphi'(\theta_H - \widetilde{c}(\mu(\alpha, x))) \\ +\frac{\lambda(1-\alpha)}{\alpha(1+\lambda)}\frac{\alpha}{\alpha(1-v)+xv(1-\alpha)}\Phi'(\widetilde{c}(\mu(\alpha, x))) \end{array}\right] \frac{\partial}{\partial x}\widetilde{c}(\mu(\alpha, x)) \\ +v(1-\alpha)[(\varphi(\theta_H - \widetilde{c}(\mu(\alpha, x))) + \widetilde{c}(\mu(\alpha, x))) - (\varphi(e^*) + \theta_H - e^*)] \end{array}\right\}\right|_{\alpha=\alpha_1}.
\end{aligned}
$$

But we have that

$$
\begin{aligned}
& 1-\varphi'(\theta_H - \widetilde{c}(\mu(\alpha, x))) + \frac{\lambda(1-\alpha)}{\alpha(1+\lambda)}\frac{\alpha}{\alpha(1-v)+xv(1-\alpha)}\Phi'(\widetilde{c}(\mu(\alpha, x)))\\
\geq & 1-\varphi'(\theta_H - \widetilde{c}(\mu(\alpha, x))) + \frac{(1-\alpha)}{\alpha}\frac{\lambda}{(1+\lambda)}\Phi'(\widetilde{c}(\mu(\alpha, x))) = 0, \ \ \forall x \in [0,1].
\end{aligned}
$$

where the first inequality follows from the fact that $\frac{\alpha}{\alpha(1-v)+xv(1-\alpha)} < 1$ and $\Phi'(\widetilde{c}(\mu(\alpha, x))) < 0$; and the second equality follows from the fact that, by the definition of $\widetilde{c}(\mu)$, the first order condition of the ex post optimal contract given that the posterior is Γ_H is zero at $\widetilde{c}(\mu(\alpha, x))$. It follows that $\frac{\partial}{\partial x}[A(x, c_H) + \delta B(\widetilde{c}(\mu(\alpha, x)))] > 0$ for any $x \in [0,1]$ at $\alpha = \alpha_1$. Since the objective function is continuous with continuous derivative, there must be a

$$
\alpha_2 \equiv \inf\left\{\alpha \in (\alpha_1, 1] \left| \frac{\partial}{\partial x}[A(0, c_H) + \delta B(0, \widetilde{c}(\mu(\alpha, 0)))] < 0\right.\right\}
$$

as required such that $x = 0$ is a corner solution in $(\alpha_1, \alpha_2]$. ■

VII.8 Proof of Proposition 4

Since $\alpha > \alpha_1$, $\frac{\partial}{\partial \widetilde{c}(\mu(\alpha,x))} B\left(x, \widetilde{c}\left(\mu\left(\alpha, x\right)\right)\right) < 0$ at $x = 0$. As it can be seen from, (13) as $\alpha \to 1$, $\frac{\partial}{\partial x} B\left(x, \widetilde{c}\left(\mu\left(\alpha, x\right)\right)\right) < 0$ at $x = 0$. It follows that there is an $\widehat{\alpha} < 1$ and a $\overline{\delta}$ such that $\alpha > \widehat{\alpha}$ and $\delta > \overline{\delta}$ implies $\frac{\partial}{\partial x}\left[A\left(0, c_H\right) + \delta B\left(\widetilde{c}\left(0\right)\right)\right] < 0$, and the optimal x must be positive.

VII.9 Proof of Proposition 5

It is convenient to represent the marginal effect of x on second period costs as:

$$B_x(x,\alpha) = \frac{\partial}{\partial x} B\left(x, \widetilde{c}\left(\mu\left(\alpha, x\right)\right)\right) = D(x,\alpha)E(x,\alpha) + F\left(x,\alpha\right) \tag{24a}$$

where (see equation (13)):

$$\begin{aligned} D(x,\alpha) &= \frac{\eta \Delta\theta^2 \lambda^2}{1+\lambda}\left[\left(1-\alpha\right) - \left(1-x\right) v\alpha\right] && (25) \\ E(x,\alpha) &= \frac{v\left(1-v\right)\left(2\alpha-1\right)}{\left[vx\left(1-\alpha\right)+\alpha\left(1-v\right)\right]^2} \\ F\left(x,\alpha\right) &= v\left(1+\lambda\right)\left(1-\alpha\right)\left[\begin{array}{c} \left(\varphi\left(\theta_H - \widetilde{c}\left(\mu\left(\alpha,x\right)\right)\right) + \widetilde{c}\left(\mu\left(\alpha,x\right)\right)\right) \\ -\left(\varphi\left(e^*\right)+\theta_H - e^*\right) \end{array}\right] \\ &= \frac{\eta v\left(1+\lambda\right)\left(1-\alpha\right)}{2}\left(\frac{vx\alpha + \left(1-v\right)\left(1-\alpha\right)}{vx\left(1-\alpha\right)+\alpha\left(1-v\right)}\frac{\lambda}{1+\lambda}\Delta\theta\right)^2. && (26) \end{aligned}$$

Assume now that there are (at least) two solution x_1 and x_2. Since $\frac{\partial}{\partial x} A\left(x, c_H\right)$ is a positive constant independent of x, by the first order condition

$$\frac{\partial}{\partial x}\left[A\left(x_i, c_H\right) + \delta B\left(x_i, \widetilde{c}\left(\mu\left(\alpha, x_i\right)\right)\right)\right] = 0 \ \forall i = 1,2$$

we have that $B_x(x_i,\alpha) < 0$ for $i = 1,2$. Moreover by the second order condition $B_{xx}(x_i,\alpha) \geq 0 \ \forall i = 1,2$. By continuity, therefore, there exists a $\overline{x} \in (x_1, x_2)$ such that $B_x(\overline{x},\alpha) < 0$ and $B_{xx}(\overline{x},\alpha) \leq 0$ (see for example Figure 4.A, note that $\overline{x}$ may even coincide with x_2). However

$$B_x(\overline{x},\alpha) = D(\overline{x},\alpha)E(\overline{x},\alpha) + F\left(\overline{x},\alpha\right) < 0 \Rightarrow D(\overline{x},\alpha) < 0$$

since both $E(\overline{x},\alpha)$ and $F\left(x,\alpha\right)$ are strictly positive. But then:

$$B_{xx}(\overline{x},\alpha) = D_x(\overline{x},\alpha)E(\overline{x},\alpha) + D(\overline{x},\alpha)E_x(\overline{x},\alpha) + F_x\left(x,\alpha\right) > 0$$

since $E_x(x,\alpha)$ is obviously negative, $D_x(x,\alpha) > 0$ and

$$F_x(x,\alpha) = \frac{\partial}{\partial \widetilde{c}}\left[\varphi\left(\theta_H - \widetilde{c}\left(\mu\left(\alpha,x\right)\right)\right) + \widetilde{c}\left(\mu\left(\alpha,x\right)\right)\right]\frac{\partial}{\partial x}\widetilde{c}\left(\mu\left(\alpha,x_i\right)\right) > 0$$

since $\widetilde{c}\left(\mu\left(\alpha,x\right)\right) > \theta_H - e^*$, φ is convex and $\frac{\partial}{\partial x}\widetilde{c}\left(\mu\left(\alpha,x_i\right)\right) > 0$. But this is a contradiction: we conclude that there can be only a unique solution. ■

VII.10 Proof of Proposition 6

If $B_x(x,\alpha) > 0$, then we have a corner solution with $x^*(\alpha) = 0$, which is obviously non decreasing in α. Therefore it is sufficient to prove that $B_x(x,\alpha) \leq 0$ implies $B_x(x,\alpha) = \frac{\partial}{\partial x} B\left(x, \widetilde{c}\left(\mu\left(\alpha,x\right)\right)\right)$ is monotonically non-increasing in α. It is useful to distinguish two cases.

Case 1: $vx \leq 1 - v$. As in Proposition 7 we can represent the marginal effect of x on second period costs as:

$$B_x(x,\alpha) = \frac{\partial}{\partial x} B\left(x, \widetilde{c}\left(\mu\left(\alpha,x\right)\right)\right) = D(x,\alpha)E(x,\alpha) + F\left(x,\alpha\right) \tag{27a}$$

which are defined in (25). Therefore we have that

$$B_{x\alpha}(x,\alpha) = \frac{\partial^2}{\partial x \partial \alpha} B\left(x, \widetilde{c}\left(\mu\left(\alpha,x\right)\right)\right) = D(x,\alpha)E_\alpha(x,\alpha) + D_\alpha(x,\alpha)E(x,\alpha) + F_\alpha\left(x,\alpha\right). \tag{28a}$$

Observe that $E_\alpha(x,\alpha) \propto vx\alpha + (1-\alpha)(1-v) > 0$. We have

$$\begin{aligned} F_\alpha\left(x,\alpha\right) &\propto -\frac{vx\alpha + (1-v)(1-\alpha)}{\left(vx(1-\alpha) + \alpha(1-v)\right)^2} \left\{ \begin{array}{c} \left[vx\alpha + (1-v)(1-\alpha)\right] \\ +2\frac{(1-\alpha)\left[(1-v)^2 - (vx)^2\right]}{vx(1-\alpha)+\alpha(1-v)} \end{array} \right\} \\ &\propto -\left\{ \begin{array}{c} \left[vx\alpha + (1-v)(1-\alpha)\right]\left[vx(1-\alpha) + \alpha(1-v)\right] \\ +2(1-\alpha)\left[(1-v)^2 - (vx)^2\right] \end{array} \right\}. \end{aligned} \tag{29}$$

and therefore it is negative. The first term in the parenthesis is always positive, therefore the result follows from the fact that if $vx < 1-v$, then the last term is positive too. Since $D(x,\alpha)$ and $D_\alpha(x,\alpha)$ are negative, it follows that $B_{x\alpha}\left(x,\alpha\right) < 0$.

Case 2: $vx > 1 - v$. This time it is convenient to decompose $B_x(x,\alpha)$ as (see (13)):

$$B_x(x,\alpha) = \frac{\eta \Delta\theta^2 \lambda^2}{1+\lambda} vG(\alpha) \cdot H\left(\alpha\right)$$

where:

$$\begin{aligned} G(\alpha) &= \left[vx(1-\alpha) + \alpha(1-v)\right]^{-2}; \text{ and} \\ H(\alpha) &= \left(1 - \alpha - (1-x)\alpha v\right)(1-v)(2\alpha - 1) + \frac{1-\alpha}{2}\left[xv\alpha + (1-v)(1-\alpha)\right]^2. \end{aligned}$$

First note that if $H(\alpha) > 0$ then $B_x(x,\alpha) > 0$, and we have a corner solution in which $x\left(\alpha,\delta\right) = 0$, so $\frac{\partial x(\alpha,\delta)}{\partial \alpha} = 0$. Assume therefore that $H(\alpha) < 0$. We can write the second order derivative of $H(\alpha)$ as:

$$\begin{aligned} H_{\alpha\alpha}\left(\alpha\right) &= -4(1-v)\left[1 + (1-x)v\right] - \left[xv\alpha + (1-v)(1-\alpha)\right]\left[xv - (1-v)\right] \\ &\quad - \left[xv - (1-v)\right]\left[xv(2\alpha - 1) + 2(1-\alpha)(1-v)\right]. \end{aligned}$$

which is clearly non-positive given $vx > 1 - v$. Observe that

$$H(\frac{1}{2}) = \frac{1}{4}\left[xv + (1-v)\right]^2 > 0.$$

If $H(\alpha) > 0$ for any $\alpha \in \left[\frac{1}{2}, 1\right]$, then we have a corner solution $x(\alpha, \delta) = 0\ \forall \alpha$, and the result is proven. If there exists a $\alpha' \in \left(\frac{1}{2}, 1\right]$ such that $H(\alpha') < 0$, then there must be a α'' such that $H(\alpha'') > 0$ and $H_\alpha(\alpha'') < 0$. Strict concavity of $H(\alpha)$ however implies that $H_\alpha(\alpha) < 0$ for any $\alpha > \alpha''$. So $H(\alpha) < 0$ implies $B_{x\alpha}(x, \alpha) < 0$. ■

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