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COMPLEMENTARITIES AND GAMES: NEW DEVELOPMENTS

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ABSTRACT

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JEL Classification: C61, C62, C72, C73, D83, E22, F32, G21, L13 and O31

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Complementarities and Games: New Developments

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October 20, 2004

Abstract

The theory of monotone comparative statics and supermodular games is presented as the appropriate tool to model complementarities. The approach, which has not yet been fully incorporated into the standard toolbox of researchers, makes the analysis intuitive and simple, helps in deriving new results and in casting new light on old ones. The paper takes stock of recent contributions and develops applications to industrial organization (oligopoly, R&D, and dynamics), finance (currency and banking crisis) and macroeconomics (adjustment and menu costs). Particular attention is devoted to Markov games and to games of incomplete information (including global games).

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1 Introduction

Complementarities are pervasive in economics, ranging from coordination problems in macroeconomics and finance to pricing and product selection issues in industrial organization. At the heart of complementarity is the notion, due to Edgeworth, that the marginal value of an action or variable increases in the level of another action or variable.

Complementarities have been a recurrent and somewhat contentious topic of study for economic analysis. Indeed, while Paul Samuelson (1947) in his *Foundations* stated that "In my opinion, the problem of complementarity has received more attention than is merited by its intrinsic importance" (at the start of the section on complementarity, p. 183, 1979 edition), he later corrected himself in this very journal in 1974, on the occasion of the 40th anniversary of the Hicks–Allen revolution in demand theory, when he stated at the very beginning of his paper that

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The time is ripe for a fresh, modern look at the concept of complementarity. Whatever the intrinsic merits of the concept, forty years ago it helped motivate Hicks and Allen to perform their classical "reconsideration" of ordinal demand theory. And, as I hope to show, the last word has not yet been said on this ancient preoccupation of literary and mathematical economists. The simplest things are often the most complicated to understand fully.

Complementarities have a deep connection with strategic situations, and the concept of strategic complementarity is at the center stage of game-theoretic analyses. Examples abound. An arms race: If the Soviet Union increased its spending on nuclear weapons, it would pay the United States to respond by increasing its own spending. A bank run: If other customers of the bank that holds your savings withdraw their money, it might pay you to also take your money out. A currency crisis: If other investors attack the currency it might pay me to do so as well. An R&D race: If my main rivals in the pharmaceutical industry increase their R&D spending, does it pay me to increase it also? Technology adoption: My neighbors have introduced a new type of crop, does it pay me to follow? Location: If a new store locates in the shopping mall, does the mall become a more attractive location for other stores?

The modeling of complementarities, including strategic situations, has proved challenging. The reason is that the tools at hand were not attuned to deal with environments where indivisibilities, nonsmooth payoffs, and complex strategy spaces were naturally the norm rather than the exception. To make matters worse, multiple equilibria are typical in the presence of complementarities and policy analysis is left orphan. Instances of coordination failure with multiple equilibria abound: bank runs, debt runs on a country, low employment/activity equilibria, revolutions, and development traps provide some examples. In all these situations there are multiple self-fulfilling expectations of agents in the economy. A key issue is how to build coherent models that are useful for policy analysis. A challenging aspect of any crisis situation is to disentangle self-fulfilling from fundamentals-driven explanations that help answer questions such as: What is the effect of an increase in the amount of central bank reserves on the probability of a run on the currency? What is the impact of an increase in the solvency ratio on the probability of failure of a bank? What is the effect of a change in foreign short-term debt exposure on the probability of default of a small open economy?

The appropriate toolbox to deal with complementarities and, in particular, with strategic situations is the theory of monotone comparative statics and supermodular games. This theory provides the method for making the analysis of complementarity simple. The basic idea of the theory is to exploit fully both order and monotonicity properties. The achievements of this approach are as follows. First, it provides a framework for thinking rigorously about complementarities, identifying key parameters in the environment to look at (e.g., what are the critical properties of the payoffs and action spaces?). Second, it simplifies the analysis, clarifying the drivers of the results (e.g., is the regularity condition really needed to obtain the desired comparative static result?). Third, it encompasses the analysis of multiple equilibria situations by ranking equilibria and helping understand how potential equilibria move with the parameters of interest. Finally, it easily incorporates complex strategy spaces such as those arising in dynamic games and games of incomplete information.

The paper provides an introduction to the tools of supermodular games and a range of applications to industrial organization (Cournot and Bertrand markets, monopolistic competition, R&D races, multimarket oligopoly, switching costs, dynamic investment games), macroeconomic models (menu costs, search, aggregate demand externalities, technology adoption, adjustment costs), and finance (currency crisis, bank runs). The next section presents the approach in somewhat more detail as well as the plan of the paper.

2 Tools, results, and plan of the paper

The basic theory was developed by Donald Topkis (1978, 1979) and further developed and applied to economics by Xavier Vives (1985a, 1990a) and Paul Milgrom and John Roberts (1990a). Yet the theory continues to be extended at the frontier of research— for example, to dynamic games and games of incomplete information. The beauty of the approach is not its complexity but rather how much it simplifies the analysis and clarifies results. In fact, even the basic tools are not fully exploited by economists in current research.

The theory of supermodular games and monotone comparative statics, based on lattice-theoretic methods, has provided a powerful toolbox for analyzing the consequences of complementarities in economics. Monotone comparative statics analysis provides conditions under which optimal solutions to optimization problems move monotonically with a parameter. In this paper I provide an introduction to this methodology and then apply it to the study of strategic interaction in the presence of complementarities. This approach exploits order and monotonicity properties, in contrast to classical convex analysis. The central piece of attention will be games of strategic complementarities, where the best response of a player to the actions of rivals is increasing in their level.

The purpose of this paper is to bring forward some recent applications of the lattice-theoretic methodology and at the same time provide an introduction to the toolbox. I shall demonstrate the usefulness of the approach for:

- providing a common analytical frame to study complementarities;
- deriving new results; and
- casting new light on old results by simplifying proofs and discarding unnecessary assumptions.

Modeling strategic interaction presents formidable problems. A Nash equilibrium may not exist, at least in pure strategies. Or, instead, there may be multiple equilibria: How do players coordinate on one of them? How can the policy maker be sure that changing a parameter will have the desired effect?

Classical comparative statics analysis provides ambiguous results in the presence of multiple equilibria and imposes strong regularity conditions. These regularity conditions become particularly strong when applied to games with complex functional strategy spaces, such as dynamic or Bayesian games. We will see how complementarities are intimately linked to multiple equilibria and how supermodular methods provide a natural tool for characterizing them.

The class of games with strategic complementarities and the tools to analyze them have very nice properties.

- They allow very general strategy spaces, including indivisibilities and functional spaces such as those arising in dynamic or Bayesian games.
- They ensure the existence of equilibrium in pure strategies (without requiring quasiconcavity of payoffs, smoothness assumptions, or interior solutions).
- They allow a global analysis of the equilibrium set, which has an order structure with largest and smallest elements.
- Equilibria have useful stability properties, and there is an algorithm to compute extremal equilibria.
- Monotone comparative statics results are obtained with minimal assumptions by either focusing on extremal equilibria or considering best–response dynamics after the perturbation.

Furthermore, as we shall see, results can be extended beyond the class of games with strategic complementarities.

Let me highlight here some examples of how the lattice-theoretic approach either obtains new results that are hard or impossible to derive using the classical approach; or improves upon results already obtained by getting rid of unnecessary assumptions; or simplifies and deepens our understanding of the proof of known results.

- Consider an R&D race where each firm invests continuously to obtain a breakthrough and where we want to know what the effect is of increasing the number of participants n in the race (Tom Lee and Louis Wilde (1980)). Under very weak assumptions this game is one of strategic complementarities, and it will have multiple equilibria. The problem of using the classical approach is that increasing n may make some equilibria disappear while others may appear. Classical analysis will not help here, but with the lattice approach we obtain an unambiguous comparative statics result: increasing n will necessarily increase R&D effort, provided that out–of–equilibrium adjustment dynamics are of a general adaptive form.
- Consider a dynamic monopolistic competition model with menu costs where firms interact repeatedly over an infinite horizon and each firm receives an idiosyncratic demand or cost shock every period. Under what

conditions does there exist a Markov perfect equilibrium? When is the current price of a firm increasing in its past price and the distribution of prices in the market? The lattice approach provides the key assumptions needed to answer these questions (Christopher Sleet (2001) and Byoung Jun and Vives (2004)).

- Masahiro Okuno-Fujiwara, Andrew Postlewaite and Kotaro Suzumura (1990) provided conditions under which fully revealing equilibria obtain in duopoly games of voluntary disclosure of information when information is verifiable. The conditions involve restrictive regularity assumptions such as concavity of payoffs, interiority of equilibria, and independent types for the players. Our approach allows us to omit these unnecessary regularity assumptions, highlight the crucial ones (those related to monotonicity conditions), and extend the result to n-player games.

- Global games (Hans Carlsson and Eric van Damme (1993)) are games of incomplete information with types determined by each player observing a noisy signal of the state. They are proving to be a popular methodology for equilibrium selection, using iterated elimination of dominated strategies, and have wide applications to currency and banking crises and macroeconomics (Stephen Morris and Hyun Shin (2002)). Global games are Bayesian games, and the lattice approach is particularly suited to analyze them. For example, recent major advances in the difficult problem of showing the existence of Bayesian equilibrium in pure strategies have been made using the lattice-theoretic methodology. Furthermore, by realizing that global games are typically games of strategic complementarities, we understand why and how iterated elimination of dominated strategies works and why and under what conditions equilibrium selection is successful. Indeed, we will see how equilibrium is unique precisely when strategic complementarities are weak and that comparative statics results can be derived even for multiple equilibria.

The methodology of supermodular games provides the tools and an appropriate framework for satisfactorily confronting multiple equilibria and comparative statics. However, we should be aware also that the lattice-theoretic approach is not a panacea and cannot be applied to everything. (Indeed, the approach builds on a set of assumptions.) To give an obvious example, the approach cannot make equilibria appear in a game that has no equilibrium to start with.

We begin by introducing a simple class of games in Section 3, where many of the important issues are highlighted. The class includes monopolistic competition, search, and adoption games. Section 4 provides an introduction to the theory and basic results. Section 5 provides applications to oligopoly and comparative statics in the context of Cournot, Bertrand, and R&D games, including multimarket oligopoly competition. Section 6 deals with dynamic games; there we examine when increasing or decreasing dominance will obtain in investment games, and we also characterize strategic incentives in Markov games. Applications to menu, switching, and adjustment costs are provided as well. Section 7 studies Bayesian games, characterizing equilibria in pure strategies and comparative statics properties, with applications (among others) to games of voluntary disclosure and global games, including currency and banking crises. The Appendix provides a brief recollection of the most important definitions and results of the lattice-theoretic method.

3 A simple framework

Games of strategic complementarities are those in which players respond to an increase in the strategies of the rivals with an increase in their own strategy. This section presents an example suggesting the flavor of many of the results that can be obtained with the approach.

Consider a game with a continuum of players in which the payoff to a player is $\pi(a_i, \widetilde{a}; \theta_i)$. Here a_i is the action of the player, lying in a (normalized) compact interval $[0,1]$; $\widetilde{a}$ is the average or aggregate action; and θ_i is a (possibly idiosyncratic) payoff-relevant parameter.[1] I consider first the case with homogeneous players and then later the case with heterogeneous players.

3.1 Homogeneous players

Consider the symmetric case, where the payoff to a player is given by $\pi(a_i, \widetilde{a}; \theta)$. Suppose that π is smooth in all arguments and strictly concave in a_i, and let $r(\cdot)$ be the best response of an individual player to aggregate action $\widetilde{a}$. In this framework equilibria will be symmetric because, given any aggregate action $\widetilde{a}$, there is a unique best response $r(\widetilde{a}; \theta)$. For interior solutions we will have that

$$\frac{\partial \pi}{\partial a_i}\left(r(\widetilde{a}), \widetilde{a}; \theta\right) = 0.$$

If $\partial^2 \pi / (\partial a_i)^2 < 0$, then r is continuously differentiable and

$$r'(\widetilde{a}) = -\frac{\partial^2 \pi / \partial a_i \partial \widetilde{a}}{\partial^2 \pi / (\partial a_i)^2}.$$

Therefore, sign $r'(\widetilde{a})$ =sign $\partial^2 \pi / \partial a_i \partial \widetilde{a}$ and best replies are increasing if $\partial^2 \pi / \partial a_i \partial \widetilde{a} \geq 0$. A symmetric equilibrium is characterized by $r(a; \theta) = a$. Suppose also that $\partial^2 \pi / \partial a_i \partial \theta \geq 0$, so that an increase in θ increases the marginal profit of the action of a player and his best response $r(\cdot)$.

Two examples of the game are monopolistic competition and search. In monopolistic competition (see Vives (1999), Sec. 6.6) the action would Be the price of a firm with $\widetilde{a}$ the average price in the market and θ a demand or cost parameter. We have

$$\pi(a_i, \widetilde{a}; \theta) = (a_i - \theta) D(a_i, \widetilde{a}),$$

[1] The analysis with n players is similar.

with $D(\cdot)$ the demand function and θ the (common) marginal cost. As we will see in Section 4, for many demand systems $\partial^2 \log D/\partial a_i \partial \widetilde{a} > 0$ (meaning that the elasticity of demand for product i is decreasing in the average price) and therefore $\partial^2 \log \pi/\partial a_i \partial \widetilde{a} = \partial^2 \log D/\partial a_i \partial \widetilde{a} > 0$. Under this condition we will have that $r'(\widetilde{a}) > 0$ because best replies are invariant to an increasing transformation of the payoffs, such as the logarithm. In the search model (Peter Diamond (1981)), the action a_i is the effort of trader i in looking for a partner. The benefit (probability of finding a partner) is proportional to own effort and is increasing in the aggregate effort $\widetilde{a}$ of others:

$$\pi(a_i, \widetilde{a}; \theta) = \theta a_i g(\widetilde{a}) - C(a_i),$$

where $\theta > 0$ is the efficiency of the search technology and where $g(\cdot)$ and the cost of effort $C(\cdot)$ are increasing functions. In this case, $\partial^2 \pi/\partial a_i \partial \widetilde{a} = \theta g'(\widetilde{a}) \geq 0$.

In these examples it is easy to generate multiple equilibria. For instance, in the search model let $g(\widetilde{a}) \equiv \widetilde{a}$ and let C be increasing with $C(0) = 0$; then $a_i = 0$ for all i is always an equilibrium. If C is smooth and strictly convex with $C'(0) = 0$ then there are two equilibria, $a_i = 0$ and $a_i = \widehat{a} > 0$, with $\theta \widehat{a} = C'(\widehat{a})$ for all i. The latter equilibrium increases strictly with θ and is Pareto superior to the no–effort equilibrium. Another possibility is when g has an S-shaped function and $C'(a) \equiv a$; then there will be three equilibria. They will be the solutions (i.e., $\underline{a}, \widehat{a}$, and $\overline{a}$) to $\theta g(a) = a$ as depicted in Figure 1 (lower branch). In this example $r(\widetilde{a}) = \theta g(\widetilde{a})$. Obviously, equilibria are the solution to $r(a) = a$ and $r'(\widetilde{a}) = \theta g'(\widetilde{a})$.

Several properties of the equilibria in the search example are worth noticing.

1. A sufficient condition to have multiple equilibria is that strategic complementarities be sufficiently strong— namely, that $r'(a) > 1$ for some candidate equilibrium $r(a;\theta) = a$ (such as point $\widehat{a}$ in Figure 1).

2. The symmetric equilibria are ordered. There exits a largest ($\overline{a}$) and a smallest ($\underline{a}$) equilibrium (this follows trivially here given that actions are one-dimensional), and equilibria can be Pareto ranked. This is a general property whenever π is increasing in $\widetilde{a}$ (positive externalities).

3. Extremal equilibria, $\underline{a}$ and $\overline{a}$, are stable with respect to the usual best–reply dynamics. Indeed, it is immediate that best response dynamics starting at $a = 0$ (resp,. $a = 1$) will converge to $\underline{a}$ (resp., $\overline{a}$). See Figure 1.

4. Iterated elimination of strictly dominated strategies defines two sequences that converge, respectively, to $\underline{a}$ and $\overline{a}$. For example, let $\underline{a}^0 = 0$. Players will never use a strategy $a < r(0)$ because it is strictly dominated by $\underline{a}^1 = r(0)$. Now, knowing that no one will use a strategy in $[0, r(0))$, the region $[0, r(r(0)))$ will also be strictly dominated. Let $\underline{a}^2 = r\left(\underline{a}^1\right)$ and define $\underline{a}^k$ recursively. The sequence $\underline{a}^k$ is increasing and converges to $\underline{a}$ (indeed, it coincides with best–reply dynamics starting at $a = 0$). (See Figure 1.) This means that rationalizable strategies will lie in the interval

$[\underline{a}, \overline{a}]$, and if the equilibrium is unique, then the game will be dominance solvable. That is, the final outcome of the process of iterated elimination of strictly dominated strategies is both unique and an equilibrium.

5. An increase in the parameter θ will lead to an increased action in equilibrium, given out-of-equilibrium best-response dynamics, and this increase will be over and above the direct effect of the increase in the parameter. Indeed, increasing θ will move $r(\cdot)$ upward (as in Figure 1), and the equilibrium level of a will increase. Starting at $a = \overline{a}$, the direct effect will lead us to $r(\overline{a}) > \overline{a}$ and the full equilibrium impact to $\overline{a}' > \overline{a}$.[2] The consequences of a common shock (or, for that matter, an idiosyncratic shock) are amplified. Because of strategic complementarities there is a *multiplier effect*. Indeed, the direct effect of an increase in θ in the action of an agent, taking as given the average action, is amplified by the increase in the average action. This happens whether we focus on at extremal (or stable) equilibria, or rather consider best–response dynamics after the perturbation. Even starting at an unstable equilibrium, or at an equilibrium that disappears once θ increases, an increase in θ will result in an increase in a over and above the direct effect. In Figure 1 the unstable equilibrium $\widehat{a}$ disappears with the increase in θ, moving $r(\cdot)$ upward, and best–reply dynamics lead to the new equilibrium $\overline{a}'$.

With strategic substitutability among strategies, $\partial^2\pi/\partial a_i\partial\widetilde{a} < 0$, there cannot be multiple symmetric equilibria. In this case it is immediate that there is a unique symmetric equilibrium (because $\partial^2\pi_i/(\partial a_i)^2 + \partial^2\pi_i/\partial a_i\partial\widetilde{a} < 0$ and $\frac{\partial\pi_i}{\partial a_i}(a, a; \theta) = 0$ will have a unique solution). It is easy to see that, when $0 > r' > -1$ (or $|r'| < 1$), the game is dominance solvable.[3] This corresponds to the case where the symmetric equilibrium is stable according to the usual cobweb dynamics. Equivalently, in terms of iterated elimination of strictly dominated strategies, agents recognize that no one will take an action larger than $r(0)$; this starts the process of elimination of strategies, now with alternating regions on both sides of the candidate equilibrium.

Models of aggregate demand externalities and models of Keynesian effects have a similar flavor to our simple model (see Russell Cooper and Andrew John (1988)). The monopolistic competition model has been used extensively in the growth, development, regional, and international trade literatures to generate complementarities and multiplier processes (see Kiminori Matsuyama (1995) for a survey). In all those instances the presence of multiple Pareto rankable equilibria, multiplier effects, and cumulative self-reinforcing processes is central to the analysis.

[2] Indeed, at a stable equilibrium $r' < 1$ (or $\partial^2\pi_i/(\partial a_i)^2 + \partial^2\pi_i/\partial a_i\partial\widetilde{a} < 0$). At equilibrium $r(a;\theta) = a$ and therefore, in the vicinity of θ, $da/d\theta = \partial r/\partial\theta/1 - r' > \partial r/\partial\theta$ provided that $r' > 0$ (or $\partial^2\pi/\partial a_i\partial\widetilde{a} > 0$). More directly, $da/d\theta = -\left(\partial^2\pi_i/\partial a_i\right)/\left(\partial^2\pi_i/(\partial a_i)^2 + \partial^2\pi/\partial a_i\partial\widetilde{a}\right) > \partial r/\partial\theta = -\left(\partial^2\pi_i/\partial a_i\partial\theta\right)/\left(\partial^2\pi_i/(\partial a_i)^2\right) > 0$.

[3] Roger Guesnerie (1992) has shown this in a version of the model.

3.2 Heterogeneous players

A variation of the search example encompasses heterogeneous agents.[4] Suppose an agent must decide whether or not to adopt a new technology (or whether to "invest", "act", or "participate"). His action is $a_i = 0$ if there is no adoption and is $a_i = 1$ if there is adoption. The cost of adoption θ is idiosyncratic and follows a distribution function F on the interval $\left[\underline{\theta}, \overline{\theta}\right]$. The cost θ_i for agent i is an independent draw from F. The benefit of adoption is $g\left(\widetilde{a}\right)$, where $\widetilde{a}$ is the total mass adopting (which will be between 0 and 1) and no adoption yields no benefit. Therefore,

$$\pi\left(a_i, \widetilde{a}; \theta_i\right) = a_i\left(g\left(\widetilde{a}\right) - \theta_i\right).$$

The game is one of strict strategic complementarities if $g' > 0$. It is worth noticing that, because of independence, the adopting mass $\widetilde{a}$ will be nonrandom. Player i will adopt if $g\left(\widetilde{a}\right) - \theta_i \geq 0$. An equilibrium will be given by an adoption threshold $\widehat{\theta}$ and an adopting mass $\widetilde{a} = F\left(\widehat{\theta}\right)$ such that $g\left(\widetilde{a}\right) = \widehat{\theta}$ and agent i will adopt if $\theta_i \leq \widehat{\theta}$. The aggregate best reply to the adopting mass $\widetilde{a}$ is just $F\left(g\left(\widetilde{a}\right)\right)$ or, equivalently, the best reply to a threshold $\widehat{\theta}$ used by other players is $g\left(F\left(\widehat{\theta}\right)\right)$. The equilibria can be depicted as in Figure 1, where on the horizontal axis we have $\widetilde{a}$ or $\widehat{\theta}$ and along the vertical axis the best–reply $r\left(\widetilde{a}\right) = F\left(g\left(\widetilde{a}\right)\right)$ or $\hat{r}\left(\widehat{\theta}\right) = g\left(F\left(\widehat{\theta}\right)\right)$. For example, let $g\left(\widetilde{a}\right) = \widetilde{a}$, $\underline{\theta} < 0$, and $\overline{\theta} > 1$. Then for $\theta < 0$ to adopt is a dominant strategy (i.e., it pays to adopt even if no one else adopts); for $\theta > 1$ not to adopt is a dominant strategy (i.e., it does not pay to adopt even if everyone else adopts). In equilibrium, $\widetilde{a} = F\left(\widehat{\theta}\right) = \widehat{\theta}$.

The equilibrium threshold $\widehat{\theta}$ solves $g\left(F\left(\theta\right)\right) - \theta = 0$. The solution will be unique if $g'F' - 1 = g'f - 1 < 0$, where f is the density of F. It is thus immediate that if $g' < 0$ (strategic substitutability) then the equilibrium is unique. The question is: When do we have a unique equilibrium with strategic complementarities?

It is instructive to think of the case where θ_i follows a normal distribution with mean μ_θ and variance σ_θ^2 and where the costs of adoption θ_i and θ_j $(j \neq i)$ are potentially correlated with covariance $\rho\sigma_\theta^2$ for $\rho \in [0, 1)$. The case $\rho = 0$ corresponds to the independent case considered previously. Suppose that players adopt strategies with adoption threshold $\widehat{\theta}$. (In Section 7 we will see that equilibrium must be of this form.) From the point of view of player i and given θ_i, the adopting mass will be given by

$$\widetilde{a_i} \equiv \Pr[\theta_j \leq \widehat{\theta} | \theta_i] = \Phi\left(\frac{\widehat{\theta} - \left(\rho\theta_i + (1-\rho)\mu_\theta\right)}{\sigma_\theta\sqrt{1-\rho^2}}\right),$$

where $\Phi\left(\cdot\right)$ is the cumulative distribution function of the standard normal. The agent will adopt if and only if $g\left(\widetilde{a_i}\right) - \theta_i \geq 0$, and the equilibrium threshold $\widehat{\theta}$

[4] See Satyajit Chatterjee and Cooper (1989), Marco Pagano (1989), or Philip Dybvig and Chester Spatt (1983) for related examples.

will satisfy

$$g\left(\Pr[\theta_j \leq \widehat{\theta}|\theta_i = \widehat{\theta}]\right) - \widehat{\theta} = 0,$$

where

$$\Pr[\theta_j \leq \widehat{\theta}|\theta_i = \widehat{\theta}] = \Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}\frac{\widehat{\theta}-\mu_\theta}{\sigma_\theta}\right).$$

The solution will be unique if $\sqrt{(1-\rho)/(1+\rho)}g'\phi/\sigma_\theta - 1 < 0$, where $\phi = \Phi'$ is the density of the standard normal. It is then immediate that the equilibrium will be unique when $\sqrt{(1-\rho)/(1+\rho)}\overline{g}'/\sigma_\theta\sqrt{2\pi} < 1$, where $\overline{g}' \equiv \sup_{a\in[0,1]} g'(a)$.[5] There will be a unique equilibrium when the degree of strategic complementarity is not too strong. This may happen either because payoff complementarities are weak ($\overline{g}'$ low); or because each player ex ante faces a large cost uncertainty (σ_θ high); or because the correlation of the costs is high (ρ close but not equal to 1). All three factors tend to lessen the strength of strategic complementarities.

Let $g(\widetilde{a}) = \widetilde{a}$ in order to illustrate the effect of uncertainty. If costs are perfectly correlated then there are multiple equilibria for $\theta \in (0,1)$. In this case, there is complete information because a player— by knowing his own cost— knows the costs of any other player. However, a little bit of imperfect cost correlation (ρ close to 1) will yield a unique equilibrium. Note, for example, that $\Phi\left(\sqrt{(1-\rho)/(1+\rho)}\left(\widehat{\theta}-\mu_\theta\right)/\sigma_\theta\right)$ tends to 1/2 either when $\sigma_\theta \to \infty$ or as $\rho \to 1$, yielding the unique solution $\widehat{\theta} = 1/2$. In Figure 2 the case $\mu_\theta = 1/2$ is displayed, and $\widehat{\theta} = \mu_\theta = 1/2$ is the equilibrium threshold. Then, if $\sqrt{(1-\rho)/(1+\rho)}/\sigma_\theta\sqrt{2\pi} > 1$, two more equilibria appear.

Either with a diffuse prior or when the cost of a player gives very precise information about the costs of others, the (strategic) uncertainty of player i is maximal with respect to the behavior of others. This induces a best response for the player which is quite "flat", that is, not very sensitive to the threshold used by others.[6]

3.3 How general are the results?

The question arises of how far the nice results— about existence and characterization of equilibria and comparative static properties— in our simple game of Section 3.1 extend to different specifications (what if payoffs are not concave and best responses are not unique?) or more general games with strategic complementarities, or even beyond. As we will see in the next section, most of the properties generalize to multidimensional strategy spaces, discrete or continuous, and even functional spaces as well as to nonsmooth and nonconcave payoffs. The basic insight of the next section will be that, to obtain the desired results,

[5] Recall again that if $x \sim N(\mu, \sigma^2)$ then $f(\mu) = \left(\sigma\sqrt{2\pi}\right)^{-1}$, where f is the density of x.

[6] What the two cases have in common is that the player puts very little weight on prior information: when σ_θ^2 is very large because the prior is flat; when is ρ close to 1 because the type of the player predicts (almost) perfectly the types of others.

only the monotonicity properties of incremental payoffs and the order properties of strategies matter. Most of the regularity conditions typically assumed will not be crucial. In Section 7 we will study Bayesian games and see how our intuitions concerning the game with heterogeneous players of Section 3.2 generalize to equilibrium selection in global games. We will see how the consideration of threshold strategies is in fact without loss of generality and how the key to uniqueness will be an information structure (diffuse prior, or the signal of a player giving very precise information about the signals of others) that weakens strategic complementarities. The general principle is that, in order to obtain uniqueness of equilibrium in the presence of complementarities, the degree of strategic complementarities must be weak.

4 An introduction to games with strategic complementarities

In this section I provide first a brief introduction to the tools and main results of the theory and then comment on the theory's scope.

4.1 Modeling complementarities and results

Complementarities are dealt with using tools provided by the theory of monotone comparative statics and supermodular games. Those tools are based on lattice-theoretic results that exploit order and monotonicity properties of action sets and payoffs. The basis of the approach are monotone comparative statics results developed by Topkis (1978) and the application of Alfred Tarski's (1955) fixed point theorem to increasing functions. In a game situation, minimal assumptions are put on strategy sets and payoffs so that best responses are increasing and move monotonically with the parameters of interest. Then Tarski's fixed point theorem delivers existence of equilibria as well as order properties of the equilibrium set, and comparative statics results follow naturally.

This approach provides a powerful analytical tool that confronts the usual obstacles when analyzing a game: existence of pure-strategy equilibria, comparison of equilibria, and comparative statics. In particular, in games of strategic complementarities the presence of multiple equilibria need not be an obstacle to performing comparative statics analysis.

The emphasis of the exposition will be on intuition and not the technical details. This section will provide the minimal background necessary for a reader to follow the rest of the paper and the Appendix contains technical definitions and intermediate results. Examples will be developed to illustrate the methodology.[7]

We will use the intuitive concept of a game of strategic complementarities (GSC) whenever the best responses of the players in the game are increasing

[7] The reader is referred to Chapter 2 of Vives (1999) for a more thorough and general treatment of the theory as well as proofs, further references, and applications.

in the actions of rivals. The technical concept of supermodular game (to be shortly defined below) defined provides sufficient conditions for best responses to be increasing.

I will provide a definition of a supermodular game in a smooth context in order to keep the mathematical apparatus to a minimum, but this is by no means the most general way to define it. Consider the game $(A_i, \pi_i; i \in N)$, where N is the set of players, $i = 1, ..., n$; A_i is the strategy set, a compact cube in Euclidean space; and π_i the payoff of player $i \in N$ (defined on the cross product of the strategy spaces of the players A). Let $a_i \in A_i$ and $a_{-i} \in \prod_{j \neq i} A_j$ (i.e., we denote by a_{-i} the strategy profile $(a_1, ..., a_n)$ excepting the ith element). Let a_{ih} denote the hth component of the strategy a_i of player i.

We will say that the game $(A_i, \pi_i; i \in N)$ is *smooth supermodular* if, for all i,

- A_i is a compact cube in Euclidean space;
- $\pi_i (a_i, a_{-i})$ is twice continuously differentiable:
    1. supermodular in a_i for fixed a_{-i} or $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$, and
    2. with increasing differences in (a_i, a_{-i}) or $\partial^2 \pi_i / \partial a_{ih} \partial a_{jk} \geq 0$ for all $j \neq i$ and for all h and k.

The game is smooth *strictly* supermodular if the inequality in (2) is strict.

Condition (1) is the complementarity property (supermodularity) in own strategies. It means that the marginal payoff to any strategy of player i is increasing in the other strategies of the player. Condition (2) is the strategic complementarity property in rivals' strategies a_{-i}. It means that the marginal payoff to any strategy of player i is increasing in any strategy of any rival player. This property of π_i is termed *increasing differences in* (a_i, a_{-i}). In the general formulation of a supermodular game, strategy spaces need only be "complete lattices", only continuity (not differentiability) of payoffs is needed, and properties (1) and (2) are stated in terms of increments.[8]

In a supermodular game, very general strategy spaces can be allowed. These include indivisibilities as well as functional strategy spaces, such as those arising in dynamic or Bayesian games (as we will see in Section 6.2 and Section 7). Regularity conditions such as concavity and interior solutions can be dispensed with. The complementarity properties are robust in the sense that they are preserved under addition or integration, pointwise limits, and maximization (with

[8] Continuity is needed to ensure the existence of best replies, and the continuity requirement can also be weakened. See the Appendix for the general definitions of lattices, supermodularity, increasing differences, and supermodular game.

respect to a subset of variables, preserving supermodularity for the remaining variables).[9]

Two leading oligopoly models fit, in many specifications, the complementarity assumptions made. A first example is a Cournot oligopoly with complementary products. In this case, the strategy sets are compact intervals of quantities and the complementarity assumptions are natural (what Michael Spence (1976) called "strong complements"). A second example is a Bertrand oligopoly with differentiated substitutable products, where each firm produces a different variety. The demand for variety i is given by $D_i(p_i, p_{-i})$, where p_i is the price of firm i and p_{-i} denotes the vector of the prices charged by the other firms. A linear demand system will satisfy the complementarity assumptions.

The application of the theory can be extended by considering increasing transformations of the payoff (which does not change the equilibrium set of the game). We say that the game is *log-supermodular* if π_i is nonnegative and if $\log \pi_i$ fulfills conditions (1) and (2). In the Bertrand oligopoly example, the profit function, $\pi_i = (p_i - c_i) D_i(p_i, p_{-i})$ of firm i where c_i is the constant marginal cost, is log-supermodular in (p_i, p_{-i}) whenever $\partial^2 \log D_i / \partial p_i \partial p_j \geq 0$. For firm i, this holds whenever the own-price elasticity of demand η_i is decreasing in p_{-i}, as with constant elasticity, logit, or constant expenditure demand systems.[10]

The key results of the theory are obtained by a combination of monotone comparative statics results due to Topkis and Tarski's fixed point theorem. The results by Topkis (1978) deliver monotone increasing best responses, even when π_i is not quasi–concave in a_i.

The basic monotone comparative statics result states that the set of optimizers of a function $u(x,t)$ that is parameterized by t, supermodular in x, and with increasing differences in x and t has a largest and a smallest element and that both are increasing in t.[11]

In a supermodular game this means that each player i has a largest, $\overline{\Psi}_i(a_{-i}) = \sup \Psi_i(a_{-i})$, as well as a smallest, $\underline{\Psi}_i(a_{-i}) = \inf \Psi_i(a_{-i})$, best reply and that they are increasing in the strategies of the other players. If the game is strictly supermodular, then any selection from the best–reply correspondence is increasing.

Some intuition can be gained from the simple framework of Section 3.1. We observed that, if $\partial^2 \pi_i / (\partial a_i)^2 < 0$, then sign $r'(\tilde{a})$ =sign $(\partial^2 \pi_i / \partial a_i \partial \tilde{a})$. Now, even when π_i is not quasi–concave, the monotone comparative statics result implies that if $\partial^2 \pi_i / \partial a_i \partial \tilde{a} > 0$ then any selection from the best-reply correspondence of player i (which may have jumps) is increasing in the average

[9] Supermodularity and increasing differences can even be weakened to define an "ordinal supermodular" game, relaxing supermodularity to the weaker concept of quasi–supermodularity and increasing differences to a single–crossing property (see Milgrom and Chris Shannon (1994)). However, such properties (unlike supermodularity and increasing differences) have no differential characterization and need not be preserved under addition or partial maximization operations.

[10] See Chapter 6 of Vives (1999). However, this is not a universal result, as we will see in Section 4.2.

[11] See Lemma 1 in the Appendix for a precise statement of the result.

action.

We could define also the (weaker) concept of a game of strategic complementarities (GSC), under our maintained assumptions, as a game where: (a) strategy sets are compact cubes (or "complete lattices"); (b) the best reply of any player has extremal (largest and smallest) elements; and (c) those elements are increasing in the strategies of rivals. Similarly, we could define a game of strict strategic complementarities if, in addition, any selection from the best reply of any player is increasing in the strategies of the rivals.[12] All the results stated hereafter will then hold, replacing (strictly) supermodular game by GSC (game of strict SC).

The following results hold in a supermodular game. Let $\overline{\Psi} = \left(\overline{\Psi}_1, ..., \overline{\Psi}_n\right)$ and $\underline{\Psi} = (\underline{\Psi}_1, ..., \underline{\Psi}_n)$ denote the extremal best–reply maps.

Result 1. Existence and order structure (Topkis (1979)). In a supermodular game there always exist extremal equilibria: a largest element $\overline{a} = \sup\left\{a \in A : \overline{\Psi}(a) \geq a\right\}$ and a smallest element $\underline{a} = \inf\{a \in A : \underline{\Psi}(a) \leq a\}$ of the equilibrium set.

The result is shown by applying Tarski's fixed point theorem (which implies that an increasing function from a compact cube into itself has a largest and a smallest fixed point; see Appendix) to the extremal selections of the best-reply map $\overline{\Psi}$ and $\underline{\Psi}$, which are monotone by the strategic complementarity assumptions. There is no reliance on quasi–concave payoffs and convex strategy sets to deliver convex-valued best replies, as is required when showing existence using Kakutani's fixed point theorem.[13]

In the Bertrand oligopoly, for example, when the payoff is supermodular or log-supermodular then it follows that extremal price equilibria do exist. The results can be extended to convex costs and multiproduct firms and so provide a large class of Bertrand oligopoly cases for which the classical nonexistence of equilibrium problem encountered by Roberts and Hugo Sonnenschein (1977) does not arise.

Result 2. Symmetric games. For a symmetric supermodular game (exchangeable against permutations of the players), the following statements hold.

- Symmetric equilibria exist because the extremal equilibria $\overline{a}$ and $\underline{a}$ are symmetric.[14] Hence, if there is a unique symmetric equilibrium then the equilibrium is unique (since $\overline{a} = \underline{a}$). This result proves to be a very useful

[12] This definition was used in Vives (1985a) who concentrated attention on monotone increasing best responses as the defining characteristic of games with strategic complementarities. See the Appendix for a more formal definition along those lines.

[13] The equilibrium set has additional order properties (see Vives (1985a), Vives (1990a), Problem 2.5 in Vives (1999), and Lin Zhou (1994)).

[14] Indeed, if $\overline{a}_1 \neq \overline{a}_2$ then, because the game is symmetric, $(\overline{a}_2, \overline{a}_1, \overline{a}_3, ..., \overline{a}_n)$ will also be an equilibrium and therefore, because $(\overline{a}_1, \overline{a}_2, \overline{a}_3, ..., \overline{a}_n)$ is the largest equilibrium, $\overline{a}_1 \geq \overline{a}_2 \geq \overline{a}_1$ and $\overline{a}_1 = \overline{a}_2$.

tool for showing uniqueness in symmetric supermodular games. For example, in a symmetric version of a Bertrand oligopoly system with constant elasticity of demand and constant marginal costs, it is easy to see that there exists a unique symmetric equilibrium. Since the game is strictly log–supermodular, we can conclude that the equilibrium is unique.

- If the strategy spaces of the players are one–dimensional (or, more generally, completely ordered) then a symmetric strictly supermodular game has only symmetric equilibria.[15]

For one-dimensional strategy spaces, existence of symmetric equilibria can be obtained by relaxing the monotonicity requirement of best responses. It is enough then that all jumps in the best reply of a player be up. Existence follows from Tarski's intersection point theorem.[16] The result is easy to grasp considering a function f: $[0,1] \to [0,1]$ which when discontinuous jumps up but not down. The function must then cross the 45^o line at some point. Indeed, suppose that it starts above the 45^o line (otherwise, 0 is a fixed point); then it either stays above it (and then 1 is a fixed point) or it crosses the 45^o line. Versions of this fixed point theorem have been derived by McManus (1962, 1964) and Roberts and Sonnenschein (1976) to show existence of equilibria in symmetric Cournot games with convex costs.[17] More generally: In a symmetric game where the strategy space of each player is a compact interval and the payoff to a player depends only on her own strategy and the aggregate strategy of rivals, if the best reply of a player has no jumps down then symmetric equilibria exist. This implies in particular that, for the game in Section 3.1, under very weak assumptions a symmetric equilibrium will always exist.[18]

Result 3. Welfare (Milgrom and Roberts (1990a), Vives (1990a)). In a supermodular game, if the payoff to a player is increasing in the strategies of the other players (positive externalities) then the largest (resp., smallest) equilibrium point is the Pareto best (resp., worst) equilibrium. This is a very simple result that is at the base of finding equilibria that can be Pareto ranked in many games with strategic complementarities. For example, in the Bertrand oligopoly example, the profits associated with the largest price equilibrium are also the highest for every firm.

[15] See footnote 23 in Vives (1999) for a proof of the statement.

[16] See Section 2.3.1 in Vives (1999).

[17] Milgrom and Roberts (1994) also state and prove the theorem with $S = [0,1]$.

[18] The argument is simple. Consider a symmetric game with compact intervals as strategy spaces and let $\pi_i(a_i, a_{-i}) = \pi\left(a_i, \sum_{j\neq i} a_j\right)$, as in a Cournot game with homogeneous product and identical cost functions (or as in the game of Section 3 with a continuum of players). Existence of symmetric equilibria follows then from the stated result if the best-reply Ψ_i of a player (identical for all i due to symmetry) has no jumps down. This is in fact true if costs are convex in the Cournot game. Symmetric equilibria are given by the intersection of the graph of $a_i = \Psi_i\left(\sum_{j\neq i}\right) a_j$ with the line $a_i = \left(\sum_{j\neq i} a_j\right)/(n-1)$.

Result 4. Stability and rationalizability. Consider a supermodular game with continuous payoffs. Then we have the following.

1. Simultaneous response best-reply dynamics (Vives (1990a)):

   - Approach the "box" $[\underline{a}, \overline{a}]$ defined by the smallest and the largest equilibrium points of the game. Hence, if the equilibrium is unique then it is globally stable.
   - Converge monotonically downward (upward) to an equilibrium starting at any point in the intersection of the upper (lower) contour sets of the largest (smallest) best replies of the players $A^+ \equiv \left\{a \in A : \overline{\Psi}(a) \leq a\right\}$ $(A^- \equiv \{a \in A : \underline{\Psi}(a) \geq a\})$. See Figure 3.

2. The extremal equilibria $\underline{a}$ and $\overline{a}$ correspond to the largest and smallest serially undominated strategies. Therefore, if the equilibrium is unique then the game is dominance solvable (Milgrom and Roberts (1990a)).

This result implies that all relevant strategic action is happening in the box $[\underline{a}, \overline{a}]$ defined by the smallest and largest equilibrium points. For example, rationalizable outcomes (Douglas Bernheim (1984), David Pearce (1984)) and supports of mixed–strategy and correlated equilibria must lie in the box $[\underline{a}, \overline{a}]$. The argument for the second part of Result 4.1 is quite simple because, for example, starting at any point in A^+ (see Figure 3), best–reply dynamics define a monotone decreasing sequence that converges to a point that must (by continuity of payoffs) be an equilibrium. Results in 4.1 extend to a large class of adaptive dynamics, of which best–reply dynamics are a particular case.

In fact, starting at the largest (smallest) point of the strategy space A —recall it is a cube— best–reply dynamics with the largest (smallest) best–response map $\underline{\Psi}$ $(\overline{\Psi})$ will lead to the largest (smallest) equilibrium $\underline{a}$ $(\overline{a})$ (Topkis (1979)). For example, starting at $\inf A$ (see Figure 3) best–reply dynamics with the smallest best reply map $\underline{\Psi}$ define a monotone increasing sequence that converges to a point y that, by continuity of payoffs, must be an equilibrium. Furthermore, this must be the smallest equilibrium, $y = \underline{a}$. For any other equilibrium $x, x \geq \inf A$, and iterating the best reply map $\underline{\Psi}$ on both sides of the inequality yields $x \geq \underline{a}$ because $\underline{\Psi}$ is increasing.

Starting at an arbitrary point, we cannot ensure convergence because, for instance, a cycle is possible. For example, in Figure 3 starting at $a^0 = (\underline{a}_1, \overline{a}_2)$, the simultaneous response best–reply dynamics cycle between $(\underline{a}_1, \overline{a}_2)$ and $(\overline{a}_1, \underline{a}_2)$.

In the Bertrand oligopoly example with linear constant elasticity or logit demands, the equilibrium is unique and so the game is dominance solvable and globally stable.

Another interesting result is that properly mixed equilibria (i.e., Nash equilibria for which at least two players' strategies are not pure strategies) in strictly supermodular games are unstable with respect to best–reply or more general adaptive dynamics (Federico Echenique and Aaron Edlin (2003)). An example is the mixed–strategy equilibrium in the battle of the sexes.

Result 5. Comparative statics. Consider a n-player supermodular game with payoff for firm i, $\pi_i(a_i, a_{-i}; t)$, parameterized by a vector $t = (t_1, ..., t_n)$. If π_i has increasing differences in (a_i, t) ($\partial^2 \pi_i / \partial a_{ih} \partial t_j \geq 0$ for all h and j) then with an increase in t:

(i) the largest and smallest equilibrium points increase; and

(ii) starting from any equilibrium, best–reply dynamics lead to a (weakly) larger equilibrium following the parameter change.

Furthermore, the latter result can be extended to a class of adaptive dynamics, including fictitious play and gradient dynamics. Continuous equilibrium selections that do not increase monotonically with t predict unstable equilibria (Echenique (2002)). The comparative statics result is generalized in Milgrom and Shannon (1994).

An heuristic argument for the result is as follows. The largest best reply of player i is increasing in t, and from this it follows that the largest equilibrium point (as determined by the largest best replies) also increases with t. Indeed, $\overline{a} = \sup \left\{ a \in A : \overline{\Psi}(a; t) \geq a \right\}$, and $\overline{\Psi}(a; t)$ is increasing in t. Obviously, for an increase in the equilibrium to take place we need only, for example, that the payoff to firm i be affected by t_i and not by any other t_j. An increase in t leaves the old equilibrium in A^- (see Figure 3) and thus sets in motion via best reply (or more generally via adaptive dynamics) a monotone increasing sequence that converges to a larger equilibrium. Increasing actions by one player reinforce the desire of all other players to increase their actions, and the increases are mutually reinforcing (i.e., they exhibit positive feedback).

Another way to look at the feedback loop is to think in terms of multiplier effects. As stated in Section 3, a multiplier effect in the parameter t_j obtains if the equilibrium reaction of each player to a change in the parameter is strictly larger than the reaction of the player keeping the strategies of the other players constant. This will happen, for example, in a smooth strictly supermodular game with one–dimensional strategy spaces for which $\partial^2 \pi_i / \partial a_i \partial t_j \geq 0$ with strict inequality for at least one firm if either considering extremal equilibria or following best–reply adjustment dynamics after a parameter change. In Figure 4, where there is a unique equilibrium, the effect of an increase in t_1 is to move outward the best reply of player 1. If player 2 were to stay put at a_2^0 then the best response of player 1 would be $\widehat{a}_1$, but in equilibrium $a_1' > \widehat{a}_1$.[19]

In games with strategic complementarities, unambiguous monotone comparative statics obtain if we concentrate on stable equilibria. This is a multidimensional global version of Samuelson's (1979) correspondence principle, which links unambiguous comparative statics with stable equilibria and is obtained with standard calculus methods applied to interior and stable one-dimensional models.

As an example consider the (supermodular or log-supermodular) Bertrand oligopoly. There, extremal equilibrium price vectors are increasing in an excise tax t. Indeed, we have that $\pi_i = (p_i - t - c_i) D_i(p)$ and $\partial^2 \pi_i / \partial p_i \partial t = -\partial D_i / \partial p_i > 0$.

[19] Martin Peitz (2000) gives sufficient conditions for a price game to display multiplier effects.

Result 6. Duopoly with strategic substitutability (Vives (1990a)). If $n = 2$ and there is (a) strategic complementarity in own strategies, with π_i supermodular in a_i or $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$, and (b) strategic substitutability in rivals' strategies, with π_i with decreasing differences in (a_i, a_j) or $\partial^2 \pi_i / \partial a_{ih} \partial a_{jk} \leq 0$ for all $j \neq i$ and for all h and k, then the transformed game with new strategies $s_1 = a_1$ and $s_2 = -a_2$ is smooth supermodular. (See Figure 5 and note that this is the mirror image of Figure 3 with respect to the ordinate axis.) Therefore, all the results stated previously apply to this duopoly game as well. Unfortunately, the trick does not work for $n > 2$ and the extension to the strategic substitutability case for n players does not hold.

A typical example of duopoly with strategic substitutability is a Cournot market, where usually best replies are decreasing. In this case, the welfare result is as follows. If for some players payoffs are increasing in the strategies of rivals and for other players they are decreasing, then the largest equilibrium is best for the former and worst for the latter. This is the case in the Cournot duopoly with the strategy transformation yielding a supermodular game. The preferred equilibrium for a firm is the one in which its output is largest and the output of the rival lowest.

4.2 The scope of the theory: Is anything a GSC?

If not everything is a game of strategic complementarities, where are the bounds of the theory?

First of all, if we take the view that the order of the strategy spaces is part of the description of the game or that there is a "natural" order in the strategy spaces, then there are many games that are not of strategic complementarities. For example, not all Bertrand games with product differentiation are supermodular games. Roberts and Sonnenschein (1977), James Friedman (1983), and Vives (1999, Sec. 6.2) provide examples, including games with avoidable fixed costs and the classical Hotelling model where firms are located close to each other. In those cases, at some point best replies may jump down and a price equilibrium (in pure strategies) may fail to exist.[20] Indeed, with goods that are gross substitutes, prices may be strategic substitutes because the own-price elasticity of demand need not decrease in the prices charged by rivals. A price increase by rival j may lead to an *increase* in the own-price elasticity of demand for firm i because it makes consumers of brand i who do not have a strong preference for any product —that is, who are more price sensitive, more willing to switch brands. It may then pay for firm i to cut the price to gain these consumers. Steven Berry, James Levinsohn, and Ariel Pakes (1999) find some empirical support for this in certain markets. Another instance of strategic price substitutability among prices may come from the presence of strong network externalities. For example, in the logit model with network externalities (Simon Anderson, Andre de Palma, and Thisse (1992, Ch.7)), increasing

[20] See, however, the modified Hotelling game in Jacques Thisse and Vives (1992), where best responses may be discontinuous but are increasing.

the price set by a rival raises the value for consumers of the network of firm i, so it may pay this firm to cut prices in order to enlarge this lead (although this will not happen for small network externalities). To put it another way: in many games, best responses are simply nonmonotone. For example, they are increasing in some portion of the strategy space and decreasing in another.

However, we could also take the view that the order of the strategy sets of the players is a modeling choice at the convenience of the researcher. This is what we have done to extend the reach of the theory to duopolies with strategic substitutes. Then, if we allow also the construction of this order ex post, with knowledge of the equilibria of the game, then the answer to the question of the bounds of the theory is that most games *are* of strategic complementarities. This means that complementarities alone, in the weak sense stated, do not have much predictive power unless coupled with additional structure (Echenique (2004a)). Indeed, define a game with strategic complementarities as one in which there is a partial order on strategies (that can be chosen by the modeler), so that best responses are monotone increasing (and with strategy sets having a lattice structure).[21] Then: (i) a game with a unique pure strategy equilibrium is a GSC if and only if Cournot best–response dynamics (with unique or finite-valued best replies) have no cycles except for the equilibrium; and (ii) a game with multiple pure strategy equilibria is always a GSC. As a corollary: (iii) 2×2 games, generically, are either GSC or have no pure strategy equilibria (like matching pennies). Result (i) in particular means that a game with a unique and globally stable equilibrium is a GSC, according to the definition given. An example is the strategic substitutes case in the continuum of agents model of Section 2 when $r' > -1$. Note that in this case the game is dominance solvable.

Result (ii) is shown by taking one equilibrium to be the largest and another the smallest strategy profiles in a way that best responses are increasing.[22] Indeed, a game with multiple equilibria always involves a coordination problem (i.e., coordinating on one equilibrium). We can then find an order on strategies that makes the game one of strategic complementarities. However, note that this is done with a priori knowledge of the equilibria and the defined order; indeed, it may not be "natural" at all.

5 Oligopoly and comparative statics

This section reviews some of the basic applications to oligopoly, surveys very recent ones, and provides some new ones. It develops comparative statics results in Cournot markets (including entry), patent races, and multidimensional competition.

[21] In a GSC, as defined in this section, there may be no equilibrium in pure strategies because strategy sets need not be complete lattices.

[22] If a correspondence ϕ from X to X has two fixed points a and b, then define an order $\geq$ on X as follows. Let $y \geq x$ if and only if any of the following is true: $x = y, x = a$, or $y = b$. Then ϕ is actually weakly increasing in the sense that, if $y \geq x$, then there is a z in ϕ and a z' in $\phi(y)$ with $z' \geq z$. (In fact $(X, \geq)$ is a complete lattice; see Appendix.)

The analysis illustrates several points: the potential pitfalls of classical analysis, the extension of the methods to games that need not display complementarities globally, and the isolation of the crucial assumptions driving the results. As an example of the first issue, classical analysis —when studying the effects of increasing the number of firms n into a Cournot market— ignores that some equilibria may disappear (or appear) when changing n, making any "local" study meaningless (see Rabah Amir and Val Lambson (2000)). An analysis of a multimarket oligopoly coming from two-sided competition will exemplify the second issue. Using lattice-theoretic methods, conditions for "perverse" comparative statics will be derived in a context where the underlying game is not supermodular (Luis Cabral and Lluis Villas-Boas (2004)). Finally, an examination of a patent race will isolate the crucial assumptions behind the comparative statics of R&D effort with respect to the number of participants in the race (Vives (1999)). We deal in turn with comparative statics in Cournot markets, patent races, and multidimensional competition.

5.1 Comparative statics in Cournot markets

The standard Cournot game displays strategic substitutability: therefore, the game is supermodular only in the duopoly case (by changing the sign of the strategy space of one player), as discussed in Result 6. However, the lattice-theoretic approach delivers results also with n firms. I consider here a symmetric market (see Amir (1996a) and Vives (1999) for other results).

Consider a n-firm symmetric Cournot market in which the profit function of firm i is given by

$$\pi_i = P\left(Q\right) q_i - C\left(q_i\right).$$

Here $P\left(\cdot\right)$ is the smooth inverse demand with $P' < 0$, Q is total output, $C\left(\cdot\right)$ is the cost function of the firm, and q_i its output level. We parameterize the cost function of firm i by θ and let $C\left(q_i;\theta\right)$ be smooth with $\partial^2 C/\partial\theta\partial q_i \leq 0$.

In the standard approach (Jesús Seade (1980a,b), Avinash Dixit (1986)) it is assumed that payoffs are quasi–concave, and conditions

$$\left(n+1\right) P'\left(nq\right) + nP''\left(nq\right) q < 0 \;\text{ and }\; C''\left(q\right) - P'\left(nq\right) > 0$$

are imposed so that there is a unique and locally stable symmetric equilibrium q^*. Then standard calculus techniques show that an increase in θ increases q^* and that total output increases and profits per firm decrease as n increases. The comparative statics of output per firm with respect to the number of firms are ambiguous.

The classical approach has several problems. First of all, it is silent about the potential existence of asymmetric equilibria. Second, it is restrictive and may be misleading. For example, if the uniqueness condition for symmetric equilibria does not hold and there are multiple symmetric equilibria, changing n may either cause the equilibrium considered to disappear or introduce more equilibria (as in Figure 1).

In the lattice-theoretic approach (Amir and Lambson (2000), Vives (1999)), it is assumed only that $P' < 0$ and $C'' - P' > 0$. Then it can be shown[23] that

a symmetric equilibrium (and no asymmetric equilibrium) exists. Furthermore, at extremal Cournot equilibria (or following out–of–equilibrium best–reply dynamics): individual outputs are increasing in θ, total output is increasing in n, and profits per firm decrease with n. Furthermore, it can be shown that individual outputs decrease (increase) with n if demand is log-concave (log-convex and costs are zero). This approach does away with the unnecessary assumptions of the standard approach and derives new results.

The approach here also delivers the conditions (in a differentiated product environment) for Bertrand prices to be lower than Cournot prices as a corollary of the fact that Cournot prices must lie in region A^+(Figure 3) when actions are prices (Vives (1985b, 1990a); see also Vives (1999, Sec. 6.3)).

5.2 Patent races

Suppose that n firms are engaged in a memoryless patent race and have access to the same R&D technology (Lee and Wilde (1980)). An innovating firm obtains the prize V and losers obtain nothing. If a firm spends x continuously then the (instantaneous) probability of innovating is given by $h(x)$, where h is a smooth concave function with $h(0) = 0$, and $h' > 0$, $\lim_{x\to\infty} h'(x) = 0, h'(0) = \infty$ (a region of increasing returns for small x may be allowed). In the absence of innovating, the normalized profit of firms is zero. Under these conditions the expected discounted profits (at rate r) of firm i investing x_i if rival j invests x_j is given by

$$\pi_i = \frac{h(x_i)V - x_i}{h(x_i) + \sum_{j\neq i} h(x_j) + r}$$

(see Lee and Wilde (1980) and Jennifer Reinganum (1989)). Denote the best response of a firm by $x_i = R\left(\sum_{j\neq i} h(x_j) + r\right)$; this is well defined under the assumptions. Lee and Wilde (1980) restrict attention to symmetric Nash equilibria of the game and show that, under a stability condition at a symmetric equilibrium x^*, $R'((n-1)h(x^*))(n-1)h'(x^*) < 1$, x^* increases with n.

However, this approach suffers from the same problems as the comparative statics of entry in Cournot markets. It requires assumptions to ensure a unique and stable symmetric equilibrium and cannot rule out the existence of asymmetric equilibria. Nonetheless, the following mild assumptions ensure that the game is strictly log-supermodular: $h(0) = 0$ and h is strictly increasing in $[0, \overline{x}]$, with $h(x)V - x < 0$ for $x \geq \overline{x} > 0$. It follows then from Result 2 that equilibria exist and all are symmetric.

Let $x_i = x$ and $x_j = y$ for $j \neq i$. Then $\log \pi_i$ has (strictly) increasing differences in (x, n) for all y $(y > 0)$, and at extremal equilibria the expenditure

[23] See Vives (1999, pp. 42–43, 93–96, Sec. 4.3.1) for details.

intensity x^* is increasing in n. Furthermore, if h is smooth with $h' > 0$ and $h'(0) = \infty$, then $\partial \log \pi_i / \partial x_i$ is strictly increasing in n and (at extremal equilibria) x^* is strictly increasing in n. This follows because, under our assumptions, equilibria are interior and must fulfill the first–order conditions.

As before, starting at any equilibrium, an increase in n will raise the research intensity, with out-of-equilibrium adjustment according to best–reply dynamics. This will be so even if some equilibria disappear or new ones appear as a result of increasing n.

5.3 Multidimensional competition

Multidimensional competition provides another fertile ground for application of the approach, which can readily handle multidimensional strategy spaces. We will consider first an example with advertising and price as strategies and then examine multimarket oligopoly situations.

5.3.1 Advertising and prices

Consider our Bertrand oligopoly example where the demand $D_i(p; t_i)$ of firm i depends on advertising effort t_i, with $\partial D_i / \partial t_i > 0$. Suppose that goods are gross substitutes, $\partial D_i / \partial p_j \geq 0$ for $j \neq i$, and that demand is downward sloping, $\partial D_i / \partial p_i < 0$. Let

$$\pi_i = (p_i - c_i) D_i(p; t_i) - F_i(t_i);$$

here F_i is the cost of advertising, with $F_i' > 0$. The action of the firm is $a_i = (p_i, t_i)$, with natural upper bounds for p_i and t_i. Profits π_i are strictly supermodular in $a_i = (p_i, t)$ if

$$\frac{\partial^2 \pi_i}{\partial p_i \partial t_i} = (p_i - c_i) \frac{\partial^2 D_i}{\partial p_i \partial t_i} + \frac{\partial D_i}{\partial t_i} > 0.$$

A sufficient condition for the condition to hold is that $\partial^2 D_i / \partial p_i \partial t_i \geq 0$. This amounts to requiring that advertising increases the customers willingness to pay. Furthermore, π_i has increasing differences in $((p_i, t_i), (p_{-i}, t_{-i}))$ if $\partial^2 D_i / \partial p_i \partial p_j \geq 0$ for $j \neq i$ (given that $\partial D_i / \partial p_i \partial t_j = 0$, $j \neq i$). Under these assumptions, the game is supermodular and the largest (smallest) equilibrium has the feature of having high (low) prices and advertising levels. Multiple equilibria obtain with a symmetric linear demand system, where t_i increases the demand intercept if F' is concave enough. We have thus found conditions under which high prices are associated with high advertising levels.

5.3.2 Multimarket oligopoly

The previous example can be extended readily to multiproduct firms and even to pricing games are neither supermodular nor log-supermodular. For example, in the multiproduct logit model of Richard Spady (1984), best responses are increasing and there is a unique Bertrand equilibrium despite the fact that payoffs are single–peaked (not quasi-concave) and neither supermodular or log-supermodular in own actions or prices. Even so, strategic complementarity across prices of different firms holds.

A multimarket mixed oligopoly featuring products demand complements within the firm as well as substitutes across firms provides another example. This situation is typical of two-sided markets, where two groups of market participants benefit from interaction via a platform or intermediary. Intermediaries compete for business from both groups and set prices. Examples are numerous and include readers/viewers and advertisers in media markets, cardholders/consumers and merchants/retailers in payment systems such as credit cards, consumers and shops in shopping malls, authors and readers in academic journals, borrowers and depositors in banking, "subscription to a network" and "number of calls made to a network" in telecom markets, and in general buyers and sellers put together with the help of intermediaries (in real estate, financial products, or auction markets). The interaction between the two sides gives rise to complementarities or externalities between groups that are not internalized by end users. For example, a consumer who uses a credit card does not internalize the benefit that it confers to the other side of the market (the merchants).

Consider a situation of two-sided exclusive intermediation with two groups of participants (say columnists and readers, dating bars, workers and firms in a single region, consumers and shops in a mall), where each participant joins one of the two existing intermediaries and where the utility derived by a member of a group from joining a particular intermediary is increasing in the number of members of the other group joining the same intermediary.[24] When linear demands arise from Hotelling-type preferences for the intermediaries, the result is that prices charged by intermediaries are strategic complements across firms but strategic substitutes within the firm. The multimarket oligopoly game is therefore not a supermodular game as defined in Section 4. However, best replies will be increasing as long as the demand complementarity among the products of the same firm/intermediary is not very strong. In the context of the linear demand game with small and symmetric network effects, best replies are increasing and there is a unique symmetric equilibrium.

An interesting result in the Hotelling game, where total demand is inelastic, is that an increase in the cross-group network effect (i.e., an increase in the degree of demand complementarity of the "products" of the intermediary) reduces equilibrium profits. An increase in the impact of the benefits that one side of the market confers on the other when they go to an intermediary has, in fact, a detrimental equilibrium effect on profits. The reason is that the externality

[24] See Mark Armstrong (2002) for a survey of two-sided competition.

increase has no positive direct impact on demand at a symmetric equilibrium in which the whole market is covered, and it motivates each intermediary to cut prices. Since total demand stays constant (because it is price inelastic), equilibrium profits decrease. This result can be generalized whenever (a) the direct effect of the externality on demand at symmetric equilibria is small, so that profits for any intermediary have decreasing differences in the price charged to a group (and in consequence the externality parameter and best replies shift inward as the externality parameter increases) and (b) total demand is fixed (or it is quite price inelastic), so that the equilibrium price decrease translates into a profit decrease. In those circumstances the strategic pricing effect dominates the direct effect (Cabral and Villas-Boas (2004)). Economies of scope have a similar effect than demand externalities.[25]

6 Dynamic games

This section will take a look at dynamic issues building on comparative statics results for supermodular games (like Result 5) that predict movements of equilibrium variables when a parameter changes.

I examine first the conditions under which increasing or decreasing dominance occurs in oligopoly —that is, whether leaders or laggards have more incentives to invest. This is particularly relevant in situations where investment today, which could be a larger firm size if there are learning effects and/or adjustment costs, affects competitive conditions tomorrow. This application will illustrate the power of the approach to isolate the drivers of results and extend them beyond GSC (Susan Athey and Armin Schmutzler (2001)). I deal afterwards with full-blown dynamic Markov games and Markov perfect equilibria (MPE). First I tackle how static complementarities translate into dynamic complementarities and use the methodology to characterize MPE. Conditions are given that enable contemporaneous (intra-period) strategic complementarity (SC) and intertemporal (inter-period) SC to obtain. The relationships between static and dynamic strategic substitutability and complementarity are studied in alternating move games (Eric Maskin and Jean Tirole (1987, 1988a,b)) and in games with adjustment costs (Jun and Vives (2004)). Finally, the problem of existence and characterization of Markov perfect equilibria is addressed (Lauren Curtat (1996), Sleet (2001)).

The outcome of the analysis are new results uncovered (characterization of dynamic strategic complementarity; linkage between static and dynamic complementarity concepts; existence of MPE) and isolation of crucial assumptions in known results (increasing dominance; monotonicity of dynamic reaction functions in alternating move games).

6.1 Increasing or decreasing dominance?

[25] Peitz (2003) uses supermodular methods to study the effects of asymmetric access price regulation in telecom markets.

Suppose that the payoff to player i is given by $\pi_i(a_i, a_{-i}; t)$ with $t = (t_1, ..., t_n)$. The parameter t_i is to be interpreted as the state variable or initial conditions of player i in the game. Let both actions and state variables be one-dimensional. We would like to find conditions under which, if two firms differ only in their state variables if and $t_i > t_j$, then at any equilibrium we have $a_i(t) \geq a_j(t)$, with the interpretation being that an initial dominance is reinforced by actions of the firms. For example, in the presence of a learning curve, the firm that has accumulated more output has incentive to produce more.

Suppose that $\pi_i(a_i, a_{-i}; t)$ has increasing differences in (a_i, a_j) for $i \neq j$ (strategic complementarity) and increasing differences in $(a_i, (t_i, -t_{-i}))$. Assume also that all the players have the same strategy set and that the payoffs are exchangeable (players do not care about the identity of their opponents, only about their actions and payoff–relevant parameters or state variables). This means that the payoffs of two players are the same if actions and state variables are exchanged among them. Suppose also that payoffs are strictly quasi–concave, so there is a unique best–response function for any player, and that we have an equilibrium for which (without loss of generality) $a_1 < a_2$ with $t_1 > t_2$. Fix the actions of the players $n = 3$,..., n at their equilibrium levels. Because of strict quasi–concavity and exchangeability, we can write the best response of firm 1 as $r(a_2; t_1, t_2)$ and that of firm 2 as $r(a_1; t_2, t_1)$. Because of strategic complementarity and $a_1 < a_2$, we have $a_1 = r(a_2; t_1, t_2) \geq r(a_1; t_2, t_1)$. Since $t_1 > t_2$ and since $\pi_i(a_i, a_{-i}; t)$ has increasing differences in $(a_i, (t_i, -t_{-i}))$, it follows that $r(a_1; t_1, t_2) \geq r(a_1; t_2, t_1) = a_2$, contradicting the supposition that $a_1 < a_2$. We conclude, as desired, that $a_1 \geq a_2$ if $t_1 > t_2$. (See Athey and Schmutzler (2001))[26]

To help the intuition, just think of the case $n = 2$ in Figure 6 starting from a symmetric equilibrium at $t_1 = t_2$ and increasing t_1. We see how the best reply of firm 1 shifts outward while the best reply of player 2 shifts inward and the equilibrium moves to a region with $a_1 \geq a_2$.

An example is provided by the Bertrand oligopoly model via product differentiation with learning by doing or, alternatively, with production adjustment costs, or even with switching costs. With learning by doing the profit function of firm i is

$$\pi_i = (p_i - (c - f(t_i)) D_i(p),$$

where t_i is the accumulated output of the firm. Letting $a_i = -p_i$, we have that $\partial^2 \pi_i / \partial a_i \partial t_i > 0$ and $\partial^2 \pi_i / \partial a_i \partial t_j < 0$. With production adjustment costs, the profit function is

$$\pi_i = (p_i - c) D_i(p) - F(D_i(p) - D_i(t)),$$

where t_i is the price of the firm in the previous period and F is the increasing and convex production adjustment cost with $F(0) = 0$. Then $\partial^2 \pi_i / \partial p_i \partial t_i > 0$ and $\partial^2 \pi_i / \partial p_i \partial t_j < 0$. In both cases, the firm starting with a higher output

[26] Without requiring quasi–concavity we could make the same argument with the extremal best replies. The result would then be true for extremal equilibria.

level (lower price) has an incentive to set lower prices in equilibrium. However, this does not mean that there is increasing dominance. Even though in any period the larger firm sets a lower price, it may well be that the price difference between the firms disappears over time. In fact, this is exactly what happens at the MPE of an infinite–horizon version of the model (Jun and Vives (2004)).

In the switching costs model (Alan Beggs and Paul Klemperer (1992)), firms compete in prices and t_i is the loyal customer base of firm i. In this case we have that $\partial^2\pi_i/\partial p_i \partial t_i > 0$, because lowering prices is more costly to a firm with a larger customer base, and $\partial^2\pi_i/\partial p_i \partial t_j < 0$. It then follows that a firm with a larger customer base will be softer in pricing. This is to be interpreted as decreasing dominance (and indeed the authors show that, at an MPE of the full-blown dynamic game, initial asymmetries in market shares are eroded). However, the reader is warned that in a dynamic game firms are forward looking and the continuation payoffs need not look like the static payoffs. Therefore, the static dominance need not translate in dominance in the dynamic game. We will show in the next section the relationships between static and dynamic properties of payoffs.

The result can be extended to the strategic substitutes case (where $\pi_i(a_i, a_{-i}; t)$ has decreasing differences in (a_i, a_j), $i \neq j$) with the restriction that $-\partial^2\pi_i/(\partial a_i)^2 > \left|\partial^2\pi_i/\partial a_i \partial a_j\right|$, $i \neq j$ (this implies that the profit function of any player is concave and that a duopoly game would have a unique equilibrium). The conditions in the result for strategic substitutes are typically met when actions are investments in cost reduction and also in some models of quality enhancement.[27] Then profits at the market stage as a function of those investments display strategic substitutability both in Cournot and Bertrand models. The result can also be used to show that learning by doing in a Cournot market leads to increasing dominance. That is, the firm that is ahead of the learning curve remains ahead because it has incentives to produce more. Actions are current rates of output, and state variables are the inherited accumulated production of each firm. Let the profit function of firm i be given by

$$\pi_i = (P(Q) - (c - f(t_i)))\, q_i.$$

Here $P(\cdot)$ is the inverse demand; Q is total output; $f(\cdot)$ is the learning curve, a differentiable and concave function of total accumulated output of the firm t_i with $f' > 0$; and q_i is its current output level. If inverse demand is log-concave then best replies are downward sloping (strategic substitutes). Furthermore, $\partial^2\pi_i/\partial q_i \partial t_i = f' > 0$ and $\partial^2\pi_i/\partial q_i \partial t_j = 0$. As a consequence, $t_i > t_j$ implies that, at the (unique) Cournot equilibrium, $q_i(t) \geq q_j(t)$.[28]

[27] This is so in the Avner Shaked and John Sutton (1982) model of vertical quality differentiation when the market is covered. However, in the classical linear Bertrand duopoly with product differentiation, investments in quality that raise the intercept of demand for the own product (Vives (1985a)) or that increase the willingness to pay by lowering the absolute value of the slope of demand $|\partial D_i/\partial p_i|$(Vives (1990b)) are strategic complements.

[28] A similar example with product differentiation and network demand externalities (Michael Katz and Carl Shapiro (1986)) would have $\pi_i = (P_i(q) - (c - f(t_i)))\, q_i$, where q is the vector of output levels of the firms and t_i is the accumulated sales of product i.

6.2 Markov games

An important issue is how static complementarities translate (or not) into dynamic complementarities. In this section we will explore the issue in the context of discrete time Markov games. A Markov strategy depends only on (state) variables, denoted y, that condense the direct effect of the past on the current payoff. Let the current payoff of player i be $\pi_i(x, y)$, where x is the current action profile vector and y is the state evolving according to $y = f(x^-, y^-)$, where x^- and y^- are (respectively) the lagged action profile vector and the lagged state. A Markov perfect equilibrium (MPE) is a subgame-perfect equilibrium in Markov strategies. That is, an MPE is a set of strategies optimal for any firm, and for any state of system, given the strategies of rivals.

What do we mean by dynamic strategic complementarity (SC) or dynamic strategic substitutability (SS)? We can think of "contemporaneous" SC when the value function at an MPE $V_i(y)$ displays SC (V_i has increasing differences in (y_i, y_{-i})). We can think of "intertemporal" SC when dynamic best replies, or the policy function at an MPE, are monotone. There is intertemporal SC (SS) when a player raising his state variable today increases (decreases) the state variable of his rival tomorrow. I will investigate these properties for a class of simple dynamic Markov games that admits two-stage games, simultaneous move games with adjustment costs, and alternating moves games.

The class of simple dynamic Markov games is defined as follows. Consider the n-player game in which the actions of player i in any period lie in A_i, a compact cube of Euclidean space; here $\pi_i(x, y)$ is the current payoff for player i, with $y \in A$ the action profile in the previous period (state variables) and $x \in A$ the current action profile. This simple class of games encompasses two-stage games and infinite–horizon games of simultaneous moves with adjustment costs or of alternating moves. In a two–stage game, $y \in A$ is the action profile in the first stage, $x \in A$ the action profile in the second stage, and $\pi_i(x, y)$ the payoff for player i. Consider now an infinite–horizon, discrete time game with discount factor δ. With simultaneous moves and adjustment costs, the payoff to player i is given by

$$\pi_i(x, y) = u_i(x) + F_i(x, y),$$

where $u_i(x)$ is the current profit in the period and $F_i(x, y)$ is the adjustment cost in going from past actions (y) to current actions (x). Assume that $F_i(x, x) = 0$, $i = 1, 2$; that is, when actions are not changed, there is no adjustment cost. With alternating moves in a duopoly, x is the action of the player moving now and y is the action of the player who moved last period.

We take in turn the issues of contemporaneous SC in two–stage games and intertemporal SC or SS in infinite–horizon games. We end the section with some remarks on the existence of MPE.

6.2.1 Contemporaneous SC in two-stage games

The contemporaneous SC property obtains (a) if at the second stage, for any actions y in the first stage, payoffs $\pi_i(x,y)$ display SC and (b) if the SC property is preserved when payoffs are folded back at the first stage in a subgame-perfect equilibrium.[29]

Suppose that $\pi_i(x,y)$ displays increasing differences (or is supermodular) in any pair of variables. Let $V_i(y) \equiv \pi_i(x^*(y), y)$, where $x^*(y)$ is an extremal equilibrium in the second stage. Extremal equilibria exist at the second–stage for any y because the second stage game is supermodular. A particular case is when, contingent on y, a unique Nash equilibrium $x^*(y)$ obtains at the second stage. $V_i(y)$ is thus the first–period reduced form payoff for player i. I claim that $V_i(y)$ is supermodular in y.

The argument is simple. We have

$$V_i(y) \equiv \pi_i(x^*(y), y) = \max_{x_i} \pi_i\left(x_i, x^*_{-i}(y), y\right).$$

Note that $x^*_j(y)$ increases in y because π_i has increasing differences in (x_i, y). It follows that $V_i(y)$ is supermodular in y because (i) π_i is supermodular in all arguments, (ii) $x^*_j(y)$ is increasing in y, (iii) supermodularity is preserved by increasing transformations of the variables, and (iv) supermodularity is preserved under the maximization operation.

The result can be readily generalized to finite–horizon multistage games with observable actions (as defined, e.g., by Fudenberg and Tirole (1991)), where the payoff to each player displays increasing differences in any two variables.[30]

An example of the result is provided by the Bertrand oligopoly with advertising. Under the assumptions made (Section 5.3.1),

$$\pi_i = (p_i - c_i)\, D_i(p; z_i) - F_i(z_i)$$

is supermodular in any pair of arguments, and the first–stage value function at extremal equilibria is supermodular. That is, advertising expenditures are strategic complements. The assumptions are fulfilled in the classical linear differentiated product Bertrand competition model with constant marginal costs when either advertising or investment in product quality raises the demand intercept of the firm exerting the effort (Vives (1985a)) or increases the willingness to pay for the product of the firm by lowering the absolute value of the slope of demand $|\partial D_i / \partial p_i|$ (Vives (1990b)). In this case, for a given advertising effort

[29] This section draws on Vives (2004).

[30] However, the result cannot be extended to the case where each payoff function $\pi_i(x,y)$ fulfills the ordinal complementarity conditions (or single–crossing property SCP) in any pair of variables. Indeed, it is easy to construct examples where each payoff fulfills the SCP for all pairs of variables while the property is not preserved in the reduced form first–period payoffs (Echenique (2004b)). Neither supermodularity nor increasing differences can be weakened to the ordinal SCP, even though the simultaneous move ("open loop") game would be an ordinal GSC and even though the second–period equilibrium is monotone in first–period choices.

there is a unique price equilibrium at the second stage.[31]

The result can be extended easily to a duopoly case in which, for all i, $\pi_i(x, y)$ has increasing differences in $(x_i, -x_j)$, $(y_i, -y_j)$, and $(x_i, (y_i, -y_j))$, $j \neq i$. An example is provided by a Cournot duopoly in which outputs are strategic substitutes and y_i is the cost–reduction effort by firm i. Let

$$\pi_i = P(x_1, x_2)x_i - C_i(x_i, y_i)$$

with $\partial^2 C_i/\partial x_i \partial y_i \leq 0$. Then the assumptions are fulfilled because $\partial^2 \pi_i/\partial x_i \partial y_i \geq 0$, and $\partial^2 \pi_i/\partial x_i \partial y_j = \partial^2 \pi_i/\partial z_i \partial y_j = 0$ for $j \neq i$. We then have that cost reduction investments are strategic substitutes at the first stage. With linear demand there is a unique equilibrium at the second stage (see Vives (1990b) for a computed example where investment reduces the slope of marginal costs and for a reinterpretation in terms of firms that invest in expanding their own market).[32]

6.2.2 Intertemporal strategic complementarity

Consider a stationary MPE of an infinite–horizon simultaneous move game with discount factor δ, and let $V_i(y)$ be the value function associated to player i at the MPE. Player i solves

$$\max_{x_i} \{\pi_i(x, y) + \delta V_i(x)\}.$$

Let $x^*(y)$ be the (assumed unique) contemporaneous Nash equilibrium, given y and the MPE policy functions for the players. From Result 5 we have that, for all i, if

1. $\pi_i(x, y) + \delta V_i(x)$ has increasing differences in (x_i, x_{-i}) and
2. π_i has increasing differences in (x_i, y),

then $x^*(y)$ is increasing in y (i.e., we have intertemporal SC: x_i^* increases with y_j for $j \neq i$). In order for (1) to hold it is sufficient that both π_i and V_i have increasing differences in (x_i, x_{-i}).

Likewise, we have the corresponding result for a duopoly with strategic substitutability. For all i, if

1. $\pi_i(x, y) + \delta V_i(x)$ has increasing differences in $(x_i, -x_j)$, $j \neq i$, and
2. π_i has increasing differences in $(x_i, (y_i, -y_j))$,

then x_i^* increases in $(y_i, -y_j)$ (i.e., we have intertemporal SS: x_i^* decreases with y_j for $j \neq i$).

The question is: When will the assumptions be fulfilled? We will consider in turn the adjustment cost model and the alternating move duopoly.

[31] However, if firms invest in cost reduction, the second–stage SC is transformed into a first–stage SS. The same happens with product enhancement investments in the Shaked and Sutton (1982) model of vertical quality differentiation when the market is covered.

[32] It is worth noting that, with high enough spillovers, firms' R&D cost–reduction investments are SC in the two-stage game (Claude d'Aspremont and Alexis Jacquemin (1988)).

Simultaneous moves with adjustment costs Consider the adjustment cost model and interpret actions as either prices or quantities. Let production or price bear the convex adjustment cost F. Models with price adjustment costs, or "menu costs", are commonly used in macroeconomics. It is easy to see that, with price competition (with static SC) and menu costs, the marginal profit for firm i is increasing in the price y_i charged by the firm in the previous period and decreasing in the price y_j charged by the rival in the previous period. This case falls in the domain of the foregoing general result, provided that the value function V_i displays SC (this is true in the linear-quadratic specification). With quantity competition (static SS) and production adjustment costs, the marginal profit for firm i is increasing in the production y_i of the firm in the previous period and decreasing in the production y_j of the rival in the previous period. This case falls in the domain of the duopoly result with SS, provided the value function displays SS (as in the linear-quadratic specification).

In these two cases, static SC or SS is transformed into intertemporal SC or SS. However, this need not be always so. Jun and Vives (2004) have fully characterized the linear and stable MPE in a symmetric differentiated duopoly model with quadratic payoffs and adjustment costs in a continuous time infinite–horizon differential game by building on the work of Stanley Reynolds (1987) and Robert Driskill and Stephen McCafferty (1989). Jun and Vives (2004) consider both SC (Bertrand) and SS (Cournot) competition with production or price (menu) adjustment costs. It is found that contemporaneous (dynamic) SC or SS are inherited from static SC or SS. Indeed, V_i displays increasing (decreasing) differences in (y_i, y_j) when there is static SC (SS). Intertemporal SC or SS then obtains depending on what variable bears the adjustment cost. We know already from the previous paragraph that, under price competition with menu costs (quantity competition with production adjustment costs), static SC (SS) is transformed into intertemporal SC (SS).

In contrast, for the mixed case of price competition with production adjustment costs, Jun and Vives show that the static SC is transformed into intertemporal SS. Then we have that the marginal profit for firm i is increasing in the price y_i of the firm in the previous period and decreasing in the price of the rival y_j in the previous period. The reason —much as in the learning curve model with price competition— is that a firm wants to make the rival small today in order to induce it to price softly tomorrow. Indeed, a smaller rival will face a stiff cost of increasing its output. A cut in price today will therefore bring a price increase by the rival tomorrow. The result is that if production is costly to adjust then intertemporal SS obtains, whereas if price is costly to adjust then intertemporal SC obtains.

Having intertemporal SC or SS matters because it governs strategic incentives at the MPE with respect to innocent behavior at the open-loop equilibrium. Indeed, Jun and Vives show that with intertemporal SC (SS), steady–state prices at the MPE will be above (below) the stationary open–loop equilibrium prices, which coincide with the static equilibrium prices with no adjustment costs. In fact, this provides a generalization of the taxonomy of strategic behavior in two-stage games of Drew Fudenberg and Tirole (1984) to the full-blown infinite–horizon game.

Alternating move duopoly Consider a duopoly game in which the payoff to firm i $(i = 1, 2)$ is $\pi_i(a_1, a_2)$ and the action set available to the firm is a compact interval. Two players in a duopoly interact repeatedly, with player 1 moving in odd periods $t = 1, 3, ...$ and player 2 in even periods $t = 0, 2, ...$ (Richard Cyert and Morris DeGroot (1970), Maskin and Tirole (1987, 1988a,b)). The action (e.g., price or quantity) of player i is fixed for one period. Denote by x the action of the player moving now and by y the action of the player who moved last period. The state variable for firm i is therefore the action taken in the previous period by firm j. A (pure) Markov strategy for firm i is a function $R_i(\cdot)$ that maps the past action of firm j into an action for firm i. This is truly a dynamic reaction function, in contrast with the best–response functions derived in the static games considered in Section 3 (in which best-response functions are useful in finding equilibria and characterizing stability properties).

A Markov perfect equilibrium (MPE) is a pair of dynamic reaction functions $(R_1(\cdot), R_2(\cdot))$ such that, for any state, a firm maximizes its present discounted profits given the strategy of the rival. The pair $(R_1(\cdot), R_2(\cdot))$ is an MPE if and only if there exist value functions $(V_1(\cdot), V_2(\cdot))$ such that player 1 solves $R_1(y) \in \arg\max_x \left\{\pi_1(x, y) + \delta\widetilde{V}_1(x)\right\}$, where

$\widetilde{V}_1(x) = [\pi_1(x, R_2(x)) + \delta V_1(R_2(x))]$

$V_1(y) = \max_x \left\{\pi_1(x, y) + \delta\widetilde{V}_1(x)\right\}$,

and similarly for player 2. That is, given the state variable y (the current action of firm 2) for firm 1, $V_1(y)$ gives the present discounted profits when it is firm 1's to move and both firms use the dynamic reaction functions $(R_1(\cdot), R_2(\cdot))$.

Suppose that MPE dynamic reaction functions exist. Then, according to our result, they will be monotone increasing (decreasing) if the underlying one-shot simultaneous move game is strictly supermodular (supermodular in $(x, -y)$), that is, if $\pi_i(x, y)$ has strictly increasing differences in (x, y) (in $(x, -y)$). Then any selection $R_1(\cdot)$ from the set of maximizers of $\pi_1(x, y) + \delta\widetilde{V}_1(x)$ is increasing (decreasing). No other property is needed.

Existence of an MPE can be established easily with quadratic payoff functions: Cournot with homogeneous products (Maskin and Tirole (1987)) and Bertrand with differentiated products (Vives 1999, Ex. 9.12)). It can be shown that for any δ there is a unique linear MPE that is symmetric and (globally) stable and that the steady–state action is increasing in δ and equals the static Nash equilibrium when $\delta = 0$.

The strategic incentives for a firm in the Cournot case are to increase its output in order to reduce the output of the rival and in the Bertrand case (with product differentiation) to increase price in order to induce the rival to be softer in pricing. Thus, in the Cournot (Bertrand) case, the static strategic substitutability (complementarity) translates into intertemporal strategic substitutability (complementarity). In other words, in the Cournot (Bertrand) case, both static and dynamic reaction functions are downward (upward) sloping. An increase in the weight firms that put into the future (a larger δ) increases the strategic incentives, with the result of a higher output (price) in the Cournot (Bertrand) market. In any case, the equilibrium action is larger than the static equilibrium when $\delta > 0$.

With homogenous products and price competition ,dynamic reaction functions are no longer monotone. This is so because, with a homogeneous product, the marginal profit of a firm is not monotone in the price charged by the rival. For example, if the rival sets a (strictly) lower price than firm i, then firm i's marginal profit of changing its price is zero; and if the rival sets a (strictly) larger price then firm i's marginal profit is positive, provided that its price is below the monopoly price. However, if the prices of both firms are equal, the marginal profit is negative. A consequence of this lack of monotonicity is the finding of multiple equilibria (including equilibria of the "kinked demand curve" type and price cycles; see Maskin and Tirole (1988b)).

6.2.3 Existence of MPE

Until now we have not dealt with the existence problem for MPE, only with the characterization of equilibria. Lattice-theoretic methods can be used when there is enough monotonicity in the problem under study.

Laurent Curtat (1996) shows existence of MPE of stochastic games with complementarities in discrete time and infinite horizon.[33] He considers multidimensional action spaces and a multidimensional state evolving according to a transition probability as a function of the current state and action profile. Payoffs are smooth and display per–period complementarities and positive externalities or spillovers (the payoff to a player is increasing in the actions of rivals and the state); the transition distribution function is smooth, displays complementarities, and is stochastically increasing in actions and states. Furthermore, the payoff to a player as well as the transition distribution function fulfil a strict dominant diagonal condition. These strong assumptions allow us to collapse the multiperiod problem to a reduced form static game (with continuation value functions increasing in the state variable) which is shown to be supermodular. An equilibrium can then be found with value functions increasing in the state.

Examples of games fulfilling the assumptions are a dynamic version of the search game considered in Section 3 (where the parameter θ evolves stochastically in a monotone increasing way with the average search effort of the population: the higher the average effort $\widetilde{a}_t$ in period t, the higher the θ_{t+1} in expected terms) and a dynamic version of a Cournot oligopoly with complementary products and learning by doing, where a high level of accumulated output by one firm yields stochastically higher levels of cumulated experience and lower production costs to the firm (learning by doing) and to the rivals (spillovers).

Another successful application of the techniques used to prove the existence of an MPE is provided by Sleet (2001). Sleet considers a version of the adjust-

[33] See Amir (1996b) for another application.

ment cost model of the previous section in an infinite–horizon discrete game with a continuum of players and symmetric payoffs. Firms set prices and prices are costly to adjust. The payoff to a player in any period is given by $\pi(x, y, G, \theta) = R(x, G, \theta) - F(x, y)$, where x is the current price of the firm, y the price in the previous period, G the distribution of prices chosen by other firms, θ a firm–specific shock (i.i.d. across firms and dates), R the net revenue function, and F the adjustment cost. The payoff is increasing in G and has increasing differences in $(x, (G, \theta))$, and $-F(x, y)$ is supermodular. Under some further technical restrictions, the existence of a symmetric monotone MPE is shown in which each firm uses the same increasing MPE policy function; this yields an action in the current period that is contingent on last period's action, last period's distribution of actions, and the player's specific shock θ. This is done by showing the existence of a fixed point of an increasing function that maps (increasing) policy functions onto themselves. The problem is simplified because, with a continuum of firms, no firm can influence any aggregate and each firm faces a dynamic programming problem. Furthermore, and as usual with the lattice-theoretic approach, an algorithm to compute the largest or the smallest equilibrium policy functions can be provided.

The model corresponds to a monopolistic competition model where no firm influences the market aggregates yet each retains some market power, the demand or technology firms are subject to a period-specific shock, and prices are subject to continuous adjustment costs. For example, the demand for the product of a firm may depend on the average price charged in the market or on a price index. The assumptions are fulfilled with linear or constant elasticity demands and quadratic or constant elasticity production costs (subject to a multiplicative shock) and with quadratic costs of price adjustment.[34]

7 Bayesian games

Bayesian games provide a fertile ground for applications of the lattice-theoretical approach. The reason is that they allow for general strategy spaces and payoff functions. In Section 7.1 I present the setup of the Bayesian game and basic approaches to the difficult issue of existence of equilibrium in pure strategies —together with some examples and applications to oligopoly, teams, and games of voluntary disclosure. Most recent advances are based on the lattice-theoretic approach, be it with supermodular games (Vives (1990a)), single-crossing properties (Athey (2001)), or "monotone supermodular" games (Timothy Van Zandt and Vives (2004)). The last two approaches deliver conditions for equilibria to be monotone in type, a desirable property in auctions and global games, for example. In fact, global games typically belong to the monotone supermodular class. Section 7.2 deals with global games and applications to currency and banking crisis. It makes clear that dominance solvability in the standard global game (Morris and Shin (2002)) obtains because the underlying game is

[34] See Olivier Blanchard and Stanley Fischer (1989, Sec. 8.1) and Julio Rotemberg (1982).

one of strategic complementarities, and the key to uniqueness is precisely that the strength of the strategic complementarities is not too large.

7.1 Bayesian Nash equilibrium: Existence and characterization

In a Bayesian game, the type of a player embodies all the decision-relevant private information. Let T_i be a subset of Euclidean space and the set of possible types t_i of player i. The types of the players are drawn from a common prior distribution μ on $T = \prod_{i=0}^{n} T_i$, where T_0 is residual uncertainty not observed by any player. The action space of player i is a compact cube of Euclidean space A_i, and his payoff is given by the (measurable and bounded) function $\pi_i : A \times T \to \mathbb{R}$, where $A = \prod_{i=1}^{n} A_i$. The (ex post) payoff to player i when the vector of actions is $a = (a_1, ..., a_n)$ and when the realized types $t = (t_1, ..., t_n)$ is thus $\pi_i\,(a; t)$. Action spaces, payoff functions, type sets, and the prior distribution are common knowledge. The Bayesian game is then fully described by $(A_i, T_i, \pi_i; i \in N)$.

A (pure) strategy for player i is a (measurable) function $\sigma_i : T_i \to A_i$ that assigns an action to every possible type of the player. Let Σ_i denote the strategy space of player i and identify strategies σ_i and τ_i if they are equal with probability 1. Let $\sigma = (\sigma_1, ..., \sigma_n)$. Denote the expected payoff to player i, when agent j uses strategy σ_j, by $U_i\,(\sigma) = E\pi_i\,(\sigma_1\,(t_1)\,, ..., \sigma_n\,(t_n)\,; t)$.

A Bayesian Nash equilibrium is a Nash equilibrium of the game $(\Sigma_i, U_i, i \in N)$ where the strategy space and payoff function of player i are denoted Σ_i and U_i, respectively. Denote by $\beta_i \colon \Sigma_{-i} \to \Sigma_i$ player i's best-reply correspondence in terms of strategies. Then a Bayesian Nash equilibrium is a strategy profile σ such that $\sigma_i \in \beta_i(\sigma_{-i})$ for $i \in N$. We can define a natural order in the strategy space $\Sigma_i : \sigma_i \leq \sigma_i'$ if $\sigma_i\,(t_i) \leq \sigma_i'\,(t_i)$, in the usual componentwise order, with probability 1 on T_i.

This formulation of a Bayesian game is general and encompasses common and private values as well as perfect or imperfect signals. With "pure" private values, allowing for correlated types, we have $\pi_i\,(a; t_i)$. For example, types are private cost parameters of firms. A "common value" case is $\pi_i\,(a; t) = v_i(a; \Sigma_i t_i)$. For instance, there is a common demand shock in an oligopoly and firm i observes component t_i only. As an example of imperfect signals, suppose firms observe with noise their cost parameters. Then t_0 could represent the n-vector of firms' cost parameters and t_i the private cost estimate of firm i. The various cost parameters as well as the error terms in the private signals may be correlated.

7.1.1 Equilibrium existence in pure strategies

Existence of pure-strategy Bayesian equilibria in games with a continuum of types and/or actions has proved to be a difficult issue. Typical sufficient conditions for existence of pure-strategy Bayesian equilibria include conditionally independent types, finite action spaces, and atomless distributions for types (see Roy Radner and Richard Rosenthal (1982) and Milgrom and Robert Weber (1985)).[35] Under these assumptions the authors show first the existence of mixed strategy equilibria and then obtain a purification result. In order for this approach to work, independence (or at least conditional independence) of the distribution of types is needed.

The lattice-theoretic method has provided three types of results:

1. for supermodular games with general action and type spaces (Vives (1990a));
2. for games in which each player uses a strategy increasing in type in response to increasing strategies of rivals (Athey (2001)); and
3. for "monotone" supermodular games with general action and type spaces (Van Zandt and Vives (2004)).

Supermodular games In the first approach (Vives (1990a) and Vives (1999, Sec. 2.7.3,)), existence of pure–strategy Bayesian equilibria follows from supermodularity of the underlying family of games defined with the ex post payoffs for given realizations of the types of the players. A key observation is that supermodularity of this underlying family of games is inherited by the Bayesian game.

Let π_i be supermodular in a_i and have increasing differences in (a_i, a_{-i}). Then $U_i(\sigma)$ is supermodular in σ_i and has increasing differences in (σ_i, σ_{-i}), because supermodularity and increasing differences are preserved by integration. Furthermore, strategy spaces in the Bayesian game Σ_i can be shown to have the appropriate order structure (i.e., they are complete lattices). Then the game $(\Sigma_i, U_i, i \in N)$ is a GSC an,d for all $\sigma_{-i} \in \Sigma_{-i}$, $\beta_i(\sigma_{-i})$ contains extremal elements $\bar{\beta}_i(\sigma_{-i})$ and $\underline{\beta}_i(\sigma_{-i})$. Existence of extremal pure strategy Bayesian equilibria then follows from the general versions of the results in Section 2 (see also Vives (1990a; 1999, Sec. 2.7.3)). This existence result holds for multidimensional action spaces and requires no distributional restrictions. The driving assumption is strategic complementarities.

Applications of this approach can be found in oligopoly games and team theory (as we shall see below), Diamond's (1982) search model, natural resource exploration games with private information (see Kenneth Hendricks and Dan Kovenock (1989) and Milgrom and Roberts (1990a)), and global games (see Section 7.3).

Single-crossing properties In the second approach (Athey (2001)), conditions are imposed so that an equilibrium in monotone increasing strategies (in types) can be found. Suppose that both action A_i and types sets T_i for any player i are compact subsets of the real line and that types have a joint density μ that is bounded, atomless, and log-supermodular (i.e., types are affiliated). Suppose also that $\pi_i(a, t)$ is continuous and supermodular in a_i and

[35] Ali Khan and Yeneng Sun (1995) show existence of pure-strategy equilibria when types are independent, payoffs continuous, and action sets countable.

has increasing differences in (a_i, a_{-i}) and (a_i, t) or, alternatively, that $\pi_i(a, t)$ is nonnegative and log-supermodular in (a, t). Then the Bayesian game has a pure-strategy equilibrium in increasing strategies. Note that in the first case the first approach outlined already delivers existence of a pure-strategy equilibrium.

The proof of these results relies on the standard Kakutani fixed point theorem, which relies on convex-valued correspondences. It turns out that with discrete action spaces and under the prevailing assumptions, best–response correspondences are convex valued. A key step in the proof is to show that, under our assumptions, if the rivals of player i use increasing strategies then the payoff to player i is log-supermodular or has increasing differences (or, in general, fulfills an appropriate single–crossing property) in action and type. This ensures that a player uses a strategy that is increasing in his type as a best response to increasing strategies of rivals.[36] The existence result for discrete action spaces can then be used to show existence with a continuum of actions via a purification argument.

An example of the result is our differentiated Bertrand oligopoly in which firm i has random marginal cost t_i. Then it is immediate that $E(D_i(p_i, p_{-i}(t_{-i}) \mid t_i)$ is log-supermodular in (p_i, t_i) if both $D_i(p_i, p_{-i})$ and the joint density of $(t_1, ..., t_n)$ are log-supermodular and if the strategies of rivals, $p_j(\cdot)$, $j \neq i$, are increasing in types. It follows that

$$E(\pi_i \mid t_i) = (p_i - t_i)E(D_i(p_i, p_{-i}(t_{-i}) \mid t_i)$$

is log-supermodular in (p_i, t_i) and that the best–reply map of player i is increasing in t_i.

The approach can be used also in games that are not of SC and with discontinuous payoffs. For example, in auctions the existence of monotone equilibria in pure strategies can be shown for:

- first-price auctions with heterogeneous (weakly) risk–averse bidders characterized by private affiliated values or common value and conditionally independent signals (Athey (2001)); and
- uniform price auctions featuring multiunit demand with nonprivate values and independent types (McAdams (2003)).[37]

Monotone supermodular games Combining both approaches, Van Zandt and Vives (2004) show a stronger result for "monotone" supermodular games

[36] The result follows directly from the assumptions by noting that (a) if $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular (or log-supermodular) then so is the function $f(h_1(x_1), ..., h_n(x_n))$ provided the h_i functions are increasing and (b) that if $g(x, t)$ is supermodular (log-supermodular) in (x, t) then $E\{g(x, t) \mid t_i\}$ is supermodular (log-supermodular) in (x, t_i) provided that (the random vector) t is affiliated. See Vives (1999, pp.69, 229–230) and Athey (2001) for details.

[37] The result is shown using an intermediate result, allowing multidimensional types and multidimensional (Euclidean) action spaces, that puts assumptions on nonprimitives. The conditions are: atomless types and interim (conditional on type) expected payoff of each player (quasi)-supermodular in his action; and with single-crossing in own action and type given that other players use strategies increasing in types.

with multidimensional action spaces and type spaces. Let $\Delta(T_{-i})$ be the set of probability distributions on T_{-i} and let player i's posteriors be given by the (measurable) function $p_i : T_i \to \Delta(T_{-i})$, consistent with the prior μ, where $p_i(\cdot \mid t_i) \in \Delta(T_{-i})$ denotes i's posteriors on T_{-i} conditional on t_i.[38] A monotone supermodular game is defined by the following properties.

1. *Supermodularity and complementarity between action and type*: π_i supermodular in a_i, and with increasing differences in (a_i, a_{-i}) and in (a_i, t).
2. *Monotone posteriors*: $p_i : T_i \to \Delta(T_{-i})$ increasing with respect to the partial order on $\Delta(T_{-i})$ of first–order stochastic dominance (a sufficient but not necessary condition is that μ be affiliated).

Under these conditions, there is a largest and a smallest Bayesian equilibrium and each one is in monotone strategies. There might be other equilibria that are in nonmonotone strategies but, if so, they will be "sandwiched" between the largest and the smallest one, which are monotone in type. The assumptions on action and type spaces can be considerably weakened but the result cannot be extended to log-supermodular payoffs.

The argument for the result is powerful yet simple. First, the Bayesian game is of strategic complementarities, as in the first approach to existence. This means that the extremal best–reply maps are well–defined for each player and are increasing in the strategies of rivals. Second, the extremal best replies to monotone (in type) strategies are monotone (in type). This follows because π_i is supermodular in a_i and has increasing differences in $(a_i, (a_{-i}, t))$ and because posteriors are monotone. The result is that a higher type for i chooses a higher action because a shift in beliefs (the posterior p_i is increasing and higher types believe that other players are more likely to be of higher types as well) and also because the induced expected payoff has increasing differences in (a_i, t_i).[39] Third, if the largest best–reply map $\overline{\beta}_i(\sigma_{-i})$ is increasing, the largest best reply to monotone strategies is monotone, and payoffs are continuous, then there is a largest equilibrium and it is in monotone strategies. This follows by starting a Cournot tâtonnement with strategies for each player i equal, for any type, to the largest element in the action set A_i (it exists because A_i is a cube in Euclidean space). Then the Cournot tâtonnement defines a decreasing sequence of monotone strategies; its limit must be an equilibrium owing to the continuity of payoffs, and the limit is also in monotone strategies. Furthermore, it is easy to see that the limit must be the largest equilibrium.

[38] The assumptions on type and action spaces can be weakened considerably (out of the realm of Euclidean spaces), and there is no need to assume a common prior (see Van Zandt and Vives (2004)).

[39] Let $V_i(a_i, t_i, P_{-i}) \equiv \int_{T_{-i}} \pi_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i})\, dP_{-i}(t_{-i})$. Then V_i has increasing differences in (a_i, t_i) and in (a_i, P_{-i}) because $\pi_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i})$ has increasing differences in (a_i, t) (since π_i has increasing differences in $(a_i, (a_{-i}, t))$ and σ_{-i} is increasing) and because of the monotone posteriors condition. Furthermore, V_i is supermodular in a_i because π_i is supermodular in a_i.

7.1.2 Monotone supermodular games: Applications

Monotone supermodular games fit a variety of problems. Van Zandt and Vives (2004) present an application to the discrete setup of an adoption game on a graph with local network effects. Section 7.2 provides an application to global games. I provide here illustrative examples of multimarket oligopoly and a team problem, as well as a comparative statics result that can be applied to extending the results of games of voluntary disclosure.

Multimarket oligopoly. Consider a Bertrand multimarket oligopoly where firm $i = 1, ..., n$ produces potentially H varieties, $h = 1, ..., H$, and has profits

$$\pi_i = \sum_{h=1}^{H} (p_{ih} - c_{ih}) D_{ih} (p_i, p_{-i}; \theta_h),$$

where θ_h is a random demand shock for market h. Let the type of firm i be $t_i = (c_i, s_i)$, where s_i is a multidimensional signal about the random vector θ and $c_i = (c_{i1}, ..., c_{iH})$. (In the notation used, the vector θ is part of T_0.) The payoff u_i is supermodular in the prices and has increasing differences in p_i and (c_i, θ) if, for example, D_{ih} is linear and increasing in θ_h and if all the goods are gross substitutes (both across markets and across brands). For instance, if θ and $(c_i, s_i)_{i \in N}$ are affiliated then the increasing posteriors condition is satisfied. Extremal equilibria will then be monotone, so prices at extremal equilibria will increase in cost and demand signals.

Teams. Consider a team problem (Radner (1962)) in which the common function to be optimized is supermodular, there are increasing differences between actions and types, and the distribution of types yields monotone posteriors. Each member of the team chooses a decision rule or strategy that is contingent on his private information (type) in order to maximize the common objective. We know that the team optimum will be a Bayesian equilibrium of the game among team members (Radner (1962)). Suppose there is a unique equilibrium. We then conclude that there is a team optimum and at the optimum players use decision rules that are monotone in type. For example, in a multidivisional firm in which the total profit of the firm has been internalized by the division's managers, a_j could be the vector of actions or "efforts" under the control of manager j and s_j his private information relating to cost and demand conditions for division j.

Comparative statics and strategic information revelation. Monotone supermodular Bayesian games have a useful comparative statics property: Extremal equilibria are increasing in posteriors. A consequence is that, if payoffs display positive externalities (π_i is increasing in a_{-i}), then increasing posteriors increases the equilibrium expected payoffs. Therefore, in a game with positive externalities, the expected payoff of each player in an extremal equilibrium is increasing in the posteriors of the other players (ordered by first–order stochastic dominance). This result can easily be strengthened to "strictly increasing" under certain regularity assumptions (including some smooth strict complementary conditions[40] and requiring π_i to be strictly increasing in a_j).

This comparative statics result has a ready application to games of voluntary disclosure. Okuno-Fujiwara, Postlewaite and Suzumura (1990) have provided conditions under which fully revealing equilibria obtain in duopoly games of voluntary disclosure of information when information is verifiable. The conditions involve restrictive regularity assumptions such as one–dimensional actions, concavity of payoffs, uniqueness and interiority of equilibrium, and independent types for the players. The basic intuition of the result is that in equilibrium inferences are skeptical: if a player reports a set of types others believe the worst (i.e., others believe that the player is of the most unfavorable type in the reported set). This unravels the information. For example, consider a Cournot duopoly in which types are the (constant marginal) costs of firms, which can be high or low. Then a firm reporting nothing (the full set) will be assumed to have high costs because if the firm had low costs it would have said so.

Noting that the authors work in the context of a monotone supermodular game, our approach dispenses with these unnecessary regularity assumptions and highlights the crucial ones, those related precisely to the complementarity and monotonicity assumptions of a strict version of the monotone supermodular game: that the marginal payoff of an action of a player is strictly increasing in the actions of rivals and in the types of players. The outcome is an extension of the result to n-player GSC games or to a duopoly with strategic substitutability, multidimensional actions, affiliated types, and possibly multiple noninterior equilibria (provided they are extremal). (See Van Zandt and Vives (2004) for the details.)

7.2 Global games

Global games were introduced by Carlsson and van Damme (1993) as games of incomplete information with types determined by each player observing a noisy signal of the underlying state. The aim is to select an equilibrium with a perturbation of a complete information game. The basic idea is that, when analyzing a complete information game with potentially multiple equilibria, players must entertain the "global picture" of slightly different possible games being played. Each player has a noisy estimate of the game being played and knows that the other players are also receiving noisy estimates.

Carlsson and van Damme (1993) show that in 2×2 games if each player observes a noisy signal of the true payoffs and if ex ante feasible payoffs include payoffs that make each action strictly dominant, then as noise becomes small an iterative strict dominance selects one equilibrium. The equilibrium selected is the John Harsanyi and Reinhard Selten (1988) risk-dominant one if there are

[40] With $\partial u_j / \partial a_{jh}$ strictly increasing in t_j and t_i for all h and with $\partial u_j / \partial a_{jh}$ strictly increasing in a_{ih} for all h.

two equilibria in the complete information game. Carlsson and van Damme do not explicitly consider supermodular games but, in the interesting case of two equilibria in a complete information game, the game is one of strategic complementarities.

I will analyze a standard symmetric binary action global game with the tools of supermodular games, provide some applications to currency and financial crises, and conclude with some robustness considerations.

7.2.1 A binary action game of strategic complementarities

Consider a version of the game with a continuum of players in the simple framework of Section 3. The action set of player i is $A_i \equiv \{0,1\}$, with $a_i = 1$ interpreted as "acting" and $a_i = 0$ "not acting" (and let $a_i = 1$ be "larger" than $a_i = 0$). To act may be to invest, adopt a technology or standard, revolt, attack a currency, or run on a bank. The fraction of people acting is $\widetilde{a}$ and the state of the world is θ. There is a critical fraction of people $h(\theta)$ above which it pays to act , with $h(\cdot)$ strictly increasing and crossing 0 at $\theta = \underline{\theta}$ and 1 at $\theta = \overline{\theta}$. See Figure 7.

Let $\pi^1 = \pi(a_i = 1, \widetilde{a}; \theta)$ and $\pi^0 = \pi(a_i = 0, \widetilde{a}; \theta)$. The differential payoff to acting is given by the following chart.

	$\widetilde{a} \geq h(\theta)$	$\widetilde{a} < h(\theta)$
$\pi^1 - \pi^0$	$B > 0$	$-C < 0$

For any given state of the world θ, this defines a supermodular game in which $\pi^1 - \pi^0$ is increasing in $\widetilde{a}$ and $-\theta$ or, equivalently, $\pi(a_i, \widetilde{a}; \theta)$ has increasing differences in $(a_i, (\widetilde{a}, -\theta))$. It is immediate that if $\theta \leq \underline{\theta}$ then it is a dominant strategy to act; if $\theta \geq \overline{\theta}$ then it is a dominant strategy not to act; and for $\theta \in (\underline{\theta}, \overline{\theta})$ there are multiple equilibria: either everyone acting or no one acting. We know also, according to Result 5 in Section 3, that extremal equilibrium strategies will be monotone (decreasing) in θ. Indeed, the largest equilibrium is $a_i = 1$ for all i if $\theta \leq \overline{\theta}$, and $a_i = 0$ for all i if $\theta > \overline{\theta}$, and it is (weakly) decreasing in θ.

Consider now the incomplete information game where players have a normal prior on the state of the world $\theta \sim N(\mu_\theta, \tau_\theta^{-1})$ and where player i observes a private signal $s_i = \theta + \varepsilon_i$ with normally distributed noise $\varepsilon_i \sim N(0, \tau_\varepsilon^{-1})$, i.i.d. across players.[41] Morris and Shin (2002) show that iterated elimination of dominated strategies then leads to a unique outcome provided that $\tau_\theta / \sqrt{\tau_\varepsilon}$ is small. Thus we have a unique Bayesian equilibrium. We show here how the tools of supermodular games can be used to conclude that the game is dominance solvable without actually having to go through the elaborate process of iterated elimination of dominated strategies. Furthermore, we shall see, in a very transparent way, how the approach brings intuition behind the uniqueness result.

[41] In the notation of Section 7.1 we have $t_0 = \theta$ and $t_i = \theta + \varepsilon_i$.

Note first that the game is monotone supermodular since $\pi(a_i, \widetilde{a}; \theta)$ has increasing differences in $(a_i, (\widetilde{a}, -\theta))$ and since signals are affiliated. This means that extremal equilibria exist, are symmetric (because the game is symmetric), and are in monotone (decreasing) strategies (according to the results in Section 7.1.1). Because of binary action, the strategies must then be of the threshold form: $a_i = 1$ if and only if $s_i < \hat{s}$. Therefore, the extremal equilibrium thresholds $\overline{s}$ and $\underline{s}$ bound the set of rationalizable strategies.

Now, an equilibrium will be characterized by two thresholds (s^*, θ^*) with s^* yielding the acting signal threshold and θ^* the state–of–the–world threshold, below which the acting mass is successful and an acting player obtains the payoff $B - C > 0$ (the currency falls, the bank fails, or the revolt succeeds). The critical thresholds must fulfill two equations:

1. $\widetilde{a}(\theta^*, s^*) = \Pr(s \leq s^* \mid \theta^*) = h(\theta^*)$;
2. $\Pr(\theta \leq \theta^* \mid s^*) = \gamma$, where $\gamma \equiv C/(B + C) < 1$.

The first equation states that at the critical state of the world, in equilibrium, the fraction of acting players must equal the critical fraction above which it pays to act. The second equation states that at the critical signal threshold the expected payoff of acting and not acting is the same:

$$\begin{aligned} & E\{\pi(1, \widetilde{a}(\theta); \theta) - \pi(0, \widetilde{a}(\theta); \theta) \mid s = s^*\} \\ = \; & \Pr(\theta \leq \theta^* \mid s^*) B + \Pr(\theta > \theta^* \mid s^*)(-C) = 0. \end{aligned}$$

Equations (1) and (2) may have multiple solutions. However, it can be shown that if (and only if) $\tau_\theta / \sqrt{\tau_\varepsilon}$ is small enough the solution is unique, in which case the equilibrium is unique and the game is dominance solvable because then $\overline{s} = \underline{s}$.

If $\tau_\theta / \sqrt{\tau_\varepsilon}$ is not small enough then typically there are three equilibria. The main reason why the equilibrium is unique with small noise in the signals is that decreasing the amount of noise decreases the strength of the strategic complementarity among the actions of the players. Indeed, multiple equilibria come about when the strategic complementarity is strong enough.

It is instructive to sketch the proof of uniqueness in order to bring forward the intuition. Suppose that $h(\cdot)$ is continuously differentiable with $h' > 0$. Let $\underline{h'} \equiv \min_{\theta \in [\underline{\theta}, \overline{\theta}]} h'(\theta) > 0$. Let $P(s, \hat{s})$ be the probability that the acting players succeed if they use a threshold $\hat{s}$ and the player receives a signal s. That is,

$$P(s, \hat{s}) \equiv \Pr[\theta < \widehat{\theta}(\hat{s}) | s] = \Phi\left(\sqrt{\tau_\theta + \tau_\epsilon}\left(\widehat{\theta}(\hat{s}) - \frac{\tau_\theta \mu_\theta + \tau_\epsilon s}{\tau_\theta + \tau_\epsilon}\right)\right).$$

Here $\widehat{\theta}(\hat{s})$ is the critical θ below which there is success when players use a strategy with threshold $\hat{s}$, and Φ is the cumulative distribution of the standard normal random variable $N(0, 1)$.

It is immediate that P is strictly decreasing in s, $\partial P / \partial s < 0$, and nondecreasing in $\hat{s}$, $\partial P / \partial \hat{s} \geq 0$. Given that other players use a strategy with threshold $\hat{s}$, the best response of a player is to use a strategy with threshold $\tilde{s}$ where

$P(\tilde{s},\hat{s})=\gamma$: act if and only if $P(s,\hat{s})>\gamma$ or, equivalently, if and only if $s<\tilde{s}$. This defines a best–response function

$$r(\hat{s})=\frac{\tau_\theta+\tau_\epsilon}{\tau_\epsilon}\widehat{\theta}(\hat{s})-\frac{\tau_\theta}{\tau_\epsilon}\mu_\theta-\frac{\sqrt{\tau_\theta+\tau_\epsilon}}{\tau_\epsilon}\Phi^{-1}(\gamma).$$

The game is of strategic complementarities, and we have that

$r'=-(\partial P/\partial\hat{s})/(\partial P/\partial s)\geq 0$: a higher threshold $\hat{s}$ by others induces a player to use also a higher threshold. Furthermore, it is easily checked[42] that $\widehat{\theta}'(\hat{s})\leq\left[1+\sqrt{2\pi/\tau_\epsilon}\underline{h}'\right]^{-1}$. As a consequence ,

$$\frac{\tau_\theta}{\sqrt{\tau_\varepsilon}}\leq\sqrt{2\pi}\underline{h}'\implies r'(\hat{s})=\frac{\tau_\theta+\tau_\epsilon}{\tau_\epsilon}\widehat{\theta}'(\hat{s})\leq 1,$$

with equality only when $h(\theta)=1/2$. This ensures that $r(\hat{s})$ crosses the 45^o line only once and that the equilibrium is unique. If $h(\theta)=\theta$ then $\underline{h}'=1$, and if $\tau_\theta/\sqrt{\tau_\varepsilon}>\sqrt{2\pi}$ then $r'(\hat{s})>1$ for $h(\theta)=\theta=1/2$. Thus, for example, for γ such that $\theta^*=1/2$ there are three equilibria (similarly as in Figure 2 in Section 3.2).[43]

With small noise the strategic complementarity is lessened, and $r'\leq 1$, because then a player faces greater uncertainty about the behavior of others. Indeed, consider the limit cases $\tau_\varepsilon\to+\infty$ (or, equivalently, a diffuse prior $\tau_\theta=0$). Then it is not hard to see that the distribution of the proportion of acting players $\widetilde{a}(\theta,s^*)$ is uniformly distributed over $[0,1]$ conditional on $s_i=s^*$. This means that players face maximal strategic uncertainty and cannot coordinate on different equilibria.[44] In contrast, with complete information there are multiple equilibria when $\theta\in(\underline{\theta},\overline{\theta})$. Indeed, at any of the equilibria players face no strategic uncertainty. For example, in the equilibrium in which everyone acts, a player has a point belief that all other players will act.

The intuition for the uniqueness result should be familiar to the reader from our simple framework with heterogenous agents (Section 3.2 and Figure 2). There the cost of adoption ($a_i=1$) for player i is θ_i and follows a normal distribution with mean μ_θ, variance σ_θ^2, and covariance with the adoption cost for $j\neq i$, θ_j, $\rho\sigma_\theta^2$ with $\rho\in[0,1)$. The benefit of adoption is $g(\widetilde{a})$, where $\widetilde{a}$ is the total mass adopting and no adoption yields no benefit. The payoff is therefore $\pi(a_i,\widetilde{a};\theta_i)=a_i(g(\widetilde{a})-\theta_i)$. Now, if $g'>0$ then the game is monotone

[42] We have that $\widehat{\theta}(\hat{s})$ is the solution in θ of $\Pr(s\leq\hat{s}\mid\theta)$ $(=\Phi(\sqrt{\tau_\epsilon}(\hat{s}-\theta)))=h(\theta)$. From this equation we can solve for the inverse function and obtain $\widehat{s}(\theta)=\theta+(1/\sqrt{\tau_\epsilon})\Phi^{-1}(h(\theta))$ with derivative $\widehat{s}'=1+(1/\sqrt{\tau_\epsilon})h'(\theta)\left[\phi\left(\Phi^{-1}(h(\theta))\right)\right]^{-1}$, where ϕ is the density of the standard normal. Since ϕ is bounded above by $1/\sqrt{2\pi}$, it follows that $\widehat{s}'$ is bounded below: $\widehat{s}'(\theta)\geq 1+\sqrt{2\pi/\tau_\epsilon}\underline{h}'$, where $\underline{h}'=\min_{\theta\in[\underline{\theta},\overline{\theta}]}h'(\theta)>0$. Hence, $\widehat{\theta}'(\hat{s})\leq\left[1+\sqrt{2\pi/\tau_\epsilon}\underline{h}'\right]^{-1}$(with strict inequality except when $h(\theta)=1/2$ because then $\Phi^{-1}(1/2)=0$ and ϕ attains its maximum: $\phi(0)=1/\sqrt{2\pi}$).

[43] As γ ranges from 0 to 1, θ^* goes from $\overline{\theta}=1$ to $\underline{\theta}=0$.

[44] It is worth noting that the uniqueness argument made is robust to general distributions for the uncertainty as long as the noise in the signals is small. Indeed, with very precise signals all priors "look uniform" (Morris and Shin (2002)).

supermodular because $\pi(a_i, \widetilde{a}; \theta)$ has increasing differences in $(a_i, (\widetilde{a}, -\theta))$ and types are affiliated. This means that extremal equilibria exist, are symmetric (because the game is symmetric), and are in monotone (decreasing) strategies of the form $a_i = 1$ if and only if $\theta_i \leq \widehat{\theta}$. This provides the rationale for concentrating on threshold strategies (something that was assumed in Section 3.2). From Section 3.2 we know that the equilibrium will be unique if

$$\sqrt{\frac{1-\rho}{1+\rho}} \sqrt{\frac{\tau_\theta}{2\pi}} \overline{g}' < 1,$$

where $\overline{g}' \equiv \sup_{a \in [0,1]} g'(a)$. This may happen because of weak payoff complementarities ($\overline{g}'$ low); because of a diffuse prior (τ_θ low); or because the correlation of the costs is high (ρ and the signal of each player is very precise). As before, in all these situations the degree of strategic complementarity is not too strong and we are in the "flat" best –response case of Figure 2.

In summary, by using the theory of supermodular games we bring the intuition for the uniqueness result, clarify the role of the assumptions, and we obviate the necessity of solving for iterated elimination of dominated strategies. We can start by noting that the game is monotone supermodular. This means that extremal equilibria exist and are in monotone (threshold) strategies. Those extremal equilibria can be found starting at extremal points of the strategy sets of players ($\overline{s} = \infty$ and $\underline{s} = -\infty$) and iterating using best responses (Vives (1990a)). We must make sure that the process is not stuck at extremal points of strategy space (e.g., the boundary assumptions on h guarantee this since if $1 > h(\theta) > 0$, or $\overline{\theta} = \infty$ and $\underline{\theta} = -\infty$, then both to act and not act coexist as equilibria no matter what signal is realized). The extremal equilibrium thresholds $\overline{s}$ and $\underline{s}$ bound the set of rationalizable strategies, and if the equilibrium is unique then the game is dominance solvable. The condition for equilibrium uniqueness is precisely that strategic complementarities are not too strong, and this holds when the signals are precise enough or if the prior is diffuse. In this situation, each player faces a lot of (strategic) uncertainty about the aggregate action of the other players.

7.2.2 Applications

It is well known that multiple equilibria make comparative statics and policy analysis difficult. The uniqueness of an equilibrium delivered by the global game approach comes to the rescue. In the region where the equilibrium is unique, we can obtain several useful results as follows:

- When $\theta < \theta^*$, the acting mass of players succeeds. In the range $[\theta^*, \overline{\theta})$ there is coordination failure from the point of view of players, because if all them were to act then they would succeed.
- Both θ^* and s^* (and the probability that the acting mass succeeds) is decreasing in the relative cost of failure $\gamma \equiv C/(B+C)$ and in the expected

value of the state of the world μ_θ.[45]

- There is a multiplier effect of public information. An increase in μ_θ will have an effect on the equilibrium threshold s^* over and above the direct impact on the best response of a player $\partial r/\partial\mu_\theta = -\tau_\theta/\tau_\epsilon$. Indeed, the prior mean μ_θ of θ can be understood as a public signal of precision τ_θ and, exactly as in the simple framework of Section 3.1,

$$\left|\frac{ds^*}{d\mu_\theta}\right| = \frac{|\partial r/\partial\mu_\theta|}{1-r'} > \left|\frac{\partial r}{\partial\mu_\theta}\right|$$

whenever the uniqueness condition ($r' < 1$) is met and the game is a GSC ($r' > 0$). The multiplier is largest when r' is close to 1 —that is, when we approach the multiplicity of equilibria region. The multiplier effect of public information is emphasized by Morris and Shin (2002), who interpret it in terms of the coordinating potential of public information beyond its strict information content. The reason is that public information becomes common knowledge and affects the equilibrium outcome. Every player knows that an increase in μ_θ will shift downward the best replies of the rest of the players, thereafter everyone will be more cautious in acting. This phenomenon, for example, may be behind the apparent overreaction of financial markets to Fed announcements.[46]

The approach is useful for policy analysis because it links the probability of occurrence of a "crisis" (successful mass action) at the unique equilibrium with the state of the world: $\Pr(\theta \le \theta^*)$. This is in contrast with the complete information model, where multiple self-fulfilling equilibria arise in the range $(\underline{\theta}, \overline{\theta})$. Hence the theory builds a bridge between the self-fulfilling theory of crisis (e.g., Diamond and Dybvig (1983)) and the theory that links crisis to the fundamentals (e.g., Gary Gorton (1985, 1988)).[47]

The uniqueness property is nice in a game, but we can still perform comparative statics analysis in a GSC even if there are multiple equilibria. Suppose we are in the multiple equilibrium region and that μ_θ increases. The comparative statics result that the critical thresholds θ^*, and s^* decrease still holds for extremal equilibria or for reasonable out-of-equilibrium dynamics that eliminate the middle "unstable" equilibrium. Indeed, we know that extremal equilibria of monotone supermodular games are increasing in the posteriors of the players. A sufficient statistic for the posterior of a player under normality is $E(\theta \mid s) = \tau_\theta\mu_\theta + \tau_\epsilon s/\tau_\theta + \tau_\epsilon$, which is increasing in μ_θ. It follows then

[45] This follows immediately because (θ^*, s^*) solves the system $\Phi\left(\sqrt{\tau_\epsilon}(s-\theta)\right) = h(\theta)$ and $\Phi\left(\sqrt{\tau_\theta+\tau_\epsilon}\left(\theta - \frac{\tau_\theta\mu_\theta+\tau_\epsilon s}{\tau_\theta+\tau_\epsilon}\right)\right) = \gamma$; from this it follows that θ^* solves $\varphi(\theta) \equiv \tau_\theta(\theta-\mu_\theta) - \sqrt{\tau_\epsilon}\Phi^{-1}(h(\theta)) - \sqrt{\tau_\theta+\tau_\epsilon}\Phi^{-1}(\gamma) = 0$, and $\varphi' < 0$ when $\tau_\theta/\sqrt{\tau_\epsilon}$ is small enough. Furthermore, θ^* and s^* move together.

[46] Michael Chwe (1998) provides evidence of the per–viewer price of advertising in TV; it is higher for big sports events, consistent with the multiplier effect of public information.

[47] An early model with incomplete information that obtains a unique Bayesian equilibrium with a positive probability of a crisis is Postlewaite and Vives (1987).

that extremal equilibrium thresholds $(-\theta^*, -s^*)$ increase with μ_θ. The out-of-equilibrium adjustment can take the form of best-reply dynamics where, at any stage after the perturbation, a new state of the world θ is drawn independently and a player best-responds to the strategy threshold used by other players at the previous stage.

We present here two applications that illustrate the power of the approach. Suppose in all of them that the uniqueness condition is fulfilled (i.e., $\tau_\theta/\sqrt{\tau_\varepsilon}$ is small enough).[48]

The first application is a modified version of the currency attacks model of Morris and Shin (1998). This is a highly streamlined model of currency attacks. Let θ be the reserves of the central bank (with $\theta \leq 0$ meaning that reserves are exhausted). There is a continuum of speculators and speculator i has one unit of resources to attack the currency ($a_i = 1$) at a cost C after receiving a signal about the level of resources of the central bank. Let $h(\theta) = \theta$ and let the attack succeed if $\widetilde{a} \geq \theta$. The (capital) gain if there is a depreciation is $\hat{B}$ (and it is fixed). Let $B = \hat{B} - C$. The result is that the probability of a currency crisis is decreasing in $C/\hat{B}$ and in the expected value of the reserves of the central bank. In the region $\left[\theta^*, \overline{\theta}\right)$, if speculators were to coordinate their attack then they would succeed, but in fact the currency holds.

The second application is an instance of coordination failure in the interbank market, providing a rationale for a Lender of Last Resort (LLR) intervention (Jean Charles Rochet and Vives (2004)). Consider a market with three dates: $\tau = 0, 1, 2$. At date $\tau = 0$, the bank possesses own funds E and collects uninsured wholesale deposits (CDs for example) for some amount $D_0 \equiv 1$. These funds are used to finance some investment I in risky assets (loans), the rest being held in cash reserves M. Under normal circumstances, the returns θI on these assets are collected at date $\tau = 2$, the CDs are repaid at their face value D, and the stockholders of the bank get the difference (when it is positive). However, early withdrawals may occur at an interim date $\tau = 1$, following the observation of private signals on the future realization of θ. If the proportion $\widetilde{a}$ of these withdrawals exceeds the cash reserves M of the bank, then the bank is forced to sell some of its assets. A continuum of fund managers make investment decisions in the interbank market. At $\tau = 1$ each fund manager, after receiving a private signal about θ, decides whether to cancel ($a_i = 1$) or renew his CD ($a_i = 0$). Let $m \equiv M/D$ be the liquidity ratio, $\underline{\theta} \equiv D - M/I$ the solvency threshold of the bank, $\lambda > 0$ the fire sales premium of early sales of bank assets, and $\overline{\theta} \equiv (1 + \lambda)\underline{\theta}$ the "supersolvency" point where a bank does not fail even if no fund manager renews his CDs. The bank fails if $\widetilde{a} \geq h(\theta)$, where

$$h(\theta) \equiv m + \frac{1-m}{\lambda}\left(\frac{\theta}{\underline{\theta}} - 1\right)$$

for $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and $h(\theta) = 0$ for $\theta \leq \underline{\theta}$. A fund manager is rewarded for making the right decision. The equilibrium failure threshold of the bank is $\theta^* \in \left[\underline{\theta}, \overline{\theta}\right]$, and in the range $[\underline{\theta}, \theta^*)$ the bank is solvent but illiquid. This provides a rationale for

[48] See Morris and Shin (2002) for other applications.

a LLR intervention with the discount window. Comparative statics results are also easily obtained. The critical θ^* (and probability of failure) is a decreasing function of the liquidity ratio m and the solvency (E/I) of the bank, of the critical withdrawal probability γ, and of the expected return on the bank's assets μ_θ; it is an increasing function of the fire–sale premium λ and of the face value of debt D.

7.2.3 Robustness and extensions

David Frankel, Morris, and Ady Pauzner (2003) obtain a generalization of the limit uniqueness result to games of strategic complementarities. The authors consider a Bayesian game (A_i, T_i, π_i) for $i \in N$, where A_i is a compact interval and where $\pi_i(a_i, a_{-i}; \theta)$ is continuous and has increasing differences in $(a_i, (a_{-i}, \theta))$. The state θ is drawn from a continuous density with connected support, and player i receives a private signal $s_i = \theta + \kappa\varepsilon_i$ with $\kappa > 0$, where ε_i is drawn from an atomless density with compact support (and the error terms are i.i.d. across players). The authors also assume that, for extreme values of θ, extreme actions in A_i are strictly dominant (this is the equivalent of the assumption that $h(\cdot)$ crosses 0 and 1 at finite values) and make the technical assumption that $\pi_i(a, \theta)$ has sensitivity to actions with a Lipschitz bound. The result is that if θ is uniformly distributed over a large interval or for κ tending to 0, there is a (essentially) unique Bayesian equilibrium in pure strategies (and it is increasing in type). Under the given assumptions, we are in the frame of monotone supermodular games (Section 7.1) and hence the existence of extremal equilibria monotone in type is guaranteed.

The framework can be extended:

- To include large players (see Giancarlo Corsetti, Amil Dasgupta, Morris and Shin (2004) on currency attacks as well as Corsetti, Bernardo Guimaraes, and Nouriel Roubini (2003) and Morris and Shin (2002) on the impact of the IMF as provider of "catalytic finance").

- To relax the strategic complementarity condition of actions to a single–crossing condition and obtain a uniqueness result in switching strategies, assuming that signals fulfill the monotone likelihood ratio property. However, then it cannot be guaranteed that there are no other equilibria in non-monotone strategies (see Athey (2001)); Itay Goldstein and Ady Pauzner (2003) apply a similar strategy to model bank runs when the depositor's game is not one of strategic complementarities.

- To consider dynamic settings modeling —for example, contagion (Dasgupta (2003)) and dynamic speculative attacks (Christophe Chamley (2003)).

8 Concluding remarks

In the paper I have surveyed the theory and several applications of the lattice-theoretic approach in the study of complementarities in games. The survey has been by no means exhaustive. Indeed, the method, as has been made clear in the text, can be applied fruitfully to comparative statics analysis and so is useful in practically all domains of economic theory. For example, it has been applied to demand analysis, to the theory of the firm and organizations, and to dynamic optimization problems (see, e.g., the applications in Milgrom and Roberts (1990b) and Milgrom and Shannon (1994)); cooperative games (see Topkis (1998) for a survey, and Robert Shimer and Lones Smith (2000) for a model of assortative matching with frictions); and evolutionary games (see, e.g., Carlos Alós-Ferrer and Ana Ania (2002)). Despite these applications, it is safe to say that the approach promises to deliver much more when the tools become part of the standard methods in economics and when empirical analysis develops and interacts with model building. The empirical analysis of complementarities, with implications for the new methods, is taking off in the study of innovation (Eugenio Miravete and Jose Pernías (2004) and Pierre Mohnen and Lars-Hendrik Röller (2003)) and of markets with potentially multiple equilibria (Andrew Sweeting (2004) on the timing of radio commercials; Federico Ciliberto and Elie Tamer (2004) on airline markets).

In summary, the strength of the approach is its simplicity, its capacity to generate new results, and the power it has to make results transparent. The challenges are multiple:

- continue pushing the frontier of the theory with a view toward applications in dynamic games and games of incomplete information;
- incorporate the methodology fully in the standard toolbox of economists; and
- develop the empirical analysis.

9 Appendix: Summary of lattice-theoretic methods

For the convenience of the reader I include a few definitions and results of lattice methods. More complete treatments can be found in Vives (1999, Ch. 2) and Topkis (1998).

A binary relation $\geq$ on a nonempty set X is a *partial order* if $\geq$ is reflexive, transitive, and antisymmetric. An upper bound on a subset $A \subset X$ is $z \in X$ such that $z \geq x$ for all $x \in A$. A greatest element of A is an element of A that is also an upper bound on A. Lower bounds and least elements are defined analogously. The greatest and least elements of A, when they exist, are denoted max A and min A, respectively. A supremum (resp., infimum) of A is a least upper bound (resp., greatest lower bound); it is denoted $\sup A$ (resp., $\inf A$).

A *lattice* is a partially ordered set $(X, \geq)$ in which any two elements have a supremum and an infimum. A lattice $(X, \geq)$ is *complete* if every nonempty subset has a supremum and an infimum. A subset L of the lattice X is a *sublattice* of X if the supremum and infimum of any two elements of L belong also to L.

Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. A function $f\colon X \to T$ is *increasing* if, for x, y in X, $x \geq y$ implies that $f(x) \geq f(y)$.

A function $g\colon X \to \mathbb{R}$ on a lattice X is *supermodular* if, all x, y in X, $g(\inf(x, y)) + g(\sup(x, y)) \geq g(x) + g(y)$. It is *strictly supermodular* if the inequality is strict for all pairs x, y in X that cannot be compared with respect to $\geq$ (i.e., neither $x \geq y$ nor $y \geq x$ holds). A function f is (*strictly*) *submodular* if $-f$ is (strictly) supermodular; a function f is (*strictly*) *log-supermodular* if $\log f$ is (strictly) supermodular.

Let X be a lattice and T a partially ordered set. The function $g\colon X \times T \to R$ has (*strictly*) *increasing differences* in (x, t) if $g(x', t) - g(x, t)$ is (strictly) increasing in t for $x' > x$ or, equivalently, if $g(x, t') - g(x, t)$ is (strictly) increasing in x for $t' > t$. Decreasing differences are defined analogously. If X is a convex subset of $\mathbb{R}^n$ and if $g\colon X \to R$ is twice-continuously differentiable, then g has increasing differences in (x_i, x_j) if and only if $\partial^2 g(x)/\partial x_i \partial x_j \geq 0$ for all x and $i \neq j$.

Supermodularity is a stronger property than increasing differences: if T is also a lattice and if g is (strictly) supermodular on $X \times T$, then g has (strictly) increasing differences in (x, t). The two concepts coincide on the product of linearly ordered sets: if X is such a lattice, then a function $g\colon X \to \mathbb{R}$ is supermodular if and only if it has increasing differences in any pair of variables.

The main comparative statics tool for our purposes is the following.

Lemma 1 *Let X be a compact lattice and let T be a partially ordered set. Let $u\colon X \times T \to \mathbb{R}$ be a function that (a) is supermodular and continuous on the lattice X for each $t \in T$ and (b) has increasing differences in (x, t). Let $\varphi(t) = \arg\max_{x \in X} u(x, t)$. Then:*

1. *$\varphi(t)$ is a non-empty compact sublattice for all t;*
2. *φ is increasing in the sense that, for $t' > t$ and for $x' \in \varphi(t')$ and $x \in \varphi(t)$, we have $\sup(x', x) \in \varphi(t')$ and $\inf(x', x) \in \varphi(t)$; and*
3. *$t \mapsto \max \phi(t)$ and $t \mapsto \min \phi(t)$ are well-defined increasing functions.*

Remark If u has strictly increasing differences in (x, t), then all selections of φ are increasing.

Remark If $X \subset \mathbb{R}^m$, solutions are interior, and $\partial u/\partial x_i$ is strictly increasing in t for some i, then all selections of φ are strictly increasing (Edlin and Shannon (1998)).

The basic fixed point theorem in the lattice-theoretic approach is Tarski (1955).

Theorem 2 *(Tarski (1955)) Let A be a complete lattice (e.g., a compact cube in $\mathbb{R}^m$). Then an increasing function $f : A \to A$ has a largest* $\sup\{a \in A : f(a) \geq a\}$ *and a smallest* $\inf\{a \in A : a \geq f(a)\}$ *fixed point.*

Supermodular game The game $(A_i, \pi_i; i \in N)$ is *supermodular* if, for all i, the following statements hold.

- A_i is a compact lattice.
- $\pi_i\,(a_i, a_{-i})$ is continuous:
    1. is supermodular in a_i; and
    2. has increasing differences in (a_i, a_{-i}).

Game of strategic complementarities Given a set of players N, strategy spaces A_i, and (nonempty) best–reply maps Ψ_i, $i = 1, ..., n$, we define a *game of strategic complementarities* (GSC) as one in which, for each i, A_i is a complete lattice and Ψ_i is increasing and has well-defined extremal elements.

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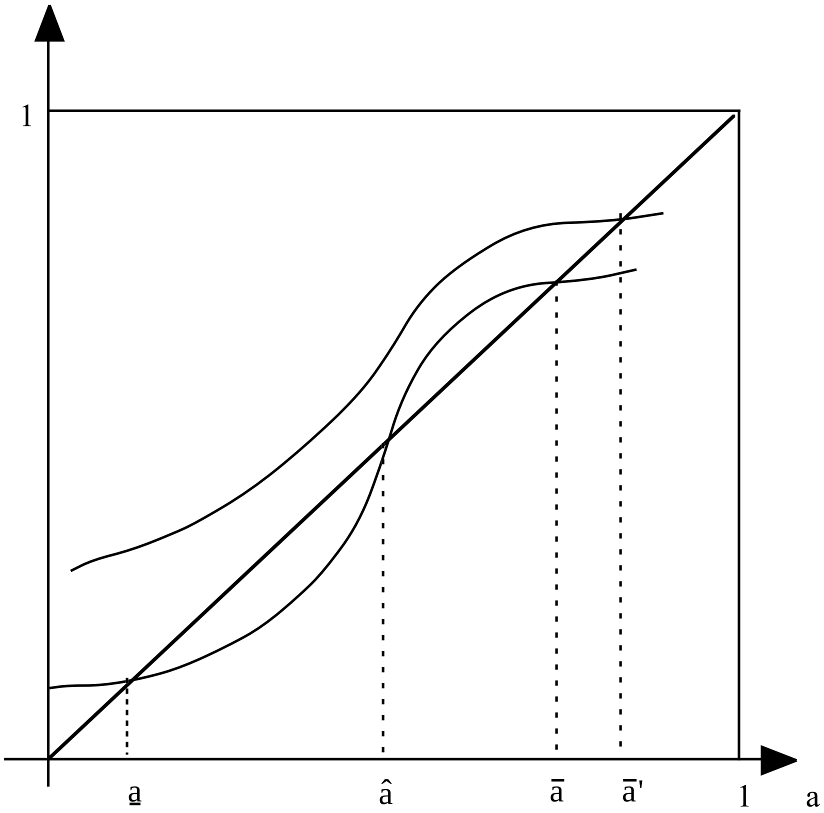


Figure 1

Best response (with homogenoues players)
and multiple equilibria.

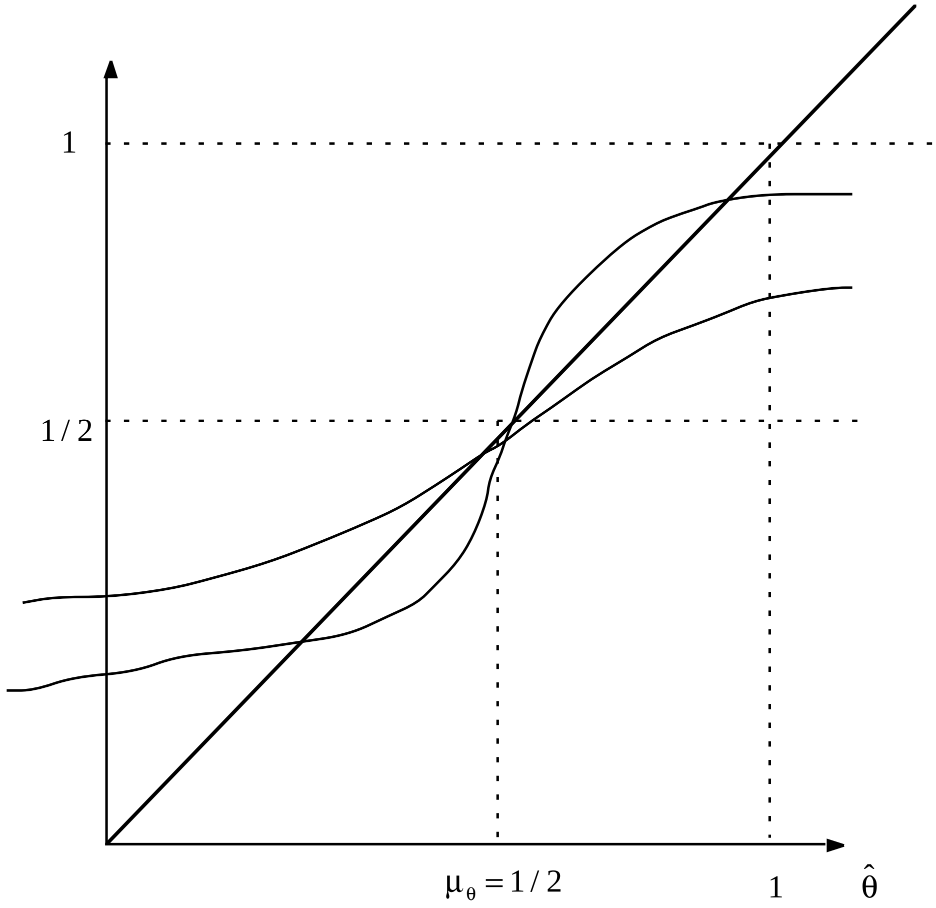
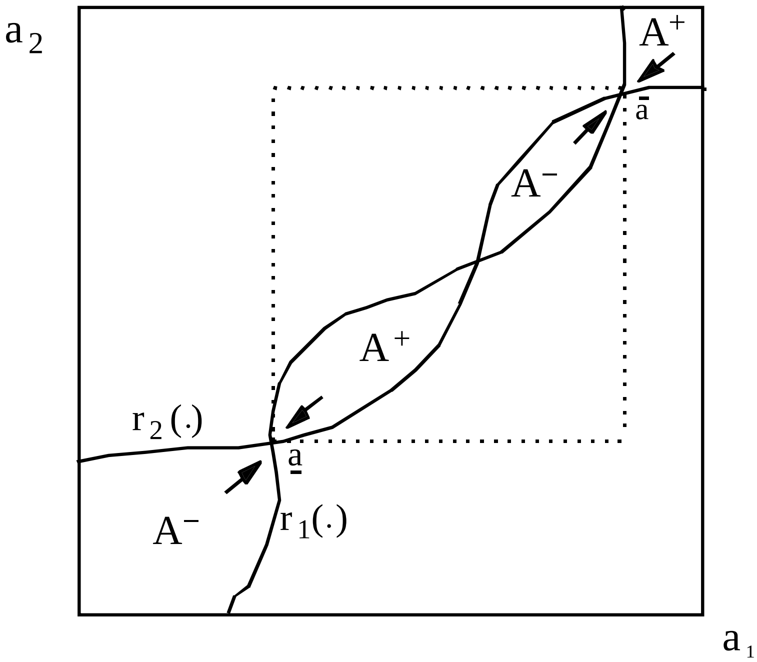


Figure 2

Equilibrium thresholds (with heterogeneous agents)
and multiple equilibria



Cournot tatônnement in a supermodular game

with best reply functions $r_1(\cdot), r_2(\cdot)$

Figure 3

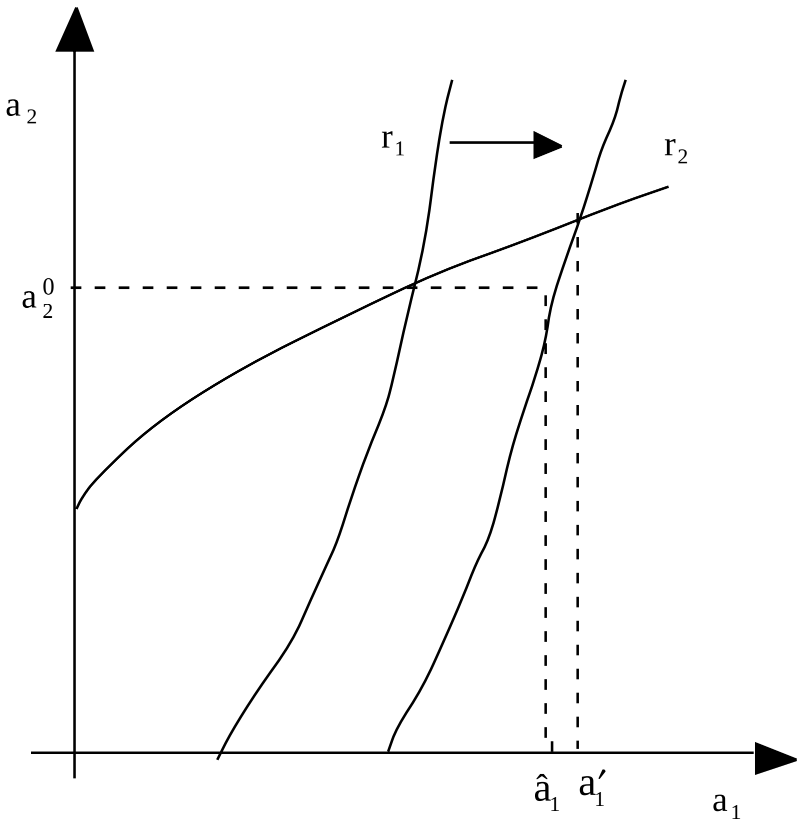


Figure 4

Effect of an increase in parameter t_1

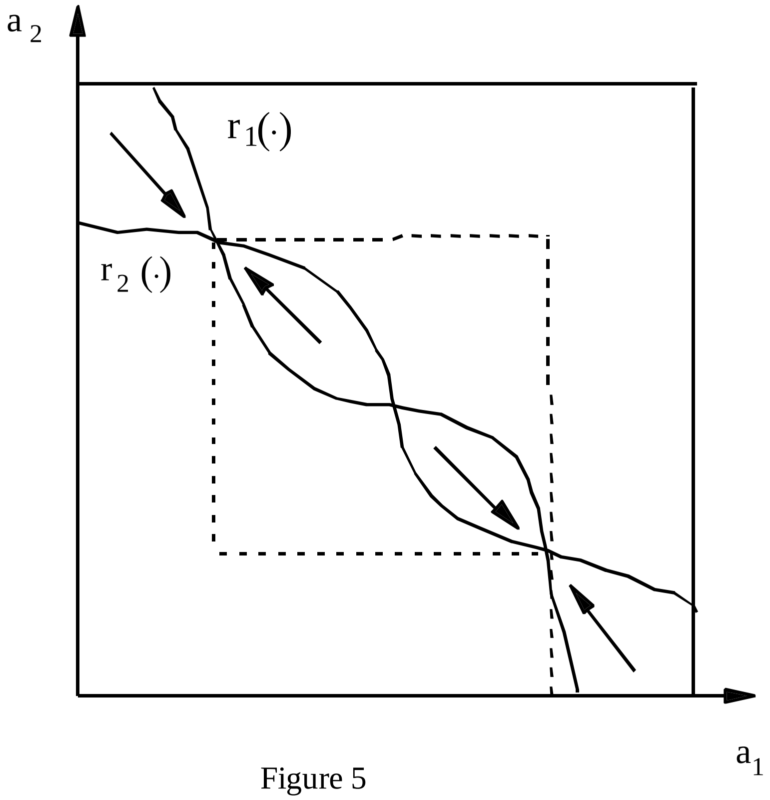


Figure 5

A duopoly game with decreasing best replies

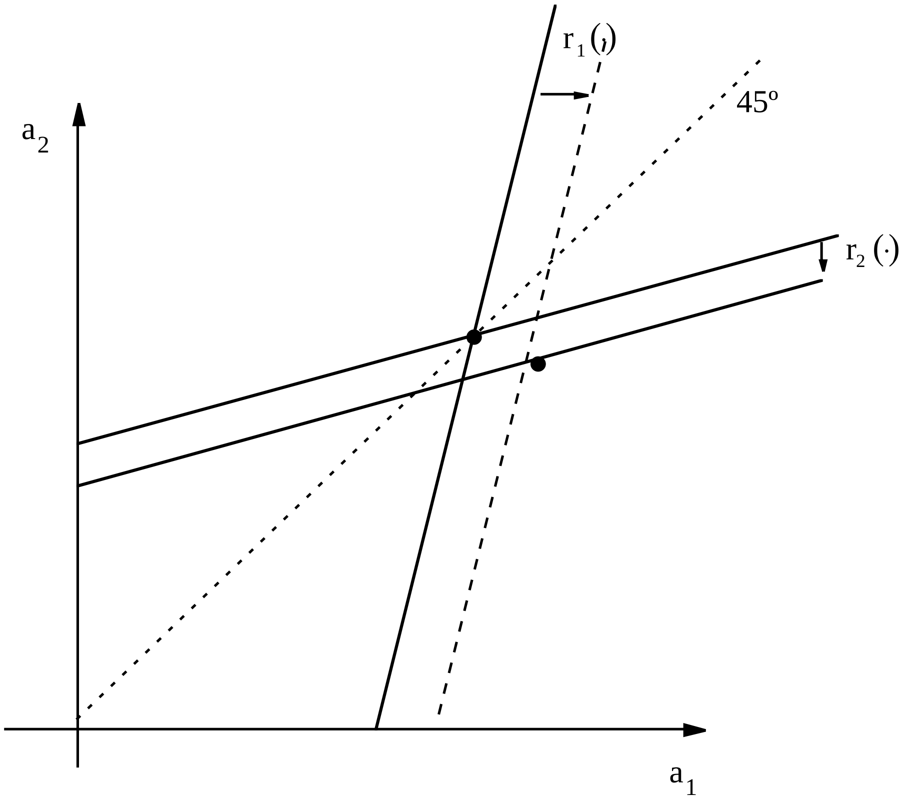


Figure 6

Effects of going from $t_1 = t_2$ to $t_1 > t_2$

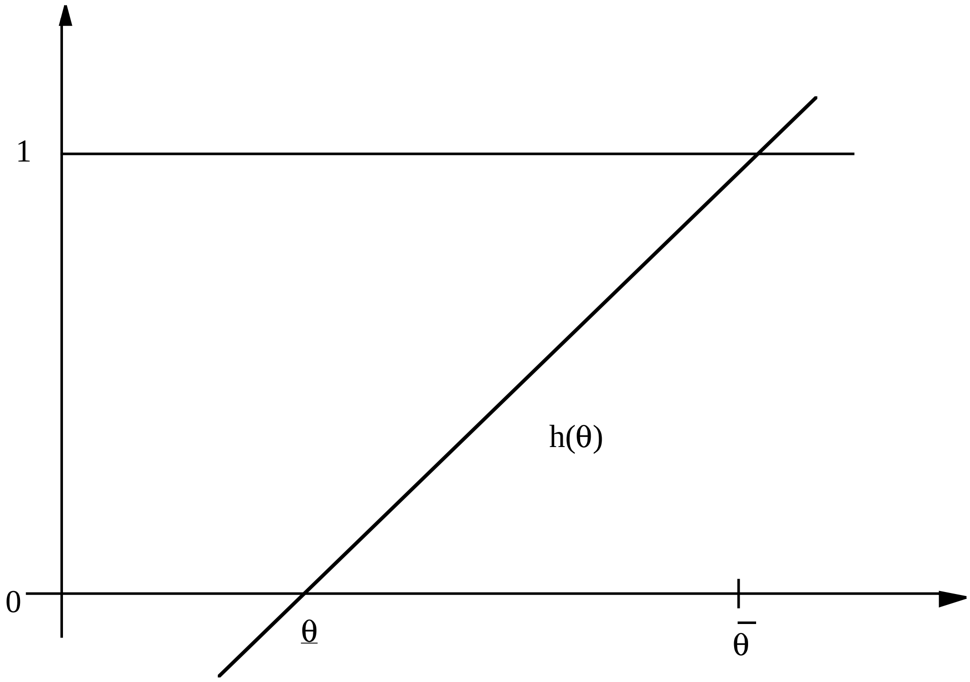


Figure 7

Critical fraction of players above which it pays to act