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**HOLD-UP PROBLEMS AND FIRM FORMATION**

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# ABSTRACT

## Hold-Up Problems and Firm Formation*

Agents from a homogeneous population organize themselves into productive partnerships and are confronted with a hold-up problem when making relation-specific investments in those partnerships. The problem is mitigated if agents can leave a partnership in which they have invested, bear the costs yet forego the benefits of the investment, join another partnership, invest there anew, and appropriate the surplus created by the new investment. To capture the idea we introduce the notion of reinvestment-proof equilibria in which no agent has an incentive to reinvest or to change his investment in the current firm. We show that the presence of a small inefficient firm causes substantial efficiency gains in all larger firms.

JEL Classification: D20, D60 and L20
Keywords: efficient firm structure, firm formation, hold-up problem and reinvestment-proof equilibria

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# 1 Introduction

Since the seminal contributions by Williamson (1985), Grout (1984), Grossman and Hart (1986) and Hart and Moore (1988), many authors have examined the hold-up problem. The problem arises when a group of agents share a surplus and when an agent making a relation-specific investment cannot reap all the benefits from that investment. Then under-investment can occur. For instance, knowledge of firm-specific software may not be worth much in the external labor market. While a person with that knowledge may be very valuable to the current employer, that person's bargaining power vis-à-vis the employer can be quite limited if the skill is not widely sought after in the market place. Thus a forward-looking individual may not be interested in acquiring this sort of firm-specific skill and rather learn more portable skills. In a nutshell, this incentive problem constitutes the "hold-up problem". Hart (1995) and Holmström (1999) provide an extensive discussion of the problem.

The traditional literature on the hold-up problem starts from the bilateral relationship of two parties that make relation-specific investments in isolation and in the absence of complete contingent contracts.[1] An important strand of this literature has identified institutional[2] or contractual[3] devices to reduce or, possibly, eliminate any under-investment associated with the hold-up problem.[4] A novel strand of the literature, e.g. Felli and Roberts (2000), assumes two sides of the market, say a population of potential employers and a population of potential employees. A member from one side can form

[1]See Williamson (1985), Grout (1984), Grossman and Hart (1986), Hart and Moore (1988).

[2]Grossman and Hart (1986], Hart and Moore (1990), Aghion and Tirole (1997).

[3]Aghion, Dewatripont, and Rey (1994), Nöldeke and Schmidt (1995), Maskin and Tirole (1999), Tirole (1999), Segal and Whinston (2000, 2002).

[4]On the foundations of incomplete contracts see Che and Hausch (1999), Hart and Moore (1999), Segal (1999), Segal and Whinston (2002) and Watson (2003).

an exclusive bilateral relationship with a member from the other side of the market. Then the question is to what extent and under what circumstances competition for partners (and against agents of one's own type) mitigates the hold-up problem without recourse to contractual commitments.[5]

The purpose of this paper is to examine the efficiency of investment in an equilibrium model that incorporates the firm formation process.

Our model is complementary to the recent literature in that we assume a homogeneous population of agents, i.e. only one type of agents, who can organize themselves into productive partnerships, called firms in the sequel, and make relation-specific investments. We assume that there is an optimal firm size. We derive necessary conditions for an efficient industry structure, that is an efficient partition of the agent population into firms, and show by example that it is not always efficient to create as many firms of optimal size as possible.

There exists a free-rider problem and a corresponding hold-up problem once a firm is formed. One would expect that the problem is mitigated if firm-specific investments do not severely impede a partner's mobility and marketability, if partners are resourceful enough to make valuable firm-specific investments at the new firm in the case that they move. To explore this idea, we adopt the essential innovation that individuals who have already invested in one firm may leave, join another firm and invest there again. While such an individual still loses the benefits in the old firm and must invest anew, the individual may find an inefficient firm which can be greatly improved and generate a high surplus for the joining individual. In equilibrium no individual wants to switch firms. In order to deter other firm members from leaving,

[5]Another novel strand of literature identifies organizational devices to alleviate the hold-up problem, see Rajan and Zingales (1998) and Halonen-Akatwijuka (2004).

partners must sink a sufficient amount of relation-specific investments. As a result, investment will be above the free-riding level.

To capture the idea formally, we develop the notion of a reinvestment-proof equilibrium where no agent has an incentive to reinvest or to change his investment in the current firm. We explore the existence of such equilibria and focus on typical equilibrium phenomena: The industry structure consists of a number of larger firms (of optimal size) and one smaller firm. The investment levels in the smaller firm are well below the efficient level. However, the mere existence of a small inefficient firm causes substantial efficiency gains in the large firms relative to free riding. Individuals in large firms must choose high investment levels, since otherwise, other firm members would exit and reinvest in the small firm. It is precisely the inefficiency of the small firm that allows an individual to generate a large surplus by reinvesting there. As a result, investment in all other firms must be sufficiently high to deter firm members from reinvesting somewhere else. Although as a rule, the firm formation process will not yield Pareto-efficient allocations, the hold-up problem is significantly alleviated.

Our paper suggests that a general equilibrium perspective of the firm formation process provides insights complementary to those from the existing rich microeconomic literature on the hold-up problem. The paper is organized as follows. In the next section we introduce the basic structure of the model. In the third section we characterize the efficient industry structures and investment levels. In section four we study endogenous firm formation. We characterize stable allocations and define the concept of a reinvestment-proof equilibrium. We provide equilibria with smaller firms and discuss robustness and comparative statics. Section five concludes.

## 2 Agent Characteristics and Allocations

In this section, we describe the basic structure of the model: individuals, industry structures, commodities, endowments, production functions, investments, allocations, and preferences.

**Individuals, Firms, and Industry Structures.** We consider a finite population of individuals, represented by a set $I = \{1, \ldots, N\}$. A generic individual is denoted $i$ or $j$. Firms in our context are commercial or productive partnerships. Thus a firm is a collection of individuals deciding to work together, to engage in joint production. A generic firm is denoted $f$ or $g$ and is a non-empty subset of $I$. The population $I$ is partitioned into firms. Thus, there exists a partition $P$ of $I$ into non-empty subsets, interpreted as firms. We call any such partition $P$ an **industry structure** in $I$. At the prevailing industry structure $P$, each individual $i$ belongs to exactly one firm $f \in P$. If $P$ consists of $F$ firms, we frequently label them $f = 1, \ldots, F$, provided this causes no confusion. Let $\mathcal{P}$ denote the set of all partitions of $I$ into non-empty subsets, that is the set of all industry structures in $I$. We shall treat the industry structure as an object of endogenous choice.

**Endowments of Individuals.** Each individual $i$ is endowed with an amount $R_i \geq 0$ of a **non-tradeable asset** which can be used for relation-specific investments. In the simplest and most standard case $R_i$ is the human capital individual $i$ has acquired in the past which can be used to enhance the production possibilities of a firm. We assume $R_i = R > 0$ for all individuals $i$.

**Relation-Specific Investments.** An individual $i$ invests only in the particular firm $f$ of which it is a member. The amount of relation-specific investment chosen by individual $i$ is denoted by $r_i$. Set $\mathcal{R} = [0, R_1] \times \ldots \times [0, R_N]$.

Then the relation-specific investments undertaken by the entire population are summarized by a vector $\mathbf{r} = (r_i)_{i\in I} \in \mathcal{R}$. We call $\mathbf{r}$ an **investment profile**. Given an investment profile $\mathbf{r} = (r_i)_{i\in I}$, firm $f$ ends up with an array of relation-specific investments undertaken by firm members, $\mathbf{r_f} = (r_i)_{i\in f}$. We treat the constraints $r_i \leq R_i$ as non-binding in the sequel. Still, the notation $R_i, i \in I$, and $\mathcal{R}$ proves useful.

**Production.** A firm $f$ has a production function of the form

$$y_f = g_f(\mathbf{r_f}) = k_{|f|} \cdot \sum_{i\in f} r_i$$

where $y_f$ is the amount of output produced by the firm and $\mathbf{r_f}$ is the vector of relation-specific investments by the members of the firm. Thus the output of each firm is solely determined by the amount of relation-specific investments undertaken by firm members. The coefficient $k_{|f|}$ depends only on firm size (the number of firm members), $|f|$, and measures the firm's marginal productivity of relation-specific investments. We assume

$$k_1 < k_2 < \cdots < k_{n-1} < k_n > k_{n+1} > k_{n+2} > \ldots$$

Hence $n \in \{2, 3, \ldots\}$ is the most efficient or optimal firm size.

**Consumption.** Each consumer $i \in I$ consumes an amount $x_i \geq 0$ of **tradeable commodity** in $X_i = \mathbb{R}_+$. Moreover, the consumer makes a specific investment $r_i$ in firm $f$, if $i$ belongs to firm $f$. A **consumption profile** is an element of $\mathcal{X} \equiv \prod_{j\in I} X_j$. Generic elements of $\mathcal{X}$ are denoted $\mathbf{x} = (x_i)_{i\in I}$. The set of consumption profiles for a potential firm $f$ (joint consumption plans for members of firm $f$) is $\mathcal{X}_f = \prod_{i\in f} X_i$. $\mathcal{X}_f$ has generic elements $\mathbf{x_f} = (x_i)_{i\in f}$. If $\mathbf{x} = (x_i)_{i\in I} \in \mathcal{X}$ is a consumption profile, then the consumption profile for firm $f$ is the restriction of $\mathbf{x} = (x_i)_{i \in I}$ to $f$, $\mathbf{x_f} = (x_i)_{i \in f}$.

**Allocations.** An **allocation** is a tuple $(\mathbf{x}; \mathbf{r};\ P) \in \mathcal{X} \times \mathcal{R} \times \mathcal{P}$ specifying a consumption profile, an investment profile, and an industry structure. We call an allocation $(\mathbf{x}; \mathbf{r}; P) \in \mathcal{X} \times \mathcal{R} \times \mathcal{P}$ **feasible** if

$$\sum_{i \in I} x_i = \sum_{f \in P} g_f(\mathbf{r_f}). \tag{1}$$

Upon the specification of individual preferences, by means of utility representations, an allocation determines the welfare of each and every member of society.

**Consumer Preferences.** An individual $i \in I$ cares about pairs $(x_i, r_i) \in X_i \times [0, R_i]$ of individual consumptions bundles and firm-specific investments. For individual $i$, we assume that:

- $i$ has preferences on $X_i$ represented by the utility function $V_i : X_i \longrightarrow \mathbb{R}$, $V_i(x_i) = x_i$.
- $i$ has the (dis)utility function $W_i : [0, R_i] \rightarrow \mathbb{R}$, $W_i(r_i) = \frac{1}{2}\, r_i^2$.
- $i$'s total utility is given by $U_i(x_i, r_i) = V_i(x_i) - W_i(r_i) = x_i - \frac{1}{2}\, r_i^2$.

# 3 Efficient Industry Structure and Investments

In our specific model, there are two potential sources for inefficiencies. First, the industry structure may be inefficient. It may be possible to regroup the population and reduce the number of firms of non-optimal size. Second, given a particular industry structure, the investments within the existing firms may not be efficient. Next we introduce notions of efficiency at various levels, starting at the firm level. Notice that the concept of aggregate welfare or social surplus is appealing in our context, since individual utility functions are quasi-linear and, hence, utility is transferable.

## 3.1 Definitions

Given a firm $f$ of size $|f|$, an array of $f$-specific investments, $\mathbf{r_f} = (r_i)_{i\in f}$, is $f$-**efficient** if it maximizes the aggregate welfare of firm members, which amounts to

$$g_f(\mathbf{r_f}) - \frac{1}{2}\sum_{i\in f} r_i^2 = \sum_{i\in f}(k_{|f|}r_i - \frac{1}{2}r_i^2),$$

if the firm's output is distributed to the partners. Obviously, $\mathbf{r_f}$ is $f$-efficient if and only if $r_i = k_{|f|}$ for all $i \in f$.

Given an industry structure $P$, an array of investments $\mathbf{r} = (r_i)_{i\in I} \in \mathcal{R}$ is $P$-**efficient** if $\mathbf{r_f} = (r_i)_{i\in f}$ is $f$-efficient for all $f \in P$. This is equivalent to $r_i = k_{|f|}$ for all $f \in P, i \in f$. A $P$-efficient $\mathbf{r} \in \mathcal{R}$ is constrained optimal. It maximizes social surplus given $P$.

A combination $(\mathbf{r}; P) \in \mathcal{R} \times \mathcal{P}$ of an array of investments and an industry structure is **efficient** or **Pareto-efficient** if it maximizes social surplus

$$\sum_{f\in P} g_f(\mathbf{r_f}) - \frac{1}{2}\sum_{i\in I} r_i^2.$$

Evidently, if $(\mathbf{r}; P)$ is efficient, then $\mathbf{r}$ is $P$-efficient, that is $r_i = k_{|f|}$ for all $f \in P, i \in f$.

Finally, an industry structure $P$ is **efficient** if it gives rise to the maximal social surplus. To be precise, an industry structure is efficient if there exists an array of investments $\mathbf{r}$ so that the combination $(\mathbf{r}; P)$ is efficient. To find the efficient industry structures, determine for each industry structure $P$ the social surplus $S(P)$ generated by any $P$-efficient $\mathbf{r} \in \mathcal{R}$. The efficient industry structures are the maximizers of $S(P)$.

## 3.2 The Structure of Efficient Industry Structures

Since there is an optimal firm size, one could simply form as many firms of optimal size as possible plus eventually one firm of less than optimal size. This simple and promising procedure yields an efficient industry structure, if $N \leq n$; then the efficient industry structure consists of a single firm of size $N$. The procedure also creates an efficient industry structure if $N$ is a multiple of $n$, if there exists a positive integer $q$ such that $N = q \cdot n$; then an efficient industry structure consists of $q$ firms of optimal size. However, it turns out that it is not always efficient to create as many firms as possible of optimal size $n$. Next we collect and prove this and a few other facts about efficient industry structures.

**Fact 1** *In general, an industry structure with $q$ firms of size $n$ and one or no firm of size $m < n$ is not optimal.*

This follows from the following example. However, we also obtain

**Proposition 1** *An optimal industry structure contains at most $n - 1$ firms whose size is different from $n$ and no firm of size larger than $2n$.*

This follows from Facts 2,3,5, and 6 below.

**Example 1.** Let $k_s = 20s - s^2$ for $s \leq 20$ so that $n = 10$, $k_n = 100$. Consider $N = 12$ so that $m = 2$, $k_m = 36$, $q = 1$. Optimal investment in the firm of size $n$ yields $r_n = k_n = 100$ and a total surplus of $n \cdot \frac{1}{2} \cdot 100^2 = 50,000$. Optimal investment in the firm of size $m$ yields $r_m = k_m = 36$ and a total surplus of $m \cdot \frac{1}{2} \cdot 36^2 = 1,296$. Hence the social surplus obtainable with this industry structure is 51,296.

Alternatively, consider a single firm of size $N$. Optimal investment in that firm yields $r_N = k_N = 96$ and a social surplus of $N \cdot \frac{1}{2} \cdot 96^2 = 55,296$. Hence the industry structure $P = \{\{1, \ldots, 10\}, \{11, 12\}\}$ is inefficient. Any allocation of surplus (utility allocation) that can be achieved under $P$ is dominated by some utility allocation that is feasible under $P^* = \{I\} = \{\{1, \ldots, 12\}\}$. Further numerical comparisons show that the industry structure $P^*$ is efficient in the sense that it gives rise to the maximal social surplus. $\square\square$

**Fact 2** *An efficient industry structure cannot contain both firms of size less than $n$ and firms of size larger than $n$.*

For otherwise, take one firm of size less than $n$ and one firm of size larger than $n$ and move one person from the larger firm to the smaller firm. This would yield a better industry structure than the original one.

**Fact 3** *An efficient industry structure does not contain more than $n-1$ firms of size less than $n$.*

For otherwise, take one firm of minimal size, say of size $m$, and $m$ other firms of size less than $n$, say of sizes $s_1, \ldots, s_m$ and replace these $m + 1$ firms by $m$ firms of respective sizes $s_1 + 1, \ldots, s_m + 1$. The industry structure thus obtained would be better than the original one.

**Fact 4** *If an efficient industry structure contains exactly $T$ firms of size less than $n$, say sizes $n_1, \ldots, n_T$, and $T > 1$, then $\sum_{t=1}^{T} n_t > n$.*

For otherwise, combining these $T$ firms into a single firm would yield a better industry structure.

**Fact 5** *An efficient industry structure does not contain any firm $f$ of size $|f| \geq 2n$.*

For breaking up such a firm into two firms of sizes $n$ and $|f| - n$ would create a better industry structure.

**Fact 6** *If an efficient industry structure contains $T$ firms of size larger than $n$ and smaller than $2n$, say sizes $n + n_1, \ldots, n + n_T$, then $\sum_{t=1}^{T} n_t < n$. In particular, $T < n$.*

For otherwise, one could reduce these firm sizes to $n + m_1, \ldots, n + m_T$ with $0 \leq m_t \leq n_t$ for all $t = 1, \ldots, t$ and $m_t < n_t$ for some $t$ and form an additional firm of size $n$. This would create a better industry structure.

## 3.3 The Incentive and Hold-up Problem

If the industry structure were fixed, individuals were locked into their current firms or partnerships, and contracts did not stipulate otherwise, a member $i$ of firm $f$ would choose his $f$-specific investment as the solution to

$$\max_{r_i} \left\{ \frac{1}{|f|} k_{|f|} \sum_{j \in f} r_j - \frac{1}{2} r_i^2 \right\} .$$

Here the presumption is that outcome is shared equally among firm members, unless stipulated otherwise. This yields $r_i = k_{|f|}/|f|$ for all $i \in f$ whereas $f$-efficiency requires $r_i = k_{|f|}$. As a rule, therefore, an inefficiency results because of under-investment. The inefficiency of non-contractible relation-specific investments for a given industry structure is reminiscent of the standard hold-up problem in the theory of incomplete contracts. In the following, we explore endogenous firm formation.

# 4 Endogenous Firm Formation

In this section, we endogenize the industry structure, the relation-specific investments, and the consumption profile. Thus the outcome of the equilibrating process is a feasible allocation $(\mathbf{x}; \mathbf{r}; P)$. In general, the equilibrium allocation will not be efficient. First we explore a notion of weak stability of an allocation meaning that no new firms can form and improve the welfare of their members. It turns out that weakly stable allocations rarely exist. Second we consider a simple equilibrium concept based on reinvestment-proofness. The latter means that no agent has an incentive to leave the current firm, forego the benefits of the current investment, and join another firm (or form a one-person firm) and reinvest again in newly formed firm. In the latter case, we assume that the new firm member can act like a turnaround expert and reap all (or most) of the incremental surplus he generates. The old members of the enlarged firm are minimally affected by such a change. They can keep their previous investment levels and enjoy the same or slightly more consumption than before. All the efficiency gain is due to the extra effort of the joining member while a more drastic turnaround might induce the old members to increase their investments. Still, in order to prevent a member from leaving, the relation-specific investments in a firm have to be sufficiently high, which alleviates the hold-up problem without completely eliminating it.

## 4.1 Stable Allocations

Since a feasible allocation includes a multilateral matching in the form of an industry structure, it is natural to ask which allocations are stable in the sense of the literature on matching and assignment games. A further step would be to investigate the existence and properties of core allocations. But this might go to far, because it would assume away rather than address the hold-up problem. However, we shall consider a **weak individual rationality requirement**, $U_i(x_i, r_i) \geq 0$, which is justified by the fact that an individual $i$ can always form the firm $f = \{i\}$ and choose $x_i = 0, r_i = 0$. Of course, the one-person firm does not face a hold-up problem and, therefore, the individual may even make the $f$-efficient choice $r_i = k_1, x_i = k_1 r_i = k_1^2$ and end up with $U_i(x_i, r_i) = \frac{1}{2}k_1^2$.

We call a feasible allocation $(\mathbf{x}; \mathbf{r};\, P)$ **weakly stable** if there does not exist a potential firm $f$, an array of relation-specific investments $\mathbf{r'_f} = (r'_i)_{i\in f}$, and a consumption profile $\mathbf{x'_f} = (x'_i)_{i\in f}$ such that $f \notin P$, $\sum_{i\in f} x'_i = k_{|f|} \sum_{i\in f} r'_i$, and $U_i(x'_i, r'_i) > U_i(x_i, r_i)$ for $i \in f$. Without the stipulation $f \notin P$, one would obtain the definition of a weak core allocation. It turns out that weak stability proves already a very demanding requirement.

**Proposition 2** *Suppose that $N = qn$ for some integer $q > 1$ and $(\mathbf{x}; \mathbf{r};\, P)$ is a weakly individual rational allocation. Then $(\mathbf{x}; \mathbf{r};\, P)$ is weakly stable if and only if $|g| = n$ for all $g \in P$ and $(x_i, r_i) = (k_n^2, k_n)$ for all $i \in I$.*

The proof is given in the appendix. We also obtain:

**Proposition 3** *Suppose that $N > 2n$ and $N \neq qn$ for every integer $q$. Then there does not exist a feasible allocation which is weakly individual rational and weakly stable.*

The proof is relegated to the appendix.

## 4.2 A Simple Equilibrium Model

Let us emphasize from the outset that the following equilibrium concept does not postulate any efficiency properties *per se*, even though we are looking for desirable equilibrium outcomes. One basic premise is that an individual leaving a firm loses any benefits from prior relation-specific investments. Another fundamental premise is that bargaining power differs significantly depending on whether individuals form a new firm together or an individual can offer a better production plan to an existing firm. In the former case, individuals may have equal bargaining power when they collectively decide on the distribution of the firm's output. In the latter case, we assume that a new firm member can act like a turnaround expert and appropriate most or all of the incremental surplus he generates. More specifically, the newcomer can offer a sizeable individual investment in the new firm and ask for most of the additional surplus. In a reinvestment-stable equilibrium, no agent wants to leave his current firm. This requires that the relation-specific investments in existing firms are sufficiently large.

Given a feasible allocation $(\mathbf{x}; \mathbf{r}; P)$, agent $i$ **has an incentive to reinvest** if there exist $f \in P \cup \{\emptyset\}$ with $i \notin f$, $r'_i \in [0, R_i]$, and $x'_j \geq 0$ for $j \in f \cup \{i\}$ such that

$$\sum_{j \in f \cup \{i\}} x'_j = k_{|f|+1}(r'_i + \sum_{j \in f} r_j); \tag{2}$$

$$U_i(x'_i, r'_i) - \frac{1}{2} r_i'^2 > U_i(x_i, r_i); \tag{3}$$

$$U_j(x'_j, r_j) \geq U_j(x_j, r_j) \text{ for all } j \in f. \tag{4}$$

Condition (2) means that the new consumption plans $x'_j$ for the newly formed firm $f \cup \{i\}$ are feasible, if the newcomer $i$ makes the investment $r'_i$ and the old members $j$ keep their old investments $r_j$. Condition (3) means that $i$ is better off reinvesting in the newly formed firm $f \cup \{i\}$ than staying with her old firm, even though the costs $\frac{1}{2}r_i^2$ of the relation-specific investment in the old firm are sunk and cause a hold-up problem. The formulation (3) implies that after reinvesting in the newly formed firm, $i$ incurs the total costs $\frac{1}{2}r_i^2 + \frac{1}{2}(r_i')^2$. This separable form makes sense if, for example, agents invest human capital which is reusable. Condition (4) means that an old member consumes at least as much in the newly formed firm $f \cup \{i\}$ as as a member of $f$ while keeping the previous investment level $r_j$. In the sequel, we treat the constraint as binding, i.e. $x_j' = x_j$, to simplify the analysis.

**Definition.** We call a feasible allocation $(\mathbf{x}; \mathbf{r}; P)$ a **reinvestment-proof equilibrium** if

(i) no agent has an incentive to reinvest;

(ii) in the actual firms $f \in P$, each member receives consumption $g_f(\mathbf{r_f})/|f|$;

(iii) given (ii), no partner has an incentive to increase his investment in the current firm;

(iv) given (ii), if $i \neq j$ belong to the same firm $f \in P$ and $i$ lowers his investment to the hold-up level $k_{|f|}/|f|$, then $j$ has an incentive to reinvest.

## 4.3 A Specific Equilibrium

Notwithstanding Example 1, we are looking for an equilibrium of the firm formation process that results in $q$ firms of size $n$ and one firm of size $N - qn$ where the number $q = \lfloor N/n \rfloor$ is the largest integer such that $qn \leq N$. This does not necessarily yield an efficient industry structure as demonstrated by

Example 1. But if $N$ is large relative to $n$, it comes close to an efficient industry structure on a per capita basis, as Proposition 1 shows.

In general, if $N > n$, then $N = qn+m$ with uniquely determined integers $q \geq 1$ and $m \in \{1, \ldots, n\}$. Given such a composition, we call a reinvestment-proof equilibrium $(\mathbf{x}; \mathbf{r}; P)$ a **reinvestment-proof $m$-equilibrium**, if $P$ consists of $q$ firms of size $n$ and one firm of size $m$ and $f, g \in P$, $|f| = |g|$, $i \in f, j \in g$ implies $(x_i, r_i) = (x_j, r_j)$, that is there is equal treatment within firms and across firms of equal size.

## 4.4 Equilibrium with a Smaller Firm

Here we study existence of properties of a reinvestment-proof $m$-equilibrium with one firm of smaller size $m = N - qn < n$. Moreover, we denote the equilibrium values of relation-specific investment by an individual in firms of size $n$ and $m$ by $r_n$ and $r_m$, respectively. Finally, we index firms of size $n$ by $f$ and the smaller firm by $g$.

Recall that if the industry structure were fixed, individuals would choose their relation-specific investment as the solution to

$$\max_{r_i} \left\{ \frac{1}{n} k_n \sum_{j \in f} r_j - \frac{1}{2} r_i^2 \right\}, \text{ if } i \in f;$$
$$\max_{r_i} \left\{ \frac{1}{m} k_m \sum_{j \in g} r_j - \frac{1}{2} r_j^2 \right\}, \text{ if } i \in g.$$

This yields

$$\begin{aligned} r_i &= \frac{1}{n} k_n \quad \text{if } i \in f; \\ r_j &= \frac{1}{m} k_m \text{ if } j \in g. \end{aligned}$$

$P$-efficiency would require

$$r_i = k_n \quad \text{if } i \in f;$$

$$r_j = k_m \quad \text{if } j \in g.$$

If, however, the industry structure is not fixed and firm members have attractive outside options, then individuals might want to increase their investments in order to keep their partners from leaving. It turns out that this motivation is not strong enough to restore efficiency in general. But it can reduce the inefficiency considerably.

With an endogenous industry structure associated with a reinvestment-proof $m$-equilibrium, $1 \leq m < n$, we obtain the following equilibrium conditions:

$$r_n \geq \frac{1}{n} k_n \tag{5}$$

$$r_m \geq \frac{1}{m} k_m \tag{6}$$

$$(k_{m+1} - k_m)(m\, r_m) + k_{m+1}^2 - \frac{1}{2} k_{m+1}^2 - k_n\, r_n \leq 0 \tag{7}$$

$$(k_{n+1} - k_n)(n\, r_n) + k_{n+1}^2 - \frac{1}{2} k_{n+1}^2 - k_m r_m \leq 0 \tag{8}$$

$$(k_{n+1} - k_n)(n\, r_n) + k_{n+1}^2 - \frac{1}{2} k_{n+1}^2 - k_n\, r_n \leq 0 \tag{9}$$

The first two inequalities state that individuals do not want to increase their investments in the firms they enter at the beginning. The third inequality in (7) states that an individual in a firm $f$ does not gain anything by joining firm $g$ and offering a better consumption plan to all firm members. Note that such a deviating individual would choose its relation-specific investment according to:

$$\max_{r_i} \left\{ k_{m+1} \left( m\, r_m + r_i \right) - k_m m\, r_m - \frac{1}{2} r_i^2 - k_n r_n \right\}$$

which yields

$$r_i = k_{m+1}.$$

The net gains from switching from firm $f$ to firm $g$ amount to

$$k_{m+1}^2 + (k_{m+1} - k_m)\, m\, r_m - \frac{1}{2}\, k_{m+1}^2 - k_n r_n$$

which yields inequality (7) as non-switching condition.

Similarly, the last two inequalities state that switching from firm $f$ to $g$ or from one firm with size $n$ to another firm of size $n$ is not beneficial for an individual. Note that in these two cases the marginal productivity of relation-specific investments decreases and, thus, other firm members must be compensated accordingly so that they are not harmed by the entrant.

Finally, in order to be in equilibrium, no individual should be able to reduce its relation-specific investment without causing other firm members to leave. For $r_n \leq \frac{1}{n}\, k_n$ and $r_m \leq \frac{1}{m} k_m$, individuals have no incentive to reduce investment. If $r_n > \frac{1}{n}\, k_n$ or $r_m > \frac{1}{m}\, k_m$, we obtain three inequalities involving the net utility effect of other firm members leaving when the individual under consideration reduces its relation-specific investment to $r_n' \in [k_n/n, r_n)$ or $r_m' \in [k_m/m, r_m)$, respectively, i.e. below the equilibrium level, but at most to the level it would choose under a fixed industry structure. These inequalities are:

$$(k_{m+1} - k_m)\,(m\, r_m) + \frac{1}{2}\, k_{m+1}^2 - k_n \left( \frac{(n-1) r_n + r_n'}{n} \right) > 0. \tag{10}$$

$$(k_{n+1} - k_n)\,(n\, r_n) + \frac{1}{2}\, k_{n+1}^2 - k_n \left( \frac{(n-1) r_n + r_n'}{n} \right) > 0. \tag{11}$$

$$(k_{n+1} - k_n)\,(n\, r_n) + \frac{1}{2}\, k_{n+1}^2 - k_m \left( \frac{(m-1) r_m + r_m'}{m} \right) > 0. \tag{12}$$

For instance, the first inequality states that an individual in a firm of size $n$ leaves and joins the firm with size $m$ if another member of the firm under consideration reduces investment to $r'_n \in [k_n/n, r_n)$. In equilibrium, the first or the second inequality must hold if $r_n > \frac{1}{n}k_n$. The third one must hold if $r_m > \frac{1}{m}k_m$ and $r'_m \in [k_m/m, r_m)$.

## 4.5 Existence Result

To construct an equilibrium, we set $r_m = \frac{1}{m}k_m$ and posit inequality (7) with equality:

$$(k_{m+1} - k_m)\,(m\,r_m) + \frac{1}{2}\,k_{m+1}^2 - k_n r_n = 0$$

which implies

$$r_n = \frac{k_m(k_{m+1} - k_m) + \frac{1}{2}\,k_{m+1}^2}{k_n}.$$

We obtain:

**Proposition 4** *Suppose $N > n$ and $1 < m < n$. Then there exists a reinvestment-proof $m$-equilibrium with*

$$\begin{aligned} r_m &= \frac{1}{m}k_m, \\ r_n &= \frac{k_m(k_{m+1} - k_m) + \frac{1}{2}\,k_{m+1}^2}{k_n} \end{aligned}$$

*if and only if the following two conditions hold:*

$$n\Big\{k_m(k_{m+1} - k_m) + \frac{1}{2}k_{m+1}^2\Big\} > k_n^2 \tag{13}$$

$$k_{n+1} \leq k_{n+1}^* \equiv -n\,r_n + \sqrt{n^2 r_n^2 + 2k_n n\,r_n + \frac{2k_m^2}{m}} \tag{14}$$

The proof is given in the appendix. Note that for $m = 1 < n$, the argument requires in addition that $\frac{1}{2}k_2^2 \geq k_1^2$ holds.

Comparison of the resulting equilibrium allocation with the Pareto-efficient allocations reveals a clear inefficiency in the small firm. There cannot be full efficiency in the large firms either. But one can achieve substantial efficiency gains. In particular, if $m + 1 = n$, then $\frac{1}{2}k_n \leq r_n \leq \frac{3}{4}k_n$ and $1/2 \leq \lambda_n \leq 3/4$ where $\lambda_n \equiv r_n/k_n$ measures the relative efficiency in large firms.

It remains to be seen if an equilibrium as depicted in Proposition 4 ever exists. Notice that condition (14) holds whenever $k_{n+1}$ is sufficiently small, especially if there is a drastic drop in productivity beyond the optimal firm size. Notice further that condition (13) is always met for $m + 1 = n$. Both conditions are satisfied with $m + 1 = n$ in the following example.

**Example 2.** Let $N = 17$ and $n = 4$. Hence $m = 3$ and $m+1 = n$. Moreover, let $k_s = 1/(1 + |s - n|)$ for all $s$. Then $k_n = k_{m+1} = 1$ and $k_m = k_{n+1} = 1/2$. It is easy to check that conditions (13) and (14) hold. Finally, $k_n/n = 1/4$ and $r_n = 3/4$. □□

**Example 1 reconsidered.** Now let us revisit Example 1. In that example, $n + 1 = 11$ with $k_{n+1} = 99$ and $m + 1 = 3$ with $k_m = 36$, $k_{m+1} = 51$. Hence the resulting value for $r_n$ in Proposition 4 is $r_n = 18.405$. Moreover, with $n = 10$ and $k_n = 100$, (13) amounts to $18,405 > 10,000$ which holds true. On the other hand, (14) is violated. Hence the specific equilibrium does not exist. Similarly, one can verify that (14) is violated for $m = n - 1 = 9$ as well while (13) holds. □□

**Example 3.** Next let us modify Example 1 by setting $k_s = 18s - s^2$ for

$11 \leq s < 18$. Then $k_{n+1} = 77$ and both (13) and (14) are satisfied. For the large firm, $k_n/n = 10$ and $k_n = 100$ whereas $r_n = 18.405$ and $\lambda_n = 0.18405$. Hence relative to free riding, there is a significant efficiency gain while this gain appears moderate when compared to $f$-efficiency. Instead of $m = 2$, let us further consider the case $m = n - 1 = 9$. In this case, $k_m = 99$ and $k_{m+1} = k_n = 100$. Again, both (13) and (14) are satisfied. Now $\lambda_n = 0.5099$, which constitutes quite a substantial efficiency gain relative to free riding. $\square\square$

## 4.6 Robustness and Comparative Statics

The equilibrium analysis until now is based on the fact that $N = qn + m$ with $1 \leq m < n$. This raises the question if and how a change in population size would affect existence and welfare properties of a reinvestment-stable $m$-equilibrium. Towards an answer, we first elaborate on the case $m = n$ where the efficient industry structure does not contain a small firm. Subsequently, we shed some light on the comparative statics with respect to $m$.

Suppose $N = qn$ with $q \geq 2$ so that the population can be partitioned into $q$ optimal partnerships. Let $P$ be such a partition. We are interested in the existence of an investment level $r_n > k_n/n$ such that $(\mathbf{x}; \mathbf{r}; P)$ constitutes a reinvestment-stable equilibrium where $\mathbf{x}; = (k_n r_n, \dots, k_n r_n)$ and $\mathbf{r} = (r_n, \dots, r_n)$. To this end, let us make (9) binding which is equivalent to

$$r_n(nk_{n+1} - (n+1)k_n) = -\frac{1}{2}k_{n+1}^2.$$

Since $(n+1)k_n - nk_{n+1} > 0$, one can obtain this equality by setting

$$r_n = \frac{1}{2}k_{n+1}^2/((n+1)k_n - nk_{n+1}).$$

Now let $\epsilon \in (0,1)$ be such that $k_{n+1} = (1-\epsilon)k_n$. Then

$$r_n = \frac{1}{2} \cdot \frac{(1-\epsilon)^2}{1+n\epsilon} \cdot k_n.$$

Hence this choice of $r_n$ implies $r_n \leq \frac{1}{2}k_n$. Moreover, $r_n \to \frac{1}{2}k_n$ as $\epsilon \to 0$. We shall elaborate on these two implications later. Our immediate concern is whether the candidate investment level satisfies the constraint (5) which amounts to $(1-\epsilon)^2 \geq 2(1/n + \epsilon)$. The latter inequality can be simplified to $4\epsilon - \epsilon^2 \leq 1 - 2/n$. Hence a necessary condition for (5) is $\epsilon < 1/3 - 2/(3n)$ and a sufficient condition is $\epsilon \leq 1/4 - 1/(2n)$.

**Remark 1.** Since up to a permutation of individuals, a specific reinvestment-proof $m$-equilibrium is unique (provided it exists), we are going to refer to THE REINVESTMENT-PROOF $m$-EQUILIBRIUM.

**Remark 2.** The reinvestment-stable $n$-equilibrium exists if and only if the ratio $k_{n+1}/k_n$ exceeds a threshold $1-\epsilon^*$, that is if $k_{n+1}$ is large enough relative to $k_n$. In contrast, the reinvestment-stable $(n-1)$-equilibrium exists only if (14) holds, that is only if $k_{n+1}$ is sufficiently small. This observation suggests that simultaneous existence of both equilibria is very unlikely, although not impossible.

**Remark 3.** In the reinvestment-stable $(n-1)$-equilibrium, $\frac{1}{2}k_n < r_n \leq \frac{3}{4}k_n$ holds, whereas in the reinvestment-stable $n$-equilibrium, $r_n \leq \frac{1}{2}k_n$ obtains. Hence the presence of a firm of suboptimal size $m = n-1$ has necessarily a beneficial effect on the investment level of firms of optimal size $n$. But would that effect get even greater, if $m$ were smaller? It turns out that $\lambda_n = r_n/k_n$ is not always strictly decreasing in $m$. If for instance the reinvestment-stable $(n-2)$-equilibrium and the reinvestment-stable $(n-1)$-equilibrium both exist, then the former may, but need not give rise to a higher investment level $r_n$.

We are going to illustrate and examine the various possibilities suggested in the foregoing remarks, by (re)considering the previous examples and two new ones. Examples 1 to 3 confirm the conjecture that the reinvestment-stable $n$-equilibrium and $(n-1)$-equilibrium rarely coexist. In Example 4, $n = 3$, the reinvestment-stable $m$-equilibrium exists for all $m$, and $r_n/k_n$ is strictly decreasing in $m$. In Example 5, $n = 4$ and the reinvestment-stable $m$-equilibrium exists for $m = 2, 3$. The value of $r_n$ associated with $m = 3$ is greater than the one associated with $m = 2$.

**Example 1 continued.** With $n = 10$, $k_n = 100$ and $k_{n+1} = 99$. Hence $\epsilon = 1/100$ and $\lambda_n \approx 0.4455 \gg 0.10 = 1/n$. Thus the reinvestment-stable $n$-equilibrium exists and leads to a significant efficiency gain. On the other hand, we have already established that the reinvestment-stable $m$-equilibrium does not exist for $m \in \{2, 9\}$. □□

**Example 2 continued.** We found that the reinvestment-stable $(n-1)$-equilibrium does exist. But $n = 4$, $k_n = 1$ and $k_{n+1} = 1/2$ yield $\epsilon = 1/2$ and $r_n = \lambda_n = 1/24 < 1/n$. Therefore, the reinvestment-stable $n$-equilibrium does not exist. □□

**Example 3 continued.** We found that the reinvestment-stable $m$-equilibrium exists for $m \in \{2, 9\}$. Regarding $m = n$, one obtains $\epsilon = 0.23$ and $\lambda_n < 1/n$ and consequently non-existence of the reinvestment-stable $n$-equilibrium. □□

**Example 4.** Let $c \geq 0$, and $k_s = c + 2ns - s^2$ for $s = 1, \ldots, n+1$. Then $k_n = c + n^2$, $k_{n-1} = k_{n+1} = c + n^2 - 1$,

$k_{n+1}/k_n = 1 - 1/(c + n^2)$, $\epsilon = 1/(c + n^2)$.
For $n \geq 4$, $\epsilon + 1/(2n) \leq 1/16 + 1/8 = 3/16 < 1/4$. Therefore, the reinvestment-stable $n$-equilibrium exists. For instance, $c = 0, n = 10$ yields $\lambda_n = 0.4455$ and the reinvestment-stable $n$-equilibrium leads to a considerable efficiency gain relative to free riding.

Next let us focus on the special case $c = 3, n = 3$ and all possible values for $m$.

m=3: $\epsilon = 1/12$, $\lambda_n = 1/3 + 1/360 = 1/n + 1/360 = 0.336111$.

m=2: (15) holds, since $m = n - 1$. (16) holds, since $m = 2$, $k_m = k_{n+1} < k_n$. One obtains $r_n = \{1/2 + 1/144\}k_n = 0.506944k_n$.

m=1: (15) is easily verified. With $r_n = 7.041667$ and $k_{n+1} = 11$, one obtains $(k_{n+1} + nr_n)^2 \approx 1032 < 1081 \approx (nr_n)^2 + 2k_n nr_n + 2k_m^2/m$, hence (16). Moreover, $r_n = 0.586806k_n$.

It follows that the reinvestment-stable $m$-equilibrium exists for $m = 1, 2, 3$ and that $r_n$ is strictly decreasing in $m$. □□

Notice that if a rent-seeking agent $j$ joins a partnership of original size $m$, $j$ chooses $r_j = k_{m+1}$ and each prior member $i$ keeps the prior investment level $r_i = k_m/m$, then two effects occur. First, $j$ directly generates the surplus $d_m = \frac{1}{2}k_{m+1}^2$. Secondly, the change in firm size creates the incremental surplus $i_m = (k_{m+1} - k_m) \cdot k_m$. Clearly, $d_n < d_{n-1}$ and $i_n < 0 < i_{n-1}$, in accordance with our earlier observation that the reinvestment-stable $(n-1)$-equilibrium induces a higher investment level $r_n$ than the reinvestment-stable $n$-equilibrium, if both exist. But $d_m$ is increasing in $m < n$ while the fluctuation of $i_m$ depends on the numerical specification of the model and the overall effect can go either way. In Example 4, $i_m$ is strictly decreasing in $m < n$ and that effect dominates. Now suppose that $k_{m+1} - k_m$ is constant or increasing in $m < n$. Then $d_m$ and $i_m$ are strictly increasing in $m < n$. It remains to be shown that under this assumption, the reinvestment-stable $m$- and $(m+1)$-equilibria can exist for some $m+1 < n$. To this end, we present the following example.

**Example 5.** Let $k_s = s$ for $s \leq n$ and let $k_{n+1}$ so that (16) is satisfied for $m = 1, \ldots, n-1$. Then (15) is satisfied if and only if $m + 2 > \sqrt{3+2n}$. Specifically, let $n = 4$. Then (15) and (16) are satisfied for $m = 2, 3$. Hence the reinvestment-stable $m$-equilibrium exists for $m = 2, 3$. Furthermore, the reinvestment-stable 3-equilibrium induces a higher investment level $r_n$ than the reinvestment-stable 2-equilibrium. □□

# 5 Final Remarks

Our analysis confirms that the hold-up problem can be significantly alleviated if partners are resourceful enough to make valuable firm-specific investments at the new firm in the case that they move and are able to appropriate the surplus generated by those investments. Reinvestment-proof equilibria where agents choose investment levels above the free-riding level frequently, though not always, exist. In contrast, weakly stable allocations akin to stable matchings, which are immune against the formation of new firms where all members may alter there investments relative to the status quo, rarely exist because of three extra destabilizing effects: First, new firms are not only created by one agent switching firms. Second, all members of a new firm may alter their investments. Third, an individual who is switching firms can reverse the prior investment and totally avoid the hold-up problem — which, therefore, is assumed away rather than being resolved.

We conclude that it is more appropriate to use reinvestment-proofness than weak stability to address the hold-up problem in partnerships. To the extent that the desired reinvestment-proof equilibria exist, always some and sometimes impressive efficiency gains compared to the free-riding outcome are achieved. If such an equilibrium does not exist, then free riding prevails or individuals have incentives to change firms. If an additional individual appears on the scene (or somebody disappears), then depending on the model parameters, equilibrium may be replaced by disequilibrium or vice versa or an equilibrium may be followed by a different one.

Finally, we hope that the present framework offers directions for future research. First, it will be useful to explore the firm formation process when some agents have ideas, i.e. access to production technologies, while others can only make relation-specific investments. This would make it possible to distinguish between entrepreneurs and employers. Second, in a more general equilibrium setting one would like to introduce competitive exchange across firms and households, thereby distinguishing between consumption goods and investment goods. The current state of the general equilibrium approach to group formation in the presence of competitive markets has mainly focused on pure exchange.[6] The current paper suggests that such a line of research might provide new insights into the working of market economies.

[6]The collective decision approach in the presence of competitive exchange has been pursued in Gersbach and Haller (2001).

# 6 Appendix

**Proof of Proposition 2:**

Let $N = qn$ and $(\mathbf{x}; \mathbf{r};\ P)$ as hypothesized.
"If": Suppose $(x_i, r_i) = (k_n^2, k_n)$ for all $i \in I$. Then for any potential firm $f$, any array of relation-specific investments $\mathbf{r'_f} = (r'_i)_{i\in f}$, and any consumption profile $\mathbf{x'_f} = (x'_i)_{i\in f}$ such that $\sum_{i\in f} x'_i = k_{|f|} \sum_{i\in f} r'_i$, one obtains

$$\begin{aligned}&\textstyle\sum_{i\in f} U_i(x'_i, r'_i) = \sum_{i\in f}[x'_i - \frac{1}{2}(r'_i)^2] = \sum_{i\in f}[k_{|f|}r'_i - \frac{1}{2}(r'_i)^2]\\ &\textstyle\leq \sum_{i\in f} \frac{1}{2}k_{|f|}^2 \leq \sum_{i\in f} \frac{1}{2}k_n^2 = \sum_{i\in f} U_i(x_i, r_i).\end{aligned}$$

Consequently, $U_i(x'_i, r'_i) > U_i(x_i, r_i)$ cannot hold for all $i \in f$. This shows that $(\mathbf{x}; \mathbf{r};\ P)$ is weakly stable.

"Only if": Suppose $|g| \neq n$ for some $g \in P$. Then

$$\sum_{i\in I} U_i(x_i, r_i) < N \cdot \frac{1}{2}k_n^2. \tag{15}$$

Without loss of generality, let us assume
$U_1(x_1, r_1) \leq U_2(x_2, r_2) \leq \ldots \leq U_N(x_N, r_N)$. Then

$$\sum_{i\leq j} U_i(x_i, r_i) < j \cdot \frac{1}{2}k_n^2 \text{ for all } j \in I. \tag{16}$$

Now consider the firm $h = \{1, 2, \ldots, n\}$ and distinguish two cases.

In case $h \notin P$, let $f = h$. Then by (16),

$$\Delta \equiv n \cdot \frac{1}{2}k_n^2 - \sum_{i\in f} U_i(x_i, r_i) > 0.$$

For $i \in f$, set $r'_i = k_n$, $x'_i = U_i(x_i, r_i) + \frac{1}{2}k_n^2 + \Delta/n$. Then for each $i \in f$, $U_i(x'_i, r'_i) = x'_i - \frac{1}{2}(r'_i)^2 = U_i(x_i, r_i) + \Delta/n > U_i(x_i, r_i)$ and $x'_i \geq 0$ because $(\mathbf{x}; \mathbf{r};\ P)$ is weakly individual rational. Moreover, $\sum_{i\in f} x'_i = nk_n^2 = k_{|f|} \sum_{i\in f} r'_i$. Thus $(\mathbf{x}; \mathbf{r};\ P)$ is not weakly stable.

In case $h \in P$, let $f = \{1, 2, \ldots, n-1, n+1\}$. Then

$$\sum_{i \in f} U_i(x_i, r_i) < n \cdot \frac{1}{2} k_n^2.$$

For otherwise, one would obtain

$$\sum_{i \leq 2n} U_i(x_i, r_i) \geq 2 \sum_{i \in f} U_i(x_i, r_i) \geq 2n \cdot \frac{1}{2} k_n^2,$$

a contradiction to (16). From here we can proceed as in the previous case to show that $(\mathbf{x}; \mathbf{r};\ P)$ is not weakly stable. ■■

**Proof of Proposition 3:**

Let $N$ be as hypothesized and let $(\mathbf{x}; \mathbf{r};\ P)$ be a feasible allocation. Since $N \neq qn$ for every integer $q$, there exists $g \in P$ with $|g| \neq n$. Therefore, one can relabel the members of population $I$ so that inequalities (15) and (16) hold. Then one can proceed as in the "Only if" part of the previous proof and show that weak individual rationality of $(\mathbf{x}; \mathbf{r};\ P)$ implies that $(\mathbf{x}; \mathbf{r};\ P)$ is not weakly stable. ■■

**Proof of Proposition 4:**

By construction, inequalities (6) and (7) are satisfied with equality. To satisfy inequality (5), inserting the equilibrium value yields

$$n\{k_m(k_{m+1} - k_m) + \frac{1}{2}\, k_{m+1}^2\} > k_n^2$$

which is the first condition (13) in the proposition. Next we consider inequality (10). Because (7) holds with equality and $r_n > \frac{1}{n}\, k_n$, condition (10) is satisfied. Therefore, no individual in a large firm with size $n$ can reduce its level of relation-specific investment without causing other members to leave and to join the small firm. To be in equilibrium, inequality (11) need not be fulfilled in addition to (10). Moreover, inequality (12) is irrelevant because of $r_m = \frac{1}{m}\, k_m$.

It remains to verify the equilibrium conditions (8) and (9). Observe that

$$\begin{aligned} k_n\, r_n &= k_m\, k_{m+1} - k_m^2 + \frac{1}{2}\, k_{m+1}^2 \\ &\geq \frac{1}{2}\, k_{m+1}^2 \geq \frac{k_m^2}{m} = k_m\, r_m \end{aligned}$$

as long as $m > 1$. Therefore, if inequality (8) is fulfilled, (9) holds automatically. Consider now condition (8):

$$(k_{n+1} - k_n)(n\, r_n) + \frac{1}{2}\, k_{n+1}^2 - \frac{k_m^2}{m} \leq 0.$$

For $k_{n+1} = 0$ the condition is trivially fulfilled. For $k_{n+1} = k_n$ the condition becomes

$$\frac{1}{2}\, k_n^2 - \frac{k_m^2}{m} \leq 0$$

which is impossible to fulfill, since $m > 1$ and $k_n > k_m$. Since the expression on the left-hand side of condition (8) is continuous and monotonically increasing in $k_{n+1}$, there exists a critical value $k_{n+1}^*$ such that (8) holds if $k_{n+1} \leq k_{n+1}^*$. The critical value is given by

$$k_{n+1}^* = -n\, r_n + \sqrt{n^2\, r_n^2 + 2k_n n\, r_n + \frac{2k_m^2}{m}}.$$

This completes the proof. ■■

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