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ABSTRACT

Joint Production in Teams*

Consider Holmström's moral hazard in teams problem when there are *n* agents, each agent *i* has an a_i-dimensional strategy space and output can be *m*-dimensional. We show that a compensation mechanism that satisfies budget balance, limited liability and implements an efficient allocation generically exists if and only if $\Sigma a_i/(n-1) < m$. When this condition is satisfied, the optimal mechanism discourages collusive behaviour and, under a weak condition, filters out inefficient equilibria.

JEL Classification: D23, D82, J33 and L23
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I Introduction

In a seminal work, Holmström [1982] has shown that under standard assumptions, when a production function $y = f(x_1, ..., x_n)$ depends on the effort x_i of many agents, it is not possible to design a compensation system that is contingent only on observable output y, provides incentives for an efficient level of effort to more than one agent and satisfies budget balance. Budget balance means that the mechanism cannot commit to 'throw away' surplus or to give it to someone who does not participate in production. Since budget balance can be interpreted as a constraint imposed by renegotiation-proofness, this result is relevant not only for those organizations in which it seems a natural assumption from an institutional or empirical point of view (for example, a partnership), but for all the organizational forms.[1]

For this reason, a rich literature has explored the conditions that are necessary for this impossibility result: from the properties of the production function (continuity, for example), to conditions on observability of effort among agents. This line of research is not only important because it highlights sufficient conditions to design efficient mechanisms, but also because it helps to understand the real nature of the problem: which are the essential assumptions and which assumptions are dispensable.

In this work, we generalize the classical moral hazard in teams problem to the case when heterogeneous goods are jointly produced and therefore output is not necessarily single-valued, but may take the form:

$$\mathbf{y} = \begin{cases} y_1 = f_1(\mathbf{x}_1, ..., \mathbf{x}_n) \\ ... \\ y_m = f_m(\mathbf{x}_1, ..., \mathbf{x}_n) \end{cases} . \tag{1}$$

In particular, we assume that output is a vector in $\mathbb{R}^m$ with $m \geq 1$, and each agent i can affect output in many different ways, so that $\mathbf{x}_i$ is a vector in $\mathbb{R}^{a_i}$ with $a_i \geq 1$. The key assumption is that the system of equations (1) is indeterminate: for any dimensionality of output, the number of free variables $\{\mathbf{x}_i\}_{i=1}^n$ may be *arbitrarily* large. This, therefore, is a straightforward generalization of Holmström's model, which corresponds to the particular case when $m = a_i = 1$ for any i.

In this paper we characterize the necessary and sufficient condition under which a compensation mechanism that implements an efficient allocation exists in this more general environment.

This extension of the basic model is intuitive and one may find many real world examples. Consider the case of a merchant bank which has two offices ('Trading Office' and 'Corporate Finance Office'); one team works in the first office ($\{\mathbf{x}_i\}_{i=1}^l$ $l < n$-1) and another team works in the other ($\{\mathbf{x}_i\}_{i=l+1}^n$). The activities of the merchant bankers in the 'Corporate Finance Office', however, have externalities for the production of the first office; similarly, traders' activity generates externalities on the corporate finance office. In this case, output (the product of the

[1] Indeed, Alchian and Demsetz [1972] were the first to highlight this issue in their investigation of the theory of the firm.

two offices) would be two-dimensional and take a form like (1).[2] In general, joint production as in (1) arises whenever synergies in the production function induce economies of scope. An early discussion of these cases can be found in Marshall [1920], who first underscored their importance in the theory of production.

To highlight the key components of our analysis, it is useful to present the results in two steps. Consider first a standard moral hazard in teams model in which each agent i controls an effort level $x_i \in \mathbb{R}$, and the output space is two (or higher) dimensional. The common interpretation of the teams' problem is that the principal wants to control too many variables ($n > 1$) with only one instrument ($y \in \mathbb{R}$). The intuition, therefore, should generalize when the number of variables is larger (especially if arbitrarily larger) than the number of instruments. However, this is not the case, and there is a 'discontinuity' between the cases $y \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^2$ (or a higher dimensional space). Indeed, we prove that for any generic, differentiable production function, any number $n > 2$ of agents (possibly arbitrarily large) and any $m \geq 2$, there is a compensation scheme that satisfies budget balance, respects limited liability of the agents and such that the resulting game played by the agents has an efficient Strong Nash Equilibrium; moreover, under weak conditions, the equilibrium sustained by this scheme is unique in the class of Coalition-Proof Equilibria.[3] This result seems interesting because, as we said, it holds for a generic production function: it fails only in a 'pathological' case. But also because, by showing that the number n of agents to control is irrelevant for efficiency, it highlights a feature of Holmström's approach that may contribute to a better understanding of the moral hazard in teams problem.

To investigate this point we enlarge the agents' strategy space as well. We show that the key aspect of the teams' problem is the relationship between the dimensionality of the instrument and the *average* dimensionality of the agents' strategy spaces. In particular, efficiency is generically possible when

$$\frac{\sum_{i=1}^{n} a_i}{n-1} < m. \tag{2}$$

When the output space is two-dimensional, this implies that the classic inefficiency result is restored only if at least n-1 agents have strategy spaces with the same dimensionality of output. When $\frac{\sum_{i=1}^{n} a_i}{n-1}$ is larger than m efficiency is not generically possible, even if limited liability is relaxed.

Taken together, therefore, these results change the common interpretation of the moral hazard in teams problem since they show that it is not the size of the team which is relevant, but only an aggregate measure of the dimensionality of the agents' strategy spaces; and that

[2] In this case, a natural specification would be that $y_1 = \psi_1(x_1, .., x_l, y_2)$ and $y_2 = \psi_2(x_{l+1}, .., x_N, y_1)$, which implies that $y_i = f_i(x_1, .., x_N)$ for $i = 1, 2$ as in (1). An equivalent interpretation of the model is that output is unidimensional but team 1 works in period t and team 2 in period $t + 1$ and the activity at time t affects productivity at time $t + 1$ (I thank Ken Binmore for this interpretation).

[3] Notice that a Strong Nash equilibrium is also a Coalition Proof equilibrium. Since the set of Strong Nash equilibria is included in the set of Coalition Proof equilibria, a fortiori the mechanism achieves uniqueness in the class of Strong Nash eqilibria.

with multidimensional output, the budget balance constraint is often not binding.

In the second part of the paper we introduce noise in production and show that the mechanism can be extended to cover the case in which effort levels are subject to stochastic perturbations. In particular, we consider the situation where each agent's effort level affects only the probability distribution of his own actual contribution in the production function (x_i), and only the agent can observe the realization of his own x_i. When there are these stochastic perturbations, the problem is considerably more complicated because, although the designer of the compensation scheme may calculate the *ex ante* optimal level of effort in equilibrium, the realizations of the actual contributions are stochastic: information on the realizations of the agents' efforts cannot be used directly, or can be used directly only if the mechanism makes it incentive compatible for them to reveal this information. The extraction of this information, however, is made possible by adding a round of communication after production takes place and before compensations are paid. What makes the communication round more elaborate is the requirement of budget balance both in and out-of-equilibrium. To the extent that a declaration affects any agent's compensation (say agent j), it may affect the compensation of the sender since it increases or decreases the budget for compensating all the agents other than j. Given a weak condition, however, it is possible to eliminate the incentives to misrepresent their realizations. The idea is simple: we group all of the agents into particular subsets and allocate to each of these subsets a share of the realized output that is fixed *ex ante* (i.e., it is contingent neither on the realized level of output nor on any other variable observable after production takes place). The allocation of this surplus among the agents in the set, however, is contingent on the realizations of output and the declarations of the agents that do not belong to the set. In this way, the incentives of the agents that are not in the set are isolated from the allocation of surplus among agents in the set.

These results may contribute to the understanding of the internal organization of a partnership because they highlight the role of the balanced budget requirement in shaping the optimal design of incentives. As mentioned earlier, the efficient mechanism described above requires an analysis of the internal flow of information (the communication stage) and generates empirical predictions on the internal organization of the partnership which allows this flow of information: in particular, the fact that the internal allocation of surplus is not completely contingent on observable signals in the efficient mechanism, but is determined, in part, *ex ante.* These are all empirically testable hypotheses.

The following analysis proceeds as follows. Section II presents the model and a simple example in order to give the intuition of the main ideas. In Section III, we develop the general case and show the efficiency result with vector outputs. In Section IV we study the relationship between efficiency, the dimensionality of output and of the strategy spaces of the agents. In Section V, we consider the case of stochastic perturbations and their implications for the internal organization of the partnership. Section VI concludes. The following subsection reviews the related literature.

I.1 Related literature

As we said, the starting point of our analysis is the model proposed by Holmström [1982] in which the inefficiency result described above is first proven. In another important contribution, Legros and Matthews [1993] extend the model and show that, if the production function is not continuous and differentiable, efficiency may sometimes be achieved and, in general, nearly efficient equilibria are obtainable in a large class of situations using mixed strategies if we allow for unlimited liability. However, they also confirm that the fundamental insight in Holmström [1982] remains true proving that with a differentiable production function efficiency, budged balance and limited liability are incompatible.

To obtain efficiency in more general cases, a recent line of research relaxes the assumption on observability of effort.[4] Miller [1997] assumes that at least one agent observes the effort of one other agent and the identity of this informed agent is common knowledge. He shows a message game that supports an efficient equilibrium with budget balance and limited liability in this case. Strautz [1999] further develops this approach assuming that production is sequential and that the agents can see the output levels in the different stages (i.e., can monitor the effort of the other agents). Again, given these assumptions it is shown that efficiency is not incompatible with budget balance and limited liability.

Our approach is clearly related to this literature since the assumption that the output space is vector valued can be interpreted as an extension of the signal space. However, we do not assume that the agents have information on the effort level of any other agent but only on final output as in Holmström [1982]: therefore the information enlargement is minimal. Interestingly, although in our model agents act simultaneously, one way to interpret it is to assume that two teams produce sequentially and affect each others outcomes with externalities (see footnote 2). In this sense, our model can also be seen as a generalization of Strautz [1999] to the case in which a partnership is sequential but at any stage more than one agent is active.[5]

All these models consider partnerships in which output is deterministic. In the second part of the paper, as we said, we extend our framework to verify robustness to noise. Stochastic partnerships are studied in few papers. Williams and Radner [1988] is the first paper to present a model in which the production function is stochastic. In this framework it is generally possible to satisfy the first order necessary conditions of incentive compatibility; however, except in some particular cases, it is not possible to show that these conditions are sufficient. Matsushima [1991], Legros and Matsushima [1991] and Fudenberg, Maskin and Levine [1994] extend this approach and present conditions for efficiency that are easier to satisfy. Note, however, that these conditions do not apply to our framework since efficiency can be achieved only if we

[4] There are other important papers that have studied efficiency in partnerships but that are somewhat less related to our work. Among these, Rasmusen [1987] exploits risk aversion of agents to achieve efficiency; and Radner [1986], Radner, Myerson and Maskin [1986] embed the partnership game in a repeated setting. Itoh [1991] obtains teams production as the optimal choice of task structure when agents can help each other.

[5] In Strautz's model agents are active sequentially one at a time and output is observable after each agent's action.

introduce the round of communication described above, a feature that is novel to this paper.[6]

The issue of uniqueness of the equilibrium is not generally studied in the partnership literature. Only two papers discuss it. Ma [1988] shows a mechanism with a unique sub-game perfect equilibrium, but he does not require budget balance and does not consider moral hazard[7]. Strautz [1999] shows that in the sequential partnership model that he considers there is a mechanism with budget balance such that the resulting game played by the agents has a unique sub-game perfect efficient equilibrium. However, this construction exploits the fact that players are assumed to be active one at a time, and the fact that changes in output can be observed at any stage. The problem in which more than one agent acts simultaneously is harder. Indeed we show that there are natural examples in which uniqueness in Nash equilibrium is impossible; however, even in these cases, we can find a mechanism such that the resulting game has a unique Coalition-Proof-Nash efficient equilibrium.

II The model and a simple example

In this section we define the model and present a simple example to highlight the main intuition of the results. The model that we consider is a simple generalization of the standard 'moral hazard in teams' problem. We analyze a joint production problem in which n agents choose effort $\mathbf{x}_i$ at a cost $c_i(\mathbf{x}_i)$, where $i \in N$, and N is the set of all agents. However, our environment extends the standard model in two ways. First, we consider a production function with the form as in (1) so that output is a vector in $\mathbb{R}^m$, $m \geq 1$. Second, we allow each agent to affect output in many ways, so that effort can be a vector $\mathbf{x}_i$ in a compact set $X_i = [0, M]^{a_i} \subset \mathbb{R}^{a_i}$, where the dimensionality $1 \leq a_i \leq m$ can differ across agents. We will indicate the vector of effort as $\mathbf{x}_i \equiv \{x_{i,1}...x_{i,a_i}\}$ for any $i = 1...n$, where $x_{i,l}$ is the generic l^{th} element of $\mathbf{x}_i$. We denote $X = \prod_{i=1}^n X_i$ the set of possible efforts and $\mathbf{x} \in X$ is the collection of the agents' efforts. As in Holmström [1982], we assume that $c_i : X_i \to \mathbb{R}$ are convex, differentiable and increasing (not necessarily strictly) in all arguments with $c_i(\mathbf{0}) = 0$ $\forall i$. The production function is $\mathbf{f} = \{f_1, ..., f_m\}$, where the functions $f_j : \prod_{i=1}^n \mathbb{R}^{a_i} \to \mathbb{R}$ are increasing (not necessarily strictly), concave and $C^1(X)$ (i.e., differentiable with continuous derivatives) with $f_j(\mathbf{0}) = 0$. Let $\mathbf{y} \in \mathbb{R}^m$ denote the vector of outputs. These assumptions may be relaxed: we make them only to put this work in direct comparison with Holmström's results (i.e., the case $y \in \mathbb{R}$). In particular, assuming that the production function is $C^1(X)$ makes the results more clear-cut, since we know from Legros and Matthews [1993] that when it is differentiable, efficiency is impossible when output is a real number. Legros and Matthews [1993] call this environment the *neoclassical teams' problem.*

To complete the description of the environment, we introduce an appropriate generalization

[6] This literature assumes that actions and possible output levels can assume only a finite number of values. Among other conditions, in order to obtain efficiency, it is required that the number of possible values that the output level can assume is higher than the sum of the number of possible actions available to any agent.

[7] He focuses on an adverse selection model.

of the concept of budget balance. One natural way to do it is to assume that there are m prices of outputs p_i with $i = 1...m$ such that the budget is $b(\mathbf{y}) = \sum_{i=1}^{m} p_i y_i$; in general we may consider the case in which $b(\mathbf{y})$ is any differentiable, weakly concave function $b : \mathbb{R}^m \to \mathbb{R}$ with $b(\mathbf{0}) = 0$. When it does not generate ambiguities, we will use the notation $b(\mathbf{x}) = b(\mathbf{f}(\mathbf{x}))$. A compensation mechanism satisfies *budget balance* if $\sum_{i=1}^{n} w_i(\mathbf{y}) = b(\mathbf{y})$, where $(w_i(\mathbf{y}))_{i=1}^{n}$ are wages contingent on output. To continue with the example above, it is typical for partners in a merchant bank to share the profits of the entire firm, not only the profits of their own offices (implicitly recognizing the existence of externalities). A compensation mechanism satisfies *limited liability* if $w_i(\mathbf{y}) \geq 0$ for any agent i and any output $\mathbf{y}$. We define:

Definition 1 A compensation mechanism is *feasible* if it satisfies budget balance and limited liability.

We define $\mathbf{x}_i^*$ as the first best level of effort for each agent,

$$\mathbf{x}^* = (\mathbf{x}_1^*, ..., \mathbf{x}_n^*) \in \arg\max_{\mathbf{x} \in X} \left\{ b\left(\mathbf{f}(\mathbf{x}_1..\mathbf{x}_n)\right) - \sum_{j \in N} c_j(\mathbf{x}_j) \right\}$$

and the resulting efficient output as $\mathbf{y}^* = \mathbf{f}(\mathbf{x}_1^*, ..., \mathbf{x}_n^*)$.
Given a production function $\mathbf{f}$ and a compensation mechanism $\mathbf{w}$, the agents play a game $\Gamma(\mathbf{f}, \mathbf{w})$ in which the payoffs are specified by the utilities $u_i(\mathbf{x}) = w_i(\mathbf{f}(\mathbf{x})) - c_i(\mathbf{x}_i)$, and the strategies are the effort levels $\mathbf{x}_i \in X_i$. For a given equilibrium concept used to solve the game, a compensation mechanism is said to (uniquely) *implement an efficient outcome* if $\Gamma(\mathbf{f}, \mathbf{w})$ has an (a unique) efficient equilibrium. We can therefore define:

Definition 2 A production function *sustains efficiency in equilibrium* if a feasible compensation mechanism that implements an efficient outcome exists.

Definition 3 A production function *uniquely sustains efficiency in equilibrium* if a feasible compensation mechanism that uniquely implements an efficient outcome exists.

Whenever a production function $\mathbf{f}$ does not sustain efficiency in equilibrium we might wonder if a mechanism that satisfies budget balance and implements an efficient outcome (but might violate limited liability) exists. We therefore define:

Definition 4 A production function *weakly sustains efficiency in equilibrium* if a compensation mechanism that satisfies budget balance and implements an efficient outcome exists.

The goal of this paper is to characterize conditions under which a production function sustains efficiency in Nash equilibrium or in a stronger equilibrium concept;[8] and whenever efficiency is possible, characterize conditions under which it uniquely sustains efficiency in equilibrium, at least for an appropriate equilibrium concept.[9]

[8] Indeed we will show conditions under which a production function sustains efficiency in Strong Nash equilibrium. See footnote 16 for a formal definition of a Strong Nash equilibrium.

[9] We will show conditions under which a production function uniquely sustains efficiency in Coalition Proof Equilibrium.

In order to illustrate the main idea, we start with a simple example. Let us assume that output is two-dimensional, effort is a real variable x_i for each agent i, and production functions are linear in efforts ($y_1 = \sum_i a_i x_i$, $y_2 = \sum_i b_i x_i$). In the following section, we generalize the result to the case of a generic (possibly non-linear) production function and higher dimensionality of both outputs and inputs. Here, we assume that the coefficients $\mathbf{a}$ and $\mathbf{b}$ are not linearly dependant (I use the notation $\mathbf{a}$ as the vector $\{a_i\}_{I=1}^{n}$; the same for $\mathbf{b}$), this is a very weak assumption since it only requires the vectors not to be multiples one of the other. Given this, we can display a simple mechanism that satisfies budget balance, limited liability such that, independently from the number of agents, the resulting game has an efficient equilibrium. We first describe the mechanism and then give a simple graphical interpretation.

After production takes place, if a deviation has been detected because $(y_1, y_2) \neq (y_1^*, y_2^*)$, we identify the set of agents who may be 'guilty'. We start defining the set of "suspected" agents $G(y_1, y_2)$ as:

$$G(y_1, y_2) := \left\{ i \left| [y_1 - \sum_{j \neq i} a_j x_j^*] = \frac{a_i}{b_i}[y_2 - \sum_{j \neq i} b_j x_j^*] \right. \right\}.$$

Besides the fact that $G(y_1, y_2)$ can always be computed given the observable variables because it depends only on (y_1, y_2), this set has two relevant characteristics. First, it can be easily verified that if agent i unilaterally deviates from the first best, then $i \in G(y_1, y_2)$. Second, and more importantly, *if there is unilateral deviation, then there is always at least one non-suspected agent.* To see this, assume, without loss of generality, that agent i unilaterally deviates from the first best and exerts effort $x_i \neq x_i^*$ and, by contradiction, $N \cap G(y_1, y_2) = N$. By the definition of the set $G(y_1, y_2)$, we have that for any agent l in N it must be that $[y_1 - \sum_{j \neq l} a_j x_j^*] = \frac{a_l}{b_l}[y_2 - \sum_{j \neq l} b_j x_j^*]$, which implies:

$$\begin{aligned}
[a_i x_i + \sum_{j \neq i} a_j x_j^* - \sum_{j \neq l} a_j x_j^*] = \frac{a_l}{b_l}[b_i x_i + \sum_{j \neq i} b_j x_j^* - \sum_{j \neq l} b_j x_j^*] \\
\Leftrightarrow a_i(x_i - x_i^*) + a_l x_l^* = \frac{a_l}{b_l} b_i (x_i - x_i^*) + a_l x_l^* \\
\Leftrightarrow a_l = \frac{a_i}{b_i} b_l \qquad \forall l \in N.
\end{aligned}$$

But this is in contradiction with the assumption that the vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Therefore, we know that there is an agent $k \in N$ such that $k \notin G(y_1, y_2)$. This property follows from the fact that, although for any observed deviation $\mathbf{y} \neq \mathbf{y}^*$ the system is indeterminate, for each agent we can generate a *determinate system* of equations using the equilibrium conditions requiring that all the other agents are not deviating. This sequence of determinate systems is not enough to identify who is 'guilty', but it is enough to identify at least one agent who is not guilty.

Given this, a mechanism that implements the efficient outcome can be easily constructed. Let us denote $|A|$ as the number of agents in a set A. A mechanism that satisfies budget balance and efficiency is:

$$w_i(y_1, y_2) = \begin{cases} 0 & i \in G(y_1, y_2) \text{ and } N \setminus G(y_1, y_2) \neq \emptyset \\ \frac{b(y_1, y_2)}{|N \setminus G|} & i \notin G(y_1, y_2) \text{ and } N \setminus G(y_1, y_2) \neq \emptyset \\ \frac{b(y_1, y_2)}{n} & y_1, y_2 \neq y_1^*, y_2^* \text{ and } N \setminus G(y_1, y_2) = \emptyset \\ c_i(x_i^*) + \frac{1}{n}[b(y_1^*, y_2^*) - \sum_{i \in N} c_i(x_i^*)] & y_1, y_2 = y_1^*, y_2^* \text{ and } N \setminus G(y_1, y_2) = \emptyset \ . \end{cases} \quad (3)$$

In this mechanism, all the agents in $G(y_1, y_2)$ are punished (receiving zero): this is enough to preserve incentives since if j has deviated, then $j \in G(y_1, y_2)$; moreover, we can preserve budget balance since, if there is only a unilateral deviation, $N \setminus G(y_1, y_2) \neq \emptyset$. In the case where there is more than a deviation (i.e., by *ii*, $N \setminus G(y_1, y_2) = \emptyset$), we just divide equally the budget: but this is irrelevant for incentives. Finally, it is easy to see that both budget balance and limited liability are satisfied.

The interesting aspect of this example is not really the fact that to solve the efficiency problem we need only to identify an agent who has not deviated instead of the agent who has deviated. Indeed, it is always true that *if* we find a "non-deviator", then the moral hazard in teams problem is solved: in this case, we can give this agent all the surplus and so we can punish a unilateral deviation maintaining budget balance. The key question, however, is *when* it is possible to identify an agent who has certainly not deviated. The example shows a case in which this is generally possible, and therefore efficiency can generally be achieved in equilibrium.

In the next section, we will not only extend this result to the general non-linear case and higher dimensional input and output spaces, but we will also show that, with a slightly more elaborate mechanism, it is possible to achieve the result with a stronger equilibrium concept and with a mechanism that filters out other undesired equilibria.

III Generic efficiency

In the example presented in the previous section, we only required linear independence of the coefficients of the production function. Therefore, in the class of linear production functions, the result is *generic.*[10] The natural extension of this example is to show that the result holds for any generic, possibly non-linear production function. This is the goal of this section.

In order to understand the general case, it is useful to identify what it is special about the linear production function considered above. Consider a graphical representation of the problem. For each agent $i \in N$, let us define Y_i as the set of outputs that agent i can induce with a unilateral deviation: $Y_i := \left\{ \mathbf{y} \,\middle|\, \exists\, \mathbf{x}_i \in X_i \text{ s.t. } \mathbf{y} = \mathbf{f}(\mathbf{x}_i, \mathbf{x}^*_{-i}) \right\}$, where $\mathbf{x}^*_{-i}$ is the vector of the levels of efficient effort of all agents except i. Given the two-dimensional example of the previous section, we can also define the correspondence $Y_i\left(y_1\right) := \left\{ y_2 \,\middle|\, \exists\, x_i \in X_i \text{ s.t. } \mathbf{y} = \mathbf{f}(x_i, x^*_{-i}) \right\}$ for any agent $i \in N$ and $\mathbf{y} = (y_1, y_2)$, so that $Y_i\left(y_1\right)$ maps from y_1 to the levels of y_2 that are feasible with a unilateral deviation by agent i.

[10] We use here the term generic in its intuitive meaning. See Definition 4 below for a formal definition of genericity.

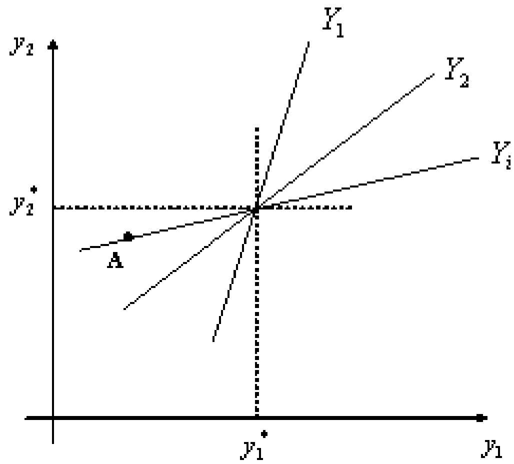

Figure 1: A special case: linear production functions.

Consider now Figure 1, where the two axes represent the variables that are observable and on which the mechanism can be contingent (outputs y_1, y_2). When the production function is linear, Y_i is also linear and, by definition, passes through $\mathbf{y}^* = (y_1^*, y_2^*)$ for each agent i. If we have a unilateral deviation by i, the output must be on Y_i (see, for example, point A in Figure 1). It is not necessarily true that two different agents i and k will have $Y_i \neq Y_k$[11] but, *by the property of the set* $G(y_1, y_2)$ *proven in the previous section*, we know that we can always find at least one agent k' with $Y_{k'} \neq Y_i$. We may therefore divide the agents into (at least) two sets, and punish them accordingly: thanks to the non-emptiness of $G(y_1, y_2)$, this can always be done while preserving budget balance.

When we consider the general non-linear case, the sets Y_is are still well-defined lines, but, as we can see from Figure 2.A, they need not be linear in y_1, y_2-space.[12] By definition, these sets will all cross at y_1^*, y_2^* but now they may (and typically they will) cross at other points. From Figure 2.B, we see what may be the problem: in this case, not only do the sets Y_i cross in more than one point, but they *all* cross at a point different from (y_1^*, y_2^*). For example, consider the situation in which agent i deviates and reduces his own effort generating point $\widetilde{\mathbf{y}}$: in this case, there may be no way to distinguish which agent actually deviated. Clearly this "pathological" case is impossible if the intersection of the unilateral deviation sets Y_is is empty. When this condition is satisfied we can always find a non-guilty agent who can act as a budget breaker: we can therefore punish all the "suspected agents" (among which there is certainly the deviator), and obtain efficiency.

The real question, therefore, is: when is this condition satisfied? We do not need to work

[11] Actually it is easy to see that if $\frac{a_i}{b_i} = \frac{a_j}{b_j}$, then $i \in G \Rightarrow j \in G$.

[12] Indeed, it is easy to show that for any agent i, $Y_i(y_1)$ defines an upper-hemicontinuous correspondence that is: non-decreasing in y_1 ($y' \geq y''$ implies that $\min Y_i(y') \geq \max Y_i(y'')$); and not "thick" (for no $i \in N$ there is an open set X in $\Re^2$ such that $X \subseteq I_i$).

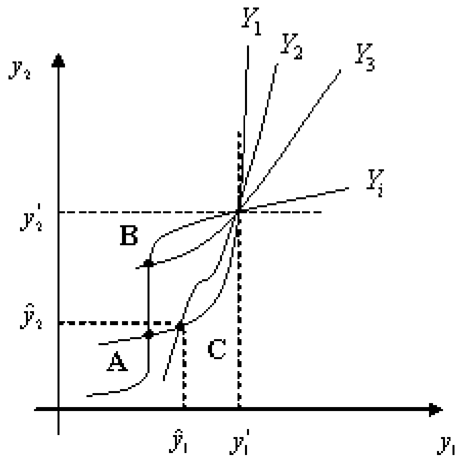

Fig 2.A

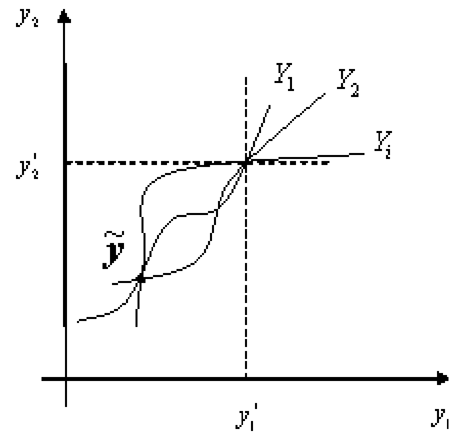

Fig 2.B

Figure 2: Figure 2.A represents the general non linear case. Figure 2.B represents a "pathological" case.

in a multidimensional environment to define unilateral deviations: even in a unidimensional world, if the intersection of the set that can be generated by unilateral deviations is empty, then efficiency is achievable. In their seminal work, Legros and Matthews [1993] define the set $Y = \cap_{i=1}^{n} Y_i$, and show an example in which agents have only finite actions and efficiency is possible even in Holmström's unidimensional environment because Y contains only $\mathbf{y}^*$. In the example, actions are assumed to be discrete, and each action of each agent induces a different level of output, therefore it is possible to perfectly identify who has deviated, or, if necessary, who has not deviated.[13] But the main result of their analysis is precisely that this condition is vacuous, because never satisfied in the standard neoclassical teams problem. Indeed, Legros and Matthews [1993] prove that, in the standard case, (almost) efficiency is achievable only if we relax the limited liability constraint and we allow for unbounded punishments.

In the present environment, on the contrary, the condition for efficiency is extremely weak. We are not ruling out a situation in which some of the sets Y_i cross at some point different from $\mathbf{y}^*$ as in Figure 2; we are requiring only that these sets do not *all* cross at the *same* point. Therefore, we may have a situation in which, for example, all the sets except one cross at a point $(\widehat{y}_1, \widehat{y}_2)$. The result that we intend to prove is precisely that when the environment is multidimensional, the intersection condition is *generically* satisfied. The intuition for this is evident from Figures 2.A and 2.B: if we are in the case depicted in Figure 2.B, then any arbitrarily small perturbation of $\mathbf{f}$ will shift at least one Y_i and make the intersection of the unilateral deviation sets empty.

To formalize this argument, we need a preliminary definition. Consider the set $\mathcal{F}$ of $C^1(X)$

[13] With finite actions, however, it is generically not necessary to identify an agent who has not deviated because the agent who has deviated is perfectly identified with a generic production function.

production functions that we have considered so far. The set $\mathcal{F}$ is endowed with a Whitney C^1 topology by letting a sequence of functions $f_n \in C^1(X)$ converge to f if and only if $f-g$ as well as the Jacobian of $f-g$ converge uniformly to zero in the space of continuous functions with euclidean norm (cf. Golubitsky and Guillemin [1973]). We define:

Definition 4 A set of production functions in $F \subseteq \mathcal{F}$ is *generic* in $\mathcal{F}$ if it contains a set that is open and dense in $\mathcal{F}$.

Given this definition, the set of non generic production functions is at most a countable union of closed sets: the Baire Category theorem guarantees that these sets have empty interior and therefore are topologically "small" (cf. Royden [1988, §7]). This is the standard definition of genericity in this environment.[14]

The next result characterizes the condition sufficient for the moral hazard in teams game to have an efficient Strong Nash equilibrium.[15]

Theorem 1 *If $\sum_{i=1}^{n} \frac{a_i}{n-1} < m$, then the set of production functions that sustains efficiency in Strong Nash Equilibrium is generic in $\mathcal{F}$.*

The basic intuition of this result is the same as the intuition presented above for the examples of Figures 1 and 2. Here, we complete the discussion with a few comments on the role of the dimensionality of the strategy space and the formula $\sum_{i=1}^{n} \frac{a_i}{n-1} < m$. A unilateral deviation by agent i generates a manifold in $\mathbb{R}^m$ with dimensionality a_i. In Figures 1 and 2, for example, agent i induces a line (a unidimensional manifold) in $\mathbb{R}^2$; in Figure 3.A, agent k induces a plane (a two-dimensional manifold) in $\mathbb{R}^3$. The intersection of manifolds in $\mathbb{R}^m$ is a manifold itself, but with lower dimensionality. In Figure 2, for instance, the intersection $Y_{ij} = Y_i \cap Y_j$ is a manifold with zero dimensionality (at most, a set of points); in Figure 3.A, Y_{ij} and Y_k are two-dimensional manifolds, and their intersection is a line. Transversality theory guarantees that, in general, if not empty, the intersection of two generic manifolds with dimensionality a_i and a_j, is a manifold with dimensionality $m-(m-a_i)-(m-a_j) = a_i + a_j - m$.[16] Suppose now that we have tree agents, i, j and k. The intersection of their unilateral deviation sets $(Y_i \cap Y_j) \cap Y_k = Y_{ij} \cap Y_k$ is either empty, or it has dimensionality larger or equal than zero. For

[14] The space of smooth production function is infinitely dimensional. In this case there is no natural analog of the Lebesgue measure, and therefore the usual measure theoretic notion of genericity is not available. There are different ways to solve this problem. Definition 3 follows the standard topological approach adopted in the literature (for a general discussion, and application to general equilibrium theory, see Mas-Colell [1985, §8]). Anderson and Zame [2004] have recently proposed a new definition of genericity based on the measure theoretic notion of "prevalence" (Hunt, Sauer and Yorke [1992]).

[15] A strategy profile $\widehat{\mathbf{x}} \in \prod_{i=1}^{n} X_i$ is a Strong Nash equilibrium if and only if for all subset of agents $J \subseteq N$ and for all joint deviations $\mathbf{x}_J \in \prod_{i\in J} X_i$, there is an agent $l \in J$ such that $u_l(\mathbf{x}) \geq u_l(\mathbf{x}_J, \widehat{\mathbf{x}}_{-J})$, where $u_l(\cdot)$ is agent j's utility and $\widehat{\mathbf{x}}_{-J}$ is the strategy used in equilibrium by the agents that do not belong to J. A Strong Nash equilibrium, therefore is a Nash equilibrium, but the converse is not generally true. Note moreover, that even under the conditions that guarantee the existence of a Nash equilibrium, a Strong Nash equilibrium may not exist. This equilibrium concept was introduced by Aumann [1959].

[16] For this result see, for example, Balasko [1988]. For extensive treatments of Differential Topology and Transversality theory, see also Milnor [1997].

a generic production function, this occurs only if $m-(m-a_k)-(m-a_i-a_j+m)\geq 0$, that is if $\sum_l a_l - 2m = \sum_l a_l - (n-1)\,m \geq 0$, which is the condition in Theorem 1. For example, if $m=3$, $a_1=a_2=2$ and $a_3=1$, then $\sum_l a_l = 5 < (n-1)m = 6$: and indeed $Y_{12}\cap Y_3$ would be the intersection of two unidimensional lines floating in $\mathbb{R}^3$, which (except at their origin $\mathbf{y}^*$) do not generically intersect in a three-dimensional space. The larger the number of agents and the lower the dimensionality a_i agent i's strategy space, the lower the dimensionality of the intersection Y. When $\sum_{i=1}^{n}\frac{a_i}{n-1} < m$, the deviation sets do not generically intersect, and we obtain the condition of Theorem 1.

As mentioned, one interpretation of the teams problem is that it is impossible to control the effort of n variables when only one variable (output) is observable. Theorem 1, however, shows that we have a *discontinuity* in the efficiency result, regardless of the number of agents and the (generic) choice of the production function. From Holmström [1982], we know that when output is one-dimensional, then efficiency is impossible if there is more than one agent; on the contrary, Theorem 1 proves that when output is at least two-dimensional, then efficiency is generically possible for any number of agents if strategies are real variables: indeed, when $n>2$ and $a_i=1$ $\forall\ i$, then the condition $\sum_{i=1}^{n}\frac{a_i}{n-1} < m$ is always satisfied for any $m\geq 2$. Theorem 1, however, extends the example of the previous section not only with respect to the dimensionality of the output space but also with respect to the strategy space.

In the next section we will prove that the sufficient condition for a generic production function to sustain efficiency in Nash equilibrium in Theorem 1 is actually necessary. Before exploring the exact relationship between the dimensionality of the strategy space and efficiency, however, we now discuss two properties of the efficient mechanism: its *robustness to collusion*; and the conditions under which the production function *uniquely supports efficiency* in equilibrium.

A Nash equilibrium guarantees that no agent finds it optimal to unilaterally deviate from the equilibrium strategy; but it does not guarantee that a subset of agents find it profitable to coordinate their actions and play a joint deviation. When agents can communicate, it is indeed realistic to imagine that a subset of agents may form a coalition to maximize the joint utilities of its members at the expenses of the remaining agents: it is therefore important to make sure that the compensation mechanism does not leave incentives for these joint deviations. There are two approaches to model collusion-proofness. On the one hand, we can imagine that agents can write binding agreements among themselves and transfer monetary payments.[17] This does not seem the appropriate approach in the present environment because if the agents could observe each others effort levels and write binding commitments, then they would also be able to implement the efficient outcome. The moral hazard in teams problems is precisely the consequence of the inability to write such a rich compensation mechanism. Moreover, it is difficult to enforce secret (and "illegal") agreements among agents in front of a court, even if these contracts are contingent only on observable output.

In the second approach to model collusion, agents cannot commit to side payments, and

[17] This is the approach adopted in the seminal work on collusion by Tirole [1992].

only deviations that are incentive compatible are considered: given the behavior of the agents not in the group, the joint deviation of the colluding agents must be a Nash equilibrium. This property is guaranteed when the equilibrium is Coalition-Proof.[18] Theorem 1 proves not only that efficiency is robust to collusion in this sense, but also that efficiency can be implemented in a more compelling sense, as a Strong Nash Equilibrium. In a Strong Equilibrium, the agents in the deviating coalition cannot commit to monetary transfers among themselves, so side payments are not allowed among the deviating agents; but contrary to a Coalition-Proof Equilibrium, in a Strong Equilibrium the deviating agents can *commit* to a jointly optimal deviation profile.

The second important property of an efficient mechanism is the uniqueness of the equilibrium in the resulting game. The literature on the moral hazard in teams problem has focused attention on the *possibility* of efficiency. However, when efficiency can be implemented by some mechanism, it is important to avoid that the agents coordinate on a "bad" equilibrium with an inefficient outcome. Indeed, even if equilibria may be ranked in terms of aggregate surplus, it is not necessarily true that the most efficient equilibrium is also Pareto superior: given the compensation mechanism, some agent may strictly be better off in a less efficient equilibrium. In these cases, even if agents can communicate ex ante, we would not necessarily expect that they would coordinate on the most efficient equilibrium. The mechanism, however, may be designed to avoid these problems.

For any two vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^{a_i}$ we say that $\mathbf{x} > \mathbf{y}$ if $x_l \geq y_l$ for all $l = \{1, .., a_i\}$ and there is a k such that $x_k > y_k$. The budget function $b(\mathbf{x})$ has *strict increasing differences* if for all agents i we have that $b(x_i, x_{-i}) - b(x_i', x_{-i}) > b(x_i, x_{-i}') - b(x_i', x_{-i}')$ for all $x_i, x_i' \in X_i$ and $x_{-i}, x_{-i}' \in X_{-i}$ such that $x_i > x_i'$ and $x_{-i} > x_{-i}'$ (cf. Topkis [1998]). Increasing differences says that an increase in the effort level of any agent $j \neq i$ raises the desirability of exerting a high level of effort for agent i. This is therefore a natural assumption that captures the complementarities that typically characterize team work. Under the most natural assumption on the function $b(\cdot)$, when the budget is the monetary value of output $b(\mathbf{x}) = p_1 f_1(\mathbf{x}) + p_1 f_1(\mathbf{x})$, $b(\mathbf{x})$ has increasing differences if and only if the production function has increasing differences;[19] for a general function $b(\mathbf{x}) = b(\mathbf{f}(\mathbf{x}))$, this property, however, depends both on $\mathbf{f}$ and b. For a given b, let $\mathcal{F}^{id} \subseteq \mathcal{F}$ be the set of production functions such that $b(\mathbf{f}(\mathbf{x}))$ has increasing differences. We have:

Proposition 1 *If* $\sum_{i=1}^{n} \frac{a_i}{n-1} < m$, *then the set of production functions that uniquely sustains efficiency in pure strategy Coalition-Proof Nash equilibrium is generic in* $\mathcal{F}^{id}$.

Theorem 1 and Proposition 1 are complementary. Since a Strong Nash Equilibrium is also a Coalition-Proof-equilibrium, in Theorem 1 we could also write "Coalition-Proof equilibrium"; similarly, if an equilibrium is unique in Coalition-Proof equilibrium, then *a fortiori* it must be

[18] See Bernheim, Peleg and Whinston [1986] for a formal definition. This approach to model collusion is also used in Battaglini [2002].

[19] Note moreover that if $b(\mathbf{x})$ is supermodular in $\mathbf{x}$, then $b(\mathbf{x})$ has increasing differences. The reverse, however, is not necessarily true.

unique as a Strong Nash equilibrium, since the set of Strong Nash equilibria is included in the set of Coalition-Proof equilibria.

The main reason why there may be multiplicity of equilibria are complementarities in production functions; when these are very strong we always have a Nash equilibrium with low effort. In these cases, the ability to collude that is implicit in the definition of a Coalition-Proof equilibrium is key to kill these equilibria, since it allows the agents to collectively deviate from inefficient outcomes. Consider the following very simple example where there exists no compensation system that would guarantee that the resulting game has a unique Nash equilibrium, but where the conditions of Proposition 1 are satisfied, so we have uniqueness in Coalition-Proof equilibrium:

Example 1 There are three agents. Assume that the production function is a Cobb-Douglass: $y_l = x_1^{\alpha_l} x_2^{\beta_l} x_3^{\eta_l}$ with $l = 1, 2$ and the cost functions of the agents are such that a positive level of effort is required from all of them at the optimum.

Clearly, in this case the efforts of agents 1 and 2 are complementary and $\bigcap_{i=1}^{n} Y_i = \{\mathbf{y}^*\}$ is satisfied for a generic choice of parameters α_l, β_l and η_l: so, by Proposition 1, this production function uniquely sustains efficiency in Coalition-Proof pure strategy equilibrium. However, we can easily verify that it does not uniquely sustain efficiency in Nash equilibrium.[20]

The assumption of increasing differences is not necessary and can be relaxed but, among the sufficient conditions, it is the one that has the most straightforward intuition behind it. Indeed, the requirement of increasing differences of the production function is particularly natural in this context: one of the reasons why agents form partnerships are precisely complementarities in the production function. It is interesting to note that while increasing differences without Coalition-Proofness may be a source of multiplicity, increasing differences together with Coalition-Proofness induces the uniqueness result.

IV Dimensionality of the strategies and efficiency

In this section, we complete the analysis on the role of the dimensionality of the action space, proving that the condition in Theorem 1 is necessary to obtain generic efficiency. To this goal, it is useful to distinguish two cases: when $\sum_{i=1}^{n} \frac{a_i}{n-1} > m$, and when $\sum_{i=1}^{n} \frac{a_i}{n-1} = m$.

The following theorem extends Holmström's impossibility result to the case of vector-valued production functions, and shows that the dimensionality of the agents' strategy sets is essential for generic efficiency:

Theorem 2 *If $\sum_{i=1}^{n} \frac{a_i}{n-1} > m$, then the set of production functions that does not weakly sustain efficiency in Nash Equilibrium is generic in $\mathcal{F}$.*

[20] If $x_1 = 0$ then the wage that agent 2 receives cannot depend on the level x_2 since it depends only on output and output does not depend on x_2 when $x_1 = 0$. But then also agent 2 will exert zero effort. So for any wage schedule, we have an equilibrium with $x_1 = x_2 = x_3 = 0$.

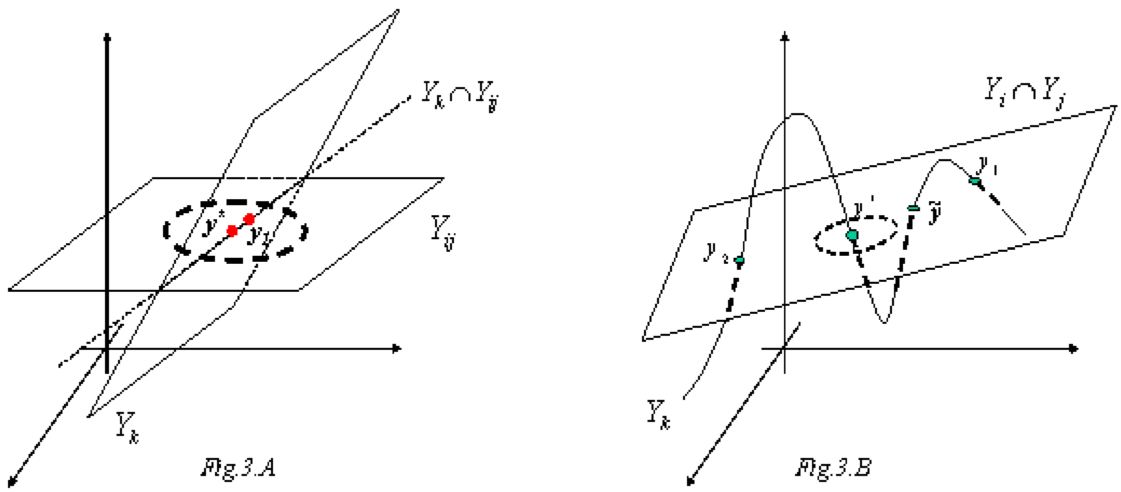

Figure 3: Fig 3.A represents the case in which $\sum_{i=1}^{n} \frac{a_i}{n-1} > m$. When, as in Fig 3.B, $\sum_{i=1}^{n} \frac{a_i}{n-1} = m$ and the intersection is a zero dimensional manifold, efficiency is possible, but not generically.

The intuition of this result can be seen from Figure 3.A. Suppose that there are three agents i, j, and k. As we have seen above, the intersection $Y_{ij} = Y_i \cap Y_j$ has generically dimensionality $a_i + a_j - m$; and $Y = Y_{ij} \cap Y_k$ is a manifold with dimensionality $q = \sum_{i=1}^{n} a_i - (n-1)\, m$ (where, in this case, $n = 3$). Therefore, when $\sum_{i=1}^{n} a_i > (n-1)\, m$, $Y = Y_{ij} \cap Y_k$ is a manifold with dimensionality $q \geq 1$. In Figure 3.A, for example, Y_{ij} and Y_k are two-dimensional, so Y is one-dimensional. When $q \geq 1$, the set of deviations that can be induced by any agent in the production team (for example, $\mathbf{y}_1$ in Figure 3) is a manifold passing through $\mathbf{y}^*$: a line, as in Figure 3.A, when $\sum_{i=1}^{n} a_i - (n-1)\, m = 1$; a higher dimensional surface when $\sum_{i=1}^{n} a_i - (n-1)\, m > 1$. In particular, there is a subsets of inefficient outcomes which can be achieved by a unilateral deviation by i, which can also be achieved by a unilateral deviation by any of the other agents $-i$. To guarantee incentives for efficient production, we need to make sure that none of the $N \setminus i$ agents finds it profitable to induce a point in this set. For this reason, the payoff that all agents except i receive in case any of these points is realized must be low enough. However, by budget balance, this implies a lower bound for the payoff of this agent i: Theorem 2 proves that this bound is high enough to make the deviation profitable for this agent, independently from the compensation mechanism.

Although Theorem 2 extends Holmström's theorem to the case with vector outputs, the fact that we need a significantly high lower-bound on the "average" dimensionality of the agents' strategies considerably weakens the inefficiency result (since, as proven above, when $\sum_{i=1}^{n} \frac{a_i}{n-1} < m$ efficiency is generically feasible).

One last case needs to be studied to complete the analysis of the efficiency possibility frontier in the teams problem: when $\sum_{i=1}^{n} \frac{a_i}{n-1}$ is exactly equal to m. In this case, efficiency is possible, but not *generically* possible. To see this, imagine a case in which all agents except agents i, j and k have full dimensional strategy at the efficient level $\mathbf{x}^*$. If Y_i, Y_j and Y_k intersect at a point $\widetilde{\mathbf{y}}$ in a neighborhood of $\mathbf{y}^*$, then any agent could induce such deviation $\widetilde{\mathbf{y}}$ (see Figure 3.B). If we perturb the production function, Y_i, Y_j and Y_k would still generically intersect at a point very near to $\widetilde{\mathbf{y}}$. By continuity, if a deviation to $\widetilde{\mathbf{y}}$ is optimal,[21] then a deviation to the new intersection point after the perturbation would still be optimal.

In this case, however, we can extend the techniques developed by Legros and Matthews [1993] for the unidimensional case to complete the analysis and pin down the necessary and sufficient condition for this particular case too. Given $\mathbf{f}$, for any $\mathbf{y} \in Y_i$ define:

$$\varphi_i(\mathbf{f}, \mathbf{y}) = \inf \left\{ c_i(\mathbf{x}_i) \,|\, \mathbf{f}(\mathbf{x}_i, \mathbf{x}_{-i}) = \mathbf{y},\, \mathbf{x}_i \in X_i \right\},$$

and $\psi(\mathbf{f}, \mathbf{y}) = b(\mathbf{y}) - b(\mathbf{y}^*) - \sum_{i=1}^{n} [\varphi_i(\mathbf{f}, \mathbf{y}) - \varphi_i(\mathbf{f}, \mathbf{y}^*)]$.

Theorem 3 *If $\sum_{i=1}^{n} \frac{a_i}{n-1} = m$, then a production function $\mathbf{f}$ weakly sustains efficiency in Nash equilibrium if and only if $\psi(\mathbf{f}, \mathbf{y}) \leq 0$ for any $\mathbf{y} \in Y$*

When $\sum_{i=1}^{n} \frac{a_i}{n-1}$ is exactly equal to m, therefore, the fact that output is vector-valued provides little advantage with respect to the unidimensional case characterized by Legros and Matthews [1993]: and, in particular, it is no longer true that efficiency can be sustained in a generic set of production functions. In addition, as again in the corresponding result by Legros and Matthews [1993], Theorem 3 does not guarantee that there exists a mechanism that implements an efficient outcome and, besides budget balance, satisfies limited liability.[22] A condition for this stronger result requires additional assumptions on the environment. A simple sufficient condition is that the cost functions are not too heterogeneous.[23]

Despite these similarities with the unidimensional case, there is an important difference. In a unidimensional environment efficiency is impossible because the necessary condition in Theorem 3 is never satisfied with a $C^1(X)$ production function. In a multidimensional world, however, we can find examples in which it is easily satisfied, even if the intersection Y contains a $\mathbf{y} \neq \mathbf{y}^*$. To prove that the condition is never satisfied in a unidimensional world, Legros and Matthews [1993] find a $\mathbf{y}$ that violates it in the neighborhood of $\mathbf{y}^*$. In a unidimensional world with a $C^1(X)$ production function, this can always be done without loss of generality. As it can be seen from Figure 3.B, however, when output is multidimensional it is generically impossible to find a $\widetilde{\mathbf{y}}$ in the intersection Y which is arbitrarily near $\mathbf{y}^*$ (in a neighborhood of $\mathbf{y}^*$ the function is approximately linear, therefore the unilateral deviation sets Y_is do not generically intersect). When $\widetilde{\mathbf{y}}$ is not in the neighborhood of $\mathbf{y}^*$, then the condition of Theorem 3 may be satisfied and we may continue to be able to support an efficient outcome.[24]

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[21] It is not difficult to write examples in which the deviation sets $Y^{ij} \cap Y^k$ intersect at a point that is arbitrarily near y^*. In this case, an argument similar to the proof of Proposition 3 can be used to prove that this point would be a profitable deviation for some agent in any compensation mechanism.

[22] Legros and Matthews [1993] do not require wages to be non negative. In their terminology, a production function "sustains efficiency" if, in our terminology, it "weakly sustains efficiency."

[23] For example, this condition is always satisfied when the agents have the same cost functions. It can also be proven that even if limited liability is violated, the wages are never lower than a finite lower bound.

[24] It is easy to construct such examples if we assume that the cost functions $c_i(\cdot)$ increase steeply as we move away from $\mathbf{x}_i$.

V Noise and communication

One assumption that is important for the previous analysis is that it is possible to predict the equilibrium optimal effort levels $\mathbf{x}_i^*$s. Thanks to this assumption, we may condition on the event in which only agent i is deviating and calculate the sets Y_is. This assumption is made not only in this work, but is typical in the literature on partnerships.

Consider, however, the following perturbation of the problem. Each agent takes an action $x_i \in \mathbb{R}$ but the actual level of effort which enters the production function is $\widetilde{x}_i$: a random variable that with probability $1-\varepsilon$ is equal to x_i, and with probability ε is distributed according to a continuous distribution $f(\widetilde{x}_i)$ with support $[0, x_i]$. Assume that the perturbations are independent across agents. The interpretation of this perturbation is straightforward: x_i is, as before, an agent i's effort; but now, some event may reduce the actual contribution; for example, due to a flat tire, a worker arrives two hours late to work. In this case, regardless of how small ε is, if the number of agents is very large, the number of 'deviators' (i.e., agents with $x_i < x_i^*$) is arbitrarily large, even if the effort levels of all agents are at first best. This situation is interesting because it describes an environment in which, while the probability of under-performance for a single agent may be low if effort is high, the probability of many agents under-performing is high in large partnerships even in equilibrium. This makes a considerable difference from a theoretical point of view. In the previous sections we exploited the dimensionality of output and the 'one deviation' condition to identify an agent who did not deviate. Now, even if the perturbations are very small, because we have a large number of them, this identification is lost: so the environment is qualitatively different. This situation is problematic for the mechanism designed in the previous section. For example, in the linear case of Section II, it is no longer generically true that the set $N \setminus G(y_1, y_2)$ is empty: which is necessary in that mechanism to satisfy budget balance. The purpose of this section is to show that the basic idea presented in the previous sections may be exploited even in this situation with noise. For simplicity, in this section we focus on the case with $\mathbf{y} \in \mathbb{R}^2$ and $x_i \in \mathbb{R}$.

Before presenting the main result, we can observe that this model of noise is novel in this literature. One alternative way to introduce noise is to assume that output is equal to the value of the production function plus a perturbation ϵ: $\widetilde{f}(\cdot) = f(\cdot) + \epsilon$, therefore noise is attached to the technology. In our approach, instead, noise is attached to the effort levels of the agents, not to the technology, which still deterministically maps the realized efforts to the output level. It is probably possible to argue that, in reality, every variable is subject to perturbation; but the real question is which is the primary source of noise. The first approach assumes that agents can perfectly control effort, but technology is noisy. In the model presented above we assume that the technology is predictable, at least in the short-term, but agents are subject to perturbations. This seems a natural assumption, especially in partnerships with many members. Moreover, as we aim to show, this type of noise structure, which is certainly relevant in partnerships, can be essentially eliminated by the design of an appropriate communication and compensation system.[25]

To extend the mechanism studied in the previous section and study efficiency, we make two assumptions. The first is:

Assumption 5.1 (*Minimal Observability*) Each agent observes the realization of his own actual contribution $\widetilde{x}_i$.

This seems natural: to remain in the example of the "flat tire", partner i wakes up early in the morning to be at work on time (effort is efficient); he has a flat tire and arrives two hours late (realization); but he knows it. Theoretically this situation is in between the standard moral hazard in teams problem and the environment studied by Miller [1997]. Miller, in fact, studies a situation in which at least one agent can observe the effort of another agent besides his own:[26] in the present work, instead, we assume that each agent observes only his own contribution.[27]

Given minimal observability, we might consider a two stage mechanism where in the first stage agents make a report on their realized contributions $\widetilde{x}_i$; and in the second stage, the mechanism allocates surplus on the basis of observable output and these declarations. The problem is to design a mechanism that gives the agents' incentives to be truthful in the communication stage. This is more difficult in this environment because we need to respect budget balance: if the declaration of agent i is used to determine the payoff of agent j, then we have to make sure that the change in payoff in agent j due to the declaration of i will not affect the payoff of agent i. Since we have to do this for all agents and keep budget balance, the analysis is substantially complicated. A sufficient condition for the construction of this mechanism to be possible is that each agent i has a *comparative advantage* in the production of one good with respect to some other agent j:

Assumption 5.2 (*Single Crossing*) For each agent i there is an agent $D(i)$ such that:

$$\frac{\partial f_l(x_1...x_N)}{\partial x_i} \Big/ \frac{\partial f_{-l}(x_1...x_N)}{\partial x_i} > \frac{\partial f_l(x_1...x_N)}{\partial x_{D(i)}} \Big/ \frac{\partial f_{-l}(x_1...x_N)}{\partial x_{D(i)}}$$

for some $l = 1, 2$ and any point $x_1...x_N$.[28]

[25] All the results presented below can be extended to the case in which we add noise also to the technology, if this noise is small, as it is perhaps realistic to assume at least from a short term point of view: in this case, the redistribution of surplus would not be contingent on a particular level of output but on a region of possible output realizations. Clearly, incentives would be less accurate, a problem that is unavoidable by any mechanism; but when noise is not too large, output would work as a precise enough signal for the mechanism. However, we find the logic of such an extension less interesting and therefore we have omitted it. This type of argument is also suggested by Strausz [1998] to justify his deterministic model.

[26] Miller, moreover, studies a partnership without noise.

[27] An alternative interpretation of the model with $\widetilde{f}(\cdot) = f(\cdot) + \epsilon$ is that the technology is not noisy, but the agents unconsciously commit mistakes in spite of their good intentions: and therefore they do not know the realization $\widetilde{x}_i$. In this case, the difference with our model is that agents are (even ex post) not aware of their own mistakes.

[28] This inequality may not be defined if the production function is weakly increasing and the denominator of one of the sides is zero. Note, however, that it could be easily generalized. Read in this case this assumption as: if both the right hand side and the left hand side are bounded, then the inequality holds; if the right hand side is unbounded, then the left hand side is bounded.

From a graphical point of view, this condition implies that for any i, there is an agent j such that if we fix the realized effort of all other agents $\mathbf{x}_{-i,j}$, and a level of output y, the resulting isoquants defined by $f_l(x_i, x_j, \mathbf{x}_{-i,j}) = y_l$ $l = 1, 2$ intersect at most once. In the case of a linear production function, for example, these isoquants are two straight lines in the x_i, x_j space with (negative) slopes that, except in the non-generic case in which they coincide, intersect at most once: so that the condition is automatically satisfied. In the general case, this condition is sufficient but not necessary to have at most one intersection; however, it has a simple and natural interpretation. The condition is naturally satisfied if agent i affects directly one production function and, indirectly the other through externalities: any agent who affects directly the production function of the second output would have a comparative advantage in it with respect to i. Note also that this condition is not very strong for an other reason: we are not requiring that $D(i)$ is unique, we may have cases where $D(i)$ is a set of partners and cases where $i \neq j$ but $D(i) = D(j)$.

We can now state the main result of this section. A measure of the profitability of the partnership which is independent with the number of agents is the rate of return of 'a dollar' spent in effort at the first best:

$$r = \frac{Eb(x^*) - \sum_{j \in N} c_j(x_j^*)}{\sum_{j \in N} c_j(x_j^*)},$$

where $Eb(x^*)$ is the expected revenue of the partnership at the first best level of effort. We have:

Proposition 2 *Assume that minimal observability and single crossing hold. In this case, there is a threshold $r^*(\varepsilon)$ such that if the return of the partnership is $r \geq r^*(\varepsilon)$, then the the production function sustains efficiency in Nash equilibrium. The threshold $r^*(\varepsilon)$ is independent from the number of agents N, and $r^*(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

This result shows that when the partnership is profitable enough, efficiency is possible despite the budget balance and limited liability constraints; indeed if individual noise is small, efficiency is feasible in almost all partnerships, independently from the number of agents: independently of the aggregate level of noise, the threshold level $r^*(\varepsilon)$ converges to zero as the noise ε of a single agent's performance converges to zero. Since N is arbitrary, we can have any large number of deviations with probability close to one even if ε is small: but for efficiency, *only the level of individual noise is relevant.* This happens because with the communication stage we can isolate individual incentives from aggregate noise. We describe the idea of the mechanism here (details may be found in the Appendix).

First, we associate each agent with his companion $D(i)$ and form a set $\{i, D(i)\}$. We may therefore construct a family of sets $S = \{\{i, D(i)\}, \{j, D(j)\}, \{k, D(k)\}...\}$ such that each agent is at least in one of them. The mechanism assigns to each of these sets an *ex ante* fixed share of the surplus that is produced by the partnership: this share will be independent of the declarations and the realized output. We then have two stages.

The first stage takes place after output is produced: each agent i is asked to report the realized level of his own contribution $\widetilde{x}_i$. The declaration of i (denote it d_i) will be used to allocate the fixed share of surplus among the members of each group $s \in S$ in which i is not a member. Since the share associated to each group is fixed and i does not belong to any group in which his declaration is influential, agent i has no incentives to report untruthfully.

Given this, the wage may be contingent on a richer set of signals: y_1, y_2 and the (endogenous) d_is. For any i, j such that $\{i, j\} \in S$ we may consider the system:

$$\begin{aligned} y_1 &= f_1(d_1, ..., d_{i-1,} x_i, d_{i+1,}.., d_{D(i)-1,}, x_{D(i)}, d_{D(i)+1}..d_n) \\ y_2 &= f_2(d_1, ..., d_{i-1,} x_i, d_{i-1,}.., d_{D(i)-1,}, x_{D(i)}, d_{D(i)+1}..d_n) \end{aligned} \tag{4}$$

The single crossing assumption guarantees that this system has a unique solution which depends on the declaration of the agents other than i and j (that is: $d_{-\{i,j\}}$). Given the observable y_1, y_2, in fact, each equation in (4) defines the isoquants obtainable through a deviation by agent i or j, given that the realized effort levels of all the remaining agents are equal to their declarations. The single crossing assumption is a condition on their relative slope which makes sure that the isoquants cross only once and therefore that there is a unique pair x_i, x_j that may generate y_1, y_2. If, given the declarations of agents $-ij$, it turns out that agent i has deviated and agent j has not, then we allocate all of the surplus to agent j. Since the mechanism can use the true realizations of the effort levels of the agents that are not in the group, i and j do not face the risk to be punished because of the noise in the performance of others: this is the reason why the result holds for any N and for any degree of aggregate uncertainty. Moreover, because there is no spillover of surplus from the group in favor of the agents outside the group, incentives for truthful revelation in the communication stage are preserved.

Despite the fact that the agents have an obvious conflict of interest in the redistribution of the surplus, the mechanism is successful in neutralizing the conflicts that naturally arise in these situations, but it provides weak incentives to reveal the truth in the communication stage. If the agents are indifferent, there might be exogenous reasons for an agent to sabotage the mechanism (he might be jealous of other agents' performance, for example): the mechanism takes care of the incentives that are endogenous in the model. These considerations on exogenous motives, however, would be true even if there were strict incentives, since the exogenous motives might be stronger than the incentives: so the only issue is the multiplicity of equilibria (which would exist even if the equilibria were strict). On the other hand, there might be many reasons to expect that if agents do not have direct (endogenous) incentives to sabotage the mechanism, they would not do it: they might be evaluated on the basis of the success of the partnership, there might be a probability of repetition of the game, or there might be a positive probability of an inspection that reveals the true realizations.

On the other hand, the mechanism of this section has some other valuable characteristics. First, it guarantees that one can identify agents who have not deviated; but, in particular, it enables one to isolate the risk faced by each agent from the aggregate uncertainty of the partnership. In addition, the mechanism is independent from the order in which agents report in the communication stage. This seems to be a particularly interesting property. It would not be realistic to imagine that in a large partnership all of the partners would be able to report simultaneously, and that no information on the reports of other agents is observed.

VI Conclusions

In this paper we have presented a generalization of the moral hazard in teams problem introduced by Holmström [1982] to study the case when there are multiple outputs. We have characterized the necessary and sufficient condition for the existence of an efficient equilibrium and shown that it is easily satisfied in natural environments. For example, if $n > 2$ agents control an effort level variable $x_i \in \mathbb{R}$, and output is in $\mathbb{R}^m$ with $m > 1$, then for any (possibly arbitrarily large) number of agents and a generic production function, there is a wage schedule that satisfies budget balance, limited liability and such that the resulting game played by the agents has an efficient equilibrium. This result runs against the common interpretation of the standard moral hazard in teams problem: i.e., that the principal wants to control too many variables ($n > 1$) with only one instrument, observable output y. Indeed, if this were true, the intuition would generalize to the case when the number of variables is larger (especially if arbitrarily larger) than the instruments. What really matters is not the number of variables that the principal desires to control (i.e., the number of x_is) but the relationship between the *aggregate* dimensionality of the instrument and the dimensionality of these variables. Efficiency is not generically possible when the "average" dimensionality of the agents' strategies $\sum_{i=1}^{n} \frac{a_i}{n-1}$ is larger than the dimensionality of output; but when $\sum_{i=1}^{n} \frac{a_i}{n-1} < m$, then efficiency can be achieved for a generic production function.

Appendix

A. Proof of Theorem 1

The result is proven by the following lemmata.

Lemma 1 *If $b(\mathbf{x})$ has strict increasing differences, then there exists a sequence $\{\alpha_i\}_{i=1}^n$ such that $\alpha_i \in (0,1)$, $\sum_i \alpha_i = 1$ and $\forall i$*

$$\alpha_i[b(\mathbf{x}_1^*,..,\mathbf{x}_n^*) - \sum_j c_j(\mathbf{x}_i^*)] > \max_{\mathbf{x}_i}[b(\mathbf{0},\mathbf{0},..,\mathbf{x}_i,..,\mathbf{0}) - c_i(\mathbf{x}_i)]. \tag{A.1}$$

Proof. Define

$$\alpha_i = \frac{\max_{\mathbf{x}_i}[b(\mathbf{x}_i,\mathbf{x}_{-i}=\mathbf{0}) - c_i(\mathbf{x}_i)]}{[b(\mathbf{x}_1^*,...\mathbf{x}_n^*) - \sum_j c_j(\mathbf{x}_j^*)]} + \frac{1}{N}[1 - \frac{\sum_j \max_{\mathbf{x}_j}[b(\mathbf{x}_j,\mathbf{x}_{-j}=\mathbf{0}) - c_j(\mathbf{x}_j)]}{[b(\mathbf{x}_1^*,...\mathbf{x}_n^*) - \sum_j c_j(\mathbf{x}_j^*)]}.$$

Clearly, we have that $\sum_i \alpha_i = 1$. Let us define $\pi(\mathbf{x}_1,...\mathbf{x}_n) = b(\mathbf{x}_1,...\mathbf{x}_n) - \sum_j c_j(\mathbf{x}_j)$. It can be verified if $b(\mathbf{x})$ has strict increasing differences, then $\pi(\mathbf{x}_i,\mathbf{x}_{-i})$ has strict increasing differences in $\mathbf{x}_i,\mathbf{x}_{-i}$ $\forall i$ as well; and clearly $\pi(\mathbf{0},...\mathbf{0}) = 0$. Working inductively on the number of agents, we can prove that $\sum_j \pi(\widetilde{\mathbf{x}}_j,\mathbf{x}_{-j}=\mathbf{0}) < \pi(\widetilde{\mathbf{x}}_1,...\widetilde{\mathbf{x}}_n)$ is true for any vector $\widetilde{\mathbf{x}} > 0$ (for future reference we call this inequality A.2). Assume now, by contradiction, that $\sum_j \max_{\mathbf{x}}[b(\mathbf{x}_j,\mathbf{x}_{-j} = \mathbf{0}) - c_j(\mathbf{x}_j)] \geq [b(\mathbf{x}_1^*,...\mathbf{x}_n^*) - \sum_j c_j(\mathbf{x}_j^*)]$ then by inequality (A.2):

$$[b(\mathbf{x}_1^{**},...,\mathbf{x}_n^{**}) - \sum_j c_j(\mathbf{x}_j^{**})] > [b(\mathbf{x}_1^*,...\mathbf{x}_n^*) - \sum_j c_j(\mathbf{x}_j^*)], \tag{A.3}$$

where $\mathbf{x}_i^{**} \in \arg\max_{\mathbf{x}_i}[b(\mathbf{x}_i,\mathbf{x}_{-i}=\mathbf{0}) - c_i(\mathbf{x}_i)]$; but (A.3) is in contradiction with the definition of $\{\mathbf{x}_i^*\}_{i=1}^n$. Therefore, $\sum_j \max_{\mathbf{x}_j}[b(\mathbf{x}_j,\mathbf{x}_{-j}=\mathbf{0}) - c_j(\mathbf{x}_j^*)] < [b(\mathbf{x}_1^*,...\mathbf{x}_n^*) - \sum_j c_j(\mathbf{x}_j^*)]$. It follows that $1 > \alpha_i > \frac{\max_{\mathbf{x}_i}[b(\mathbf{x}_i,\mathbf{x}_{-i}=\mathbf{0})-c_i(\mathbf{x}_i)]}{[b(\mathbf{x}_1^*,...\mathbf{x}_n^*)-\sum_j c_j(\mathbf{x}_i^*)]} \geq 0$ for any i, and (A.1) holds for any $i \in N$. ■

Let $Y = \bigcap_{i=1}^n Y_i$. We now show that if $Y = \{\mathbf{y}^*\}$, then we can find a mechanism with the properties described in Theorem 1.

Lemma 2 *If $Y = \{\mathbf{y}^*\}$, then efficiency is feasible in Strong Nash Equilibrium.*

Proof. Define a function $\iota : \mathbb{R}^m \to N(\mathbf{y})$ that maps from output to one agent in the set

$$N(\mathbf{y}) = \begin{cases} N \setminus G(\mathbf{y}) & \text{if non empty} \\ N & \text{else} \end{cases} \tag{A.2}$$

where $G(\mathbf{y})$ is the set of suspected agents: $\{i \,|\mathbf{y} \in Y_i\}$. Define:

$$w_i(\mathbf{y}) = \begin{cases} b(\mathbf{y}) & \mathbf{y} \neq \mathbf{y}^* \text{ and } i = \iota(\mathbf{y}) \\ 0 & \mathbf{y} \neq \mathbf{y}^* \text{ and } i \neq \iota(\mathbf{y}) \\ \widetilde{w}_i(\mathbf{y}) & \text{else} \end{cases}$$

where $\widetilde{w}_i(\mathbf{y}) = c_i(\mathbf{x}_i^*) + \alpha_i^*[b(\mathbf{x}_1^*,...\mathbf{x}_n^*) - \sum_j c_j(\mathbf{x}_j^*)]$ and α_i^* is such that $\sum_i \alpha_i^* = 1$, $\alpha_i^* \geq 0$ for any i, and, if $b(\mathbf{x})$ has strict increasing differences, it satisfies the conditions of Lemma 1. We now show that the mechanism satisfies budget balance and such that the resulting game has an efficient Strong Nash-equilibrium (SN). By definition, we have $\sum_{i\in N} w_i(\mathbf{y}) \equiv b(\mathbf{y})\ \forall y$, so budget balance is satisfied. Since $\mathbf{y}(\mathbf{x}) \geq 0$ (and therefore $b(\mathbf{x}) \geq 0$ as well), it follows that $w_i(\mathbf{y}) \geq 0\ \forall i$ and $\forall y$: and therefore limited liability is also satisfied. Consider now a unilateral deviation by agent j. Since $Y = \{\mathbf{y}^*\}$, we know that $N \setminus G(\mathbf{y}) \neq \emptyset$; moreover $j \notin N \setminus G(\mathbf{y})$ implies that $j \neq \iota(\mathbf{y})$, and $w_j(f(\mathbf{x}_j, \mathbf{x}^*_{-j})) = 0$. Since $\mathbf{x}^* \in \arg\max b(\mathbf{f}(\mathbf{x})) - \sum_{i\in N} c_i(\mathbf{x}_i)$ and $\mathbf{x}_i = \mathbf{0}$ is feasible for each $i \in N$, it must be that $b(\mathbf{f}(\mathbf{x}^*)) - \sum_{i\in N} c_i(\mathbf{x}^*_i) \geq 0$. For any $\widehat{\mathbf{x}}_i$, therefore:

$$\begin{aligned} w_j(\mathbf{f}(\mathbf{x}^*_j, \mathbf{x}^*_{-j})) - c_j(\mathbf{x}^*_j) &= \alpha^*_i[b(\mathbf{y}^*) - \sum_{i\in N} c_i(\mathbf{x}^*_i)] \\ &\geq 0 \geq w_j(\mathbf{f}(\widehat{\mathbf{x}}_j, \mathbf{x}^*_{-j})) - c_j(\widehat{\mathbf{x}}_j). \end{aligned}$$

Therefore agent j has no unilateral incentive to deviate. To see that in any coalition $J \subseteq N$ with at least 2 agents there is (at least) one agent that would veto a deviation note that for any $\mathbf{y} \neq \mathbf{y}^*$ any agent $i \in J \setminus \iota(\mathbf{y})$ receives zero and so would veto; therefore the efficient level of effort is a Strong Nash equilibrium. ■

We now show that when $\sum_{i=1}^n \frac{a_i}{n-1} < m$, then the set of production functions such that $Y = \{\mathbf{y}^*\}$ is generic in $\mathcal{F}$. Let $\mathcal{F}$ be the set of $C^1(X)$ production functions that maps $X = \prod_{i=1}^n X_i$ into $\mathbb{R}^m$. For any $\mathbf{f} \in C^1(X)$, define $Y_i^{\mathbf{f}} = \left\{\mathbf{y} \in \mathbb{R}^m \,\middle|\, \exists \mathbf{x}_i \in X_i \text{ s.t. } \mathbf{y} = \mathbf{f}(\mathbf{x}_i, \mathbf{x}^*_{-i}(\mathbf{f}))\right\}$, where $\mathbf{x}^*_{-i}(\mathbf{f})$ is the efficient level of effort for all agents except i when the production function if $\mathbf{f}$; and similarly we define $x^*_{il}(\mathbf{f})$ to be the lth component efficient level of effort vector $\mathbf{x}^*_i(\mathbf{f})$ of agent i when the production function if $\mathbf{f}$ (in both cases, the cost functions $\{c_i(\mathbf{x}_i)\}_{i=1}^n$ are constant, and we only consider changes in $\mathbf{f}$). We denote $Y^{\mathbf{f}} = \bigcap_{i=1}^n Y_i{}^{\mathbf{f}}$. Let $\mathcal{F}^* := \left\{\mathbf{f} \in \mathcal{F} \,\middle|\, Y^{\mathbf{f}} = \{\mathbf{f}(\mathbf{x}^*(\mathbf{f}))\}\right\}$ be the subset of production functions that satisfy the desired property. In the following, as in the main text above, when the function $\mathbf{f}$ is unambiguous we omit it from these expressions. For future reference, given a function $\mathbf{g}(\mathbf{x}, \mathbf{y})$ of variables $\mathbf{x}, \mathbf{y}$, the Jacobian of $\mathbf{g}(\mathbf{x}, \mathbf{y})$ is denoted $J\mathbf{g}(\mathbf{x}, \mathbf{y})$; the partial Jacobian with respect the subset of variables $\mathbf{x}$ (respectively $\mathbf{y}$) is denoted $J_{\mathbf{x}}\mathbf{g}(\mathbf{x}, \mathbf{y})$ (respectively $J_{\mathbf{y}}\mathbf{g}(\mathbf{x}, \mathbf{y})$).

Lemma 3 *If $\sum_{i=1}^n \frac{a_i}{n-1} < m$, then the set $\mathcal{F}^*$ generic in $\mathcal{F}$.*

Proof. For any $\mathbf{f} \in \mathcal{F}$ and $\widehat{\mathbf{x}} \in \mathbb{R}^m$, define the linear function $L_{\widehat{\mathbf{x}}}[\mathbf{f}(\mathbf{x})] = \mathbf{f}(\widehat{\mathbf{x}}) + J\mathbf{f}(\widehat{\mathbf{x}}) \cdot (\mathbf{x} - \widehat{\mathbf{x}})$. Let $\mathcal{D} = \left\{\mathbf{f} \in \mathcal{F}^* \,\middle|\, Y^{L_{\widehat{\mathbf{x}}(\mathbf{f})}[\mathbf{f}(\mathbf{x})]} = \{\mathbf{f}(\mathbf{x}^*(\mathbf{f}))\}\right\}$. This is the set of functions $\mathbf{f}$ such that the hyperplanes tangent to the deviations sets $Y_i^{\mathbf{f}}$ at the efficient output level have only one point of intersection (the efficient level). To prove that $\mathcal{F}^*$ is generic in $\mathcal{F}$ we only need to prove that $\mathcal{D}$ is open and dense in $\mathcal{F}$. We first prove that $\mathcal{D}$ is dense in $\mathcal{F}$. It suffices to prove that for any $\mathbf{f} \in \mathcal{F}$, we can always find a $\mathbf{f}' \in \mathcal{D}$ which is arbitrarily close to it. Consider an arbitrary production function $\mathbf{f} \in \mathcal{F}$. Let $\left(\left(\boldsymbol{\phi}_i^j\right)_{j=1}^m\right)_{i=1}^n$ a collection of vectors $\boldsymbol{\phi}_i^j$ in $\mathbb{R}^{a_i}$ indexed by i and j. For any agent i we can stack these vectors and generate a $m \times a_i$ vector $\boldsymbol{\phi}_i$ in which the first a_i components are equal to $\boldsymbol{\phi}_i^1$, the second a_i components are $\boldsymbol{\phi}_i^2$, etc.; we can then stack the vectors associated with the agents and define a $m \times \sum_{i=1}^n a_i$ vector $\Omega = \left(\begin{array}{ccc} \boldsymbol{\phi}_1, & ... & ,\boldsymbol{\phi}_n \end{array} \right)^T$. Define a function $\mathbf{f}^\epsilon : X \times \Omega \to \mathbb{R}^m$:

$$\mathbf{f}^\epsilon(\mathbf{x},\Omega) = \begin{cases} f_1(\mathbf{x}_1, ..., \mathbf{x}_n) \; - \frac{\epsilon}{2} \sum_{i=1}^n \sum_{l=1}^{a_i} \left(\phi_{il}^1\right)^2 (x_{il} - x_{il}^*)^2 \\ ... \\ f_m(\mathbf{x}_1, ..., \mathbf{x}_n) - \frac{\epsilon}{2} \sum_{i=1}^n \sum_{l=1}^{a_i} \left(\phi_{il}^m\right)^2 (x_{il} - x_{il}^*)^2 \end{cases}, \tag{A.4}$$

where x_{il}^*, x_{il} and ϕ_{il}^j are the lth component of, respectively, vectors $\mathbf{x}_i^*$, $\mathbf{x}_i$ and $\boldsymbol{\phi}_i^j$, and $\epsilon > 0$ is a scalar. The new function (A.4) is identical to $\mathbf{f}$ at the efficient level of effort $\mathbf{x}^*$; the new addends, however, perturb the set of outcomes that can be achieved in a unilateral deviation: the parameters in Ω control the "shape" of the deviation; and ϵ controls its "size." When we keep the parameters in Ω constant, $\mathbf{f}^\epsilon(\mathbf{x},\Omega)$ is denoted $\mathbf{f}_\Omega^\epsilon(\mathbf{x})$. The Jacobian of $\mathbf{f}^\epsilon(\mathbf{x},\Omega)$ with respect to $\boldsymbol{\phi}_i$, therefore, is the $m \times (m \cdot a_i)$ matrix:

$$J_{\boldsymbol{\phi}_i}\mathbf{f}^\epsilon(\mathbf{x},\Omega) = \epsilon \begin{bmatrix} \phi_{i1}^1 s_{i1} & .. & \phi_{ia_i}^1 s_{ia_i} & 0 & .. & 0 & .. & 0 & .. & 0 \\ 0 & .. & 0 & \phi_{i1}^2 s_{i1} & .. & \phi_{ia_i}^2 s_{ia_i} & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & 0 & 0 & .. & 0 & .. & \phi_{i1}^m s_{i1} & .. & \phi_{ia_i}^m s_{i2} \end{bmatrix},$$

where $s_{il} = (x_{il} - x_{il}^*)^2$. The vectors $\boldsymbol{\phi}_i^j$ are chosen to be generic (and so with no zeros) therefore, since $a_i \geq 1$, $J_{\boldsymbol{\phi}_i}\mathbf{f}^\epsilon(\mathbf{x},\Omega)$ has rank equal to m. For any agent i, define $\mathbf{f}_i^\epsilon(\mathbf{x}_i,\Omega) = \mathbf{f}^\epsilon(\mathbf{x}_i, \mathbf{x}_{-\mathbf{i}}^*;\Omega)$. Consider now a neighborhood $B_{\mathbf{x}^*}^r$ centered at $\mathbf{x}^*$ (with radius r) and define the set $C_r = X \setminus B_{\mathbf{x}^*}^r$. By the Inverse Function Theorem (Milnor [1997, §1]) we have that, for a generic Ω, $\mathbf{f}_i^\epsilon$ maps each neighborhood of $\mathbf{x}_i$ in C_r dieffomorphically onto a subset in $Y_i^{\mathbf{f}_\Omega^\epsilon}$, implying that $Y_i^{\mathbf{f}_\Omega^\epsilon}$ is a a_i-dimensional manifold.[29] Let us define $Y^{\mathbf{f}^\epsilon} = \bigcap_{l=1}^n Y_l^{\mathbf{f}^\epsilon}$. We now show that $Y^{\mathbf{f}^\epsilon}$ must be empty. Let us define the functions $\boldsymbol{\varphi}_1 = \mathbf{f}_1^\epsilon - \mathbf{f}_2^\epsilon$, $\boldsymbol{\varphi}_2 = \mathbf{f}_1^\epsilon - \mathbf{f}_3^\epsilon$, and so on up to $\boldsymbol{\varphi}_{n-1} = \mathbf{f}_1^\epsilon - \mathbf{f}_n^\epsilon$; and the system $\Psi(\mathbf{x}, \Omega) = \mathbf{0}$, where $\Psi : X \times \Omega \to \mathbb{R}^{m \times (n-1)}$ described by the $m \times (n-1)$ column vector $\left(\begin{array}{cccc} \boldsymbol{\varphi}_1(\mathbf{x},\Omega), & \boldsymbol{\varphi}_2(\mathbf{x},\Omega), & ... & \boldsymbol{\varphi}_{n-1}(\mathbf{x},\Omega) \end{array} \right)^T$. The set of $x \in C_r$ that maps into the intersection $Y^{\mathbf{f}^\epsilon}$ is characterized by the system $\Psi(\mathbf{x}, \Omega) = \mathbf{0}$.

Consider the Jacobian of $\Psi(\mathbf{x}, \Omega)$ with respect to Ω:

$$J_\Omega \Psi(\mathbf{x}, \Omega) = \begin{bmatrix} J_{\boldsymbol{\phi}_1}\mathbf{f}_1^\epsilon & -J_{\boldsymbol{\phi}_2}\mathbf{f}_2^\epsilon & 0 & ... & 0 \\ J_{\boldsymbol{\phi}_1}\mathbf{f}_1^\epsilon & 0 & -J_{\boldsymbol{\phi}_3}\mathbf{f}_3^\epsilon & ... & 0 \\ ... & ... & ... & ... & ... \\ J_{\boldsymbol{\phi}_1}\mathbf{f}_1^\epsilon & 0 & 0 & ... & -J_{\boldsymbol{\phi}_n}\mathbf{f}_n^\epsilon \end{bmatrix}.$$

Since the $J_{\boldsymbol{\phi}_j}\mathbf{f}_j^\epsilon$ $\forall j = 1..n-1$ are generic, $J_\Omega \Psi(\mathbf{x}, \Omega)$ has rank equal to the number of equations, $(n-1)m$. The Transversality Theorem (cf. Mas-Colell [1985, §8]) implies that for any $\mathbf{x} \in Y^{\mathbf{f}_\Omega^\epsilon}$ and a given generic Ω, $J_{\mathbf{x}}\Psi(\mathbf{x}, \Omega)$ has rank $(n-1)m$ as well: this, however, is impossible since $J_{\mathbf{x}}\Psi(\mathbf{x}, \Omega)$ has only $\sum_{l=1}^n a_l$ columns. From this contradiction, we conclude that the intersection

[29] A map $f : X \to Y$ is called a diffeomorphism if f carries X homeomorphically onto Y and if both f and f^{-1} are smooth. A map $f : X \to Y$ is a homeomorphism if is a one to one mapping from X onto Y, and f and f^{-1} are continuous.

$Y^{\mathbf{f}^{\epsilon}_{\Omega}}$ is empty in $X \setminus B^{r}_{\mathbf{x}^*}$. Since the radius of $B^{r}_{\mathbf{x}^*}$ can be chosen to be arbitrarily small, it follows that for a generic Ω, $Y^{\mathbf{f}^{\epsilon}_{\Omega}}$ contains only the efficient level $\mathbf{y}^*$. The same (identical) argument can be applied to the linear function $L_{\mathbf{x}^*}[\mathbf{f}^{\epsilon}_{\Omega}(\mathbf{x})]$ to prove that $Y_n^{L_{x^*(\mathbf{f})}[\mathbf{f}^{\epsilon}_{\Omega}]} = \{\mathbf{y}^*\}$. We conclude, therefore, that $\mathbf{f}^{\epsilon}_{\Omega}(\mathbf{x}) \in \mathcal{D}$ for any ϵ and generic Ω. Since X is compact, we can verify that for any $\mathbf{x} \in X$, $\|\mathbf{f}^{\epsilon}_{\Omega}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| \to 0$ as $\epsilon \to 0$: therefore for any $\mathbf{f} \in \mathcal{F}$ and $\delta > 0$, there is a is a $\mathbf{f}^{\epsilon}_{\Omega} \in \mathcal{D}$ such that $\|\mathbf{f} - \mathbf{f}^{\epsilon}_{\Omega}\| < \delta$.

We now prove that $\mathcal{D}$ is open. Assume that $\mathbf{f} \in \mathcal{D}$, we want to show that any function in a (small enough) neighborhood of $\mathbf{f}$ is also in $\mathcal{D}$. Consider an open Neighborhood $B_{\mathbf{x}^*}$ centered at $\mathbf{x}^*$. Its complement $B^{C}_{\mathbf{x}^*} = X \setminus B_{\mathbf{x}^*}$ is compact. Consider an agent i. For any point $\mathbf{x}$ in $B^{C}_{\mathbf{x}^*} \cap Y^{\mathbf{f}}_i$, we can find an $\varepsilon_{\mathbf{x}} > 0$ and a hyperplane $Y^{\mathbf{f}}_j$ (correspondent to the set of outcomes that can be induced by some agent $j \neq i$ with a unilateral deviation) such that the distance between this point and the hyperplane is at least $\varepsilon_{\mathbf{x}}$ (if this were not true, then $\mathbf{f} \notin \mathcal{D}$, a contradiction). Define ε as the minimal $\varepsilon_{\mathbf{x}}$ in $B^{C}_{\mathbf{x}^*}$: since this set is compact, $\varepsilon > 0$. For any sequence of functions $\mathbf{f}^n$ which converges to $\mathbf{f}$ in the Whitney C^1 topology, both $\mathbf{f}^n$ and $J_x\mathbf{f}^n$ converge uniformly to, respectively, $\mathbf{f}$ and $J_x\mathbf{f}$ in $\langle C^0(X), \|\cdot\|\rangle$ (the space of continuous function defined over X with the euclidean norm $\|\cdot\|$). For any ε, therefore, there must be a n_1 such that $\|\mathbf{f} - \mathbf{f}^n\| < \frac{\varepsilon}{2}$ for any $n > n_1$, implying $Y^{\mathbf{f}^n}$ to be empty in $B^{C}_{\mathbf{x}^*}$ for any $n > n_1$. We can make the radius of the neighborhood $B_{\mathbf{x}^*}$ as small as we want, so, without loss of generality, $\mathbf{f}$ can be assumed to be linear in the set $B_{\mathbf{x}^*}$ (and in its closure $\overline{B}_{\mathbf{x}^*}$), with coefficients given by $J_{\mathbf{x}}\mathbf{f}$. Since $\mathbf{f} \in \mathcal{D}$, there is no point in $\overline{B}_{\mathbf{x}^*}$ which belongs to all (linear) hyperplanes $Y_i^{L_{\mathbf{x}^*(\mathbf{f})}[\mathbf{f}]}$. As $\mathbf{f}^n \to \mathbf{f}$, the Jacobian of $\mathbf{f}^n$ also converges uniformly to the Jacobian of $\mathbf{f}$ in $\langle C^0(X), \|\cdot\|\rangle$, implying that there is a $n_2 > 0$ such that for $n > n_2$ the same property must be true for $\mathbf{f}^n$ as well. Since $B^{C}_{\mathbf{x}^*} \cup \overline{B}_{\mathbf{x}^*} = X$, for any sequence of functions $\mathbf{f}^n$ converging to $\mathbf{f}$ in the Whitney C^1 topology, there must be a $\overline{n} > \max\{n_1, n_2\}$ such that $\mathbf{f}^n \in \mathcal{D}$ for $n > \overline{n}$. ■

B. Proof of Proposition 1

Consider the same mechanism constructed in Lemma 2. Assume that there is a pure strategies Nash equilibrium $\{\widetilde{\mathbf{x}}_i\}_{i=1}^n$ such that $y_i(\{\widetilde{\mathbf{x}}_i\}_{i=1}^n) \neq y_i^*$. Since all surplus is going to agent $\iota(y)$ it must be that $\widetilde{\mathbf{x}}_i = 0$ for $i \neq \iota(y)$. The pay off of $\iota(y)$ is not larger then $b(\mathbf{0}, \mathbf{0}, ..\mathbf{x}_{\iota(\mathbf{y})} = \widetilde{\mathbf{x}}_{\iota(\mathbf{y})}, \mathbf{0}..) - c_{\iota(\mathbf{y})}(\widetilde{\mathbf{x}}_{\iota(\mathbf{y})})$. This is not larger than $\max_{\mathbf{x}}[b(\mathbf{0}, \mathbf{0}, ..\mathbf{x}_{\iota} = \mathbf{x}, \mathbf{0}..) - c_{\iota(\mathbf{y})}(\mathbf{x})]$ and therefore, given A.1, is strictly smaller than $\alpha^*_{\iota(\mathbf{y})}[b(\mathbf{x}^*_1, ...\mathbf{x}^*_n) - \sum_j c_j(\mathbf{x}^*_i)]$. So the profile $\{\widetilde{\mathbf{x}}_i\}_{i=1}^n$ can not be a CP-equilibrium since the outcome is dominated by the outcome of the equilibrium profile $\{\mathbf{x}^*_i\}_{i=1}^n$. The result, then, follows from Lemma 3 and the fact that, since $b(\cdot)$ is continuous, $\mathcal{D} \cap \mathcal{F}^{id}$ is generic in $\mathcal{F}^{id}$. ■

C. Proof of Theorem 2

Consider a neighborhood $B_{\mathbf{x}^*}$ centered at $\mathbf{x}^*$, an its image $B_{\mathbf{y}^*}$ projected by $\mathbf{f}$. From the Inverse Function Theorem (cf. Milnor [1997, §1], and Lemma 3 above), the sets Y_is are transversal manifolds with dimensionality a_i on $B_{\mathbf{y}^*}$. From Balasko [1988, § Math. 1.9], $Y = \cap_{j=1}^n Y_j$ is generically a manifold with dimensionality $q = \sum_{i=1}^{n} a_i - (n-1)\,m$. Since we assume $\sum_{i=1}^{n} a_i > (n-1)\,m$, Y is generically (at least) a one-dimensional manifold on $B_{\mathbf{y}^*}$ passing through $\mathbf{y}^*$. For any i there exists a collection of functions $\{\mathbf{g}_j : X_i \to X_j\}_{j \in N \setminus i}$ such that $\mathbf{f}(\mathbf{x}^*_{-ij}, \mathbf{g}_j(\mathbf{x}_i), \mathbf{x}^*_i) - \mathbf{f}(\mathbf{x}^*_{-ij}, \mathbf{x}^*_j, \mathbf{x}_i) = 0$ for any agent $j \in N \setminus i$ at any $\mathbf{x}_i$ in $B_{\mathbf{x}^*} \cap \mathbf{f}_i^{-1}(Y)$, where $\mathbf{f}_i^{-1}(Y)$ is the set of unilateral deviations of agent i that induces an outcome in Y. Indeed, let us define $\mathbf{f}_j(\mathbf{x}_j) = \mathbf{f}_j(\mathbf{x}_j, \mathbf{x}^*_{-j})$ the function that maps $B_{\mathbf{x}^*} \cap \mathbf{f}_j^{-1}(Y)$ diffeomorphically onto Y:[30] we have that $\mathbf{g}_j(\mathbf{x}_i) = \mathbf{f}_j^{-1} \circ\ \mathbf{f}_i(\mathbf{x}_i)$, where $\mathbf{f}_j^{-1}$ is the inverse of $\mathbf{f}_j$, which is obviously well defined and differentiable since $\mathbf{f}_j$ is a diffeomorphism (this also follows from the Inverse Function Theorem). By convention, we can also define $\mathbf{g}_i(\mathbf{x}_i) = \mathbf{x}_i$. Therefore any agent j can imitate the output induced by a deviation of i, at least if the deviation is small. (See, for example, point $\widehat{\mathbf{y}}$ in Figure 3.A that represents an example in which $m = 3$, $a_j = a_k = 2$ and $a_i = 3$). Moreover, by a standard application of the chain rule (cf. Milnor (1997, §1), the Jacobian of $\mathbf{g}_j$ is given by $J_{\mathbf{x}_i}\mathbf{g}_j(\mathbf{x}_i) = J_{\mathbf{x}_j}\mathbf{f}_j^{-1}(\mathbf{x}_j) \cdot J_{\mathbf{x}_i}\mathbf{f}_i(\mathbf{x}_i)$, where $J_{\mathbf{x}_i}\mathbf{f}_i(\mathbf{x}_i)$ (respectively $J_{\mathbf{x}_j}\mathbf{f}_j^{-1}(\mathbf{x}_j)$) is the Jacobian of $\mathbf{f}_i$ with respect to $\mathbf{x}_i$ (respectively, $\mathbf{f}_j^{-1}$ with respect to $\mathbf{x}_j$) calculated at $\mathbf{x}_i$ (resp. $\mathbf{x}_j$). Because Y is at least one-dimensional, $\mathbf{f}_i^{-1}(Y)$ is also at least one-dimensional, and we can find a sequence $\mathbf{x}_i^k$ such that $\mathbf{f}(\mathbf{x}_i^k, \mathbf{x}^*_{-i}) \in Y$ and $\mathbf{x}_i^k \to \mathbf{x}_i^*$. We can define the a_i-dimensional column vector $\mathbf{d}_i^k = \left(\left| x_1^* - x_{i,1}^k \right|, \ \ldots \ , \left| x_{a_i}^* - x_{i,a_i}^k \right| \right)^T$, with norm $\left\| \mathbf{d}_i^k \right\|$. By the incentive compatibility condition of agent j, we have that for any $\mathbf{x}_i$ in $B_{\mathbf{x}^*}$:

$$\begin{aligned} w_j\left(\mathbf{f}\left(\mathbf{x}^*\right)\right) - c_j\left(\mathbf{x}_j^*\right) &\geq w_j\left(\mathbf{f}\left(\mathbf{x}^*_{-j}, \mathbf{g}_j(\mathbf{x}_i^k)\right)\right) - c_j\left(\mathbf{g}_j(\mathbf{x}_i^k)\right) \\ &= w_j\left(\mathbf{f}\left(\mathbf{x}^*_{-i}, \mathbf{x}_i^k\right)\right) - c_j\left(\mathbf{g}_j(\mathbf{x}_i^k)\right), \end{aligned}$$

where the equality follows by the definition of $\mathbf{g}_j$. Summing up over $j \in N \setminus i$ and using the budget balance condition we have a lower bound on the profitability of a deviation by i:

$$\begin{aligned} &\left[w_i\left(\mathbf{f}\left(\mathbf{x}^*_{-i}, \mathbf{x}_i^k\right)\right) - c_i\left(\mathbf{x}_i^k\right)\right] - \left[w_i\left(\mathbf{f}\left(\mathbf{x}^*\right)\right) - c_i\left(\mathbf{x}_i^*\right)\right] \\ \geq\ &\sum_{j=1}^{n}\left[c_j\left(\mathbf{x}_j^*\right) - c_j\left(\mathbf{g}_j(\mathbf{x}_i^k)\right)\right] - \left[b\left(\mathbf{f}\left(\mathbf{x}^*\right)\right) - b\left(\mathbf{f}\left(\mathbf{x}^*_{-i}, \mathbf{x}_i^k\right)\right)\right]. \qquad \text{(C.3)} \end{aligned}$$

The right hand side can be written as $\sum_{j=1}^{n} J_{\mathbf{x}_j} c_j(\mathbf{x}^*) \cdot J_{\mathbf{x}_i}\mathbf{g}_j(\mathbf{x}_i) \cdot \mathbf{d}_i^k - J_{\mathbf{y}} \cdot b(\mathbf{y}^*) J_{\mathbf{x}_i}\mathbf{f}(\mathbf{x}^*) \cdot \mathbf{d}_i^k + o\left\|\mathbf{d}_i^k\right\|$, where $\frac{o\left\|\mathbf{d}_i^k\right\|}{\left\|\mathbf{d}_i^k\right\|} \to 0$ as $k \to \infty$. Since $\{\mathbf{x}_i^*\}_{i=1}^{n}$ is efficient, however, the first order necessary condition implies $J_{\mathbf{x}_j} c_j(\mathbf{x}^*) = J_{\mathbf{y}} b(\mathbf{y}^*) \cdot J_{\mathbf{x}_j}\mathbf{f}(\mathbf{x}^*)$. Therefore, using the formula for $J_{\mathbf{x}_i}\mathbf{g}_j(\mathbf{x}_i)$ derived above, we have:

$$\begin{aligned} &\sum_{j=1}^{n} J_{\mathbf{y}} b(\mathbf{y}^*) \cdot J_{\mathbf{x}_j}\mathbf{f}(\mathbf{x}^*) \cdot J_{\mathbf{x}_j}\mathbf{f}^{-1}(\mathbf{x}^*) \cdot J_{\mathbf{x}_i}\mathbf{f}(\mathbf{x}^*) \cdot \mathbf{d}_i^k \qquad \text{(C.4)} \\ &- J_{\mathbf{y}} b(\mathbf{y}^*) \cdot J_{\mathbf{x}_i}\mathbf{f}(\mathbf{x}^*) \cdot \mathbf{d}_i^k \\ =\ &(n-1)\, J_{\mathbf{x}_i} c_i(\mathbf{x}^*) \cdot \mathbf{d}_i^k > 0. \qquad \text{(C.5)} \end{aligned}$$

[30] See footnote 32 for the definition of a diffeomorphism.

From (C.3) and (C.4) it follows that there is a $\overline{k}$ such that for $k > \overline{k}$, $w_i\left(\mathbf{f}\left(\mathbf{x}^*_{-i}, \mathbf{x}^k_i\right)\right) - c_i\left(\mathbf{x}^k_i\right) > w_i\left(\mathbf{f}\left(\{\mathbf{x}^*_i\}_{i=1}^n\right)\right) - c_i\left(\mathbf{x}^*_i\right)$. We conclude that, for any wage schedule, the efficient level of effort would not be optimal for agent i. ■

D. Proof of Theorem 3

This result follows, with minor variations, from the argument in Theorem 1 of Legros and Matthews [1993]. ■

E. Proof of Proposition 2

As discussed in Section V, for each agent i we may associate the agent $D(i)$ and construct a set $\{i, D(i)\}$. We may therefore construct a collection of sets $S = (\{i, D(i)\}, \{j, D(j)\}, \{k, D(k)\}...)$ such that $\cup_{s \in S} s = N$. For each i define n_i as the number of sets $s \in S$ such that $i \in s$. Assume that the partnership has return r, as defined in Section V. Let us define $\widehat{\alpha}_i = \frac{(1+r)c_i(x_i^*)}{Eb(y|x_i = x_i^* \ \forall i)}$ for any $i \in N$; it can be verified that $\widehat{\alpha}_i \geq 0$ and $\sum_{i \in N} \widehat{\alpha}_i = 1$. To each group $s = \{i, j\} \in S$ we associate a share of revenues $\omega_{i,j} = [\frac{\widehat{\alpha}_i}{n_i} + \frac{\widehat{\alpha}_j}{n_j}]b(y_1, y_2)$. The share associated to each subset (i.e., $\frac{\widehat{\alpha}_i}{n_i} + \frac{\widehat{\alpha}_j}{n_j}$) is fixed and not contingent on y; moreover, $\sum_{\{i,j\} \in S} \omega_{i,j} = b(y_1, y_2)$. Consider now the following mechanism. After effort is chosen and the $\widetilde{x}_i$s are realized, each agent reports his actual contribution $\widetilde{x}_i$. This declaration is denoted d_i because agents may lie. Given Assumption 5.2, the system of equations (4) has at most a unique solution which clearly depends on the declaration of the agents other than i and j:[31] call it $\widehat{x}_i(d_{-\{i,j\}}), \widehat{x}_j(d_{-\{i,j\}})$. So we may identify which of the two agents has deviated and punish him accordingly. For each $s = \{i, j\} \in S$, therefore, the wage schedule is defined as:

$$w^i_{i,j}(\mathbf{y}, d_{-\{i,j\}}) = \begin{cases} 0 & \widehat{x}_i(d_{-\{i,j\}}) \neq x_i^* \text{ and } \widehat{x}_j(d_{-\{i,j\}}) = x_j^* \\ [\frac{\widehat{\alpha}_i}{n_i} + \frac{\widehat{\alpha}_j}{n_j}]b(y_1, y_2) & \widehat{x}_i(d_{-\{i,j\}}) = x_i^* \text{ and } \widehat{x}_j(d_{-\{i,j\}}) \neq x_j^* \\ \frac{\widehat{\alpha}_i}{n_i}b(y_1, y_2) & else \end{cases}.$$

Since $w^i_{i,j}(\mathbf{y}, d_{-\{i,j\}}) \geq 0$ for any agent i and any possible $\mathbf{y}$, limited liability is satisfied. For any declarations and output and for any realization $\{\widetilde{x}_i\}_{i=1}^n$, moreover, $\sum_{i \in N} \sum_{\{s_i | i \in s_i, \ s_i \in S\}} w^i_{s_i}$ is equal to $b(y_1, y_2)$, so the mechanism satisfies budget balance too. We now verify that the game induced by this mechanism has an efficient Nash equilibrium. A deviation in the first step does not yield a strict improvement for any agent; in fact, the wage received by each agent $i \in N$ is equal to $\sum_{\{s_i | i \in s_i, \ s_i \in S\}} w^i_{s_i}$: since d_i does not enter in (4), $w^i_{s_i}$s are independent on d_i and so it is the sum of them. Consider now a unilateral deviation in effort by agent i. Since the perturbations are independently distributed, conditional on the realization of agent's i effort, for any s_i such that $i \in s_i$ and $s_i \in S$ the wages $w^i_{s_i}$ depend only on the realization of effort of

[31] Out of equilibrium, the system may have no solution since agents may lie in Stage 1. In equilibrium, however, there is always a solution.

the other agent j in s_i. If agent i exert the efficient level of effort the expected wage is

$$\begin{aligned} Ew^i_{s_i}(x^*) &= (1-\varepsilon)^2 \frac{\widehat{\alpha}_i}{n_i} E[b(y_1,y_2)\,|\widetilde{x}_i = x_i^* \ \forall i] \\ &\quad + (1-\varepsilon)\,\varepsilon \left(\frac{\widehat{\alpha}_i}{n_i} + \frac{\widehat{\alpha}_j}{n_j}\right) E[b(y_1,y_2)\,\left|\widetilde{x}_i = x_i^*,\ \widetilde{x}_j < x_j^*\right.] \\ &\quad + \varepsilon^2 \frac{\widehat{\alpha}_i}{n_i} E[b(y_1,y_2)\,\left|\widetilde{x}_i < x_i^*,\ \widetilde{x}_j < x_j^*\right.]. \end{aligned}$$

If i deviates and chooses $\widehat{x}_i \neq x_i^*$, he receives no more than:

$$\begin{aligned} Ew^i_{s_i}(\widehat{x}_i, x^*_{-i}) &= (1-\varepsilon)\,\varepsilon \frac{\widehat{\alpha}_i}{n_i} E[b(y_1,y_2)\,\left|\widetilde{x}_i = \widehat{x}_i,\ \widetilde{x}_j < x_j^*\right.] \\ &\quad + \varepsilon^2 \frac{\widehat{\alpha}_i}{n_i} E[b(y_1,y_2)\,\left|\widetilde{x}_i < \widehat{x}_i,\ \widetilde{x}_j < x_j^*\right.] \\ &< (1-\varepsilon)\,\varepsilon \left(\frac{\widehat{\alpha}_i}{n_i} + \frac{\widehat{\alpha}_j}{n_j}\right) E[b(y_1,y_2)\,\left|\widetilde{x}_i = x_i^*,\ \widetilde{x}_j < x_j^*\right.] \\ &\quad + \varepsilon^2 \frac{\widehat{\alpha}_i}{n_i} E[b(y_1,y_2)\,\left|\widetilde{x}_i = x_i^*,\ \widetilde{x}_j < x_j^*\right.]. \end{aligned}$$

Summing over all the sets s_i to which i belongs, the net benefit of a deviation, therefore, is not larger than:

$$\begin{aligned} Ew(\widehat{x}_i, x^*_{-i}) - c_i(\widehat{x}_i) - Ew(x^*) + c_i(x_i^*) &< \varepsilon^2 \widehat{\alpha}_i \left\{ \begin{array}{c} E[b(y_1,y_2)\left|\widetilde{x}_i = x_i^*,\ \widetilde{x}_j < x_j^*\right.] \\ -E[b(y_1,y_2)\left|\widetilde{x}_i < x_i^*,\ \widetilde{x}_j < x_j^*\right.] \end{array} \right\} \\ &\quad - (1-\varepsilon)^2 \widehat{\alpha}_i E[b(y_1,y_2)\,|\widetilde{x}_i = x_i^* \ \forall i] + c_i(x_i^*) \\ &< (2\varepsilon - 1)\,\widehat{\alpha}_i E[b(y_1,y_2)\,|\widetilde{x}_i = x_i^* \ \forall i] + c_i(x_i^*) \\ &= [(2\varepsilon - 1)(1+r) + 1]\, c_i(x_i^*). \end{aligned}$$

A deviation, therefore, is never profitable if $(1-2\varepsilon)(1+r) \geq 1$, i.e. if $r \geq r^*(\varepsilon) = \frac{2\varepsilon}{1-2\varepsilon}$. Clearly $r^*(\varepsilon)$ is independent of N and $r^*(\varepsilon) \to 0$ as $\varepsilon \to 0$. ■

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