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ABSTRACT

On Sustainable Pay-As-You-Go Systems*

An unfunded Social Security system faces a major risk, sometimes referred to as 'political risk'. In order to account properly for this risk, the paper considers a political process in which the support to the system is asked from each newborn generation. The analysis is conducted in an overlapping generations economy that is subject to macroeconomic shocks. As a consequence, the political support varies with the evolution of the economy. The impact of various factors – intra-generational redistribution, risk aversion, financial markets, governmental debt – on the political sustainability of a pay-as-you-go system is discussed.

JEL Classification: C72 and C78
Keywords: intra-generational redistribution, overlapping generations, pay-as-you-go, political economy, risk and social security system

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1 Introduction

In most developed countries, the first pillar of the social security system is unfunded, financed through compulsory contributions. This compulsory aspect allows the system to implement redistribution within a generation. Indeed, the choice of a system is not only determined by market forces, but also by political considerations. The interaction with financial markets puts however some constraints on political choices : stocks and bonds can also be used by individuals to transfer income into their retirement period. The recent controversies on Social Security are partly due to this interaction : until recently, the returns on the stock market appeared to many young individuals much more attractive than those expected from Social Security. The collapse of the stock market, the unexpected rise in life expectancy, and the decline in fertility rates all make also clear that uncertainty at a macro level cannot be neglected in an assessment of social security systems. The aim of this paper is to analyze the political choice of a system while taking into account these three important features : the intra-generational redistribution performed by social security, the financial opportunities faced by individuals, and the inherent uncertainty on the evolution of the economy.

As many countries are experiencing, an unfunded system faces a major risk, sometimes referred to as "political risk". Whereas the future generations, as future contributors, are concerned with the current design of the system, they do not participate in this choice. The risk that they do not agree to contribute is difficult to assess, especially in an uncertain environment. To address this issue, this paper considers a political process in which each new-born generation must support the current social security design. Hence, the process involves no commitment for the future. Sustainability defines an equilibrium for such political process. In some economies, no pension system with positive contribution rates is sustainable[2]. The main goal of this paper is to identify the conditions that favor the existence of a sustainable system in a risky environment under various financial environments.

Social security systems can vary in several dimensions. Two are most important : the redistribution that is performed *within* a generation through social security and the "size" of the system (as measured by the contribution rate or the share of social security expenditure of GDP). In most countries, the size of social security has much evolved overtime (actually has increased up to now). In contrast, regarding the redistribution aspect, a striking fact is that most systems can still be classified as they were when set up. Some are mainly "bismarckian" or earnings-related, with individual's benefits linked

[2]From now on, by sustainable system will mean a system in which contributions are positive.

to previous contributions and earnings, while others are "beveridgean", with almost "flat" pension benefits. Indeed, the intra-generational redistribution is not much discussed.

To account of these stylized features, and admittedly to make the analysis tractable, this paper assumes that the redistribution performed by a system is specified by a benefit rule. More precisely, I consider pay as you go systems (payg) that levy contributions on workers through a tax bearing on labor earnings. The collected amount is shared among the retirees in function of their past contributions according to some fixed benefit rule (for instance, it is shared in proportion to individuals' contribution as in a pure bismarckian system). Given the benefit rule, the current contribution rate is subject to political approval.

The analysis is conducted in an overlapping generations economy. Individuals live for two periods, and may differ in their preferences and productivity/income. At each period, the economy is subject to shocks on labor productivity, the rate of return on capital, and the population growth rate. Political support is modeled by requiring that a young individual, called decisive, agrees on the current Social Security tax (in a sense to be made precise). Most results do not qualitatively depend on who the decisive voter is, but to fix the idea he (or she) may be thought of as a median voter, or as the poorest agent. The decisive individual's agreement is conditioned on the current information on realized shocks. Crucially, it also depends on the expectations on pension benefits, which are affected both by the (exogenous) evolution of the economy and the (endogenous) future contribution rates. The sustainability condition accounts for this forward looking aspect : pension benefits are correctly expected. In other words, under sustainability, the current generation supports the system knowing that future generations will also support the system even though they are not committed to do so.

It should be stressed that contributions rates, and pension benefits, are allowed to vary with the state of the economy. From a theoretical perspective, it should be clear that there is no reason to exclude contingent contribution rates. Also, this assumption is in line with the recent reforms that implement the so-called *notional* pay as you go systems. In Sweden for instance, annuities are indexed on growth according to an explicit and agreed upon formula. Thus, our approach may shed some light on how a system must be designed so as to cope with uncertainty and be politically viable.

Sustainability is influenced by the available financial opportunities. I first study an economy without any other investment opportunity than the physical technology. The existence of a sustainable system is characterized and some comparative statics exercises are performed. The risk characteristics of the shocks and risk aversion turn out to play a significant role. In particular, under some conditions, the more risk averse the decisive voter is, the more likely a sustainable system to exist. This result supports the view that risk sharing considerations may favor the sustainability of an unfunded social security systems. The intuition is that, by providing benefits that are linked to labor earnings, a payg system is a tool for improving risk sharing across generations, risk sharing that cannot be provided by markets. To ascertain the role of risk sharing *across* generations, the analysis is extended to a setup in which full opportunities of exchanges *within* a generation are available through short term complete markets.

That risk sharing of macroeconomic risks may favor a payg system (in top of the well known argument of dynamical efficiency) has been shown from an *ex ante* efficiency point of view (see e.g. Bohn (1998)). A planner can design the system so as to improve risk sharing and increase the *ex ante* welfare of all generations. From a political perspective, the planned system may run into serious difficulties. At the time a new born generation is asked to contribute, information on the current state of the economy is available. On the basis of this information[3], it may altogether refuse to contribute the amount that was planned previously. According to our results, intergenerational risk sharing nevertheless promotes the sustainability of a payg system even if information and the associated political constraints are taken into account.

The interaction between governmental debt and sustainable payg is examined. In a two period lived overlapping generations economy, an unfunded system and governmental rolled-over debt perform similar intergenerational transfers, as pointed out by Diamond (1965) : the newly issued debt is bought by young individuals, and the collected amount is used to reimburse the mature debt, which is held by the old generation. Indeed, in a riskless economy, because the returns on social security and debt are both comparable, sustainability imposes very strong conditions. In a stochastic framework, debt payoffs are not *a priori* comparable with (i.e. proportional to) pension benefits, which are indexed on current wages. The availability of rolled-over governmental debt, however, is shown to severely restrict the possibilities of redistribution of a sustainable payg at an equilibrium without short sales constraints (theorem 3).

The political approach to Social Security, initiated by Browning (1975), aims to explain the factors determining the size of a system and to assess the impact of redistribution. Most works have been conducted in a deterministic setup using a median voter model, as surveyed in Galasso and Profeta (2002). The median voter model does not explain well the stylized fact that the size of Social Security and the level of within-cohort redistribution would be negatively correlated across countries. The reason is that the median voter is

[3]This is the *interim* perspective. Similar difficulties arise in decentralizing *ex ante* optimal allocations, see Demange (2002).

an individual with median income, at least if individuals are no credit constrained[4]. Since the median income is lower than the average one, more redistributive systems, favorable to poor people, get a larger support under majority. This motivates the choice of leaving undetermined the decisive voters' characteristics.

Strategic "pension games", as first proposed by Hammond (1975), also address the sustainability of intergenerational transfers. Pension games have a prisoner dilemma flavor : if each generation has only the choice between defecting or performing a prescribed transfer, and uses no punishment, the dominant strategy (non cooperative) equilibrium is defection. To support intergenerational transfers, punishment may help (but is not necessarily very appealing in that context), as well as reputation (see among others Kandori (1992), Cooley and Soares (1999)). Finally the availability of institutions that are costly to change also promote the viability of intergenerational transfers (Esteban and Sakovicks (1993)). Instead, this paper excludes commitment while assuming away the strategic aspects raised by the just mentioned literature : each individual, in particular the decisive voter, does not take into account any equilibrium or signaling effect.

The paper is organized as follows. Section 2 sets up the model and defines the concept of sustainable payg as used here. Section 3 studies the existence and properties of a sustainable payg without any financial securities. The analysis is carried out in a stationary set up in which the evolution of the economy is described by a Markov state. Section 4 provides some extensions; it considers partially funded systems and analyzes the impact of financial short term markets. The interaction between sustainability and governmental debt is studied in Section 5. Proofs are gathered in the final Section.

2 The model

2.1 The economy

I consider the simplest overlapping generations economy that allows for macroeconomic uncertainty and heterogeneity in individuals income. There is a single good that can be either consumed or invested and the investment and labor productivities are exogenous.

Individuals Each generation is composed with I-types of individuals indexed by i, $i = 1, \ldots, I$ that grow at the same rate, which is the population growth rate. Individuals live for two periods, supply a fixed quantity of labor when young, and retire when old.

[4] If there are credit constraints, poor people may join the "rich" and favor a low contribution rate, in which case the median voter is an individual whose income is larger than the median one, as shown by Casamatta, Cremer and Pestieau (2000).

Each i is characterized by a productivity parameter, θ^i, and a utility function U^i. The characteristic determines the individual's wage in conjunction with the macroeconomic variable as explained below. Function U^i represents i's preferences over life time consumption plans[5] $(c_y, \tilde{c}_o)$, where c_y and c_o denote consumption when young and old respectively (individuals are not altruistic). Each utility function can be of the Von Neumann Morgenstern type or of the recursive type, that is, dropping index, of the form $E[u(c_y, \tilde{c}_o)]$ or $u(c_y) + \delta u(v^{-1}(Ev(\tilde{c}_o))$. Functions u and v are defined over positive consumption levels, concave, strictly increasing in each argument and continuously differentiable. Furthermore, to avoid corner solutions, Inada conditions are assumed : $\lim_{c \to 0} U_c^{'}(c_y, c_o) = \infty$ for $c = c_y$ or c_o.

Macroeconomic variables. The economy is subject to exogenous macro-economic shocks on the rate of return on capital, ρ, the growth rate of population, γ, and the average labor productivity ω.

The good may be transferred from one period to the next through a linear random technology with rate of return $\tilde{\rho}$: s_t units invested at date t yield $\tilde{\rho}_{t+1} s_t$ in period $t+1$.

The rate of growth of the population between dates $t-1$ and t is denoted by $\tilde{\gamma}_t$. Owing to the inelastic supply of "young" individuals, the ratio workers to retirees at t is also equal to γ_t. The assumption that $\tilde{\gamma}_t$ is possibly perceived as random at date $t-1$ is far from unrealistic. In most developed countries, especially european, the decline in the fertility rate and the sharp increase in life expectancy were not expected to be so severe. Also labor participation is rather unpredictable, owing to changes in behavior or legislation affecting the choice of retirement date, or women working decision, or the number of working hours for instance[6].

Labor productivity[7] at date t, denoted by w_t, determines the level of labor earnings at that date : The wage income of a worker of characteristic θ, for short a $\theta-$worker, at t is given by θw_t. Normalizing the distribution of the parameter θ across workers to 1, w_t stands for the *average* wage income at date t.

The law of the macro economic variables $(\tilde{\gamma}_t, \tilde{w}_t, \tilde{\rho}_t)$ is exogenous, and supposed to be known by individuals. In particular, young individuals correctly expect the distribution of the shocks at the subsequent period conditional on the observed realizations. We shall conduct most of the analysis in a stationary framework, assuming that $(\tilde{\gamma}_t, \tilde{w}_t, \tilde{\rho}_t)$ follows a first order Markovian process (assumption made precise later on). In such a framework,

[5]We denote by $\tilde{x}$ a random variables and by x its realization.

[6]This point also suggests that the ratio workers to retirees is in part endogenous, sensitive to some policies, especially in Social Security. This aspect is not addressed here.

[7]Productivity growth can be handled with by considering γ_t as the growth of effective labor. Then w_t stands for the transitory shocks.

the last realization is sufficient for correct prediction.

Individuals' budget constraints. It is important to describe the constraints and risks faced by individuals. They are affected by Social Security and the available financial instruments.

We consider Social Security systems that are compulsory, levied through a tax bearing on labor income. Let τ_t be the Social Security tax at date t. Given the labor productivity shock, w_t, each $\theta-$worker contributes $\tau_t \theta w_t$ to the system.

In addition to the constant return to scale technology, individuals may have access to financial markets. Several cases will be considered (no security or contingent short term securities, or government bonds). At this point, we only need to specify the current securities price q_t and their payoffs next period $\tilde{a}_{t+1}$ (both are vectors if there are multiple securities).

A $\theta-$individual who expects next average pension benefits $\tilde{\pi}^a_{\theta,t+1}$ faces the budget constraints :

$$\begin{cases} c^y + s + q_t b = (1-\tau_t)\theta w_t, s \geq 0 \\ \tilde{c}^o = s\tilde{\rho}_{t+1} + b.\tilde{a}_{t+1} + \tilde{\pi}^a_{\theta,t+1} \end{cases} \tag{1}$$

In the first period, labor income is used to consume, to invest in the technology, to buy (or sell) financial assets, and finally to contribute $\tau_t \theta w_t$ to social security. In the second period, the individual retires and consumes all of his resources.

Different sources of risks affect income at retirement - risk on investment return, financial securities payoffs, and pension benefits. They determine young individuals' saving behavior and support to Social Security. A crucial point is to assess how expectations on pension benefits are formed. Quite naturally the return and risk characteristics of pensions are shaped by the design of the Social Security system. This paper considers unfunded systems (see however section 4.2 on mixed systems).

2.2 Social Security

In an unfunded Social Security system, the collected amount at any date is fully transferred to retirees. At date t, given the labor productivity shock, w_t, and the current tax level, τ_t, workers contribute on average $\tau_t w_t$. Therefore, if population has grown by the factor γ_t between $t-1$ and t, the per head *average* pension benefits are equal to

$$\pi_t = \gamma_t \tau_t w_t. \tag{2}$$

The pension benefits received by a particular retiree may differ from the average level π_t, except in a beveridgean system. In a bismarckian (also called purely contributive) system for instance, pension benefits are proportional to contributions, equal to $\theta\pi_t$ for a $\theta-$individual. More generally, a rule determines the pension benefits in relation to previous contributions. This rule is described here through the *redistributive factors*, $\mu(\theta)$, which give the distortion with respect to a bismarckian system. More precisely, the benefits $\pi_{\theta,t}$ received by a $\theta-$retiree are given by[8] :

$$\pi_{\theta,t} = \theta\mu(\theta)\pi_t = \theta\mu(\theta)\gamma_t\tau_t w_t. \tag{3}$$

The function μ is positive, nonincreasing (to describe redistribution), and satisfies $\sum_i \theta_i\mu(\theta_i) = 1$ so as to ensure budget balance.[9]

As said in the introduction, the redistribution of payg systems is not much discussed in most countries. Therefore, we shall assume that *the benefit rule, as specified by the redistribution factors μ, is taken as given and only the contribution rates τ are subject to political approval.* Contribution rates are likely to vary. Hence, at date t, the contribution rate at the following period, $\tilde{\tau}_{t+1}$, may be perceived as random, possibly correlated with the macro-economic variables that are realized at $t+1$. How are the contribution rates determined ?

2.3 Political support

Since a major risk faced by an unfunded system is that contributors refuse to pay, we shall assume that a political process determines contribution rates. A reasonable process must satisfy two requirements : first that each new generation is asked to support the system (if the current contribution rate is positive), second that no generation can choose the contribution rates that will apply in the future. Here, the support at a given date is described by the agreement of a "decisive" voter, who is a member of the new born generation. The decision process by which the decisive voter is chosen is supposed to be known (see for example the section in which the decisive voter is the median voter). As will be clear, no form of commitment is presumed : agreement is based on some expectations on the contribution rates that will be chosen by the future generations. The concept of sustainability precisely requires that the future generations will indeed agree to pay as was expected.

[8]The replacement ratio depends both on the redistribution factor and the contribution levels. If one considers that wages are adjusted so that at the end of his working period a θ-worker born at $t-1$ gets wages θw_t, the replacement ratio is equal to $\gamma_t\tau_t\mu(\theta)$.

[9]In Casamatta *et alii* for instance, a system combines a bismarckian and beveridgean systems in fixed proportions α and $1-\alpha$, which gives : $\mu(\theta) = (\alpha\theta + (1-\alpha))/\theta$.

Political support at a given date : Decisive individual's agreement The decisive voter is asked whether he would like to change the level of the *current* positive contribution rate, with the understanding that the level of his benefits next period would be changed in the same proportion. The answer to this question primarily depends on how much he expects to receive from the system. Thanks to the benefit formula (3), an individual only needs to form expectations on the distribution of average per head pension benefits next period. Omitting dates and decisive individual's index, let $\widetilde{\pi^a}$ denote these expectations. A decisive θ-individual contributes $\tau\theta w$ and expects to receive pension benefits equal to $\theta\mu(\theta)\widetilde{\pi^a}$. He faces the question of whether he would rather contribute $\lambda\tau\theta w$ and receive $\lambda\theta\mu(\theta)\widetilde{\pi^a}$, for some λ different from 1.

The scale level λ modifies the successive budget constraints as given by (1). Therefore, through optimal behavior, a decisive θ-voter anticipates by choosing λ the indirect utility level $V(\lambda)$ defined by :

$$\begin{cases} V(\lambda) = \max_{c_y,c_o,s\geq 0,b} \quad U(c^y,\tilde{c}^o) \\ c^y + s + qb = (1-\lambda\tau)\theta w, \\ \tilde{c}^o = s\tilde{\rho} + b\tilde{a} + \lambda\theta\mu(\theta)\widetilde{\pi^a} \end{cases} \tag{4}$$

This leads to the following definition.

Definition 1 *Given a strictly positive contribution rate τ and expectations on the level of average per head pensions $\widetilde{\pi^a}$ next period, a θ-individual agrees on τ if the indirect utility function $V(\lambda)$ defined by (4) is maximized*[10] *at $\lambda = 1$.*

Sustainability The decisive voter does not choose the contribution rate for the next generation. Instead his behavior is based on some expectations on this rate, via the expectations on average pensions. To be sustainable, future generations must agree on the expected rate. More precisely, consider a process of contribution rates $(\tilde{\tau}_t)$. Assume that individuals have correct expectations on the dynamics of the economy, that is, on the stochastic evolution of both the exogenous variables (macro-economic shocks) and endogenous ones (such as contribution rates, prices, and decisive voters' characteristics). An individual must expect the average per head benefit at $t+1$ to be distributed as $\tilde{\gamma}_{t+1}\tilde{\tau}_{t+1}\tilde{w}_{t+1}$ by budget balance (2). This gives the following definition :

[10] The indirect utility function V is concave in λ. Hence, agreement is equivalent to the first order condition $V'(1) = 0$. Thanks to the envelope theorem, this condition writes as

$$\tau w U'^{d}_{c_y} - \mu(\theta)[U'^{d}_{c_o}\widetilde{\pi^a}] = 0 \tag{5}$$

(marginal utility is evaluated at the optimal consumption plan for $\lambda = 1$).

Definition 2 *A process of contribution rates* $(\tilde{\tau}_t)$ *is said to be sustainable if at each date* t *whenever* $\tau_t > 0$ *the decisive voter agrees on* τ_t*, his expectations on next average pension benefits being distributed as* $\tilde{\gamma}_{t+1}\tilde{\tau}_{t+1}\tilde{w}_{t+1}$*, given the available information at* t *on the dynamics of the economy.*

Therefore, under sustainability, a voter agrees on the current rate at t if next generation is expected to contribute according to $\tilde{\tau}_{t+1}$, and generation $t+1$ will indeed agree to contribute that much, in each possible state, given that generation $t+2$ is expected to contribute according to $\tilde{\tau}_{t+2}$, and so on. Sustainability does not presume any commitment device.

Remark. Consider a deterministic stationary economy, in which each exogenous variable and decisive voters' characteristic are constant over time. In such an economy, Casametta et al. consider a voting process with commitment in which a constant rate is decided forever. In that setup, the equilibrium outcomes of the processes with or without commitment coincide[11]. The reason is that if $\tau_t = \tau_{t+1} = \tau$ for any t, choosing the scale level λ amounts to choose the rate $\lambda\tau$ that applies at both the current and next periods. Therefore the agreement of the decisive voter is like asking him to choose *the* contribution rate that applies to his generation *and* to the future one. Furthermore, since the environment is not changing overtime, if agreement holds at one period, it always holds.

Consider now a stochastic environment. On one hand, contribution rates should not be restricted to be identical to the current ones. On the other hand, if the current decisive voter could choose freely the future contribution rate without any link with the current rate, he would simply set the next period rate at its maximum, which does not make much sense. This drawback is avoided by linking the successive rates through the proportional factor λ.

2.4 Stationary framework

I shall focus on a stationary framework. This framework is the simplest one that extends a deterministic setup (in which there is a unique state). First, the shocks are assumed to follow a first order Markovian process with a unique invariant distribution. The realization of the shock $e = (\tilde{\gamma}, \tilde{w}, \tilde{\rho})$ at some date is called the *state* of the economy. For simplicity, the state space E is assumed to be finite. The transition probability from state e to state $e_+ = (\gamma_+, w_+, \rho_+)$ is denoted by $\Pr(e_+|e)$. Second, I restrict attention to

[11] This is true for constant contribution rates over time. In a riskless economy, a stationary payg system amounts to money with constant price. It is well known that other equilibria, say with sunspots, may exist. For an analysis of this type of equilibrium see Azariadis and Galasso (2002).

situations in which all economic variables, social security and individuals decisions, are stationary, that is are time invariant functions of the state[12] :

- a payg system specifies how the contribution rate is adjusted in function of the state of the economy : it is described by $\boldsymbol{\tau} = (\tau(e))$. Thus, contribution rate at time t is given by : $\tau_t = \tau(e_t)$,

- individual i's consumption and portfolio decisions when young are described by functions of the state e at birth, c_y^i, s^i, and b^i, and when old by a function c_o^i of both states e and e_+ that realize during i's lifetime,

- the decisive voter characteristic in state e is denoted by $\theta^d(e)$, and to simplify notation we write $\mu^d(e) = \mu(\theta^d(e))$.

The function $\boldsymbol{\tau}$ can be interpreted as the pension system designed and by the social security institution. Individuals, who are assumed to know the design $\boldsymbol{\tau}$, can derive the distribution of next contribution rates, hence of their pensions, conditional on the current state. The design is sustainable if the decisive voter agrees in each possible state. This means that the design of a sustainable system may give some insight on how a system must be designed to cope with uncertainty so as to be politically viable.

We are now ready to analyze the conditions under which a sustainable system exists. Most of the analysis considers contribution rates that are strictly positive in each state (written as $\boldsymbol{\tau} > 0$) except in section 4.1. We start with the situation in which no securities are available.

3 Sustainability without financial assets

In this section, individuals can only invest in the technology, which yields the exogenous rate of return $\tilde{\rho}$. At a stationary payg described by contribution rates $\boldsymbol{\tau}$, the average per head benefit is equal to $(\gamma w \tau)(e_+)$ if e_+ is realized. This gives the following budget constraints for a young $\theta-$worker born in state e who forms correct expectations :

$$\begin{cases} c_y + s = (1 - \tau(e))\theta w(e) \\ c_o(e_+) = \rho(e_+)s + \theta\mu(\theta)(\gamma\tau w)(e_+) \text{ for each } e_+ \end{cases} \tag{6}$$

Given $\boldsymbol{\tau}$, each individual chooses to invest some non negative amount s so as to maximize his utility conditional on the current state e under (6). The market for the consumption

[12] Most results extend if the state of the economy at some date is enlarged by including some components of the past history, as in Demange and Laroque (1999). To describe the feasibility conditions, the state necessarily includes the current values e of the shocks. Everything goes through provided that the state space, say E^*, is finite and that there is a unique invariant distribution the support of which is the whole space E^*.

good is automatically balanced.

To state the agreement conditions, consider the optimal consumption plan of the decisive voter born in state e. At this plan, let $\mathrm{mrs}^d_{\boldsymbol{\tau}}(e, e_+)$ denote the marginal rate of substitution between current consumption and consumption next period contingent on state[13] e_+. Using that agreement is equivalent to the first order condition on the scale level (see footnote 10) gives :

The payg system $\boldsymbol{\tau} > 0$ *is sustainable if (and only if) in each state* e *:*

$$\tau(e)w(e) = \mu^d(e) \sum_{e_+} mrs^d_{\boldsymbol{\tau}}(e, e_+)(\gamma\tau w)(e_+). \tag{7}$$

Note that without a payg system (taking $\boldsymbol{\tau} = 0$) there are no transfers at all between generations : an "autarky" equilibrium is obtained. It turns out that the existence of a sustainable payg system is related to some property at this autarky equilibrium.

3.1 Sustainable payg : a characterization

It is convenient to introduce the positive matrix $A_{\boldsymbol{\tau}}$ defined by

$$A_{\boldsymbol{\tau}}(e, e_+) = \mu^d(e)\gamma(e_+)\frac{w(e_+)}{w(e)}\mathrm{mrs}^d_{\boldsymbol{\tau}}(e, e_+) \tag{8}$$

Conditions (7) that characterize a (strictly positive) sustainable payg $\boldsymbol{\tau}$ can be put in matrix form :

$$\boldsymbol{\tau} = A_{\boldsymbol{\tau}}\boldsymbol{\tau} \tag{9}$$

(the system is not linear since $\boldsymbol{\tau}$ affects the matrix).

Theorem 1 *Consider a stationary economy without financial assets. Let* $A_{\mathbf{0}}$ *be the matrix of the decisive voters' weighted intertemporal rates of substitution defined in (8) computed at the autarky equilibrium. There is a sustainable payg system if and only if the largest eigenvalue of* $A_{\mathbf{0}}$ *is above* 1.

[13] With an expected utility function u for instance,

$$\mathrm{mrs}^d_{\boldsymbol{\tau}}(e, e_+) = \frac{u'^d_{c_o}(c_y(e), c_o(e, e_+))}{E[u'^d_{c_y}(c_y(e), c_o(e, e_+))|e]} Pr(e|e_+).$$

Note that the transition probability is included in the marginal rate.

In a deterministic economy the eigenvalue condition is easy to understand. At autarky, individuals can only invest in the technology to get some income at retirement. Therefore, the marginal rate of substitution is equalized to the inverse of ρ, so that the unique element of matrix $A_{\mathbf{0}}$ is equal to $\mu^d\gamma/\rho$. Hence the eigenvalue condition simply says that, from the point of view of the decisive voter, the (risk-less) return of the payg dominates the technology return[14]. In a stochastic environment, returns cannot be compared so easily : they are risky, and are moreover endogenous through the variations in contribution rates. To understand and interpret the eigenvalue condition, consider the introduction of small contribution rates "in the direction" of $\boldsymbol{\tau}_0$, i.e. of the form $\epsilon\boldsymbol{\tau}_0$ for $\epsilon > 0$. Consumption levels are changed but, from the envelope theorem, taking ϵ small enough, their impact is negligible. The impact on the decisive voters' utility born in state e is therefore proportional to

$$\mu^d(e)\sum_{e_+}\text{mrs}_0^d(e,e_+)\gamma(e_+)\tau_0(e_+)w(e_+) - \tau_0(e)w(e),$$

which, up to $w(e)$, is equal to $(A_{\mathbf{0}}\boldsymbol{\tau}_0 - \boldsymbol{\tau}_0)(e)$. Therefore small contribution rates in the direction of $\boldsymbol{\tau}_0$ make decisive voters better off than at autarky *whatever the state at birth* if the inequality $A_{\mathbf{0}}\boldsymbol{\tau}_0 > \boldsymbol{\tau}_0$ is satisfied. From a well known result on positive matrices, such direction exists if and only if the matrix $A_{\mathbf{0}}$ has an eigenvalue larger than 1. This readily gives an interpretation of the eigenvalue condition : there is a payg system that is Pareto improving for decisive voters over autarky. Such a payg however has few chances to be sustainable (that is $\boldsymbol{\tau}_0$, or any system proportional to it, does not satisfy $\boldsymbol{\tau} = A_{\boldsymbol{\tau}}\boldsymbol{\tau}$ in general). The proof of existence, under the eigenvalue condition, relies on a fixed point argument[15].

Why is the eigenvalue condition necessary ? Since there are no financial assets, changing the level of the payg only affects the distribution of endowments and has no price effect. Therefore, whatever $\boldsymbol{\tau}$ the decisive voter can choose to get his autarky consumption level simply by setting λ equal to 0. This readily implies that whatever the state at birth, the decisive voter is better off at a positive sustainable system $\boldsymbol{\tau}$ than at the autarky equilibrium. By a concavity argument, decisive voters are also made better off by the introduction of small contribution rates "in the direction" of $\boldsymbol{\tau}$, and the previous argument applies (i.e. $A_{\mathbf{0}}\boldsymbol{\tau} > \boldsymbol{\tau}$).

[14]A positive sustainable rate maximizes $U^d(\theta^d w(1-\tau), \mu^d\theta^d\gamma\tau w)$. If $\mu^d\gamma < \rho$, the decisive voter prefers to invest in the technology, hence he chooses $\tau = 0$.

[15]Formally one has to show that if A_0 has an eigenvalue larger than 1, then there is a positive $\boldsymbol{\tau}$ that satisfies $\boldsymbol{\tau} = A_{\boldsymbol{\tau}}\boldsymbol{\tau}$, i.e. that is an eigenvector with eigenvalue 1 of $A_{\boldsymbol{\tau}}$.

The characterization in theorem 1 is useful to understand the determinants that favor sustainability. Increasing any element of the matrix $A_{\mathbf{0}}$ increases the maximal eigenvalue. Hence, not surprisingly, whether a sustainable payg exists depends positively on the decisive voter redistributive factor, and on the population growth rate (note that the marginal rates of substitution at autarky are independent of γ). Before examining the impact of risk aversion, Example 1 illustrates that sustainability is also much related to the underlying stochastic process.

Example 1 This simple example illustrates the influence of the stochastic process on sustainability. There are only two states, corresponding to low and high population growth rates for instance, γ_l and γ_h. The technology return is riskless, equal to ρ. Take also $\mu^d(e) = 1$ (the analysis is directly transposed to a constant μ^d since only the ratio μ^d/ρ matters). At autarky, individuals save in the technology, since otherwise they would not consume when old. Therefore marginal rates of substitution are equalized to the inverse of ρ whatever state at birth. This yields the matrix $A_{\mathbf{0}}$:

$$A_{\mathbf{0}} = \frac{1}{\rho} \begin{pmatrix} \gamma_h p_h & \gamma_l(1-p_h) \\ \gamma_h(1-p_l) & \gamma_l p_l \end{pmatrix}$$

in which p_h denotes $Pr(\gamma_h|\gamma_h)$ the probability of a future high growth rate conditional on a current high rate, and similarly for p_l. The existence of a sustainable payg is much related to the stochastic process[16]. To see this, consider the two extreme cases of either persistent or switching states.

(1) States are persistent if p_h and p_d are close to 1. The sustainability condition is close to $\gamma_h > \rho$. Notice that this condition is compatible with $\gamma_l < \rho$. The design of a sustainable payg is as follows. The contribution rate in the low state is positive, but much smaller than in the high state. Accordingly, if the current state is low, whereas the return on the payg can be smaller than the technology return (if $\gamma_l < \rho$) with high probability p_l, this is compensated by the large return obtained if next state is high. If the current state is high, the return on the payg is larger than ρ with a large probability.

(2) In the case of a perfect switch between the states (p_h and p_l are null), a sustainable payg exists if

$$\sqrt{\gamma_h \gamma_l} > \rho. \tag{10}$$

Assume that $\gamma_l < \rho$. Since in state h next population growth is γ_l, a first thought could be that the return of the payg for the decisive voter is γ_l, lower than investment return, which

[16] The maximal eigenvalue writes : $\frac{\mu^d}{\rho}[\mathcal{S}/2 + \sqrt{(S/2)^2 - det})]$ where $S = \gamma_h p_h + \gamma_l p_l$ and $det = \gamma_h \gamma_l (1 - p_h - p_l)$

would suggest that no payg is sustainable. A sustainable system may nevertheless exist if γ_h is large enough so that condition (10) is fulfilled. To understand why, consider small contribution rates in the "direction" of $(\tau_0(h), \tau_0(l)) = (\sqrt{\gamma_l}, \sqrt{\gamma_h})$. We show that they improve the decisive voter's welfare over autaky in each state. In state h, the return on the payg, $\gamma_l \tau_0(l)/\tau_0(h)$, is equal to $\sqrt{\gamma_h}\sqrt{\gamma_l}$., which is larger ρ if (10) holds : even though next population growth is smaller for sure than the technology return, the current contribution is sufficiently low compared to the next one so that the payg becomes attractive. Similarly in state l, the decisive voter faces the return $\gamma_h \tau_0(h)/\tau_0(l)$, which is also equal to $\sqrt{\gamma_h}\sqrt{\gamma_l}$, larger than ρ.

This case shows that a "myopic" comparison at a given date between population growth and investment return, even riskless, is not sufficient to assess sustainability. It should be clear that allowing contribution rates to be adjusted with the economic state is essential.

3.2 The impact of risk aversion

To analyze the role of risk aversion, it is convenient to assume voters' utility functions to be recursive, given by

$$U(c_y, \tilde{c}_o) = u(c_y) + \delta u(v^{-1}(Ev(\tilde{c}_o)) \tag{11}$$

Intertemporal substitution, which depends on u, is independent of risk aversion, which depends on v. This independence allows us to perform comparative statics on the sustainability condition with respect to risk aversion only. Indeed, under some additional assumptions the maximal eigenvalue can be computed and interpreted[17].

Proposition *Assume the decisive voter's utility to be recursive with v homothetic, states to be independent across periods, and $\mu^d(e)$ to be constant across states equal to μ^d. Then*

1. *the maximal eigenvalue of $A_{\mathbf{0}}$ is given by*

$$\mu^d E[\frac{\tilde{\gamma}}{\tilde{\rho}} \frac{v'(\tilde{\rho})\tilde{\rho}}{E[v'(\tilde{\rho})\tilde{\rho}]}] \tag{13}$$

[17] Define $\hat{c}_o$ as the certain equivalent of $\tilde{c}_o$ under the Von Neumann Morgenstern utility $v : v(\hat{c}_o) = Ev(\tilde{c}_o)$. The utility level derived from $(c_y, \tilde{c}_o)$ can be written as $u(c_y) + \delta u(\hat{c}_o)$, the intertemporal utility derived from the risk-free consumption plan $(c_y, \hat{c}_o)$. The marginal rate of substitution writes as

$$mrs(e, e_+) = \delta \frac{u'(\hat{c}_o(e))}{u'(c_y(e))} \frac{v'(c_o(e, e_+))\Pr(e_+|e)}{v'(\hat{c}_o(e))} \tag{12}$$

where $\hat{c}_o(e)$ is the certain equivalent of $\tilde{c}_o(e, e_+)$ knowing current state e.

2. *Assume in addition that $E[\tilde{\gamma}|\rho]/\rho$ is non-increasing, The eigenvalue increases with risk aversion : the more risk averse the decisive individual, the more likely a sustainable payg to exist.*

From *1*, with a risk neutral individuals, the eigenvalue is equal to $\mu^d E[\tilde{\gamma}]/E[\tilde{\rho}]$, which means that the criteria for sustainability in a riskless economy is simply transposed by considering expected values for population growth and investment return. With a risk averse individuals, the maximal eigenvalue is μ^d times the expected value of the ratio $\tilde{\gamma}/\tilde{\rho}$ under a "risk-neutral" probability that accounts for risk aversion (since the function $v'(\rho)\rho/E[v'(\tilde{\rho})\tilde{\rho}]$ defines a density). Increasing risk aversion distorts this density by putting more weight on low values of ρ. Note that the additional condition ($E[\tilde{\gamma}|\rho]/\rho$ non-increasing in ρ) is very plausible. It holds under independence between $\tilde{\rho}$ and $\tilde{\gamma}$ for instance. That, under this condition, risk aversion favors the existence of a sustainable payg is quite easy to interpret : The introduction of a payg provides pension benefits that allow for a partial hedge against investment risk, which encourages risk averse individuals to favor it.

Example 2 To illustrate further this result, take an isoelastic function u with constant elasticity β and a constant relative risk aversion function v with coefficient α :

$$u(c) = \frac{1}{1-\beta}c^{1-\beta} \text{ and } v(c) = \frac{1}{1-\alpha}c^{1-\alpha}$$

Assume also that $(ln\tilde{\rho}, ln\tilde{\gamma})$ is a gaussian vector. Easy computation[18] yields

$$E[\frac{\tilde{\gamma}}{\tilde{\rho}}\frac{v'(\tilde{\rho})\tilde{\rho}}{E[v'(\tilde{\rho})\tilde{\rho}]}] = \frac{E[\tilde{\gamma}]}{E[\tilde{\rho}]}\exp\alpha[\ \text{var}(ln\tilde{\rho}) - \ \text{cov}(ln\tilde{\rho}, ln\tilde{\gamma})]. \tag{14}$$

The exponential term reflects the impact of risk aversion. *It may substantially affect a criteria based on the simple comparison between expected growth and expected return*[19]. If $\tilde{\rho}$ and $\tilde{\gamma}$ are independent for instance. the correcting term, $\exp\alpha[\ \text{var}(ln\tilde{\rho})]$, may be far from negligible. As expected, the larger the risk aversion coefficient and the more risky the investment return, the more chance a sustainable payg to exist.

3.3 Welfare properties

Interim optimality is a relevant criterion to compare feasible allocations. An allocation

[18]Setting $X = \gamma\rho^{-\alpha}$ and $Y = \rho^{1-\alpha}$, one has to compute $E\tilde{X}/E\tilde{Y}$. Since each variable is log normal, this ratio is given by $\exp[Eln\tilde{X} - Eln\tilde{Y} + 0.5\ \text{var}(ln\tilde{X}) - 0.5\ \text{var}(ln\tilde{Y})]$, and the result follows.

[19]The assumption that $\frac{E[\tilde{\gamma}|\rho]}{\rho}$ is non-increasing is precisely satisfied if the argument in the exponential is positive.

interim dominates another one if it gives a larger expected utility at birth whatever the state[20]. As said previously, the impact of a sustainable payg on decisive voters' welfare is always positive : a decisive voter is better off than at the autarky equilibrium whatever the state at birth. Accordingly, without intra-generational heterogeneity, a sustainable payg leads to a Pareto improvement over autarky[21]. Therefore, in a representative agent set up, the interest of intergenerational transfers should not be judged on the sole basis of expected return on investment and expected population growth, as made clear by expression (14) for instance.

With intra-generational heterogeneity, given the limited tax tools and the absence of financial markets, there is little hope that a Pareto improvement over autarky can be reached. A natural question is whether individuals who earn less than the decisive voter also benefit from the set up of a sustainable payg. Without uncertainty *nor* liquidity constraints, the answer is positive. To see this, note that a θ-individual's utility level increases with his lifetime income, which is equal to $\theta w[(1-\tau)+\tau\mu(\theta)\gamma/\rho]$. Sustainability means that $\mu(\theta^d)\gamma > \rho$, which ensures that the payg system increases the lifetime income of any individual with lower wage. The argument does not extend to the uncertainty framework. The basic reason is that individual lifetime income cannot be defined. More precisely, the pension benefits that a young $\theta-$worker expects to receive cannot be valued, or equivalently be hedged[22] (except in the unlikely situation of a perfect correlation with $\tilde{\rho}$).

Under specific conditions such as identical homothetic preferences for instance, a sustainable payg system indeed makes every worker whose wage is lower than the decisive one better off. The argument is the following. At autarky, consumption and saving decisions of the various individuals are proportional. This is no longer true if a payg is implemented : because of the redistribution factor, individuals do not face the same investment opportu-

[20] Notice that an *interim* Pareto improvement is *ex ante* Pareto improvement as well. For various developments on interim optimality, in particular on the impact of the state space and the stationarity assumption see for example Demange and Laroque (1999), Chattopadhyay and Gottardi (1999), and Demange (2002).

[21] Without intra-generational heterogeneity, we fall back on an economy with a representative agent set up. Then, not only the eigenvalue condition ensures that intergenerational transfers lead to an interim Pareto improvement, but also that an *interim* optimal allocation is obtained at a rational expectations equilibrium with money with positive value or with voluntary contributions to a payg system, see Peled (1984), Demange and Laroque (1999) for instance.

[22] Whenever an investor expects to receive a random income at a future date that can be perfectly hedged through a portfolio of traded assets, one can assume that he receives all his wealth at the initial date, equal to the sum of his current income and the current value of the portfolio that duplicates his future income.

nities. In particular their preferred contribution rates may differ. However, if the system is no more nor less beneficial to a θ-individual than to the decisive voter, $\mu(\theta) = \mu(\theta^d)$, this $\theta-$individual would choose the same contribution rate as the decisive voter. Hence he would be made better off over autarky. If he benefits more from redistribution than the decisive voter, that is if $\mu(\theta) > \mu(\theta^d)$, he can only be made better off.

4 Extensions

The previous analysis relies on some assumptions that can be relaxed quite easily. First let us mention that the model applies to a median decisive voter. In line with our approach, consider a voting game in which individuals vote on the scale λ on the contribution rates. As previously, voters take the system, redistribution factor and contribution rates, as given. A median voter exists. Old individuals trivially prefer the maximal scale level defined by $\lambda\tau = 1$. As for young individuals, their preferences over scale levels are represented by the indirect utility function V defined in (4), which is concave hence single-peaked : a median voter exists. If, whatever the situation, the median voter is an individual with median income, the previous results directly apply by taking θ^d equal to the median value of θ. Of course there is not much hope to get such a result without assuming identical preferences. Furthermore, even under identical preferences, the lack of financial markets as the presence of credit constraints may prevent this too be true. An interesting empirical question would be to derive the characteristics of this median voter, and to assess the impact of risk.

I now consider three extensions : the possibility of null contribution rates, partially funded systems, and the impact of financial securities.

4.1 Incomplete payg

The question of whether a payg system can vanish from time to time and be set up again can be answered by extending the analysis. Given the importance of this question, our answer is not satisfactory. It should however be compared with the theoretical literature, which has mostly focused on deterministic economies (see however Boldrin and Rustichini (2000)). Without uncertainty, a payg cannot be rationally expected to disappear : otherwise, by a simple backward induction argument it would never have been accepted in the first place. In a stochastic set up, the backward induction argument does not work, as long as the decisive voter expects to receive pension benefits with some positive probability.

To study more precisely sustainability for a non null payg, recall that the decisive voter agreement is needed only if he is asked to contribute. Thus, given a non zero vector $\boldsymbol{\tau}$, the agreement conditions (7) must be satisfied only in the states fin which contributions are positive, i.e. $\tau(e) > 0$. Let E_p be the set of states in which contributions are positive, and $A^d_{\mathbf{0}|\mathbf{E_p}}$ be the sub-matrix of $A_\mathbf{0}$ obtained by keeping the entries associated with the states in E^p only. The following corollary to theorem 1 is easily obtained.

Corollary *Consider a stationary economy without financial assets. There is a sustainable stationary payg in which contributions are strictly positive for the states in* E_p *if and only if the maximal positive eigenvalue of the sub-matrix* $A^d_{\mathbf{0}|\mathbf{E_p}}$ *is larger than* 1.

Let us illustrate this corollary with example 1, and consider a system that asks for contributions in state h only. The diagonal term readily gives the sustainability condition : $\mu^d \gamma_h p_h > \rho$. The condition is easy to interpret : if contributions are positive in the high state only, not only the population growth must be large enough compared to the investment return (as in a deterministic framework), but also the probability for a contributor to get a pension, here the probability p_h, must be sufficiently high. As this example also shows, complete and incomplete sustainable payg may or may not simultaneously exist (consider persistent and switching states).

4.2 Mixed systems

The analysis has so far been restricted to fully unfunded systems. A natural question is whether a more flexible pension system, partly funded, would significantly change the political support to the system. Since individuals have the opportunity of investing into the technology, one can think that they can undo the funded part, so that the previous analysis applies to the unfunded part. This is not true for two reasons. First, in the absence of financial markets, short selling the technology return is impossible. As a consequence, individuals may end up with a (compulsory) investment through the funded part of the system that is too large with respect to their needs. Second, even funded, the system may nevertheless perform some redistribution.

It is quite easy to incorporate into the analysis mixed systems in which a fixed proportion of the contributions is invested into the technology. Let a system be said $\alpha-$mixed if the proportion α of the contributions is invested at each period and redistributed at the subsequent period. Budget balance then gives that the level of the per head average pension is equal to

$$\pi_t = (1-\alpha)\gamma_t \tau_t w_t + \alpha \tau_{t-1} w_{t-1} \rho_t. \tag{15}$$

The first term on the right hand side corresponds to the part of the contributions directly paid to the retirees, namely the unfunded part, and the second term to the payoff from the amount invested the previous period.

One easily shows that a sustainable $\alpha-$mixed system surely exists if a fully unfunded one exists. The intuition is the following one. Since the decisive voter is assumed not to be harmed by redistribution, he obtains a return on the funded part of a mixed system that is not smaller than by direct investment. This element favors his support to the system[23]. The design of a sustainable system however depends on α.

4.3 Complete markets

In the previous section, we have shown that, without financial markets, a payg system may be valuable because it provides some risk sharing opportunities within a generation. A natural question is whether this result is driven by the lack of financial markets. Indeed owing to differences in revenues and tastes -in particular in preferences for present consumption and attitudes towards risk- young individuals within a generation may benefit from exchanging among themselves. Without financial markets, no exchanges are possible so that a payg system may be a substitute to them. If this drives its sustainability, the pension system does not play the "natural" role it is assigned to, and organizing intra-generational risk sharing is a better solution. To get some insight on this question, we consider here financial markets.

Securities are one period lived : they are traded at one date in exchange of a (final) payoff at the subsequent date. As a consequence, only individuals within a generation exchange these securities. A natural benchmark is the situation in which the maximal opportunities of borrowing and lending and exchange of future risks are possible : mar-

[23] Arguments similar to those used for an unfunded system show that a sustainable $\alpha-$mixed system is characterized by

$$\tau(e)w(e) = \mu^d(e) \sum_{e_+} \mathrm{mrs}^d_{\boldsymbol{\tau}}(e, e_+)[(1-\alpha)(\gamma\tau w)(e_+) + \alpha\tau(e)w(e)\rho(e_+)].$$

and that such a system exists if for some positive $\boldsymbol{\tau}_0$

$$\tau_0(e)w(e) < \mu^d(e) \sum_{e_+} \mathrm{mrs}^d_{\mathbf{0}}(e, e_+)[(1-\alpha)(\gamma\tau_0 w)(e_+) + \alpha\tau_0(e)w(e)\rho(e_+)] \text{ in each sate } e.$$

Recall that if an unfunded system exists, a positive $\boldsymbol{\tau}_0$ that satisfies $\boldsymbol{\tau}_0 < A_{\mathbf{0}}\boldsymbol{\tau}_0$. This vector satisfies the above inequalities because at autarky individuals invest, so that $\sum_{e_+} \mathrm{mrs}^d_{\mathbf{0}}(e, e_+)\rho(e_+) = 1$, and furthermore $\mu^d(e) \geq 1$. This proves the claim that a sustainable $\alpha-$mixed system exists whenever an unfunded one does. The impossibility of short selling does not matter for the existence condition, owing to the fact that individuals invest at autarky.

kets are (short-term) complete. To be more precise, in each period, there are enough financial securities so that any consumption plan contingent on next state can be reached through an appropriate portfolio (the spanning condition). As a consequence, equilibrium determines a set of Arrow Debreu prices.

More formally, given a state e, let $q(e_+|e)$ be the contingent price in terms of present good to be given in exchange of one unit of the good next period if state e_+ is realized. If the contribution rates are $\boldsymbol{\tau}$, a young $\theta-$worker faces the budget constraints :

$$\begin{cases} c_y + s + q(e_+|e)b(e_+) = (1-\tau(e))\theta w(e) \\ c_o(e_+) = \rho(e_+)s + b(e_+) + \theta\mu(\theta)(\gamma\tau w)(e_+) \text{ for each } e_+ \end{cases} \tag{16}$$

Each individual chooses to invest and consume so as to maximize his utility conditional on the current state e. An equilibrium is obtained if the aggregate demand for the contingent securities is null, that is if $\sum_i b^i(e_+) = 0$. Using standard arguments there is an equilibrium for any $\boldsymbol{\tau}$ with $0 \leq \tau(e) < 1$. At equilibrium, intertemporal rates of substitution for all individuals are equalized to contingent prices. Denoting by $q_{\boldsymbol{\tau}}(e_+|e)$ the contingent price vector given $\boldsymbol{\tau}$: $\text{mrs}^i(e, e_+) = q_{\boldsymbol{\tau}}(e_+|e)$ for each i.

The agreement conditions can be stated in terms of these state prices. The reason is that, thanks to complete markets, an individual's lifetime income can be defined as the value of all incomes evaluated at the contingent prices, and that welfare increases with it. Here lifetime income for a $\theta-$individual is given by [24]

$$[w(e) - \tau(e)w(e) + \mu(\theta)\sum_{e_+} q_{\boldsymbol{\tau}}(e_+|e)(\gamma\tau w)(e_+)]\theta, \tag{17}$$

which is equal to the sum of the wage and the net value of the pension system, that is the value of the future benefits less the contribution. By choosing the scale λ, the decisive voter multiplies by λ the net value he derives from the pension system. The agreement condition in state e follows :

$$\tau(e)w(e) = \mu^d(e)\sum_{e_+} q_{\boldsymbol{\tau}}(e_+|e)(\gamma\tau w)(e_+). \tag{18}$$

[24] More precisely, the budget equations (16) are equivalent to the single lifetime budget constraint which says that the value of consumption levels and net investment are equal to the value of incomes :

$$c_y + s + \sum_{e_+} q_{\boldsymbol{\tau}}(e_+|e)(c_o(e_+) - \rho(e_+)s) = W$$

where W is the lifetime income defined by (17).

The sustainability of $\boldsymbol{\tau}$ requires these conditions to be met in all states. This gives in matrix form : $\boldsymbol{\tau} = C_{\boldsymbol{\tau}}\boldsymbol{\tau}$ in which $C_{\boldsymbol{\tau}}$ is the positive square matrix defined by

$$C_{\boldsymbol{\tau}}(e, e_+) = \mu^d(e)\gamma(e_+)\frac{w(e_+)}{w(e)}q_{\boldsymbol{\tau}}(e_+|e). \tag{19}$$

Since the intertemporal rates of substitution are equalized to contingent prices, $C_{\boldsymbol{\tau}}$ is, in a complete markets setup, analogous to matrix $A_{\boldsymbol{\tau}}$ defined by (8). Finally, note that the autarky equilibrium without intergenerational transfers, ($\boldsymbol{\tau} = 0$), is affected by financial markets, implying that $A_{\mathbf{0}}$ and $C_{\mathbf{0}}$ differ except if financial markets are useless (such as under identical homogeneous preferences).

Theorem 2 *Consider a stationary economy in which young individuals have access to a complete set of contingent securities. Assume the equilibrium to be unique in each state e for each $\boldsymbol{\tau}$, $\boldsymbol{\tau} < 1$ and let $C_{\mathbf{0}}$ be the matrix of weighted contingent prices defined in (19) computed at the autarky equilibrium.*

If the largest eigenvalue of $C_{\mathbf{0}}$ is above 1 *there is a sustainable payg.*

If the autarky equilibria with and without markets coincide, the converse is true : a sustainable payg with complete markets exists only if the largest eigenvalue of $C_{\mathbf{0}}$ is above 1.

We cannot assert that a decisive voter is surely better off at a sustainable payg than at the autarky equilibrium with complete markets : he evaluates the system at the contingent prices without taking into account equilibrium effects[25] (that is that $q_{\boldsymbol{\tau}}$ may differ from $q_{\mathbf{0}}$). This explains why the eigenvalue condition is only sufficient.

The optimal exchange of risks allowed by complete markets does not affect drastically the results in the sense that redistribution is not constrained (in contrast with the case of rolled over governmental debt considered in next section). A comparison of theorems 2 and 1 helps us to assess the importance of financial markets on sustainability. Sustainability may be possible without financial markets but fail with complete ones only if the largest eigenvalue of $A_{\mathbf{0}}$ is above 1 but that of $C_{\mathbf{0}}$ is less than 1. To illustrate this point, consider an economy as in section 3.2, with intertemporal independent shocks and recursive preferences. Without complete markets, the eigenvalue at autarky is given by (13), that is equal to $\mu^d E[\frac{\tilde{\gamma}}{\tilde{\rho}}\frac{v'(\tilde{\rho})\tilde{\rho}}{E[v'(\tilde{\rho})\tilde{\rho}]}]$ in which v represents the decisive voter's attitudes towards risk.

[25] On the other hand, he is surely better off than at the autarky without financial assets, since he can choose a null scale level and not to trade : this explains why the eigenvalue condition is necessary when both autarky equilibria coincide.

With complete markets, the autarky equilibrium depends on the risk aversion of all individuals. For simplicity consider only two individuals. If the non decisive individual has the same utility v than the decisive one, no trade occurs at autarky. The sustainability conditions are identical with or without financial markets. If instead the non decisive individual is risk neutral complete markets allow the risk averse individual to be him to be fully insured at a fair price : the decisive voter gets a constant consumption level him to fully insure whatever the realized shock. Easy computation now gives that the eigenvalue at autarky is equal to $\mu^d E[\tilde{\gamma}]/E[\tilde{\rho}]$. Most likely, this value is smaller than the corresponding one without markets, (if population growth and technology return are not too much positively correlated, $E[\tilde{\gamma}]/E[\tilde{\rho}] < E[\frac{\tilde{\gamma}}{\tilde{\rho}} \frac{v'(\tilde{\rho})\tilde{\rho}}{E[v'(\tilde{\rho})\tilde{\rho}]}]$). Therefore a sustainable system has more chances to exist without financial markets. The intuition should be clear. As explained in the previous section, the risk profile of the payg system provides an insurance against the risky investment return. This is why a sustainable payg may exist even though $\mu^d E[\tilde{\gamma}] > E[\tilde{\rho}]$. If however insurance against investment risk is provided for free by contemporaneous risk neutral individuals, then only expected returns matter.

5 Sustainability with rolled over debt

This section analyzes sustainability when governmental debt is issued and rolled over. Governmental debt also performs intergenerational transfers. These transfers depend on the price of debt, which is endogenous. The price effect, and its interaction with a payg, makes the analysis more complex than previously.

The government is assumed to issue at each date bonds that mature at the subsequent date. At t, a unit of bond promises a (possibly random) revenue next period denoted by $\tilde{a}_{t+1}$. The total amount of debt is normalized at each date so that the number of shares is equal to the size of the young generation. Equivalently the number of units of bonds per young is equal to one. Furthermore debt is rolled over, without using any tax instrument[26]. Therefore, at any point in time, the payments to bondholders are covered by the newly issued debt. This gives the balance equation at time t : $n_{t-1}a_{t-1} = n_t q_t$, in which n_t is the current size of the population, and q_t the price of one unit of bond. Dividing by the population size of generation $t-1$, this yields

$$a_t = \gamma_t q_t. \tag{20}$$

This equality says that *without taxation the payoff promised by debt is constrained to be equal to the future price of debt multiplied by population growth.* Hence, in a stochastic

[26] This is in contrast with the analysis of Gale (1994).

economy, debt cannot promise a sure payoff (since the price of debt is determined by equilibrium forces, there are few chances for $\gamma_t q_t$ to be risk-less).

We focus on stationary situations in which both the contribution rate τ_t and the price of debt q_t are time invariant functions of the current state e, respectively $\boldsymbol{\tau} = (\tau(e))$ and $\boldsymbol{q} = (q(e))$. We look for an equilibrium in which expectations are correct : the young agents form correct expectations on the distribution of the future state $\tilde{e}_+$, that is on wage, population growth and gross interest rate, conditional on the current state, and they infer the distribution of the endogenous variables, debt price and contribution rate.

Given the price function $\boldsymbol{q}$ for debt, b units of debt yield $b\gamma(e_+)q(e_+)$ in state e_+ to its owner according to (20). So, under correct expectations, the budget constraints of a young $\theta-$worker born in state e are given by :

$$\begin{cases} c_y + s + bq(e) = (1-\tau(e))\theta w, \;\; s \geq 0, \, b \geq \underline{b} \\ \tilde{c}_o = s\rho(e_+) + b\gamma(e_+)q(e_+) + \theta\mu(\theta)(\gamma\tau w)(e_+) \text{ each } e_+ \text{ in } E \end{cases} \tag{21}$$

in which $\underline{b}$ is set equal to $-\infty$ if there are no short sale constraints, and to 0 if there are. This yields the following definition.

Definition 3 *: An equilibrium with rolled over debt and sustainable payg is defined by debt prices $\boldsymbol{q} = (q(e))$, contribution rates $\boldsymbol{\tau} = (\tau(e))$ both nonnegative, and consumption plans $c_y^i(e), (c_o^i(e, e_+))_{e_+ \in E}$ for each i, each e in E, satisfying the following conditions in each state e :*

1. *for each i, $c_y^i(e), (c_o^i(e, e_+))_{e_+ \in E}$ maximizes $E[u^i(c_y, \tilde{c}_o)|e]$ over the constraints (21) for $\theta = \theta^i$*
2. *the bond market clears : $\sum_i b^i(e) = 1$*
3. *the decisive voter agrees on $\tau(e)$, his expectations on average next period pensions being given by $(\gamma\tau w)(\tilde{e}_+)$ conditional on e*

Condition *1* states standard rational behavior under correct expectations. By condition *2* the market for debt clears since the total number of shares is equal to the population size. By Walras law, the market for the good clears as well. Conditions *1* and *2* together say that given contribution rates $\boldsymbol{\tau}$ a rational expectations equilibrium with debt obtains. Condition *3* states the decisive voters' agreement, still under correct expectations.

Note that a sustainable positive payg without financial assets, as considered in section 3, gives rise to an equilibrium in which debt has no value. We are interested here in situations in which both contribution rates and debt prices are positive (if any). If debt has a value, the transfer of consumption good from young to old agents is endogenous through the (non zero) price of debt : it is equal on average per young to $(\tau w + q)(e)$ in state e.

Theorem 3 *Assume that debt is rolled over and that there are no short sales constraint on debt ($\underline{b} = -\infty$). At an equilibrium with positive sustainable payg and positive debt prices :*

1. *the decisive voter is not subsidized, i.e. $\mu^d(e) = 1$ in each state,*
2. *the returns of the payg and debt are identical for any non subsidized individual : for some positive k, $q(e) = k\boldsymbol{\tau}(e)w(e)$ in each state e.*

Thus, without short sales constraints, the returns of the two infinite lived assets, debt and payg, must be equalized on average at an equilibrium with sustainable payg. The result is trivial in a deterministic set up : from the point of view of the decisive agent, payg and debt offer both risk-free returns, respectively γ and $\mu\gamma$ which should coincide to avoid arbitrage opportunities. The result is much more surprising in a stochastic set up in which there is a priori room for two assets with different returns. That debt and payg returns are equalized from the point of view of the decisive voter does not preclude redistribution, even if quite limited : workers with lower income than the decisive one may be subsidized, and those with larger income be taxed.

6 Concluding remarks

The model is clearly too simple in some aspects. For example, a payg system provides retirees with an annuity, thereby insuring them against the risk of living old[27]. Making insurance compulsory avoids the usual problems encountered in markets with asymmetric information. As documented by various studies, the premium associated to the longevity risk is roughly 5% (see Brown, Mitchell, and Poterba (2001). To take account of this premium, an extra return on a payg could be introduced. Clearly this would be favorable to sustainability. On the other hand, because of inelastic labor, the analysis neglects the standard distortionary effects of taxation on labor supply.

Whatever restrictions, we think that the main results are quite robust. Macroeconomic risks modify substantially the analysis of unfunded systems, from the political sustainability and welfare point of views. Governmental debt limits seriously the possibility of redistribution in the absence of short sales constraints. Finally, the design of the system,

[27]For an analysis of this type of insurance in a dynastic framework, see Fuster, Imrohoroglu, and Imrohoroglu (2003).

in particular allowing contributions rates to be contingent on the state of the economy, plays an essential role in promoting sustainability.

7 Proofs

Proof of theorem 1. In a stationary economy without financial assets, the choice of the decisive voter with characteristic $\theta = \theta^d(e)$ takes the following form. Consider a positive vector $\boldsymbol{\tau}$. In each state e with $\tau(e) > 0$, let $V(\lambda, \boldsymbol{\tau}, e)$ be the maximum of the voter expected utility if he chooses optimal consumption levels and portfolio under the budget constraints :

$$\begin{cases} c^y + s = (1 - \lambda\tau(e))\theta w, s \geq 0 \\ c^o(e_+) = s\rho(e_+) + \lambda\theta\mu(\theta)(\gamma\tau w)(e_+) \text{ in each state } e_+ \end{cases} \tag{22}$$

Sufficient condition For each state e, let $\lambda(\boldsymbol{\tau})(e)$ be the value that maximizes $V(\lambda, \boldsymbol{\tau}, e)$ for the decisive voter over nonnegative λ (a maximizer exists and is unique by standard arguments). By definition, the strictly positive contribution rate vector $\boldsymbol{\tau}$ is sustainable if and only if $\lambda(\boldsymbol{\tau})(e) = 1$ in any state e. Furthermore, if $\lambda(\boldsymbol{\tau})(e)$ is constant across states, say equal to ℓ, the contribution rate vector $\ell\boldsymbol{\tau}$ is sustainable. This remark allows us to work on normalized contribution vectors.

We construct a correspondence F from the simplex to itself such that at a fixed point $\boldsymbol{\tau}^*$, $\lambda(\boldsymbol{\tau}^*)(e)$ is constant over states. Let us consider contribution rates in the simplex :

$$\Delta = \{\boldsymbol{\tau} = (\tau(e)), \tau(e) \geq 0, \sum_e \tau(e) = 1.\}$$

We first define F on the interior of the simplex. Given a strictly positive vector $\boldsymbol{\tau} >> 0$, the eigenvalue condition implies that $\lambda(\boldsymbol{\tau})(e)$ is strictly positive for at least one state e, that is, the vector $\lambda(\boldsymbol{\tau})$ is not null. To see this, remark that by choosing $\lambda = 0$ the individual faces the same budget constraints as at the autarky equilibrium. So $\lambda(\boldsymbol{\tau})(e) = 0$ is optimal if

$$-\tau(e)w(e) + \mu^d(e) \sum_{e_+} \text{mrs}^d_{\mathbf{0}}(e, e_+)(\gamma\tau w)(e_+) \leq 0 \tag{23}$$

where $\text{mrs}^d_{\mathbf{0}}(e, e_+)$ is computed at the autarky allocation. If $\lambda(\boldsymbol{\tau})$ is the null vector, inequality (23) holds in each state e. This writes in matrix form as

$$(A_{\mathbf{0}} - I)\boldsymbol{\tau} \leq 0.$$

By assumption, the positive matrix $A_{\mathbf{0}}$has its largest eigenvalue strictly larger than 1. By a well known theorem, the matrix $(A_{\mathbf{0}} - I)$ is invertible and its inverse is positive.

Multiplying the above inequality by $(A_{\mathbf{0}} - I)^{-1}$ yields that $\boldsymbol{\tau}$ is a negative vector, a contradiction. Thus, for any $\boldsymbol{\tau} >> 0$, we may define :

$$F(\boldsymbol{\tau}) = \{t \in \Delta, t(e) = 0 \text{ if } \lambda(\boldsymbol{\tau})(e) < max_{f\in E}[\lambda(\boldsymbol{\tau})(f)]\} \tag{24}$$

Therefore, each contribution vector in $F(\boldsymbol{\tau})$ assigns a null rate to each state in which the decisive voter does not choose the maximum value for λ.

For $\boldsymbol{\tau}$ in the boundary of the simplex, with some null components, let E_0 (resp. E_p) be the set of states with null (resp. positive) component. Define $F(\boldsymbol{\tau})$ as the subset of the simplex composed with the vectors whose components in E_p are null :

$$F(\boldsymbol{\tau}) = \{t \in \Delta, t(e) = 0 \text{ if } \boldsymbol{\tau}(e) > 0\}. \tag{25}$$

The correspondence F is clearly convex-valued. It is also upper hemicontinuous. Let $(\boldsymbol{\tau}_n)$ be a sequence converging to $\boldsymbol{\tau}$. If $\boldsymbol{\tau}$ belongs to the interior of Δ, $\boldsymbol{\tau}_n >> 0$ for n sufficiently large, and upper hemicontinuity immediately follows from the continuity of $\lambda(\boldsymbol{\tau})$ over the interior of Δ. If $\boldsymbol{\tau}$ belongs to the boundary, the sequence $(\boldsymbol{\tau}_n(e)))$ converges to $\boldsymbol{\tau}(e) = 0$ for each e in E_0, and $\boldsymbol{\tau}_n(e) > \boldsymbol{\tau}(e)/2$ for e in E_p and n sufficiently large. It follows, using the first order condition (5) on λ, that $\lambda(\boldsymbol{\tau}_n(e))$ converges to $+\infty$ for e in E_0, and that $\lambda(\boldsymbol{\tau}_n(e))$ is bounded for each e in E_p (in a state of E_0, the contribution rate is almost null in exchange of a non negligible pension benefits with positive probability, when a state in E_p is realized. Therefore surely $t_n(e) = 0$ for e in E_p and each $t_n \in F(\boldsymbol{\tau}_n)$. This proves that any adherence point of a sequence $(t_n,\, t_n$ in $(F(\boldsymbol{\tau}_n)))$ belongs to $F(\boldsymbol{\tau})$.

By Kakutani's theorem, F has a fixed point. Let $\boldsymbol{\tau}^* \in F(\boldsymbol{\tau}^*)$. Note first that, from definition (25), $\boldsymbol{\tau}^*$ does not belong to the boundary of the simplex. Hence, $\boldsymbol{\tau}^*$ is a strictly positive vector and $F(\boldsymbol{\tau}^*)$ defined by (24). Now, $F(\boldsymbol{\tau}^*)$ contains the strictly positive vector $\boldsymbol{\tau}^*$ only if the components of $\lambda(\boldsymbol{\tau}^*)$ are all identical, say to ℓ^*. Since ℓ^* is strictly positive, (otherwise $\lambda(\boldsymbol{\tau}^*)$ would be null) the vector $\ell^*\boldsymbol{\tau}^*$ is sustainable.

Necessary condition It remains to prove that, conversely, the existence of sustainable $\boldsymbol{\tau}$ implies the eigenvalue condition. This follows from concavity arguments on the indirect utility achieved by an individual facing contribution rates $\boldsymbol{\tau}$. Let $V(e, \boldsymbol{\tau})$ be the expected utility of an individual born in state e when the rates are given by $\boldsymbol{\tau} = (\tau(e))_{e\in E}$. To simplify notation, we drop the individual's index. We use an auxiliary result, which was introduced in Demange and Laroque (1999).

Lemma A1 *The functions $V(e, \boldsymbol{\tau})$ are concave in $\boldsymbol{\tau}$ for all e. Furthermore :*

$$\frac{1}{E(u'_y|e)\theta w(e)} \frac{\partial V(e, \boldsymbol{\tau})}{\partial \tau(e')} = [A_{\boldsymbol{\tau}}(e, e') - \mathbb{1}_{e=e'}]$$

Proof : By definition, $V(e,\boldsymbol{\tau})$ is equal to the maximum of $Eu(c_y,c_o)|e$ over the budget constraints. Since the constraints are linear in $\boldsymbol{\tau}$, V is concave in $\boldsymbol{\tau}$. By the envelope theorem one immediately gets :

$$\frac{\partial V(e,\boldsymbol{\tau})}{\partial \tau(e)} = [-E(u'_y|e)w(e) + u'_o[e,e]\gamma(e)\mu(e)w(e)\Pr(e|e)]\theta(e)$$

and for $e_+ \neq e$:

$$\frac{\partial V(e,\boldsymbol{\tau})}{\partial \tau(e_+)} = u'_o[e,e_+]\gamma(e_+)\theta(e)\mu(e)w(e_+)\Pr(e_+|e).$$

The result follows from the definition of the matrix $A_{\boldsymbol{\tau}}$. ■

Now, consider a sustainable $\boldsymbol{\tau}$. The decisive voter is better off whatever state, so $V(e,\boldsymbol{\tau}) - V(e,0) > 0$. By the concavity of V, we have :

$$\sum_{e'} \frac{\partial V(e,0)}{\partial \tau(e')}\boldsymbol{\tau}(e') \geq V(e,\boldsymbol{\tau}) - V(e,0) > 0.$$

Dividing inequality in state e by $E(u'_y|e)\theta w(e)$ and using lemma A1, yields

$$(A_0 - I)\boldsymbol{\tau} > 0.$$

By a standard property of non negative matrices (see e.g. in Debreu Herstein [1953, p.601]), the maximal eigenvalue of A_0 is strictly larger than 1. ■

Proof of Proposition 1. I first show that the general term of matrix $A_{\mathbf{0}}$ is

$$A_{\mathbf{0}}(e,e_+) = \mu^d \frac{w(e_+)}{w(e)} \frac{v'(\rho(e_+))\gamma(e_+)\Pr(e_+)}{E[v'(\tilde{\rho})\tilde{\rho}]}. \tag{26}$$

Recall from (12) that

$$mrs(e,e_+) = \delta \frac{u'(\hat{c}_o(e))}{u'(c_y(e))} \frac{v'(c_o(e,e_+))\Pr(e_+|e)}{v'(\hat{c}_o(e))}$$

At autarky, saving are invested in the risky technology only. Moreover investment must be positive to get some consumption when old. Therefore investment $s(e)$ in state e satisfies the first order condition $\sum_{e_+} \mathrm{mrs}(e,e_+)\rho(e_+) = 1$. This gives

$$1 = \delta \frac{u'(\hat{c}_o(e))}{u'(c_y(e))} \frac{E[v'(c_o(e,\tilde{e}_+))\rho(\tilde{e}_+)|e]}{v'(\hat{c}_o(e))} \text{ and } \mathrm{mrs}(e,e_+) = \frac{v'(c_o(e,e_+))\Pr(e_+|e)}{E[v'(c_o(e,\tilde{e}_+)\rho(\tilde{e}_+)|e]}$$

Now it suffices to use that $c_o(e, e_+) = \rho(e_+)s(e)$, that v is homothetic, and states are independent to get

$$\text{mrs}(e, e_+) = \frac{v'(\rho(e_+))\Pr(e_+)}{E[v'(\rho(e_+))\rho(e_+)]}$$

which gives (26).

All rows of $A_{\mathbf{0}}$ are proportional among each other. Hence the eigenvalues of $A_{\mathbf{0}}$ are all null except one, which is equal to the sum of the diagonal terms $\sum_e A_{\mathbf{0}}(e, e)$. This gives a maximal eigenvalue equal to $\mu^d E[v'(\tilde{\rho})\tilde{\gamma}]/E[v'(\tilde{\rho})\tilde{\rho}]$, which can also be written as

$$\mu^d E[\frac{\gamma}{\rho}\frac{v'(\rho)\rho}{E[v'(\tilde{\rho})\tilde{\rho}]}].$$

This proves *1.*

It remains to show property *2*, i.e. that increasing risk aversion increases the maximal eigenvalue. Let $g_v(\rho) = \frac{v'(\rho)\rho}{E[v'(\tilde{\rho})\tilde{\rho}]}$, and denote by G_v the distribution of ρ with density g_v with respect to the initial one. From the property of conditional expectation :

$$E[\frac{\gamma}{\rho}g_v(\rho)] = E[\frac{E[\gamma|\rho]}{\rho}g_v(\rho)],$$

the term on the right hand side being the expectation of $E\frac{[\gamma|\rho]}{\rho}$ under G_v. Now take w more concave than $v : w = f(v)$ where f is concave. It suffices to show that the expectation of $E\frac{[\gamma|\rho]}{\rho}$ under G_w is larger than under G_v. Since the function $E\frac{[\gamma|\rho]}{\rho}$ is assumed to be nonincreasing this is true if distribution G_v first-order dominates distribution G_w. To show this note that

$$g_w(\rho) = \frac{f'(v(\rho))v'(\rho)\rho}{E[f'(v(\rho))v'(\tilde{\rho})\tilde{\rho}]}.$$

By the intermediate values theorem $E[f'(v(\rho))v'(\tilde{\rho})\tilde{\rho}] = f'(v(\rho^*))E[v'(\tilde{\rho})\tilde{\rho}]$ for some value ρ^* in the support of the distribution of $\tilde{\rho}$. So

$$g_w(\rho) = \frac{f'(v(\rho))}{f'(v(\rho^*))}g_v(\rho).$$

Since $f'(v(\rho))$ is decreasing with ρ, the difference $[g_w(\rho) - g_v(\rho)]$ is positive for $\rho < \rho^*$ and negative for $\rho > \rho^*$: this gives the result. ■

Proof of Theorem 2 :

An equilibrium for financial securities always exists in each state e for $\boldsymbol{\tau}$ with $0 \leq \boldsymbol{\tau} < 1$. To show this, consider in each state e the economy with young workers. There are $1+\#E$ goods -the good available today, and the good available next period in each possible state-and a technology that transforms the first good into the other ones according to a linear technology. Each young θ-individual is endowed with $(1-\tau(e))\theta w(e)$ of the first good and $(\theta\mu(\theta)(\gamma\tau w)(e_+)$ of the good next period contingent on e_+. If $\boldsymbol{\tau}(e) < 1$, the aggregate amount of the good today is positive. Note that the goods available next period can be obtained by transforming the first one through investment. So, even if endowments are null for these goods, standard arguments show the existence of an equilibrium.

The uniqueness assumption allows us to define the function

$$W(\boldsymbol{\tau})(e) = \mu^d(e) \sum_{e_+} q_{\boldsymbol{\tau}}(e_+|e)(\gamma\tau w)(e_+)/w(e) = (C_{\boldsymbol{\tau}}\boldsymbol{\tau})(e),$$

which is continuous on $0 \leq \boldsymbol{\tau} < 1$. The agreement condition is satisfied in state e if $W(\boldsymbol{\tau})(e) = \boldsymbol{\tau}(e)$. Thus $\boldsymbol{\tau}$ is sustainable if $W(\boldsymbol{\tau}) = \boldsymbol{\tau}$, that is $\boldsymbol{\tau}$ is a positive fixed point of $\boldsymbol{\tau}$.

Since the null vector is a fixed point we first need to restrict W to strictly positive rates. Thanks to the eigenvalue assumption, there exists a contribution vector $\boldsymbol{\tau}_0$ such that $W(\boldsymbol{\tau}_0) > \boldsymbol{\tau}_0$. To see this note that there is $\boldsymbol{\tau}_1 > 0$ such that $C_{\mathbf{0}}\boldsymbol{\tau}_1 > \boldsymbol{\tau}_1$. By continuity, for ϵ positive small enough $C_{\epsilon\boldsymbol{\tau}_1}\boldsymbol{\tau}_1 > \boldsymbol{\tau}_1$. It suffices to choose $\boldsymbol{\tau}_0 = \epsilon\boldsymbol{\tau}_1$.

Let us consider contribution rates in the set

$$T = \{\boldsymbol{\tau} = (\tau(e)), \tau_0(e) \leq \tau(e) \leq 1\}.$$

T is compact and convex. The function W, defined for contribution rates smaller than 1 can be extended by continuity to any $\boldsymbol{\tau}$ in T by setting $W(\boldsymbol{\tau})(e) = 0$ if $\boldsymbol{\tau}(e) = 1$. To see this let $\boldsymbol{\tau}(e) = 1$, and a sequence $(\boldsymbol{\tau}_n)$, $\boldsymbol{\tau}_n < 1$ converging to $\boldsymbol{\tau}$. In the economy starting in state e, the aggregate endowment for the good at the initial date tends to zero but is positive in any subsequent state e_+ (since $\tau_0(e_+) \leq \tau(e_+)$). This implies that contingent prices $q_{\boldsymbol{\tau}_n}(e_+|e)$ along the sequence converge to zero, hence also $W(\boldsymbol{\tau}_n)(e)$.

We define a correspondence F from T to itself as follows :

- If $W(\boldsymbol{\tau})(e) < \boldsymbol{\tau}(e)$ for a state e then $F(\boldsymbol{\tau}) = \boldsymbol{\tau}_0$.
- Otherwise, that is if $W(\boldsymbol{\tau}) \geq \boldsymbol{\tau}$, define

$$F(\boldsymbol{\tau}) = \{t \in T, s.t. t(e) = \min(W(\boldsymbol{\tau})(e), 1) \text{ if } W(\boldsymbol{\tau})(e) > \boldsymbol{\tau}(e)\}$$

Therefore, $t(e)$ can take any value in $[\boldsymbol{\tau}_0(e), 1]$ in a state e for which $W(\boldsymbol{\tau})(e) = \boldsymbol{\tau}(e)$. The correspondence is clearly convex-valued. It is also upper hemicontinuous, thanks to the continuity of the function $W(\boldsymbol{\tau})$. It suffices to argue as for theorem 1. Let $(\boldsymbol{\tau}_n)$ be a sequence converging to $\boldsymbol{\tau}$ and (t_n) with t_n in $F(\boldsymbol{\tau}_n)$. If the inequality $W(\boldsymbol{\tau})(e) < \boldsymbol{\tau}(e)$ is met in state e, the same inequality is also met for $\boldsymbol{\tau}_n$ with n sufficiently large, hence $t_n =$

$t = \boldsymbol{\tau}_0$. Similarly by reversing the inequality. Since $t(e)$ is unrestricted if $W(\boldsymbol{\tau})(e) = \boldsymbol{\tau}(e)$, surely any limit point of (t_n) belongs to $F(\boldsymbol{\tau})$: upper hemicontinuity follows.

By Kakutani's theorem, F has a fixed point, $\boldsymbol{\tau}^* \in F(\boldsymbol{\tau}^*)$.

Surely $W(\boldsymbol{\tau}^*) \geq \boldsymbol{\tau}^*$. If not, by definition of F, $F(\boldsymbol{\tau}^*)$ is the singleton $\boldsymbol{\tau}_0$. Hence $\boldsymbol{\tau}^*$ must be equal to $\boldsymbol{\tau}_0$. This is impossible since by construction $W(\boldsymbol{\tau}_0) > \boldsymbol{\tau}_0$).

Therefore $W(\boldsymbol{\tau}^*) \geq \boldsymbol{\tau}^*$. For a state e such that $W(\boldsymbol{\tau}^*)(e) > \boldsymbol{\tau}^*(e)$, $\boldsymbol{\tau}^*(e)$ must be equal to $\min(W(\boldsymbol{\tau}^*)(e), 1)$ which gives $\boldsymbol{\tau}^*(e) = 1$. But$\boldsymbol{\tau}^*(e) = 1$ implies $W(\boldsymbol{\tau}^*)(e) = 0$, which contradicts $W(\boldsymbol{\tau}^*)(e) > \boldsymbol{\tau}^*(e) = 1$, This proves that $W(\boldsymbol{\tau}^*) = \boldsymbol{\tau}^*$. ∎

Proof of Theorem 3 :

Consider an equilibrium in which the price of debt and contribution rates are both positive. In the absence of short sale constraints, the first order condition on i's debt holding is

$$q(e) = \sum_{e_+} mrs^i(e, e_+)\gamma(e_+)q(e_+) \tag{27}$$

This condition holds for the decisive voter in state e. Using the definition of $A_{\boldsymbol{\tau}}(e, e_+)$, it can be rewritten as

$$\frac{q(e)}{w(e)}\mu^d(e) = \sum_{e_+} A_{\boldsymbol{\tau}}(e, e_+)\frac{q(e_+)}{w(e_+)} \tag{28}$$

Define the vector $\boldsymbol{q}'$ by $q'(e) = q(e)/w(e)$. Recall that $\mu^d(e) \geq 1$ in each state. So $\boldsymbol{q}'$ satisfies

$$\boldsymbol{q}' \leq A_{\boldsymbol{\tau}}\boldsymbol{q}' \text{ and } \boldsymbol{q}' = A_{\boldsymbol{\tau}}\boldsymbol{q}' \text{ only if } \mu^d(e) = 1 \text{ in all states } e \tag{29}$$

Moreover, by sustainability, $\boldsymbol{\tau}$ satisfies :

$$\boldsymbol{\tau} = A_{\boldsymbol{\tau}}\boldsymbol{\tau} \text{ and } \boldsymbol{\tau} > 0 \tag{30}$$

We now use well-known results on positive matrices. First (29) implies that $A_{\boldsymbol{\tau}}$ has a positive eigenvector with eigenvalue strictly larger than 1 whenever $\boldsymbol{q}'$ differs from $A_{\boldsymbol{\tau}}\boldsymbol{q}'$. Second, (30) implies that the maximal eigenvalue of $A_{\boldsymbol{\tau}}$ is 1 and that all positive eigenvectors are proportional to $\boldsymbol{\tau}$. Hence (29) and (30) imply that $\mu^d(e)$ is identically equal to 1, and that $\boldsymbol{\tau}$ and $\boldsymbol{q}'$ are proportional. This means that the decisive voter is never subsidized and furthermore that, from his point of view, the return on the payg system and on debt are equalized. ∎

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