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# A RATIONAL EXPECTATIONS THEORY OF THE KINK IN EARNINGS REPORTS

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# ABSTRACT

## A Rational Expectations Theory of the Kink in Earnings Reports*

Empirical evidence suggests that the distribution of earnings reports exhibits kinks. Managers manage earnings as if to meet exogenously pre-specified targets, such as avoiding losses and avoiding a decrease in earnings. This is puzzling because the compensation to managers at these pre-specified targets seems to be smooth. We propose a game theoretic model explaining this phenomenon. In our model, investors form expectations of such a manipulative behaviour by the manager. Given these expectations, the best response of the manager is to fulfil the investors' expectations, resulting in a discontinuity in the distribution of earnings reports. Our model generates several new empirical predictions regarding the existence of the kink, its size and its location relative to the distribution of earnings reports.

JEL Classification: D82 and G30
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# 1 Introduction

Managers of publicly traded firms possess information useful to the valuation of their firms. Lacking access to privately produced information, investors must determine the stock price primarily based on public information. Consequently, mandatory periodical earnings announcements by the managers serve as the predominant source of information for the investors. Earnings announcements have been studied extensively, and were recently the subject of intense scrutiny by the public and the regulators. We know that managers have some leeway in reporting earnings within the accounting conventions, and that managerial compensation is frequently related to earnings either directly, or indirectly via the stock price. Thus managers have incentives and ability to manage (manipulate) earnings, and the empirical evidence suggests that they frequently do so.[1] There is also evidence that the compensation of managers is a determinant of the extent of earnings management.[2]

While it is reasonable to assume that the true earnings of a firm are drawn from a continuous distribution, the empirical distribution of earnings reports is discontinuous. Burgstahler and Dichev (1997) provide evidence on such discontinuity. They show that managers manage earnings as if to meet exogenously pre-specified targets, such as avoiding losses and avoiding a decrease in earnings. Degeorge, Patel and Zeckhauser (1999) provide evidence of such a discontinuity close to analysts' earnings forecasts consensus (see also Abarbanell and Lehavy (2000)). The resulting discontinuity manifests itself as a kink in the distribution of reports slightly below certain predetermined levels. This phenomenon raises two puzzling questions:

1. *Why* do managers insist on reporting last year's earnings (earnings decrease avoidance), or positive earnings (loss avoidance)? This is a puzzle because they are not explicitly compensated for meeting these "targets", since their compensation is typically smooth at these points. Despite the smoothness of the compensation and of the earnings distribution we observe a kink at 0 and at last year's earnings. A similar argument applies to the kink observed near analysts' consensus expectation. These kinks are generated endogenously. Why?

2. *How* do managers manage earnings to avoid losses and earnings decrease and

[1] The empirical literature on earnings management is voluminous. See Healy and Wahlen (1999) for a review.

[2] See for instance: Healy (1985), Bergstresser and Philippon (2002), and Kedia (2003).

to meet analysts' expectations? In particular, what are the inter-temporal considerations of managers? What is the relation between managerial flexibility at any particular time and the ability to meet these targets? Do managers use accruals or real actions in meeting these targets? Which accruals are used to manage the earnings?

The existing answers to the first question are based on behavioral arguments either on the investor's utility (see Burgstahler and Dichev 1997) or on their information processing heuristics (see Degeorge, Patel and Zeckhauser 1999). The goal of this paper is to propose a game-theoretic, rational model addressing this question. Doing so, we are able to provide new insights to the discontinuity phenomenon, and suggest several new testable implications.

The second question is primarily of an empirical nature, and we do not address it in this paper. It has been addressed to some extent by Burgstahler and Ditchev themselves, and later on by Roychowdhury (2003) and Choy (2004).

We use a signaling based earnings management model similar to Fischer and Verrecchia (2000). In our model, the manager trades off the costs of earnings management against his private benefits from such management due to stock based compensation in the face of a rational response by uninformed investors. This trade-off determines the optimal level of manipulation. Had there been an exogenous bonus paid at a prespecified target, the kink in reports would have been easily explained: all managers with earnings slightly below the target would have manipulated the earnings upward to meet the target as long as the marginal cost of earnings manipulation is lower than the bonus.

Our model shows that such a behavior is possible even when an exogenous bonus does not exist. Essentially, it follows from self-fulfilling investors' expectations. Investors expect managers to manage earnings (they form "beliefs"). Specifically, they expect all managers who have earnings in a specific interval to report the upper end point of this interval. Thus, all managers with earnings in this interval "pool" and provide the same report. Given these expectations, the manager's best response is to comply and pool the reports. If the manager does not follow this pooling strategy, he is punished in terms of stock price. Given the manager's strategy, the investors' expectations are fulfilled, and the report together with investors' beliefs constitute an equilibrium. Thus, our model shows that even when both the earnings distribution and the compensation to the manager are smooth, the distribution of earnings reports can exhibit a discontinuity. The driving force of the manipulative behavior of the managers is not an exogenous bonus paid at a pre-specified target, but the fact that *investors expect* the manager to manipulate in this way, and given these expectations, his best response is to do so.

This rational expectations equilibrium generates several new testable implications that deepen our understanding of the kink in earnings reports and its sources. Our model suggests that:

- The size of the kink is increasing in the extent of stock based compensation, and in growth opportunities of the firm, while it is decreasing in the quality of the enforcement of accounting standards.

- The location of the kink will be to the left of the mean of the distribution of reported earnings. The kink will be empirically pronounced only if it is close to the mean.

- The kink will be more pronounced empirically in firms with a higher extent of stock based compensation and with less volatile earnings.

We use these predictions to suggest ways to empirically refine the results of Burgstahler and Dichev (1997), and get a better understanding of the origins of this phenomenon.

Our model adds to a group of *one-period* models studying earnings management. These models are surveyed in Lambert (2001), and include papers such as Verrecchia (1986), Baiman, Evans and Noel (1987), Newman and Sansing (1993), Evans and Sridhar (1996) and Demski and Dye (1999). Most relevant to our paper is Fischer and Verrecchia (2000) offering a one period model of earnings bias. Of course, studying earnings management in a one-period framework is not optimal, because many aspects of earnings management are of an inter-temporal nature. This includes earnings smoothing, balancing and rebalancing of discretionary accruals across time, and reputation maintenance. We do not address those issues in this paper.

The paper is organized as follows. In Section 2 we present the model. In Section 3 we present our newly suggested equilibrium. In Section 4 we study the properties of the equilibrium and suggest testable implications. Section 5 provides several extensions to the model. In Section 6 we check the robustness of the equilibrium, while Section 7 concludes. Technical proofs are in the Appendix.

## 2 Model

We assume that the true earnings of the firm, $x$, are drawn from a normal distribution with mean $x_0$ and variance $\sigma^2$. The cumulative distribution is denoted by $F$, and the density is denoted by $f$. The parameters of the distribution are common knowledge; however, only the manager observes the realization $x$.[3] The manager is mandated to publish an earnings report, $x^R$, which the investors observe and use to price the stock. The utility of a manager who reports $x^R$ after observing $x$ is given by

$$U^M(x, x^R) = \alpha P(x^R) - \beta(x - x^R)^2, \tag{1}$$

where $\alpha \neq 0$, $\beta > 0$, and $P(x^R)$ is the market price of the firm given the report.

The second term of the manager's utility function represents the cost of manipulation, which is central to our paper. Positive $\beta$ implies that higher deviations from the truth carry higher legal, regulatory, or psychic costs for the manager. An alternative interpretation of this term is that the manager can take real actions (e.g., asset sales, suboptimal investments, aggressive sales efforts) to generate earnings that are different from the true earnings under the optimal policy. These actions are costly in terms of the future earnings of the firm, and therefore are costly for the manager.

The first term represents the fact that managerial compensation depends on the stock price, which makes the manager potentially biased.[4] Usually, managers prefer to see higher stock prices, but it well may be that temporarily they may be interested in reducing the price of the stock, e.g. to buy back shares, or to make the next period price increase more dramatic. Thus, $\alpha$ may be positive or negative, and is likely to vary across firms. For clarity of presentation we provide a detailed analysis of the case $\alpha > 0$ (a positive bias). The case $\alpha < 0$ is parallel, and is presented in Section 5. All parameters are common knowledge.

We assume that investors are risk neutral and hence price the stock proportionally to the expected value conditional on all the available information. We denote by $c > 0$

[3] While $x$ denotes the realization of earnings known only to the manager, we denote by $\tilde{x}$ the random variable of earnings observed by the uninformed investores.

[4] The pay-for-performance sensitivity of compensation has been increasing over the years. See Hall and Liebman (1998) compared to the findings of Jensen and Murphy (1990).

the Price-Earnings (P/E) ratio for this firm. Thus, the price of the stock prior to the manager's report is $p_0 = cx_0$ - the prior mean, while the post-report price is $p_1 = P(x^R) = cE(\tilde{x}|x^R)$.[5] We conclude that for all pairs $x$ and $x^R$ we have

$$U^M(x, x^R) = \alpha c E(\tilde{x}|x^R) - \beta(x - x^R)^2. \tag{2}$$

The manager has two conflicting interests: on the one hand, he would like to boost (if $\alpha > 0$) the stock price by manipulating his report; on the other hand, he does not want to manipulate his report too much, because the marginal cost of manipulation is increasing. The relative weight that the manager assigns to each of these two incentives is determined by the ratio $\frac{\alpha}{\beta}$. The higher this ratio is, the more inclined is the manager to deviate from the truth. The combination of the normal prior distribution of earnings with a quadratic cost function yields a tractable model.[6]

A reporting strategy for the manager is a real function $\rho : \mathbb{R} \to \mathbb{R}$ that maps true earnings into reports: $x^R = \rho(x)$. A pricing function for the investors is a function $P : \mathbb{R} \to \mathbb{R}$ that maps the manager's report into a price. A perfect Bayesian equilibrium is defined as a reporting strategy $\rho^*$ for the manager, joint with a pricing function $P^*$ for the investors such that:

1. The pricing function $P^*$ is consistent with the strategy $\rho^*$, by applying Bayes rule whenever possible.

2. For all $x \in \mathbb{R}$, $\rho^*(x) \in \arg\max_{x^R} U^M(x, x^R)$.

Observe first that truthful reporting, i.e., $\rho(x) = x$, is not an equilibrium. Indeed, if $\rho(x) = x$ for all $x \in \mathbb{R}$, then investors adjust their beliefs to reflect this strategy; thus, $P(x^R) = x^R$ for all reports $x^R$. If a manager who observes real earnings of $x$ reports truthfully, he obtains $\alpha c x$. If this manager raises his report to $x + \varepsilon$ ($\varepsilon > 0$), he obtains $\alpha c(x + \varepsilon) - \beta\varepsilon^2$. Thus, deviation is beneficial for all sufficiently small $\varepsilon$. It is also easy to verify that a perfect pool (a babbling equilibrium), i.e., $\rho(x)$ *is constant* for all $x \in \mathbb{R}$, cannot be an equilibrium for any pricing function.

[5] Note that in our model, $c$ stands for the "true" P/E ratio of the firm - the ratio between price and *true* earnings. Earnings management renders this ratio somewhat different from the implied P/E ratio - the ratio between price and *reported* earnings.

[6] This tractability does not come for free. Indeed, the cost of earnings management in our model depends only on the extent of earnings management $x^R - x$, but not directly on $x^R$ or $x$. Thus, in the model, any level of earnings management is just as costly for large firms as it is for small firms.

We show below the existence of two types of equilibria in this model. The first and conventional equilibrium is the perfectly separating (fully revealing) one. This equilibrium serves as a benchmark for our analysis.

**Proposition 1** *There exists a unique perfectly separating, continuously differentiable equilibrium. The equilibrium strategy of the manager is linear in x:* $\rho_s^*(x) = x + \frac{\alpha c}{2\beta}$ *for all* $x \in \mathbb{R}$*. The pricing function is linear in the report:* $P_s^*(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$ *for all* $x^R \in \mathbb{R}$*.*

**Proof:** In the Appendix.

This perfectly separating equilibrium is characterized by a constant earnings manipulation. Regardless of the realization of true earnings, the manager inflates his report by $\frac{\alpha c}{2\beta}$. Naturally, the investors are not fooled and price the stock correctly. This kind of equilibrium is standard in the continuous type, costly signalling literature (e.g., Riley (1979) and Stein (1989)). Notice that this equilibrium is inefficient. The manager cannot avoid the costly earnings management, and pays the costs that it imposes on him. Unfortunately, he gains nothing from this behavior, because it is correctly interpreted by the investors.[7]

# 3 A Partially Pooling Equilibrium

The separating equilibrium yields a smooth distribution of earnings reports, and hence no kink. However, there exists another equilibrium in this model. To demonstrate it we transform the fully separating equilibrium in Proposition 1 into a partially pooling, discontinuous equilibrium. We conjecture the existence of an interval $[a, b]$ such that the following partially pooling strategy is optimal for the manager:

$$\rho_p^*(x) \equiv \begin{cases} b & x \in [a, b] \\ x + \frac{\alpha c}{2\beta} & \text{otherwise} \end{cases} \tag{3}$$

The conjectured strategy $\rho_p^*(x)$ is a simple modification of the fully revealing equilibrium strategy $\rho_s^*(x)$. For high or low values of earnings outside the interval $[a, b]$ the manager sticks to the same strategy as in Proposition 1: $\rho_p^*(x) = \rho_s^*(x) = x + \frac{\alpha c}{2\beta}$.

[7] The empirical evidence on whether investors are actually fooled by earnings management is mixed. Rangan (1998) and Teoh, Welch and Wong (1998) claim that managers succeed in fooling investors by manipulating reports. On the contrary, Shivakumar (2000) concludes that investors are not misled and account correctly for the manipulative behavior of managers.

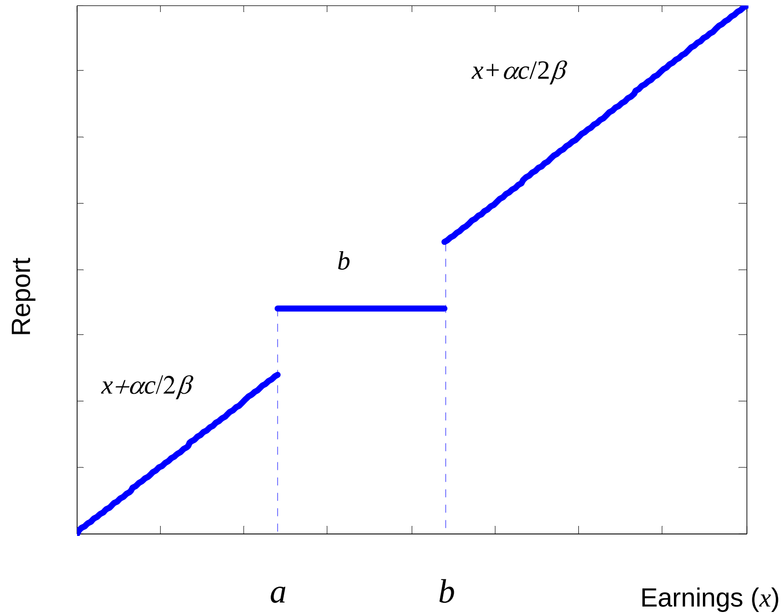


Figure 1: The Partially Pooling Equilibrium Reporting Strategy

He does not report truthfully (this is suboptimal), but does reveal his true type. For the intermediate values that fall inside the interval $[a, b]$, the manager always reports $b$, the upper bound of the interval. Figure 1 depicts this partially pooling strategy.

We would like to show that such a partially pooling equilibrium exists in our model. The first step in proving existence is to find necessary conditions for $\rho_p^*(\cdot)$ to be an equilibrium. First, note that if $\rho_p^*(\cdot)$ is an equilibrium then the type is fully revealed for all reports $x^R < a + \frac{\alpha c}{2\beta}$, or $x^R > b + \frac{\alpha c}{2\beta}$. In contrast, when the investors observe a report of $b$, they can only deduce that the type is somewhere in the interval $[a, b]$. Using Bayes rule, it follows that conditional on a report of $b$, the posterior beliefs of the investors are distributed according to a truncated normal distribution on $[a, b]$.[8] Thus, the first necessary condition for a partially pooling equilibrium is that the pricing function of the investors must satisfy

$$P_p^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a + \frac{\alpha c}{2\beta} \;\; or \;\; x^R > b + \frac{\alpha c}{2\beta} \\ cd(a, b) & x^R = b, \end{cases} \qquad (4)$$

[8] This means that for all $s \in [a, b]$, $\Pr(\tilde{x} \leq s | \pi^R = b) = \Pr(\tilde{x} \leq s | \tilde{x} \in [a, b]) = \frac{F(s) - F(a)}{F(b) - F(a)}$.

where $d(a,b) \equiv E(\tilde{x}|\tilde{x} \in [a,b])$ is the mean of true earnings conditional on the information that they are in $[a,b]$. Next, notice that in order for $\rho_p^*(\cdot)$ to be an equilibrium, a manager with type $x = a$ must be indifferent between reporting $a + \frac{\alpha c}{2\beta}$ and reporting $b$. Similarly, the manager with type $x = b$ must be indifferent between reporting $b + \frac{\alpha c}{2\beta}$ and reporting $b$. Evaluating the manager's utility given by (2) at these points, and using (4) we obtain a system of equations:

$$\begin{aligned} \alpha ca - \beta(\frac{\alpha c}{2\beta})^2 &= \alpha cd(a,b) - \beta(b-a)^2 \\ \alpha cb - \beta(\frac{\alpha c}{2\beta})^2 &= \alpha cd(a,b). \end{aligned} \tag{5}$$

Solving it yields

$$b = a + \frac{\alpha c}{\beta}, \tag{6}$$

and

$$d(a,b) \equiv E(\tilde{x}|\tilde{x} \in [a,b]) = a + \frac{3\alpha c}{4\beta}. \tag{7}$$

The intuition behind these necessary conditions is as follows. Both the '$a$' and the '$b$' types must be indifferent between the two alternatives they face. While the '$a$' type increases his earnings manipulation from $\frac{\alpha c}{2\beta}$ to $\frac{\alpha c}{\beta}$ (he reports $b = a + \frac{\alpha c}{\beta}$ instead of $a + \frac{\alpha c}{2\beta}$), the '$b$' type gains by reducing his earnings manipulation costs to zero (he reports truthfully, instead of reporting $b + \frac{\alpha c}{2\beta}$). Thus, the increase in earnings manipulation by the '$a$' type is exactly identical to the decrease in earnings manipulation by the '$b$' type. However, because the manipulation cost is convex, the compensation in terms of price required by the '$a$' type exceeds the price concession made by the '$b$' type. Thus, $cd(a,b)$, the price given a report of $b$, must be strictly higher than $c$ times the midpoint of the interval $[a,b]$, namely $c\frac{a+b}{2}$. This implies that $d(a,b)$ lies to the right of the midpoint of the interval $[a,b]$. Given the quadratic cost function assumption, this conditional expectation lies exactly three quarters of the way between $a$ and $b$ (see (7)).

The pricing function in (4) is formed using Bayes rule given the conjectured equilibrium $\rho_p^*(\cdot)$. However, there are potential reports that never appear in this equilibrium. Specifically, no type will ever publish a report that lies in $[a+\frac{\alpha c}{2\beta}, b) \cup (b, b+\frac{\alpha c}{2\beta}]$. To complete the picture, and fully delineate the partially pooling equilibrium, we have to specify the out-of-equilibrium pricing. Since Bayes rule does not apply we have some leeway in this choice. Actually, there exists a continuum of out-of-equilibrium

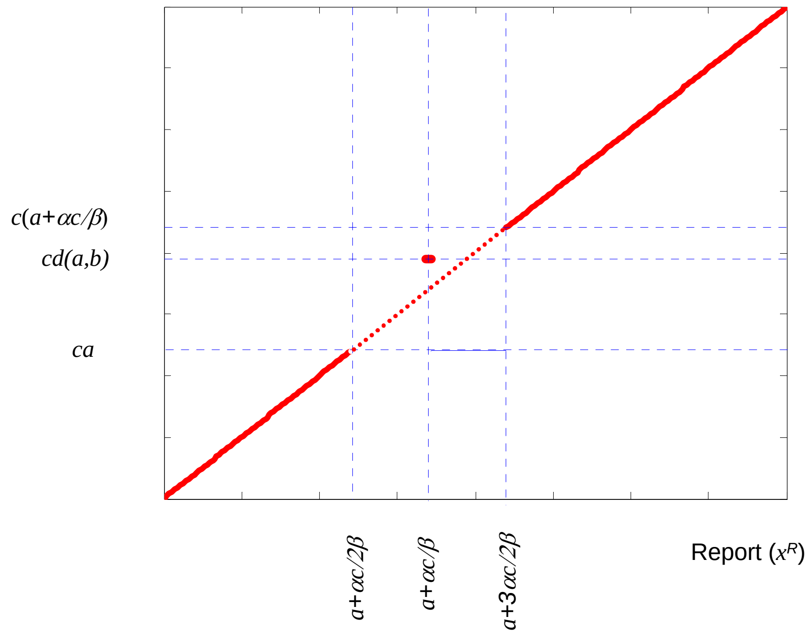


Figure 2: The Pricing Function in the Partially Pooling Equilibrium

pricing functions that support this equilibrium. By way of introduction we use the following: for all reports $x^R \in [a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta}]$, the price is $P(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$. Thus, if investors observe an unexpected report, they conclude that the manager is "mistakenly" playing the benchmark separating equilibrium $\rho_s^*(\cdot)$. These out-of-equilibrium beliefs are used here for simplicity. In Section 6 we fully characterize the set of out of equilibrium pricing functions that support this equilibrium. Figure 2 depicts the pricing function of the investors in the partially pooling equilibrium. The dotted line describes the out-of-equilibrium pricing, while the bold dot is $cd(a, b)$: the price conditional on observing a report of $b = a + \frac{\alpha c}{\beta}$.

In the next proposition we prove the existence of the conjectured partially pooling equilibrium.

**Proposition 2** *There exists a unique interval* $[a, b]$ *such that the reporting strategy*

$$\rho_p^*(x) \equiv \begin{cases} b & x \in [a, b] \\ x + \frac{\alpha c}{2\beta} & \textit{otherwise} \end{cases}$$

*joint with the pricing function*

$$P_p^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a + \frac{\alpha c}{2\beta} \;\; or \;\; x^R > b + \frac{\alpha c}{2\beta} \\ cd(a,b) = c\left(a + \frac{3\alpha c}{4\beta}\right) & x^R = b \\ c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R \in [a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta}] \end{cases}$$

*constitute a Perfect Bayesian Equilibrium.*

**Proof**: In the Appendix we prove that there exists a unique interval $[a, b]$ such that the necessary conditions (6) and (7) are satisfied. In particular, this interval makes the types $x = a$ and $x = b$ indifferent between reporting $b$ and reporting $x + \frac{\alpha c}{2\beta}$. We claim that $\rho_p^*(\cdot)$ applied to this interval is an equilibrium strategy. The pricing function on the equilibrium path satisfies Bayes rule, given the manager's strategy by construction. Since $\rho_s^*(\cdot)$ is an equilibrium, and $\rho_p^*(\cdot)$ differs from $\rho_s^*(\cdot)$ only on $[a, b]$, we only have to rule out deviations to and from the pooling interval $[a, b]$.

Consider first a type $\hat{x} \in (a, b)$. The conjectured equilibrium strategy $\rho_p^*(\cdot)$ specifies that he should report $b$. Since the out-of-equilibrium pricing is identical to the pricing given $\rho_s^*(\cdot)$, Proposition 1 implies that his best possible deviation is to report $\rho_s^*(\hat{x}) = \hat{x} + \frac{\alpha c}{2\beta}$. However, using the facts that $d(a, b) = a + \frac{3\alpha c}{4\beta}$, and $b = a + \frac{\alpha c}{\beta}$ we obtain

$$\begin{aligned} U^M(\hat{x}, \rho_p^*(\hat{x})) - U^M(\hat{x}, \rho_s^*(\hat{x})) &= U^M(\hat{x}, b) - U^M(\hat{x}, \hat{x} + \frac{\alpha c}{2\beta}) \\ &= \alpha c d(a, b) - \beta(b - \hat{x})^2 - [\alpha c \hat{x} - \beta(\frac{\alpha c}{2\beta})^2] \\ &= \beta(\hat{x} - a)(b - \hat{x}) > 0, \end{aligned} \tag{8}$$

where the inequality follows since $\hat{x} \in (a, b)$. Therefore, type $\hat{x}$ is better off reporting $b$ as required.

Consider now a type $\hat{x} \notin [a, b]$. According to $\rho_p^*(\cdot)$ he should report $\hat{x} + \frac{\alpha c}{2\beta}$. Since the out-of-equilibrium pricing is identical to the pricing function given $\rho_s^*(\cdot)$, Proposition 1 implies that this type would not deviate to any report in $[a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta}]$. Thus, his only potential beneficial deviation is to report $b$. However, using the facts that $d(a,b) = a + \frac{3\alpha c}{4\beta}$, and $b = a + \frac{\alpha c}{\beta}$ we obtain:

$$\begin{aligned} U^M(\hat{x}, \hat{x} + \frac{\alpha c}{2\beta}) - U^M(\hat{x}, b) &= \alpha c \hat{x} - \beta(\frac{\alpha c}{2\beta})^2 - \alpha c d(a,b) + \beta(\hat{x} - b)^2 \\ &= \alpha c \hat{x} - \beta(\frac{\alpha c}{2\beta})^2 - \alpha c(a + \frac{3\alpha c}{4\beta}) + \beta(\hat{x} - a - \frac{\alpha c}{\beta})^2 \\ &= \beta(\hat{x} - a)(\hat{x} - b) > 0, \end{aligned}$$

where the inequality follows since $\hat{x} \notin [a, b]$. Therefore, no deviation is beneficial. ■

It is important to note that for any given set of parameter values our model does not create a "dent" or a "kink" in the distribution of reports, rather it creates a "hole". No manager will ever provide a report in the range $[a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta}]$ in equilibrium. This, however, can be reconciled with the observed empirical results: in a cross-sectional empirical implementation of our equilibrium, even if there were holes in the distribution of earnings reports for a given firm, those holes would not be likely to occur at exactly the same spot across firms. Therefore, in the data we should observe a drop in the frequency of certain reports rather than a complete elimination of reports in an interval. To demonstrate this, Figure 3 presents the results of 10,000 rounds of simulation based on the partially pooling equilibrium. In order to take into account the variation in distribution parameters across firms we perturb both the mean and the variance in each simulation round as follows. The mean $x_0$ is drawn from a normal distribution with mean 0 and standard deviation 0.1, and the standard deviation is drawn from a normal distribution (truncated at 0) with mean 0.7 and standard deviation 0.1. We use $\frac{\alpha c}{\beta} = 1.5$ (constant across firms). For each simulation round we calculate the pooling interval $[a, b]$ by numerically solving the set of equations (6) and (7). From Proposition 2 we know that this set of equations has a unique solution. Figure 3 depicts the distribution of *earnings reports* resulting from the simulation. The kink in the distribution of reports is a direct consequence of the pool located just below the mean of the distribution.

# 4 Properties of the Partially Pooling Equilibrium

In this section we present several properties of the partially pooling equilibrium. We start by studying the size and the location of the pooling interval relative to the

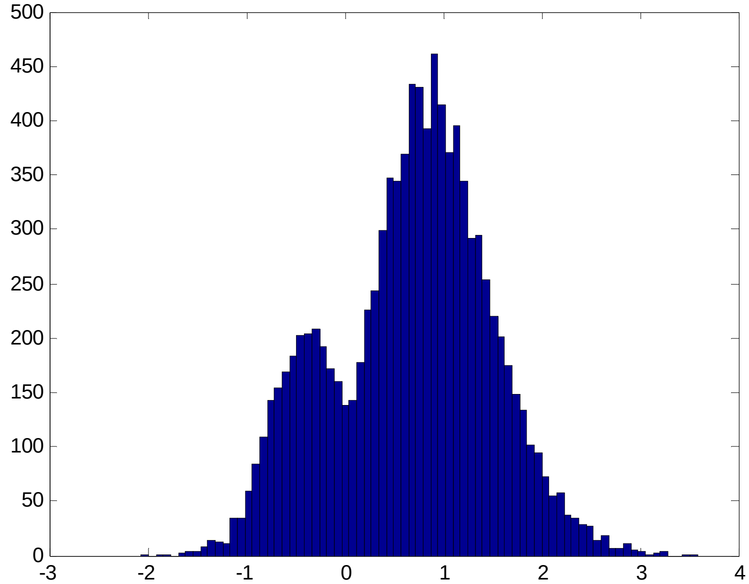


Figure 3: A Simulation of the Partially Pooling Equilibrium

earnings distribution. Then, we study the probability of pooling, which is a measure of how pronounced is the kink in an empirical study. Next, we use these results to suggest new empirical implications. Finally, we conduct a welfare analysis comparing the separating and the partially pooling equilibria.

## 4.1 Size and Location of the Pooling Interval

In our model, both the size and the location of the pool relative to the earnings distribution are determined endogenously.[9] The following corollary is a direct consequence of Proposition 2. It shows how the size of the pooling interval depends on the exogenous parameters. It also shows that the pooling interval lies either entirely or mostly to the left of the unconditional mean of the true earnings $x_0$.

**Corollary 1** *The following holds:*

1. *The size of the pooling interval is:* $b - a = \frac{\alpha c}{\beta}$.

[9] Notice that while our model determines the location of the pool relative to the distribution it does not determine the absolute location of the pool. Namely, a shift in $x_0$ induces an identical shift in the location of the pooling interval $[a, b]$. Thus, the model can support kinks at different points such as 0 and last year's earnings. In Section 4.3 we show how this property of the model can be utilized for empirical purposes.

2. *The lower bound of the pooling interval, a, is always to the left of the mean of true earnings* $x_0$. *Moreover,* $a < x_0 - \frac{3\alpha c}{4\beta}$.

3. *The upper bound of the pooling interval, b, can be to the left or to the right of the the unconditional mean of true earnings* $x_0$, *but is always smaller that* $x_0 + \frac{\alpha c}{4\beta}$.

4. *The conditional mean satisfies:* $d(a, b) = a + \frac{3\alpha c}{4\beta} < x_0$.

**Proof:** The facts that $b - a = \frac{\alpha c}{\beta}$ and $d(a, b) = a + \frac{3\alpha c}{4\beta}$ is a restatement of (6) and (7). To see that $d(a, b) < x_0$ note that given that $d(a, b) = a + \frac{3\alpha c}{4\beta}$, it must be that the distribution mass on the right hand side of the pooling interval outweighs the distribution mass on the left hand side of the interval. Under the normal distribution (as well as under any unimodal symmetric distribution) this is possible only if the conditional mean lays strictly to the left of the unconditional mean. This together with (6) and (7) implies that $a < x_0 - \frac{3\alpha c}{4\beta}$ and $b < x_0 + \frac{\alpha c}{4\beta}$. ■

Figure 4 illustrates this result. It shows the pooling interval compared to the distribution of earnings (not the reports). The figure demonstrates a case where the pooling interval straddles the unconditional mean. The pooling interval can also lie entirely to the left of $x_0$ depending on parameter values. Due to the pooling, the distribution of reported earnings is not normal, and it is shifted to the right compared to the earnings distribution. The shift is equal to $\frac{\alpha}{2\beta}$ outside of the pooling interval, and is smaller that $\frac{a}{2\beta}$ in the pooling interval. For this reason, the pooling interval is located to the left of the mean of the distribution of *reported earnings*. Figure 3 demonstrates this.

From Corollary 1 we conclude that the size of the pooling interval is increasing in the amount of stock based incentives given to the manager, and in the P/E ratio, whereas it is decreasing in the quality of enforcement of the accounting standards. On the other hand, the size of the pooling interval is independent of the volatility of earnings.

In the separating equilibrium, a change in the volatility of earnings has no effect on the equilibrium. In particular, the earnings management is equal to $\frac{\alpha c}{2\beta}$ regardless of $\sigma$. In contrast, in the partially pooling equilibrium, a change in $\sigma$ moves the pooling interval $[a, b]$ and hence affects the probability of pooling. In order to study the impact of $\sigma$ on the location of the pooling interval relative to the earnings distribution we

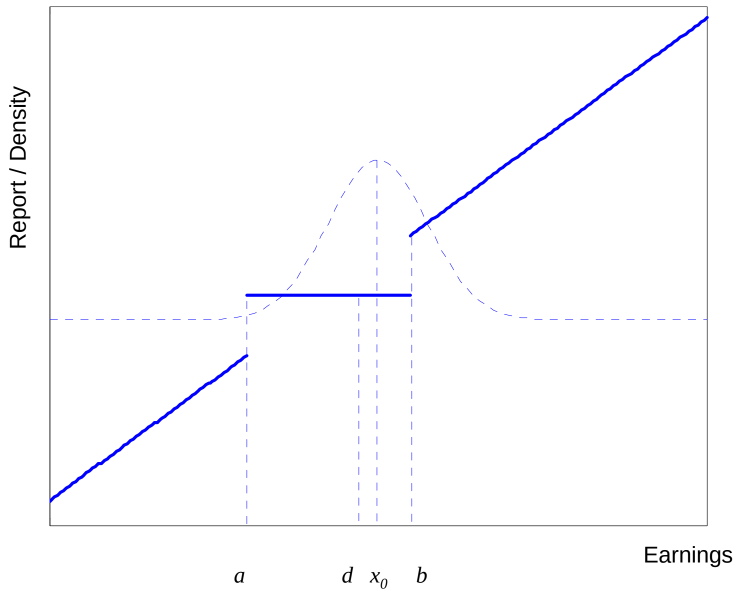


Figure 4: The Partially Poolling Equilibrium and the Underlying Distribution

consider $a$, the left hand side of the interval, as a function of $\sigma$: $a(\sigma)$. The location effect of $\sigma$ is studied in the next proposition.

**Proposition 3** *A decrease in the underlying volatility moves the pooling interval to the right:* $\frac{\partial a(\sigma)}{\partial \sigma} < 0$*. Moreover,* $\lim_{\sigma \to 0} a(\sigma) = x_0 - \frac{3\alpha c}{4\beta}$*, and* $\lim_{\sigma \to \infty} a(\sigma) = -\infty$*.*

**Proof**: In the Appendix.

Proposition 3 shows that the pooling interval's location relative to the unconditional mean of true earnings is sensitive to the underlying volatility of earnings. Higher $\sigma$ moves the pooling interval to the far left tails of the distribution. On the other hand, as $\sigma$ declines, the pooling interval moves to the right, eventually straddling the unconditional mean. In the limit, as $\sigma$ tends to 0, the unconditional mean $x_0$ and the conditional mean $d(a, b) = a + \frac{3\alpha c}{4\beta}$ coincide. To demonstrate this result, Figure 5 plots the effect of a change in $\sigma$ on the location of the pooling interval. We use the following parameter values: $\frac{\alpha c}{\beta} = 1$, $x_0 = 0$, and let $\sigma$ vary between 0.1 and 1.5. As the volatility of earnings becomes larger, the pooling interval moves to the left, but its size is unchanged. As $\sigma$ tends to 0 the pooling interval straddles the undconditional mean $x_0$, the conditional mean $d(a, b)$ tends to 0, and the pooling

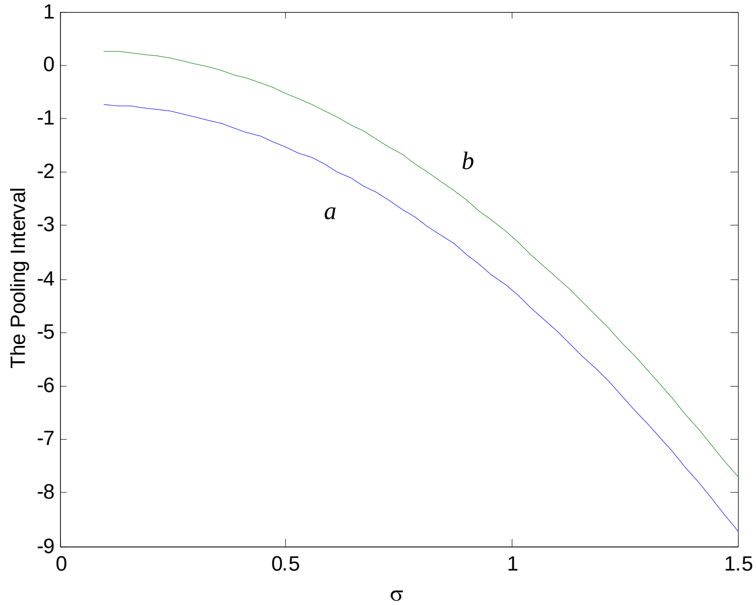


Figure 5: The Effect of $\sigma$ on the Location of the Pooling Interval

interval tends to $[-0.75, 0.25]$ as expected.

The incentive parameters $\alpha$, $\beta$, and $c$ enter the model only as a ratio $\frac{\alpha c}{\beta}$. This ratio affects the location of the pooling interval. Figure 6 demonstrates this effect. It uses the following parameter values: $x_0 = 0$, $\sigma = 0.7$, and $\frac{\alpha c}{\beta}$ varies between 0.5 and 3. Recall (Corollary 1) that the vertical difference between the two curves in Figure 6 is $b - a = \frac{\alpha c}{\beta}$. The figure shows that as the value of $\frac{\alpha c}{\beta}$ increases, $b$ increases, while $a$ increases for low values of $\frac{\alpha c}{\beta}$ and decreases for high values. Extensive numerical calculations show that this behavior is representative; however, we are not able to prove it analytically for all values of $\frac{\alpha c}{\beta}$. Intuitively, when the pooling interval straddles the unconditional mean $x_0$ (this is what happens when $\frac{\alpha c}{\beta}$ is large) an increase in $\frac{\alpha c}{\beta}$ has a small effect on the conditional mean $d(a, b)$, which is approximately equal to $x_0$ regardless of $\frac{\alpha c}{\beta}$. Thus, an increase in $\frac{\alpha c}{\beta}$ of one unit, is translated to an increase in $b$ of one quarter of this unit and a decrease in $a$ of three quarter of a unit. This is the only way to maintain $d(a, b)$ at three quarters of the way between $a$ and $b$ without moving it. For this reason, Figure 6 shows an increase in $b$ and a decrease of $a$ for high values of $\frac{\alpha c}{\beta}$. In contrast, when $\frac{\alpha c}{\beta}$ is small, the pooling interval lies on the far left of the distribution, and a one unit increase in $\frac{\alpha c}{\beta}$ is translated to approximately

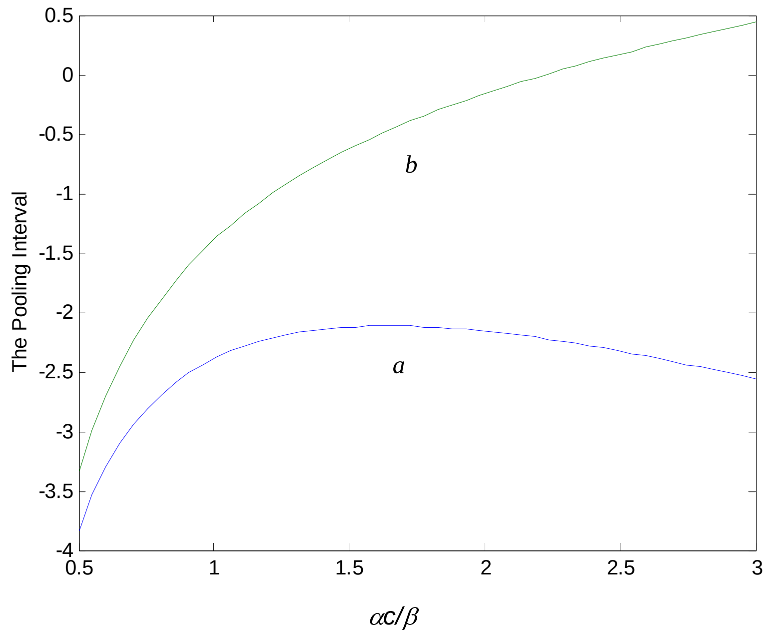


Figure 6: The Effect of $\frac{\alpha c}{\beta}$ on the Location of the pooling interval

one unit of increase in both $a$ and $b$.

## 4.2 Probability of Pooling

We have shown that the pooling interval always exists. However, the probability of pooling may still be very low, and in some occasions this will preclude the identification of the kink in empirical studies. For this reason, it is important to understand the effect of the exogenous parameters on the probability of pooling.

While we have established the impact of the change in $\sigma$ on the location of the pooling interval, it is not sufficient in order to establish that the probability of observing the pooling behavior declines in $\sigma$. This probability is given by $F_\sigma(b(\sigma)) - F_\sigma(a(\sigma))$. The subscript $\sigma$ reminds us that a change in $\sigma$ affects the probability of pooling in two ways: first, it affects the location of $a(\sigma)$ as described in Proposition 3. But it also affects the probability distribution itself. A higher variance puts more mass in the tails of the distribution. The two effects act in opposite directions, thus the overall effect is ambiguous. We have not been able to derive the net effect analytically, but in all of our extensive numerical calculations the location effect

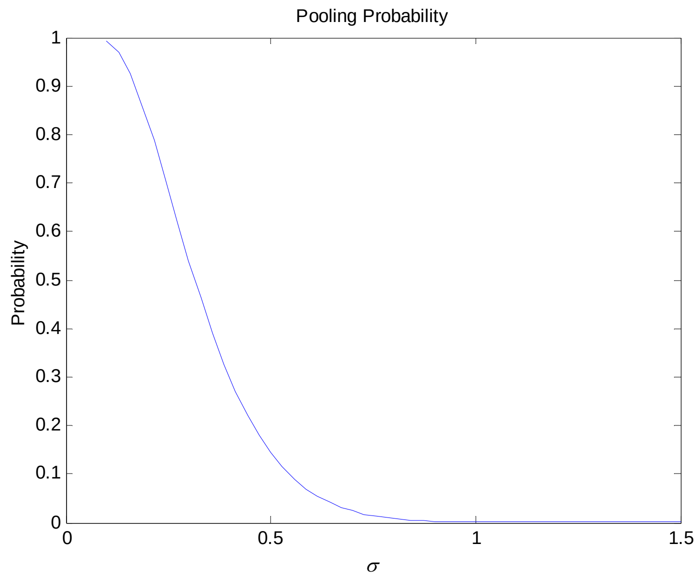


Figure 7: Earnings Volatility and the Probability of the Pooling Interval

dominates: the probability of observing the pooling behavior, $F_\sigma(b(\sigma)) - F_\sigma(a(\sigma))$, declines in $\sigma$. Figure 7 depicts the effect of a change in earnings volatility on the probability of pooling. The parameter values are: $\frac{\alpha c}{\beta} = 1$, $x_0 = 0$, and we let $\sigma$ vary between 0.1 and 1.5. As $\sigma$ becomes higher the probability of pooling declines, since the pooling interval lies in less relevant parts of the distribution.

The incentive parameters $\alpha$ and $\beta$, and the P/E ratio, $c$, enter the model only as a ratio $\frac{\alpha c}{\beta}$. A higher $\frac{\alpha c}{\beta}$ has two effects on the probability of pooling: (i) it induces a larger pooling interval, which tends to increase the probability of pooling; (ii) it moves the position of the pooling interval as demonstrated previously in Figure 6. The net effect of these two is hard to sign analytically. However, extensive numerical calculations show that an increase in $\frac{\alpha c}{\beta}$ *always* increases the probability of pooling. Namely, an increase in the size of the pooling interval is associated with an increased probability of pooling despite the change in the location of the pool. This is demonstrated in Figure 8. We use $x_0 = 0$, and $\sigma = 0.7$ and let $\frac{\alpha c}{\beta}$ vary between 0.5 and 3. The pooling probability is close to 0 when the ratio $\frac{\alpha c}{\beta}$ is small, making the partially pooling equilibrium essentially identical to the separating one. As $\frac{\alpha c}{\beta}$ increases, the probability of pooling increases monotonically.

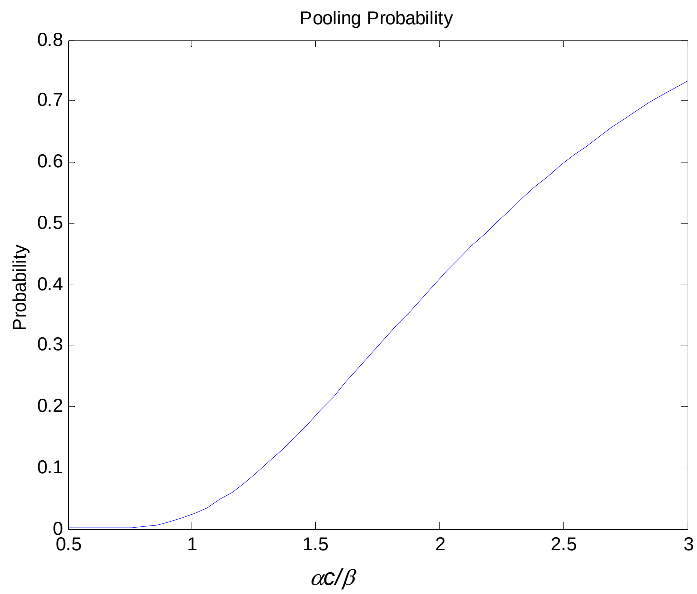


Figure 8: The Effect of a Change in $\frac{\alpha c}{\beta}$ on the Probability of Pooling

### 4.3 Empirical Implications

Our model can be used to deepen the understanding of the discontinuity of earnings reports phenomenon. A first empirical application of the model is to provide a theory regarding the size of the pool and the probability of pooling. Indeed, the above comparative statics suggest that:

- Controlling for the incentive schemes, we expect to observe more pronounced kinks in firms and industries with lower volatility of earnings.
- Controlling for the volatility of earnings, stronger stock-based incentives imply a larger degree of pooling behavior (a larger and more pronounced kink). Firms with a higher proportion of managerial ownership should exhibit this effect. Reliance on stock based compensation differs across countries and industries; and in the US the use of stock-based compensation has been rapidly increasing over time. Cross-sectional variations in the P/E ratio generate the same predictions as variations in $\alpha$.
- The empirical interpretation of $\beta$ is not straightforward since it involves both the enforcement of accounting standards and also to some extent the accounting

standards themselves. La Porta et al. (1998) suggest such a combined measure of "legal enforcement" that takes into account both the level of enforcement and the accounting standards. Leuz, Nanda and Wysocki (2003) use the La Porta et al. measure as one of their explanatory variables to study earnings management around the world. They show that legal enforcement is negatively correlated with earnings management. Our model suggests that a similar measure of legal enforcement should be negatively correlated with the size and significance of the kink.

A second empirical implication of our model touches the location of the kink at well documented positions such as 0 (loss avoidance), and last year's earnings (earnings decrease avoidance). Specifically, our model suggests that the pool cannot be located to the right of the mean of the distribution of earnings reports, and will not be pronounced empirically if it is located to the far left of the mean. For concreteness, consider the case of a kink around last year's earnings (earnings decrease avoidance). The above rationale leads us to argue that mangers will manipulate earnings to avoid earnings decrease only if market expectations are that the current year's earnings will be close to last year's earnings.

This prediction allows us to better understand and further investigate the results of Burgstahler and Dichev (1997). It implies that the kink in the distribution of reports close to last year's earnings comes primarily from those firms that were expected by the market to report last year's earnings (or close). Thus, our model suggests the following refinement of the Burgstahler and Dichev results.

1. Divide the Burgstahler and Dichev sample into three sub-samples. In the first sub-sample collect only the observations in which the expected earnings are close to last year's earnings. This can be done by using analysts' expectations as a proxy for investors expectations. In the second sub-sample collect the observations in which the expected earnings (analysts' consensus) is much higher than last year's earnings. In the third sub-sample collect the observations in which the expected earnings are much lower than last year's earnings.

2. Our model suggests that the first sub-sample will exhibit a strong kink near last year's earnings stemming from the market expectations. Our model also suggests that the second sub-sample will exhibit a weak kink close to last year's earnings, while the third sub-sample should exhibit no kink.

A similar argument can be made regarding the kink at 0 (earnings decrease avoidance). Essentially, this argument suggests that analysts' expectations being a proxy for investors' expectations are a main source of the kink in the distribution in earnings reports. If these expectations happen to coincide with 0 or with last year's earnings then we will observe a pronounced kink at these points too. If expectations fall far from 0 and far from last year's earnings, then either the kink will be less pronounced or there will not be a kink at all depending on the location of 0 (or last year's earnings) relative to the expectations.

### 4.4 Welfare Analysis

Earnings management is inevitable in equilibrium since truth telling is not an equilibrium. However, the pooling behavior allows the manager to manage earnings less on average compared to the separating equilibrium.

**Proposition 4** *The expected earnings management in the partially pooling equilibrium is lower than the expected earnings management in the separating equilibrium.*

**Proof:** The expected earnings management in the separating equilibrium is $\frac{\alpha c}{2\beta}$. The expected earnings management in the pooling equilibrium is:

$$
\begin{aligned}
& \int_{-\infty}^{a} \frac{\alpha c}{2\beta} f(x)dx + \int_{a}^{b} (b-x) f(x)dx + \int_{b}^{\infty} \frac{\alpha c}{2\beta} f(x)dx \qquad (9) \\
= & \frac{\alpha c}{2\beta} - \frac{\alpha c}{2\beta}(F(b)-F(a)) + \int_{a}^{b} (b-x) f(x)dx \\
= & \frac{\alpha c}{2\beta} - \frac{\alpha c}{2\beta}(F(b)-F(a)) + b(F(b)-F(a)) - d(a,b)(F(b)-F(a)) \\
= & \frac{\alpha c}{2\beta} - \frac{\alpha c}{4\beta}(F(b)-F(a)),
\end{aligned}
$$

where the second equality follows since by definition: $d(a,b) = \frac{\int_a^b x f(x)dx}{F(b)-F(a)}$, and the last equality follows from the fact that $b - d(a,b) = \frac{\alpha c}{4\beta}$ (Equations (6) and (7)). This implies that the expected earnings management in the pooling equilibrium is strictly lower than in the separating equilibrium. ■

By pooling all types in the interval $[a, b]$, the manager introduces vagueness into his reports. This enables him to lower the extent of earnings management and reap ex-post economic rents compared to the fully separating equilibrium. Indeed, if the real earnings $x$ fall outside of the pooling interval, then both equilibria are identical. However, if $x \in [a, b]$, the pooling report dominates. Formally:

**Lemma 1** *For all* $x \in [a, b]$, $U^M(x, \rho_p^*(x)) > U^M(x, \rho_s^*(x))$.

**Proof**: Follows directly from equation (8). ■

Intuitively, the vagueness in the manager's report has two effects on his utility compared to the fully separating equilibrium. On the one hand, it changes the extent of the earnings management by the manager. On the other hand it affects the pricing of the stock given this manipulation. The net effect is always positive within the pooling interval $[a, b]$ and is zero outside of this interval. If the manager is forced to manipulate more, then he is more than compensated by price increase, while if he manipulates less, then the price decline is not sufficient to offset the reduction in cost.

From the point of view of the uninformed investors, this pooling behavior can be either ex-post beneficial or harmful, compared to the fully separating equilibrium. Specifically, if $x \in [a, d(a, b))$, the investors end up paying $cd(a, b)$ for a stock that is worth less, but for $x \in (d(a, b), b]$ the investors pay $cd(a, b)$ and get a stock that is worth more. The probability mass on the right half of the pooling interval is larger than the probability mass on the left half to such an extent that risk neutral investors are ex-ante indifferent between the two reporting strategies. We obtain

**Proposition 5** *The partially pooling equilibrium ex-ante Pareto dominates the separating equilibrium.*

**Proof**: For the manager: from Lemma 1, we have $U^M(x, \rho_p^*(x)) > U^M(x, \rho_s^*(x))$ for all $x \in [a, b]$. As the probability mass of $[a, b]$ is positive we obtain immediately that: $E_x U^M(x, \rho_p^*(x)) > E_x U^M(x, \rho_s^*(x))$.

For the investors: the investors' payoff is the price they pay less the actual value of the stock: $P(\rho(x)) - cx$. In the fully separating equilibrium we have $P(\rho_s^*(x)) = cx$; hence the payoff is identically 0. In the partially pooling equilibrium we have $P(\rho_p^*(x)) = cE(\tilde{x}|\rho_p^*(x))$. Hence, from an ex-ante point of view, the payoff to the investors is

$$E_x(P(\rho_p^*(x)) - cx) = cE_x(E(\tilde{x}|\rho_p^*(x)) - x) = cE_x E(\tilde{x}|\rho_p^*(x)) - cE_x(x) = 0.$$

Thus, ex-ante the risk neutral investors are indifferent between these two equilibria. ■

A caveat is in place. Our model focuses on just a part of the real life story of stock based compensation - the reporting part. In reality, stock based compensation has a motivating aspect (as captured in standard principal agent frameworks). Thus, an increase in $\alpha$ while increases earnings management, and the extent of pooling in our model, has an additional positive effect of aligning the incentives of managers and investors and inducing a higher realization of true earnings. For this reason, our welfare analysis in this section draws an incomplete picture of the division of surplus between market participants.

Actually, had managers been allowed not to report at no cost, then two new equilibria would emerge in our model.[10] A first equilibrium is a "perfect pool". No manager ever reports, and the price always stays at the unconditional mean. The out of equilibrium beliefs that support this equilibrium are such that if a manager ever reports something, the investors believe that his type is not higher than the mean. Another equilibrium in this case is a "lemons" equilibrium: there exists a threshold type $a \in \mathbb{R}$ such that all the types below the threshold type do not report and hence are pooled by the investors, while all the types above the threshold level follow the separating equilibrium. The out of equilibrium beliefs that support this equilibrium are such that if a manager ever publishes a report lower than $a + \frac{\alpha c}{\beta}$, the investors price him at $cE(\tilde{x}|\tilde{x} < a)$, the conditional mean given that the manager's type is lower than $a$.[11]

Both these equilibria are highly efficient in our framework because a large portion of earnings manipulation costs is saved. Is it wise then to mandate earnings reports? Obviously, earnings reports are an important tool in allowing investors to monitor the performance of managers. As a result, earnings reports seem to be necessary in order to facilitate raising capital by corporations. For this reason, allowing managers not to report seems out of the question in reality, although it saves the costs of earnings distortions demonstrated in our model.

[10] We thank an anonymous referee for suggesting this point.

[11] The proof that this is indeed an equilibrium is similar to the proof of Proposition 2.

# 5 Extensions

In this section we analyze two extensions of our base model. First we consider earnings based compensation and then we study the case of $\alpha < 0$, representing a downward bias by managers.

## 5.1 Earnings-Based vs. Stock-Based Compensation

Managers are frequently paid bonuses based directly on their reported earnings. We would like to know how does this compensation scheme affect the incentive of the managers to manipulate the reported earnings.[12] We assume for simplicity that such bonuses are linear in reported earnings: the manager maximizes the following utility function

$$U^M(x, x^R) = \alpha P(x^R) + \gamma x^R - \beta(x - x^R)^2,$$

where $\gamma > 0$ represents the strength of the earnings-based compensation. The fully revealing equilibrium is similar: the optimal earnings report is

$$x^R = x + \frac{\alpha c + \gamma}{2\beta}.$$

The addition of the earnings-based compensation affects the incentive to inflate earnings, and changes the manipulation cost for the manager.

Using the same reasoning as in our base model we can show that the partially pooling equilibrium is determined by the two equations:

$$\begin{aligned} \alpha ca + \gamma a + \gamma \frac{\alpha c + \gamma}{2\beta} - \beta(\frac{\alpha c + \gamma}{2\beta})^2 &= \alpha cd(a, b) + \gamma b - \beta(b - a)^2 \\ \alpha cb + \gamma b + \gamma \frac{\alpha c + \gamma}{2\beta} - \beta(\frac{\alpha c + \gamma}{2\beta})^2 &= \alpha cd(a, b) + \gamma b \end{aligned}$$

Solving these yields the following generalization to Equations (6) and (7):

$$\begin{aligned} b &= a + \frac{\alpha c + \gamma}{\beta}, \text{ and} \\ d(a, b) &\equiv E(\tilde{x} | \tilde{x} \in [a, b]) = a + \frac{(3\alpha c + \gamma)}{4\alpha c}(b - a). \end{aligned}$$

It is important to note that the pooling equilibrium exists if and only if the conditional expectation $d(a, b)$ is located in the pooling interval (between $a$ and $b$). This implies that

$$\alpha c > \gamma > -3\alpha c. \tag{10}$$

[12] The effect of earnings based compensation on earnings management has been studied empirically in several papers. See for instance Healy (1985).

Condition (10) imposes a bound on the extent of earnings based compensation that is consistent with the pooling equilibrium. To understand this condition suppose first that $\alpha = 0$, namely that compensation depends on reported earnings only. In this case, Condition (10) implies that the pooling equilibrium cannot exist. The reason is that the pooling equilibrium depends on investors beliefs, represented by the pricing function. In equilibrium, investors beliefs must support the strategy of the manager. When the manager's compensation does not depend on the stock price, investors beliefs are of no interest to him, and the first order condition makes the separating equilibrium the only one. In a less polar case, the manager's compensation is based on both earnings reports and stock price. Condition (10) then tells us that in order to get pooling, the extent of earnings based compensation relative to stock based compensation cannot be too extreme. Thus, the manager's compensation must strongly depend on the stock price for him to pool. Incidentally, the reliance on stock-based compensation has been steadily increasing over time (see Jensen and Murphy (1990) and Hall and Liebman (1998)), which adds relevance to the proposed equilibrium.

## 5.2 Downward Bias

So far we have restricted our attention to the case of $\alpha > 0$. If $\alpha < 0$ we obtain symmetric results. The perfectly separating equilibrium in Proposition 1 is unaffected; however, since $\alpha < 0$, managers bias their earnings downwards by a constant $\frac{\alpha c}{2\beta}$. As for the partially pooling equilibrium we obtain the following parallel to Proposition 2:

**Corollary 2** *Suppose $\alpha < 0$. There exists a unique interval $[b, a]$ such that the reporting strategy*

$$\rho_p^*(x) \equiv \begin{cases} b & x \in [b, a] \\ x + \frac{\alpha c}{2\beta} & otherwise \end{cases}$$

*joint with the pricing function*

$$P_p^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R > a + \frac{\alpha c}{2\beta} \ \ or \ \ x^R < b + \frac{\alpha c}{2\beta} \\ cd(a,b) = c\left(a + \frac{3\alpha c}{4\beta}\right) & x^R = b \\ c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R \in [b, a + \frac{\alpha c}{2\beta}) \cup (b + \frac{\alpha c}{2\beta}, b] \end{cases}$$

*is an equilibrium.*

In this case, the price conditional on observing a report of $b$ is higher than the unconditional price, namely, $cd(a,b) > p_0 = cx_0$.

# 6 Robustness

In this section we test the robustness of the partially pooling equilibrium. First, we characterize the set of out of equilibrium pricing functions that support this equilibrium. Then, we demonstrate the existence of other partially pooling equilibria, and show that they all possess the same attributes. This family of equilibria nests our suggested partially pooling equilibrium as a special case.

## 6.1 Out-of-Equilibrium Beliefs

Contrary to the separating equilibrium, the partially pooling, discontinuous equilibrium relies strongly on out of equilibrium pricing. Some reports will never appear in equilibrium. Since Bayes rule does not apply, the modeler has some leeway in prescribing the beliefs associated with these reports.[13] In Section 3 we assumed that if investors observe an out-of-equilibrium report $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$ then they believe that the manager is "mistakenly" playing the benchmark linear equilibrium. These out of equilibrium beliefs, while sufficient to support the equilibrium, are not necessary, namely, they are too strong. Below, we provide a necessary and sufficient condition for out of equilibrium beliefs to support the partially pooling equilibrium. The fundamental idea here is to find the pricing function that will make types '$a$' and '$b$' just indifferent between their equilibrium action of reporting $b$, and providing an out of equilibrium report. It turns out that there exists a unique pricing function of this type.

[13] The game theoretic literature on equilibrium refinements offers a multitude of concepts to limit the freedom of the modeler in choosing "reasonable" out of equilibrium beliefs in signalling games. One prevalent concept is the "Sequential Equilibrium" of Kreps and Wilson (1982). Our equilibrium is trivially sequential because the objective function in (1) satisfies the Spence-Mirrless Single Crossing Property, and hence, the manager never randomizes between two actions. Another prevalent criterion for refinement is the "Intuitive Criterion" of Cho and Kreps (1987). It is straightforward to verify that our partially pooling equilibrium, as specified in Proposition 2 survives this criterion. Actually, the Intuitive Criterion works best in models with just two types of informed parties, therefore it doesn't impose much restriction on out of equilibrium beliefs in our continuous type framework. Other criteria such as the "Divinity Criterion" (Banks and Sobel (1987)) were developed for finite type games, and we find them hard to interpret in our framework.

Our first step is the next lemma showing that a sufficient condition for an out-of-equilibrium pricing function to support our partially pooling equilibrium, is that the types '$a$' and '$b$' are indifferent between following the equilibrium strategy and deviating from it to an out of equilibrium report.

**Lemma 2** *Consider any out-of-equilibrium report $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$ combined with an out-of-equilibrium pricing function $P(x^R)$. The following holds:*

1. *If $x^R \in (a + \frac{\alpha c}{2\beta}, b)$, and if type '$a$' is indifferent between the equilibrium report of $b$, and the out-of-equilibrium report of $x^R$, then all other types $x' \neq a$ strictly prefer the equilibrium report $b$ over the out-of-equilibrium report $x^R$.*

2. *If $x^R \in (b, b + \frac{\alpha c}{2\beta})$, and if type '$b$' is indifferent between the equilibrium report of $b$, and the out-of-equilibrium report of $x^R$, then all other types $x' \neq b$ strictly prefer the equilibrium report $b$ over the out-of-equilibrium report $x^R$.*

**Proof:** In the Appendix.

Based on Lemma 2, the partially pooling equilibrium strategy $\rho_p^*(\cdot)$ is said to be supported by a *tight* pricing function $P(x^R)$, if for all $x^R \in (a + \frac{\alpha c}{2\beta}, b)$, type '$a$' is indifferent between the equilibrium strategy and deviating to $x^R$, and for all $x^R \in (b, b + \frac{\alpha c}{2\beta})$, type '$b$' is indifferent between following the equilibrium strategy and deviating to $x^R$. The next proposition shows that there exists a unique tight pricing function. Moreover, a necessary and sufficient condition for any pricing function to support the partially pooling equilibrium is that the out of equilibrium pricing function will lie weakly below the tight pricing function. To see this intuitively, consider Figure 9. The out of equilibrium pricing function used in Proposition 2 is represented in this figure by the dotted straight line connecting points A and B, except for a report of $b$ where the price is $cd(a, b)$. The tight pricing function is the dotted curve ADB. The original pricing function lies below the tight one, meaning that under the original out of equilibrium pricing, the investors "punish" the manager more severely then necessary in order to maintain this equilibrium. In general, any out of equilibrium pricing that lies below the curve ADB will support the partially pooling equilibrium. Thus, the tight pricing function is the least restrictive one that still supports this equilibrium. Formally,

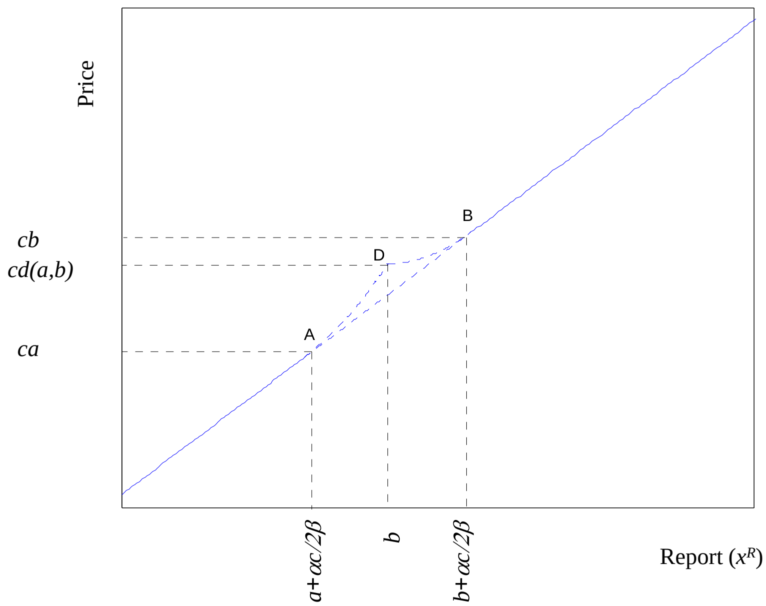


Figure 9: The Tight Pricing Function

**Proposition 6** *There exists a unique tight pricing function that supports the partially pooling strategy $\rho_p^*(\cdot)$. This pricing function is given by*

$$P_t^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a + \frac{\alpha c}{2\beta} \text{ or } x^R > b + \frac{\alpha c}{2\beta} \\ c \cdot d(a,b) & x^R = b \\ c\left(a - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}(x^R - a)^2\right) & x^R \in [a + \frac{\alpha c}{2\beta}, b) \\ c\left(b - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}(x^R - b)^2\right) & x^R \in (b, b + \frac{\alpha c}{2\beta}] \end{cases}.$$

*Moreover, a necessary and sufficient condition for any pricing function $P^*(x^R)$ to support the partially pooling equilibrium is that: $P^*(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$ if $x^R < a + \frac{\alpha c}{2\beta}$ or $x^R > b + \frac{\alpha c}{2\beta}$, $P^*(x^R) = c \cdot d(a,b)$ if $x^R = b$, and $P^*(x^R) \leq P_t^*(x^R)$ for all $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$.*

**Proof:** In the Appendix.

## 6.2 Other Partially Pooling Equilibria

In Section 3 we have shown the existence of a partially pooling equilibrium, in which the manager reports the upper bound of the pooling interval. That equilibrium, however, is not the only partially pooling equilibrium in this model. Actually, there exists a continuum of similar partially pooling equilibria. Equilibria in this family differ by the pooling report the manager makes when the realized earnings are in the interval $[a, b]$. While in our original equilibrium all managers with earnings in $[a, b]$ report the upper bound $b$, we can create other equilibria in which managers report $b + \eta$, for all $\eta \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$. Our original equilibrium corresponds to the case $\eta = 0$. This is demonstrated in the next proposition.

**Proposition 7** *For all $\eta \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$, there exists a unique interval $[a_\eta, b_\eta]$ such that the reporting strategy*

$$\rho_p^\eta(x) \equiv \begin{cases} b_\eta + \eta & x \in [a_\eta, b_\eta] \\ x + \frac{\alpha c}{2\beta} & \textit{otherwise} \end{cases},$$

*joint with the pricing function*

$$P^\eta(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a_\eta + \frac{\alpha c}{2\beta} \;\; \textit{or} \;\; x^R > b_\eta + \frac{\alpha c}{2\beta} \\ c\left(b_\eta - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}\eta^2\right) & x^R = b_\eta + \eta \\ c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R \in [a_\eta + \frac{\alpha c}{2\beta}, b_\eta + \eta) \cup (b_\eta + \eta, b_\eta + \frac{\alpha c}{2\beta}] \end{cases},$$

*constitute a Perfect Bayesian Equilibrium.*

**Proof:** Similar to the proofs of Propositions 2. ■

The family of equilibria described above nests our original partially pooling equilibrium as a special case. We have chosen to focus on this equilibrium because it is the simplest one analytically, and completely represents the family. All the welfare implications, comparative statics, and empirical predictions carry on to the entire family of equilibria. The choice of $\eta$ just changes the length, and position of the pooling interval.

# 7 Conclusion

Managers have some leeway in the way they report earnings, however, earnings management is not costless for them. Managers' compensation is based on the stock price, which depends on investors' beliefs about the true earnings. The result is that managers can and do manipulate earnings, and investors take this into account when pricing the stock. Given this, we show that a discontinuity in the distribution of earnings reports may emerge even when both the distribution of true earnings and the compensation to the managers are smooth.

Our model generates new empirical predictions that stem from the self-fulfilling expectations nature of the equilibrium. These predictions relate the size of the kink, its location relative to the mean of the earnings report distribution, and its empirical pronouncement to the extent of stock based compensation and to the volatility of earnings. We also offer ways to refine and better understand our current knowledge of the kink phenomenon. These predictions call for future empirical scrutiny of our results.

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# Appendix

Proof of Proposition 1

Let $\rho_s(\cdot)$ be a perfectly separating, continuously differentiable reporting strategy. Since $\rho_s(\cdot)$ is perfectly separating it can be inverted; thus, let $\varphi_s = \rho_s^{-1}$. It follows that the only pricing function consistent with $\rho_s(\cdot)$ is given by $P_s(\cdot) = c\varphi_s(\cdot)$. The utility of the manager given earnings of $x$ and a report of $x^R$ is given by

$$U^M(x, x^R) = \alpha c\varphi_s(x^R) - \beta(x^R - x)^2. \tag{11}$$

The first order condition of (11) with respect to $x^R$ is

$$\frac{d}{dx^R}\varphi_s(x^R) - \frac{2\beta}{\alpha c}x^R + \frac{2\beta}{\alpha c}x = 0.$$

Since in equilibrium $x = \varphi_s(x^R)$, we obtain the following linear, first-order differential equation for an equilibrium

$$\frac{d}{dx^R}\varphi_s(x^R) = -\frac{2\beta}{\alpha c}\varphi_s(x^R) + \frac{2\beta}{\alpha c}x^R.$$

All potential solutions of this equation are given by

$$\varphi_s(x^R) = x^R - \frac{\alpha c}{2\beta} + Ke^{-\frac{2x^R\beta}{c\alpha}},$$

where $K$ is a constant. We claim that $K = 0$. Indeed, suppose on the contrary that $K > 0$, then a simple calculation shows that $\varphi_s(x^R)$ is strictly convex and has a unique minimum at $x^R = -\frac{c\alpha}{2\beta}\ln\frac{c\alpha}{2k\beta}$. This implies that $\varphi_s(x^R)$ is bounded from below, contrary to the fact that $x$ can take any value in $\mathbb{R}$ (it is drawn from a normal distribution). Similarly, we can rule out the case $K < 0$. Therefore, we obtain $\varphi_s(x^R) = x^R - \frac{\alpha c}{2\beta}$, $P_s(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$ and $\rho_s(x) = x + \frac{\alpha c}{2\beta}$, as required. ■

Proof of Proposition 2

It is sufficient to show that there exists a unique $a \in \mathbb{R}$, such that $E(\tilde{x}|\tilde{x} \in [a, a + \frac{\alpha c}{\beta}]) = a + \frac{3\alpha c}{4\beta}$. As $b = a + \frac{\alpha c}{\beta}$ we shall denote the conditional expectation by $d(a) = E(\tilde{x}|\tilde{x} \in [a, a + \frac{\alpha c}{\beta}])$ instead of $d(a, b)$. Thus, we will show that there exists a unique $a \in \mathbb{R}$, such that $d(a) = a + \frac{3\alpha c}{4\beta}$. The conditional expectation $d(a)$ is the expectation of a truncated normal random variable over the interval $[a, a + \frac{\alpha c}{\beta}]$. It is well known (see Johnson, Kotz and Balakrishnan (1994)) that $d(a)$ may be expressed using the following formula:

$$d(a) = x_0 - \sigma^2\frac{f(a + \frac{\alpha c}{\beta}) - f(a)}{F(a + \frac{\alpha c}{\beta}) - F(a)} \quad a \in \mathbb{R}. \tag{12}$$

Also, notice that the first derivative of the normal density satisfies:

$$f'(x) = -\frac{x - x_0}{\sigma^2}f(x). \tag{13}$$

The following two lemmas are needed in order to establish the existence of the required $a$.

**Lemma 3** *The following holds for any $s > 0$:*

$$\begin{aligned}\lim_{x\to\infty}\frac{f(x+s)}{f(x)} &= 0\\ \lim_{x\to-\infty}\frac{f(x+s)}{f(x)} &= \infty.\end{aligned}$$

**Proof.** For any $s > 0$, and $x \in \mathbb{R}$ we have

$$\frac{f(x+s)}{f(x)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp-\frac{(x+s-x_0)^2}{2\sigma^2}}{\frac{1}{\sqrt{2\pi\sigma^2}}\exp-\frac{(x-x_0)^2}{2\sigma^2}} = \exp-\frac{s(2x-2x_0+s)}{2\sigma^2}.$$

The result follows by taking the appropriate limits. ■

**Lemma 4** *The following holds:*

$$\begin{aligned}\lim_{a\to\infty}[d(a)-a] &= 0\\ \lim_{a\to-\infty}[d(a)-a] &= \frac{\alpha c}{\beta}.\end{aligned}$$

**Proof.** By (12), and using L'Hopital's rule we have

$$\begin{aligned}\lim_{a\to\infty} d(a) &= x_0 - \sigma^2\lim_{a\to\infty}\frac{f(a+\frac{\alpha c}{\beta})-f(a)}{F(a+\frac{\alpha c}{\beta})-F(a)} = x_0 - \sigma^2\lim_{a\to\infty}\frac{f'(a+\frac{\alpha c}{\beta})-f'(a)}{f(a+\frac{\alpha c}{\beta})-f(a)}\\ &= x_0 + \sigma^2\lim_{a\to\infty}\frac{\frac{a+\frac{\alpha c}{\beta}-x_0}{\sigma^2}f(a+\frac{\alpha c}{\beta})-\frac{a-x_0}{\sigma^2}f(a)}{f(a+\frac{\alpha c}{\beta})-f(a)}\\ &= x_0 + \lim_{a\to\infty}\frac{(a+\frac{\alpha c}{\beta}-x_0)f(a+\frac{\alpha c}{\beta})-(a-x_0)f(a)}{f(a+\frac{\alpha c}{\beta})-f(a)}\\ &= \lim_{a\to\infty}\left[a+\frac{\alpha c}{\beta}\frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta})-f(a)}\right]\end{aligned}$$

Now, by plugging $s = \frac{\alpha c}{\beta}$ in Lemma 3 it follows that

$$\lim_{a\to\infty}[d(a)-a] = \frac{\alpha c}{\beta}\lim_{a\to\infty}\frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta})-f(a)} = \frac{\alpha c}{\beta}\lim_{a\to\infty}\frac{1}{1-\frac{f(a)}{f(a+\frac{\alpha c}{\beta})}} = 0,$$

as required.

As for the second part, repeating the previous analysis we obtain

$$\lim_{a\to-\infty} d(a) = \lim_{a\to-\infty}\left[a+\frac{\alpha c}{\beta}\frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta})-f(a)}\right].$$

Using Lemma 3 it follows that

$$\lim_{a\to-\infty}[d(a)-a]=\frac{\alpha c}{\beta}\lim_{a\to-\infty}\frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta})-f(a)}=\frac{\alpha c}{\beta}\lim_{a\to-\infty}\frac{1}{1-\frac{f(a)}{f(a+\frac{\alpha c}{\beta})}}=\frac{\alpha c}{\beta},$$

as required. ■

It is now easy to prove the existence of a required $a$. Indeed, define $H(a)\equiv d(a)-a-\frac{3\alpha c}{4\beta}$. From Lemma 4 it follows that $\lim\limits_{a\to-\infty}H(a)=\frac{\alpha c}{4\beta}>0$, and $\lim\limits_{a\to\infty}H(a)=-\frac{3\alpha c}{4\beta}<0$. Thus, from the continuity of $H(a)$ we conclude that there exists an $a\in\mathbb{R}$ such that $H(a)=0$.

Our next step is to prove the uniqueness of the chosen $a$. We accomplish this by showing that $H(a)$ is strictly decreasing, namely, that $d'(a)<1$. This part of the proof is less straightforward and is omitted here for brevity. It is available upon request. ■

## Proof of Proposition 3

For brevity we assume $x_0=0$.[14] Since we are interested in the impact of $\sigma$, we view $a$ and $d$ as functions of $\sigma$. Define

$$H(a,\sigma)\equiv d(a,\sigma)-a-\frac{3\alpha c}{4\beta}.$$

The relation between $a$ and $\sigma$ is given by the implicit equation $H(a,\sigma)=0$. In the proof of Proposition 2 we have shown that $\frac{\partial H(a,\sigma)}{\partial a}<0$ for all $a,\sigma\in\mathbb{R}$. By the implicit function theorem we have

$$\frac{\partial a(\sigma)}{\partial\sigma}=-\frac{\frac{\partial H(a,\sigma)}{\partial\sigma}}{\frac{\partial H(a,\sigma)}{\partial a}}.$$

Thus, to show that $\frac{\partial a(\sigma)}{\partial\sigma}<0$ it is sufficient to show that $\frac{\partial H(a,\sigma)}{\partial\sigma}<0$. We have

[14] A different choice of $x_0$ would shift $a(\sigma)$ by a constant, and thus it has no effect on $\frac{\partial a}{\partial\sigma}$.

$$
\begin{aligned}
\frac{\partial H(a,\sigma)}{\partial\sigma} &= \frac{\partial d(a,\sigma)}{\partial\sigma} = \frac{\partial}{\partial\sigma}\frac{\int_a^{a+\frac{\alpha c}{\beta}} xf(x)dx}{F(a+\frac{\alpha c}{\beta})-F(a)} = \frac{\partial}{\partial\sigma}\frac{\frac{1}{\sqrt{2\pi\sigma^2}}\int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\frac{1}{\sqrt{2\pi\sigma^2}}\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx} \\
&= \frac{\int_a^{a+\frac{\alpha c}{\beta}} \frac{x^3}{\sigma^3}e^{-\frac{x^2}{2\sigma^2}}dx\cdot\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx - \int_a^{a+\frac{\alpha c}{\beta}} \frac{x^2}{\sigma^3}e^{-\frac{x^2}{2\sigma^2}}dx\cdot\int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\left[\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx\right]^2} \\
&= \frac{1}{\sigma^3}\left[\frac{\int_a^{a+\frac{\alpha c}{\beta}} x^3e^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx} - \frac{\int_a^{a+\frac{\alpha c}{\beta}} x^2e^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx}\cdot\frac{\int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{\pi^2}{2\sigma^2}}dx}\right] \\
&= \frac{1}{\sigma^3}\left[\frac{\int_a^{a+\frac{\alpha c}{\beta}} x^3f(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)d\pi} - \frac{\int_a^{a+\frac{\alpha c}{\beta}} x^2f(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}\cdot\frac{\int_a^{a+\frac{\alpha c}{\beta}} xf(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}\right] \\
&= \frac{1}{\sigma^3}\left[E(\tilde{x}^3|a\le\tilde{x}\le a+\frac{\alpha c}{\beta}) - E(\tilde{x}^2|a\le\tilde{x}\le a+\frac{\alpha c}{\beta})\cdot E(\tilde{x}|a\le\tilde{x}\le a+\frac{\alpha c}{\beta})\right] \\
&= \frac{1}{\sigma^3}Cov(\tilde{x}^2,\tilde{x}|a\le\tilde{x}\le a+\frac{\alpha c}{\beta}).
\end{aligned}
$$

Thus, the sign of $\frac{\partial H(a,\sigma)}{\partial\sigma}$ is equal to the sign of $Cov(\tilde{y},\ \tilde{y}^2)$, where $\tilde{y}$ is a random variable obtained from a truncation of $\tilde{x}$ between $a$ and $a+\frac{\alpha c}{\beta}$. It can be shown that this covariance is strictly negative, as required.[15]

Given that $a(\sigma)$ is decreasing in $\sigma$, we know that $a_0 \equiv \lim_{\sigma\to 0} a(\sigma)$ exists. It is easy to see that for any fixed $a < x_0$ and $b > a$ we have

$$
\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a,b]) = \begin{cases} b & x_0\notin[a,b] \\ x_0 & x_0\in[a,b] \end{cases}. \tag{14}
$$

We claim first that there exists an $a_0 > 0$ such that $a_0+\frac{\alpha c}{\beta} > x_0$. Indeed, suppose on the contrary that $a_0+\frac{\alpha c}{\beta}\le x_0$. This implies by (14) that $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a(\sigma),a(\sigma)+\frac{\alpha c}{\beta}]) = a_0+\frac{\alpha c}{\beta}$, contradicting the fact that for all $\sigma$, $E(\tilde{x}|\tilde{x}\in[a(\sigma),a(\sigma)+\frac{\alpha c}{\beta}]) = a(\sigma)+\frac{3\alpha c}{4\beta}$. Now, for all $\varepsilon > 0$ sufficiently small we have: $a_0-\varepsilon+\frac{\alpha c}{\beta} > x_0$. Thus, by (14) we have: $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a_0-\varepsilon,a_0-\varepsilon+\frac{\alpha c}{\beta}]) = x_0$. From the continuity of the conditional expectation and since $\varepsilon$ is arbitrary we conclude that $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a(\sigma),a(\sigma)+\frac{\alpha c}{\beta}]) = x_0$. And, hence $\lim_{\sigma\to 0} a(\sigma) = x_0-\frac{3\alpha c}{4\beta}$, as required.

[15] The proof is technical. It applies to any symmetric and continuous distribution, and not only to the normal distribution. It relies on the fact that the truncation interval $[a, a+\frac{\alpha}{\beta}]$ is tilted to the left-hand side of the distribution. We omit the proof here for brevity, but it is available upon request.

As for the case of $\sigma \to \infty$. For all fixed $a$ and $b$ we have: $E(\tilde{x}|\tilde{x} \in [a,b] \to \frac{a+b}{2}$. Indeed, by applying L'Hopital's law we obtain

$$\begin{aligned}
\lim_{\sigma\to\infty}\left[x_0 - \sigma^2\frac{f(b)-f(a)}{F(b)-F(a)}\right] &= x_0 - \lim_{\sigma\to\infty}\sigma^2\frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}} - e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\int_a^b e^{\frac{-(x-x_0)^2}{2\sigma^2}}dx} \\
&= x_0 - \frac{1}{b-a}\lim_{\sigma\to\infty}\frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}} - e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\frac{1}{\sigma^2}} \\
&= x_0 - \frac{1}{b-a}\lim_{\sigma\to\infty}\frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}} - e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\frac{1}{\sigma^2}} \\
&= x_0 - \frac{1}{b-a}\lim_{\sigma\to\infty}\frac{\frac{(b-x_0)^2}{\sigma^3}e^{\frac{-(b-x_0)^2}{2\sigma^2}} - \frac{(a-x_0)^2}{\sigma^3}e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{-\frac{2}{\sigma^3}} \\
&= x_0 + \frac{(b-x_0)^2-(a-x_0)^2}{2(b-a)} = \frac{a+b}{2}
\end{aligned}$$

This calculation implies that if $a_\infty \equiv \lim_{\sigma\to\infty} a(\sigma)$ were finite, we would have that $d(a(\sigma)) \to a_\infty + \frac{\alpha c}{2\beta}$ - a contradiction to the fact $d(a(\sigma)) = a(\sigma) + \frac{3\alpha c}{4\beta}$ for all $\sigma$. ■

**Proof of Lemma 2**

We shall prove Part 1 of the lemma. The proof of Part 2 is symmetric.

Suppose $x^R \in (a + \frac{\alpha c}{2\beta}, b)$ is an out-of-equilibrium report, and let $P(x^R)$ be the price in case a report of $x^R$ is observed. We claim that in this case, if the '$a$' type is indifferent between submitting a report of $b$ (equilibrium report) or $x^R$ (deviating), then all other types $x' \neq a$ strictly prefer to stick to their equilibrium report. We shall consider three cases.

**Case 1:** $x' \in (a,b]$. Since the '$a$' type is indifferent between submitting $b$, and deviating to $x^R$, we obtain

$$\alpha cd - \beta(b-a)^2 = \alpha P(x^R) - \beta(x^R - a)^2. \tag{15}$$

The payoff to type $x' \in (a,b]$ from reporting $x^R$ is: $\alpha P - \beta(x^R - x')^2$. It follows that the largest benefit from deviating to a report of $x^R$ is incurred when the type is equal to the report, namely: $x' = x^R$. In this case, the payoff in case of deviation is $\alpha P$, while the payoff on the equilibrium path is: $\alpha cd - \beta(x^R - b)^2$. By (15), the difference between the payoff on the equilibrium path, and the payoff in case of deviation is

$$\begin{aligned}\alpha cd - \beta(x^R - b)^2 - \alpha P &= -\beta(x^R - b)^2 + \beta(b-a)^2 - \beta(x^R - a)^2 \\ &= 2\beta(x^R - a)(b - x^R) > 0.\end{aligned}$$

Thus, type $x'$ strictly prefers to stick to his equilibrium report.

**Case 2:** $x' < a$. Since the '$a$' type is indifferent between submitting $a + \frac{\alpha c}{2\beta}$, and deviating to $x^R$ we obtain

$$\alpha ca - \beta(\frac{\alpha c}{2\beta})^2 = \alpha P(x^R) - \beta(x^R - a)^2. \tag{16}$$

Now, if type $x'$ follows the equilibrium he obtains: $\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2$. If on the other hand he deviates to $x^R$ he obtains: $\alpha P(x^R) - \beta(x^R - x')^2$. Using (16) we obtain that the difference is

$$\begin{aligned}\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2 - \alpha P(x^R) + \beta(x^R - x')^2 &= \alpha cx' - \alpha ca - \beta(x^R - a)^2 + \beta(x^R - x')^2 \\ &= \beta(a - x')(2x^R - x' - a - \frac{\alpha c}{\beta}) \\ &> \beta(a - x')(2(a + \frac{\alpha c}{2\beta}) - x' - a - \frac{\alpha c}{\beta}) \\ &= \beta(a - x')^2 > 0,\end{aligned}$$

where the penultimate inequality follows since $x^R > a + \frac{\alpha c}{2\beta}$. Thus, type $x'$ is better off sticking to the equilibrium strategy.

**Case 3:** $x' > b$. In Case 1, we have shown that if type '$a$' is indifferent between the two alternatives, then type '$b$' strictly prefers to stick to the equilibrium. Thus

$$\alpha cb - \beta(\frac{\alpha c}{2\beta})^2 > \alpha P(x^R) - \beta(x^R - b)^2.$$

Therefore

$$\alpha P(x^R) + \beta(\frac{\alpha c}{2\beta})^2 < \alpha cb + \beta(x^R - b)^2.$$

We conclude that

$$\begin{aligned}\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2 - \alpha P(x^R) + \beta(x^R - x')^2 &> \alpha cx' - \alpha cb - \beta(x^R - b)^2 + \beta(x^R - x')^2 \\ &= \beta(x' - b)(x' + b + \frac{\alpha c}{\beta} - 2x^R) \\ &> \beta(x' - b)(x' - b + \frac{\alpha c}{\beta}) > 0,\end{aligned}$$

where the penultimate inequality follows since $x^R < b$. Thus, the deviation is not profitable. This concludes the proof. ■

**Proof of Proposition 6**

The cases $x^R < a + \frac{\alpha c}{2\beta}$, $x^R > b + \frac{\alpha c}{2\beta}$, and $x^R = b$ are identical to these cases in Proposition 2, and are determined uniquely using Bayes rule. As for the pooling region: for all $x^R \in [a + \frac{\alpha c}{2\beta}, b)$, we look for a pricing function $P_t^*(x^R)$ that makes type '$a$' indifferent between deviating to $x^R$ and sticking to the equilibrium. This indifference implies that this pricing function must satisfy

$$\alpha ca - \beta(\frac{\alpha c}{2\beta})^2 = \alpha P_t^*(x^R) - \beta(x^R - a)^2.$$

Solving for $P_t^*(x^R)$ yields the required result. A similar calculation applies for the case $x^R \in (b, b+\frac{\alpha c}{2\beta})$. Lemma 2 implies that this out-of-equilibrium pricing guarantees that no type will be willing to deviate from the partially pooling strategy $\rho_p^*(\cdot)$.

Now, let $P^*(x^R)$ be any other pricing function. Obviously, it must coincide with $P_t^*(x^R)$ if $x^R < a + \frac{\alpha c}{2\beta}$ or $x^R > b + \frac{\alpha c}{2\beta}$, or $x^R = b$. Now, if $P^*(x^R) \leq P_t^*(x^R)$ for all $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$ then types '$a$' and '$b$' (weakly) prefer to provide the equilibrium report $b$ compared to any out of equilibrium report. By an argument similar to Lemma 2 this implies that all other types strictly prefer not to deviate from the equilibrium. This establishes sufficiency. To show necessity, suppose on the contrary that $P^*(x^R) > P_t^*(x^R)$ for some $x^R \in (a+\frac{\alpha c}{2\beta}, b)$. Then type '$a$' would prefer deviating and reporting $x^R$ instead of $b$. Similarly, if $P^*(x^R) > P_t^*(x^R)$ for some $x^R \in (b, b + \frac{\alpha c}{2\beta})$ then type '$b$' would deviate. This concludes the proof. ■