DISCUSSION PAPER SERIES

No. 4480

GROUP DECISION-MAKING IN THE SHADOW OF DISAGREEMENT

Kfir Eliaz, Debraj Ray and Ronny Razin

PUBLIC POLICY

Centre for Economic Policy Research

www.cepr.org

Available online at: www.cepr.org/pubs/dps/DP4480.asp

ISSN 0265-8003

GROUP DECISION-MAKING IN THE SHADOW OF DISAGREEMENT

Kfir Eliaz, New York University and CEPR
Debraj Ray, New York University and CISC
Ronny Razin, New York University and CEPR

Discussion Paper No. 4480
July 2004

Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR, UK
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: www.cepr.org

CEPR Discussion Paper No. 4480

July 2004

ABSTRACT

Group Decision-Making in the Shadow of Disagreement*

A model of group decision-making is studied, in which one of two alternatives must be chosen. While group members differ in their valuations of the alternatives, everybody prefers agreement to disagreement. Our model is distinguished by three features: *private information* regarding valuations, *varying intensities* in the preference for one outcome over the other, and the option to declare *neutrality* in order to avoid disagreement. We uncover a variant on the 'tyranny of the majority': there is always an equilibrium in which the majority is more aggressive in pushing its alternative, thus enforcing their will via both numbers and voice. Under general conditions, however, an aggressive minority equilibrium inevitably makes an appearance, provided that the group is large enough. The notable exception is the special case of unanimity rule: we show that aggressive minority equilibria may never exist irrespective of group size. Aggressive minority equilibria invariably display a 'tyranny of the minority': it is always true that the increased aggression of the minority more than compensates for smaller numbers, leading to the minority outcome being implemented with larger probability than the majority alternative. We fully characterize the asymptotic behaviour of this model as group size becomes large, and show that all equilibria must converge to one of three possible limit outcomes.

JEL Classification: D72 and D78

Kfir Eliaz
Economic Department
New York University
269 Mercer Street
New York, NY 10003
USA
Tel: (1 212) 998 8912
Fax: (1 212) 995 4186
Email: kfir.eliaz@nyu.edu

Debraj Ray
Economic Department
New York University
269 Mercer Street
New York, NY 10003
USA
Email: debraj.ray@nyu.edu

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=153104

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=133751

Ronny Razin
Economic Department
New York University
269 Mercer Street
New York, NY 10003
USA
Tel: (1 212) 998 8904
Fax: (1 212) 995 3932
Email: ronny.razin@nyu.edu

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=152435

*We thank Eddie Dekel, Wolfgang Pesendorfer and Muhamet Yildiz for useful comments. Ray acknowledges support from the National Science Foundation.

Submitted 07 June 2004

1. Introduction

We study situations in which a group must decide between alternative courses of action. Our particular interest is in decision-making processes which exhibit two critical features. First, disagreement (or not implementing any of the choices on the table) is the worst possible outcome for all concerned. Second, each group member has the option to declare "neutrality"; effectively, to say that she does not care strongly about either alternative and will support any outcome that may be more forcefully espoused by others with more intense preferences. To be sure, the two features are closely related: declaring neutrality is a means of avoiding disagreement.

There are, obviously, several examples of such situations. A government may need to formulate a long-run response to terrorism: individuals may disagree — often vehemently — over the nature of an appropriate response, but everyone might agree that complete inaction is the worst of the options. Jury members in the process of deliberation may disagree on whether or not the defendent is guilty; however, in most cases they all prefer to reach an agreement than to drag the deliberations on endlessly. An investigative committee looking into the causes of a riot, or a political assassination, or a corruption scandal, may be under significant pressure to formulate *some* explanation, rather than simply say they don't know. And of course, the bargaining literature presents a plethora of examples in which disagreement is universally regarded as a bad outcome.

Moreover, in each of these cases the option to declare some form of neutrality is typically built into the deliberations. For example, think of an academic department that meets to make an offer to one of several candidates. Different faculty members may disagree over the ranking of the candidates. Moreover, some faculty members may feel more strongly towards their favorite candidate than others. However, no member wants to see the slot taken away by the Dean because the department could not agree on an offer. Because faculty members may be uncertain as to the rankings and intensities of their colleagues, those faculty members who do not feel strongly about the issue will be less vocal and willing to go with whatever decision is reached, while those who feel strongly about their favorite candidate will argue aggressively in her favor.

Likewise, in the jury example, members may disagree over whether or not the defendent is guilty. Moreover, some jury members would have stronger feelings on the issue of conviction than others. However, in most cases, all would want to reach some unanimous decision than end up with a hung jury.[1] Consequently, those jurors who feel strongly towards conviction or acquital would be more vocal during deliberation, while those who feel less strongly on the issue might be willing to support either side in order to facilitate an agreement.[2]

To be sure, one could analyze each of these diverse examples separately using a specific model tailor-made to that situation. An alternative approach is to propose a somewhat more abstract model that captures the basic ingredients common to all the examples, and this is the route

[1] A particular case in point is the recent trial of Lee Malvo, the younger of the two men accused in the D.C. sniper case. According to the interviews conducted with some of the jury members who sat on that trial, the jury was split between conviction and acquital. Even though conviction could mean the death penalty for the accused, some of the jurors who opposed conviction remarked that they felt it was more important to reach a unanimous decision then end up in a hung jury.

[2] While these examples refer to formal institutions that follow written rules, we also have in mind more mundane examples where *ad hoc* groups meet to reach some decision. Think of a group of colleagues who want to meet for lunch. Each would probably have a different ranking of available restaurants. In addition, some may feel more strongly than others about some of the available restaurants. While there is no written rule on how to reach such a decision, the group would feel uncomfortable to eat at a restaurant to which at least one member objected. Since each individual may be unsure about the rankings and intensities of his colleagues, those with strong feeling on the issue would try to be the first ones to suggest a restaurant, while others who do not feel as strongly would be willing to go to any place agreed on by the others.

we take in this paper. Our goal is to study a particular formulation of group decision-making in the shadow of disagreement, one which we believe to be representative of many real-world scenarios.[3] We proceed as follows.

A group of n agents must make a joint choice from a set of two alternatives. Each agent must either name an alternative — A or B — or she can declare "neutrality", in that she agrees to be counted, in principle, for either side. Once this is accomplished, we tally declarations for each alternative, *including the number of neutral announcements*. If, for an alternative, the resulting total is no less than some exogenously given supermajority, we shall call that alternative *eligible*.

Because neutral annoucements are allowed for and tallied on both sides, all sorts of combinations are possible: exactly one alternative may be eligible, or both, or neither. If *exactly* one alternative is eligible, that alternative is implemented. If both are eligible — as will typically be the case when there are a large number of neutrals — one alternative is picked and implemented at random. If neither is eligible — which will happen if there is a fierce battle to protect one's favorite alternative — then no alternative is picked: the outcome is disagreement.

The objective of the paper is to set up this model and study its equilibria.

Several features of the model deserve comment. First, while the specific formulation is cast in terms of a voting model, we do not necessarily have voting in mind. The exogenously given supermajority may or may not amount to full consensus or unanimity, and in any case is to be interpreted as some preassigned degree of consensus or social norm that the group needs to achieve. For instance, in many informal situations, it may be considered socially undesirable to choose an option objected to by at least one person.

Second, the option to remain neutral is a novel feature of our model. At the same time, it is a natural ingredient in the examples discussed above. We only add here that the neutrality option may be interpreted in several ways. One formal institution that corresponds directly is approval voting: members of the group submit an "approval" or "disapproval" for each alternative. A voter who approves both alternatives is effectively declaring neutrality. We have already discussed several examples in which neutrality is an informal yet central feature of the decision-making process. In addition, one could imagine several quasi-formal mechanisms that help individuals to avoid disagreement by allowing their vote to be counted in a way that ensures a win to one of the alternatives. For example, one could delegate his ballot to an impartial arbitrator, who appreciates the anxiety of all concerned to avoid disagreement, and is therefore interested in implementing some outcome. Therefore, one could interpret the neutrality declaration as the reduced form of some unspecified procedure, which is used to help avoid unnecessary disagreements.

Third, we are interested in the "intensity" of preference for one alternative over the other, and how this enters into the decision to be neutral, or to fight for one's favorite outcome. Specifically, we permit each person's valuations to be independent (and private) draws from a distribution, and allow quite generally for varying cardinal degrees of preference. A corollary of this formulation is that *others* are not quite sure of how strongly a particular individual might feel about an outcome and therefore about how that individual might behave. This is one way in which uncertainty enters the model.

Uncertainty also plays an additional role, in that no one is sure how many people favor one given alternative over the other. To be sure, we assume that there is a common prior — represented by an independent probability p — that an individual will favor one alternative (call it A) over the other (call it B). Without loss of generality take $p \leq 1/2$. If, in fact, $p < 1/2$, one might say that it is commonly known that people of "type A" are in a minority, or more precisely in a stochastic minority.

[3]Thus it is not an axiomatic description of a normative or quasi-normative solution that we are after, as in Nash bargaining, nor so we seek to implement a particular solution correspondence by the choice of a mechanism.

We will see that these two types of uncertainty are very important for the results we obtain.

A major goal of the paper is to study equilibria that "favor" one side: either the minority or the majority. It is intuitive — and we develop this formally in the analysis — that in any equilibrium, each individual will use a cutoff rule: there will exist some critical relative intensity of preference (for A over B or *vice versa*) such that the individual will announce her favorite outcome if intensities exceed this threshold, and neutrality otherwise. If the cutoff is lower, then a type may be viewed as being more "aggressive": she announces her own favorite outcome more easily (and risks disagreement with greater probability). Thus, equilibria in which an individual of the majority type uses a lower cutoff than an individual of the minority type may be viewed as favoring the majority: we call them *majority equilibria*. Likewise, equilibria in which the minority type uses a lower cutoff will be called *minority equilibria*.

One might use a parallel from the Battle of the Sexes (after all, in some sense, our model is an enriched version of that game) to search for particular majority or minority equilibria. For instance, might one not be able to sustain an equilibrium in which all members of a particular type are "fully aggressive" (using the lowest possible cutoff) while their opponents all timidly declare neutrality, regardless of valuation? The answer is that such a configuration is indeed an equilibrium. But, as we argue in detail in Section 4.2.2, this equilibrium fails a weak robustness or stability criterion. If the compatriots of, say, a type-A individual *do* announce neutrality for a huge range of relative valuations (rather than the *entire* range), it will push an individual type-A person to announce A for a large range of valuations, thus rendering the "perfect neutrality" cutoff unstable to the tiniest perturbations. As we shall see in Section 4.2.2, uncertainty about group sizes plays a central and indispensable role in this result, though this is not the only indispensable role played by uncertainty in this model.

Nevertheless, Proposition 1 establishes that a *majority* equilibrium — one satisfying the robustness criterion just described — always exists. In this equilibrium, both sides use "interior" cutoffs, but the majority uses a more aggressive cutoff than the minority. This is an interesting manifestation of the "tyranny of the majority".[4] Not only are the majority greater in number (or at least stochastically so), they are also more vocal in expressing their opinion. In response — and fearing disagreement — the minority are more cowed towards neutrality. So in majority equilibrium, group outcomes are doubly shifted towards the majority view, once through numbers, and once through greater voice.

We then turn to minority equilibria. Given the refinement described two paragraphs ago, such equilibria may not exist; indeed, it is easy enough to find examples of nonexistence. Yet Proposition 2 establishes the following result: if the required supermajority μ is not unanimity (i.e., $\mu < 1$), and if the size of the stochastic minority p exceeds $1-\mu$, then for all sufficiently large population sizes, a minority equilibrium must exist.

How large is large? To be sure, the answer must depend on the model specifics, but our computations suggest that in reasonable cases, population sizes of 8–10 (certainly less than the size of a jury!) are enough for existence.

From one point of view this result seems intuitive, yet from others it is remarkable. Intuitively, as population size increases, the two types of uncertainty that we described — uncertainty about type and uncertainty regarding valuation intensity — tend to diminish under the strength of the Law of Large Numbers. This would do no good if $p < 1-\mu$, for then the minority would neither be able to win, nor would it be able to block the majority. [Indeed, Proposition 3 in Section 5.2 shows that if $p < 1-\mu$, then for large population sizes a minority equilibrium cannot exist.] But

[4] It is possible that our use of this term constitutes a slight abuse of terminology, given that the phrase is typically invoked in the context of simple majority rule. We deal with supermajorities, so the term "tyranny" (of either majority or minority) here is used in the sense of more strident use of *voice*.

if p exceeds $1-\mu$, the minority acquires "credibility" to block the wishes of the majority, or at least does so when the population is large enough.

For two reasons, however, this notion of "credible blocking" does not form a complete explanation. First, credible blocking is not tantamount to a credible *win*. Indeed, it is easy to see that as μ goes up, the minority find it easier to block but also harder to win. So the previous result must *not* be viewed as an assertion that the minority is "better protected" by an increase in μ. As an example in Section 5.1 makes clear, this is not true. [Nevertheless, insofar as existence is concerned, the fact that $p > 1-\mu > 0$ guarantees existence for large population sizes.]

Second — and this extends the discussion of the previous paragraph — the case of unanimity ($\mu = 0$) is special. Proposition 4 shows that there are conditions (on the distribution of valuations) under which a minority equilibrium *never* exists, no matter how large the population size is. So blocking credibility alone does not translate into the existence of a minority equilibrium in the unanimity case. In short, any "intuitive explanation" for Proposition 2 must also account for these observations.

Given the general existence result for minority equilibrium (in all non-unanimity scenarios), we turn to a study of its properties. Recall that in the majority equilibrium, the majority group will have a greater chance of implementing its preferred outcome on two counts: greater voice, and greater number. Obviously, this synergy is reversed for the minority equilibrium: there, the minority have greater voice, yet they have smaller numbers. One might expect the net effect of these two forces to result in some ambiguity. The intriguing content of Proposition 5 is that in a minority equilibrium, *the minority must always implement its favorite action with greater probability than the majority*. Voice more than compensates for number.

Our paper thus suggests that in group decision-making the outcomes tend to be invariably biased in one direction or another. In majority equilibrium this is obvious. But it is also true of minority equilibrium. This lends some support to the view that group decision-making tends to have extreme characteristics intrinsically built into the process.

Next, in Section 6, we study the set of equilibria as population size grows large. Proposition 6 shows that there are exactly three limit outcomes to which all equilibria must converge. Two of these outcomes are "limit minority equilibria". Of the two, one exhibits a zero cutoff for the minority, and the other exhibits a positive minority cutoff which is nevertheless lower than the majority cutoff. The third outcome is a "limit majority equilibrium" in which the cutoff used by the majority is zero.

The two "corner" and the unique interior limit outcomes suggest once again that despite the refining-away of corner equilibrium as in the Battle of the Sexes, in the limit (as group size grows large) a similar structure comes to prevail. There are several reasons why such a parallel is misleading. In the first place, we are not only interested in "large" group sizes, and the structure we have is richer than the Battle of the Sexes for finite group sizes. Second, even if we concentrate exclusively on the limit situation, the corner equilibria (in which one side always fights for its favorite) possess a special structure: *the other side does not necessarily yield fully*. That is, they may use an interior cutoff even in the limit. [Proposition 6 actually provides a complete characterization in that it gives necessary and sufficient conditions for such interior cutoffs and describes exactly what they are.] Thus, to coin a phrase, these may be "semi-corner" equilibria, in contrast to the "full corners" always to be found in the Battle of the Sexes. Finally, it should also be remembered that both the corner equilibria do *not* exist, in general, for the unanimity case.

Finally, Proposition 7 takes up the question of disagreement as group size grows large. To motivate this, imagine that μ is very close to unity. With large populations, might disagreement probabilities not rise quickly? Of course, this sort of query neglects the strategic nature of decision-making in this model. Individual cutoffs vary endogenously with population size. Indeed, Proposition 7 establishes (again in the non-unanimity case) that the probability of disagreement must converge to zero along all equilibrium sequences that converge to the semi-corners identified above. For those equilibria that converge to the remaining minority outcome, we show that the probability of disagreement is bounded away from one as the population size goes to infinity.

2. Related Literature

One central result in our paper is that minorities may fight more agressively and win. Of course, the well-known Pareto-Olson thesis (see Pareto (1906) and Olson (1965)) suggests that minorities might put up a stronger fight when voting is costly. This intuition is confirmed in some complete-information models with private voting costs (see Araki and Börgers (1996) and Haan and Kooreman (2003)), though in other variants with incomplete information (e.g., Ledyard (1984), Palfrey and Rosenthal (1983) and Campbell (1999)), the majority still wins more frequently even when the minority fights harder, assuming that preference intensities do not differ across groups.[5]

Our model also features a “cost of voting": it is the expected loss caused by disagreement. But this cost is a *public* bad, and it canot be shifted from one voter to another. [In addition, the magnitude of this cost is determined endogenously in equilibrium.]

An important feature of our model is that individuals base their decision on how strongly they prefer one alternative to another. This feature is shared with several papers that investigate different mechanisms in which intensity of preferences determine individual voting behavior. Vote-trading mechanisms, in which voters can trade their votes with one another, have been analyzed in Buchanan and Tullock (1962) and have more recently been revisited by Philipson and Snyder (1996) and Piketty (1994). Cumulative voting mechanisms in which each voter may allocate a fixed number of votes among a set of candidates has been analyzed as early as in Dodgson (1884) and more recently revisited by Gerber, Morton and Rietz (1998), Jackson and Sonnenschein (2003) and Hortala-Vallve (2004). In a related vein, Casella (2003) introduces a system of storable votes, in which voters can choose to store votes in order to use them in situations that they feel more strongly about.

These papers take a normative approach to group decision making in an attempt to design optimal procedures. Our approach is different. We take a positive approach and focus on existing institutions that rely on supermajority rules. We argue that a threat of disagreement may push individuals to base their decisions not only on their ordinal preferences, but also on their preference intensities. At the same time, we do not claim that the decision protocol we analyze — a supermajority rule coupled with a neutrality option and a threat of disagreement — necessarily leads to an efficient outcome (though mechanism design in our context would certainly be an interesting research project).

In particular, our analysis highlights the importance of consensus and the fear of gridlock as a mechanism through which intensities of preferences are translated into the decision making process. In this context, Ponsati and Sákovicz (1996) is also related to the present paper. Indeed, their model is more ambitious in that they explicitly attempt to study the dynamics of capitulation in an ambient environment similar to that studied here. This leads to a variant on the war of attrition, and their goal is to describe equilibria as differential equations for capitulation times, at which individuals cease to push their favorite alternative.

[5]Certainly, if minorites are sufficiently more zealous in the espousal of their favorite issue, they may fight more aggressively *and* win more often, as Campbell (1999) also shows.

Our paper is also connected to experimental literature on jury behavior. Several studies performed by Kahneman, Schkade and Sunstein (1998a, 1998b, 1999) on mock juries have arrived at the following conclusions: (1) different juries are likely to reach similar conclusions about the relative severity of different cases, and (2) juries do not produce less erratic and more predictable awards than individuals. Although these studies were performed on cases of punitive damages, rather than criminal cases (where the decision is binary), the above findings can be interpreted in a manner consistent with our results. First, the relative composition of the jury, whether the majority are white, black, poor or rich, does not affect the jury's decision in an unambiguous way. Put differently, a jury may decide on the same punitive damages to a black plaintiff when the majority of jury members are black as well as when the majority are white. Second, if decisions made by juries were more predictable than ones made by individuals, then one would expect that a randomly selected jury will most likely make a decision, which conforms with the views of the majority in the population. Hence, the unpredictablity of jury decisions can be interpreted as multiple equilibria: it is just as likely that the minority will influence the decision of the jury, as it is that the majority will influence its decision.

3. The Model

3.1. **The Group Choice Problem.** A group of n agents must make a joint choice from a set of two alternatives, which we denote by A and B. The rules of choice are described as follows:

[1] Each agent must either name an alternative — A or B — or she can declare "neutrality", in that she agrees to be counted, in principle, for either side.

[2] If the total number of votes for an alternative plus the number of neutral votes is no less than some exogenously given supermajority m ($> n/2$), then we shall call that alternative *eligible*.

[3] If no alternative is eligible, no alternative is chosen: a state D (for "disagreement") is the outcome.

[4] If a single alternative is eligible, then that alternative is chosen.

[5] If *both* alternatives are eligible, A or B are chosen with equal probability.

Notice that we view eligibility as a "zero-one" characteristic: either an alternative is eligible or it is not. There is no sense in which one alternative is more eligible than another. An alternative is eligible if it is socially fit to be implemented. The social fit is either determined by a written rule or by social norms.[6] One could think of an alternative formulation in which the option with the most votes wins in case both options pass the supermajority requirement. Such a formulation would make the analysis of the pivotal event more complicated, but we suspect that it would not alter any of our results.

3.2. **Valuations.** Normalizing the value of disagreement to zero, each individual will have valuations (v_A, v_B) over A and B. These valuations are random variables, and we assume they are private information. Use the notation (v, v'), where v is the valuation of the favorite outcome ($\max\{v_A, v_B\}$), and v' is the valuation of the remaining outcome ($\min\{v_A, v_B\}$). An individual will be said to be of *type* A if $v = v(A)$, and of *type* B if $v = v_B$. [The case $v_A = v_B$ is unimportant as we will rule out mass points below.]

Our first restriction is

[6] According to this view there is no distinction between the following two scenarios in supermajority rule other than unanimity: (i) one person declares A while all others remain neutral, and (ii) one person declares A, another declares B, and all others remain neutral. In both scenarios B has a positive chance of being implemented. Even though A is the only alternative declared in the first scenario, there are not enough objections to prevent B from being implemented.

[A.1] Each individual prefers either outcome to disagreement. That is, $(v, v') \gg 0$ with probability one.

In Section 7.3 we remark on the consequences of dropping the assumption that disagreement is worse than either alternative.

In what follows we shall impose perfect symmetry across the two types *except* for the probability of being one type or the other, which we permit to depart from $1/2$. [The whole idea, after all, is to study majorities and minorities.]

[A.2] A person is type A with (iid) probability $p \in (0, 1/2]$, and is type B otherwise. Regardless of specific type, however, (v, v') are chosen independently and identically across agents.

3.3. **The Game.** First, each player is (privately) informed of her valuation (v_A, v_B). Conditional on this information she decides to announce either A or B, or simply remain neutral and agree to be counted in any direction that facilitates agreement. Because an announcement of the opposite alternative (to a player's type) is weakly dominated by a neutral stance, we presume that each player either decides to vote her own type, or to be neutral.[7] The rules in Section 2.1 then determine expected payoffs.

4. Equilibrium

4.1. **Cutoffs.** Consider a player of a particular type, with valuations (v, v'). Define $q \equiv n - m$. Notice that our player only has an effect on the outcome of the game (that is, she is pivotal) in the event that there are *exactly* q other players announcing her favorite outcome. For, suppose there are more than q such announcements, say for A. Then B cannot be eligible, and whether or not A is eligible, our player's announcement cannot change this fact. So our player has no effect on the outcome. Likewise, if there are strictly less than q announcements of A, then B is eligible whether or not A is, and our player's vote (A or neutral) cannot change the status of the latter.

Now look at the pivotal events more closely. One case is when there are precisely q announcements in favor of A, and $q+1$ or more announcements favoring B. In this case, by staying neutral our agent ensures that B is the only eligible outcome and is therefore chosen. By announcing A she guarantees that neither outcome is eligible, so disagreement ensues. In short, by switching her announcement from neutral to A, our agent creates a personal loss of v'.

In the second case, there are q announcements or less in favor of B. In this case, by going neutral our agent ensures that A and B are both eligible, so the outcome is an equiprobable choice of either A or B. On the other hand, by announcing A, our agent guarantees that A is the *only* eligible outcome. Therefore by switching in this instance from neutral to announcing A, our agent creates a personal gain of $v - (v + v')/2$.

To summarize, let P^+ denote the probability of the former pivotal event (q compatriots announcing A, $q + 1$ or more announcing B) and P^- the probability of the latter pivotal event (q compatriots announcing A, q or less announcing B). It must be emphasized that these probabilities are not exogenous. They depend on several factors, but most critically on the strategies followed by the other agents in the group. Very soon we shall look at this dependence more closely, but notice that even at this preliminary stage we can see that our agent must follow a *cutoff rule*. For announcing A is weakly preferred to neutrality if and only if

$$P^-[v - (v + v')/2] \geq P^+ v'.$$

[7] For a similar reason we need not include the possibility of abstention. Abstention (as opposed to neutrality) simply increases the probability of disagreement, which all players dislike by assumption.

Define $u \equiv \frac{v-(v+v')/2}{v'}$. Note that (by [A.1]) u is a well-defined random variable. Then the condition above reduces to

$$P^{-}u \geq P^{+}, \tag{1}$$

which immediately shows that our agent will follow a cutoff rule using the variable u.

Notice that we include the extreme rules of always announcing neutrality (or always announcing one's favorite action) in the family of cutoff rules. [Simply think of u as a nonnegative extended real.] If a cutoff rule does not conform to one of these two extremes, we shall say that it is *interior*.

By [A.2], the variable u has the same distribution no matter which type we are referring to. We assume

[A.3] u is distributed according to the atomless cdf F, with strictly positive density f on $(0, \infty)$.

4.2. **Symmetric Equilibrium.** In this paper, we study symmetric equilibria: those in which individuals of the same type employ identical cutoffs.

4.2.1. *Symmetric Cutoffs.* Assume, then, that all A-types use the cutoff u_A and all B-types use the cutoff u_B. We can now construct the probability that a *randomly chosen* individual will announce A: she must be of type A, which happens with probability p, and she must want to announce A, which happens with probability $1 - F(u_A)$. Therefore the overall probability of announcing A, which we denote by λ_A, is given by

$$\lambda_A \equiv p[1 - F(u_A)].$$

Similarly, the probability that a randomly chosen individual will announce B is given by

$$\lambda_B \equiv (1-p)[1 - F(u_B)].$$

With this notation in hand, we can rewrite the cutoff rule (1) more explicitly. First, add P^{-} to both sides to get

$$P^{-}(1+u) \geq P^{+} + P^{-}.$$

Assuming that we are studying this inequality for a person of type A, the right-hand side is the probability that exactly q individuals announce A, while the left-hand side is the joint probability that exactly q individuals announce A *and* no more than q individuals announce B. With this in mind, we see that the cutoff u_A must solve the equation

$$\binom{n-1}{q}\lambda_A^q \sum_{k=0}^{q} \binom{n-1-q}{k} \lambda_B^k (1-\lambda_A-\lambda_B)^{n-1-q-k}(1+u_A) = \binom{n-1}{q}\lambda_A^q(1-\lambda_A)^{n-1-q}. \tag{2}$$

Likewise, the cutoff u_B solves

$$\binom{n-1}{q}\lambda_B^q \sum_{k=0}^{q} \binom{n-1-q}{k} \lambda_A^k (1-\lambda_A-\lambda_B)^{n-1-q-k}(1+u_B) = \binom{n-1}{q}\lambda_B^q(1-\lambda_B)^{n-1-q}. \tag{3}$$

We will sometimes refer to these cutoffs as “best responses", though it should be clear that u_A embodies not just a “response" by an individual but also an equilibrium condition: that this individual response is equal to the cutoff employed by all compatriots of the same type.

4.2.2. *A Simple Refinement* . At this stage, an issue arises which we would do well to deal with immediately. It is that a symmetric cutoff of ∞ is always a best response for any type to any cutoff employed by the other type, provided that $q > 0$. This is easy enough to check: if no member in group A is prepared to declare A in any circumstance, then no A-type will find it in her interest to do so as well. This is because (with $q > 0$) no such individual is ever pivotal.

Hence the "full neutrality cutoff" $u = \infty$ is always a best response. But it is an unsatisfactory best response. The reason is that if the compatriots of, say, a type-A individual *do* announce A for a tiny range of very high u-values, it will push an individual type-A person to announce A for a large range of u-values, thus rendering the cutoff $u_A = \infty$ "unstable".

First let us give an intuitive argument for this. Consider an individual of type A, and let us entertain a small perturbation in the strategy of her compatriots: they use a very large cutoff, but not an infinite one. Now, in the event that our agent is pivotal, it must be that her group is very large with high probability, because her compatriots are only participating to a tiny extent, and yet there are q participants in the pivotal case. This means that group A is likely to win (conditional on the pivotal event), and our individual will want to declare A for a large range of her u-values. This shows the "instability" of the cutoff $u_A = \infty$.

This argument has a clean counterpart in the formal analysis. Once we allow for compatriots (say, of type A) to use any interior cutoff u_A, we have $\lambda_A > 0$, so that (2) reduces to the simpler form

(4) $$\sum_{k=0}^{q} \binom{n-1-q}{k} \lambda_B^k (1-\lambda_A-\lambda_B)^{n-1-q-k}(1+u'_A) = (1-\lambda_A)^{n-1-q}.$$

where we're denoting our individual's cutoff by u'_A as a reminder that we haven't imposed the symmetry condition yet.

If we divide λ_B by $1-\lambda_A$, we form the probability that a randomly chosen person announces B conditional on her not announcing A. Let's call this probability π:

$$\pi \equiv \frac{\lambda_B}{1-\lambda_A}.$$

With this notation, (4) may be rewritten as

(5) $$\frac{1}{1+u'_A} = \sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k},$$

where m, it will be recalled, is the size of the supermajority ($n-q$ in other words). Now imagine that all compatriots have a very large cutoff, so that u_A is very big. Then λ_A is close to zero, so that $\pi \simeq \lambda_B$. So, by (5), u'_A is bounded. This means that the full-neutrality response is not robust to small perturbations away from full neutrality.

These arguments are *a fortiori* true in the special case of unanimity: $q = 0$. Indeed, it is easy to check that full neutrality is *never* a best response in this case, so no robustness arguments need to be invoked.

Note that invoking weak dominance does not rule out full neutrality. To see this consider the profile in which both groups use a cutoff of zero and so are always voting their type. In this case, when a voter of type A is pivotal, he knows for sure that there are more than q declarations of B. Therefore, this voter has a strict incentive to claim neutrality. Note however, that the above profile is the only profile against which neutrality is a strict best response for *every* type.

4.2.3. *Equilibrium Conditions.* In summary, then, the arguments of the previous section permit us to rewrite the equilibrium conditions (2) and (3) as follows:

$$\alpha(u_A, u_B) \equiv (1+u_A)\sum_{k=0}^{q}\binom{m-1}{k}\pi^k(1-\pi)^{m-1-k} = 1, \tag{6}$$

and

$$\beta(u_A, u_B) \equiv (1+u_B)\sum_{k=0}^{q}\binom{m-1}{k}\sigma^k(1-\sigma)^{m-1-k} = 1, \tag{7}$$

where $m = n - q$, $\pi \equiv \lambda_B/1{-}\lambda_A$, and $\sigma \equiv \lambda_A/1{-}\lambda_B$.

We dispose immediately of a simple subcase: the situation in which there is simple majority and n is odd, so that q precisely equals $(n-1)/2$. The following result applies:

Observation **1.** *If $q = (n-1)/2$, there is a unique equilibrium which involves $u_A = u_B = 0$.*

To see why this must be true, consult (6) and (7). Notice that when $q = (n-1)/2$, it must be that $m - 1 = n - q - 1 = q$. So the best responses must equal zero no matter what the size of the other group's cutoff. In words, there is no cost to announcing one's favorite outcome in this case. Recall that the only conceivable cost to doing so is that disagreement might result, but in the pivotal case of concern to any player, there are q compatriots announcing the favorite outcome, which means there are no more than $n - 1 - q = q$ opposing announcements. So disagreement is not a possibility.

In the remainder of the paper, then, we concentrate on the case in which a genuine supermajority is called for:

[A.4] $q < (n-1)/2$.

The following observations describe the structure of response functions in this situation. [A.1]–[A.4] hold throughout.

Observation **2.** *A symmetric response u_i is uniquely defined for each u_j, and declines continuously as u_j increases, beginning at some positive finite value when $u_j = 0$, and falling to zero as $u_j \to \infty$.*

Observation **3.** *Consider the point at which type A's response crosses the 45^0 line, or more formally, the value $\bar{u}$ at which $\alpha(\bar{u}, \bar{u}) = 1$. Then type B's best response cutoff to $\bar{u}$ is lower than $\bar{u}$, strictly so if $p < 1/2$.*

While the detailed computations that support these observations are relegated to the Appendix, a few points are to be noted. First, complete neutrality is never a (robust) best response even when members of the other group are *always* announcing their favorite alternative. The argument for this is closely related to the remarks made in Section 4.2.2 and we shall not repeat them here. On the other hand, "full aggression" — $u = 0$ — is *also* never a best response except in the limiting case as the other side tends to complete neutrality. These properties guarantee that every equilibrium (barring those excluded in Section 4.2.2) employs interior cutoffs.

Observation 3 requires some elaboration. It states that *at the point where the best response of Group A leaves both sides equally aggressive* (so that $u_A = u_B = \bar{u}$), group B's best response leads to greater aggression. The majority takes greater comfort from its greater number, and therefore are more secure about being aggressive. There is less scope for disagreement. However, note the emphasized qualification above. As we shall see later, it will turn out to be important.

Figure 1 provides a graphical representation. Each response function satisfies observation 2, and in addition observation 3 tells us that the response function for B lies above that for A at the 45^0 line. We have therefore established the following proposition.

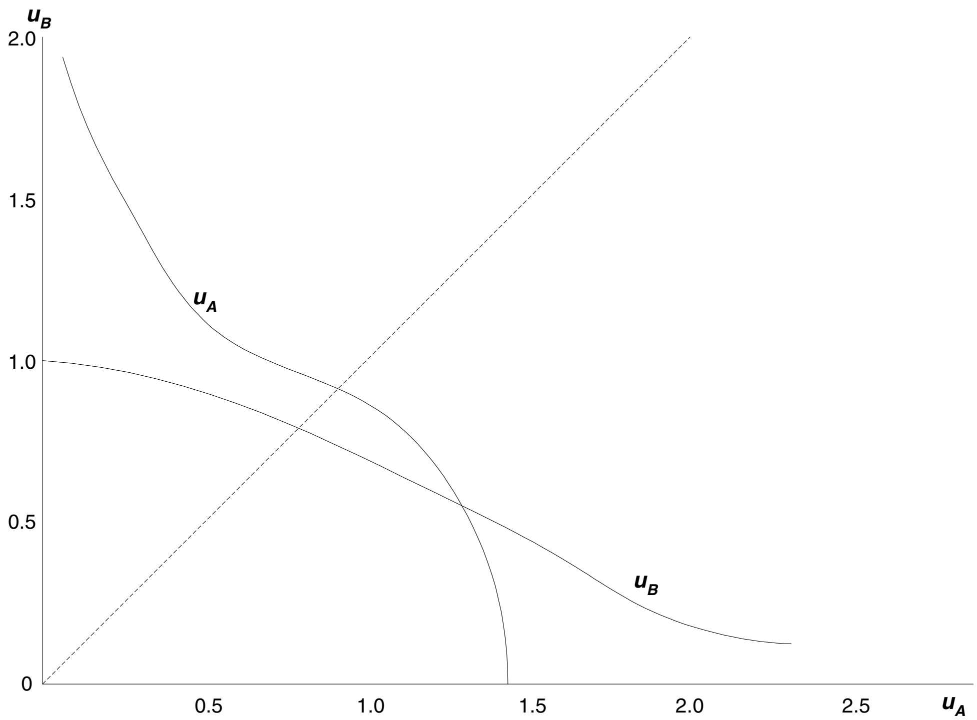

Figure 1. Existence of a Majority Equilibrium

Proposition **1.** *An equilibrium exists in which members of the stochastic majority — group B — behave more aggressively than their minority counterparts: $u_B < u_A$.*

Proposition 1 captures an interesting aspect of the "tyranny of the majority". Not only are the majority greater in number (at least stochastically so in this case), they are also more vocal in expressing their opinion. So group outcomes are doubly shifted — *in this particular equilibrium* — towards the majority view, once through numbers, and once through greater voice.[8] We will call such an equilibrium a *majority equilibrium*.

5. Minority Equilibria

5.1. **Existence.** Figure 1, which we used in establishing Proposition 1, is drawn from actual computation. We set $n = 4$, $p = 0.4$, $\nu = 1/4$, and chose F to be gamma with parameters (3,4). Under this specification, there is, indeed, a unique equilibrium and (by Proposition 1) it must be the majority equilibrium.

Further experimentation with these parameters leads to an interesting outcome. When n is increased, the response curves appear to "bend back" and intersect yet again, this time above the 45^0 line (see Figure 2). A *minority equilibrium* (in which $u_A < u_B$, so that the minority are more aggressive) makes its appearance. For this example, it does so when there are 12 players.

The bending-back of response curves to generate a minority equilibrium appeared endemic enough in the computations, that we decided to probe further. To do this, we study large populations in which the ratio of q to n is held fixed at $\nu \in (0, 1/2)$. More precisely, we look at

[8] Notice that this model has no voting costs so that free-riding is not an issue. Such free-riding is at the heart of the famous Olson paradox (see Olson (1965)), in which small groups may be more effective than their larger counterparts.

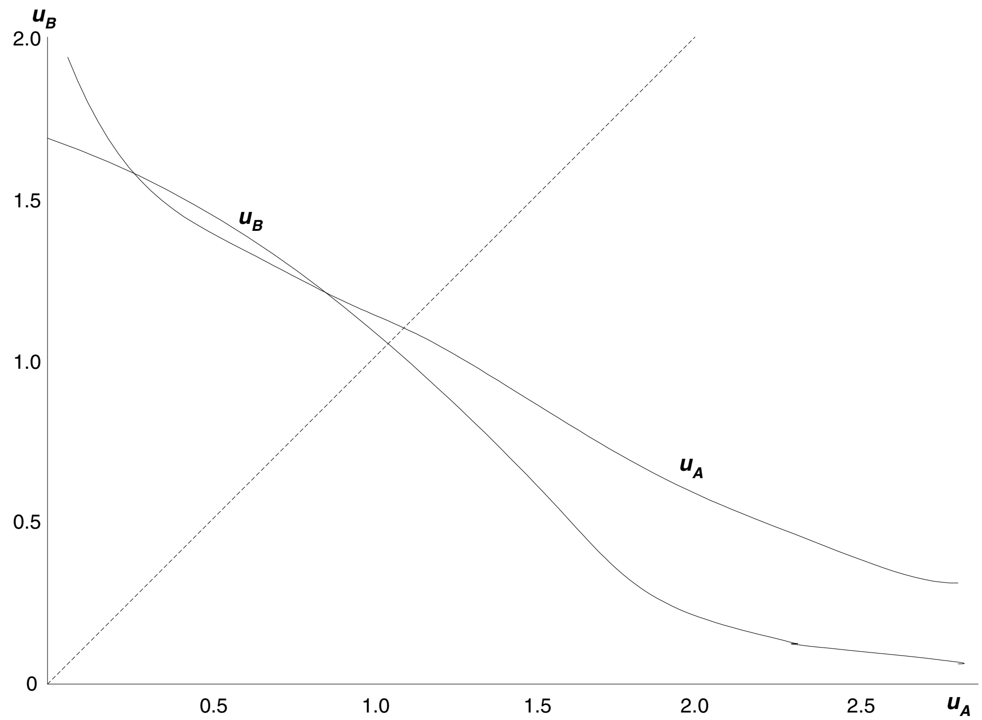

Figure 2. Minority Equilibrium

sequences $\{n, q\}$ growing unboundedly large so that q is one of the (at most) two integers closest to νn. We obtain the following analytical confirmation of the simulations:

Proposition **2.** *Assume that $0 < \nu < p \leq 1/2$. Consider any sequence $\{n, q\}$ such that $n \to \infty$ and q is one of the (at most) two integers closest to νn. Then there exists a finite N such that for all $n \geq N$, a minority equilibrium must exist.*

Several comments are in order. First, if there is a minority equilibrium, there must be at least two of them, because of the end point restrictions implied by Observations 2 and 3. Some of these equilibria will suffer from stability concerns similar to those discussed in Section 4.2.2. But there will always be other minority equilibria that are "robust" in this sense.[9]

Second, it might be felt that the threshold N described in Proposition 2 may be too large for "reasonable" group sizes. Our simulations reveal that this is not true. For instance, within the exponential class of valuation distributions, the threshold at which a minority equilibrium appears is typically around $N = 10$ or thereabouts, which is by no means a large number.

Third, the qualification that $\nu > 0$ is important. The unanimity case, with $q = 0$ is delicate. We return to this issue in Section 7. The case $p \leq \nu$, which we also treat in Section 7, is of interest as well.

Finally, as an aside, note that Proposition 2 covers the symmetric case $p = 1/2$, in which case the content of the proposition is that an asymmetric equilibrium exists (for large n). To be sure, the proposition is far stronger than this assertion, which would only imply (by continuity) that a minority equilibrium exists (with large n) if p is sufficiently close to $1/2$.

[9]Once again, this follows from the end-point restrictions.

5.2. **Discussion of the Existence Result.** We can provide some intuition as to why minority existence is guaranteed for large n but not so for small n. Observe that when n is "small", there are two sorts of uncertainties that plague any player. She does not know how many people there are of her type, and she is uncertain about the realized distribution of valuations. Both these uncertainties are troublesome in that they may precipitate costly disagreement. The possibility of disagreement is lowered by more and more people adopting a neutral stance, though after a point it will be lowered sufficiently so that it pays individuals to step in and announce their favorite outcome. For a member of the stochastic majority, this point will be reached earlier, and so a majority equilibrium will always exist.

On the other hand, when n is large, these uncertainties go away or at any rate are reduced. Now the expectation that the minority will be aggressive can be credibly self-fulfilling, because the expectation of an aggressive strategy can be more readily transformed into the expectation of a winning outcome. This intuition suggests that when the proportion of the minority is smaller than the superminority ratio, then minority equilibria do not exist for large n. This is confirmed in the following proposition.

Proposition **3.** *Assume that $0 < p < \nu < \frac{1}{2}$. Consider any sequence $\{n, q\}$ such that $n \to \infty$ and q is one of the (at most) two integers closest to νn. Then there exists a finite N such that for all $n \geq N$, a minority equilibrium does not exist.*

Taken together, Propositions 2 and 3 may suggest a monotonic relation between the supermajority requirement and the "power" of the minority. Common intuition suggests that a higher supermajority requirement facilitates the emergence of a minority equilibrium. Indeed, the comparative politics literature compares different political systems and motivates what has been termed "consensus systems" (Lijphart (1999)) by the desire to protect minorities from the tyrany of the majority.

However, this is generally false in our model. To see why, consider an individual of type A and her best response condition. As q decreases, A's cutoff increases (holding B's cutoff fixed), i.e., the group fights less aggressively. This follows from the fact that as q decreases, the probability that the B-types might block A increases. Because the above effect of lowering q applies to both groups, it is not clear which group benefits from this change.

To demonstrate the ambiguous effect of lowering q consider the following example: let $n = 1,000$ (in light of Proposition 5 we intentionally pick a large n), $p = 0.4$ and consider the distribution function $F(u) = 1 - \frac{1}{\sqrt{\ln(u+e)}}$. For $q = 300$ there exists a minority equilibrium $u_A \simeq 1.35$ and $u_B \simeq 80$. However, for $q = 10$ there exists no minority equilibrium.

The above example seems to suggest that for some distribution functions a minority equilibrium may not exist when the supermajority requirement is *at* unanimity. Indeed, this is true.

Proposition **4.** *Suppose that the distribution of u, $F(u)$, satisfies the condition*

$$\frac{f(x)}{1 - F(x)} \leq \frac{1}{(1+x)\ln(1+x)} \tag{8}$$

for all $x > 0$. Then in the case where $m = n$ — i.e., unanimity — a minority equilibrium cannot exist for any n.

Note that cdf from the above example, $F(u) = 1 - \frac{1}{\sqrt{\ln(u+e)}}$, satisfies the sufficient condition (8). Moreover, while conceivably not necessary, *some* condition is needed to rule out minority equilibria in the unanimity case: there do exist cdf's for which minority equilibria exist for all large n.[10]

[10] One example of such a cdf is the exponential distribution $F(u) = 1 - e^{-u}$.

Finally, compare and contrast our findings with the asymmetric equilibria in the Battle of the Sexes (BoS). Recall that analogues of those equilibria exist in this model as well, *but they have already been eliminated by the refinement introduced in Section 4.2.2*. One might suspect that the equilibria of our model converge (as n grows large) to the equilibria of the BoS game. In this sense, the equilibria could be perceived as purification of the BoS equilibria. However, Proposition 4 establishes that this is not the case. Indeed, in some cases, minority equilibria do not exist for any n. Hence, uncertainty plays a crucial role in our model. This conclusion will be further strengthened when we study limit outcomes in Section 6.

5.3. **Minorities Win in Minority Equilibrium.** In this section we address the distinction between an equilibrium in which one group *behaves* more aggressively, and one in which that group *wins* more often. For instance, in the majority equilibrium the majority fights harder *and* wins more often than the minority does. [It cannot be otherwise, the majority are ahead both in numbers and aggression.] But there is no reason to believe that the same is true of the minority equilibrium. The minority may be more aggressive, but the numbers are not on their side.

However, a remarkable property of this model is that a minority equilibrium *must involve the minority winning with greater probability than the majority*. Provided that a minority equilibrium exists, aggression must compensate for numbers.

Proposition **5.** *In a minority equilibrium, the minority outcome is implemented with greater probability than the majority outcome.*

This framework therefore indicates quite clearly how group behavior in a given situation may be swayed both by majority and minority concerns. When the latter occurs, it turns out that we have some kind of “tyranny of the minority": they are so vocal that they actually swing outcomes (in expectation) to their side.

The proof of this proposition is so simple that we provide it in the main text, in the hope that it will serve as its own intuition.

Proof. Recall (6) and (7) and note that $u_A < u_B$ in a minority equilibrium. It follows right away that $\sum_{k=0}^{q}\binom{m-1}{k}\pi^k(1-\pi)^{m-1-k} > \sum_{k=0}^{q}\binom{m-1}{k}\sigma^k(1-\sigma)^{m-1-k}$, so that $\pi < \sigma$. Expanding this inequality, we conclude that $\lambda_B(1-\lambda_B) < \lambda_A(1-\lambda_A)$. Because $\lambda_A < 1/2$, this can only happen in two ways: either $\lambda_B > 1-\lambda_A$, or $\lambda_B < \lambda_A$. The former case is impossible, because λ_A and λ_B describe mutually exclusive events, so the latter case must obtain. But this implies the truth of the proposition. ∎

6. Limit Equilibria

In Section 5.1 we established the existence of a minority equilibrium. Existence was guaranteed for large n and for all supermajority rules except for unanimity. As we've already remarked, there must be at least two such equilibria, while in addition we know that there is at least one majority equilibrium. This raises the question of what the set of equilibria look like as the group size grows without bound.

The purpose of this section is to prove that despite the possibly large multiplicity of equilibria for finite group size, there are exactly three limit outcomes. Two of these outcomes are “limit minority equilibria". Of the two, one exhibits a zero cutoff for the minority, and the other exhibits a positive minority cutoff which is nevertheless lower than the majority cutoff. The third outcome is a “limit majority equilibrium" in which the cutoff used by the majority is zero.

Moreover, the two corner equilibria (in which one side always fights for its favorite) possess a special structure: *the other side does not necessarily yield fully*. That is, the rival side may use an interior cutoff even in the limit, and we will characterize this cutoff exactly.

We will also study disagreement probabilities along any sequence of equilibria.

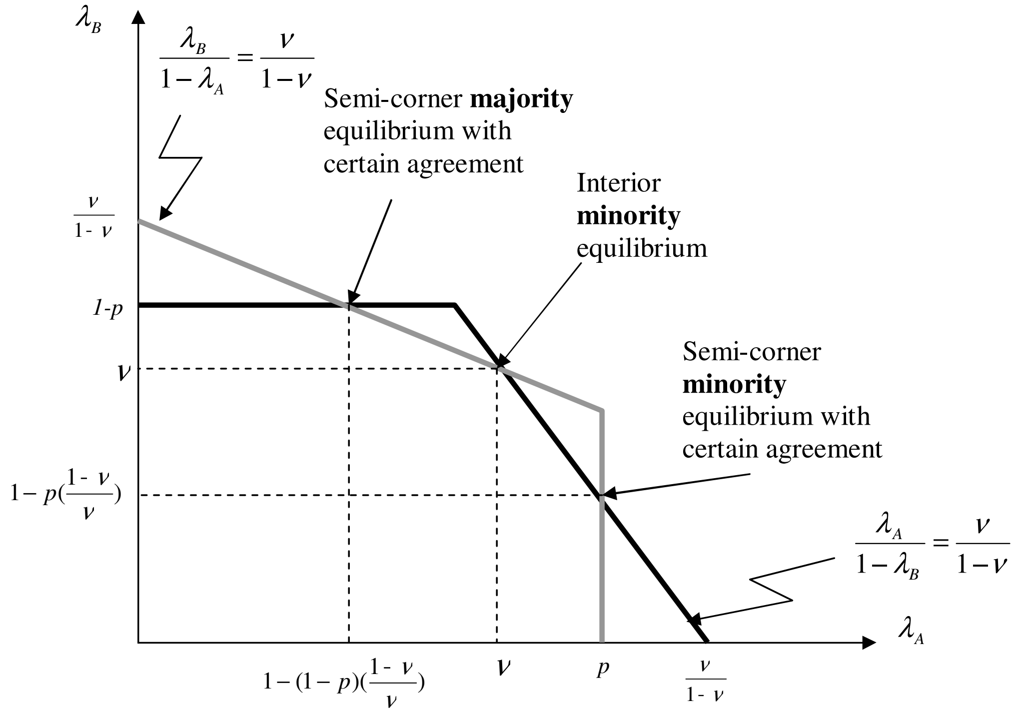

Figure 3. Limit Equilibrium Cutoffs

6.1. **A Characterization of Limit Outcomes.** We now study the various limit points of equilibrium cutoff sequences. We will denote a generic limit point by (u_A^*, u_B^*).

Proposition **6.** *Assume that* $\nu > 0$.

[1] *Suppose that* $(u_A^*, u_B^*) \gg 0$. *Then both limits must be finite, and solve*

$$p[1 - F(u_A^*)] = (1-p)[1 - F(u_B^*)] = \nu. \tag{9}$$

[2] *Suppose that* $u_A^* = 0$. *Then* $u_B^* < \infty$ *if and only if* $p < (1-\nu)/\nu$, *and in that case* u_B^* *is given by the condition*

$$F(u_B^*) = \frac{p(1-2\nu)}{(1-p)\nu}. \tag{10}$$

[3] *Likewise, suppose that* $u_B^* = 0$. *Then* $u_A^* < \infty$ *if and only if* $1 - p < (1-\nu)/\nu$, *and in that case* u_A^* *is given by the condition*

$$F(u_A^*) = \frac{(1-p)(1-2\nu)}{p\nu}. \tag{11}$$

[4] *Moreover, if* $p > \nu$, *each of the three configurations described above are limits for some sequence of equilibria.*

Proposition 6 is best understood by looking at Figure 3, which is drawn for the "semi-corner case" in which $\nu < p < 1-p < \nu/(1-\nu)$. This figure depicts the loci $\lambda_B/(1-\lambda_A) = \nu/(1-\nu)$ and $\lambda_A/(1-\lambda_B) = \nu/(1-\nu)$, suitably truncated to respect the constraints that $\lambda_A \leq p$ and $\lambda_B \leq 1-p$. We claim that limit equilibrium cutoffs must simultaneously lie on *both* these truncated loci. To see this, suppose that some cutoff sequence $\{\lambda_A^n, \lambda_B^n\}$ lies below the locus $\lambda_B/(1-\lambda_A) = \nu/(1-\nu)$ (along some subsequence, but retain the original index n). Then the equilibrium condition (6), coupled with the strong law of large numbers, assures us that $u_A^n \to 0$, or that $\lambda_A^n \to p$, which pulls the system back on to the locus. If, on the other hand, the cutoff sequence $\{\lambda_A^n, \lambda_B^n\}$ lies *above* the locus $\lambda_B/(1-\lambda_A) = \nu/(1-\nu)$, we have a contradiction as follows. First, by using (6) again, we may conclude that $\lambda_A^n \to 0$. Next, recall that $\lambda_B^n \leq 1-p < \nu/(1-\nu)$ (by assumption), but this and the previous sentence contradict the presumption that $\lambda_B^n/(1-\lambda_A^n) > \nu/(1-\nu)$ for all n.

Of course, the same sort of argument applies to both loci, so we may conclude that equilibrium cutoffs must converge to one of three intersections displayed in Figure 3.[11]

The last part of the proposition asserts that when minority equilibria exist for large n, each of the three cases indeed represent “bonafide" limit points, in that each case is an attractor for some sequence of equilibria. For the majority corner, this is obvious, as majority equilibria always exist and no sequence of majority equilibria can ever converge to a minority outcome. That the other two limits are also non-vacuous follow from the proof of existence of minority equilibria (the reader is invited to study the formal arguments in Section 9).

6.2. **Disagreement.** One important implication of Proposition 6 is that even when there is little uncertainty regarding the size of each faction, both sides may still put up a fight. In particular, when $1-p < \frac{1-\nu}{\nu}$ *all* limit equilibria consist of “fighting" on both sides. This raises the question of whether disagreement is bound to occur in large populations.

Proposition **7.** *Assume $\nu > 0$.*

[1] *Suppose that $\nu < p < \frac{1-\nu}{\nu}$ and let u_B^* be the limit cutoff value that solves (10). Then in the limit semi-corner equilibrium $(0, u_B^*)$ both sides agree with certainty.*

[2] *Assume $1-p < \frac{1-\nu}{\nu}$ and let u_A^* be the limit cutoff value that solves (11). Then in the limit semi-corner equilibrium $(u_A^*, 0)$ both sides agree with certainty.*

[3] *Consider any sequence of equilibria $(u_A^n, u_B^n) \to (u_A^*, u_B^*)$ where u_A^* and u_B^* solve (9). Then the probability of disagreement along that sequence is bounded away from one.*

The proofs of [1] and [2] follow immediately by looking at Figure 3. At the semi-corner minority equilibrium the proportion of A votes is simply p, which is strictly greater than ν. The proportion of B votes is $1-p[(1-\nu)/\nu]$, which is strictly smaller than ν. It follows that in the limit A is the unique eligible alternative, and hence that A will be implemented with certainty. Analogous arguments show that in the semi-corner majority equilibrium, B is the unique eligible alternative.

The proof of [3] is more involved. Recall that in this case the proportion of A and B votes converges to the superminority requirement ν. One may be tempted to conclude that the probability of disagreement in this case must converge to $\frac{1}{4}$. A closer examination reveals that this may not be the case. Indeed, what is important in determining the probability of disagreement is not the mere convergence of λ_A and λ_B to ν, but their *rate* of convergence. So far, the equilibrium conditions do not allow us to pin down the probability of disagreement in this case. Still, we establish that this probability is bounded away from one.

The intuition for this result is the following. Suppose that the probability of disagreement is high. Then the probability that each group is blocking the supermajority of its rival is also high. In particular, this means that group cutoffs are not wandering off to infinity. On the other hand,

[11] It is also possible to construct versions of this diagram for the other cases, such as $1-p > \nu/(1-\nu)$ but $p < \nu/(1-\nu)$.

we can see that if group A, for example, is blocking group B, then the latter will be discouraged from making a B announcement. Doing so will most likely lead to disagreement, while casting a neutral vote ensures an agreement on A. This argument makes for high cutoffs, a contradiction to the bounded group cutoffs that were asserted earlier in this paragraph.

In part, the formalization of the above intuition is easy, but the simultaneous movements in population size and cutoffs necessitate a subtle argument. In particular, the last implication — that cutoffs become large with population size — rests on arguments regarding *rates* of change as a function of population. The reader is referred to the formal proof for details.

What allows individuals to agree, even when there are great many of them, is the option to remain neutral. This can be seen if we analyze a restricted version of our model in which individuals have only two options: A or B. We carry out this analysis in Section 7.1. There, we show that Proposition 7 ceases to hold.

Finally, note that the case of unanimity is *not* covered here. This question remains open.

7. Extensions

7.1. **No Neutrality .** In our opinion, when faced with impending disagreement, the option of a neutral stance is very natural. This is why we adopted this specification in our basic model. [As discussed already, neutrality is not to be literally interpreted as a formal announcement.] Nevertheless, it would be useful to see if the insights of the exercise are broadly preserved if announcements are restricted to be either A or B.

We can quickly sketch such a model. An individual is now pivotal under two circumstances. In the first event, the number of people announcing her favorite outcome is exactly q, which we assume to be less than $(n-1)/2$.[12] By announcing her favorite, then, disagreement is the outcome, while an announcement of the other alternative would lead to that alternative being implemented. The loss, then, from voting one's favorite in this event is precisely v' (recall that the disagreement payoff is normalized to zero). In the second event, the number of people announcing the alternative is exactly q. By announcing her favorite, she guarantees its implementation, while the other announcement would lead to disagreement. So the gain from voting one's favorite in this event is v. Consequently, an individual will announce her favorite if

$$\Pr(\text{exactly } q \text{ others vote for alternative})v \geq \Pr(\text{exactly } q \text{ others vote for favorite})v'.$$

Define $w \equiv v/v'$. Then equilibrium cutoffs w_A and w_B are given by the conditions

$$w_A \Pr(|B| = q) \geq \Pr(|A| = q) \tag{12}$$

and

$$w_B \Pr(|A| = q) \geq \Pr(|B| = q) \tag{13}$$

where $|A|$ and $|B|$ stand for the number of A- and B-announcements out of $n-1$ individuals, and where equality must hold in each of the conditions provided the corresponding cutoff strictly exceeds 1, which is the lower bound for these variables.

In this variation of the model, it is obvious that at least one group must be "fully aggressive" (i.e., its cutoff must equal one).[13] Moreover, as long as we are in the case $q < (n-1)/2$, *both* groups cannot *simultaneously* be "fully aggressive": one of the cutoffs must strictly exceed unity.

So, in contrast to our model, in which all (robust) equilibria are fully interior, the equilibria here are at "corners" (full aggression on one side, full acquiescence on the other) or "semi-corners"

[12] The case $q = (n-1)/2$ is exactly the same as in Observation 1 for the main model. No matter what the valuations are, each individual will announce her favorite outcome.

[13] Simply examine (12) and (13) and note that both right-hand sides cannot strictly exceed one.

(full aggression on one side, interior cutoffs on the other). The semi-corner equilibria are always robust in the sense of Section 4.2.2, and we focus on these in what follows.[14]

In particular, to examine possible minority equilibria, set $w_A = 1$. Then use the equality version of (13) to assert that

$$w_B = \left(\frac{p + (1-p)H(w_B)}{(1-p)[1-H(w_B]} \right)^{n-1-2q} \tag{14}$$

in any such equilibrium, where H is the (assumed atomless) cdf of w, distributed on its full support $[1, \infty)$.

It is easy to use (14) to deduce

Observation **4.** [1] *A semi-corner minority equilibrium exists if (n, q) are sufficiently large.*

[2] *In any minority equilibrium, the minority outcome is implemented with greater probability than the majority outcome.*

So the broad contours of our model can be replicated in this special case. This is reassuring, because it reassures us of the robustness of the results. At the same time this variation allows us to highlight the main implication of allowing voters to remain neutral: absent neutrality voters may be locked into situations in which they are almost certain to disagree. This is formalized in the next result.

Observation **5.** *Assume $0 < \nu < p < \frac{1}{2}$. Consider any sequence $\{n, q\}$ such that $n \to \infty$ and q is one of the two integers closest to νn. Then there exists a sequence of semi-corner minority equilibria for which the probability of disagreement coverges to one.*

The above result demonstrates the importance of being neutral: neutrality allows the players to avoid disagreement. Recall that Proposition 7 establishes that with neutrality, the probability of disagreement at every interior equilibrium is bounded away from one. Once the option of neutrality is taken away, the probability that players reach a disagreement (at any interior equilibrium) must go to one along some sequence of minority equilibria.

7.2. **Known Group Size.** Our model as developed has the potential drawback that the instance of a known group size is not a special case. More generally, individuals may have substantial information regarding the ordinal stance of others (though still remaining unsure of their cardinal preferences).[15]

One way to accommodate this concern is to amend the model to posit a probability distribution $\theta(n_A)$ over the number n_A of A-types in the population. [The current specification of cardinal intensities may be retained.] This has the virtue of nesting our current model as well as known group size as special cases.[16] In addition, the basic structure of our model is easily recreated in this more general setting. For instance, if θ exhibits full support, a similar robustness argument applies to eliminate the "coordination-failure" corner equilibria, and downward-sloping "reaction functions", as in Figure 1, may be constructed just as before. The concept of a stochastic minority can also be easily extended. However, there are interesting conceptual issues involved in *changing* group size: in particular, we will need to specify carefully how θ alters in the process.

While a full analysis of this model is "beyond the scope of the current paper", we provide some intuition by studying the extreme case in which group size is known; i.e., $\theta(n_A) = 1$ precisely at some integer $n_A < n/2$. We retain all our other assumptions.

[14] In contrast to our setup, the "full corner" equilibria may or may not be robust. We omit the details of this discussion.

[15] In our current model, such "substantial information" is only possible if p is close to either 0 or 1.

[16] In the current model, $\theta(n_A) = \binom{n}{n_A} p^{n_A} (1-p)^{n-n_A}$ for some $p \in (0, 1/2)$.

Of course, θ no longer has full support, so the arguments in Section 4.2.2 do not apply to this case. To see why, consider the case when all B types are voting for B, whereas only extreme A-types are voting for A. When an A-type knows exactly how many B-types there are, he realizes that he can only create a disagreement by voting for A. Therefore, when group sizes are known, the two corner equilibria are robust (in the sense of Section 4.2.2). This suggests that the corner equilibria are unnatural in the following sense: when faced with some uncertainty about group sizes, some individuals may still put up a fight.

A further observation relates to the importance of group size in the emergence of minority equilibria. Potentially, the existence of minority equilibria in our original model may be due to two types of uncertainties that are relaxed in large groups. First, as the number of individuals in the group increases, voters have a more accurate estimate of the proportion of their types in the group. Second, as the population increases, each individual has a better picture of the distribution of intensities among his compatriots.

What if group sizes are known? Then it can easily be shown that the equilibrium cutoff for one type depend only on the equilbrium cutoff of the other type. More precisely, an equilibrium (u_A, u_B) satisfies the following equations,

$$(1+u_A)\sum_{k=0}^{q}\binom{n_B}{k}(F(u_B))^{n_B-k}(1-F(u_B))^k = 1$$
$$(1+u_B)\sum_{k=0}^{q}\binom{n_A}{k}(F(u_A))^{n_A-k}(1-F(u_A))^k = 1$$

where $n_A < n_B$ are the number of individuals of type A and B respectively.

It is straightforward to construct examples in which there does not exist a minority equilibrium for small n_A and n_B. For instance, take $F(u) = 1 - \frac{1}{\sqrt{\ln(u+e)}}$, $n_A = 2, n_B = 3$ and $q = 1$. For these values there exists a unique interior majority equilibrium, $u_A \approx 250$ and $u_B \approx 0.22$. However, using arguments similar to those employed in Proposition 2 and 4, one can show that for large n a minority equilibrium exists and the probability of disagreement is bounded away from one. By simple stochastic dominance arguments, it can be shown that in any minority equilbrium the minority wins more often.

We conclude that certainty regarding the numbers of A and B types is not sufficient to generate a minority equilibrium; even when the numbers of A and B types are known, we still need n to be sufficiently large for the minority to prevail.

7.3. **Types who Prefer Disagreement to the Rival Alternative.** Suppose there exist types who rank disagreement above their second best alternative. Clearly, voting for the preferred alternative is weakly dominant for these types. Hence, in any interior equilibrium these individuals would vote their type. In this sense, incorporating these voters into our model is equivalent to adding aggregate noise. We believe that if the proportion of such types is sufficiently low, all of our results continue to hold.

8. Summary

We study a model of group decision-making in which one of two alternatives must be chosen. While group members differ in their valuations of the alternatives, everybody prefers some alternative to disagreement.

We uncover a variant on the “tyranny of the majority": there is always an equilibrium in which the majority is more aggressive in pushing its alternative, thus enforcing their will via both numbers and voice. However, under very general conditions an aggressive minority equilibrium inevitably makes an appearance, provided that the group is large enough. This equilibrium displays a “tyranny of the minority": it is always true that the increased aggression of the minority more than compensates for smaller number, leading to the minority outcome being implemented with larger probability than the majority alternative.

These equilibria are not to be confused with “corner" outcomes in which a simple failure of coordination allows any one group to be fully aggressive and another to be completely timid, without regard to group size. Indeed, one innovation of this paper is to show how such equilibria are entirely non-robust when confronted with varying intensities of valuations, and some amount of uncertainty regarding such valuations. In fact, as we emphasize in the paper, minority equilibria don’t always exist: they don’t exist, in general, for low population sizes and in the unanimity case they may not exist for *any* population size.

We also fully characterize limit outcomes as population size goes to infinity. We show that there are exactly three limit outcomes to which all equilibria must converge. Two of these outcomes are “limit minority equilibria". Of the two, one exhibits a zero cutoff for the minority, and the other exhibits a positive minority cutoff which is nevertheless lower than the majority cutoff. The third outcome is a “limit majority equilibrium" in which the cutoff used by the majority is zero. The two corner equilibria which display full aggression on one side do not, in general, force complete timidity on the rival side. We provide a complete characterization by providing necessary and sufficient conditions for the interiority of such cutoffs and describing exactly their values.

Finally, we address the question of disagreement as group size grows large. We show that the probability of disagreement must converge to zero along all equilibrium sequences that converge to the semi-corners identified above. For those equilibria that converge to the remaining interior minority outcome, we show that the probability of disagreement is bounded away from one as the population size goes to infinity. The option to remain neutral is crucial in obtaining this result. Observation 5 in Section 7 considers an extension in which the neutrality option is removed, and proves that there is always a sequence of equilibria (in group size) along which the probability of disagreement must converge to one.

9. Proofs

Proof of Observation 2. For concreteness, set $i = A$ and $j = B$. Fix any $u_B \in [0, \infty)$. Recall that

$$\pi = \frac{\lambda_B}{1-\lambda_A} = \frac{(1-p)[1-F(u_B)]}{1-p[1-F(u_A)]},$$

so that π is continuous in u_A, with $\pi \to 1 - F(u_B)$ as $u_A \to 0$, and $\pi \to (1-p)[1-F(u_B)]$ as $u_A \to \infty$. Consequently, recalling (6) and noting that $q < (n-1)/2$, we see that $\alpha(u_A, u_B)$ converges to a number strictly less than one as $u_A \to 0$, while it becomes unboundedly large as $u_A \to \infty$. By continuity, then, there exists some u_A such that $\alpha(u_A, u_B) = 1$, establishing the existence of a cutoff.

To show uniqueness, it suffices to verify that α is strictly increasing in u_A. Because the expression $\sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k}$ must be decreasing in π, it will suffice to show that π itself is declining in u_A, which is a matter of simple inspection.

To show that the response u_A strictly decreases in u_B, it will therefore be enough to establish that α is also increasing in u_B. Just as in the previous paragraph, we do this by showing that π is decreasing in u_B, which again is a matter of elementary inspection.

Finally, we observe that $u_A \downarrow 0$ as $u_B \uparrow \infty$. Note that along such a sequence, $\pi \to 0$ regardless of the behavior of u_A. Consequently, $\sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k}$ converges to 1 as $u_B \uparrow \infty$. To maintain the equality (6), therefore, it must be the case that $u_A \downarrow 0$.

Of course, all these arguments hold if we switch A and B. ∎

Proof of Observation 3. Let $\bar{u}$ be defined as in the statement of this Observation. Define $\bar{\lambda}_A \equiv p[1 - F(\bar{u})]$ and $\bar{\lambda}_B \equiv (1-p)[1 - F(\bar{u})]$. Then

$$(1+\bar{u}) \sum_{k=0}^{q} \binom{m-1}{k} \bar{\pi}^k (1-\bar{\pi})^{m-1-k} = 1, \tag{15}$$

where $\bar{\pi} \equiv \bar{\lambda}_B/(1-\bar{\lambda}_A)$. Now recall that σ in (7) is defined by $\sigma = \frac{\lambda_A}{1-\lambda_B}$, so that if we consider the corresponding value $\bar{\sigma}$ defined by setting $u_A = u_B = \bar{u}$, we see that

$$\bar{\sigma} \le \bar{\pi} \text{ if and only if } \bar{\lambda}_A(1-\bar{\lambda}_A) \le \bar{\lambda}_B(1-\bar{\lambda}_B).$$

But $\lambda_A \le 1/2$ (because $p \le 1/2$), so that the second inequality above holds if and only if $\bar{\lambda}_A \le \bar{\lambda}_B$, and this last condition follows simply from the fact that $p \le 1/2$.

So we have established that $\bar{\sigma} \le \bar{\pi}$. It follows that

$$\sum_{k=0}^{q} \binom{m-1}{k} \bar{\pi}^k (1-\bar{\pi})^{m-1-k} \le \sum_{k=0}^{q} \binom{m-1}{k} \bar{\sigma}^k (1-\bar{\sigma})^{m-1-k}$$

and using this information in (15), we must conclude that

$$\beta(\bar{u},\bar{u}) = (1+\bar{u}) \sum_{k=0}^{q} \binom{m-1}{k} \bar{\sigma}^k (1-\bar{\sigma})^{m-1-k} \ge 1, \tag{16}$$

Recalling that β is increasing in its first argument (see proof of Observation 2), it follows from (16) that type B's best response to $\bar{u}$ is no bigger than $\bar{u}$.

Finally, observe that all these arguments apply with strict inequality when $p < 1/2$. ∎

Proof of Proposition 1. For each $u_B \ge 0$, define $\phi(u_B)$ by composing best responses: $\phi(u_B)$ is B's best response to A's best response to u_B. By Observation 2, we see that A's best response is a positive, finite value when $u_B = 0$, and therefore so is B's response to this response. Consequently, $\phi(0) > 0$. On the other hand, A's best response is precisely $\bar{u}$ when $u_B = \bar{u}$, and by Observation 3 we must conclude that $\phi(\bar{u}) < \bar{u}$. Because ϕ is continuous (Observation 2 again), there is $u_B^* \in (0,\bar{u})$ such that $\phi(u_B^*) = u_B^*$. Let u_A^* be type A's best response to u_B^*. Then it is obvious that (u_A^*, u_B^*) is an equilibrium. Because $u_B^* < \bar{u}$, we see from Observation 2 that $u_A^* > \bar{u}$. We have therefore found a majority equilibrium. ∎

Proposition 2 and some subsequent arguments rely on the following lemma.

Lemma **1.** *Consider any sequence $\{n,q\}$ such that $n \to \infty$ and q is one of the two integers closest to νn. For any $\bar{u}_A$ satisfying*

$$p\left[1 - F\left(\bar{u}_A\right)\right] > \nu, \tag{17}$$

there exists a finite N such that for all $n \ge N$, $\hat{u}_B^n > u_B^n > \bar{u}_A$ where u_B^n solves (6) with $u_A = \bar{u}_A$, and $\hat{u}_B^n$ solves (7) with $u_A = \bar{u}_A$.

Proof. Consider any sequence $\{n,q\}$ as described in the statement of the lemma. Because $p > \nu$, there exists a range of positive cutoff values satisfying inequality (17). Consider any such value $\bar{u}_A$ and denote $\bar{\lambda}_A \equiv p\left[1 - F\left(\bar{u}_A\right)\right]$. There exists a finite n^* such that for all $n \ge n^*$,

$$\bar{\lambda}_A > \frac{q}{n-1} \simeq \nu$$

Note that there is also an associated sequence $\{m\}$ defined by $m_n \equiv n - q$.[17]

We break the proof up into several steps.

Step 1. We claim that there exists an integer M such that for each $m \geq M$ there is $u_B^m < \infty$ that solves the following equation:

$$\sum_{k=0}^{q} \binom{m-1}{k} (\pi_m)^k (1-\pi_m)^{m-1-k} = \frac{1}{1+\bar{u}_A} \tag{18}$$

where

$$\pi_m \equiv \frac{\lambda_B^m}{1-\bar{\lambda}_A}$$

and

$$\lambda_B^m \equiv (1-p)\left[1 - F\left(u_B^m\right)\right].$$

We prove this claim. Note that for all $n \geq n^*$, $1-p \geq p > q/(n-1)$, so that

$$\bar{\pi} \equiv \frac{(1-p)(n-1)}{m-1} > \frac{q}{m-1} \simeq \frac{\nu}{1-\nu}$$

for all $n \geq n^*$. Consequently, by the Strong Law of Large Numbers (SLLN),

$$\sum_{k=0}^{q} \binom{m-1}{k} \bar{\pi}^k (1-\bar{\pi})^{m-1-k} \to 0$$

as m and q grow to infinity. It follows that there exists M such that for all $m \geq M$ (and associated q),

$$\sum_{k=0}^{q} \binom{m-1}{k} \bar{\pi}^k (1-\bar{\pi})^{m-1-k} < \frac{1}{1+\bar{u}_A}. \tag{19}$$

For such m, provisionally consider $u_B^m = 0$. Then

$$\frac{\lambda_B^m}{1-\bar{\lambda}_A} = \frac{1-p}{1-p\left[1-F\left(\bar{u}_A\right)\right]},$$

and using this in (17), we conclude that

$$\pi_m = \frac{\lambda_B^m}{1-\bar{\lambda}_A} = \frac{1-p}{1-p\left[1-F\left(\bar{u}_A\right)\right]} > \frac{(1-p)(n-1)}{m-1} = \bar{\pi}.$$

Combining this information with (19), we see that if $u_B^m = 0$, then

$$\sum_{k=0}^{q} \binom{m-1}{k} \pi_m^k (1-\pi_m)^{m-1-k} < \frac{1}{1+\bar{u}_A}. \tag{20}$$

Next, observe that if u_B^m is chosen very large, then λ_B^m and consequently π_m are both close to zero, so that $\sum_{k=0}^{q} \binom{m-1}{k} \pi_m^k (1-\pi_m)^{m-1-k}$ is close to unity. It follows that for such u_B^m,

$$\sum_{k=0}^{q} \binom{m-1}{k} \pi_m^k (1-\pi_m)^{m-1-k} > \frac{1}{1+\bar{u}_A}. \tag{21}$$

Combining (20) and (21) and noting that the LHS of (18) is continuous in u_B^m, it follows that for all $m \geq M$ there exists $0 < u_B^m < \infty$ such that the claim is true.

[17] While correct notation would demand that we denote this sequence by m_n, we shall use the index m for ease in writing.

Step 2. One implication of (18) in Step 1 is the following assertion: as $(m,q)\to\infty$,

(22) $$\pi_m \to \nu/(1-\nu)\in(0,1),\ \text{and in particular, } u_B^m \text{ is bounded.}$$

To see why, note that $\frac{1}{1+\bar{u}_A}\in(0,1)$. Using (18) and SLLN, it must be that $\pi_m\to\nu/(1-\nu)\in(0,1)$ as $(m,q)\to\infty$. Recalling the definition of π_m it follows right away that u_B^m must be bounded.

Step 3. Next, we claim there exists an integer M^* such that

(23) $$\text{For all } m\geq M^*, u_B^m>\bar{u}_A.$$

To establish this claim, note first, using (17), that

$$p\left[1-F\left(\bar{u}_A\right)\right]>\frac{q}{n-1}=\frac{\frac{q}{m-1}}{1+\frac{q}{m-1}}\geq\frac{\frac{q}{m-1}}{\frac{1-p}{p}+\frac{q}{m-1}}$$

where the last inequality follows from the assumption that $p\in\left(0,\frac{1}{2}\right]$, so that $\frac{1-p}{p}\geq 1$. A simple rearrangement of this inequality shows that

(24) $$\frac{(1-p)\left[1-F\left(\bar{u}_A\right)\right]}{1-p\left[1-F\left(\bar{u}_A\right)\right]}>\frac{q}{m-1}\simeq\frac{\nu}{1-\nu}$$

Now suppose, contrary to the claim, that $u_B^m\leq\bar{u}_A$ along some subsequence of m. Then on that subsequence,

(25) $$\pi_m=\frac{\lambda_B^m}{1-\bar{\lambda}_A}=\frac{(1-p)\left[1-F\left(u_B^m\right)\right]}{1-p\left[1-F\left(\bar{u}_A\right)\right]}\geq\frac{(1-p)\left[1-F\left(\bar{u}_A\right)\right]}{1-p\left[1-F\left(\bar{u}_A\right)\right]}$$

Combining (24) and (25), we may conclude that along the subsequence of m for which $u_B^m\leq\bar{u}_A$,

$$\inf_m\pi_m>\frac{\nu}{1-\nu},$$

which contradicts (22) of Step 2.

To prepare for the next step, let $\hat{u}_B^m$ denote the best response of the B-types to $u_A=\bar{u}_A$. That is,

(26) $$\frac{1}{1+\hat{u}_B^m}=\sum_{k=0}^{q}\binom{m-1}{k}\sigma_m^k\left(1-\sigma_m\right)^{m-1-k},$$

where

$$\sigma_m\equiv\frac{\bar{\lambda}_A}{1-\hat{\lambda}_B^m}$$

and

$$\hat{\lambda}_B^m\equiv(1-p)\left[1-F\left(\hat{u}_B^m\right)\right]$$

Step 4. There is an integer M^{**} such that for all $m\geq M^{**}$, $\hat{u}_B^m>u_B^m$.

To prove this claim, suppose on the contrary that $\hat{u}_B^m\leq u_B^m$ along some subsequence of m. [All references that follow are to this subsequence.] Then

(27) $$\sigma_m=\frac{\bar{\lambda}_A}{1-\hat{\lambda}_B^m}=\frac{p\left[1-F\left(\bar{u}_A\right)\right]}{1-(1-p)\left[1-F\left(\hat{u}_B^m\right)\right]}\geq\frac{p\left[1-F\left(\bar{u}_A\right)\right]}{1-(1-p)\left[1-F\left(u_B^m\right)\right]}=\frac{\bar{\lambda}_A}{1-\lambda_B^m}.$$

Recall from (22), Step 2, that $\frac{\lambda_B^m}{1-\bar{\lambda}_A}\to\frac{\nu}{1-\nu}$. Therefore $\lambda_B^m\to\bar{\lambda}_B$, where $\bar{\lambda}_B\equiv\frac{\nu}{1-\nu}\left(1-\bar{\lambda}_A\right)$. Recall from (17) that $\bar{\lambda}_A>\nu$, so that $\bar{\lambda}_B<\nu$ and in particular $\bar{\lambda}_B<\bar{\lambda}_A$. Because $p\leq 1/2$, so is $\bar{\lambda}_A$, and these last assertions permit us to conclude that $\bar{\lambda}_A\left(1-\bar{\lambda}_A\right)>\bar{\lambda}_B\left(1-\bar{\lambda}_B\right)$, or equivalently, that

$$\frac{\bar{\lambda}_A}{1-\bar{\lambda}_B}>\frac{\bar{\lambda}_B}{1-\bar{\lambda}_A}.$$

Using this information in (27) and recalling that $\lambda_B^m \to \bar{\lambda}_B$, we may conclude that

$$\lim \inf_{m\to\infty} \sigma_m \geq \frac{\bar{\lambda}_A}{1-\bar{\lambda}_B} > \frac{\bar{\lambda}_B}{1-\bar{\lambda}_A} = \frac{\nu}{1-\nu},$$

where the last equality is from (22). It follows from (26) that $\hat{u}_B^m \to \infty$. But this contradicts our supposition that $\hat{u}_B^m \leq u_B^m$ (that along a subsequence) because the latter is bounded; see (22) of Step 2. ∎

Proof of Proposition 2. Consider any sequence $\{n, q\}$ as described in the statement of the proposition. Choose some cutoff $\bar{u}_A$ that satisfies (17). By Lemma 1, there is an integer N such that for all $n \geq N$, $\hat{u}_B^n > u_B^n > \bar{u}_A$. Define, for each $n \geq N$ and each $u_A \in (0, \bar{u}_A]$, $\psi^n(u_A)$ as the *difference* between B's best response to u_A and the value of u_B to which u_A is a best response. By Lemma 1 and Observation 2, ψ^n is well-defined and continuous on this interval. Using Observation 2 yet again, it is easy to see that (for each n) $\psi^n(u_A) < 0$ for small values of u_A, while the statement of Lemma 1 assures us that $\psi^n(\bar{u}_A) > 0$. Therefore for each n, there is $\tilde{u}_A^n \in (0, \bar{u}_A)$ such that $\psi^n(\tilde{u}_A^n) = 0$. If we define $\tilde{u}_B^n$ to be the best response to $\tilde{u}_A^n$, it is trivial to see that $(\tilde{u}_A^n, \tilde{u}_B^n)$ constitutes an equilibrium.

Finally, note that

$$\tilde{u}_A^n < \bar{u}_A < u_B^n < \hat{u}_B^n < \tilde{u}_B^n,$$

where the second and third inequalities are a consequence of Lemma 1, and the last inequality comes from the fact that the best response function is decreasing (Observation 1). This means that $(\tilde{u}_A^n, \tilde{u}_B^n)$ is a minority equilibrium. ∎

Proof of Proposition 3. Suppose on the contrary that a minority equilibrium (u_A^n, u_B^n) exists along some subsequence of n (all references that follow are to this subsequence). Then $\lim_{n\to\infty}(u_A^n, u_B^n)$ is either (∞, ∞), $(0, \infty)$ or a pair of strictly positive but finite numbers (u_A^*, u_B^*). To prove that our supposition is wrong, we show that none of these limits can apply.

Assume $(u_A^n, u_B^n) \to (\infty, \infty)$. Then $\lambda_A^n \to 0$ and $\lambda_B^n \to 0$. This implies that $\pi^n \to 0$ and $\sigma^n \to 0$. But this implies, by equations (6) and (7) and using SLLN, that $(u_A^n, u_B^n) \to (0, 0)$, a contradiction.

Assume $(u_A^n, u_B^n) \to (0, \infty)$. Then $\lambda_A^n \to p$ and $\lambda_B^n \to 0$, so that $\sigma^n \to p < \nu < \frac{q}{m-1}$. But using (7) and SLLN, this implies that $u_B^n \to 0$, a contradiction.

Assume $(u_A^n, u_B^n) \to (u_A^*, u_B^*)$, where both u_A^* and u_B^* are strictly positive and finite. Using SLLN and equations (6) and (7, it follows that π^n and σ^n must both converge to $\frac{q}{m-1}$. This means that $\lambda_A^n \to \lambda_A^*$ and $\lambda_B^n \to \lambda_B^*$ such that

$$\frac{\lambda_B^*}{1-\lambda_A^*} = \frac{\lambda_A^*}{1-\lambda_B^*}$$

This equlity holds only if $\lambda_A^* = \lambda_B^*$, or if $\lambda_A^* = 1 - \lambda_B^*$. Suppose the former is true. Then $\pi^n \to \pi^*$ where

$$\pi^* = \frac{\lambda_B^*}{1-\lambda_A^*} < \frac{\nu}{1-\nu} \simeq \frac{q}{m-1}$$

But the above inequality implies, by (6) and SLLN, that $u_A^n \to 0$, a contradiction. Suppose next that $\lambda_A^* = 1 - \lambda_B^*$. But $1 - \lambda_B^* > p > \lambda_A^*$, a contradiction. ∎

Proof of Proposition 4. Under unanimity, (6) and (7) reduce to

$$\frac{1}{1+u_A} = (1-\pi)^{n-1} \tag{28}$$

and

$$\frac{1}{1+u_B} = (1-\sigma)^{n-1}. \tag{29}$$

For any given n and $k = A, B$, define $y_k \equiv (1 + u_k)^{1/(n-1)}$. Then $y_k \geq 1$, and (28) and (29) may be rewritten as

$$1 - \pi = \frac{1}{y_A} \tag{30}$$

and

$$1 - \sigma = \frac{1}{y_B}. \tag{31}$$

Recalling that $\pi = \lambda_B/(1-\lambda_A)$ and $\sigma = \lambda_A/(1-\lambda_B)$, we may use (30) and (31) to solve explicitly for λ_A and λ_B. Doing so and writing out λ_k for $k = A, B$, we see that

$$\lambda_A = p[1 - F(u_A)] = \frac{y_B - 1}{y_A + y_B - 1}, \tag{32}$$

while

$$\lambda_B = (1 - p)[1 - F(u_B)] = \frac{y_A - 1}{y_A + y_B - 1}. \tag{33}$$

By multiplying both sides of (32) by $1 - F(u_B)$ and both sides of (33) by $1 - F(u_A)$ and using the fact that $p < 1 - p$, we may conclude that

$$[1 - F(u_B)] \left[(1 + u_B)^{1/(n-1)} - 1\right] < [1 - F(u_A)] \left[(1 + u_A)^{1/(n-1)} - 1\right] \tag{34}$$

We will now prove that $u_A > u_B$. Given (34), it will suffice to prove that

$$[1 - F(x)] \left[(1 + x)^{1/(n-1)} - 1\right]$$

is nondecreasing in x. This, in turn, is implied by the stronger observation that

$$\frac{d}{dx}[1 - F(x)] \left[(1 + x)^{1/(n-1)} - 1\right] \geq 0$$

for every $x > 0$, or equivalently, that

$$\frac{f(x)}{1 - F(x)} \leq \frac{\theta(1 + x)^{\theta - 1}}{(1 + x)^{\theta} - 1}, \tag{35}$$

where $\theta \equiv \frac{1}{n-1} \in (0, 1]$.

To this end, we demonstrate that for all $x > 0$ and $\theta \in (0, 1]$,

$$\frac{\theta(1 + x)^{\theta - 1}}{(1 + x)^{\theta} - 1} \geq \frac{1}{(1 + x)\ln(1 + x)} \tag{36}$$

To establish (36), note that for fixed $x > 0$, $h(\theta) \equiv (1 + x)^{\theta}$ is differentiable and convex in x. By a standard property of differententiable convex functions, $h(\theta_1) - h(\theta_2) \leq h'(\theta_1)(\theta_1 - \theta_2)$ for all θ_1 and θ_2. Applying this inequality to the case $\theta_1 = \theta$ and $\theta_2 = 0$, we may conclude that

$$h(\theta) - h(0) = (1 + x)^{\theta} - 1 \leq h'(\theta)\theta = (1 + x)^{\theta} \ln(1 + x)\theta,$$

and a quick rearrangement of this inequality produces (36).

To complete the proof, combine (8) and (36) to obtain (35). ∎

Proof of Proposition 6. Recall the conditions describing equilibrium cutoffs:

$$\frac{1}{1 + u_A} = \sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1 - \pi)^{m-1-k}$$

and

$$\frac{1}{1+u_B} = \sum_{k=0}^{q} \binom{m-1}{k} \sigma^k (1-\sigma)^{m-1-k}.$$

For each integer n (with associated m and q) and every $u \geq 0$, define a function $h(u,n)$ by the condition that

$$\sum_{k=0}^{q} \binom{m-1}{k} h(u,n)^k (1-h(u,n))^{m-1-k} \equiv \frac{1}{1+u}.$$

Note that h is well-defined for each (u,n). With this in hand, we may rewrite the equilibrium conditions more succintly as

$$\frac{\lambda_B^n}{1-\lambda_A^n} = \pi^n = h(u_A^n, n) \equiv \alpha^n \tag{37}$$

and

$$\frac{\lambda_A^n}{1-\lambda_B^n} = \sigma^n = h(u_B^n, n) \equiv \beta^n, \tag{38}$$

where we are now starting to index all endogenous variables by n in order to prepare for sequences of equilibria. Solving these two equations for λ_A^n and λ_B^n, we see that

$$\lambda_A^n = p[1-F(u_A^n)] = \frac{\beta^n(1-\alpha^n)}{1-\alpha^n\beta^n} \tag{39}$$

and

$$\lambda_B^n = (1-p)[1-F(u_B^n)] = \frac{\alpha^n(1-\beta^n)}{1-\alpha^n\beta^n}. \tag{40}$$

We now study various limits of equilibrium cutoff sequences. We will denote the limits in all cases by (u_A^*, u_B^*). The following lemma summarizes simple properties of h and will be used throughout.

Lemma **2.** [1] *For every n, h is strictly increasing in u, with $h(0,n)=0$ and $h(u,n)\to 1$ as $n\to\infty$.*

[2] *If u^n converges to u with $0<u<\infty$, then $\lim_{n\to\infty} h(u^n,n) = \nu/(1-\nu)$.*

[3] *If u^n converges to 0 then $\limsup_{n\to\infty} h(u^n,n) \leq \nu/(1-\nu)$.*

[4] *If $u^n\to\infty$, then $\liminf_{n\to\infty} h(u^n,n) \geq \nu/(1-\nu)$.*

The proof of this lemma follows from routine computations and the use of the law of large numbers, and is omitted.

Now we prove part [1] of the proposition. First, we claim that u_A^* and u_B^* are finite. For suppose, say, that $u_A^* = \infty$ (the argument in the other case is identical). It follows from (39) that either α^n has a limit point at 1, or that β^n has a zero limit point. The latter possibility is ruled out by Lemma 2, because $u_B^* > 0$ by assumption. It follows that $\limsup_{n\to\infty}\alpha^n = 1$, but but then Lemma 2 assures us that $u_B^* = \infty$ as well.

The first of the two conclusions in the preceding sentence implies that $\limsup_{n\to\infty}\lambda_B^n = 1$ (use (38)), but the second conclusion implies that $\lim_{n\to\infty}\lambda_B^n = 0$ (use (40)). These two implications contradict each other.

So $0 \ll (u_A^*, u_B^*) \ll \infty$, but we know then from lemma 2 that $(\alpha^n, \beta^n) \to (\nu,\nu)$ as $n\to\infty$. Simple computation using (39) and (40) then yields (9). It should be noted that this limit (which is unique in the class of strictly positive limits) has $u_A^* < u_B^*$; that is, it is a “limit" minority equilibrium.

Next, we prove part [2]; the proof of part [3] is completely analogous. Suppose, then, that $u_A^* = 0$. We first prove the sufficiency of the restriction on p. To this end, assume that $u_B^* = \infty$.

Consider some subsequence in which α^n and β^n converge (to some α^* and β^*). Then (39) implies that

$$\frac{\beta^*(1-\alpha^*)}{1-\alpha^*\beta^*} = p. \tag{41}$$

while at the same time, (40) implies that

$$\frac{\alpha^*(1-\beta^*)}{1-\alpha^*\beta^*} = 0, \tag{42}$$

(42) implies either that $\alpha^* = 0$ or that $\beta^* = 1$. But the latter cannot happen, for then (41) cannot be satisfied (note that the LHS of (41) is well-defined even when $\beta^* = 1$, because $\alpha^* < 1$ by Lemma 2). So it must be that $\alpha^* = 0$. But then (41) implies that $p = \beta^*$. Lemma 2 tells us that $\beta^* \geq \nu/(1-\nu)$, so that $p \geq \nu/(1-\nu)$.

Conversely, suppose that $u_B^* < \infty$. Again, consider some subsequence in which α^n and β^n converge to some α^* and β^*. Therefore (39) implies that

$$\frac{\beta^*(1-\alpha^*)}{1-\alpha^*\beta^*} = p. \tag{43}$$

while (40) implies that

$$\frac{\alpha^*(1-\beta^*)}{1-\alpha^*\beta^*} = (1-p)[1-F(u_B^*)]. \tag{44}$$

We can eliminate α^* from this system. We also note that by Lemma 2, β^* must equal $\nu/(1-\nu)$. Using these observations along with some routine computation, we obtain precisely (10).

We also know that $F(u_B^*) < 1$. Using this information in (10), we may conclude that $p < \nu/(1-\nu)$.

Finally, we establish part [4]. Assume, to the contrary, there exists no sequence of equilibria whose limit is given by the first configuration. By parts [2] and [3] of the proposition, the limit of any sequence of minority equilibria has either $u_A^* = 0$ or $u_A^* > u_B^*$. To reach a contradiction, pick any $u_A > 0$ satisfying (17). By Lemma 1, there exists an integer N such that for all $n \geq N$, there exists a minority equilibrium (u_A^n, u_B^n) with $u_A^n > u_A$. From Proposition 1 it follows that for any $p < \frac{1}{2}$ and for any n, there does not exist a pair of numbers (u, u) that solve the equilibrium conditions (6) and (7). We therefore conclude that for all $n \geq N$, there exists a minority equilibrium (u_A^n, u_B^n) with $0 < u_A < u_A^n < u_B^n$, in contradiction to our initial assumption.

Suppose next that there exists no sequence of equilibria whose limit is given by the second configuration. Then by parts [1] and [3] of the proposition, the limit of any minority equilibrium must satisfy that $u_A^* \geq \nu > 0$. Let $\epsilon \in (0, \nu)$. By Lemma 1, there exists a finite $N > 0$ such that for all $n \geq N$ there exists a minority equilibrium (u_A^n, u_B^n) with $u_A^n < \epsilon$. But this means that the limit of any such sequence cannot satisfy that $u_A^* \geq \nu$, a contradiction.

Finally, assume there exists no sequence of equilibria whose limit is given by the third configuration. This implies, by [1] and [2], that the limit of any sequence of equilibrium cutoffs has $u_A^* < u_B^*$. But this contradicts Proposition 1, which states that for every n there exists an equilibrium with $u_A^n > u_B^n$. ∎

Proof of Proposition 7. The proofs of [1] and [2] are given in the discussion following the statement of the proposition in the text. We now proceed to prove [3]. Assume that $q < \frac{n-1}{2}$ (When $q = \frac{n-1}{2}$ the probability of disagreement is zero). Note that the probability of disagreement is equal to $\Pr(|A| > q, |B| > q)$, where $|.|$ stands for cardinality. Because

$$\Pr(|A| > q, |B| > q) \leq \min\{\Pr(|A| > q), \Pr(|B| > q)\},$$

it suffices to show that $\Pr(A > q)$ and $\Pr(B > q)$ cannot both converge to one along some subsequence of n.

Suppose, on the contrary, that $\Pr(A > q)$ and $\Pr(B > q)$ do converge to one along some subsequence of n (retain notation). The proof proceeds in two steps. In the first step we show that for large n both λ_A and λ_B are strictly above ν. Moreover, if either λ_A or λ_B converges to ν, then it converges at a rate slower than $\frac{1}{\sqrt{n}}$. In the second step we show that this implies that the equilibrium cutoffs, u_A and u_B, must be growing to infinity, in contradiction to step 1.

Step 1. $\lim_{n\to\infty} \frac{(\lambda_A-\nu)\sqrt{n}}{\sqrt{\lambda_A(1-\lambda_A)}} = \infty$ and $\lim_{n\to\infty} \frac{(\lambda_B-\nu)\sqrt{n}}{\sqrt{\lambda_B(1-\lambda_B)}} = \infty$.

We prove $\lim_{n\to\infty} \frac{|\lambda_A-\nu|\sqrt{n}}{\sqrt{\lambda_A(1-\lambda_A)}} = \infty$; similar arguments hold for λ_B.

Assume to the contrary that there exists a subsequence for which $\lim_{n\to\infty} \frac{(\lambda_A^{k_n}-\nu)\sqrt{n}}{\sqrt{\lambda_A(1-\lambda_A)}} = c$, where $-\infty \le c < \infty$.

Let X_n denote the number of A announcements (i.e., $|A|$). By the Berry-Esséen Theorem (see, for example, Feller (1986, Chapter XVI.5, Theorem 1)), for some $\varepsilon < \Phi(-c)$, there exists an N such that for $n > N$

$$\Pr(X_n > q) = \Pr\Big(\frac{X_n - n\lambda_A^{k_n}}{\sqrt{n\lambda_A^{k_n}(1-\lambda_A^{k_n})}} > \frac{-(\lambda_A^{k_n}-\nu)\sqrt{n}}{\sqrt{\lambda_A^{k_n}(1-\lambda_A^{k_n})}}\Big) < 1 - \Phi(-c) + \varepsilon < 1$$

and this contradicts our premise that $\lim_{n\to\infty} \Pr(|A| > q) = 1$.

Recalling that $\pi = \frac{\lambda_B}{1-\lambda_A}$ and $\sigma = \frac{\lambda_A}{1-\lambda_B}$, it follows from step 1 that $\lim_{n\to\infty} \frac{(\pi-\frac{\nu}{1-\nu})\sqrt{n}}{\sqrt{\pi(1-\pi)}} = \infty$ and $\lim_{n\to\infty} \frac{(\sigma-\frac{\nu}{1-\nu})\sqrt{n}}{\sqrt{\sigma(1-\sigma)}} = \infty$.

Step 2. If $\lim_{m\to\infty} \frac{(\pi-\frac{\nu}{1-\nu})\sqrt{m-1}}{\sqrt{\pi(1-\pi)}} = \infty$ and $\lim_{m\to\infty} \frac{(\sigma-\frac{\nu}{1-\nu})\sqrt{m-1}}{\sqrt{\sigma(1-\sigma)}} = \infty$, then $u_A \longrightarrow \infty$ and $u_B \longrightarrow \infty$.

As in step 1 we provide a proof for u_A and similar arguments follow for u_B.

Let Y_n be the sum of successes from a binomial distribution with probability of success π and with $m-1$ draws. Then

$$\sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k} = \Pr(Y_n \le q) \le \Pr(|Y_n - (m-1)\pi| \ge (m-1)\pi - q)$$
$$< \frac{Var(Y_n)}{((m-1)\pi - q)^2} = \frac{1}{\Big(\frac{(\pi-\frac{q}{m-1})\sqrt{m-1}}{\sqrt{\pi(1-\pi)}}\Big)^2} \to 0,$$

where the last inequality is by Chebyshev's inequality and the limit follows from the premise. Therefore, by (6) it must be that $u_A \to \infty$. This implies that $\lambda_A \to 0$, in contradiction to step 1. ∎

Proof of Observation 4. To prove part [1], define $\delta \equiv 1/(n-1-2q)$, and rewrite (13) as

$$(1 + w_B^{\delta})[1 - H(w_B)] = 1/(1-p). \tag{45}$$

Notice that when $w_B = 1$, the LHS of (45) equals 2, while the RHS is strictly smaller than 2 (because $p < 1/2$).

Now suppose that there is some w such that the LHS of (45), evaluated at $w_B = w$, is strictly less than $1/(1-p)$. In this case, consider some intersection $x = w_B$ of the function $(1+x^\delta)[1-H(x)]$ with the value $1/(1-p)$, along with the value $w_A = 1$. It can be verified that such an intersection constitutes a semi-corner minority equilibrium.

It remains to show that the condition in the first line in the previous paragraph is satisfied for all (n,q) large enough. To this end, fix some w such that $1-H(w) < 1/2(1-p)$. Now take (n,q) to infinity and notice that $\delta \to 0$. Therefore w^δ converges to 1. It follows that for large (n,q),

$$(1+w^\delta)[1-H(w)] < 1/(1-p),$$

and we are done.

Note that part [2] is trivially true for corner minority equilibria. To prove part [2] for semi-corners, note that the probability that the minority outcome is implemented is given by

$$\Pr\left(|A| \geq m\right) = \sum_{k=m}^{n} \binom{n}{m} \left[p + (1-p)H\left(w_B\right)\right]^k \left[(1-p)\left(1-H\left(w_B\right)\right)\right]^{n-k}$$

Similarly,

$$\Pr\left(|B| \geq m\right) = \sum_{k=m}^{n} \binom{n}{m} \left[(1-p)\left(1-H\left(w_B\right)\right)\right]^k \left[p + (1-p)H\left(w_B\right)\right]^{n-k}$$

Thus, $\Pr\left(|A| \geq m\right) > \Pr\left(|B| \geq m\right)$ if and only if $(1-p)\left(1-H\left(w_B\right)\right) < p + (1-p)H\left(w_B\right)$, which may be rewritten as

$$\frac{1}{2(1-p)} > 1 - H\left(w_B\right) \tag{46}$$

Now (45) tells us that

$$1 - H\left(w_B\right) = \frac{1}{(1-p)\left(1+w_B^\delta\right)}$$

where $w_B > 1$. Hence, $(1-p)\left(1+w_B^\delta\right) > 2(1-p)$, which implies (46). ∎

Proof of Observation 5. Let w_B^* be the solution to the following equation:

$$p + (1-p)H(w_B^*) = (1-p)\left[1 - H(w_B^*)\right]$$

Notice that w_B^* is well-defined and greater than 1, as long as $p < 1/2$. We now proceed in two steps.

Step 1. There exists a sequence of semi-corner minority equilibria that converges to $(1, w_B^*)$. To see this, note that when $w_B = w_B^*$ the RHS of (14) is smaller than the LHS. For any $\varepsilon > 0$, set $w_B = w_B^* + \varepsilon$. Because $\frac{p+(1-p)H\left(w_B^*+\varepsilon\right)}{(1-p)\left[1-H\left(w_B^*+\varepsilon\right)\right]} > 1$, there exists $N(\varepsilon) < \infty$ such that for all $n \geq N(\varepsilon)$, the LHS of (14) is strictly greater than its RHS. It follows that for all $n \geq N(\varepsilon)$, there exists an equilibrium $(1, w_B^n)$ where $w_B^n \in (w_B^*, w_B^* + \varepsilon)$.

Step 2. By Step 1, as $n \to \infty$, the probabilities with which a random voter votes for A or for B (along the above sequence of semi-corner minority equilibria) both converge to $1/2$. In particular, there exists an N above which these probabilities are bounded below by $\bar{\nu} > \nu$ and above by $1-\bar{\nu}$. The probability of disagreement is equal to $1 - \Pr\left(|A| \geq m\right) - \Pr\left(|B| \geq m\right)$. We now show that $\Pr\left(|A| \geq m\right)$ goes to zero as $n \to \infty$. By essentially the same argument, $\Pr\left(|B| \geq m\right)$ also goes to zero as $n \to \infty$.

Recall that

$$\Pr\left(|A| \geq m\right) = \sum_{k=m}^{n} \binom{n}{m} \left[p + (1-p)H\left(w_B\right)\right]^k \left[(1-p)\left(1-H\left(w_B\right)\right)\right]^{n-k}$$

Note that $\left|\frac{m}{n} - (1-\nu)\right| < \frac{1}{n}$. Because $1-\bar{\nu} < 1-\nu$ it follows that for large enough n,

$$1 - \bar{\nu} < \frac{m}{n} - \eta \tag{47}$$

for some $\eta > 0$. By stochastic dominance,

$$\Pr\left(|A| \geq m\right) \leq \sum_{k=m}^{n} \binom{n}{m} (1-\bar{\nu})^k \left(\bar{\nu}\right)^{n-k} \tag{48}$$

By inequality (47) and the SLLN, the RHS of (48) goes to zero. ∎

References

(1) Araki, K. and T. Börgers (1996), “How Minorities can Win Majority Votes," mimeo.
(2) Buchanan, J. M. and G. Tullock (1962), *The Calculus of Consent*, Ann Arbor: University of Michigan Press.
(3) Casella, A. (2003), "Storable Votes," NBER Working Paper 9189.
(4) Campbell, C. M. (1999), “Large Electorates and Decisive Minorities," *Journal of Political Economy* **107**, 1199–1217.
(5) Dodgson, C. (1884), *The Principles of Parliamentary Representation*, London: Harrison and Sons (supplement, 1885).
(6) Feller, W. (1986), *An Introduction to Probability Theory and its Applications*, Volume II, New Delhi: Wiley Eastern Limited.
(7) Gerber, E., Morton, R. and T. Rietz (1998), “Minority Representation in Multi-Member Districts," *American Political Science Review* **92**, 127–144.
(8) Haan, M. and P. Kooreman (2003), “How Majorities can Lose the Election: Another Voting Paradox," *Social Choice and Welfare* **20**, 509–522.
(9) Hortala-Vallve, R. (2004), “Qualitative Voting," mimeo.
(10) Jackson, M. and H. Sonnenschein (2003), “Overcoming Incentive Constraints," mimeo., Caltech.
(11) Kahneman D., D. Schkade and C.R. Sunstein (1996), “Shared Outrage and Erratic Awards: The Psychology of Punitive Damages,” *Journal of Risk and Uncertainty* **49**.
(12) Ledyard, J. O. (1984), “The Pure Theory of Large Two Candidate Elections," *Public Choice* **44**, 7–41.
(13) Lijphart, A. (1999), *Patterns of Democracy: Government Forms and Performance in Thirty-Six Countries*, New Haven and London: Yale University Press.
(14) Olson, M. (1965), *The Logic of Collective Action: Public Goods and the Theory of Groups*, Cambridge, MA: Harvard University Press.
(15) Palfrey, T. R. and H. Rosenthal (1983), “A Strategic Calculus of Voting," *Public Choice* **41**, 7–53.
(16) Pareto, V. (1906) 1927. *Manual of Political Economy*, New York: A.M. Kelley.
(17) Philipson, T. and J. Snyder (1996), “Equilibrium and Efficiency in an Organized Vote Market," *Public Choice* **89**, 245–265.
(18) Piketty, T. (1999), “Information Aggregation through Voting and Vote Trading," mimeo., MIT.

(19) Ponsati C. and J. Sàkovics (1996), "Multiperson Bargaining over Two Alternatives," *Games and Economics Behavior* **12**, 226–244.

(20) Schkade, D., C.R. Sunstein and D. Kahneman (1999), "Are Juries Less Erratic than Individuals? Deliberation, Polarization and Punitive Damages," *Columbia Law Review*.

(21) Sunstein, C.R., D. Kahneman and D. Schkade (1998), "Assessing Punitive Damages," *Yale Law Journal*.