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ABSTRACT

Indeterminacy and Unemployment Fluctuations with Constant Returns to Scale in Production*

We extend the finance-constrained economy proposed by Woodford (1986) to incorporate imperfectly insured unemployment, by introducing unions and unemployment benefits financed by labour taxation. We show that this simple extension of the Woodford model changes drastically its stability conditions and local dynamics around the steady state. In fact, in contrast to related models in the literature, we find that under constant returns to scale in production: (i) indeterminacy always prevails in the case of a unitary elasticity of substitution between capital and labour; (ii) flip and Hopf bifurcations occur for empirically credible elasticities of substitution between capital and labour, so that a rich set of dynamics may emerge at 'realistic' parameters' values.

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1 Introduction

In this paper, we study the consequences of labor market frictions and imperfect unemployment insurance, in an economy with constant returns to scale in production, where workers are financed constrained. We investigate whether these features affect the emergence of local deterministic and stochastic sunspot-driven fluctuations, by analyzing the occurrence of local bifurcations and local indeterminacy.

To do so, we introduce wage-employment bargaining between unions and firms and unemployment benefits in the financed constrained economy proposed by Woodford (1986), which considers two classes of agents: "workers" and "capitalists". The crucial assumption of his model is that, while capital is accepted as collateral to secure a loan, workers have difficulties in borrowing against labor income, facing therefore a cash in advance constraint. If capitalists discount the future less than workers, and the cash in advance is binding, capitalists simply live of capital rents, accumulating capital through a traditional consumption-saving choice, while workers do not accumulate capital and only consume out of past wage earnings. Although this framework constitutes a relevant starting point to model real world distortions in financial markets, the assumption of financially constrained workers becomes more plausible if workers are assumed to face real income uncertainty, due notably to the risk of being unemployed.

We consider therefore a setup in which unemployment emerges as an equilibrium result and being unemployed is welfare costly. Indeed, in contrast to many models in the literature, in our framework there are no perfect insurance/redistributive mechanisms that allow workers to completely insure themselves against the revenue losses incurred when unemployed.[1] As in Lloyd-Braga and Modesto (2004), we introduce union bargaining power, but do not consider, as they do, that unemployed workers engage in home production and that unions use redistributive mechanisms between their members. Instead, we assume that the government guarantees a constant minimum real income to those unemployed, financed by taxing employed workers. Since unions set wages above the reservation wage, these assumptions are equivalent to assuming an *imperfect unemployment insurance scheme.* In our view,

[1] Obviously, in the presence of a perfect insurance scheme, the existence of unemployment is irrelevant from a worker welfare point of view, since optimal diversification of risk would prevent a sharp fall in earnings under these circumstances (Hansen (1985), Rogerson, 1988)).

this model provides a description of the functioning of the labour market more in line with what we observe in the real world.

We find that this simple extension of the Woodford model changes drastically its local dynamics around the steady state. By contrast to most related models in the literature, we find that deterministic and stochastic endogenous fluctuations, driven by self-fulfilling volatile expectations, can emerge in this economy under fairly plausible values for the elasticity of input substitution, without requiring either increasing returns to scale in production or a sufficiently high share of government expenditures. In particular, in the case of a unitary elasticity of substitution between capital and labor, we find that indeterminacy *always* prevails in this economy. Furthermore, provided union power is sufficiently strong, flip and Hopf bifurcations are shown to occur for values of the elasticity of input substitution that are relatively close to one and, therefore, in accordance with the empirical literature. This leads to the emergence of a rich set of dynamics under "realistic" parameters' values, and implies that small differences in the value of the elasticity of substitution can account for considerably different stability properties of the equilibrium. Therefore, similar countries with slightly different elasticities of substitution may experience drastically different business cycles patterns.

The rest of the paper is organized as follows. In the next section we describe the model and obtain the (deterministic perfect foresight) dynamic equilibrium equations. Section 3 analyzes the local dynamics properties of the model and the occurrence of local bifurcations. Finally, in section 4, we discuss our results and compare them with those obtained in the related literature.

2 The Model

The economy we consider is composed of 5 types of agents: workers, capitalists, firms, unions and the government. All markets are assumed to be perfectly competitive, with the exception of the labour market where union power will prevent the wage from falling to its walrasian level.

2.1 The agents

Workers There is a continuum of identical infinitely lived workers of mass big enough. Preferences are described by the following utility function:

$\sum_{t=0}^{\infty} \gamma^t u(c_t^w)$, where $0 < \gamma < 1$ is the constant discount factor and c_t^w is consumption in period t.[2] Workers face, as in Woodford (1986), a consumption-saving choice in the presence of liquidity constraints and, in our case, also under income uncertainty.[3]

In each period t, a worker may be either employed (state e) - receiving in cash, at the beginning of next period, a nominal wage w_t - or unemployed (state u). We assume that the government provides a minimum guaranteed income program, ensuring to all period t unemployed workers a constant real income $b > 0$.[4] As in the case of employed workers, these resources only become available for consumption at the beginning of next period. These transfers are financed by taxing period t employed workers at the beginning of next period (i.e., when their labour income becomes available). Since the government balances its budget, the real tax τ_t, paid by each worker employed in period t, is determined endogenously by the balanced-budget condition, see (6) below. Workers may use money to transfer resources intertemporally. Therefore, the typical budget constraint of a worker who was in state $i \in \{e, u\}$ in period $t-1$ can be written as:

$$m_{t+1}^{w,i} = m_t^w + y_t^i - p_t c_t^{w,i}, \tag{1}$$

where m_t^w denotes money held at the beginning of period t, p_t is the price of output, and where $y_t^i \in \{w_{t-1} - p_t \tau_{t-1}, p_t b\}$ is a state-dependant revenue, conditioned on being in state $i \in \{e, u\}$ in period $t-1$.

Additionally, workers are subject to a cash in advance constraint implying that they can only buy the consumption good out of net earnings already received plus previously accumulated money.[5] Hence, the following constraint

$$m_t^w + y_t^i - p_t c_t^{w,i} \geq 0 \qquad i = \{e, u\} \tag{2}$$

must hold for each period t.

We also assume that, when deciding how much to consume in t and how much to save in the form of money, the worker does not know yet

[2]We assume that u satisfies the usual properties, namely: $u(c_t)$ is a continuous real valued function in $c_t \geq 0$, with $u'(c) > 0$ and $u''(c) \leq 0$ for $c_t > 0$.

[3]Formally, our description of the consumption behavior of workers can be seen as a direct application of the general framework studied in Deaton (1991).

[4]Note that in most European countries, where such minimum guaranteed income programmes exist, they are indeed indexed to inflation, in order to ensure real purchasing power of the poor.

[5]Or "cash-on-hand", in the terminology of Deaton (1991).

whether he will be employed or unemployed in t. However, he can put a probability distribution over the two states which, given that all workers are treated anonymously, consists in period t employment (l_t) and unemployment rates ($1 - l_t$), respectively. The problem of the representative worker is then to choose the positive levels of $(c_t^{w,i}, m_{t+1}^{w,i})_{t=0,1,\ldots,\infty}$, $i = \{e, u\}$, in order to maximize expected utility subject to the constraints (1)-(2). In Appendix A.1 we give the conditions under which the cash in advance constraint is binding every period for both employed and unemployed workers.[6] In this case, it is easy to see that, independently of the utility function $u(c^w)$ considered, all workers will always choose the following amounts of consumption and money holdings:

$$c_{t+1}^{w,i} = \frac{y_{t+1}^i}{p_{t+1}} \tag{3}$$

$$m_{t+1}^{w,i} = 0, \qquad i = \{e, u\} \tag{4}$$

that is, at each period, workers rationally choose to save no money and to spend all their available income on current consumption.

Unions Each period t, identical unions bargain with identical firms over wages and employment. We assume that all workers are unionized and that unions are firm-specific, i.e., we have one union per firm. Workers are matched exogenously and uniformly with unions and cannot move between firms or unions, so that each union represents the same mass of workers. Assuming that unions wish to maximize the sum of discounted consumptions of their members, we obtain (see (3)) the following objective function for the representative union:

$$\Omega_t = \sum_{i=0}^{\infty} \gamma^{i+1} \left\{ \frac{w_{t+i} l_{t+i}}{p_{t+i+1}} + b(1 - l_{t+i}) - \tau_{t+i} l_{t+i} \right\}, \tag{5}$$

where l denotes employment at the firm level, and where we have normalized the mass of workers per firm to 1.[7]

[6] This means that we focus on equilibria that are sufficiently close to a steady state equilibrium in which the condition $u'(w/p - \tau) > \gamma l u'(w/p - \tau) + \gamma(1 - l)u'(b)$ is verified. See Appendix A.1

[7] Note that at a symmetric equilibrium l is also the employment rate in the economy.

Government The government guarantees a minimal amount of real income to each unemployed worker, taxing employed individuals in order to run a balanced budget. Hence, we have that:[8]

$$\tau_t = b(1 - l_t)/l_t. \tag{6}$$

Capitalists As in Woodford (1986) capitalists are identical and maximize $\sum_{t=0}^{\infty} \beta^t Log c_t^c$, subject to $p_t c_t^c + p_t k_{t+1}^c = p_t R_t k_t^c$, where c_t^c is consumption in period t, k_t^c is the capital stock at the outset of period t and $R_t = (\rho_t + 1 - \delta)$ is the real gross rate of return on capital, ρ_t being the real rental rate of capital. The discount factor β is such that $\gamma < \beta < 1$, and $0 \leq \delta \leq 1$ is the capital depreciation rate. The solution to this problem may be written as:

$$c_t^c = (1 - \beta) R_t k_t^c \tag{7}$$

$$k_{t+1}^c = \beta R_t k_t^c. \tag{8}$$

Firms Firms are identical and operate under a constant returns to scale technology, $A l_t f(x_t)$, where $x \equiv k/l$ is the capital labor ratio and $A > 0$ is a scaling factor.[9] Firms wish to maximize the present value of discounted profits, Π_t, but must negotiate wages and employment with unions. Also, since period t wages are paid in cash at the beginning of next period, firms will have to hold, at the end of period t, $m_{t+1}^f \geq w_t l_t$. At each period t the sequence of decisions is the following. First, firms pay last period wages out of their money holdings and rent capital, k_t, on the economy-wide capital market, at a given nominal rental rate $p_t \rho_t$. Next, wages, w_t, and employment, l_t, are determined through the bargaining process, and finally firms decide the level of money holdings and then production takes place.[10] In order to ensure

[8] Note that these assumptions are equivalent to assume that the government provides an insurance mechanism (*imperfect*, due to the existence of unions, as we shall see). To participate in this programme each worker pays a real premium τ, receiving in the event of unemployment a real amount b (net of the premium). As usual, the premium must cover the expected value of payments, i.e., $\tau = (b + \tau)(1 - l)$, that we can rewrite as $b(1 - l) = l\tau$.

[9] We also make the following standard assumptions on technology: $f(x)$ is a real, continuous function for $x \geq 0$, positively valued and differentiable as many times as needed for $x > 0$, with $f^{'}(x) > 0$, $f''(x) < 0$, so that $f(x) - f^{'}(x)x > 0$.

[10] As usually done in the literature, we are assuming that workers cannot sign binding wage contracts, so that the wage and employment are determined after the capital stock decision has been made.

time consistency of the equilibrium, the problem of the firm must be solved backwards, starting with the decision on money holdings. In Appendix A.2 we show that the cash constraint is always binding. We proceed now with the wage-employment bargain and then with capital decisions.

2.2 Wage, employment and capital decisions

Wages and employment are determined through an efficient bargaining procedure. This implies that l_t and w_t must solve the generalized Nash bargaining problem:

$$\underset{(w_t,l_t)\in\Re^2_{++}}{Max} \quad \left(\Pi_t - \overline{\Pi}_t\right)^{\alpha}\left(\Omega_t - \overline{\Omega}_t\right)^{(1-\alpha)} \quad s.t. \quad l_t \leq 1 \tag{9}$$

where $0 < \alpha \leq 1$ represents the firm's power in the bargain and $(\overline{\Pi}_t, \overline{\Omega}_t)$ are the fallback payoffs of each party if no agreement in period t is reached.[11] Using (5), the fallback payoff of a union is given by $\overline{\Omega}_t = \gamma b + \gamma\Omega_{t+1}$, so that $\Omega_t - \overline{\Omega}_t = \gamma l_t\left(\frac{w_t}{p_{t+1}} - b - \tau_t\right)$. Given the sequence of decisions of firms, it can be shown (see Appendix A.2) that $\Pi_t - \overline{\Pi}_t = p_t A l_t f(x_t) - w_t l_t$.

We assume that the solution l_t of problem (9) always satisfies $l_t < 1$ so that the first order conditions are:[12]

$$(b+\tau_t)\frac{p_{t+1}}{p_t} = A\left[f(x_t) - f'(x_t)x_t\right] \tag{10}$$

$$\frac{w_t}{p_t} = A\left[f(x_t) - \alpha f'(x_t)x_t\right]. \tag{11}$$

From (10) we can see that, whatever the union's bargaining power, the level of employment is determined by the intersection of the marginal productivity of labour (MPL) curve, $A\left[f(x_t) - f'(x_t)x_t\right]$, with the real reservation wage schedule, $(b+\tau_t)p_{t+1}/p_t$. See Figure 1. Using also (11), we see that when unions have no power in the bargain, $\alpha = 1$, we recover the perfectly competitive labor market case, where real wages are identical to the marginal productivity of labour and, thereby, to the real reservation wage. By contrast, when $\alpha < 1$, the real wage is set above the MPL (and so above the reservation wage), with a markup which is increasing in the bargaining

[11] If negotiations fail, production does not take place and all workers are unemployed.

[12] Obviously workers, unions and firms being small take b and τ as given.

power of unions $(1-\alpha)$. Given the absence of perfect redistributive schemes, unemployed individuals are thus clearly worse off. Finally, note also that the level of employment is influenced by expectations of future prices (shifting the reservation wage locus) which constitutes a potential channel for the emergence of expectations driven fluctuations.

The firm, anticipating the result of the bargaining process, chooses consequently $k_t > 0$ to maximize profits, Π_t, or, equivalently, current profits, $(p_t A l_t f(x_t) - p_t \rho_t k_t - w_t l_t)$. See Appendix A.2. Using (11), current profits can be rewritten as $p_t \alpha A f'(x_t) k_t - p_t \rho_t k_t$, where l_t satisfies (10), and we obtain the following first order condition:[13]

$$\alpha A f'(x_t) = \rho_t. \tag{12}$$

2.3 Equilibrium

Assuming an identical number of capitalists and firms, equilibrium in the capital services market requires that $k_{t+1} = k_{t+1}^c$. Also, money market clearing requires that $l_t w_t = M$, where, as in Woodford (1986), M is the constant amount of outside money (per firm) in the economy. Using the definition of R, and using equations (6), (8), (10), (11) and (12) we can obtain the equilibrium dynamic system in terms of k and l, as given below in (13) and (14). This motivates the following definition.

Definition 1 *An intertemporal equilibrium with perfect foresight is a sequence* $(k_t, l_t) \in \Re^2_{++}$, $t = 0, 1, ...\infty$ *that solves the two-dimensional dynamic system, with* $x_t \equiv k_t / l_t$

$$k_{t+1} = \beta \left[\alpha A f'(x_t) + (1-\delta) \right] k_t \tag{13}$$

$$l_{t+1} A \left[f(x_{t+1}) - \alpha f'(x_{t+1}) x_{t+1} \right] = b \frac{\left[f(x_t) - \alpha f'(x_t) x_t \right]}{\left[f(x_t) - f'(x_t) x_t \right]}. \tag{14}$$

[13] Note that, because firms operate under constant returns to scale, profits are zero at equilibrium.

3 Local dynamics and bifurcation analysis

To study the local stability properties of our two dimensional dynamic system (13) and (14), around its interior steady state solution (k, l),[14] we follow the usual procedure of linearizing it around the steady state. The linearized dynamic system can be written as:[15]

$$\begin{bmatrix} k_{t+1} - k \\ l_{t+1} - l \end{bmatrix} = [J] \begin{bmatrix} k_t - k \\ l_t - l \end{bmatrix}$$

where J is the 2x2 Jacobian matrix of the system evaluated at the steady state, whose elements are given in Appendix A.3. The determinant, D, and the trace, T, of matrix J correspond respectively to the product and sum of the two roots (eigenvalues) of the associated characteristic polynomial $Q(\lambda) \equiv \lambda^2 - \lambda T + D$, and are given by the following expressions:

$$D = -\frac{(1 - s_L)(1 - \alpha)(\sigma - 1)}{(\alpha - 1 + s_L)(\sigma - 1 + s_L)} \tag{15}$$

$$T = 1 + D - \frac{\theta s_L}{(\sigma - 1 + s_L)} \tag{16}$$

where $0 < \theta \equiv 1 - \beta(1 - \delta) < 1$, and where the labor share of output, $0 < s_L = \frac{f(x) - \alpha f'(x)x}{f(x)} < 1$, and the elasticity of substitution between capital and labor, $\sigma = -\frac{f'(x)\left[f(x) - f'(x)x\right]}{f(x)f''(x)x} > 0$, are both evaluated at the steady state.

To analyze the local stability properties of our model we use the geometrical method proposed by Grandmont et al. (1998). This method amounts to locate how the trace and determinant of our system evolve in the space (T, D), when some parameters are made to vary continuously in their admissible range. In Figure 2, we have represented three lines relevant for this purpose: the *line AC*, defined by $D = T - 1$, where a local eigenvalue is

[14] It is easy to check that, under the assumptions made on technology (see footnote 9), a unique interior steady state solution of (13) and (14), (k, l) such that $l < 1$, exists as long as A and b satisfy the following conditions: $\frac{1-\beta(1-\delta)}{\beta\alpha \lim_{x\to 0} f'(x)} < A < \frac{1-\beta(1-\delta)}{\beta\alpha \lim_{x\to\infty} f'(x)}$; $0 < b < A\left[f(x) - f'(x)x\right]$.

[15] Note that (13) and (14) define locally a two dimensional dynamic system of the form $(k_{t+1}, l_{t+1}) = G(k_t, l_t)$, provided the elasticity of $l_{t+1}A\left[f(x_{t+1}) - \alpha f'(x_{t+1})x_{t+1}\right]$ with respect to l_{t+1} does not vanish at the steady state under analysis.

equal to 1; the *line* AB, defined by $D = -T - 1$, where one eigenvalue is equal to -1; and the *segment* BC, defined by $D = 1$ and $|T| < 2$, where two eigenvalues are complex conjugates of modulus 1. It is easy to verify that when T and D fall in in the interior of triangle ABC, both eigenvalues have modulus lower than one, and the steady state is a *sink*, i.e., is locally stable. In all other cases, the steady state is locally unstable. It is a *saddle* when $|T| > |D + 1|$ (one eigenvalue with modulus higher than one and one eigenvalue with modulus lower than one), and a *source* in the remaining regions (both eigenvalues with modulus higher than one).

In the context of our model, where only capital is a predetermined variable, the steady state is locally indeterminate when it is a sink. In this case, as shown in Grandmont et al. (1998), there are infinitely many nondegenerate stochastic equilibria driven by self fulfilling expectations (sunspots equilibria), that stay arbitrarily close to the steady state. By direct inspection of (15) and (16), it is straightforward to see that when $\sigma = 1$, $D = 0$ and $T = 1 - \theta \in (0, 1)$, so that point (T, D) falls within the triangle ABC, the steady state being indeterminate. Since a Cobb-Douglas technology is often taken as a benchmark case in the literature, we highlight this result in the following proposition:

Proposition 1 *The Cobb-Douglas case*
For $\sigma = 1$, $D = 0$ and $T = 1 - \theta$, and the steady state is locally indeterminate.

It is however important to understand the robustness of this result by evaluating how the local stability properties change as σ varies. This amounts to study the occurrence of local bifurcations, which can still be done with reference to the above diagram. When, by slightly changing a (bifurcation) parameter, the values of T and D cross the interior of the segment BC, a pair of complex conjugate eigenvalues crosses the unit circle and a Hopf bifurcation generically occurs. Similarly, when the values of T and D cross the AB line, one eigenvalue goes through -1, and a flip bifurcation occurs. In nonlinear models, bifurcations are important since they are consistent with structurally stable deterministic cycles (periodic or quasi periodic in the case of a Hopf bifurcation, periodic of period two in the case of a flip bifurcation) surrounding the steady state in the state space. Moreover, theses cycles even appear when the steady state is locally determinate, provided Hopf (flip) bifurcations are supercritical. In this case, as shown in Grandmont et al.

(1998), there are also infinitely many bounded stochastic equilibria driven by extrinsic uncertainty, remaining in a compact set that contains in its interior the stable cycle.

In our analysis we will take σ, α, θ and s_L, as parameters characterizing our economies.[16] Since empirical values for σ and α are either not well known or not constant across countries, we shall organize our discussion in terms of these two last parameters, considering that θ and s_L take some fixed value in their admissible range, i.e., $\theta \in (0,1)$ and $s_L \in (0,1)$. We first fix α and analyze how the trace and determinant evolve as σ, the bifurcation parameter, is made to vary continuously within its domain. From (15) and (16), it is easy to show that this locus of points (T_σ, D_σ) is defined by the following linear expression, the Δ line:

$$D = -\frac{(1-s_L)(1-\alpha)\theta}{[(\alpha-1+s_L)\theta-(1-s_L)(1-\alpha)]} + \Delta'(T-1) \tag{17}$$

$$\text{where}$$

$$\Delta' = -\frac{(1-s_L)(1-\alpha)}{[(\alpha-1+s_L)\theta-(1-s_L)(1-\alpha)]}$$

From now on, to ease the exposition, we will assume that $\sigma > 1 - s_L$.[17] Therefore, only the part of the Δ line corresponding to these values of σ is relevant to our analysis. Using (15), we see that as σ increases from $1-s_L$ to $+\infty$, D decreases from $D_{1-s_L} = +\infty$ to $D_\infty = \frac{(1-\alpha)(1-s_L)}{(\alpha-1+s_L)}$. Using (16), we can also note that $D_\infty = T_\infty - 1$. Hence, the relevant part of the Δ line is just a half-line Δ whose origin (T_∞, D_∞), for $\sigma = +\infty$, lies on the line AC, and as σ decreases to $1 - s_L$, points upwards to the right or to the left, depending on whether its slope is positive or negative (see Figure 2).

We shall now study how this half line Δ shifts in the space (T, D) as α changes from 1 to $1-s_L$.[18] It is easy to see that as α changes, the origin of the

[16] We think that it is more interesting to study the dynamics in terms of the labor share of output s_L, which is an economic meaningful 'parameter' for which there are empirical estimations, than in terms of the technological 'parameter', $f'(x)\,x/f(x)$. Of course, doing so implies that when we consider different configurations for α, while keeping fixed the value of s_L, we implicitly assume that the elasticity of $f(x)$ adjusts, so that s_L can indeed remain constant.

[17] As we discuss later, this assumption covers all the empirically interesting cases. The reader interested in the general case can refer to the longer version of this paper, which is available from the authors upon request.

[18] Note that since we keep $0 < s_L < 1$ fixed, the assumptions made on technology (see

half line Δ moves downwards along line AC, taking the values $(T_\infty, D_\infty) = (1, 0)$ for $\alpha = 1$ and $(T_\infty, D_\infty) = (-\infty, -\infty)$ when α tends to $1 - s_L$. Also, the slope of the half line Δ decreases from zero to -∞ as α decreases from 1 to some critical value, and then increases gradually until 1 as α increases further towards $1 - s_L$. Note finally that, given Proposition 1, the half line Δ crosses point $P = (1 - \theta, 0)$ for any value of α when $\sigma = 1$. All this implies that the half line Δ, lying on the left of line AC, rotates in the clockwise direction around point P as α decreases from 1 to $1 - s_L$, being horizontal for $\alpha = 1$, becoming vertical for some α included between $1 - s_L$ and 1, its slope tending to 1 as α tends to $1 - s_L$.

Using figure 2, it is then straightforward to see that several critical values for α have to be considered: α_3, the value of α such that the half line Δ crosses point B, α_2 the value of α for which the slope of the half line Δ becomes -1, and α_1 the value of α such that the half line Δ crosses point A. Indeed, using Figure 2, one can then easily check that for $\alpha_3 < \alpha < 1$, as σ continuously increases from $(1 - s_L)$ to $+\infty$, the steady-state is first a saddle and changes to a sink trough a flip bifurcation at $\sigma = \sigma_F$ (σ_F being the value of σ at which the half line Δ crosses line AB). When $\alpha_2 < \alpha < \alpha_3$, a change in stability conditions through a Hopf bifurcation appears. Indeed, the steady state is first a saddle, undergoes a flip bifurcation for $\sigma = \sigma_F$, becomes a source for $\sigma_F < \sigma < \sigma_H$, undergoes a Hopf bifurcation at $\sigma = \sigma_H$ (σ_H being the value of σ at which the half line Δ crosses segment BC) and then turns to a sink. For $\alpha_1 < \alpha < \alpha_2$, the flip bifurcation disappears. The steady state is first a source, undergoes a Hopf bifurcation when $\sigma = \sigma_H$, and then becomes a sink. Finally, for $1 - s_L < \alpha < \alpha_1$, the steady state is first a source, undergoes a Hopf bifurcation for $\sigma = \sigma_H$, becomes a sink for $\sigma_H < \sigma < \sigma_F$, a flip bifurcation occurs for $\sigma = \sigma_F$, and then turns into a saddle. All these results can be summarized in Proposition 2 below.

Proposition 2 *For $\sigma > 1 - s_L$ and defining $\alpha_1 = \frac{2(1-s_L)}{2-s_L}$, $\alpha_2 = \frac{(2+\theta)(1-s_L)}{(2+\theta-2s_L)}$, $\alpha_3 = \frac{4(1-s_L)}{4(1-s_L)+\theta}$, $\sigma_F = \frac{2[(\alpha-1+s_L)-(1-s_L)(1-\alpha)]-(\alpha-1+s_L)s_L(2-\theta)}{2[(\alpha-1+s_L)-(1-s_L)(1-\alpha)]}$ and $\sigma_H = \frac{(1-s_L)}{\alpha}$, the following generically holds:*

- ***(i)*** *if $1 - s_L < \alpha < \alpha_1$, the steady state is a source for $\sigma < \sigma_H$, undergoes a Hopf bifurcation for $\sigma = \sigma_H$, becomes a sink for $\sigma_H < \sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, and becomes a saddle for $\sigma > \sigma_F$.*

footnote 9) imply that $\alpha > 1 - s_L$.

(ii) *if* $\alpha_1 < \alpha < \alpha_2$*, the steady state is a source for* $\sigma < \sigma_H$*, undergoes a Hopf bifurcation for* $\sigma = \sigma_H$*, and becomes a sink for* $\sigma > \sigma_H$*.*

(iii) $\alpha_2 < \alpha < \alpha_3$*, the steady state is a saddle if* $\sigma < \sigma_F$*, undergoes a flip bifurcation for* $\sigma = \sigma_F$*, becomes a source for* $\sigma_F < \sigma < \sigma_H$*, undergoes a Hopf bifurcation for* $\sigma = \sigma_H$*, and becomes a sink for* $\sigma > \sigma_H$*.*

(iv) *if* $\alpha_3 < \alpha < 1$*, the steady state is a saddle if* $\sigma < \sigma_F$*, undergoes a flip bifurcation for* $\sigma = \sigma_F$*, and becomes a sink for* $\sigma > \sigma_F$*.*

4 Discussion of the results

In this section we discuss the results obtained with our model on local dynamics and bifurcations, and compare them with those obtained in the literature. To ease the discussion of the results, we have depicted in Figure 3 the results derived in Proposition 2, representing in the (α, σ) plane the bifurcation values (σ_H and σ_F) that divide the plane into different regions in which the steady state is either a sink, a source or a saddle.

4.1 Indeterminacy

One of the main striking features highlighted by Figure 3 is that, in our model, indeterminacy occurs for a wide range of values in the (α, σ) plane. In particular, as emphasized by Proposition 1, when the elasticity of substitution between capital and labor is unitary, the equilibrium is always a sink for *any* value of the other parameters. This is true even though we have assumed throughout this paper *constant returns to scale* in production.

In that respect, we believe our model differs substantially from related model in the literature that have also addressed the indeterminacy issue within the Woodford (1986) framework. For example, since the seminal paper by Grandmont et al (1998), it is well known that indeterminacy can only occur with constant returns to scale in production if capital and labor are highly complementary, an assumption which is not supported by the available evidence[19]. Indeed, as shown in Cazzavillan et al. (1998), indeterminacy in the traditional Woodford model is consistent with a unitary elasticity of substitution only if strongly increasing returns to scale are assumed (around

[19] See for instance Hamermesh (1993) and Duffy and Papageorgiou (2000).

30% for a quarterly calibration, see Barinci and Chéron, 2001). Moreover, in Lloyd-Braga and Modesto (2004), where wage and employment bargaining between unions and firms is also considered, but within a framework with no taxes or unemployment insurance, it is shown that, in the case of a Cobb-Douglas technology, indeterminacy still requires the same amount of increasing returns.

Hence, it is clear that taxation is the channel through which, in our model, indeterminacy can easily occur. In that respect, our model fits in the line of research that explores the role of different balanced-budget policy rules on the stability properties of the equilibrium.[20] However, we argue that our model still differs from these works in several of its implications.

Let us first start with the similarities. From Figure 3, we can see that a necessary condition for indeterminacy is that the elasticity of substitution between capital and labor must be higher than $\sigma_H = \frac{(1-s_L)}{\alpha}$. This condition, expressed in economic terms, simply states that the slope of the equilibrium (log-linearized) MPL curve $(-(1-s_L)/\alpha\sigma)$ must be higher than the slope of the equilibrium (log-linearized)"reservation wage" schedule (-1).[21] Therefore we end up with a condition which is similar to the one required in several other papers (like, for instance, in Benhabib and Farmer (1994) and in Schmitt-Grohé and Uribe, 1997), where the indeterminacy condition is expressed in terms of the slopes of the equilibrium labour demand and labour supply curves. In our model with bargaining over wages and employment, the relevant curves are of course the MPL and "reservation wage" schedules, but the formal condition required for indeterminacy is the same[22].

There remains, however, a major difference between our indeterminacy condition and those considered in these former papers. In Schmitt-Grohé and Uribe (1997), notably, indeterminacy can only emerge in an economy with

[20] See among others Aloi, Lloyd-Braga and Jacobsen (2003), Guo and Harrison (2001), Guo and Lansing (1998) and Schmitt-Grohé and Uribe (1997). In particular Schmitt-Grohé and Uribe (1997) show that countercyclical government expenditures favour indeterminacy. In our model this feature appears in a most natural way as government transfers, which ensure to all unemployed workers a guaranteed minimal amount of resources, obviously increase when the unemployment rate is higher.

[21] Indeed, once the requirement that the government balances its budget has been taken into account, i.e. using (6), the reservation wage expression at the steady state $(b+\tau)$ can be written as b/l, so that the elasticity is -1.

[22] Note however that in our model, contrarily to Schmitt-Grohé and Uribe (1997), there are some values of α for which this necessary condition is also *sufficient*. For example, if $\alpha_1 < \alpha < \alpha_3$, the equilibrium is always a sink when $\sigma > \sigma_H$.

constant returns to scale and a unitary elasticity of substitution if public spending is fixed and the steady-state labour income tax rate is higher than the capital share. In a recent paper, Pintus (2004) proves that a model with fixed public spending is in fact isomorphic to a model with increasing returns to scale, but without government. He shows that, for any given value of σ, the model may be only indeterminate if public spending as a share of GDP exceeds some lower bound. With the standard quarterly calibration of Cooley and Prescott (1985)[23], and assuming $\sigma = 1$, this lower bound is around 23%. Indeterminacy therefore only prevails if the "fixed cost" imposed to the economy by the predetermined level of public spending is sufficiently high, a mechanism which, he concludes, is similar to imposing external increasing returns to scale in the first place.[24] In our model, as we emphasized earlier, it is clear that taxation plays a key role for indeterminacy, as it decreases the slope of the equilibrium "reservation wage" curve. Note however that, in our framework, this slope is fixed (-1) and does not depend on the values of the model's parameters. Hence, in contrast to the works mentioned above, our conditions for indeterminacy are independent of the value of unemployment benefits b, nor do they depend on the total amount of "public redistribution" as a share of GDP, $b(1-l)/(Alf(x))$, i.e., indeterminacy in the end does no rely on imposing a sufficiently large fixed-cost on the economy.

4.2 Bifurcations

We have seen that, in our model, the presence of unions is not crucial for indeterminacy, since indeterminacy prevails with a Cobb-Douglas production function for any value of union bargaining power, $1-\alpha$. However, union power plays a key role in rendering the emergence of *bifurcations* more likely. From a theoretical point of view, we have seen that whenever a bifurcation occurs, both deterministic and stochastic cycles close to the steady state may appear, even when the steady-state is *determinate* (i.e., when it is a source or saddle, see section 3). However, for such situations to be considered seriously as a possible explanation of actual business cycles, the relevant issue is not only whether bifurcations are possible, but mostly if they occur for *empirically plausible* values of the parameters. For example, in Grandmont et al. (1998)

[23] Specifically, $\beta = 0.987$, $\delta = 0.012$ and $s_L = 0.6$.

[24] This formal equivalence between fixed public spending and increasing returns is also made in a recent contribution by Seegmuller (2004), where it is shown that models with markup variability or taste for variety can also be analyzed in a similar manner.

and in many related papers, bifurcations - although possible - are in a certain way a mere theoretical phenomenon, since they only emerge for very low elasticities of substitution between factors or strong increasing returns. By contrast, our model suggests that such situations may easily occur when unions are sufficiently strong. In fact, it is easy to see from Proposition 2 that, when $\alpha = 1$ (the competitive labour market case), there are no Hopf bifurcations, and, using the Cooley and Prescott (1995) calibration, the flip bifurcation occurs at $\sigma_F = 0.4$, a value of the capital-labor elasticity of substitution which is too low to be empirically credible. On the other hand, when union power is high (low α), flip and Hopf bifurcations appear for values of σ that are relatively close to one. In particular, as shown in figure 3, in the limit case where α tends to $1 - s_L$, these bifurcations emerge for σ *arbitrarily* close to one. Moreover, for $\alpha = 0.5$ (the value which is usually considered in the labor economics literature), flip and Hopf bifurcations occur for elasticities of substitution between capital and labor given by $\sigma_F = 1.59$ and $\sigma_H = 0.8$, respectively. Interestingly, both values fall within the range of estimated values in the empirical literature.[25]

What is suggested by this analysis? First, a rich set of dynamics, including endogenously persistent deterministic or stochastic cycles, may emerge in our model under plausible configurations of all parameters. Second, countries with different elasticities of substitution may have considerably different stability properties of the equilibrium. Given the large cross-country heterogeneity reported for this parameter in empirical studies, our model could therefore explain why countries similar in many respects but with slightly different elasticities of substitution between capital and labor may experience drastically different business cycles, and in particular different patterns of unemployment fluctuations. Analyzing these implications in a simulated version of the model would be a natural extension of the present paper.

[25] For example, using a panel of 82 countries, Duffy and Papageorgiou (2000) have concluded that for the entire sample of countries the assumption of a unitary elasticity of substitution may be rejected. Furthermore, they find that the elasticity of substitution is *significantly greater than unity* for the richest group of countries, and *significantly less than unity* for the poorest group of countries.

A Appendix

A.1 Binding cash-in-advance constraints with income uncertainty

Workers receive at the beginning of each period t a state-contigent revenue $y_t^i \in \{w_{t-1} - p_t\tau_{t-1}, p_t b\}$ for $i \in \{e, u\}$, and wish to maximize its expected lifetime utility $E\sum_{t=0}^{\infty}\gamma^t u(c_t)$ with respect to $\left\{c_{t,}^i m_{t+1}^i\right\}_{t=0}^{\infty}$, under the constraints $m_{t+1}^i = m_t + y_t^i - p_t c_t^i$ and $m_t + y_t^i - p_t c_t^i \geq 0$ for all t.

Denoting by λ_t^i and μ_t^i the Lagrange multipliers associated with these two constraints, the first order conditions for this problem are given by:

$$u'\left(c_t^i\right) = p_t\left(\lambda_t^i + \mu_t^i\right) \quad (18)$$

$$\lambda_t^i = \gamma\left[l_t\left(\lambda_{t+1}^e + \mu_{t+1}^e\right) + (1 - l_t)\left(\lambda_{t+1}^u + \mu_{t+1}^u\right)\right] \quad (19)$$

where l_t is the probability of being employed in t. We are looking for the conditions under which a consumer will choose to consume all available (after tax) resources $m_t + y_t^i$ under all possible states (employed or unemployed) so that the liquidity constraint is always binding. This means that we are looking for the sequences of revenues and probability distributions over employment and unemployment that are consistent with $\mu_t^i > 0$, so that $m_{t+1}^i = 0$, for all $t = 0, ..., \infty$ and all $i \in \{e, u\}$. This implies that the following inequality

$$\frac{u'\left(c_t^i\right)}{p_t} > \gamma\left[l_t\frac{u'\left(c_{t+1}^e\right)}{p_{t+1}} + (1 - l_t)\frac{u'\left(c_{t+1}^u\right)}{p_{t+1}}\right] \quad (20)$$

must hold for all $i \in \{e, u\}$, and where, for $t = 1, ..., \infty$, we have $c_t^e = (w_{t-1} - p_t\tau_{t-1})/p_t$ and $c_t^u = b$. Condition (20) is in particular verified at a steady state with $p_t = p_{t+1} = p$ (and therefore in the vicinity of this steady state) if $u'(w/p - \tau) > \gamma[lu'(w/p - \tau) + (1 - l)u'(b)]$ and $u'(b) > \gamma[lu'(w/p - \tau) + (1 - l)u'(b)]$. As long as $w/p - \tau \geq b$ (a condition that is implied by the wage bargaining process), only the first condition is actually required.

A.2 The firms problem

Firms wish to maximize the present value of profits, given by

$$\Pi_t = m_t^f + p_t A l_t f(x_t) - p_t\rho_t k_t - w_{t-1}l_{t-1} - m_{t+1}^f + \varphi\Pi_{t+1} \quad (21)$$

where $0 < \varphi < 1$ is the constant discount factor and m_t^f is money held by firms at the beginning of period t. Since wages must be paid in cash, we have that $m_{t+1}^f \geq w_t l_t$. Given the sequence of events, we have to solve the firm problem backwards, starting with the money holdings decision. This means that, at this stage, firms choose the level of money holdings that maximize (21) subject to $m_{t+1}^f \geq w_t l_t$ with m_t^f given, for given values of k_t, w_t and l_t. Denoting by λ_t the lagrange multiplier associated with the constraint, the first order condition for this problem is $\lambda_t = 1 - \varphi$. We can therefore see that for any $\varphi \neq 1$ the cash in advance constraint will be binding, i.e. $m_{t+1}^f = w_t l_t$. Therefore, at the second and first stages the firms' objective becomes $\Pi_t = (p_t A l_t f(x_t) - p_t \rho_t k_t - w_t l_t) + \varphi \Pi_{t+1}$. It is then easy to see that the fallback payoff of firms is $\overline{\Pi}_t = -p_t \rho_t k_t + \varphi \Pi_{t+1,}$ so that $\Pi_t - \overline{\Pi}_t = p_t A l_t f(x_t) - w_t l_t$.

A.3 The elements of matrix J

Using (13) and (14), and the definitions of σ and s_L, we obtain the following expressions for the elements of J:

$$
\begin{aligned}
J_{kk} &= 1 - \theta \left[\frac{(\alpha - 1 + s_L)}{\alpha \sigma} \right] \\
J_{kl} &= \theta \left[\frac{(\alpha - 1 + s_L)}{\alpha \sigma} \right] \frac{k}{l} \\
J_{lk} &= \frac{l}{k} \frac{(1 - s_L)}{(\alpha - 1 + s_L)(\sigma - 1 + s_L)} \{(1-\alpha)(\sigma - 1) - [(1-\alpha)(\sigma - 1) + s_L] J_{kk}\} \\
J_{ll} &= -\frac{(1 - s_L)}{(\alpha - 1 + s_L)(\sigma - 1 + s_L)} \{(1-\alpha)(\sigma - 1) + \frac{l}{k} [(1-\alpha)(\sigma - 1) + s_L] J_{kl}\}
\end{aligned}
$$

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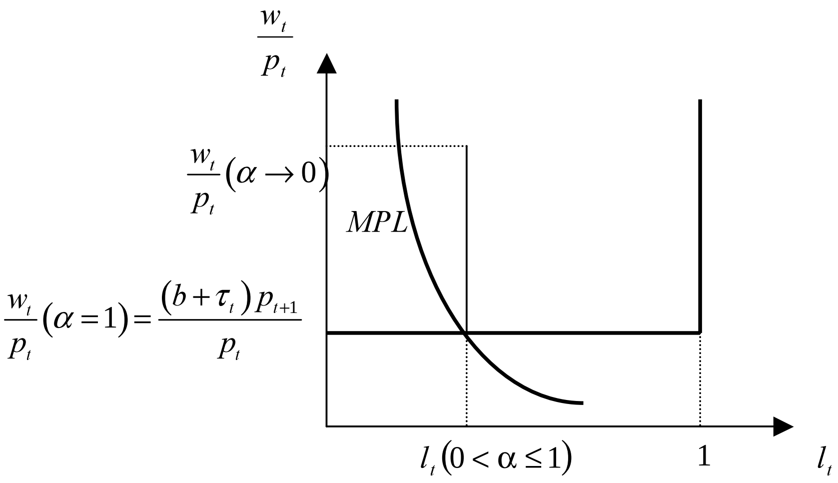

Figure 1: Labor market at temporary equilibrium

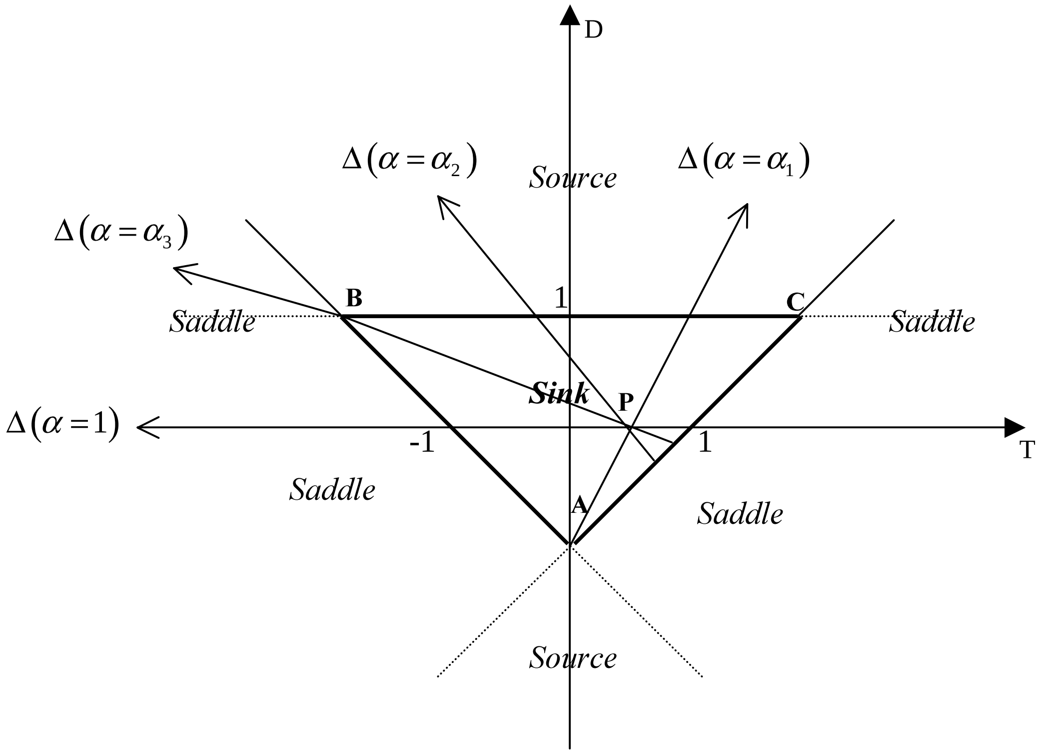

Figure 2: The local dynamics regimes in the (*T*, *D*) plane

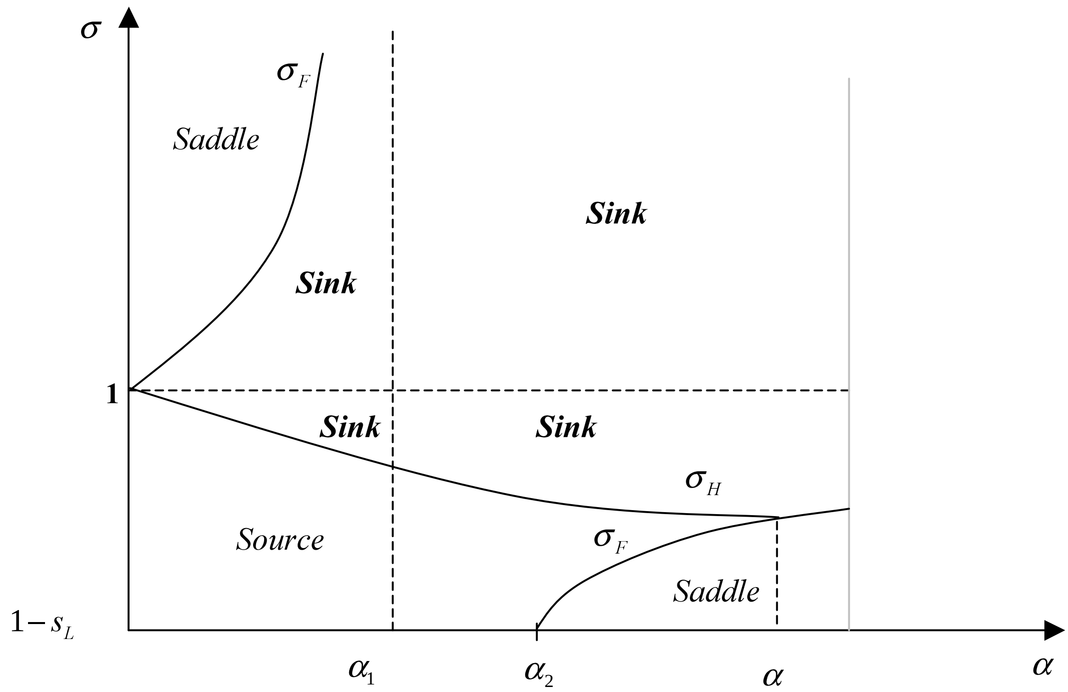

Figure 3: The local dynamics regimes in the (α,σ) plane